Indivisibilities

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Hagen Bobzin

Indivisibilities

Microeconomic Theory with Respect to Indivisible Goods and Factors

With 116 Figures

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Preface

The analysis of this volume represents an attempt to apply modern mathematical techniques to the problems arising from large and significant indivisibilities. While the classical microeconomic theory refers to assumptions about the convexity of production sets and consumer preferences, this book directs the attention to indivisible commodities. It investigates the influence of the assumed indivisibilities of factors and goods on the results of the microeconomic theory of the firm, the theory of the household and market theory.

In order to quantify the relationships between economic variables and among economic actors the theory is founded on convex analysis. Hence, many results heavily depend on the approximation of integer sets by their convex hull. As far as possible numerous figures are provided to develop the reader's geometric intuition.

My intention is to continue with FRANK's (1969) beginnings of a general, systematic and rigorous analysis of the problems of indivisibilities. The advantage of the formalized way chosen is that the numerous and detailed properties of the economic relationships can be deduced on the basis of very few assumptions. It is not surprising that at least within the market theory the most important assumption concerning indivisible goods is that at least one commodity is perfectly divisible.

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Siegen
February 1998

Hagen BOBZIN

Contents

Chapter I.

Microeconomic Theory with Respect to Indivisibilities

Indivisible goods and factors constitute a subject of economic theory associated with a series of unsolved problems. Even advanced works on microeconomic theory like VARIAN (1992) or JEHLE (1991) refrain from the consideration of indivisible goods and factors to provide a structure for the analysis where relatively simple mathematical methods can be applied. The *Handbook of Mathematical Economics* also does not contain any approach treating explicitly the integer problem. While GREEN, HELLER (1981) present the instrument of convex analysis with respect to economic applications in the first chapter, a corresponding work dealing with the indivisibility of goods and factors is missing. Even in BROWN (1991), who introduces an equilibrium analysis with nonconvex technologies in Chapter 36, the problem of indivisibility is merely of minor importance.

However, *The New Palgrave: A Dictionary of Economics*[1] contains an explicit article on indivisibilities written by BAUMOL. Apart from some hints on integer programming[2], the author cites only one work which explicitly deals with analyzing indivisibilities. In this book, which was already published in 1969, FRANK, as the first economist, presented a comprehensive analysis of the importance of indivisible goods in production theory. His approach identifies the problem of indivisibility with goods which are only available at integer amounts.

While FRANK explicitly picks up the problem of indivisibility, there is a wealth of approaches including only indirectly the problem of indivisible goods and factors. For example, ROSEN (1974) describes markets for a class of indivisible goods which can be distinguished by certain features. At the same time it is assumed that there is a sufficiently large number of these differentiated goods such that the choice of characteristics may vary continuously.

Although ROSEN does not pursue the aim to describe a market for a

[1] See EATWELL, MILGATE, NEWMAN (1987).

[2] The mentioned approaches to integer programming of GOMORY (1965) and GOMORY, BAUMOL (1960) will not be important until Chapter V (of the book).

few indivisible goods, his procedure clarifies various aspects in dealing with indivisibilities. If the solutions to optimization problems are subject to certain integer constraints, then the analytical effort ascertaining an optimal solution rises considerably in relation to "well behaved" convex problems. The alternative determination of "rounded results" ignoring the required integer values may lead to considerable errors. Hence, the decision to solve a simple but incorrect problem or an exact but costly problem depends on the return of the additional effort.[3] At the same time we expect that the return tends to depend on the relative size of the indivisible goods. The difference between the production of 100 000 or 100 001 cars is of little significance for an automobile company, whereas a household faces considerable consequences depending on whether it has got a car or not.[4]

If, like VIETORISZ (1963), we concentrate on all-or-northing decisions or on investments where we have to decide on production levels, location, and timing, then the problem of nonconvexities is not only a mathematical curiosity but it plays a rather considerable role in daily economic practice. However, microeconomic theory frequently ignores the problem of indivisible goods and factors in view of the easier analytical instruments. In particular, such favorable properties as continuity, convexity, or differentiability of functions justify this procedure as long as phenomena like indivisibilities are only of minor importance.[5] However, when indivisibilities have great importance, then we usually have to refrain from making use of the above advantages. The analysis does not only get a new look but also leads to modified results.

For example, suitable assumptions assure in economic bibliography an exact duality between the firm's production function and the cost function. Accordingly, each "well behaved" production function is associated with a unique cost function, from which we can infer back to the production function in a unique way. If we now give up the assumption of divisible goods and factors or the corresponding assumptions of convexity, then the exact one-to-one relation between the above functions is no longer valid. In view of suitable approximations it can now be examined to what extent the relationship is abolished. In Chapter III the difference to the traditional analysis by the modified behavior of the factor demand becomes most apparent. Chapter IV also indicates that many results of traditional theory are repudiated under consideration of indivisibilities. Because the existence of general equilibria bases crucially on corresponding assumptions of convexity – hence divisible goods – an existence proof considering indivisible goods only succeeds under very restrictive assumptions. Thus, the analysis concentrates on the question of how large the fault is when the requirement of integer values is ignored. If no general equilibrium exists, then these faults yield a measure of the importance of the indivisibility of goods.

The analysis refers to microeconomic issues, where the consideration of

[3]Considering the mathematical difficulties, KOOPMANS, BECKMANN (1957) suggest commencing the analysis with extremely simplified economic problems until more realistic questions are investigated.

[4]A similar characterization of indivisible goods is given by BAUMOL (1987).

[5]DIEWERT (1986) ignores completely the problem of indivisible goods and factors, although the analysis of welfare effects is based on large and significant investments in infrastructure.

indivisibilities involves two aspects. On the one hand many basic results remain unchanged; thus, they are independent of the assumptions concerning the divisibility of goods and factors. For example, each firm's profit maximizing activity has to be technically efficient, no matter whether goods and factors are perfectly divisible or not (Proposition II.10). On the other hand numerous results are no longer valid so that it has to be questioned whether these statements are simply inappropriate, whether they can be superseded by more general results, or whether completely new phenomena must be taken into account. For divisible goods a utility maximizing household can choose a commodity bundle balancing its budget, whereas there need not be such a commodity bundle for indivisible goods. Moreover, the assumption of differentiable (single-valued or vector-valued) demand functions must be abandoned. But in the more general approach with (multi-valued) demand correspondences the statement remains valid that a household tends to demand less of a commodity if the corresponding price increases.

A new phenomenon is added in form of excess capacities for indivisible goods.[6] Anyone, who buys an (indivisible) car to drive one hour per day, owns a car being unused 23 hours per day and, therefore, yielding no utility during this time. Moreover, the transport capacity of the car for four or more people is usually not exhausted. Conversely, each single trip for an individual person requires the availability of the whole capacity of the car.

Analytically, the assumption that at least one good – for example money – is perfectly divisible will be of crucial importance. In this case three cases must be distinguished: (1) All goods are divisible. (2) At least one commodity is divisible. (3) All goods are indivisible. As will be shown, the first two cases are more closely related to each other under many analytical aspects than the second and the third case. Empirically, the second case ought to be relevant.

Figure I.1 sketches the procedure of the book, where the boxes with the dotted lines emphasize the minor importance of these aspects within the presented analysis. Beginning with the description of households and firms, Chapter II introduces the fundamentals of treating indivisible goods and factors within the framework of the theory of individual economic agents. At the same time the instrument of the convex hull is introduced, which serves as a replacement for nonconvex sets.[7] Correspondingly, Appendix A introduces basic concepts of convex analysis.

Chapter III concentrates firstly on the theory of the firm and deals with dual relations in microeconomic theory. The firm's production structure introduced in Chapter II is now compared to a cost structure allowing an equivalent representation of firm's production technology. In this way, the used duality schemes highlight

[6] If the supply of an indivisible good has excess capacities, then it may be interpreted as a public good as long as no consumer can be excluded from the consumption of this good. KLEINDORFER, SERTEL (1994) refer to prisons or incinerator plants as an example for indivisible public goods, where one of several communities must supply the (locally unwanted) good. The other communities must not be excluded from using the good, but they must contribute to its costs. This aspect of a public good is not taken into account throughout this book.

[7] Analogous to BAUMOL (1987, p. 795), the problem of indivisible goods is associated immediately with integer programming. Chapter V discusses the relationship of the chosen approach with integer programming in more detail.

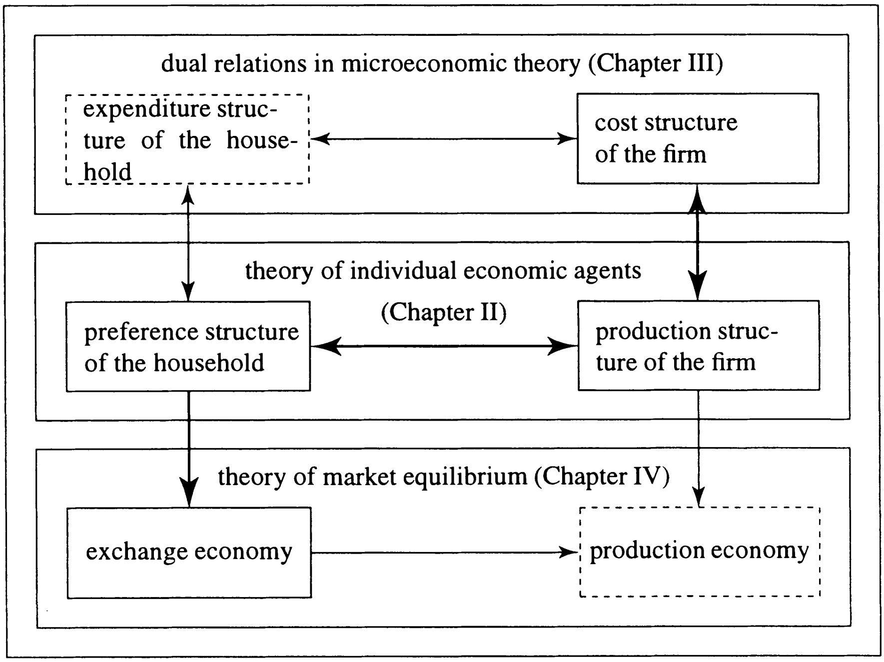

Figure I.1: Organization of the book

various aspects of optimal activities. The numerous relations in Section III.2 are summarized in Figures III.29 and III.37, the former being based on the works of SHEPHARD (1953). An example stressing the indivisibility of factors will be given accompanying the two Sections III.1 and III.2. The graphical representations do not only serve to illustrate the analytical results but also as a comparison of the findings gained in Sections III.1 and III.2. The parallel derived Appendices D.1 and D.2 refer chiefly to ROCKAFELLAR (1972) and introduce the analytical framework of the presented duality theory.

Finally, Chapter III turns again to the theory of households, where the preparation of Chapter VI is concentrated on the various aspects of market equilibria rather than the symmetric treatment regarding the theory of the firm. The derived properties of the demand for commodities are used in Chapter IV to answer the question concerning the existence of general exchange equilibria.[8] At this point the fixed-point theorems presented in Appendix C.2 will be important. The closing sections on the existence of equilibria in production economies and on alternative criteria for optimal market results serve more as an outlook than as a comprehensive treatment of the respective question.

[8]At this point no price mechanisms are discussed. For pricing rules on markets, where many persons try to attain a solitary good (auctions) or where a single good is supplied by different producers (impositions), see for example GÜTH (1995).

Chapter II.

Microeconomic Foundations

1 Axiomatic Characterization of Individual Economic Agents

1.1 The Preferences of a Household

In an economy composed of many economic agents two groups of individual economic decision units may be stressed: households and firms. The theory of the household deals with the question how to satisfy the household's needs, whereas the theory of the firm concentrates on the production of new goods.

Before an answer can be given to the question of how households and firms behave "best possibly" at any time, we have to discuss what alternatives exist and what effects the single actions have. While the household adapts its preference structure, the firm must take technical correlations into account, i.e. the production structure.

If the outcome of an action is known, then the question is raised as to which of the feasible possibilities should be taken. It is assumed that households try to satisfy their needs in the best possible way. In contrast, firms are supposed to pursue the goal of profit maximization. The analysis begins with a description of the needs of a household by the preference structure. Afterwards the firm's production technology is examined.

A household pursuing the goal of utility maximization must be capable of comparing different commodity bundles, which satisfy its needs in different ways. The result of this comparison reflects the household's preferences. Provided two commodity bundles are comparable, then one of the two commodity bundles is preferred to the other or both commodity bundles are of equal utility.

To technically record the preferences of a household, we assume firstly that each **commodity bundle** consists of n commodities $\mathbf{x} = (x_1, \ldots, x_n)^\mathsf{T}$. Thus, each commodity bundle is an element in the commodity space X. If n_d out of the n goods are divisible and if, accordingly, the other goods are indivisible, then the **commodity**

space is abbreviated to

$$X = \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}. \tag{II.1}$$

Correspondingly, it is supposed that indivisible goods can appear only in nonnegative integer units. In particular, the commodity space X contains the origin $\mathbf{x} = \mathbf{0}$. If, for example, the graphical analysis supposes a commodity space with two goods, then three cases may occur.

1. $X = \mathbb{R}_+ \times \mathbb{R}_+$ if there are two divisible goods,
2. $X = \mathbb{R}_+ \times \mathbb{Z}_+$ if there is one divisible and one indivisible good[1], and
3. $X = \mathbb{Z}_+ \times \mathbb{Z}_+$ if there are two indivisible goods.

To technically describe the household's preferences, we make use of a binary relation $\succcurlyeq$. We understand by a binary relation $\succcurlyeq$ in a set X a relation which exists between each two elements of the set X or which does not exist. In the presented case the binary relation $\succcurlyeq$ specifies all ordered pairs of commodity bundles $(\mathbf{x}, \mathbf{x}')$ such that the person concerned [2] prefers the commodity bundle $\mathbf{x}$ to $\mathbf{x}'$ or is indifferent to the two commodity bundles. Such a binary relation is called a **preference relation**. The preference relation $\succcurlyeq$ is nothing more than a subset in the Cartesian product $X \times X$, i.e. $\succcurlyeq \subset X \times X$. If the person associates a utility level with a commodity bundle $\mathbf{x}$, which is at least as large as the utility of the commodity bundle $\mathbf{x}'$, then the pair $(\mathbf{x}, \mathbf{x}')$ is an element in the preference relation $\succcurlyeq$, i.e. $(\mathbf{x}, \mathbf{x}') \in \succcurlyeq$, and more intuitively, we write $\mathbf{x} \succcurlyeq \mathbf{x}'$.

According to the preference relation $\succcurlyeq$, a person is indifferent to two commodity bundles $\mathbf{x}$ and $\mathbf{x}'$ if both $\mathbf{x} \succcurlyeq \mathbf{x}'$ and $\mathbf{x}' \succcurlyeq \mathbf{x}$ are true. The indifference is denoted by $\mathbf{x} \sim \mathbf{x}'$. If the person prefers the commodity bundle $\mathbf{x}$ to $\mathbf{x}'$, then we write $\mathbf{x} \succ \mathbf{x}'$. This relation is valid if and only if $\mathbf{x} \succcurlyeq \mathbf{x}'$ and $\neg(\mathbf{x}' \succcurlyeq \mathbf{x})$ are true.[3]

The preference relation $\succcurlyeq \subset X \times X$ is called a **preference ordering on** X if it has the following properties:[4]

[$\mathcal{P}$1] Reflexivity: Two identical commodity bundles cannot be of different value.

$$\forall \mathbf{x} \in X: \quad \mathbf{x} \succcurlyeq \mathbf{x}$$

[$\mathcal{P}$2] Completeness: The household is capable of comparing all of the commodity bundles $\mathbf{x}, \mathbf{x}' \in X$ which are different, so that at least one of the relations $\mathbf{x} \succcurlyeq \mathbf{x}'$ or $\mathbf{x}' \succcurlyeq \mathbf{x}$ is true.

$$\forall \mathbf{x}, \mathbf{x}' \in X: \quad \mathbf{x} \neq \mathbf{x}' \implies \left[\mathbf{x} \succcurlyeq \mathbf{x}' \text{ or } \mathbf{x}' \succcurlyeq \mathbf{x}\right].$$

[1] Cf. MAS-COLELL, WHINSTON, GREEN (1995, p. 19).

[2] Households and persons are considered as being equal, by which it is assumed that households consist of an individual person.

[3] The negation of a statement is abbreviated to $\neg$.
For a technical description of individual preferences see BOSSERT, STEHLING (1990, Chapter 2).

[4] A reflexive and transitive binary relation R on a set X is called a reflexive *quasi-ordering* on X. If each two different elements in X are comparable so that $(\mathbf{x}, \mathbf{x}') \in R$ or $(\mathbf{x}', \mathbf{x}) \in R$, then a reflexive quasi-ordering on X is called a reflexive *ordering* on X. Cf. BRONSTEIN, SEMENDJAJEW (1987, pp. 547–549).

[$\mathscr{P}$3] Transitivity: By a comparison in pairs the household is capable of arranging the commodity bundles $\mathbf{x}, \mathbf{x}', \mathbf{x}'' \in X$ in a unique order of nonincreasing values.

$$\forall \mathbf{x}, \mathbf{x}', \mathbf{x}'' \in X : \quad \left[\mathbf{x} \succcurlyeq \mathbf{x}' \text{ and } \mathbf{x}' \succcurlyeq \mathbf{x}''\right] \implies \mathbf{x} \succcurlyeq \mathbf{x}''.$$

If the preference relation $\succcurlyeq$ is complete for all $\mathbf{x}, \mathbf{x}' \in X$ one and only one of the following three statements is true:[5] $(i)\ \mathbf{x} \succ \mathbf{x}'$, $(ii)\ \mathbf{x}' \succ \mathbf{x}$ or $(iii)\ \mathbf{x} \sim \mathbf{x}'$. Thus, $\mathbf{x} \succcurlyeq \mathbf{x}'$ holds true if and only if either $\mathbf{x} \succ \mathbf{x}'$ or $\mathbf{x} \sim \mathbf{x}'$ is true. Analogously, the statements $\neg(\mathbf{x} \succcurlyeq \mathbf{x}')$ and $\mathbf{x}' \succ \mathbf{x}$ are equivalent to each other.

In addition to the properties of a preference ordering, further premises on the preferences are made.[6]

[$\mathscr{P}$4] Continuity: A preference ordering $\succcurlyeq$ (on the commodity space X) is said to be continuous if the level sets $\left\{\mathbf{x} \in X \mid \mathbf{x}' \succcurlyeq \mathbf{x}\right\}$ and $\left\{\mathbf{x} \in X \mid \mathbf{x} \succcurlyeq \mathbf{x}'\right\}$ are closed for every commodity bundle $\mathbf{x}' \in X$.
All continuous preference orderings are denoted by the set Π.

By Theorem A.5, p. 287, closedness of the level sets $\left\{\mathbf{x} \in X \mid \mathbf{x}' \succcurlyeq \mathbf{x}\right\}$ holds if and only if the limit $\mathbf{x}^0$ of a sequence $\{\mathbf{x}^\nu\}$ of commodity bundles with $\mathbf{x}' \succcurlyeq \mathbf{x}^\nu$ always implies $\mathbf{x}' \succcurlyeq \mathbf{x}^0$. Analogously, the **preference sets**[7]

$$\mathscr{P}(\mathbf{x}') := \left\{\mathbf{x} \in X \mid \mathbf{x} \succcurlyeq \mathbf{x}'\right\}$$

are closed if and only if

$$\mathbf{x}^\nu \to \mathbf{x}^0,\ \mathbf{x}' \succcurlyeq \mathbf{x}^\nu \quad \implies \quad \mathbf{x}' \succcurlyeq \mathbf{x}^0.$$

If the graph of the preference ordering, graph $\succcurlyeq$, summarizes all ordered pairs of commodity bundles which the preference ordering consists of,

$$\text{graph} \succcurlyeq := \left\{(\mathbf{x}^1, \mathbf{x}^2) \in X \times X \mid \mathbf{x}^1 \succcurlyeq \mathbf{x}^2\right\},$$

then closedness of the graph is equivalent to the continuity of the preference ordering.[8] Hence, the set graph $\succcurlyeq \subset X \times X$ is closed if and only if the limit $(\mathbf{x}^{10}, \mathbf{x}^{20})$ of the sequence $\{(\mathbf{x}^{1\nu}, \mathbf{x}^{2\nu})\}$ with $\mathbf{x}^{1\nu} \succcurlyeq \mathbf{x}^{2\nu}$ always implies $\mathbf{x}^{10} \succcurlyeq \mathbf{x}^{20}$. [9]

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[5]Cf. BOSSERT, STEHLING (1990, p. 16, Theorem 2.1.5).

[6]As shown by DEBREU (1959, pp. 56–59), the properties [$\mathscr{P}$1]–[$\mathscr{P}$4] guarantee that there is a utility function u being continuous in X. This outcome holds true even for the case in which all of the goods are indivisible. To prove this astonishing result, DEBREU refers to Definition C.5, p. 307, and in particular to the definition of continuous (single-valued) mappings.

[7]Note that the preference set $\mathscr{P}(\mathbf{x})$ also contains those commodity bundles which are indifferent to $\mathbf{x}$. A linguistically more accurate distinction between the set $\left\{\mathbf{x} \in X \mid \mathbf{x} \succcurlyeq \mathbf{x}'\right\}$ and the "better set" $\left\{\mathbf{x} \in X \mid \mathbf{x} \succ \mathbf{x}'\right\}$ is not taken into consideration here.

[8]Cf. NIKAIDO (1968, p. 239 f.).

Before going into the next assumption, the following notation is introduced with respect to the comparison of two vectors. Given two vectors $\mathbf{x} = (x_1, \ldots, x_n)^{\mathsf{T}}$ and $\mathbf{y} = (y_1, \ldots, y_n)^{\mathsf{T}}$ in the n-dimensional Euclidean space $\mathbb{R}^n$, then

$$\begin{aligned} \mathbf{x} > \mathbf{y} &\iff x_j > y_j \qquad j = 1, \ldots, n; \\ \mathbf{x} \geqq \mathbf{y} &\iff x_j \geqq y_j \qquad j = 1, \ldots, n; \\ \mathbf{x} \geq \mathbf{y} &\iff [\mathbf{x} \geqq \mathbf{y} \quad \text{and} \quad \mathbf{x} \neq \mathbf{y}]. \end{aligned}$$

A vector $\mathbf{x}$ with $\mathbf{x} > \mathbf{0}$ is said to be positive. The vector $\mathbf{x} \geq \mathbf{0}$ is called semi-positive and it is called nonnegative if $\mathbf{x} \geqq \mathbf{0}$.

[$\mathcal{P}$5] **Monotonicity**: A preference ordering is said to be **monotone** if the utility of a commodity bundle does not diminish for increasing quantities of goods.

$$\forall \mathbf{x}, \mathbf{x}' \in X: \quad \mathbf{x} \geq \mathbf{x}' \implies \mathbf{x} \succcurlyeq \mathbf{x}'$$

We speak of a **strongly monotone** preference ordering, even if there is a higher value.

$$\forall \mathbf{x}, \mathbf{x}' \in X: \quad \mathbf{x} \geq \mathbf{x}' \iff \mathbf{x} \succ \mathbf{x}'$$

The set of all strongly monotone continuous preference orderings is denoted by Π_{smo}.

The following assumption of convex preference orderings has a series of analytical advantages, but it is a contradiction of the assumption of indivisible goods. To what extent the advantages of convex preference sets can be transferred to the case of indivisible goods will be discussed at a later stage.

[$\mathcal{P}$6] **Convexity**: A preference ordering is called **strictly convex** if for two arbitrary commodity bundles $\mathbf{x}, \mathbf{x}' \in X$

$$\mathbf{x} \sim \mathbf{x}', \ \mathbf{x} \neq \mathbf{x}' \iff \lambda\mathbf{x} + (1-\lambda)\mathbf{x}' \succ \mathbf{x} \quad \forall \lambda \in]0, 1[\,.$$

For **convex** preference orderings only the convexity of the preference sets $\mathcal{P}(\mathbf{x})$ is required

$$\mathbf{x}', \ \mathbf{x}'' \in \mathcal{P}(\mathbf{x}) \iff \lambda\mathbf{x}' + (1-\lambda)\mathbf{x}'' \in \mathcal{P}(\mathbf{x}) \quad \forall \lambda \in [0, 1]\,.$$

The set of all strictly convex continuous preference orderings is denoted by Π_{sco}. Similarly, $\succcurlyeq \in \Pi_{co}$ denotes a convex continuous preference ordering of the person concerned.

Within the framework of this book it suffices to specify the household's preferences $\mathcal{P}$ by a family of preference sets $\big(\mathcal{P}(\mathbf{x}) \mid \mathbf{x} \in X\big)$, where this family is called the **preference structure**. Obviously, different households may possess different preferences. Accordingly, the preferences $\mathcal{P}_a$ of household a are described by the preference structure $\big(\mathcal{P}_a(\mathbf{x}) \mid \mathbf{x} \in X\big)$.

If a person has to make an individual decision on choosing a commodity bundle over a certain subset C in the commodity space X, then the subsequently defined maximal and best elements will be important.

With respect to the preference ordering $\succcurlyeq$ an element $\tilde{\mathbf{x}}$ of a set $C \subset X$ is called a **maximal element** of C if C does not contain any alternative commodity bundle which takes preference over $\tilde{\mathbf{x}}$.[10]

$$\nexists \mathbf{x} \in C: \quad \mathbf{x} \succ \tilde{\mathbf{x}}$$

Concerning the preference ordering $\succcurlyeq$, the element $\tilde{\mathbf{x}}$ is called a **best element** of C if it is at least as good as every other commodity bundle in C.

$$\forall \mathbf{x} \in C: \quad \tilde{\mathbf{x}} \succcurlyeq \mathbf{x}$$

If the person's preference ordering $\succcurlyeq$ is complete, the distinction between maximal and best elements becomes obsolete.

$$\begin{aligned} & \nexists \mathbf{x} \in C: \quad \mathbf{x} \succ \tilde{\mathbf{x}} \\ \iff & \nexists \mathbf{x} \in C: \quad \neg(\tilde{\mathbf{x}} \succcurlyeq \mathbf{x}) \\ \iff & \forall \mathbf{x} \in C: \quad \tilde{\mathbf{x}} \succcurlyeq \mathbf{x} \end{aligned}$$

Even if the preference relation $\succcurlyeq$ is incomplete, the set of best elements $D_B(C, \succcurlyeq)$ is also contained in the set of maximal elements $D_M(C, \succcurlyeq)$.

$$\begin{aligned} \tilde{\mathbf{x}} \in D_B(C, \succcurlyeq) \quad & \iff \quad \tilde{\mathbf{x}} \succcurlyeq \mathbf{x} \quad \forall \mathbf{x} \in C \\ & \iff \quad \nexists \mathbf{x} \in C \text{ with } \neg(\tilde{\mathbf{x}} \succcurlyeq \mathbf{x}) \\ & \implies \quad \nexists \mathbf{x} \in C \text{ with } \mathbf{x} \succ \tilde{\mathbf{x}} \implies \tilde{\mathbf{x}} \in D_M(C, \succcurlyeq) \end{aligned}$$

The reverse conclusion $D_M(C, \succcurlyeq) \subset D_B(C, \succcurlyeq)$ is not admissible without the preference relation $\succcurlyeq$ being complete.[11] In this first case the demand is chosen from best elements. If there are no best elements, then we try to make a choice from maximal elements – if they exist. The next theorem explains the existence of maximal elements.

Proposition II.1[12] *Given a reflexive and transitive preference relation $\succcurlyeq$ on the commodity space X. If $C \subset X$ consists of finitely many elements, then the set of maximal elements $D_M(C, \succcurlyeq)$ is not empty.*

[10] Cf. BOSSERT, STEHLING (1990, p. 31 ff.).

[11] If, for instance, $\mathbf{x} \succcurlyeq \tilde{\mathbf{x}} \iff \mathbf{x} \succ \tilde{\mathbf{x}}$, then there could be commodity bundles $\mathbf{x}'$ and $\mathbf{x}''$ which cannot be compared with each other, i.e. neither $\mathbf{x}' \succcurlyeq \mathbf{x}''$ nor $\mathbf{x}'' \succcurlyeq \mathbf{x}'$ is true. Thus, the conclusion

$$\nexists \mathbf{x} \in C: \ \mathbf{x} \succ \mathbf{x}' \implies \forall \mathbf{x} \in C: \ \mathbf{x}' \succcurlyeq \mathbf{x}$$

is wrong with respect to the commodity bundle $\mathbf{x}''$.

[12] Cf. BOSSERT, STEHLING (1990, p. 33, Theorem 2.1.13).
As shown by the proof, this proposition especially holds if the examined binary relation R is transitive and irreflexive, i.e. $\neg(\mathbf{x}R\mathbf{x})$ for all $\mathbf{x} \in X$.

An example, in which the explicit distinction between maximal and best elements with respect to a binary relation becomes clear, is given by the criterion of PARETO efficiency. A comparison of alternative allocations in the sense of PARETO will be discussed in Section IV.3.2.1.

1.2 The Production Technology of a Firm

While the household's choice of a commodity bundle depends on its preferences, the firm's production depends on the underlying production technology. This technology describes how to transform factors (or inputs) into commodities (or outputs). Usually, each firm will have an alternative production technology, which can also change in time, for example, as a result of technical progress.

Analogous to the description of preferences by the preference structure, the purpose of this section is to describe a given technology with the help of a production structure. This production structure must embrace all of the production processes which can be carried out within the firm. A production process is understood as being the transformation of an input vector $\mathbf{v} = (v_1, \ldots, v_m)^{\mathsf{T}}$ into an output vector $\mathbf{x} = (x_1, \ldots, x_n)^{\mathsf{T}}$.

It is again assumed that merely n_d of the n examined goods are divisible. Thus, according to (II.1) each output vector $\mathbf{x}$ is a point in the **commodity space** $X := \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}$. This commodity space consists of all admissible commodity bundles satisfying the required integer constraints.

The **production possibility set** $P(\mathbf{v})$ is the collection of all commodity bundles $\mathbf{x}$ capable of being produced (in each period with the given technology) by using the factors $\mathbf{v}$.

$$P(\mathbf{v}) := \{\mathbf{x} \in X \mid \mathbf{x} \text{ is producible by } \mathbf{v}\}$$

Analogous to the commodity space, the setting of the factor space must take the indivisibility of certain production factors into account. If merely m_d out of the m inputs are divisible, then the **factor space**

$$V := \mathbb{R}_+^{m_d} \times \mathbb{Z}_+^{m-m_d} \tag{II.2}$$

contains all input vectors $\mathbf{v}$ satisfying the integer constraints.

The formal assignment of the set of commodity bundles producible by inputs $\mathbf{v}$ results from the following definition of the output correspondence.

Definition II.1[13] *The multi-valued mapping* $P\colon V \to \mathfrak{P}(X)$ *assigns each input vector* $\mathbf{v}$ *in the factor space* V *to precisely one production possibility set* $P(\mathbf{v}) \in \mathfrak{P}(X)$, $\mathbf{v} \mapsto P(\mathbf{v})$, *and is called the* ***output correspondence****.*

For a more exact formal description of the production correspondence we have to establish the domain and the range of P. The domain of the output correspondence

[13]For the basics on multi-valued mappings, see Appendix C.2.

P is given by

$$\begin{aligned}\operatorname{Dom} P &= \{\mathbf{v} \in V \mid \exists \mathbf{x} \in X : \mathbf{x} \in P(\mathbf{v})\} \\ &= \{\mathbf{v} \in V \mid P(\mathbf{v}) \neq \emptyset\}\end{aligned}$$

and the range is[14]

$$\operatorname{Range} P = \{\mathbf{x} \in X \mid \exists \mathbf{v} \in V : \mathbf{x} \in P(\mathbf{v})\}.$$

Apart from the definitions of a production possibility set and an output correspondence, the concepts of both the production structure and the activity turn out to be helpful for the description of a firm.

If the **production technology**[15] P is specified by a family $\big(P(\mathbf{v}) \mid \mathbf{v} \in V\big)$ of production possibility sets satisfying the following Axioms [P1]–[P6] and possibly [P7] and [P8], then this family is called a **production structure**.

Considering the following axioms, each (admissible) input output combination $(\mathbf{v}, \mathbf{x}) \in V \times X$ is called a **possible process** or a possible activity if it is compatible with a given technology, i.e. $\mathbf{x} \in P(\mathbf{v})$. The **graph** of this technology is the union of all possible activities.[16]

$$\operatorname{graph} P := \{(\mathbf{v}, \mathbf{x}) \mid \mathbf{v} \in V,\ \mathbf{x} \in P(\mathbf{v})\}$$

All possible activities $(\mathbf{v}, \mathbf{x})$, feasible with respect to restrictions of the form $\mathbf{x} \geqq \bar{\mathbf{x}}$ (commodity commitments) and $\mathbf{v} \leqq \bar{\mathbf{v}}$ (factor constraints), are called **attainable activities**.[17]

The following axioms establish requirements which must be satisfied by a family $\big(P(\mathbf{v}) \mid \mathbf{v} \in V\big)$ to be accepted as production technology. Considering the formal treatment of production correspondences, the choice of axioms is mainly subject to economic plausibility .

[P1a] **Possibility of Inaction**: Each input vector can be used to produce a "zero output".[18]

$$\forall \mathbf{v} \in V : \quad \mathbf{0} \in P(\mathbf{v})$$

[14]Cf. FÄRE (1988, pp. 17–18).

[15]Each production technology T is assigned to an output correspondence P_T. For a given technology we may identify P with both the production technology and the output correspondence.
Note that production technologies can be equipped with different premises satisfying "more or less plausible" criteria. Cf. also SHEPHARD (1953, p. 13).

[16]The set of all possible processes is often described by the production set

$$Y := \big\{\mathbf{y} \in \mathbb{R}^{n+m} \mid \mathbf{y} \text{ is a possible activity}\big\}$$

where the components of a *netput vector* $\mathbf{y}$ represent an input if $y_i < 0$ or an output if $y_i > 0$. For a production set Y similar axioms can be formulated as for the output correspondence.
Cf. e.g. TAKAYAMA (1990, p. 51 ff.) and Section IV.3.1.

[17]Cf. FRANK (1969, p. 4). MCFADDEN calls graph P the production possibility set of the firm and $P(\mathbf{v})$ the producible output set. Cf. MCFADDEN (1978, p. 6 f.).

[18]Suppose all of the inputs would be thrown away. Cf. TAKAYAMA (1990, p. 52).

[P1b] **No Land of Cockaigne**: No positive output can be produced without any inputs.

$$\forall \mathbf{x} \in X \setminus \{\mathbf{0}\}: \quad \mathbf{x} \notin P(\mathbf{0})$$

Axiom [P1] implies on the one hand $P(\mathbf{0}) = \{\mathbf{0}\}$ since

$$\begin{aligned} \forall \mathbf{v} \in V: \mathbf{0} \in P(\mathbf{v}) &\implies \mathbf{0} \in P(\mathbf{0}) \\ \forall \mathbf{x} \geq \mathbf{0}: \mathbf{x} \notin P(\mathbf{0}) &\implies \mathbf{v} = \mathbf{0} \text{ requires } \mathbf{x} = \mathbf{0}, \end{aligned}$$

and on the other hand $[\forall \mathbf{v} \in V: \ P(\mathbf{v}) \neq \emptyset]$ since $\mathbf{x} = \mathbf{0}$ is always contained in $P(\mathbf{v})$. Accordingly, the domain of the output correspondence is

$$\text{Dom}\, P = \{\mathbf{v} \in V \mid P(\mathbf{v}) \neq \emptyset\} = V.$$

[P2] **Attainability of Each Production**:[19] For each commodity bundle $\mathbf{x} \in X$ an input vector $\mathbf{v} \in V$ exists such that the activity $(\mathbf{v}, \mathbf{x})$ is a *possible* process.

$$\forall \mathbf{x} \in X, \ \exists \mathbf{v} \in V: \quad \mathbf{x} \in P(\mathbf{v})$$

This axiom does not say that each input vector $\mathbf{v}$ is in reality feasible, but that the technology allows the production of each output combination $\mathbf{x}$ providing there are sufficiently large quantities of factors. Obviously, there are physical processes which do not allow each output and, therefore, are no possible activities. For instance, a mass cannot be speeded up to more than the velocity of light. These problems can be avoided by defining an appropriate commodity space X.[20]

[P3] **Disposability of Inputs**:[21] If the commodity bundle $\mathbf{x} \in X$ can be produced by an input vector $\mathbf{v} \in V$, then a larger input vector $\tilde{\mathbf{v}} \in V$ is also sufficient for the production of $\mathbf{x}$, $P(\mathbf{v}) \subset P(\tilde{\mathbf{v}})$.

$$\forall \mathbf{v}, \tilde{\mathbf{v}} \in V, \ \ \tilde{\mathbf{v}} \geqq \mathbf{v}: \quad \mathbf{x} \in P(\mathbf{v}) \implies \mathbf{x} \in P(\tilde{\mathbf{v}}).$$

[19] For the different notations, see PFINGSTEN (1989, p. 165) and BOL (1974, p. 92).

[20] Cf. BOL (1974, p. 104).

[21] Cf. PFINGSTEN (1989, p. 165) and FÄRE (1988, p. 6). This axiom implies at the same time the statement:

$$\forall \mathbf{v}, \tilde{\mathbf{v}} \in V, \ \ \tilde{\mathbf{v}} \geqq \mathbf{v}: \quad \mathbf{x} \notin P(\tilde{\mathbf{v}}) \implies \mathbf{x} \notin P(\mathbf{v}).$$

Frequently, in the literature on this subject a weaker form of [P3] (*Disposability of Inputs*) appears, which relates to the perfect divisibility of factors.

$$\forall \mathbf{v} \in \mathbb{R}^m_+, \ \forall \lambda \geqq 1: \quad P(\mathbf{v}) \subset P(\lambda \mathbf{v})$$

If all factors are divisible, then [P3] (Strong Disposability) implies the weaker form of [P3] (Weak Disposability) but not the converse. This statement becomes apparent when bearing in mind that only some inputs vary in [P3], but in the weaker form all of the inputs are proportionally increased. Cf. TAKAYAMA (1990, p. 52).

Axiom [P3] expresses the idea that available inputs could remain idle or could even be thrown away without disturbing the production process. Phenomena of congestion are thus explicitly ruled out. The assumption that an arbitrary number of machines can be installed in a factory building without disturbing the production process is not very realistic, but it is enough for the following explanations.

The figure opposite illustrates the graph of a production technology by the shaded area with a divisible product x and a divisible factor v. The way of illustration is adapted from TURGOT's law of returns.

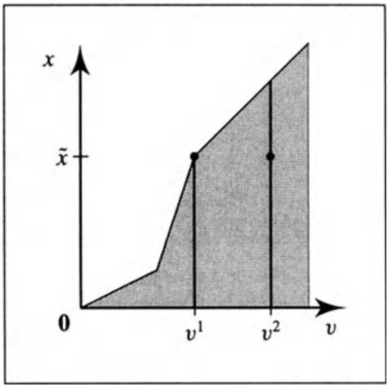

Figure II.1: Disposability of inputs (1)

Due to $\tilde{x} \in P(v^1)$, an augmentation of the input from v^1 to v^2 implies the relation $\tilde{x} \in P(v^2)$ or equivalently $P(v^1) \subset P(v^2)$, where both production possibility sets correspond to the bold vertical lines. The transfer to the case of an indivisible good and an indivisible factor is discussed in Figures II.7 and II.9.

[P4] **Disposability of Outputs**: If a commodity bundle $\mathbf{x} \in X$ can be produced by the inputs $\mathbf{v} \in V$, then each commodity vector $\tilde{\mathbf{x}} \in X$, which is not larger than $\mathbf{x}$ for any component, can be produced by using $\mathbf{v}$.

$$\forall \mathbf{x}, \tilde{\mathbf{x}} \in X,\ \ \mathbf{x} \geqq \tilde{\mathbf{x}}: \quad \mathbf{x} \in P(\mathbf{v}) \implies \tilde{\mathbf{x}} \in P(\mathbf{v}).$$

As in [P3] (*Disposability of Inputs*) previously, [P4] is influenced by the idea that an arbitrary amount of the produced goods could be thrown away. Using the example in Figure II.1, the figure opposite with $x^1 \in P(\tilde{v})$ illustrates the implication $x^2 \in P(\tilde{v})$.

In the presented form Axioms [P3] and [P4] may be applied to the case of indivisible goods and factors without any problems. Furthermore, with perfectly divisible goods and factors the two axioms serve for the characterization of boundary points of an input requirement set. Before elucidating this aspect, it is useful to introduce two more assumptions.

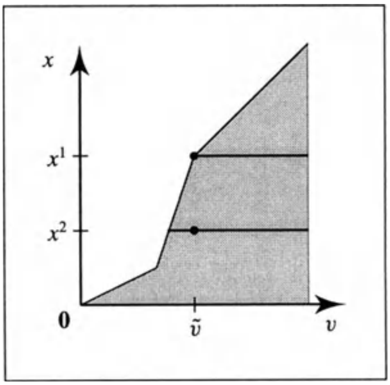

Figure II.2: Disposability of outputs (1)

[P5] **Boundedness** or impossibility of arbitrary abundance: The outputs $\mathbf{x}$ cannot be increased arbitrarily at held fixed inputs $\mathbf{v} \in V$.

$$\forall \mathbf{v} \in V: \quad P(\mathbf{v}) \text{ is bounded.}$$

Regarding nonnegative quantities of output, this requirement on a production technology can technically be described as follows: the axiom requires that an (open) ball of finite radius $r > 0$ can be defined, for example, centered at the origin such that all of the commodity bundles producible by $\mathbf{v}$ lie within this ball. Thus, using the Euclidean distance, the inequality $\|\mathbf{x}\| < r$ must hold for all $\mathbf{x} \in P(\mathbf{v})$.

In view of the optimization problems which have to be solved, the closedness of the level sets $P(\mathbf{v})$ will be very important.

[P6] **Closedness:**[22] If a sequence of possible activities $(\mathbf{v}^\nu, \mathbf{x}^\nu)$ converges to the limit $(\mathbf{v}^0, \mathbf{x}^0)$, then this limit is possible, too. When applied to the output correspondence, this means[23]

$$P\colon V \to \mathfrak{P}(X) \text{ is a closed mapping of } V \text{ into } X.$$

By Theorem C.6, p. 308, the output correspondence P is said to be closed in V if at each point $\mathbf{v}^0 \in V$ the following condition is satisfied: for any two sequences $\{\mathbf{v}^\nu\} \subset V$ and $\{\mathbf{x}^\nu\} \subset X$ it is

$$\mathbf{v}^\nu \to \mathbf{v}^0, \quad \mathbf{x}^\nu \to \mathbf{x}^0, \quad \mathbf{x}^\nu \in P(\mathbf{v}^\nu) \quad \Longrightarrow \quad \mathbf{x}^0 \in P(\mathbf{v}^0).$$

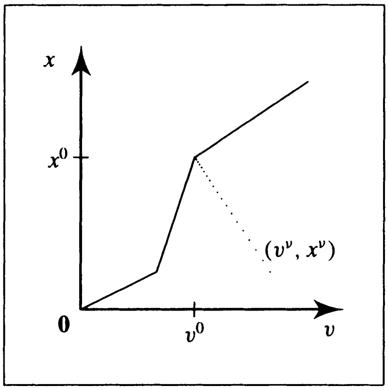

Figure II.3: Closedness of the graph graph P

With regard to economic aspects there is almost nothing to say against this implication. Essentially, the closedness means that all boundary points of the graph depicted in Figure II.3 are part of the graph. Thus, $x^0 \in P(v^0)$ represents the limit (v^0, x^0) of the illustrated sequence of possible activities (v^ν, x^ν).

The closedness is equivalent to the assumption that the graph of the output correspondence graph P is closed. Analogous to [P6], we have

$$(\mathbf{v}^\nu, \mathbf{x}^\nu) \to (\mathbf{v}^0, \mathbf{x}^0), \quad (\mathbf{v}^\nu, \mathbf{x}^\nu) \in \operatorname{graph} P \quad \Longrightarrow \quad (\mathbf{v}^0, \mathbf{x}^0) \in \operatorname{graph} P.$$

Thus, the level sets $P(\mathbf{v})$ must be closed for all $\mathbf{v} \in V$ (Theorem C.8, p. 309). Since each subset in $\mathbb{R}^n_+$ is compact if and only if it is closed and bounded, we have[24]

$$\forall \mathbf{v} \in V: \quad P(\mathbf{v}) \text{ is compact.}$$

[22] McFadden (1978, p. 7), calls production technologies satisfying Axioms [P1] and [P6] input regular.

[23] Each continuous correspondence is upper semi-continuous (Definition C.5, p. 307) and each upper semi-continuous correspondence is itself closed (Theorem C.7, p. 309). For many applications the closedness of a correspondence is enough. Semi-continuous correspondences will be of crucial importance in Chapter IV.

[24] See Appendix A.4, p. 287.

Correspondences with this property are frequently said to be compact-valued.[25]

By taking into account that the solutions to optimization problems usually lie in the boundary of the feasible region, the importance of closed sets will be apparent. The closedness of the feasible region guarantees that these optimal solutions are admissible.

Topologically, [P6] requires that each production possibility set $P(\mathbf{v})$ contains all of its boundary points. Technically, a point $\mathbf{x}$ lies in the **boundary** $\partial P(\mathbf{v})$ of the set $P(\mathbf{v})$ if each open ball centered at $\mathbf{x}$ contains a point in $P(\mathbf{v})$ and a point $\tilde{\mathbf{x}} \in \mathbb{R}^m$ with $\tilde{\mathbf{x}} \notin P(\mathbf{v})$.[26] If there is an open ball centered at $\mathbf{x}$ exclusively consisting of points in $P(\mathbf{v})$, then $\mathbf{x}$ belongs to the **interior** $\operatorname{int} P(\mathbf{v})$ of the set $P(\mathbf{v})$. Therefore, no boundary point belongs to the interior of the set $P(\mathbf{v})$ and each closed set $P(\mathbf{v})$ can be divided into

$$P(\mathbf{v}) = \partial P(\mathbf{v}) \cup \operatorname{int} P(\mathbf{v}) \quad \text{with} \quad \partial P(\mathbf{v}) \cap \operatorname{int} P(\mathbf{v}) = \emptyset.$$

The distinction between inner points and boundary points turns out to be insignificant, as soon as the analysis includes a particular indivisible good.[27] As illustrated by point B in Figure II.4, in this case $P(\mathbf{v}) = \partial P(\mathbf{v})$.

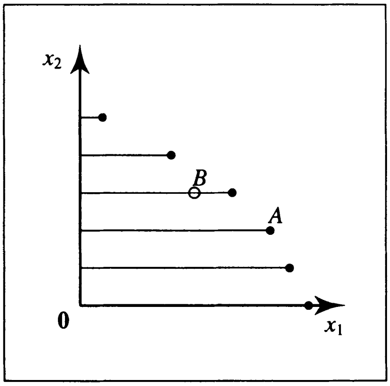

Figure II.4: The boundary of a production possibility set

Looking back at Axioms [P3] (*Disposability of Inputs*) and [P4] (*Disposability of Outputs*), two features can be seen with respect to divisible goods.

In order to describe efficient activities it is important to know what properties hold for a possible activity $(\mathbf{v}, \mathbf{x})$ lying in the boundary of the set $P(\mathbf{v})$. If a reduction of the inputs from $\mathbf{v}$ to $\tilde{\mathbf{v}}$ yields

$$\mathbf{x} > \mathbf{0},\ \mathbf{x} \in \partial P(\mathbf{v}) \implies \mathbf{x} \notin P(\tilde{\mathbf{v}}),$$

then it is not possible that the boundaries of both sets $P(\mathbf{v})$ and $P(\tilde{\mathbf{v}})$ touch each other in the positive orthant. While [P3] (*Disposability of Inputs*) merely requires $P(\tilde{\mathbf{v}}) \subset P(\mathbf{v})$, additionally, we now have $\partial P(\mathbf{v}) \cap \mathbb{R}^m_{++} \cap P(\tilde{\mathbf{v}}) = \emptyset$ for all $\tilde{\mathbf{v}} \geq \mathbf{v}$. Activities featuring the above property will be discussed later with respect to the concept of input efficiency.

[25] Cf. HILDENBRAND, KIRMAN (1988, p. 272).

[26] See Appendix A.2.

The definition at once implies $\mathbf{0} \in \partial P(\mathbf{v})$ since each open neighborhood of $\mathbf{x} = \mathbf{0}$ contains $\mathbf{0} \in P(\mathbf{v})$ and a point $\tilde{\mathbf{x}} < \mathbf{0}$ with $\tilde{\mathbf{x}} \notin P(\mathbf{v})$. Similarly, all commodity bundles lying on the coordinate axes belong to the boundary $\partial P(\mathbf{v})$.

[27] Even convex sets need not have inner points. For example, each point on a line in $\mathbb{R}^2$ is at the same time a boundary point of this line.

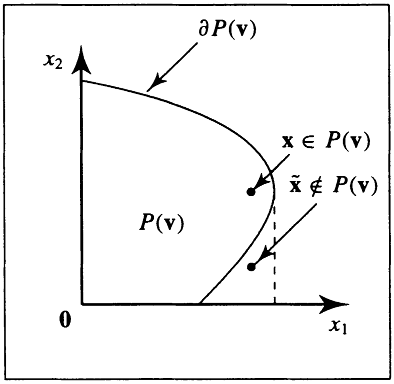

Figure II.5: Disposability of outputs (2)

The other feature applies to [P4] (*Disposability of Outputs*). For divisible goods this axiom excludes a backward bending boundary of the production possibility set $\partial P(\mathbf{v})$ over the region $\mathbf{x} > \mathbf{0}$. Therefore, the case illustrated by commodity bundle $\tilde{\mathbf{x}}$ in Figure II.5 must not occur. The importance of this remark will be evident when bearing in mind that each hyperplane tangent to the set $P(\mathbf{v})$ at a point $\mathbf{x} > \mathbf{0}$ is determined by *nonnegative* commodity prices.

Frequently, further premises are made requiring the divisibility of goods and factors.[28]

[P7] **Convexity:** For perfectly divisible goods the production possibility set $P(\mathbf{v})$ is convex for all $\mathbf{v} \in V$.

$$\forall \tilde{\mathbf{x}}, \mathbf{x} \in P(\mathbf{v}), \quad \forall \lambda \in [0, 1]: \quad \lambda\tilde{\mathbf{x}} + (1 - \lambda)\mathbf{x} \in P(\mathbf{v})$$

Besides the convexity of level sets, there is the quasi-concavity of the output correspondence whose importance will become clear afterwards.

[P8] **Quasi-concavity:** For perfectly divisible factors the output correspondence P is quasi-concave in V.

$$\forall \tilde{\mathbf{v}}, \mathbf{v} \in V, \quad \forall \lambda \in [0, 1]: \quad P(\tilde{\mathbf{v}}) \cap P(\mathbf{v}) \subset P\big(\lambda\tilde{\mathbf{v}} + (1 - \lambda)\mathbf{v}\big)$$

The premises [P6]–[P8] are rather technical than economic properties and serve for the mathematical handling of production technologies.

Note that the presented technology deals only with the static deterministic case. The randomization and dynamization of the theory desired by EICHHORN (1975) has been at least carried out with respect to a dynamic theory by SHEPHARD, FÄRE (1980).

[28] Cf. TAKAYAMA (1990, p. 54).

2 Theory of the Firm

2.1 Inverse Representation of the Production Technology

Having used a family $\big(P(\mathbf{v})|\ \mathbf{v} \in V\big)$ of production possibility sets to represent a firm's production technology, it seems plausible to compare this production structure with a household's preference structure $\big(\mathcal{P}(\mathbf{x})|\ \mathbf{x} \in X\big)$.

A preference set $\mathcal{P}(\mathbf{x})$ consists of commodity bundles which at least generate the result (i.e. the utility) $\mathbf{x}$, whereas a production possibility set $P(\mathbf{v})$ consists of those commodity bundles which can be produced by using the inputs $\mathbf{v}$.[29] Hence, the question arises whether a family equivalent to $\big(P(\mathbf{v})|\ \mathbf{v} \in V\big)$ exists such that the members consist of elements which are sufficient to produce at least a certain result (i.e. the output $\mathbf{x}$).

Up until now each vector of inputs $\mathbf{v} \in V$ is mapped into a set $P(\mathbf{v}) \subset X$ containing all commodity bundles producible by $\mathbf{v}$. This assignment ensues from the output correspondence P and is depicted in the following figure by $\mathbf{v}^1 \mapsto P(\mathbf{v}^1)$ and $\mathbf{v}^2 \mapsto P(\mathbf{v}^2)$. The inverse assignment collects all input vectors which are sufficient to produce a given commodity bundle $\mathbf{x}$.

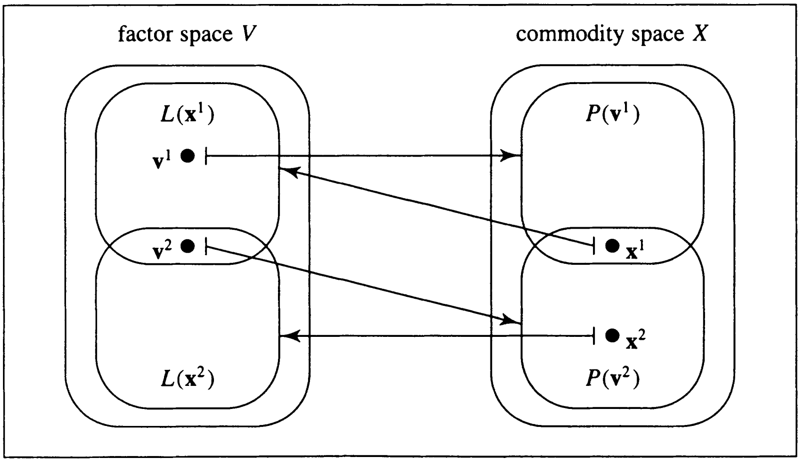

Figure II.6: Comparison of the input correspondence L and the output correspondence P

The **input requirement set** indicates those input vectors $\mathbf{v}$ permitting the production of the commodity bundle $\mathbf{x}$ (per period at a given technology).

$$L(\mathbf{x}) := \{\mathbf{v} \in V|\ \mathbf{v} \text{ permits the production of } \mathbf{x}\}$$

Definition II.2 *The* ***input correspondence*** $L\colon X \to \mathfrak{P}(V)$ *assigns precisely one*

[29]Technically, the difference can be stressed by the origin $\mathbf{x} = \mathbf{0}$. Except for $\mathcal{P}(\mathbf{0})$ no preference set contains the origin. In contrast, the origin belongs to every production possibility set; see [P1a].

input requirement set $L(\mathbf{x}) \in \mathfrak{P}(V)$ *to each commodity bundle* $\mathbf{x} \in X$, *i.e.* $\mathbf{x} \mapsto L(\mathbf{x})$.

The domain of L is a subset in the commodity space X; it is given by

$$\operatorname{Dom} L = \{\mathbf{x} \in X \mid \exists \mathbf{v} \in V : \ \mathbf{v} \in L(\mathbf{x})\} = \{\mathbf{x} \in X \mid L(\mathbf{x}) \neq \emptyset\}.$$

Analogously, the range is a subset in the factor space V,

$$\operatorname{Range} L = \{\mathbf{v} \in V \mid \exists \mathbf{x} \in X : \ \mathbf{v} \in L(\mathbf{x})\}.$$

Figure 2.1 shows the representative associations $\mathbf{x}^1 \mapsto L(\mathbf{x}^1)$ and $\mathbf{x}^2 \mapsto L(\mathbf{x}^2)$. Although this figure suggests that the production technology is described in two absolutely different ways, both forms of representation can be transformed into each other.

The equivalence of the representation of a production technology by an output correspondence P (Definition II.1) and an input correspondence L can be justified by the following statement: "The inputs $\mathbf{v}$ allow the production of the outputs $\mathbf{x}$ if and only if $\mathbf{x}$ is producible by $\mathbf{v}$." This statement, apparently lacking in content, is in opposition to a technical equivalence requiring for all possible activities $(\mathbf{v}, \mathbf{x}) \in V \times X$ [30]

$$\text{(II.3)} \qquad \begin{array}{rcl} \mathbf{v} \in L(\mathbf{x}) & \Longleftrightarrow & \mathbf{x} \in P(\mathbf{v}) \\ \text{or} \quad L(\mathbf{x}) = \{\mathbf{v} \mid \mathbf{x} \in P(\mathbf{v})\} & \Longleftrightarrow & P(\mathbf{v}) = \{\mathbf{x} \mid \mathbf{v} \in L(\mathbf{x})\}. \end{array}$$

The set of all possible activities is given by[31]

$$\operatorname{graph} L := \{(\mathbf{v}, \mathbf{x}) \mid \mathbf{x} \in X, \ \mathbf{v} \in L(\mathbf{x})\}.$$

Since both graphs graph P and graph L represent the same set of possible processes, we can abbreviate them to

$$\text{(II.4)} \qquad \mathrm{GR} := \operatorname{graph} P = \operatorname{graph} L.$$

In order to clarify this, a technology is presented, which includes the transformation of a particular indivisible factor v into a particular indivisible good x. Accordingly, each possible activity (v, x) is an element of the cross product $V \times X = \mathbb{Z}_+ \times \mathbb{Z}_+$. The set of all possible processes is given by $\mathrm{GR} = \{(v, x) \mid v \in V, \ x \in P(v)\}$; it is illustrated in Figure II.7 by the marked points. For the activity $(\tilde{v}, \tilde{x})$ we have both $\tilde{x} \in P(\tilde{v})$ and $\tilde{v} \in L(\tilde{x})$. The figure shows, moreover, the production possibility set $P(\tilde{v})$ and the later required convex hull $\operatorname{conv} P(\tilde{v})$ beside the vertical axis. Analogously, the input requirement set $L(\tilde{x})$ and its convex hull $\operatorname{conv} L(\tilde{x})$ can be read below the horizontal axis.

If the production structure $\big(P(\mathbf{v}) \mid \mathbf{v} \in V\big)$ satisfies Axioms [P1]–[P6] and possibly Axioms [P7] and [P8], then completely equivalent properties [L1]–[L8] can

[30] The equivalence relation (II.3) cannot be transferred to the later used convex hulls of $P(\mathbf{v})$ and $L(\mathbf{x})$. The implication $\mathbf{x} \in \operatorname{conv} P(\mathbf{v}) \implies \mathbf{v} \in \operatorname{conv} L(\mathbf{x})$ is as false as its inversion.

[31] Cf. OPITZ (1971, p. 242).

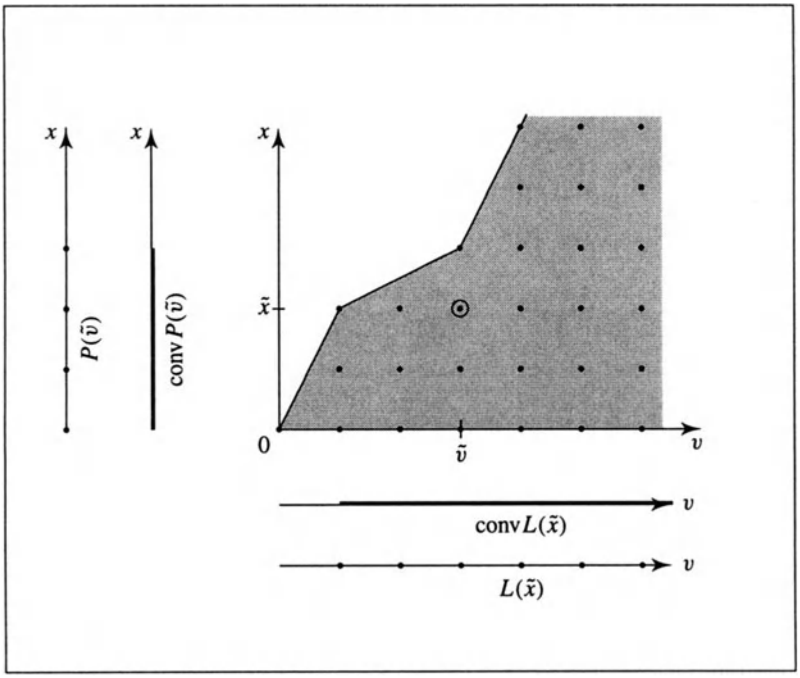

Figure II.7: The graph of a production technology

be given for the family $\left(L(\mathbf{x}) \mid \mathbf{x} \in X\right)$. At the same time the economic interpretation transmits to the alternative axioms. Before going into the equivalence (Proposition II.2), a list of the alternative axioms is given.

[L1a] Possibility of Inaction: $L(\mathbf{0}) = V$

[L1b] No Land of Cockaigne: $\forall \mathbf{x} \in X \setminus \{\mathbf{0}\} : \ \mathbf{0} \notin L(\mathbf{x})$

[L2] Attainability of Each Production: $\forall \mathbf{x} \in X : \ L(\mathbf{x}) \neq \emptyset$

[L3] Disposability of Inputs: $\forall \mathbf{v}, \tilde{\mathbf{v}} \in V, \ \mathbf{v} \leqq \tilde{\mathbf{v}} : \ \mathbf{v} \in L(\mathbf{x}) \implies \tilde{\mathbf{v}} \in L(\mathbf{x})$

[L4] Disposability of Outputs: $\forall \mathbf{x}, \tilde{\mathbf{x}} \in X, \ \mathbf{x} \geqq \tilde{\mathbf{x}} : \ \mathbf{v} \in L(\mathbf{x}) \implies \mathbf{v} \in L(\tilde{\mathbf{x}})$

[L5] Boundedness: $\|\mathbf{x}^\nu\| \to \infty : \ \bigcap_{\nu=1}^{\infty} L(\mathbf{x}^\nu) = \emptyset$

[L6] Closedness: The input correspondence L is closed in X.

[L7] Convexity: Given divisible factors, the input requirement sets $L(\mathbf{x})$ are convex for all $\mathbf{x} \in X$.

[L8] Quasi-Concavity: Given divisible outputs, the input correspondence L is quasi-concave in X.

Proposition II.2 *The output correspondence P satisfies Axioms* [P1]–[P6] *if and only if the input correspondence L satisfies Axioms* [L1]–[L6]. *For divisible goods and factors the proposition can be extended by Axioms* [P7] *and* [L8] *as well as* [P8] *and* [L7].

This can be proved as follows:

[P1a] $\Longleftrightarrow$ [L1a]: Each arbitrary input vector $\mathbf{v} \in V$ permits the production of the commodity bundle $\mathbf{x} = \mathbf{0}$.

$$\forall \mathbf{v} \in V: \mathbf{0} \in P(\mathbf{v}) \quad \Longleftrightarrow \quad \forall \mathbf{v} \in V: \mathbf{v} \in L(\mathbf{0}) \quad \overset{L(\mathbf{x})\subset V}{\Longleftrightarrow} \quad L(\mathbf{0}) = V$$

[P1b] $\Longleftrightarrow$ [L1b]: The production of each commodity bundle $\mathbf{x} \geq \mathbf{0}$ requires at least one positive input, i.e. $\mathbf{v} \geq \mathbf{0}$.

$$\forall \mathbf{x} \in X \setminus \{\mathbf{0}\}: \ \mathbf{x} \notin P(\mathbf{0}) \quad \Longleftrightarrow \quad \forall \mathbf{x} \in X \setminus \{\mathbf{0}\}: \ \mathbf{0} \notin L(\mathbf{x})$$

[P2] $\Longleftrightarrow$ [L2]: The technology assures that there is an input vector $\mathbf{v} \in V$ for each commodity bundle $\mathbf{x} \in X$ such that $\mathbf{x}$ is producible by $\mathbf{v}$.

$$\forall \mathbf{x} \in X,\ \exists \mathbf{v} \in V: \ \mathbf{x} \in P(\mathbf{v}) \quad \overset{(II.3)}{\Longleftrightarrow} \quad \forall \mathbf{x} \in X,\ \exists \mathbf{v} \in V: \ \mathbf{v} \in L(\mathbf{x})$$

Thus, the domain of the input correspondence L is

$$\operatorname{Dom} L = \{\mathbf{x} \in X \mid L(\mathbf{x}) \neq \emptyset\} = X .$$

[P3] $\Longleftrightarrow$ [L3]: If an input vector $\mathbf{v} \in V$ suffices for the production of the commodity bundle $\mathbf{x} \in X$, then each input vector $\tilde{\mathbf{v}} \geqq \mathbf{v}$ is indeed sufficient for the production of $\mathbf{x}$. Thus, for all $\mathbf{v}, \tilde{\mathbf{v}} \in V$ with $\mathbf{v} \leqq \tilde{\mathbf{v}}$ we have

$$\begin{aligned} \mathbf{v} \in L(\mathbf{x}) \quad &\overset{(II.3)}{\Longleftrightarrow} \quad \mathbf{x} \in P(\mathbf{v}) \\ &\overset{[P3]}{\Longrightarrow} \quad \mathbf{x} \in P(\tilde{\mathbf{v}}) \\ &\overset{(II.3)}{\Longleftrightarrow} \quad \tilde{\mathbf{v}} \in L(\mathbf{x}). \end{aligned}$$

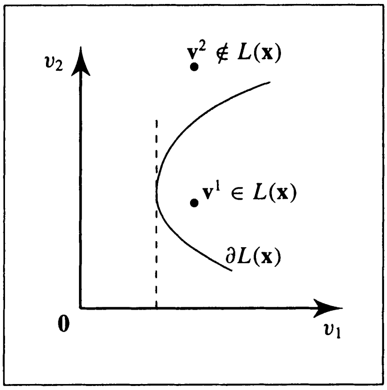

Figure II.8: Disposability of inputs (2)

[L2] (*Attainability of Each Production*) immediately implies two properties of the input requirement sets. On the one hand the sets $L(\mathbf{x})$ are not bounded. On the other hand [L3] prevents a backward bending boundary $\partial L(\mathbf{x})$ of an input requirement set $L(\mathbf{x})$ for divisible factors. Analogous to Figure II.5, the case shown in Figure II.8 may not appear.[32]

[P4] $\Longleftrightarrow$ [L4]: If an input vector $\mathbf{v} \in V$ permits the production of the commodity bundle $\mathbf{x} \in X$, then each commodity bundle $\tilde{\mathbf{x}} \leqq \mathbf{x}$ is also producible. Thus, for all $\mathbf{v} \in V$ and for all $\mathbf{x}, \tilde{\mathbf{x}} \in X$ with $\mathbf{x} \geqq \tilde{\mathbf{x}}$ we have

$$\mathbf{v} \in L(\mathbf{x}) \ \overset{(II.3)}{\Longleftrightarrow} \ \mathbf{x} \in P(\mathbf{v}) \ \overset{[P4]}{\Longrightarrow} \ \tilde{\mathbf{x}} \in P(\mathbf{v}) \ \overset{(II.3)}{\Longleftrightarrow} \ \mathbf{v} \in L(\tilde{\mathbf{x}}).$$

[32]For phenomena of congestion, where an increasing factor input implies a decreasing output, see FÄRE (1980b) and FÄRE, GROSSKOPF (1983).

[P5] $\Longleftrightarrow$ [L5]: For the necessary part as well as for the sufficient part we give a proof of contradiction.

1. If $P(\tilde{\mathbf{v}})$ is not bounded for a $\tilde{\mathbf{v}} \in V$, then there is a sequence $\{\mathbf{x}^\nu\} \subset P(\tilde{\mathbf{v}})$ with $\|\mathbf{x}^\nu\| \to \infty$. Thus, $\tilde{\mathbf{v}} \in L(\mathbf{x}^\nu)$ holds for all ν such that $\tilde{\mathbf{v}} \in \bigcap_{\nu=1}^{\infty} L(\mathbf{x}^\nu)$ contradicts [L5].
2. Supposing there is a $\tilde{\mathbf{v}}$ with $\tilde{\mathbf{v}} \in \bigcap_{\nu=1}^{\infty} L(\mathbf{x}^\nu)$ for $\|\mathbf{x}^\nu\| \to \infty$ then in contradiction to the boundedness of the set $P(\tilde{\mathbf{v}})$ for all $\nu \in \mathbb{N}$ the result is $\tilde{\mathbf{v}} \in L(\mathbf{x}^\nu) \Longleftrightarrow \mathbf{x}^\nu \in P(\tilde{\mathbf{v}})$.

[P6] $\Longleftrightarrow$ [L6]: By Theorem C.6, p. 308, the input correspondence L is said to be closed in X if

$$\mathbf{x}^\nu \to \mathbf{x}^0, \ \ \mathbf{v}^\nu \to \mathbf{v}^0, \ \ \mathbf{v}^\nu \in L(\mathbf{x}^\nu) \quad \Longrightarrow \quad \mathbf{v}^0 \in L(\mathbf{x}^0).$$

results for any two sequences $\{\mathbf{x}^\nu\} \subset X$ and $\{\mathbf{v}^\nu\} \subset V$ with the limits $\mathbf{x}^0$ and $\mathbf{v}^0$ respectively. With that the equivalence of [P6] and [L6] at once ensues from $\mathbf{x}^\nu \in P(\mathbf{v}^\nu) \Longleftrightarrow \mathbf{v}^\nu \in L(\mathbf{x}^\nu)$ and $\mathbf{x}^0 \in P(\mathbf{v}^0) \Longleftrightarrow \mathbf{v}^0 \in L(\mathbf{x}^0)$.

The input correspondence L is closed in X if and only if the graph of the input correspondence $\operatorname{graph} L = \{(\mathbf{v}, \mathbf{x}) \,|\, \mathbf{x} \in X, \ \mathbf{v} \in L(\mathbf{x})\}$ is closed. Thus, the level sets must be closed, too.

$$\forall \mathbf{x} \in X: \quad L(\mathbf{x}) \text{ is closed.}$$

Since the input requirement sets $L(\mathbf{x})$ are not bounded by [L3] (*Disposability of Inputs*), the input correspondence cannot be upper semi-continuous in the sense of Definition C.5, p. 307.[33]

Supposing again the divisibility of inputs and outputs, the subsequent axioms can be established:

[P8] $\Longleftrightarrow$ [L7]: The input correspondence L is convex-valued if and only if the following condition is satisfied for each $\mathbf{x} \in X$: [34]

$$\forall \tilde{\mathbf{v}}, \mathbf{v} \in L(\mathbf{x}), \ \ \forall \lambda \in [0, 1]: \quad \lambda\tilde{\mathbf{v}} + (1 - \lambda)\mathbf{v} \in L(\mathbf{x}).$$

By Theorem II.6, p. 78, the input correspondence L is convex-valued if and only if the inverse output correspondence P is quasi-concave in V.

[P7] $\Longleftrightarrow$ [L8]: We speak of a quasi-concave input correspondence L if it fulfills the following condition:

$$\forall \tilde{\mathbf{x}}, \mathbf{x} \in X, \ \ \forall \lambda \in [0, 1]: \quad L(\tilde{\mathbf{x}}) \cap L(\mathbf{x}) \subset L\big(\lambda\tilde{\mathbf{x}} + (1 - \lambda)\mathbf{x}\big).$$

[33]This remark is important to prevent linguistic confusion. Some authors define the upper semi-continuity of a correspondence in the sense of a closed correspondence of BERGE. Indeed, the basic idea of closed level sets is not affected. Cf. SHEPHARD (1953, p. 299) or EICHHORN, SHEPHARD, STEHLING (1979, p. 334) and contrast it to DEBREU (1982, p. 698 f.)

[34]Following SHEPHARD (1970, p. 11), the condition of a convex-valued input correspondence gets a further economic meaning if we take a dynamic production technology as basis. If λ and $1 - \lambda$ are the fractions of a unit period with the (possibly indivisible) input vectors $\tilde{\mathbf{v}}$ and $\mathbf{v}$, then $(\lambda\tilde{\mathbf{v}} + (1 - \lambda)\mathbf{v}, \mathbf{x})$ can be interpreted as an activity of a "temporally divisible" technology.

Again, the equivalence of [P7] and [L8] results from Theorem II.6.

After Proposition II.2 has been proved, another result can be delivered, which merely emphasizes the consistency of the argumentation. The domains and the ranges of the input or output correspondence satisfy[35]

$$\text{Dom}\, P = \text{Range}\, L = \{\mathbf{v} \in V \mid \exists \mathbf{x} \in X : \ \mathbf{v} \in L(\mathbf{x})\} = V,$$
$$\text{Dom}\, L = \text{Range}\, P = \{\mathbf{x} \in X \mid \exists \mathbf{v} \in V : \ \mathbf{x} \in P(\mathbf{v})\} = X.$$

The first equation says that there is no admissible input vector **v** which does not belong to at least one input requirement set $L(\mathbf{x})$. Analogously, each commodity bundle **x** belongs to at least one production possibility set $P(\mathbf{v})$ in accordance with the second equation.

Another postscript refers to the disposability of inputs and outputs. Summarizing Axioms [P3] and [P4] – or, equivalently, Axioms [L3] and [L4] – the following equivalent property with respect to the graph of the examined production technology ensues.

$$\text{(II.5)} \quad \forall\, (\mathbf{v}, \mathbf{x}), (\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \in V \times X, \ \ (-\mathbf{v}, \mathbf{x}) \geqq (-\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) : \qquad (\mathbf{v}, \mathbf{x}) \in \text{GR} \implies (\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \in \text{GR}$$

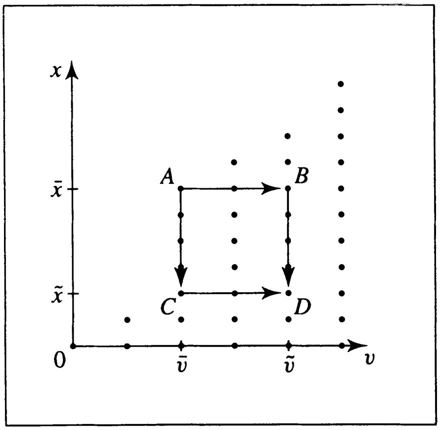

Figure II.9: Disposability of inputs and outputs

As in Figure II.7, Figure II.9 also describes the case of an indivisible input v and an indivisible output x, i.e. $V \times X = \mathbb{Z}_+ \times \mathbb{Z}_+$. Each possible activity $(v, x) \in \text{GR}$ is marked by a dot. Starting at point A with the possible activity $(\bar{v}, \bar{x})$, [P3] (*Disposability of Inputs*) says that the activity $(\tilde{v}, \bar{x})$ marked by point B is possible. Analogously, even $(\bar{v}, \tilde{x})$ or correspondingly point C describes a possible activity in accordance with [P4] (*Disposability of Outputs*). The combination of the two statements in (II.5) implies $(\tilde{v}, \tilde{x})$ or point D to be a further possible activity.

Finally, an assumption on convexity tightening Axioms [P7] and [L7] has to be stressed. Apart from convex level sets, which have been required up to now, some authors also call for the convexity of the graph GR of a production technology.[36]

[T1] **Convexity**: Given divisible goods and factors, the graph GR of the production technology is convex.

$$\forall\, (\mathbf{v}, \mathbf{x}), (\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \in \text{GR}, \ \ \forall \lambda \in [0, 1] : \quad \lambda(\mathbf{v}, \mathbf{x}) + (1 - \lambda)(\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \in \text{GR}$$

[35]Cf. e.g. SHEPHARD (1953, p. 298).

[36]Cf. e.g. TAKAYAMA (1990, p. 54) or ALIPRANTIS, BROWN, BURKINSHAW (1989, p. 69).

Both Assumptions [P7] (Convexity of Production Possibility Sets) and [L7] (Convexity of Input Requirement Sets) are necessary (but not sufficient) for the convexity of the graph. If we insert $\mathbf{v} = \tilde{\mathbf{v}} = \hat{\mathbf{v}}$ in [T1], [P7] can be obtained :

$$\begin{aligned} &\left(\hat{\mathbf{v}}, \lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\right) \in \mathrm{GR} \quad \forall \lambda \in [0,1] \\ \Longleftrightarrow \quad &\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}} \in P(\hat{\mathbf{v}}) \qquad \forall \lambda \in [0,1]. \end{aligned}$$

Analogously, Axiom [L7] results from $\mathbf{x} = \tilde{\mathbf{x}} = \hat{\mathbf{x}}$:

$$\lambda\mathbf{v} + (1-\lambda)\tilde{\mathbf{v}} \in L(\hat{\mathbf{x}}) \qquad \forall \lambda \in [0,1].$$

As illustrated by Figures II.1 and II.2, the reverse conclusion from convex input requirement sets and convex production possibility sets to the convexity of the graph GR is not allowed.

Furthermore, [P1a] (*Possibility of Inaction*), i.e. $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) = (\mathbf{0}, \mathbf{0})$, gives

(II.6) $$(\mathbf{v}, \mathbf{x}) \in \mathrm{GR} \quad \Longrightarrow \quad \lambda(\mathbf{v}, \mathbf{x}) \in \mathrm{GR} \quad \forall \lambda \in [0,1].$$

As shown below, this condition is only compatible with nonincreasing returns to scale.

2.2 Treatment of Indivisible Goods and Factors

2.2.1 The Concept of Convex Hulls

The consideration of indivisible goods is explicitly reflected by the setting of the commodity space X in accordance with (II.1). Analogously, indivisible production factors are recorded by the factor space V according to (II.2). For the subsequent expositions it is useful to extend each of the two spaces to the entire respective Euclidean space.

$$\overline{X} := \mathbb{R}^n \qquad \text{and} \qquad \overline{V} := \mathbb{R}^m$$

A vector $\mathbf{x} \in \overline{X}$ satisfying the integer constraints – i.e. satisfying $\mathbf{x} \in X$ – is called an *output* vector or commodity bundle. Analogously, a vector $\mathbf{v} \in \overline{V}$ is said to be an *input* vector if and only if it satisfies the integer constraints, $\mathbf{v} \in V$. Setting

$$\mathbf{v} \notin V: \quad P(\mathbf{v}) = \emptyset,$$

a generalized production correspondence, $P\colon \overline{V} \to \mathfrak{P}(X)$ results, assigning an empty production possibility set to each inadmissible input vector $\mathbf{v} \notin V$. In the same way

$$\mathbf{x} \notin X: \quad L(\mathbf{x}) = \emptyset$$

yields a generalized input correspondence, $L\colon \overline{X} \to \mathfrak{P}(V)$. No input vector $\mathbf{v} \in V$ suffices to produce the vector $\mathbf{x} \notin X$. The advantage of an easier analytical usage of the extended correspondences is in opposition to the disadvantage that the generalized correspondences are no longer inverse to each other. Thus, the meaning of Proposition II.2 can only be partially transferred to the generalized correspondences.

The next step asks what results of the convex analysis remain unchanged when indivisible goods and factors are taken into account. If the production possibility sets or the input requirement sets are not convex, then surrogates for these sets serving as links to convex analysis are needed. The surrogates must be chosen such that the reverse conclusion to the original sets is still possible.

For this purpose we now introduce the concept of the **convex hull**.[37] The convex hull of a production possibility set is denoted by $\operatorname{conv} P(\mathbf{v})$ and describes the smallest convex set in $\overline{X}$ containing $P(\mathbf{v}) \subset X$; see Figure II.17, p. 31. Thereupon the output correspondence $P\colon \overline{V} \to \mathfrak{P}(X)$ induces a synthetic convex-valued correspondence[38]

$$\operatorname{conv} P\colon \overline{V} \to \mathfrak{P}(\overline{X})$$

substituting each level set $P(\mathbf{v})$ by its convex hull $\operatorname{conv} P(\mathbf{v})$. Under [P7] (*Convexity*) the distinction between the production possibility set and its convex hull becomes obsolete. In this case of divisible goods

$$\operatorname{conv} P(\mathbf{v}) = P(\mathbf{v}).$$

Analogously, $\operatorname{conv} L(\mathbf{x}) \subset \overline{V}$ denotes the convex hull of the input requirement set $L(\mathbf{x}) \subset V$, and the input correspondence $L\colon \overline{X} \to \mathfrak{P}(V)$ induces a convex-valued correspondence

$$\operatorname{conv} L\colon \overline{X} \to \mathfrak{P}(\overline{V})$$

assigning each vector $\mathbf{x} \in \overline{X}$ to the convex hull of the input requirement set $L(\mathbf{x})$. Again, [L7] (*Convexity*) implies for divisible production factors

$$\operatorname{conv} L(\mathbf{x}) = L(\mathbf{x}).$$

Not forgetting that $\operatorname{conv} P(\mathbf{v})$ denotes the smallest convex set containing $P(\mathbf{v})$, we can now give a representation of each single point in $\operatorname{conv} P(\mathbf{v})$. Each point can be expressed as a convex combination of points in $P(\mathbf{v})$ and, moreover, by Theorem B.3, p. 292, the union of all convex combination of points in $P(\mathbf{v})$ equals the convex hull $\operatorname{conv} P(\mathbf{v})$, i.e.:

$$\text{(II.7)}\quad \operatorname{conv} P(\mathbf{v}) = \left\{ \sum_{i=1}^{r} \lambda_i \mathbf{x}^i \mid \mathbf{x}^i \in P(\mathbf{v}), \right.$$
$$\left. \lambda_i \geqq 0 \ (i = 1, \ldots, r), \ \sum_{i=1}^{r} \lambda_i = 1, \ r = 1, 2, \ldots \right\}.$$

[37] Cf. FRANK (1969, p. 127). For the concept of convex hull see also Appendix B.2.

[38] Note that $\operatorname{conv} \emptyset = \emptyset$.

Theorem B.4 (CARATHÉODORY), p. 293, affords information about the expression of a point $\tilde{\mathbf{x}} \in \operatorname{conv} P(\mathbf{v})$ as a convex combination of points in $P(\mathbf{v})$, namely how many points at the most are required to describe $\tilde{\mathbf{x}}$; cf. the figure opposite. Regarding the n-dimensional commodity space $\overline{X} = \mathbb{R}^n$ the theorem says that each point $\tilde{\mathbf{x}} \in \operatorname{conv} P(\mathbf{v})$ can be expressed as a convex combination of $n+1$ not necessarily distinct points in $P(\mathbf{v})$,

$$\tilde{\mathbf{x}} = \sum_{i=0}^{n} \lambda_i \mathbf{x}^i \quad \text{with} \quad \mathbf{x}^i \in P(\mathbf{v}),\ \boldsymbol{\lambda} \in \Lambda^{n+1}.$$

For the sake of brevity, Λ^{n+1} denotes the n-dimensional unit simplex according to (B.5), p. 292.

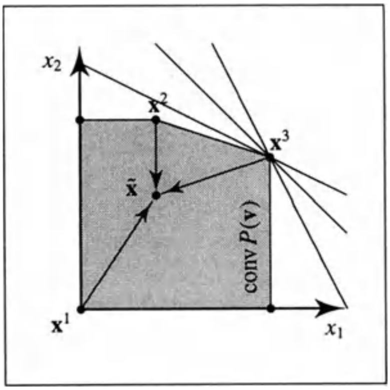

Figure II.10: Inner and outer representation of the convex hull conv $P(\mathbf{v})$

$$\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_n)^{\mathrm{T}} \in \Lambda^{n+1} \ :\Longleftrightarrow\ \left[\lambda_i \geqq 0\ (i = 0, 1, \ldots, n),\ \sum_{i=0}^{n} \lambda_i = 1\right]$$

Figure II.10 illustrates for $n = 2$ the expression of the point $\tilde{\mathbf{x}} \in \operatorname{conv} P(\mathbf{v})$ with the help of three points $\mathbf{x}^1 = \mathbf{0}$, $\mathbf{x}^2$ and $\mathbf{x}^3$ in $P(\mathbf{v})$ without explicitly giving the values of λ_i $(i = 1, 2, 3)$.

The "inner representation" of a convex hull by inner points may be reflected with "outer representations." For instance, $\operatorname{conv} P(\mathbf{v})$ may be interpreted as intersection of all convex sets containing $P(\mathbf{v})$.[39] In particular, the outer representation of a closed convex set by Theorem B.11, p. 297, will be in the focus of duality theory. Accordingly, a closed convex set $\operatorname{conv} P(\mathbf{v})$ corresponds to the intersection of all closed convex half-spaces containing $\operatorname{conv} P(\mathbf{v})$. Figure II.10 illustrates this statement using the example of some supporting hyperplanes at point $\mathbf{x}^3$.

Among the properties of the convex hull[40] it must be particularly emphasized at this point that the convex hull $\operatorname{conv} P(\mathbf{v})$ of a compact production possibility set $P(\mathbf{v})$ is compact, too. Using the theorem of KREIN-MILMAN, $\operatorname{conv} P(\mathbf{v})$ can be described as a convex hull of the set C of its extreme points.[41]

$$\text{(II.8)} \qquad \operatorname{conv} P(\mathbf{v}) = \operatorname{conv} C$$

Consequently, a point $\mathbf{x} \in \operatorname{conv} P(\mathbf{v})$ is called an **extreme point** of the set $\operatorname{conv} P(\mathbf{v})$ if the set $\operatorname{conv} P(\mathbf{v}) \setminus \{\mathbf{x}\}$ is convex or – equivalently – if it is not possible

[39]This definition may be found e.g. in NIKAIDO (1975, p. 185).

[40]See Appendix B.2.

[41]Cf. ROCKAFELLAR (1972, p. 167, Corollary 18.5.1). At the same time (II.8) implies that the set conv $P(\mathbf{v})$ must contain at least one extreme point.

to express $\mathbf{x}$ as a convex combination of two points in $\operatorname{conv} P(\mathbf{v})$ distinct from $\mathbf{x}$.[42]

$$\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2 = \mathbf{x},\ \ \lambda \in [0,1],\ \ \mathbf{x}^1, \mathbf{x}^2 \in \operatorname{conv} P(\mathbf{v}) \implies \mathbf{x}^1 = \mathbf{x}^2 = \mathbf{x}$$

Accordingly, the points $\mathbf{x}^1 = \mathbf{0}$, $\mathbf{x}^2$, and $\mathbf{x}^3$ in Figure II.10 denote three of the five extreme points.

For every extreme point $\mathbf{x}$ of $\operatorname{conv} P(\mathbf{v})$ the reverse conclusion gives $\mathbf{x} \in P(\mathbf{v})$ such that at least some points of the production possibility set can be reconstructed.[43]

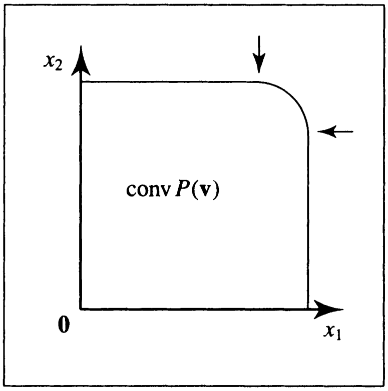

Figure II.11: Extreme points versus exposed points

The extreme points of a set $\operatorname{conv} P(\mathbf{v})$ associated with the inner representation of these sets are opposite the exposed points, which give a statement similar to (II.8) with respect to an outer representation of $\operatorname{conv} P(\mathbf{v})$. A vector $\mathbf{x}$ is called an **exposed point** of $\operatorname{conv} P(\mathbf{v})$ if there is a nontrivial[44] supporting hyperplane H with $\operatorname{conv} P(\mathbf{v}) \cap H = \{\mathbf{x}\}$. Each exposed point is also an extreme point, but the reverse conclusion does not hold; see the points marked by vectors in the figure opposite. Provided the segment corresponds to a quarter of a circle, then the criterion of exposed points cannot be fulfilled. The two points are, however, extreme points. Let C' be the set of all exposed points of $\operatorname{conv} P(\mathbf{v})$, then analogous to (II.8) for the closure of the convex hull[45] of C'

$$\text{(II.9)} \qquad \operatorname{conv} P(\mathbf{v}) = \operatorname{cl}(\operatorname{conv} C').$$

As the closure of the set $\operatorname{conv} C'$ – according to the definition of the convex hull – is the smallest of all closed sets containing $\operatorname{conv} C'$, both sets $\operatorname{conv} C'$ and $\operatorname{cl}(\operatorname{conv} C')$ can differ at the most by their boundary points. In the case of Figure II.11 both sets $\operatorname{conv} C'$ and $\operatorname{conv} P(\mathbf{v})$ differ exactly by the two marked exposed points.

With regard to the convex hull of an input requirement set $L(\mathbf{x})$ the preceding statements have to be modified since $\operatorname{conv} L(\mathbf{x})$ is unbounded. A priori, closedness of the input requirement set does no longer entail the closedness of its convex hull.[46] The following explanations serve to solve this problem.

For the moment the input requirement set $L(\mathbf{x})$ is interpreted as set of vectors and directions. A **vector** $\mathfrak{v} \in \mathbb{R}^m$ has a **direction** in $L(\mathbf{x})$ if and only if

$$\text{(II.10)} \qquad \forall \mathbf{v} \in L(\mathbf{x}),\ \forall \lambda \geqq 0: \quad \mathbf{v} + \lambda\mathfrak{v} \in L(\mathbf{x}).$$

[42]Cf. LEICHTWEISS (1980, p. 35).

[43]Cf. ROCKAFELLAR (1972, p. 165, Corollary 18.3.1).

[44]A supporting hyperplane H of a set C is said to be nontrivial when C is not entirely contained in the supporting hyperplane, $C \not\subset H$. Otherwise C could not uniquely be assigned to one of the two half-spaces generated by H.

[45]Cf. ROCKAFELLAR (1972, p. 168, Theorem 18.7).

[46]An easy counterexample is given in Appendix B.2, Figure B.1.

Figure II.12 shows two vectors $\mathbf{v}^1$ and $\mathbf{v}^2$. However, $\mathfrak{v}$ denotes a vector with a direction in $L(\mathbf{x})$. The set of all vectors with these directions is called a **recession cone** $0^+L(\mathbf{x})$.[47]

$$\text{(II.11)} \qquad 0^+L(\mathbf{x}) := \left\{\mathfrak{v} \in \mathbb{R}^m \mid \mathbf{v} + \lambda\mathfrak{v} \in L(\mathbf{x}) \ \forall\lambda \geqq 0, \ \forall \mathbf{v} \in L(\mathbf{x})\right\}$$

If the i-th input is perfectly divisible, then according to [L3] (*Disposability of Inputs*) the i-th unit vector $\mathbf{e}^i$ denotes a vector with a direction in $L(\mathbf{x})$, see Figure II.12. If all of the inputs are divisible (and if they are used as a positive amount), then the recession cone satisfies $0^+L(\mathbf{x}) = \mathbb{R}^m_+$. Finally, (II.10) and (II.11) can be combined to $L(\mathbf{x}) + 0^+L(\mathbf{x})$ taking into account the free disposability of inputs. Since the free disposability of inputs implies that $\mathbf{v} + \mathfrak{v}$ with $\mathfrak{v} \in 0^+L(\mathbf{x})$ is also an admissible input vector for each input vector $\mathbf{v} \in L(\mathbf{x})$, this set is called the free disposal hull.[48]

Keeping that in mind, a representation of the convex hull $\mathrm{conv}L(\mathbf{x})$ equivalent to (II.7) turns out to be as follows: a point $\mathbf{v}$ is an element of $\mathrm{conv}L(\mathbf{x})$ if and only if it can be expressed as a convex combination of the form

$$\text{(II.12)} \qquad \mathbf{v} = \sum_{j=1}^{k} \lambda_j \mathbf{v}^j + \sum_{j=k+1}^{r} \lambda_j \mathfrak{v}^j, \qquad \sum_{j=1}^{k} \lambda_j = 1, \ \ \lambda_j \geqq 0 \ (j = 1, \ldots, r),$$

where $\mathbf{v}^j$ are points in $L(\mathbf{x})$ and $\mathfrak{v}^j$ are arbitrary vectors with directions in $L(\mathbf{x})$, i.e. $\mathfrak{v}^j \in 0^+L(\mathbf{x})$. Figure II.12 shows the example $\mathbf{v} = {}^1\!/\!_2\mathbf{v}^1 + {}^1\!/\!_2\mathbf{v}^2 + \lambda_1\mathfrak{v}^1$ with $\mathfrak{v}^1 = \mathbf{e}^1$.

Theorem B.4 (CARATHÉODORY), p. 293, looks slightly different, i.e.: a point $\mathbf{v}$ in the m-dimensional factor space $\overline{V} = \mathbb{R}^m$ is an element of the set $\mathrm{conv}L(\mathbf{x})$ if and only if it can be expressed as a convex combination of $m + 1$ not necessarily distinct points and directions in $L(\mathbf{x})$.

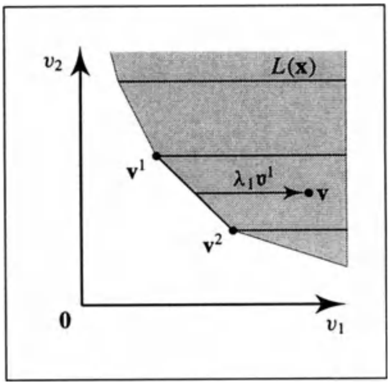

Figure II.12: Graphical representation of a point $\mathbf{v} \in \mathrm{conv}L(\mathbf{x})$ by II.12

An expression of $\mathrm{conv}L(\mathbf{x})$ analogous to (II.8) at once leads to the recognition that extreme points are not enough to represent, for instance, convex cones. As a result of this, the concept of an extreme ray is introduced. A closed half-line $h = \left\{\mathbf{v} + \lambda\mathfrak{v} \mid \lambda \geqq 0, \mathfrak{v} \in \mathbb{R}^m \setminus \{\mathbf{0}\}\right\}$ in the convex set $\mathrm{conv}L(\mathbf{x})$ is called an **extreme ray** of $\mathrm{conv}L(\mathbf{x})$ if the set $\mathrm{conv}L(\mathbf{x}) \setminus h$ is convex and if the vertex $\mathbf{v}$ of h is an extreme point of $\mathrm{conv}L(\mathbf{x})$.[49] The direction of $\mathfrak{v}$ is also called an extreme direction.

Due to $\mathrm{conv}L(\mathbf{x}) \subset \mathbb{R}^m_+$, $\mathrm{conv}L(\mathbf{x})$ contains no line such that the following

[47] A nonempty closed convex set $C \subset \mathbb{R}^n$ is bounded if and only if the recession cone 0^+C consists of the zero vector as singleton.

[48] Cf. MCFADDEN (1978, p. 8). The notation $L(\mathbf{x}) = L(\mathbf{x}) + 0^+L(\mathbf{x})$ only stresses the partition of the input requirement set into vectors and directions.

[49] Cf. LEICHTWEISS (1980, p. 39).

representation theorem can be given:[50] let $\operatorname{conv}L(\mathbf{x})$ be a closed set and let C be the set of all extreme points and extreme directions of $\operatorname{conv}L(\mathbf{x})$. Then

$$\text{(II.13)} \qquad \operatorname{conv}L(\mathbf{x}) = \operatorname{conv}C\,.$$

Analogous to exposed points of a convex set, exposed directions can be defined. A vector $\mathfrak{v}$ has an exposed direction in $\operatorname{conv}L(\mathbf{x})$ when the closed half-line $h = \{\mathbf{v} + \lambda\mathfrak{v} \mid \lambda \geqq 0,\ \mathfrak{v} \in \mathbb{R}^m \setminus \{\mathbf{0}\}\}$ can be described by a nontrivial supporting hyperplane H of $\operatorname{conv}L(\mathbf{x})$ such that $h = H \cap \operatorname{conv}L(\mathbf{x})$. Again, each extreme direction is an exposed direction where the reverse conclusion is still not admissible. If C' meets the set of all exposed points and directions, then according to (II.9)

$$\operatorname{conv}L(\mathbf{x}) = \operatorname{cl}(\operatorname{conv}C')$$

provided $\operatorname{conv}L(\mathbf{x})$ is closed.

For the sets $\operatorname{conv}P(\mathbf{v})$ and $\operatorname{conv}L(\mathbf{x})$ two properties are of importance whose meanings become clear later, especially, in dealing with gauges.

First of all, we know by the definition of the convex hull that all points on the segment between the origin $\mathbf{x} = \mathbf{0}$ and a commodity bundle $\mathbf{x} \in P(\mathbf{v})$ are elements of the convex hull of $P(\mathbf{v})$.

$$\text{(II.14)} \qquad \mathbf{v} \in V \text{ and } \mathbf{x} \in P(\mathbf{v}) \implies \lambda\mathbf{x} \in \operatorname{conv}P(\mathbf{v})\ \forall \lambda \in [0, 1]$$

Concerning divisible goods the conclusion holds good even for $P(\mathbf{v})$. This case is described as a weak disposability of outputs.[51]

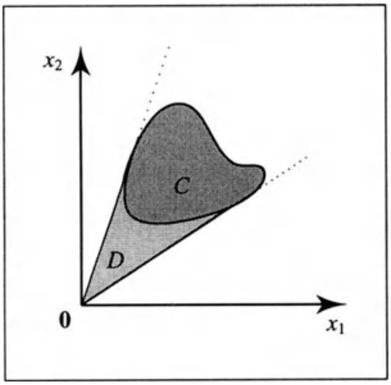

Figure II.13: Star hull, star C, of the set C

Statement (II.14) may now be generalized by defining the star hull of a set $C \subset \mathbb{R}^n$.

$$\text{(II.15)} \qquad \operatorname{star}C := \{\lambda\mathbf{x} \mid \mathbf{x} \in C,\ \lambda \in [0, 1]\}$$

The star hull of the set C depicted in Figure II.13 results from adding the set D, i.e. $\operatorname{star}C = C \cup D$. The set C is said to be **star-shaped** if $C = \operatorname{star}C$. Since each convex set containing the origin is star-shaped, the convex hulls of the production possibility sets are star-shaped for all input vectors $\mathbf{v} \in V$.

$$\operatorname{conv}P(\mathbf{v}) = \operatorname{star}\big(\operatorname{conv}P(\mathbf{v})\big)$$

The **aureoled hull** or haloed hull of a set $C \subset \mathbb{R}^n$ is defined analogous to the star hull,

$$\text{(II.16)} \qquad \operatorname{aur}C := \{\lambda\mathbf{x} \mid \mathbf{x} \in C,\ \lambda \geqq 1\}\,.$$

[50]Cf. ROCKAFELLAR (1972, p. 166, Theorem 18.5).

[51]Cf. FÄRE (1988, p. 6).

Accordingly, Figure II.14 indicates $\operatorname{aur} C = C \cup D$. Analogous to star-shaped sets, the set C is said to be **aureoled** if it meets the relation $C = \operatorname{aur} C$. Considering [L3] (*Disposability of Inputs*), the recession cone of $\operatorname{conv} L(\mathbf{x})$ is given by $0^+(\operatorname{conv} L(\mathbf{x})) = \mathbb{R}^m_+$ for all $\mathbf{x} \in X$ so that the convex hulls of the input requirement sets are aureoled for all commodity bundles $\mathbf{x} \in X$. [52]

$$\operatorname{conv} L(\mathbf{x}) = \operatorname{aur}\left(\operatorname{conv} L(\mathbf{x})\right)$$

In particular, analogous to (II.14)

$$\mathbf{v} \in L(\mathbf{x}) \implies \lambda \mathbf{v} \in \operatorname{conv} L(\mathbf{x}) \ \forall \lambda \geqq 1 .$$

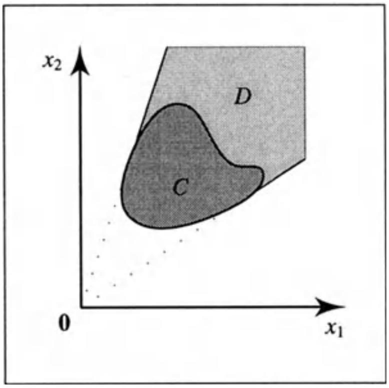

Figure II.14: Aureoled hull, aur C, of the set C

Considering divisible factors this property even holds true for $L(\mathbf{x})$, and it is called a weak disposability of factors.[53]

With regard to economic aspects the boundary points of the two convex hulls $\operatorname{conv} P(\mathbf{v})$ and $\operatorname{conv} L(\mathbf{x})$ turn out to be important. As already mentioned, the convex hull of a closed bounded set is closed as well as bounded such that each boundary point $\mathbf{x} \in \partial(\operatorname{conv} P(\mathbf{v}))$ belongs to $\operatorname{conv} P(\mathbf{v})$. Although the convex hull of a closed but unbounded set need not be closed, it is correct to presuppose the closedness of the sets $\operatorname{conv} L(\mathbf{x})$ under Axioms [L3] (*Disposability of Inputs*) and [L6] (*Closedness*). A proof of this hypothesis is given in Section 4.2, Proposition II.14. With that the boundary points $\mathbf{v} \in \partial(\operatorname{conv} L(\mathbf{x}))$ are also contained in the set $\operatorname{conv} L(\mathbf{x})$.

The structure of the sets examined in (II.15) or (II.16) permits criteria for when a point is a boundary point of the respective set. For the convex hull of the production possibility set the criterion following (II.15) is

$$\mathbf{x} \in \operatorname{conv} P(\mathbf{v}),\ \lambda \mathbf{x} \notin \operatorname{conv} P(\mathbf{v}) \ \forall \lambda > 1 \implies \mathbf{x} \in \partial\left(\operatorname{conv} P(\mathbf{v})\right).$$

In accordance with Figure II.15 each ray through the origin with a direction in $\mathbb{R}^n_+$ determines a boundary point of the convex hull $\operatorname{conv} P(\mathbf{v})$, calculating point $\tilde{\mathbf{x}}$ which is furthest away from the origin $\mathbf{x} = \mathbf{0}$ on the examined ray.[54]

It would be sensible to modify the criterion according to (II.16) for the convex hull of the input requirement set.

$$\mathbf{v} \in \operatorname{conv} L(\mathbf{x}),\ \lambda \mathbf{v} \notin \operatorname{conv} L(\mathbf{x}) \ \forall \lambda < 1 \implies \mathbf{v} \in \partial\left(\operatorname{conv} L(\mathbf{x})\right)$$

---

[52] The recession cone $0^+(\operatorname{conv} L(\mathbf{x}))$ yields for each vector $\mathbf{v} \in \operatorname{conv} L(\mathbf{x})$ and for each direction $\mathfrak{v} \in 0^+(\operatorname{conv} L(\mathbf{x}))$ the relation $\mathbf{v} + \lambda \mathfrak{v} \in \operatorname{conv} L(\mathbf{x})$ for all $\lambda \geqq 0$.
In particular, for $\mathbf{x} = \mathbf{0}$ with $\mathbf{v} = \mathbf{0} \in \operatorname{conv} L(\mathbf{0})$ we obtain $\operatorname{aur}(\operatorname{conv} L(\mathbf{0})) = \mathbb{R}^m_+$.

[53] Cf. FÄRE (1988, p. 9).

[54] The determined boundary point may coincide with the origin itself if $\mathbf{x} = \mathbf{0}$ is the only point possible on the ray concerned.

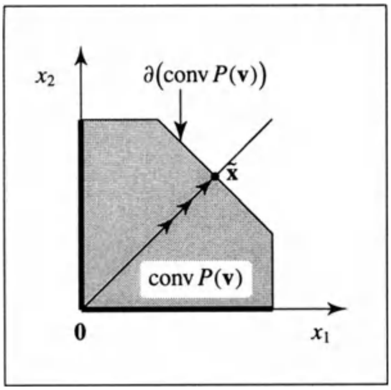

Figure II.15: Boundary points of the convex hull $\text{conv}\,P(\mathbf{v})$

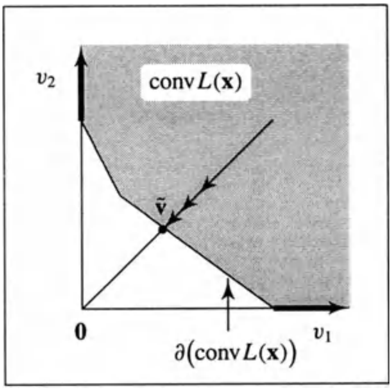

Figure II.16: Boundary points of the convex hull $\text{conv}\,L(\mathbf{x})$

Thus, a point $\tilde{\mathbf{v}}$ is a boundary point of the set $\text{conv}L(\mathbf{x})$ if there is no other point $\mathbf{v} \in \text{conv}L(\mathbf{x})$ which lies on this ray through the origin and $\tilde{\mathbf{v}}$ and which is closer to the origin $\mathbf{v} = \mathbf{0}$, see Figure II.16. Conversely, each ray through the origin with a direction in $\mathbb{R}^m_+$ is suitable to determine one of the boundary points.

Neither criteria suffice to express the entire boundary of the respective underlying set. The areas marked in bold in Figures II.15 and II.16 contain boundary points on common rays through the origin. However, only points are indicated which lie furthest away from the origin $\mathbf{x} = \mathbf{0}$ (Figure II.15) or which lie closest to the origin $\mathbf{v} = \mathbf{0}$ (Figure II.16). Corresponding phenomena may appear when the boundary of the sets $\text{conv}P(\mathbf{v})$ or $\text{conv}L(\mathbf{x})$ are backward bending; see Figure II.18. As subsequently shown, this behavior may be ignored. Intuitively, boundary points lying on the same ray through the origin cause no problems when inefficient points[55] are not considered. In particular, the points marked in bold on the axes in Figures II.15 and II.16 will not matter in economic analysis.

Transferring the statement of Axiom [P4] (*Disposability of Outputs*) to the convex hull $\text{conv}P(\mathbf{v})$ gives

Proposition II.3 *For each input vector* $\mathbf{v} \in V$ *we have under* [P1a] (*Possibility of Inaction*) *and* [P4] (*Disposability of Outputs*) *with respect to the convex hull of* $P(\mathbf{v})$*:*

$$\forall\, \hat{\mathbf{x}}, \tilde{\mathbf{x}} \in \overline{X}_+,\ \hat{\mathbf{x}} \leqq \tilde{\mathbf{x}}: \quad \tilde{\mathbf{x}} \in \text{conv}P(\mathbf{v}) \implies \hat{\mathbf{x}} \in \text{conv}P(\mathbf{v}).$$

Proof: Supposing $\tilde{\mathbf{x}} \in \text{conv}P(\mathbf{v})$, then each vector $\hat{\mathbf{x}} \in \overline{X}_+$ with $\hat{\mathbf{x}} \leqq \tilde{\mathbf{x}}$ can be expressed as a convex combination of the form

$$\text{(II.17)} \qquad \hat{\mathbf{x}} = \sum_{j=1}^{n} \theta_j \mathbf{e}^{j\mathsf{T}} \tilde{\mathbf{x}} \mathbf{e}^j + \theta_{n+1}\mathbf{0} + \theta_{n+2}\tilde{\mathbf{x}} \quad \text{with} \quad \sum_{j=1}^{n+2} \theta_j = 1,\ \theta_j \geqq 0,$$

where $\mathbf{e}^j$ $(j = 1, \ldots, n)$ are the unit vectors; see Figure II.17. Furthermore, according to Theorem B.4 (CARATHÉODORY) there are $n + 1$ vectors $\mathbf{x}^i$, which

[55] For the concept of efficiency, see Section 2.4.1.

need not necessarily be distinct, such that the following convex linear combination holds:

$$\tilde{\mathbf{x}} = \sum_{i=0}^{n} \lambda_i \mathbf{x}^i \qquad \text{with} \quad \mathbf{x}^i \in P(\mathbf{v}), \;\; \boldsymbol{\lambda} \in \Lambda^{n+1} \tag{II.18}$$

In Figure II.17 the vectors $\mathbf{x}^i$ are represented by $\mathbf{x}^1$, $\mathbf{x}^2$, and $\mathbf{x}^3$.
Due to $\mathbf{0} \in P(\mathbf{v})$ according to [P1a], it remains to be shown that $\mathbf{e}^{j\mathrm{T}}\tilde{\mathbf{x}}\mathbf{e}^j$ can also be expressed as a convex combination of points in $P(\mathbf{v})$. In this case (II.17) can be completely rewritten into a convex linear combination of points in $P(\mathbf{v})$.

With respect to Axiom [P4] we obtain

$$\mathbf{x}^i \in P(\mathbf{v}) \implies \mathbf{e}^{j\mathrm{T}}\mathbf{x}^i\mathbf{e}^j \in P(\mathbf{v})$$

for the vectors $\mathbf{x}^i$ in (II.18), for if $\mathbf{x}^i$ fulfills the integer constraints, then $\mathbf{e}^{j\mathrm{T}}\mathbf{x}^i\mathbf{e}^j = (0,\ldots,0,x^i_j,0,\ldots,0)^{\mathrm{T}}$ must also do so and, moreover, $\mathbf{e}^{j\mathrm{T}}\mathbf{x}^i\mathbf{e}^j \leqq \mathbf{x}^i$. If we put

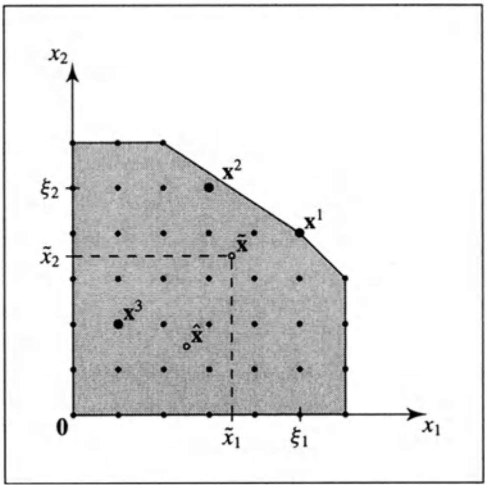

Figure II.17: Graphical representation of Proposition II.3

$$\xi_j := \max_i \left\{ \mathbf{e}^{j\mathrm{T}}\mathbf{x}^i \,|\, i = 0,\ldots,n \right\} \qquad (j = 1,\ldots,n),$$

then $\mathbf{e}^{j\mathrm{T}}\tilde{\mathbf{x}} \leqq \xi_j$ such that the required convex combinations for $j = 1,\ldots,n$ are found:

$$\exists \mu_j \in [0,1]: \quad \mathbf{e}^{j\mathrm{T}}\tilde{\mathbf{x}}\mathbf{e}^j = \mu_j \mathbf{0} + (1-\mu_j)\xi_j \mathbf{e}^j \implies \mathbf{e}^{j\mathrm{T}}\tilde{\mathbf{x}}\mathbf{e}^j \in \operatorname{conv} P(\mathbf{v});$$

see again Figure II.17. ■

Applied to the case of indivisible factors and analogous to Proposition II.3 is

Proposition II.4 *According to* [L3] *(Disposability of Inputs) for each commodity bundle* $\mathbf{x} \in X$ *with respect to the convex hull of* $L(\mathbf{x})$*:*

$$\forall\, \mathbf{v}, \tilde{\mathbf{v}} \in \overline{V}_+,\; \tilde{\mathbf{v}} \geqq \mathbf{v}: \quad \mathbf{v} \in \operatorname{conv} L(\mathbf{x}) \implies \tilde{\mathbf{v}} \in \operatorname{conv} L(\mathbf{x}).$$

Proof: According to the explanations on (II.16), based on [L3] the following recession cone of $\operatorname{conv} L(\mathbf{x})$ with $\mathbf{x} \in X$ ensues:

$$0^+\big(\operatorname{conv} L(\mathbf{x})\big) = \mathbb{R}^m_+.$$

Thus, the unit vectors $\mathbf{e}^i$ with $i = 1, \ldots, m$ are vectors with directions in conv$L(\mathbf{x})$. Hence, according to (II.12) each vector $\tilde{\mathbf{v}} \geqq \mathbf{v}$ has got a convex combination

$$\tilde{\mathbf{v}} = \mathbf{v} + \sum_{i=1}^{m} \lambda_i \mathbf{e}^i, \qquad \lambda_i \geqq 0 \ (i = 1, \ldots, m)$$

closing the proof. ∎

As shown above, Axiom [P4] (*Disposability of Outputs*) implies that the boundary of $P(\mathbf{v})$ is not allowed to bend backward for divisible goods; see Figure II.5. The same argument is valid with respect to the boundary of the convex hull $\partial(\text{conv} P(\mathbf{v}))$. In view of Proposition II.3 the case $\tilde{\mathbf{x}} \leq \mathbf{x}$ with $\mathbf{x} \in P(\mathbf{v})$ and $\tilde{\mathbf{x}} \notin \text{conv} P(\mathbf{v})$ is excluded; see the left hand part of Figure II.18. However, [P4] does not require each commodity bundle $\mathbf{x} \in X$ with $\mathbf{x} \in \text{conv} P(\mathbf{v})$ to describe an admissible activity $(\mathbf{v}, \mathbf{x})$, too; see point A in Figure II.18.

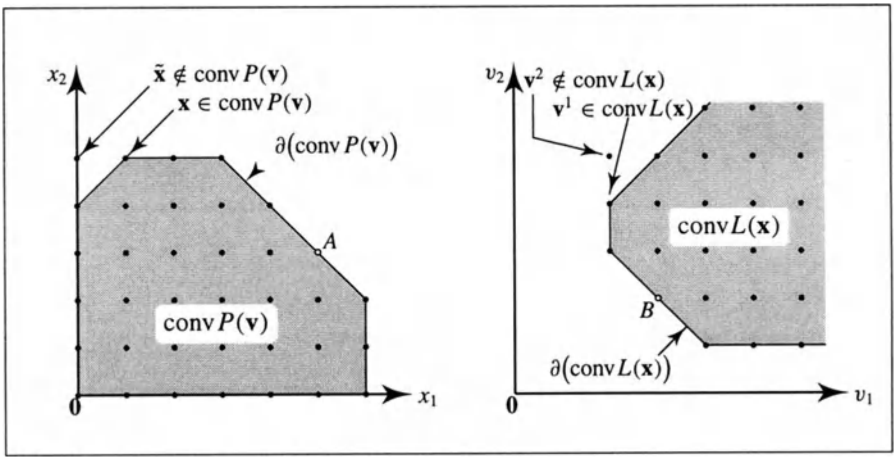

Figure II.18: Properties of the convex hull

If $(\mathbf{v}, \mathbf{x}) \in V \times X$ is a possible activity, then according to Proposition II.4 each vector $\tilde{\mathbf{v}} \in \mathbb{R}_+^m$ with $\tilde{\mathbf{v}} \geq \mathbf{v}$ must be an element of the convex hull of $L(\mathbf{x})$. Again the reverse conclusion that each pair $(\mathbf{v}, \mathbf{x}) \in V \times X$ with $\mathbf{v} \in \text{conv} L(\mathbf{x})$ is a possible process is not allowed without further assumptions. This case is illustrated by point B in Figure II.18.

2.2.2 The Assumption of Integer Convexity

The next section is dedicated to the problem that the convex hull may contain integer points which are not possible activities. Examples are shown in Figure II.18 by the points A and B. To allow the conclusion from the convex hull of a set – for instance conv$P(\mathbf{v})$ – to the original set – i.e. $P(\mathbf{v})$ – we require a weaker form of convexity for indivisible goods.

[P7a] Integer convexity:[56]

$$\forall\, \mathbf{v} \in V: \quad P(\mathbf{v}) \text{ is integer convex.}$$

For a given input vector $\mathbf{v}$ we speak of an integer convex production possibility set if the following condition is satisfied:

$$\mathbf{x} \in \operatorname{conv} P(\mathbf{v}) \text{ and } \mathbf{x} \in X \quad \Longrightarrow \quad \mathbf{x} \in P(\mathbf{v})$$

Each vector $\mathbf{x}$ in the convex hull of $P(\mathbf{v})$ fulfilling the integer constraints, $\mathbf{x} \in X$, describes, together with the held fixed input vector $\mathbf{v}$, a possible activity $(\mathbf{v}, \mathbf{x})$. Thus, cases as illustrated by point A in Figure II.18 are ruled out by assumption. In order to derive another important conclusion in connection with Proposition II.3, we now define

Definition II.3 *Given the commodity space* $X = \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}$ *with* n_d *divisible goods, the completed fractions of the indivisible goods can be determined by the residual vector* $\mathbf{x}^{\triangleleft} := \mathbf{x} - \lfloor \mathbf{x} \rfloor$ *if we define the operation* $\mathbf{x} \rightarrow \lfloor \mathbf{x} \rfloor := (\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor)^{\mathsf{T}}$ *by*

$$\lfloor x_j \rfloor = \begin{cases} x_j & \text{for } 1 \leqq j \leqq n_d \\ \{\text{greatest integer not greater than } x_j\} & \text{for } n_d < j \leqq n\,. \end{cases}$$

With the aid of this definition the following statements can be noted.

1. For each commodity bundle $\mathbf{x} \in X$ it is $\lfloor \mathbf{x} \rfloor = \mathbf{x}$ or equivalently $\mathbf{x}^{\triangleleft} = \mathbf{0}$.

2. By Axioms [P1a] (*Possibility of Inaction*), [P4] (*Disposability of Outputs*), and [P7a] (*Integer Convexity*)

   (II.19) $$\mathbf{x} \in \operatorname{conv} P(\mathbf{v}) \quad \Longrightarrow \quad \lfloor \mathbf{x} \rfloor \in P(\mathbf{v})$$

   holds true for each input vector $\mathbf{v} \in V$. The proof calls for scrutinizing an arbitrary point $\mathbf{x} \in \operatorname{conv} P(\mathbf{v})$. Due to $\lfloor \mathbf{x} \rfloor \leqq \mathbf{x}$, Proposition II.3 yields the relation $\lfloor \mathbf{x} \rfloor \in \operatorname{conv} P(\mathbf{v})$. Furthermore, since, by definition, $\lfloor \mathbf{x} \rfloor$ satisfies even the integer constraints, we have $\lfloor \mathbf{x} \rfloor \in P(\mathbf{v})$ from [P7a].

3. Under the same conditions we can conclude unequivocally from $\operatorname{conv} P(\mathbf{v})$ to $P(\mathbf{v})$.

   (II.20) $$\operatorname{conv} P(\mathbf{v}) \cap X = P(\mathbf{v})$$

   For the proof of this statement an arbitrary input vector $\mathbf{v} \in V$ is held fixed such that two cases must be examined:

   - $\mathbf{x} \in P(\mathbf{v}) \quad \Longrightarrow \quad \left[\mathbf{x} \in \operatorname{conv} P(\mathbf{v}),\ \mathbf{x} \in X\right] \quad \Longrightarrow \quad \mathbf{x} \in \operatorname{conv} P(\mathbf{v}) \cap X\,;$

[56] Cf. FRANK (1969, p. 34).

- $\mathbf{x} \in \operatorname{conv} P(\mathbf{v}),\ \mathbf{x} \in X \implies \mathbf{x} = \lfloor \mathbf{x} \rfloor \in \operatorname{conv} P(\mathbf{v}) \overset{(\text{II}.19)}{\implies} \mathbf{x} \in P(\mathbf{v})$.

The conception of integer convexity can also be applied without difficulty to the input requirement sets. For indivisible production factors [L7] (*Convexity*) is modified to

[L7a] Integer Convexity:

$$\forall \mathbf{x} \in X : \quad L(\mathbf{x}) \text{ is integer convex.}$$

As before, we speak of an integer convex input requirement set $L(\mathbf{x})$ for a given commodity bundle $\mathbf{x} \in X$ if

$$\mathbf{v} \in \operatorname{conv} L(\mathbf{x}) \text{ and } \mathbf{v} \in V \implies \mathbf{v} \in L(\mathbf{x}).$$

Every vector $\mathbf{v}$ in the convex hull of $L(\mathbf{x})$ together with the examined commodity bundle $\mathbf{x}$ constitutes a possible activity $(\mathbf{v}, \mathbf{x})$ if $\mathbf{v}$ satisfies the integer constraints, $\mathbf{v} \in V$. Therefore, cases such as point B in Figure II.18 are not allowed to appear. Analogous to the procedure in [P7a] (*Integer Convexity*), we now define

Definition II.4 *Let* $V = \mathbb{R}_+^{m_d} \times \mathbb{Z}_+^{m-m_d}$ *be the factor space with* m_d *divisible inputs. Then the fractions of indivisible factors which are lacking compared to the next larger input vector can be determined by the residual vector* $\mathbf{v}^{\triangleright} := \lceil \mathbf{v} \rceil - \mathbf{v}$ *if the operation* $\mathbf{v} \to \lceil \mathbf{v} \rceil := (\lceil v_1 \rceil, \ldots, \lceil v_m \rceil)^{\mathrm{T}}$ *is defined by*[57]

$$\lceil v_i \rceil = \begin{cases} v_i & \text{for } 1 \leqq i \leqq m_d \\ \{\text{smallest integer not smaller than } v_i\} & \text{for } m_d < i \leqq m\,. \end{cases}$$

Again three statements can be noted with this definition.

1. $\lceil \mathbf{v} \rceil = \mathbf{v}$ or equivalently $\mathbf{v}^{\triangleright} = \mathbf{0}$ for every input vector $\mathbf{v} \in V$.
2. By [L3] (*Disposability of Inputs*) and [L7a] (*Integer Convexity*)[58]

   $$\mathbf{v} \in \operatorname{conv} L(\mathbf{x}) \implies \lceil \mathbf{v} \rceil \in L(\mathbf{x}) \tag{II.21}$$

   holds for each commodity bundle $\mathbf{x} \in X$. Supposing, $\mathbf{v} \in \operatorname{conv} L(\mathbf{x})$ then due to $\lceil \mathbf{v} \rceil \geqq \mathbf{v}$, Proposition II.4 yields $\lceil \mathbf{v} \rceil \in \operatorname{conv} L(\mathbf{x})$. Since, by definition, $\lceil \mathbf{v} \rceil$ also satisfies the integer constraints, (II.21) is given by [L7a].
3. Under the same conditions we can conclude unequivocally from the convex hull $\operatorname{conv} L(\mathbf{x})$ to $L(\mathbf{x})$.

   $$\operatorname{conv} L(\mathbf{x}) \cap V = L(\mathbf{x})$$

   For the proof of this statement an arbitrary commodity bundle $\mathbf{x} \in X$ is held fixed so that two cases must be inspected:

[57] MOORE (1987) examines a production function of the form $x = f(\lfloor \mathbf{v} \rfloor)$. In this case the residual vector $\mathbf{v}^{\triangleleft} := \mathbf{v} - \lfloor \mathbf{v} \rfloor$ can be interpreted as a vector of excess capacities.

[58] For empirical tests on the extent of excess capacities in chemical industry, see LIEBERMAN (1989). Among others the author investigates a model of MANNE (1967) that assumes a demand growing in time where the examined firm is required to meet total demand. At the same time the capacity can only be extended in discrete steps.

- $\mathbf{v} \in L(\mathbf{x}) \implies [\mathbf{v} \in \operatorname{conv} L(\mathbf{x}),\ \mathbf{v} \in V] \implies \mathbf{v} \in \operatorname{conv} L(\mathbf{x}) \cap V$;
- $\mathbf{v} \in \operatorname{conv} L(\mathbf{x}),\ \mathbf{v} \in V \implies \mathbf{v} = \lceil \mathbf{v} \rceil \in \operatorname{conv} L(\mathbf{x}) \overset{(\text{II.21})}{\implies} \mathbf{v} \in L(\mathbf{x})$.

Analogous to the properties of integer convexity, [P7a] and [L7a], we now suppose

[T1a] Integer Convexity of the Graph:

$$(\mathbf{v}, \mathbf{x}) \in \operatorname{conv} \mathrm{GR} \ \text{ and } \ (\mathbf{v}, \mathbf{x}) \in V \times X \quad \implies \quad (\mathbf{v}, \mathbf{x}) \in \mathrm{GR}$$

Proposition II.5 *Given* [L3] *(Disposability of Inputs),* [L4] *(Disposability of Outputs), and* [T1a] *(Integer Convexity), then for each vector* $(\mathbf{v}, \mathbf{x})$ *in the convex hull of the graph* GR *a possible activity* $(\lceil \mathbf{v} \rceil, \lfloor \mathbf{x} \rfloor)$ *can be determined by calculating* $\lfloor \mathbf{x} \rfloor$ *and* $\lceil \mathbf{v} \rceil$ *corresponding to the Definitions II.3 and II.4.*

$$(\mathbf{v}, \mathbf{x}) \in \operatorname{conv} \mathrm{GR} \implies (\lceil \mathbf{v} \rceil, \lfloor \mathbf{x} \rfloor) \in \mathrm{GR}$$

Proof: A sketch of the proof is illustrated in Figure II.19; it consists of a combination of the Propositions II.4 and II.3.

Since $(\lceil \mathbf{v} \rceil, \lfloor \mathbf{x} \rfloor) \in V \times X$ always satisfies the integer constraints, the implication $(\lceil \mathbf{v} \rceil, \lfloor \mathbf{x} \rfloor) \in \operatorname{conv} \mathrm{GR} \implies (\lceil \mathbf{v} \rceil, \lfloor \mathbf{x} \rfloor) \in \mathrm{GR}$ holds according to [T1a]. Thus, it remains to be shown that $(\mathbf{v}, \mathbf{x}) \in \operatorname{conv} \mathrm{GR} \implies (\lceil \mathbf{v} \rceil, \lfloor \mathbf{x} \rfloor) \in \operatorname{conv} \mathrm{GR}$.

If $(\mathbf{v}, \mathbf{x}) \in \operatorname{conv} \mathrm{GR}$, then by Theorem B.4 (CARATHÉODORY) there are no more than $m + n + 1$ possible activities such that

$$(\mathbf{v}, \mathbf{x}) = \sum_{\nu=0}^{m+n} (\lambda_\nu \mathbf{v}^\nu, \lambda_\nu \mathbf{x}^\nu) \quad \text{with} \quad (\mathbf{v}^\nu, \mathbf{x}^\nu) \in \mathrm{GR},\ \boldsymbol{\lambda} \in \Lambda^{m+n+1}.$$

From Proposition II.4[59]

$$\mathbf{v}^\nu \in \operatorname{conv} L(\mathbf{x}^\nu) \implies \tilde{\mathbf{v}}^\nu \in \operatorname{conv} L(\mathbf{x}^\nu) \implies (\tilde{\mathbf{v}}^\nu, \mathbf{x}^\nu) \in \operatorname{conv} \mathrm{GR}$$

holds for all $\tilde{\mathbf{v}}^\nu \geqq \mathbf{v}^\nu$. Thus, for point $(\tilde{\mathbf{v}}, \mathbf{x})$ with $\tilde{\mathbf{v}} \geqq \mathbf{v}$ it will not be at all difficult to set a convex combination which consists of points $(\mathbf{v}^\nu, \mathbf{x}^\nu)$ supplemented by (arbitrarily many) points $(\tilde{\mathbf{v}}^\nu, \mathbf{x}^\nu)$. The implication

$$\mathbf{v} \leqq \tilde{\mathbf{v}} : \quad (\mathbf{v}, \mathbf{x}) \in \operatorname{conv} \mathrm{GR} \implies (\tilde{\mathbf{v}}, \mathbf{x}) \in \operatorname{conv} \mathrm{GR}$$

holds true especially for $\tilde{\mathbf{v}} = \lceil \mathbf{v} \rceil$. Analogously, Proposition II.3 ensues

$$\mathbf{x}^\nu \in \operatorname{conv} P(\mathbf{v}^\nu) \implies \tilde{\mathbf{x}}^\nu \in \operatorname{conv} P(\mathbf{v}^\nu) \implies (\mathbf{v}^\nu, \tilde{\mathbf{x}}^\nu) \in \operatorname{conv} \mathrm{GR}$$

for all $\mathbf{0} \leqq \tilde{\mathbf{x}}^\nu \leqq \mathbf{x}^\nu$. As before, the points $(\mathbf{v}^\nu, \mathbf{x}^\nu)$ can be supplemented by (arbitrarily many) points $(\mathbf{v}^\nu, \tilde{\mathbf{x}}^\nu)$ such that a convex combination for $(\mathbf{v}, \tilde{\mathbf{x}})$ with $\mathbf{0} \leqq \tilde{\mathbf{x}} \leqq \mathbf{x}$ exists. The implication

$$\mathbf{x} \geqq \tilde{\mathbf{x}} \geqq \mathbf{0} : \quad (\mathbf{v}, \mathbf{x}) \in \operatorname{conv} \mathrm{GR} \implies (\mathbf{v}, \tilde{\mathbf{x}}) \in \operatorname{conv} \mathrm{GR}$$

[59] The reverse conclusion $(\mathbf{v}, \mathbf{x}) \in \operatorname{conv} \mathrm{GR} \implies \mathbf{v} \in \operatorname{conv} L(\mathbf{x})$ is not admissible. For instance, $L(\mathbf{x}) = \emptyset$ holds for all $\mathbf{x} \notin X$ and therefore $\operatorname{conv} L(\mathbf{x}) = \emptyset$.

holds good especially for $\tilde{\mathbf{x}} = \lfloor \mathbf{x} \rfloor$. Summarizing we get Proposition II.5 or

$$(-\mathbf{v}, \mathbf{x}) \geqq (-\tilde{\mathbf{v}}, \tilde{\mathbf{x}}),\ \tilde{\mathbf{x}} \geqq \mathbf{0}: \quad (\mathbf{v}, \mathbf{x}) \in \operatorname{conv} \mathrm{GR} \implies (\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \in \operatorname{conv} \mathrm{GR}\ . \quad \blacksquare$$

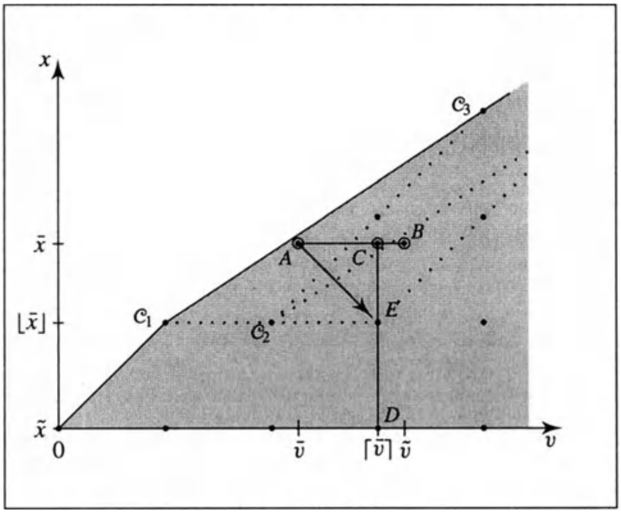

Figure II.19: Integer convexity of the graph

The graphical representation of the idea of the proof can be seen in Figure II.19, where point A is inferred from point E. The convex hull of the examined graph GR is marked by the shadowed area. The initial point A corresponds to a noninteger point $(\bar{v}, \bar{x}) \in \operatorname{conv} \mathrm{GR}$; it can be expressed as a convex combination of the points $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. Moving each of these points one unit to the right, the same convex combination yields point B with $(\tilde{v}, \bar{x})$. Now point C can be ascertained with a suitable convex combination of A and B so that the integer constraint to v is fulfilled, $(\lceil\bar{v}\rceil, \bar{x}) \in \operatorname{conv} \mathrm{GR}$. Consequently, point D with $(\lceil\bar{v}\rceil, 0) \in \mathrm{GR}$ ensues from point C. Finally, point E can be constructed from points C and D such that $(\lceil\bar{v}\rceil, \lfloor\bar{x}\rfloor) \in \mathrm{GR}$.

Disregarding the trouble in determining convex hulls, the operations $P(\mathbf{v}) \to \operatorname{conv} P(\mathbf{v})$ or $L(\mathbf{x}) \to \operatorname{conv} L(\mathbf{x})$ provide a useful tool implying only a minor loss of information under the presented assumptions. In particular, the extreme points of convex hulls – such as point $\mathcal{C}_1$ in Figure II.19 – satisfy the integer constraints and they belong to the original sets. Under the assumption of integer convex sets the original sets can even be completely reconstructed from the respective convex hull. The disadvantages unavoidably associated with the surrogate of the convex hull are outweighed by the remarkably greater advantages in handling convex sets. In particular, closed, bounded, star-shaped, or aureoled convex sets will be important for the following economic analysis.

2.3 Special Production Technologies

2.3.1 Scale Economies

The description of scale economies begins with some expositions on the homogeneity of correspondences underlying the criterion of the homogeneity of functions. The exact definitions of the used terms of convex analysis are introduced in Appendix B. Moreover, Appendices C.1 and C.2 contain explanations of the concepts of a *function as single-valued mapping* and of a *correspondence as multi-valued mapping.*

Commencing with the scalar multiplication $\lambda C = \{\lambda \mathbf{x} | \mathbf{x} \in C\}$ of a set $C \subset \mathbb{R}^n$ by a scalar λ, then for each scalar $\lambda \neq 0$ we get the equivalence relation

$$\mathbf{x} \in \lambda C \iff \mathbf{x}/\lambda \in C.$$

The scalar multiplication suggests examining sets of a certain structure. A set C is called a **cone**[60] when all $\lambda > 0$ lead to $\lambda C \subset C$, i.e.

$$\mathbf{x} \in C,\ \lambda > 0 \implies \lambda\mathbf{x} \in C.$$

Point **0** is called the **vertex** of the cone and does not need to belong to C by the presented definition.

These cones are opposite to the class of linearly homogeneous functions[61] since the epigraph, epi f, of a linearly homogeneous function f with

$$f(\lambda\mathbf{x}) = \lambda f(\mathbf{x}) \qquad \forall \lambda > 0$$

is a cone; see Figure II.20.

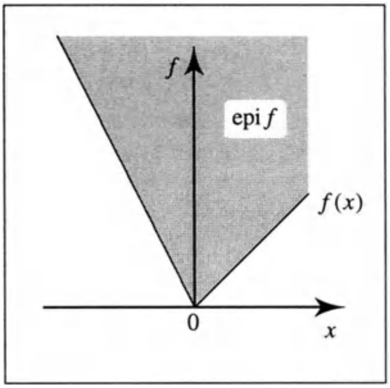

Figure II.20: Linear homogeneity of a function

Now the transference of the idea of linearly homogeneous functions to an output correspondence P or an input correspondence L causes no difficulties if all goods and factors are divisible. By a scalar multiplication of a production possibility set

$$\lambda P(\mathbf{v}) := \{\lambda\mathbf{x} | \mathbf{x} \in P(\mathbf{v})\}$$

we get as previously for $\lambda > 0$

$$\mathbf{x} \in \lambda P(\mathbf{v}) \iff \mathbf{x}/\lambda \in P(\mathbf{v}).$$

Thus, it seems reasonable to introduce a criterion of homogeneity of the following form.

Definition II.5 (Homogeneity)[62] *Suppose all of the goods and factors are perfectly divisible, then an output correspondence P is called*

subhomogeneous if	$\forall \mathbf{v} \in V,\ \forall \lambda \geqq 1:$	$P(\lambda\mathbf{v}) \subset \lambda P(\mathbf{v})$;
homogeneous of degree 1 if	$\forall \mathbf{v} \in V,\ \forall \lambda > 0:$	$\lambda P(\mathbf{v}) = P(\lambda\mathbf{v})$;
superhomogeneous if	$\forall \mathbf{v} \in V,\ \forall \lambda \geqq 1:$	$\lambda P(\mathbf{v}) \subset P(\lambda\mathbf{v})$.

60 Cf. ROCKAFELLAR (1972, p. 13).

61 See Appendix C.1.

62 Cf. for this section FÄRE (1988, p. 149).
For divisible goods the commodity space is $X = \mathbb{R}^n_+$. Divisible production factors similarly imply $V = \mathbb{R}^m_+$ with respect to the factor space.

In this definition the criteria of subhomogeneity as well as superhomogeneity can equivalently be rewritten into

P is subhomogeneous if $\quad \forall \mathbf{v} \in V, \ \forall \mu \in]0, 1]: \ \mu P(\mathbf{v}) \subset P(\mu \mathbf{v})$;
P is superhomogeneous if $\quad \forall \mathbf{v} \in V, \ \forall \mu \in]0, 1]: \ P(\mu \mathbf{v}) \subset \mu P(\mathbf{v})$.

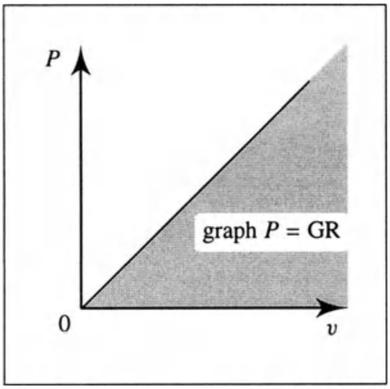

Figure II.21: Linear homogeneity of an output correspondence

In particular, the case of a linearly homogeneous output correspondence may be identified with a cone; see Figure II.21. If the proportionate variation of all inputs implies a proportionate change of all outputs by the same factor or if more precisely

$$\mathbf{x} \in P(\mathbf{v}) \iff \lambda \mathbf{x} \in P(\lambda \mathbf{v})$$

holds for all $\lambda > 0$, then we speak of **constant returns to scale**.

The output correspondence P is said to be superhomogeneous if every possible process $(\mathbf{v}, \mathbf{x})$ can be multiplied by a scalar $\lambda \geqq 1$ such that the resulting activity $(\lambda \mathbf{x}, \lambda \mathbf{v})$ is again possible.

$$\mathbf{x} \in P(\mathbf{v}) \implies \lambda \mathbf{x} \in P(\lambda \mathbf{v}) \quad \forall \lambda \geqq 1$$

Similarly, in accordance with (II.4) for a superhomogeneous output correspondence P the graph GR of the production technology concerned must be an aureoled set as defined by (II.16),[63] i.e. $\mathrm{GR} = \operatorname{aur} \mathrm{GR}$.

When the phenomenon of a proportionate increase of all factors associated with an overproportionate increase of the outputs is indicated by the term **increasing returns to scale**, then the superhomogeneity of the output correspondence P corresponds to nondecreasing returns to scale. Conversely, there are **nonincreasing returns to scale** if the production technology is subhomogeneous. We speak of a subhomogeneous output correspondence P if each possible activity can be multiplied by a scalar $\mu \in]0, 1]$ such that there again results a possible process $(\mu \mathbf{v}, \mu \mathbf{x})$.

$$\mathbf{x} \in P(\mathbf{v}) \implies \mu \mathbf{x} \in P(\mu \mathbf{v}) \quad \forall \mu \in]0, 1]$$

Thus, the output correspondence P is subhomogeneous when in accordance with (II.15) the graph of the production technology GR is a star-shaped set, $\mathrm{GR} = \operatorname{star} \mathrm{GR}$.[64] Looking at (II.6) makes it clear that a convex graph GR is only compatible with nonincreasing returns to scale.[65] In Figure II.22 the scalar is set

[63] Tthis statement can be illustrated by the parabola $x = v^2$!

[64] This statement can be illustrated by a square root $x = v^{1/2}$.

[65] Analogous to DEBREU (1959, p. 40 f.), the existence of scale economies is indicated at this point as a global property of a production technology.

to $\mu = {}^1\!/_2$ purely as an example.

$$^1\!/_2\, P(\tilde{v}) \subset P(\tilde{v}/2) \subset P(\tilde{v})$$

The treatment of the criterion of homogeneity with respect to the inverse input correspondence L ensues analogously.

Definition II.6 *Supposing that all of the goods and factors are divisible, then an input correspondence L is called*

subhomogeneous *if*	$\forall \mathbf{x} \in X,\ \forall \lambda \geqq 1:$	$L(\lambda\mathbf{x}) \subset \lambda L(\mathbf{x});$
homogeneous of degree 1 *if*	$\forall \mathbf{x} \in X,\ \forall \lambda > 0:$	$\lambda L(\mathbf{x}) = L(\lambda\mathbf{x});$
superhomogeneous *if*	$\forall \mathbf{x} \in X,\ \forall \lambda \geqq 1:$	$\lambda L(\mathbf{x}) \subset L(\lambda\mathbf{x}).$

Again there are equivalent criteria for the subhomogeneity as well as for the superhomogeneity.

L is subhomogeneous if	$\forall \mathbf{x} \in X,\ \forall \mu \in\ ]0, 1]:$	$\mu L(\mathbf{x}) \subset L(\mu\mathbf{x});$
L is superhomogeneous if	$\forall \mathbf{x} \in X,\ \forall \mu \in\ ]0, 1]:$	$L(\mu\mathbf{x}) \subset \mu L(\mathbf{x}).$

Proposition II.6 *The output correspondence P is subhomogeneous, homogeneous of degree 1, or superhomogeneous if an only if the input correspondence L is subhomogeneous, homogeneous of degree 1, or superhomogeneous respectively.*

Proof: Provided that $\lambda \geqq 1$ and that P is subhomogeneous, $^1\!/_\lambda\, P(\mathbf{v}) \subset P(\mathbf{v}/\lambda)$, then

$$\begin{aligned} L(\lambda\mathbf{x}) &= \{\mathbf{v}|\ \lambda\mathbf{x} \in P(\mathbf{v})\} \\ &= \left\{\mathbf{v}|\ \mathbf{x} \in {}^1\!/_\lambda\, P(\mathbf{v})\right\} \\ &\subset \{\mathbf{v}|\ \mathbf{x} \in P(\mathbf{v}/\lambda)\} && \text{subhomogeneity} \\ &= \{\lambda\tilde{\mathbf{v}}|\ \mathbf{x} \in P(\tilde{\mathbf{v}})\} && \text{with}\quad \tilde{\mathbf{v}} := \mathbf{v}/\lambda \\ &= \lambda L(\mathbf{x}) \end{aligned}$$

Consequently, the subhomogeneity of L results from the subhomogeneity of P. The remaining proof is given by analogous arguments. ■

As already mentioned, Figure II.22 illustrates the case of a subhomogeneous output correspondence and the case of a subhomogeneous input correspondence. Concerning the input requirement sets $L(\tilde{x})$, the scalar $\lambda = {}^1\!/_2$ results in

$$L(\tilde{x}) \subset {}^1\!/_2\, L(\tilde{x}) \subset L(\tilde{x}/2).$$

The presented criterion of homogeneity should not be mixed up with the degree of homogeneity. A function $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ is **homogeneous of degree *r*** if $\lambda^r f(\mathbf{x}) = f(\lambda\mathbf{x})$ holds good for each $\mathbf{x}$ and for each $\lambda > 0$. If we apply the analogous criterion to correspondences, then the output correspondence P is

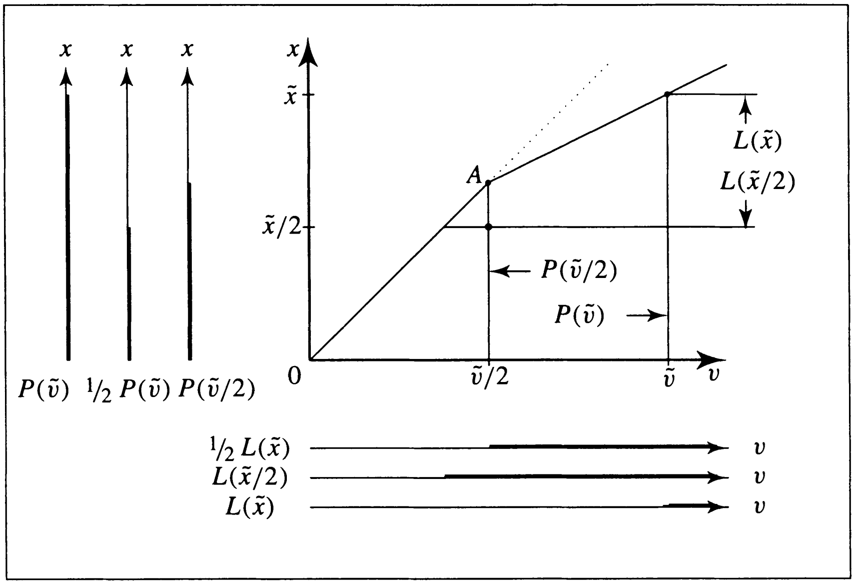

Figure II.22: Subhomogeneity of a production technology

homogeneous of degree $r \neq 0$ if and only if the input correspondence L is homogeneous of degree $^1/_r$.[66]

$$\lambda^r P(\mathbf{v}) = P(\lambda \mathbf{v}) \quad \Longleftrightarrow \quad \lambda^{1/r} L(\mathbf{x}) = L(\lambda \mathbf{x})$$

A comparison to Proposition II.6 shows that the subhomogeneity or the superhomogeneity of a correspondence must be strictly distinguished from the degree of homogeneity of this correspondence.[67]

The treatment of the phenomenon of constant returns to scale in view of indivisible goods and factors immediately implies that not all of the possible activities $(\mathbf{v}, \mathbf{x}) \in \mathrm{GR}$ can be multiplied by an arbitrary scalar $\lambda > 0$ such that the outcome $\lambda(\mathbf{v}, \mathbf{x})$ does not violate integer constraints. Even a possible proportionate increase of all inputs with $\lambda \mathbf{v} \in V$ does not imply that $\lambda \mathbf{x}$ is an admissible commodity bundle, $\lambda \mathbf{x} \notin X$ as well. Conversely, scalars exist for which the described problem cannot appear. If $(\mathbf{v}, \mathbf{x})$ fulfills the integer constraints, then, for instance, $2(\mathbf{v}, \mathbf{x})$ cannot violate these restrictions. Thus, for a possible activity $(\mathbf{v}, \mathbf{x}) \in \mathrm{GR}$ we speak of a **possible change of the production level** by the factor λ if

$$(\mathbf{v}, \mathbf{x}) \in V \times X,\ \lambda > 0 \quad \Longrightarrow \quad \lambda(\mathbf{v}, \mathbf{x}) \in V \times X .$$

After these preliminary remarks we can now establish criteria for the different forms of scale economies where the attention is directed to possible enlargements of the production level.

[66] Cf. EICHHORN (1978, p. 218 ff.).

[67] If for example the output correspondence P is homogeneous of degree $r > 1$, then the inverse input correspondence L is homogeneous of degree $1/r < 1$, although both correspondences are not subhomogeneous.

Definition II.7[68] *Let* $(\mathbf{v}, \mathbf{x}) \in \mathrm{GR}$ *be a possible activity.*
(a) *The activity* $(\mathbf{v}, \mathbf{x})$ *generates* ***decreasing economies to scale*** *if every enlargement of the production level yields*[69]

$$\forall \lambda > 1 : \quad \lambda(\mathbf{v}, \mathbf{x}) \notin \operatorname{conv} \mathrm{GR}.$$

(b) *The activity* $(\mathbf{v}, \mathbf{x})$ *generates* ***increasing returns to scale*** *if every possible increase of the production level implies a possible activity, i.e.*

$$\forall \lambda > 1 : \quad \lambda(\mathbf{v}, \mathbf{x}) \in V \times X \implies \lambda(\mathbf{v}, \mathbf{x}) \in \mathrm{GR},$$

and if for at least one possible increase of the production level $(\tilde{\lambda} > 1)$ *there is a possible activity* $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \in \mathrm{GR}$ *with* $(-\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \geq \tilde{\lambda}(-\mathbf{v}, \mathbf{x})$.

(c) *Accordingly, the activity* $(\mathbf{v}, \mathbf{x})$ *generates* ***constant returns to scale*** *if every possible increase of the production level* $(\lambda > 1)$ *implies a possible activity, but if there are no increasing returns to scale.*

For an idea of this definition we can refer to the production function $x = \lfloor 4/3 \lfloor v \rfloor \rfloor$. The indivisible output x is produced by an indivisible input v. As the last unit of the production factor becomes useless when only fractions of it are available, the productive part of the used quantity of the input v amounts to the greatest integer $\lfloor v \rfloor$ not greater than v. Similarly, the maximal quantity of output $4/3 \lfloor v \rfloor$ may include unfinished and, therefore, unusable commodity units. Thus, x denotes the number of finished commodity units.

As gathered from Figure II.23, the convex hull of the graph

$$\mathrm{GR} = \left\{(v, x) \mid x \in \mathbb{Z}_+,\ x \leqq \lfloor 4/3 \lfloor v \rfloor \rfloor \right\}$$

is an (integer) convex cone with the origin $(v, x) = (0, 0)$. The bold dots are associated with quantities of input corresponding to multiples of 3, whereas the quantities of output are multiples of 4. Since no possible activity lies above the ray through the origin, the activities marked by bold dots obey constant returns to scale.

Similarly, we can identify increasing returns to scale. Starting with the activities marked by $\odot$, each possible variation of the production level generates a possible activity lying in the adjoined ray through the origin and the points $\odot$. The smaller one of the two mentioned activities $(v, x) = (2, 2)$ generates increasing returns to scale since the greater activity $(v, x) = (4, 4)$ is dominated by $(v, x) = (3, 4)$;[70] thus, a doubling of the outputs can be realized by an underproportionate enlargement of inputs.

Before continuing the analysis, it is helpful to stress some aspects of scale economies. Constant returns to scale seem to be a plausible assumption on

[68]Cf. FRANK (1969, p. 43) or FÄRE (1988, p. 151).

[69]In particular, all possible increases of the production level imply $\lambda \mathbf{x} \notin P(\lambda \mathbf{v})$. A graphical representation of this definition results from point A in Figure II.22.

[70]For the concept of technical efficiency of an activity, see Section 2.4.1.
Furthermore, EDWARDS, STARR (1987) discuss at this point specialization effects, where the indivisibility of workers imposes an upper bound on the degree of specialization. Although specialization and cooperation of individual people as a cause for increasing returns to scale is emphasized in the literature on the division of labor (see e.g. GROENEWEGEN (1987)), the associated phenomenon conjured up by indivisible persons usually remains unmentioned.

production technologies if we assume that each possible activity may be copied.[71] This idea rules out at least decreasing returns to scale and is compatible with indivisible goods and factors. However, the reverse that each possible activity may be divided into activities of the same size turns out to be a fallacy regarding indivisible goods and factors.

Supposing again for divisible goods and factors that the production level for all possible activities may be increased *and* decreased arbitrarily, then even increasing returns to scale are excluded. Nevertheless, indivisible goods or factors become unusable by a physical division.

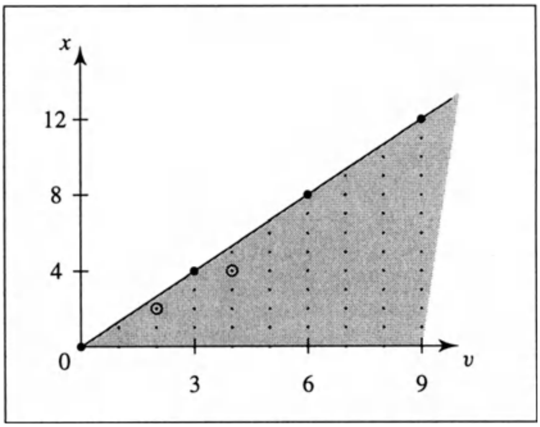

Figure II.23: Integer constant returns to scale

The criterion of increasing returns to scale loses its meaning when it is applied to inefficient activities (the concept of efficiency will be introduced in Section 2.4.1). At this point it may be enough to examine the activity $(v, x) = (2, 1)$, which is inefficient because a larger output can be produced by the same input. This activity satisfies the technical requirements of increasing returns to scale, but the waste of an additional output unit with respect to the activity concerned is the reason for this. Each copy of this activity leads to a corresponding copy of the ignored output unit such that, technically, there must be increasing returns to scale.

The described phenomenon does even not apply to the activity $(v, x) = (2, 2)$ marked by $\odot$. As shown by Figure II.23, no possible activity with the same output can be carried out at smaller inputs or permits a larger output at constant inputs. Both criteria are only satisfied because the activities marked by bold dots cannot be arbitrarily divided. Accordingly, goods and factors which are not arbitrarily divisible can be one source of increasing returns to scale.[72] On the one hand an overproportionate increase of the outputs can be achieved by joining the excess capacities; see activities[73] $(v, x) = (2, 2)$ and $(v, x) = (4, 5)$. On the other hand an underproportionate increase of the inputs is sufficient for doubling the outputs; see activities $(v, x) = (2, 2)$ and $(v, x) = (3, 4)$.

[71] Cf. SILVESTRE (1987, p. 81).

[72] KOOPMANS (1957, p. 152) states: "I have not found one example of increasing returns to scale where there is not some indivisible commodity in the surrounding circumstances."
However, VARIAN (1992, p. 15) offers a further determinant of increasing returns to scale. For example, doubling the quantity of steel in producing oil pipes by increasing the diameter of the pipe results in an overproportionate enlargement of the volume of a pipe. Thus, we speak of scale economies of the third dimension.

[73] The convention introduced in Definition II.3 reflects the problem at hand more accurately. Multiplication of the activity $(v, x) = (3, 4)$ by $\lambda = 2/3$ leads to $\lambda(v, x) = (2, 8/3)$, with the completed fraction of the last unit of the indivisible good given by $x^{\triangleleft} = x - \lfloor x \rfloor = 2/3$.

Although each possible activity can theoretically be repeated exactly, it is useful with respect to scarce resources to also permit decreasing returns to scale. Although a production technology may exhibit constant returns to scale when *all* of the inputs are increased, it is, however, usual to speak of nonconstant returns to scale with regard to the variable inputs. For instance, the production function $f(v_1, v_2, v_3) = v_1^{\alpha_1} v_2^{\alpha_2} v_3^{\alpha_3}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and with an indivisible v_3 obeys integer constant returns to scale. However, it is not be at all difficult to speak of decreasing returns to scale for $f(v_1, v_2, 1) = v_1^{\alpha_1} v_2^{\alpha_2}$ with $\alpha_1 + \alpha_2 < 1$. [74]

2.3.2 Additivity of a Production Technology

In the preceding expositions we compared activities which differ only by a positive constant factor of their production level. Both activities $(\mathbf{v}, \mathbf{x})$ and $(\lambda\mathbf{v}, \lambda\mathbf{x})$ lie on the same ray through the origin. The idea that the resulting activity is composed of two independent activities – for instance $(2\mathbf{v}, 2\mathbf{x}) = (\mathbf{v}, \mathbf{x}) + (\mathbf{v}, \mathbf{x})$ – is now stated more precisely by the assumption

[T2] Additivity: The production technology is additive.

A production technology is called additive if the sum of two activities $(\mathbf{v}, \mathbf{x})$ and $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}})$, which can be carried out separately, is again possible.

$$(\mathbf{v}, \mathbf{x}),\ (\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \in \mathrm{GR} \implies \big((\mathbf{v} + \tilde{\mathbf{v}}), (\mathbf{x} + \tilde{\mathbf{x}})\big) \in \mathrm{GR}$$

As long as this assumption refers to two processes which are each viable on their own, there is no dissent.[75] But, additivity requires that each integer multiple (integer increases to scale) of a possible activity is possible as well.

$$\forall\,(\mathbf{v}, \mathbf{x}) \in \mathrm{GR},\ \ \forall\, k \in \mathbb{Z}_+ : \quad (k\mathbf{v}, k\mathbf{x}) \in \mathrm{GR}$$

With that, Assumption [T2] *(Additivity)* immediately excludes decreasing returns to scale if all of the relevant production factors are included in the technology. Furthermore, additivity together with the possibility of inaction $(\mathbf{v}, \mathbf{x}) = (\mathbf{0}, \mathbf{0}) \in \mathrm{GR}$ implies $\operatorname{conv} \mathrm{GR}$ to be a cone; see Figure II.23. While the union of two observed activities to a particular process causes no difficulties, the assumption of the possibility to reproduce an activity arbitrarily often makes clear that the additivity of a production *technology* does not consider factor constraints.

However, scarce resources like the factor land are frequently not taken into account when describing a production technology just because the production factors are not available in an arbitrary quantity. This confusion of technology – in the sense of knowledge of feasible production methods – and factor constraints leads to different interpretations of a production technology. Since it usually seems to be impossible to duplicate a firm exactly, the assumption of additivity is often evaded. Two fundamental reasons can be offered for this.

[74] For instance, v_3 may be the number of blocks of a power station.

[75] Look for example at the case of two countries before and after taking up trade.

1. The firm's production technology cannot be completely specified.
2. Scarce resources prevent from repeating production processes. Here even the knowledge of production methods represented by skilled workers can be understood as a scarce resource.

Form the point of view of the second argument we have the following interpretation of additivity: understanding the sum of activities of individual firms as a process of an industry sector, the assumption of additivity means that in this sector prevails **free market entry**.[76] This interpretation does not refer to the free disposable knowledge of feasible production methods, which is in principle free, but it directs the attention to the restrictions subjected to each additional firm. Neither scarce resources nor bounded opportunities of sales prevent market entry.

The criterion [T2] of an additive production technology implies

$$\left.\begin{array}{ll} \mathbf{x} \in X, & \mathbf{v} \in L(\mathbf{x}) \\ \tilde{\mathbf{x}} \in X, & \tilde{\mathbf{v}} \in L(\tilde{\mathbf{x}}) \end{array}\right\} \implies \mathbf{v} + \tilde{\mathbf{v}} \in L(\mathbf{x} + \tilde{\mathbf{x}})$$

for the corresponding input correspondence or more generally[77]

$$\mathbf{x}^i \in X,\ \mathbf{v}^i \in L(\mathbf{x}^i)\ \ (i = 1, \ldots, k) \implies \textstyle\sum_i \mathbf{v}^i \in L\left(\sum_i \mathbf{x}^i\right).$$

Since the sum of two input requirement sets is defined by[78]

$$L(\mathbf{x}^1) + L(\mathbf{x}^2) = \left\{\mathbf{v}^1 + \mathbf{v}^2 \mid \mathbf{v}^1 \in L(\mathbf{x}^1),\ \mathbf{v}^2 \in L(\mathbf{x}^2)\right\},$$

we can summarize this relation by

Definition II.8[79] *An input correspondence L is called* ***superadditive*** *if it has the following property.*

$$\textstyle\sum_i L(\mathbf{x}^i) \subset L\left(\sum_i \mathbf{x}^i\right) \qquad \textit{with} \quad \mathbf{x}^i \in X \ \ (i = 1, \ldots, k)$$

Provided the existence of an input vector

$$\tilde{\mathbf{v}} \in L\left(\textstyle\sum_i \mathbf{x}^i\right) \qquad \textit{with} \quad \tilde{\mathbf{v}} \leq \mathbf{v} \quad \forall\, \mathbf{v} \in \textstyle\sum_i L(\mathbf{x}^i)$$

is guaranteed for a superadditive input correspondence L, then we speak of ***strict superadditivity****.*

[76] Cf. FRANK (1969, p. 15).

[77] A mapping, being additive as well as homogeneous of degree 1, is said to be **linear**. Cf. BERGE (1963, p. 133). EICHHORN (1978, p. 195) shows that each linear function $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ is determined by a unique $m \times n$-matrix $\mathbf{A}$, $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

[78] See Appendix B.

[79] The thought of superadditivity becomes comprehensible when we consider that an input correspondence L is said to be **additive** if it fulfills the functional equation $L(\mathbf{x}^1) + L(\mathbf{x}^2) = L(\mathbf{x}^1 + \mathbf{x}^2)$ for all $\mathbf{x}^1, \mathbf{x}^2 \in X$. Cf. EICHHORN (1978, p. 217).

Accordingly, arbitrary activities of a superadditive input correspondence can be combined in the sense of addition to a possible activity. Furthermore, the notation $L(\mathbf{x}^1)+L(\mathbf{x}^2) \subset L(\mathbf{x}^1+\mathbf{x}^2)$ states that there may be an input vector $\tilde{\mathbf{v}} \in L(\mathbf{x}^1+\mathbf{x}^2)$ not belonging to $L(\mathbf{x}^1)+L(\mathbf{x}^2)$. If such an input vector $\tilde{\mathbf{v}}$ exists, then it must have at least one smaller component than all of the input vectors $\mathbf{v}^1+\mathbf{v}^2$ with $\mathbf{v}^1 \in L(\mathbf{x}^1)$ and $\mathbf{v}^2 \in L(\mathbf{x}^2)$ since the opposite case is excluded by [L2] (*Attainability of Each Production*). If two different activities can be integrated in a particular activity permitting the production of the same outputs at smaller inputs, then we also speak of a **synergy effect**.[80] The assumption of strict superadditivity says that such a synergy effect emerges for each arbitrary combination of possible activities.

Analogous to the superadditivity of the input correspondence L, the superadditivity of the output correspondences P results from [T2] (*Additivity*).

$$\sum_i P(\mathbf{v}^i) \subset P\Big(\sum_i \mathbf{v}^i\Big) \quad \text{with} \quad \mathbf{v}^i \in V \;\; (i = 1, \ldots, k)$$

A graphical representation of a superadditive output correspondence P is given by Figure II.24,[81] where for the sake of simplicity two divisible goods x_1 and x_2 are supposed. If we carry the production possibility set $P(\mathbf{v}^1)$ tangentially to the set $P(\mathbf{v}^2)$ rotated by 180°, then a region marked by $0ABC$ results.

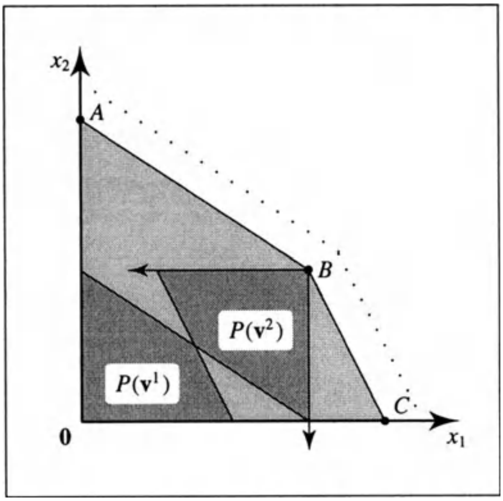

Figure II.24: Superadditivity of the output correspondence P

If $P(\mathbf{v}^1) + P(\mathbf{v}^2) = P(\mathbf{v}^1 + \mathbf{v}^2) = 0ABC$ holds exactly, then additivity results as a special case of a superadditive output correspondence P. Furthermore, if there is the possible output combination above ABC, then there is a further synergy effect[82] and we speak of a superadditive output correspondence P. Strict superadditivity requires the transformation curve corresponding to $P(\mathbf{v}^1+\mathbf{v}^2)$ to lie completely above ABC; see the dotted line.

A further conclusion from [T2] (*Additivity*) is drawn from Proposition II.5.

$$\begin{aligned} (\mathbf{v},\mathbf{x}),\ (\tilde{\mathbf{v}},\tilde{\mathbf{x}}) \in \operatorname{conv} GR &\Longrightarrow \big(\lceil\mathbf{v}\rceil, \lfloor\mathbf{x}\rfloor\big),\ \big(\lceil\tilde{\mathbf{v}}\rceil, \lfloor\tilde{\mathbf{x}}\rfloor\big) \in GR && \text{Proposition II.5} \\ &\Longrightarrow \big(\lceil\mathbf{v}\rceil + \lceil\tilde{\mathbf{v}}\rceil, \lfloor\mathbf{x}\rfloor + \lfloor\tilde{\mathbf{x}}\rfloor\big) \in GR && \text{Assumption [T2]} \end{aligned}$$

[80] Cf. HAMPDEN-TURNER (1970, p. 190).

[81] An equivalent representation is used in the theory of international trade to deduce the common transformation curve from the transformation curves of two countries. Cf. e.g. SIEBERT (1994, p. 160).

[82] A reason for such a synergy effect has been described in Figure II.23 by the activity $(v, x) = (2, 2)$.

The question whether $(\lceil \mathbf{v}+\tilde{\mathbf{v}}\rceil, \lfloor \mathbf{x}+\tilde{\mathbf{x}}\rfloor) \in \mathrm{GR}$ is also satisfied is answered by the following proposition. This supposition is interesting since it takes two effects into account. On the one hand the sum of completed fractions of a commodity j exceeds unity, $x_j^{\triangleleft}+\tilde{x}_j^{\triangleleft} = x_j - \lfloor x_j \rfloor + \tilde{x}_j - \lfloor \tilde{x}_i \rfloor \geqq 1$. The advantage of combining the two activities is reflected with regard to good j by the inequality $\lfloor x_j + \tilde{x}_j \rfloor \geqq \lfloor x_j \rfloor + \lfloor \tilde{x}_j \rfloor$. On the other hand it is possible that the sum of the "missing capacities" with regard to factor i is larger than one, $v_i^{\triangleright} + \tilde{v}_i^{\triangleright} = \lceil v_i \rceil - v_i + \lceil \tilde{v}_i \rceil - \tilde{v}_i \geqq 1$. Probably, the combination of both activities gets by with a lower quantity of input, $\lceil v_i + \tilde{v}_i \rceil \leqq \lceil v_i \rceil + \lceil \tilde{v}_i \rceil$. [83]

Proposition II.7 (Joint Production and Use) *Under the assumptions of Proposition II.5 and under* [T2] *(Additivity) the elements* $(\mathbf{v},\mathbf{x})$ *and* $(\tilde{\mathbf{v}},\tilde{\mathbf{x}})$ *in the convex hull of the graph* GR *of the production technology satisfy*

$$\big(\lceil \mathbf{v}+\tilde{\mathbf{v}}\rceil, \lfloor \mathbf{x}+\tilde{\mathbf{x}}\rfloor\big) \in \mathrm{GR}\ .$$

Proof: Let $(\mathbf{v},\mathbf{x})$ and $(\tilde{\mathbf{v}},\tilde{\mathbf{x}})$ be two points in the convex hull of the graph GR, then by Theorem B.4 (CARATHÉODORY), p. 293, there are, respectively, $m+n+1$ possible activities such that

$$(\mathbf{v},\mathbf{x}) = \sum_{\nu=0}^{m+n} (\lambda_\nu \mathbf{v}^\nu, \lambda_\nu \mathbf{x}^\nu) \qquad \text{with} \quad (\mathbf{v}^\nu,\mathbf{x}^\nu) \in \mathrm{GR},\ \boldsymbol{\lambda} \in \Lambda^{m+n+1} \quad \text{and}$$

$$(\tilde{\mathbf{v}},\tilde{\mathbf{x}}) = \sum_{\mu=0}^{m+n} (\tilde{\lambda}_\mu \mathbf{v}^\mu, \tilde{\lambda}_\mu \mathbf{x}^\mu) \qquad \text{with} \quad (\mathbf{v}^\mu,\mathbf{x}^\mu) \in \mathrm{GR},\ \tilde{\boldsymbol{\lambda}} \in \Lambda^{m+n+1}.$$

The addition of both equations is carried out under consideration of [T2], i.e.

$$(\mathbf{v},\mathbf{x}) \in \mathrm{GR} \implies (\check{\mathbf{v}},\check{\mathbf{x}}) \in \mathrm{GR} \quad \text{with} \quad (\check{\mathbf{v}},\check{\mathbf{x}}) := (2\mathbf{v}, 2\mathbf{x}).$$

This yields a convex combination

$$(\mathbf{v}+\tilde{\mathbf{v}}, \mathbf{x}+\tilde{\mathbf{x}}) = \sum_{\nu=0}^{m+n} (\tfrac{\lambda_\nu}{2}\check{\mathbf{v}}^\nu, \tfrac{\lambda_\nu}{2}\check{\mathbf{x}}^\nu) + \sum_{\mu=0}^{m+n} (\tfrac{\tilde{\lambda}_\mu}{2}\check{\mathbf{v}}^\mu, \tfrac{\tilde{\lambda}_\mu}{2}\check{\mathbf{x}}^\mu),$$

where the sum of the nonnegative coefficients is 1.

$$\sum_{\nu=0}^{m+n} \frac{\lambda_\nu}{2} + \sum_{\mu=0}^{m+n} \frac{\tilde{\lambda}_\mu}{2} = 1$$

Taking $\big((\mathbf{v}+\tilde{\mathbf{v}}), (\mathbf{x}+\tilde{\mathbf{x}})\big) \in \operatorname{conv}\mathrm{GR}$ into consideration, we obtain the possible activity $\big(\lceil \mathbf{v}+\tilde{\mathbf{v}}\rceil, \lfloor \mathbf{x}+\tilde{\mathbf{x}}\rfloor\big) \in \mathrm{GR}$ in accordance with Proposition II.5. ■

Regarding Proposition II.7, two remarks are necessary.

[83] The production technology must allow such a combination. For example, the excess capacities of two parallel operating shipyards do not complement each other without further assumptions.

1. As mentioned above, [T2] *(Additivity)* excludes decreasing returns to scale.
2. However, increasing returns to scale are usually inconsistent with the assumption of an integer convex graph GR. For divisible goods and factors a convex graph even excludes increasing returns to scale.

However, Proposition II.7 suggests allowing locally the phenomenon of increasing returns to scale even in the case of a production technology with integer constant returns to scale. Not only the mutual provision of excess capacities[84], but also the capability of completing large projects by the combination of separate activities comprises the potential of increasing returns to scale.[85]

2.3.3 Factor Constraints

The previous explanations leave factor constraints out of consideration. But usually all of the relevant production factors will be available to the firm in limited quantities. As shown by the subsequent arguments, the main properties of a production technology remain unaffected even after introducing factor constraints.

Denoting the maximal available quantity of factor i by $b_i > 0 \quad (i = 1, \dots, m)$, a restricted factor space $V_\mathbf{b} \subset V$ results, where

$$V_\mathbf{b} = \{\mathbf{v} \in V \mid P(\mathbf{v}) \neq \emptyset,\ \mathbf{v} \leqq \mathbf{b}\} = \{\mathbf{v} \in V \mid \mathbf{v} \leqq \mathbf{b}\}.$$

According to [P3] (*Disposability of Inputs*) the greatest set of all feasible commodity vectors is given by $P(\mathbf{b})$ with $P(\mathbf{v}) \subset P(\mathbf{b})$ for all $\mathbf{v} \in V_\mathbf{b}$. Noting [P5] (*Boundedness*) this set is also bounded, and together with [P6] (*Closedness*) $P(\mathbf{b})$ is not only closed but also compact. Consequently, [P2] (*Attainability of Each Production*) can only be maintained when the commodity space X is understood as a subset in $P(\mathbf{b})$.

The factor space $V_\mathbf{b}$ is restricted to the factor constraints $v_i \leqq b_i \quad (i = 1, \dots, m)$ and yields the output correspondence[86]

$$\check{P}\colon V_\mathbf{b} \to \mathfrak{P}(X) \qquad \text{with} \quad V_\mathbf{b} = \{\mathbf{v} \in V \mid \mathbf{v} \leqq \mathbf{b}\}.$$

Besides the (compact) domain $\operatorname{Dom} \check{P} = V_\mathbf{b}$ we obtain the (compact) range $X_\mathbf{b} := \operatorname{Range} \check{P} = P(\mathbf{b})$ under consideration of $\check{P}(\mathbf{v}) = P(\mathbf{v}) \subset P(\mathbf{b})$ for all $\mathbf{v} \in V_\mathbf{b}$.

As before, the sets $\check{L}(\mathbf{x}) = \{\mathbf{v} \in V_\mathbf{b} \mid \mathbf{x} \in \check{P}(\mathbf{v})\}$ determine the inverse (input) correspondence $\check{L}$ of $\check{P}$ for all $\mathbf{x} \in X_\mathbf{b}$. Due to $X_\mathbf{b} = P(\mathbf{b})$ or, equivalently, $\mathbf{b} \in \check{L}(\mathbf{x})$ for all $\mathbf{x} \in X_\mathbf{b}$, we get the domain $\operatorname{Dom} \check{L} = \{\mathbf{x} \in X_\mathbf{b} \mid \check{L}(\mathbf{x}) \neq \emptyset\} = X_\mathbf{b}$ of the input correspondence

$$\check{L}\colon X_\mathbf{b} \to \mathfrak{P}(V_\mathbf{b}) \qquad \text{with} \quad X_\mathbf{b} = P(\mathbf{b}).$$

[84] The example of a building contractor using a crane in several separately undertaken building projects has already been mentioned.

[85] KANEMOTO (1990) refers to the exchange and the joint use of indivisible goods as a reason for the spatial concentration of economic activities by the firm's choice of location.

[86] Cf. BAUMOL, PANZAR, WILLIG (1982, p. 52).

Analogously, it can be shown for the range of $\check{L}$: Range $\check{L}$ = Dom $\check{P} = V_{\mathbf{b}}$. Given the inverse correspondences $\check{P}: V_{\mathbf{b}} \to \mathfrak{P}(X_{\mathbf{b}})$ and $\check{L}: X_{\mathbf{b}} \to \mathfrak{P}(V_{\mathbf{b}})$, the graph

$$\check{\mathrm{GR}} := \{(\mathbf{v}, \mathbf{x}) \mid \mathbf{v} \in V_{\mathbf{b}},\ \mathbf{x} \in \check{P}(\mathbf{v})\}$$

indicates the collection of all attainable activities under the given factor constraints $\mathbf{b}$. All established axioms can be transferred to this case by superseding the factor space V and the commodity space X by $V_{\mathbf{b}}$ and $X_{\mathbf{b}}$ respectively. The new input requirement sets $\check{L}(\mathbf{x})$ are to be determined as the intersection of the original input requirement sets $L(\mathbf{x})$ and the restricted factor space $V_{\mathbf{b}}$, $\check{L}(\mathbf{x}) = L(\mathbf{x}) \cap V_{\mathbf{b}}$ so that each input requirement set with $\check{L}(\mathbf{x}) \subset \{\mathbf{v} \mid \mathbf{0} \leqq \mathbf{v} \leqq \mathbf{b}\}$ is bounded. Axiom [P2] (*Attainability of Each Production*) now states that each commodity bundle in $X_{\mathbf{b}}$ is in fact producible. Moreover, Axiom [L5] (*Boundedness*), which is usually hard to interpret, is easier to grasp since $\check{L}(\mathbf{x}) = \emptyset$ for all $\mathbf{x} \notin \check{P}(\mathbf{b})$.

We refer to the next example to illustrate the above implications of factor constraints for the modified output correspondence $\check{P}$ and the inverse input correspondence $\check{L}$. Supposing that the examined firm is confronted by a factor space bounded by factor constraints $V_{\mathbf{b}} = \{\mathbf{v} \in V \mid \mathbf{v} \leqq \mathbf{b}\}$, then the n-dimensional restricted commodity space $X_{\mathbf{b}} = P(\mathbf{b})$ includes all output vectors producible by utilizing all of the available factor quantities $\mathbf{b}$. If the outputs x_j $(j = 1, \ldots, n)$ are indivisible building projects with $x_j = \bar{x}_j$ or $x_j = 0$, where each project $\bar{x}_j$ excludes the realization of all the remaining projects at the factor constraints $\mathbf{b}$, then the production possibility set[87] is

$$\check{P}(\mathbf{b}) = \{\mathbf{x} \mid \mathbf{x} = \mathbf{0} \text{ or } \mathbf{x} = \mathbf{e}^{j\mathrm{T}}\bar{\mathbf{x}}\,\mathbf{e}^{j}\ (j = 1, \ldots, n)\}.$$

For two alternatives with $\bar{\mathbf{x}} = \binom{\bar{x}_1}{\bar{x}_2}$ we would correspondingly have $\check{P}(\mathbf{b}) = \left\{\binom{0}{0}, \binom{\bar{x}_1}{0}, \binom{0}{\bar{x}_2}\right\}$. The set of all feasible projects is opposite to the following input requirement sets for $j = 1, \ldots, n$.

$$\check{L}(\mathbf{e}^{j\mathrm{T}}\bar{\mathbf{x}}\,\mathbf{e}^{j}) = \{\mathbf{v} \in V_{\mathbf{b}} \mid \mathbf{v} \text{ suffices for the production of alternative } j\}$$

2.4 Optimal Activities

2.4.1 Technically Efficient Production

The two Axioms [P3] (*Disposability of Inputs*) and [P4] (*Disposability of Outputs*) are based on the idea that not only factors but also goods can be wasted without impairing the production process. In both cases resources are destroyed and therefore withheld from alternatively possible uses. Before it can be settled how to discipline firms which use unnecessary amounts of scarce resources, we have to explain what is meant by a waste of resources.

A firm with the order to produce a given commodity bundle $\mathbf{x} \in X$ does not waste any inputs if for a given production technology a certain output cannot

[87] A good survey of problems where the feasible solutions must be binary (0-1 variables) is given, for example, by PADBERG (1979).

be produced with less than the chosen quantities of inputs. Technically, an input vector $\mathbf{v}$ in the input requirement set $L(\mathbf{x})$ fulfills this criterion of **technical input efficiency** if there is no input vector $\tilde{\mathbf{v}} \leq \mathbf{v}$ also being sufficient to produce the commodity bundle $\mathbf{x}$.[88] All input vectors satisfying the criterion of input efficiency are collected in the set $\text{Eff}\, L(\mathbf{x}) \subset L(\mathbf{x})$ with[89]

$$\text{Eff}\, L(\mathbf{x}) := \begin{cases} \{\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x}),\ \tilde{\mathbf{v}} \leq \mathbf{v} \implies \tilde{\mathbf{v}} \notin L(\mathbf{x})\} & \text{for } \mathbf{x} \in X \\ \emptyset & \text{for } \mathbf{x} \notin X \end{cases}$$

and the elements of $\text{Eff}\, L(\mathbf{x})$ are said to be **input efficient**. In the same way, an activity $(\mathbf{v}, \mathbf{x})$ is said to be **input efficient** if it meets the relation $\mathbf{v} \in \text{Eff}\, L(\mathbf{x})$. In particular, $\text{Eff}\, L(\mathbf{0}) = \{\mathbf{0}\}$ ensues from $\mathbf{x} = \mathbf{0}$. A firm which does not produce anything is not allowed to use any inputs.

Figure II.25 illustrates an input requirement set $L(\mathbf{x})$ including a divisible production factor 1 and an indivisible production factor 2. All of the input efficient vectors in $\text{Eff}\, L(x) \subset L(x)$ are marked by bold dots. For these input vectors there are no alternative input vectors lying "left below" the respective input efficient point; see point A.

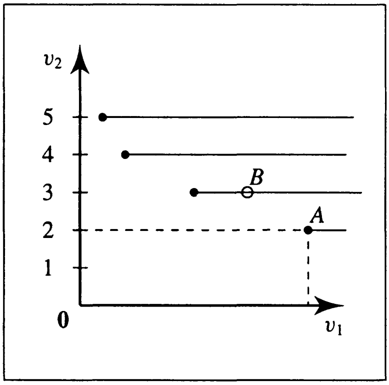

Figure II.25: Technical input efficiency

Analogous to the input efficiency with held fixed inputs $\mathbf{v}$, all output vectors $\mathbf{x}$ reflect a waste of goods (within the firm) if at least one output can be increased without increasing the inputs. Again all commodity vectors $\mathbf{x}$, satisfying this criterion of **technical output efficiency** for a given input vector $\mathbf{v}$, are gathered in the set $\text{Eff}\, P(\mathbf{v}) \subset P(\mathbf{v})$ with

$$\text{Eff}\, P(\mathbf{v}) := \begin{cases} \{\mathbf{x} \mid \mathbf{x} \in P(\mathbf{v}),\ \tilde{\mathbf{x}} \geq \mathbf{x} \implies \tilde{\mathbf{x}} \notin P(\mathbf{v})\} & \text{for } \mathbf{v} \in V \\ \emptyset & \text{for } \mathbf{v} \notin V\,. \end{cases}$$

An activity $(\mathbf{v}, \mathbf{x})$ is said to be **output efficient** if $\mathbf{x} \in \text{Eff}\, P(\mathbf{v})$.

The theory of the firm often supposes that firms produce only one commodity $x \in X$ with $X \subset \mathbb{R}_+$. In this case $\text{Eff}\, P(\mathbf{v})$ can be rewritten as

$$x \in \text{Eff}\, P(\mathbf{v}) \iff f(\mathbf{v}) := \max\{x \mid x \in P(\mathbf{v})\} = \max\{x \mid \mathbf{v} \in L(x)\}\,,$$

where the function $f\colon V \to X$ denotes a **production function**. Both the existence and the properties of a production function will be discussed later in more detail. Furthermore, the set of all input vectors producing precisely the output x is

[88] SHEPHARD (1953, p. 12) assumes furthermore the boundedness of the set of efficient input vector since no output vector can be efficiently produced by an infinitely large input vector. In this sense the COBB-DOUGLAS production function describes only a restricted production technology. For a broader discussion, see FÄRE (1972, 1980a).

[89] Apart from this efficient subset in $L(\mathbf{x})$ FÄRE defines other subsets with weaker efficiency properties. Cf. FÄRE (1988, p. 11).
As long as we assume divisible factors, efficient input vectors will always lie in the boundary $\partial L(\mathbf{x})$ of the examined input requirement set $L(\mathbf{x})$, $\text{Eff}\, L(\mathbf{x}) \subset \partial L(\mathbf{x}) \subset L(\mathbf{x})$. Cf. SHEPHARD (1953, p. 15).

called an **isoquant**.[90]

$$\begin{aligned}\mathrm{Isoq}(x) &:= \{\mathbf{v} \mid x \in P(\mathbf{v}),\ \tilde{x} > x \implies \tilde{x} \notin P(\mathbf{v})\} \\ &= \{\mathbf{v} \mid \mathbf{v} \in L(x),\ \tilde{x} > x \implies \mathbf{v} \notin L(\tilde{x})\}\end{aligned}$$

Although $\mathrm{Isoq}(x)$ and $\mathrm{Eff}\, L(x)$ are subsets in the factor space V, the isoquant is closer related to the set $\mathrm{Eff}\, P(\mathbf{v}) \subset X$. An activity $(\mathbf{v}, x)$ yields an input vector $\mathbf{v} \in \mathrm{Isoq}(x)$ if and only if the activity is output efficient, i.e. $x \in \mathrm{Eff}\, P(\mathbf{v})$.

For various reasons both concepts of efficiency may lead to different results. Not all of the output efficient activities $(\mathbf{v}, \mathbf{x})$ are input efficient at the same time et vice versa.

1. If we suppose a LEONTIEF production function $x = \min\{v_1, v_2\}$ with two divisible production factors, then each point on the right-angled isoquant except at the kink point is not input efficient. Conversely, each activity (v_1, v_2, x) is output efficient if (v_1, v_2) is an input vector on the isoquant $\mathrm{Isoq}(x)$.

2. For the case of an indivisible factor and a perfectly divisible good $x \in \mathbb{R}_+$ we offer the example of an input requirement set $L(x) = \left\{\mathbf{v} \in \mathbb{Z}^2_+ \mid v_1^\alpha v_2^\beta \geqq x\right\}$ with $\alpha, \beta \in \mathbb{R}_+$. First of all, it is to be noted that the input vector $\hat{\mathbf{v}} = \binom{1}{1}$ is input efficient for $\bar{x} = 0.5$.

$$\hat{\mathbf{v}} \in \mathrm{Eff}\, L(\bar{x}) = \left\{\mathbf{v} \mid \mathbf{v} \in L(\bar{x}),\ \tilde{\mathbf{v}} \leq \mathbf{v} \implies \tilde{\mathbf{v}} \notin L(\bar{x})\right\}$$

   Conversely, the quantity $\bar{x} = 0.5$ is not output efficient for $\hat{\mathbf{v}} = \binom{1}{1}$ because $\hat{x} = 1 > \bar{x} = 0.5$ excludes $\bar{x}$ to be an element of

$$\mathrm{Eff}\, P(\hat{\mathbf{v}}) = \left\{x \mid x \in \mathbb{R}_+,\ \tilde{x} > x \implies \tilde{x} \notin P(\hat{\mathbf{v}})\right\},$$

   where the production possibility set is given by $P(\hat{\mathbf{v}}) = \left\{x \mid 0 \leqq x \leqq \hat{v}_1^\alpha \hat{v}_2^\beta\right\}$. Thus, the activity $(\hat{v}_1, \hat{v}_2, \bar{x}) = (1, 1, 0.5)$ is input efficient, but not output efficient. However, the activity $(\hat{v}_1, \hat{v}_2, \hat{x}) = (1, 1, 1)$ is input efficient as well as output efficient.

Summing up the criteria of input and output efficiency, a possible activity $(\mathbf{v}, \mathbf{x})$ is called **technically efficient** if it is both input and output efficient.

$$\mathrm{Eff}\,\mathrm{GR} := \{(\mathbf{v}, \mathbf{x}) \mid (\mathbf{v}, \mathbf{x}) \in \mathrm{GR},\ (-\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \geq (-\mathbf{v}, \mathbf{x}) \implies (\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \notin \mathrm{GR}\}$$

In particular, the possibility of inaction $(\mathbf{v}, \mathbf{x}) = (\mathbf{0}, \mathbf{0})$ is technically efficient.[91]

[90] Besides this definition in VARIAN (1992) alternative definitions are given in FÄRE (1988).

[91] It is frequently supposed that a set of efficient activities can be described by a function t. Similar to the production function f, the implicit function t with

$$t(\mathbf{v}, \mathbf{x}) = 0 \iff (\mathbf{v}, \mathbf{x}) \in \mathrm{Eff}\,\mathrm{GR}$$

is called the **transformation function**. For given factor stocks $\mathbf{v}$ the graph of the transformation function is known as the **transformation curve**.

The search for an appropriate measure for the efficiency of an activity $(\mathbf{v}, \mathbf{x})$ begins with a look at physics. First of all, it should not be forgotten that each machine absorbs more power[92] than it delivers because losses by friction, air resistance, heating, and so on arise. The extent of the losses is measured by the (dimensionless) efficiency.

$$\text{efficiency } \eta := \frac{\text{delivered power}}{\text{supplied power}} \tag{II.22a}$$

If supply and emission of power do not need the same time, then it can be more useful to express the efficiency as a ratio of two kinds of work.

$$\text{efficiency } \eta := \frac{\text{useful work}}{\text{total work}} \tag{II.22b}$$

Due to unavoidable losses, the efficiency is always smaller than one; $\eta < 1$.

In this sense AFRIAT (1972) suggests using $\eta = \tilde{x}/f(\tilde{\mathbf{v}})$ as a measure for the output efficiency of a carried out activity $(\tilde{\mathbf{v}}, \tilde{x})$, where $f(\tilde{\mathbf{v}})$ indicates the maximal amount of the good producible by the input vector $\tilde{\mathbf{v}}$. All of the following expositions are founded on this concept. The trouble in defining a unique efficiency measure results from the observation that the input vector $\mathbf{v}$ or the output vector $\mathbf{x}$ yields no unified measures for the supplied or emitted power. After introducing prices this problem can be handled more easily.

Beginning with a measure of input efficiency of an activity $(\mathbf{v}, \mathbf{x})$, the function $\eta_I \colon V \times X \rightarrow [0, +\infty]$, which is not exactly specified for the moment, is accepted as an **efficiency measure for the input correspondence L** if it satisfies the following conditions.[93]

[EI1] The inequality $\eta_I(\mathbf{v}, \mathbf{x}) \leqq 1$ holds if and only if the activity $(\mathbf{v}, \mathbf{x})$ is possible, $\mathbf{v} \in L(\mathbf{x})$.

[EI2] The equation $\eta_I(\mathbf{v}, \mathbf{x}) = 1$ holds if and only if the activity $(\mathbf{v}, \mathbf{x})$ is input efficient $\mathbf{v} \in \operatorname{Eff} L(\mathbf{x})$.

[EI3] For $\mathbf{v}, \lambda\mathbf{v} \in L(\mathbf{x})$ with $\lambda > 0$ we have $\eta_I(\lambda\mathbf{v}, \mathbf{x}) = {}^1\!/_\lambda\, \eta_I(\mathbf{v}, \mathbf{x})$.

[EI4] Given $\mathbf{v}, \tilde{\mathbf{v}} \in L(\mathbf{x})$ and $\mathbf{v} \geq \tilde{\mathbf{v}}$ yields $\eta_I(\mathbf{v}, \mathbf{x}) < \eta_I(\tilde{\mathbf{v}}, \mathbf{x})$.

[EI5] Given $\mathbf{v} \in L(\mathbf{x})$ and $\mathbf{v} \notin \operatorname{Eff} L(\mathbf{x})$, $\eta_I(\mathbf{v}, \mathbf{x})$ should relate $\mathbf{v}$ to an $\tilde{\mathbf{v}} \in \operatorname{Eff} L(\mathbf{x})$. [94]

[92] Power is defined as work per required time. The units of work are measured in joule and the units of power are measured in watt.

[93] Cf. BOL (1983, p. 119 f.) or BOL (1986, p. 382).

[94] RUSSELL (1985, p. 123) presents a theorem whose assumptions suffice for [EI5] to be implied by [EI2], [EI3], and [EI4].

RUSSELL (1987, p. 212) discusses a further requirement which states that two "comparable" activities $(\mathbf{v}, \mathbf{x})$ and $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}})$ should have the same efficiency degree $\eta_I(\mathbf{v}, \mathbf{x}) = \eta_I(\tilde{\mathbf{v}}, \tilde{\mathbf{x}})$. The activities $(\mathbf{v}, \mathbf{x})$ and $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}})$ are said to be comparable if two vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ exist besides the unit matrix $\mathbf{I}$ such that $\tilde{\mathbf{v}} = \boldsymbol{\lambda}^{\mathsf{T}}\mathbf{I}\mathbf{v}$, $\tilde{\mathbf{x}} = \boldsymbol{\mu}^{\mathsf{T}}\mathbf{I}\mathbf{x}$, and $L(\tilde{\mathbf{x}}) = \{\tilde{\mathbf{v}} \mid \tilde{\mathbf{v}} = \boldsymbol{\lambda}^{\mathsf{T}}\mathbf{I}\mathbf{v},\ \mathbf{v} \in L(\mathbf{x})\}$ hold good.

Since all of the five conditions except the third can be interpreted without any difficulties, we need only to mention the property of homogeneity [EI3]. Doubling the quantities of an input vector $\mathbf{v}$, [EI3] requires that the resulting input vector $2\mathbf{v}$ is indicated to be half efficient. This property of the efficiency measure η_I is useful with respect to the efficiency η according to (II.22a) or (II.22b) provided the input vectors $\mathbf{v}$ or $\lambda\mathbf{v}$ are interpreted as supplied power at a constant delivered power $\mathbf{x}$.

In a first approach FÄRE, LOVELL (1978) follow FARRELL (1957) and define a (radial) measure of technical input efficiency $\mathcal{F}_I\colon V \times X \setminus \{\mathbf{0}\} \to [0, +\infty]$ with

$$\text{(II.23)} \qquad \mathcal{F}_I(\mathbf{v}, \mathbf{x}) := \begin{cases} \min\{\lambda \geqq 0 \mid \lambda\mathbf{v} \in L(\mathbf{x})\} & \text{for } \mathbf{v} \in L(\mathbf{x}) \\ +\infty & \text{for } \mathbf{v} \notin L(\mathbf{x}), \end{cases}$$

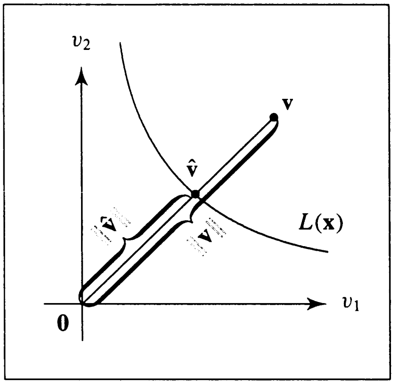

Figure II.26: FARRELL's input efficiency measure

which is called **FARRELL's input efficiency measure**; this measure will be used later in a slightly modified form as the input distance function $t_I(\mathbf{v}, \mathbf{x}) = 1/\mathcal{F}_I(\mathbf{v}, \mathbf{x})$. If $\mathcal{F}_I(\mathbf{v}, \mathbf{x}) = \hat{\lambda} > 0$, then in accordance with Figure II.26 FARRELL's measure corresponds to the distance ratio

$$\mathcal{F}_I(\mathbf{v}, \mathbf{x}) = \|\hat{\mathbf{v}}\| / \|\mathbf{v}\|$$

with $\hat{\mathbf{v}} = \hat{\lambda}\mathbf{v}$, where the input vectors $\mathbf{v}$ and $\hat{\mathbf{v}}$ lie on the same ray through the origin.[95]

Before going into the properties of this measure with regard to the requirements [EI1]–[EI5], we can establish analogous conditions for measuring the output efficiency of an activity $(\mathbf{v}, \mathbf{x})$. A function $\eta_O\colon X \times V \to [0, +\infty]$ is accepted as an **efficiency measure for the output correspondence P** if it has the following properties.[96]

[EO1] The inequality $\eta_O(\mathbf{x}, \mathbf{v}) \leqq 1$ holds if and only if the activity $(\mathbf{v}, \mathbf{x})$ is possible, $\mathbf{x} \in P(\mathbf{v})$.

[EO2] The equation $\eta_O(\mathbf{x}, \mathbf{v}) = 1$ holds if and only if the activity $(\mathbf{v}, \mathbf{x})$ is output efficient, $\mathbf{x} \in \operatorname{Eff} P(\mathbf{v})$.

[EO3] Given $\mathbf{x}, \lambda\mathbf{x} \in P(\mathbf{v})$ with $\lambda > 0$ yields $\eta_O(\lambda\mathbf{x}, \mathbf{v}) = \lambda\eta_O(\mathbf{x}, \mathbf{v})$.

[EO4] For $\mathbf{x}, \tilde{\mathbf{x}} \in P(\mathbf{v})$ and $\mathbf{x} \geq \tilde{\mathbf{x}}$ we have $\eta_O(\mathbf{x}, \mathbf{v}) > \eta_O(\tilde{\mathbf{x}}, \mathbf{v})$.

[EO5] Given $\mathbf{x} \in P(\mathbf{v})$ and $\mathbf{x} \notin \operatorname{Eff} P(\mathbf{v})$, $\eta_O(\mathbf{x}, \mathbf{v})$ should relate $\mathbf{x}$ to an $\tilde{\mathbf{x}} \in \operatorname{Eff} P(\mathbf{v})$.

[95] The idea of ascertaining efficient input vectors by rays through the origin has already been introduced in the explanations of Figure II.16, p. 30. The rays through the origin also served for the determination of boundary points of the input requirement sets.

[96] Cf. BOL (1983, p. 120).

A comparison to [EI1]–[EI5] shows that the requirements [EO1]–[EO5] indicate similar economic facts. Nevertheless, [EO3] now requires that the efficiency measure η_O is homogeneous of degree 1 in $\mathbf{x}$. Multiplication of the outputs (or the delivered power) by the factor λ at constant inputs (or supplied power) changes the efficiency degree by the same factor.

Analogously, we call the function $\mathcal{F}_O\colon X \setminus \{\mathbf{0}\} \times V \to [0, +\infty]$ with

$$\text{(II.24)} \qquad \mathcal{F}_O(\mathbf{x}, \mathbf{v}) := \begin{cases} \min\{\lambda \geqq 0 \mid \mathbf{x}/\lambda \in P(\mathbf{v})\} & \text{for } \mathbf{x} \in P(\mathbf{v}) \\ +\infty & \text{for } \mathbf{x} \notin P(\mathbf{v}) \end{cases}$$

FARRELL's output efficiency measure.[97] This measure will be seen later in a modified form as an output distance function $t_O(\mathbf{x}, \mathbf{v}) = \mathcal{F}_O(\mathbf{x}, \mathbf{v})$. A description analogous to Figure II.26 has been omitted. For $\hat{\mathbf{x}} = \mathbf{x}/\mathcal{F}_O(\mathbf{x}, \mathbf{v})$ the corresponding distance ratio becomes $\mathcal{F}_O(\mathbf{x}, \mathbf{v}) = \|\mathbf{x}\| / \|\hat{\mathbf{x}}\|$.

The relationship between FARRELL's efficiency measures becomes evident when we suppose a linearly homogeneous output correspondence P as in Figure II.27. Then by Proposition II.6 the inverse input correspondence L is linearly homogeneous such that $\lambda\mathbf{v} \in L(\mathbf{x}) \Longleftrightarrow \mathbf{x} \in P(\lambda\mathbf{v}) \Longleftrightarrow \mathbf{x}/\lambda \in P(\mathbf{v})$. Starting at FARRELL's input efficiency measure, we gain from this

$$\begin{aligned} \mathcal{F}_I(\mathbf{v}, \mathbf{x}) &= \min\{\lambda \geqq 0 \mid \lambda\mathbf{v} \in L(\mathbf{x})\} \\ &= \min\{\lambda \geqq 0 \mid \mathbf{x}/\lambda \in P(\mathbf{v})\} \\ &= \mathcal{F}_O(\mathbf{x}, \mathbf{v}) \end{aligned}$$

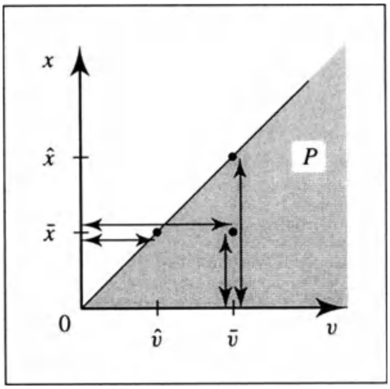

Figure II.27: Comparison of FARRELL's efficiency measures

for a possible activity $(\mathbf{v}, \mathbf{x}) \in GR$ with $\mathbf{x} \neq \mathbf{0}$. In this case both efficiency measures associate the activity $(\mathbf{v}, \mathbf{x})$ with the same efficiency degree.[98] As shown by Figure II.27, the correspondence can be expressed as distance ratios referring to the marked activity $(\bar{v}, \bar{x})$. The two efficiency measures yield

$$\mathcal{F}_I(\bar{v}, \bar{x}) = \frac{\hat{v}}{\bar{v}} = \frac{\bar{x}}{\hat{x}} = \mathcal{F}_O(\bar{x}, \bar{v})$$

or, equivalently, $\hat{v}/\bar{x} = \bar{v}/\hat{x}$.

The weakness of FARRELL's input efficiency measure can be clarified at once by two examples where the analogous problems can also be observed with respect to the output efficiency. Referring to perfectly divisible production factors, the measure at hand fails again for a LEONTIEF production function. For instance, for two production factors the function $\mathcal{F}_I$ associates each point on a right-angled

[97] Cf. FÄRE, LOVELL (1978, p. 153).
The idea of ascertaining output efficient commodity bundles by rays through the origin corresponds to the idea of representing boundary points of the production possibility set by rays through the origin. See the remarks to Figure II.13.

[98] Under similar conditions FÄRE (1988, p. 37) derives the inverse relation between the input distance function t_I and the output distance function t_O, i.e. $t_I(\mathbf{v}, \mathbf{x}) = 1/t_O(\mathbf{x}, \mathbf{v})$.

isoquant with unity, although an input vector can only be efficient if it is marked by the kink point of the isoquant. The second example refers to indivisible production factors. In contrast to [EI2] even the inefficient input vector, according to point B in Figure II.25, is associated with unity by FARRELL's measure. Also point C in Figure II.28 seems to be input efficient.[99]

Both problems result from the fact that the *radial* efficiency measure $\mathcal{F}_I$ does not compare the input vector $\mathbf{v}$ concerned to all the input vectors $\tilde{\mathbf{v}} \leqq \mathbf{v}$. An efficiency measure avoiding the above difficulties is suggested by FÄRE and LOVELL. Assuming an input vector $\mathbf{v} \in L(\mathbf{x})$, each input vector $\tilde{\mathbf{v}}$ with $\mathbf{0} \leqq \tilde{\mathbf{v}} \leqq \mathbf{v}$ can be expressed as $\tilde{\mathbf{v}} = (\lambda_1 v_1, \dots, \lambda_m v_m)^{\mathsf{T}}$ with $\lambda_i \in [0, 1]$ $(i = 1, \dots, m)$. If k denotes the number of positive coefficients λ_i, then the function $\mathcal{R}: V \times X \setminus \{\mathbf{0}\} \to [0, +\infty]$ with

$$\mathcal{R}(\mathbf{v}, \mathbf{x}) := \begin{cases} \min \left\{ {}^1\!/_k \sum_{i=1}^m \lambda_i \,\middle|\, (\lambda_1 v_1, \dots, \lambda_m v_m)^{\mathsf{T}} \in L(\mathbf{x}), \right. \\ \qquad\qquad\qquad \left. \lambda_i \in [0, 1] \;\; (i = 1, \dots, m) \right\} & \text{for } \mathbf{v} \in L(\mathbf{x}) \\ +\infty & \text{for } \mathbf{v} \notin L(\mathbf{x}) \end{cases}$$

is called **RUSSELL's input efficiency measure**; this function satisfies the premises [EI1]–[EI5].[100] Since RUSSELL's efficiency measure always considers the case $k = m$ with $\lambda_i = \lambda$ $(i = 1, \dots, m)$, we have $\mathcal{R}(\mathbf{v}, \mathbf{x}) \leqq \mathcal{F}_I(\mathbf{v}, \mathbf{x})$.

The treatment of the efficiency term with respect to the convex hull of an input requirement set $L(\mathbf{x})$ takes place by

Definition II.9[101] *Given a commodity bundle* $\mathbf{x} \in X$, *an input vector* $\mathbf{v} \in L(\mathbf{x})$ *is called an* ***input efficient with respect to the convex hull*** $\operatorname{conv}L(\mathbf{x})$ *if there is no vector* $\tilde{\mathbf{v}} \in \operatorname{conv}L(\mathbf{x})$ *with* $\tilde{\mathbf{v}} \leq \mathbf{v}$. *The set of all input vectors which are efficient with respect to the convex hull of* $L(\mathbf{x})$ *is denoted by* $\operatorname{Eff}\big(\operatorname{conv}L(\mathbf{x})\big)$.

Both FARRELL's and RUSSELL's efficiency measure lose part of their importance when the input requirement set $L(\mathbf{x})$ is substituted by its convex hull $\operatorname{conv}L(\mathbf{x})$. At least three reasons for this shortcoming can be given.

1. Efficient points need not lie in the boundary of the convex hull; see point A in Figure II.28.

2. Not every point lying on the boundary of $\operatorname{conv}L(\mathbf{x})$ is an efficient input vector; see point D in Figure II.28, which is not integer.[102]

[99] FÄRE (1975) gives similar examples. However, the reason why FARRELL's efficiency measure is of great importance is that many shortcomings cannot arise when besides input efficiency the cost efficiency introduced at a later stage is required, too.

[100] For the proof, see FÄRE, LOVELL (1978, p. 158 ff.). Furthermore, a detailed discussion of the presented efficiency measures is given by FÄRE, GROSSKOPF, LOVELL (1994). Moreover, ZIESCHANG (1984) suggests a combination of both efficiency measures $\mathcal{R}$ and $\mathcal{F}_I$.

[101] Cf. FRANK (1969, p. 45).

[102] The marked lines do not correspond to a cut through the production surface GR. FRANK (1969, p. 54) calls them iso-metric lines.

3. Only under the additional assumption [L7a] (*Integer Convexity*) we can rule out that an input vector $\mathbf{v} \in V$ with $\mathbf{v} \in \operatorname{conv}L(\mathbf{x})$ and $\mathbf{v} \notin L(\mathbf{x})$ is declared to be input efficient; see point B in Figure II.18, p. 32.

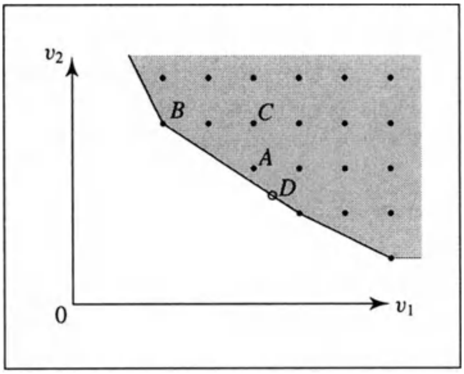

Figure II.28: Input efficiency with respect to $\operatorname{conv}L(\mathbf{x})$

Conversely, neither measures of input efficiency become absolutely useless so that they are still very important within the analysis.

1. An inefficient input vector $\mathbf{v} \in L(\mathbf{x})$ never becomes an efficient vector by forming the convex hull $\operatorname{conv}L(\mathbf{x})$. For instance, FARRELL's efficiency measure yields

$$\mathcal{F}_I(\mathbf{v},\mathbf{x}) = \min\{\lambda \geqq 0 |\ \lambda\mathbf{v} \in L(\mathbf{x})\} \geqq \min\{\lambda \geqq 0 |\ \lambda\mathbf{v} \in \operatorname{conv}L(\mathbf{x})\}.$$

2. Not all efficient input vectors $\mathbf{v} \in L(\mathbf{x})$ with $\mathcal{F}_I(\mathbf{v},\mathbf{x}) = 1$ are declared to be inefficient after the transference to the convex hull $\operatorname{conv}L(\mathbf{x})$.[103] At least the (input efficient) extreme points of $\operatorname{conv}L(\mathbf{x})$ are also declared to be input efficient. If $\tilde{\mathbf{v}}$ denotes an extreme point of $\operatorname{conv}L(\mathbf{x})$, then the relation $\mathbf{v} \notin \operatorname{conv}L(\mathbf{x})$ holds for all $\mathbf{v} \leq \tilde{\mathbf{v}}$. (Otherwise [L3] (*Disposability of Inputs*) would imply a contradiction.) With that it ensues for the extreme point $\tilde{\mathbf{v}} \in L(\mathbf{x})$:

$$\mathcal{R}(\tilde{\mathbf{v}},\mathbf{x}) = \mathcal{F}_I(\tilde{\mathbf{v}},\mathbf{x}) = 1 .$$

Remember the fact that $\operatorname{conv}L(\mathbf{x})$ contains at least one extreme point.

3. If FARRELL's efficiency measure $\mathcal{F}_I(\mathbf{v},\mathbf{x})$ is related to the set $\operatorname{conv}L(\mathbf{x})$, then a vector $\tilde{\mathbf{v}} = \mathcal{F}_I(\mathbf{v},\mathbf{x})\mathbf{v}$ is generated which can be expressed by (II.13), p. 28, as a convex combination of extreme points and extreme directions in $\operatorname{conv}L(\mathbf{x})$.

4. Another reason to retain FARRELL's efficiency measure becomes apparent after the concept of cost efficiency has been introduced.[104]

Definition II.9 can be transferred without difficulty to the output efficiency yielding analogous implications.

[103] Convex input requirement sets yield $\mathbf{v} \in \operatorname{Eff} L(\mathbf{x}) \implies \mathcal{F}_I(\mathbf{v},\mathbf{x}) = 1$ in every case.

[104] See the remarks to Figure II.31.

Definition II.10 *Let* $\mathbf{v}$ *be an admissible input vector,* $\mathbf{v} \in V$. *A commodity bundle* $\mathbf{x} \in P(\mathbf{v})$ *is said to be* ***output efficient with respect to the convex hull*** $\operatorname{conv} P(\mathbf{v})$ *if there is no vector* $\tilde{\mathbf{x}} \in \operatorname{conv} P(\mathbf{v})$ *with* $\tilde{\mathbf{x}} \geq \mathbf{x}$. *The set of all commodity bundles which are efficient with respect to the convex hull of* $P(\mathbf{v})$ *are denoted by* $\operatorname{Eff}\left(\operatorname{conv} P(\mathbf{v})\right)$.

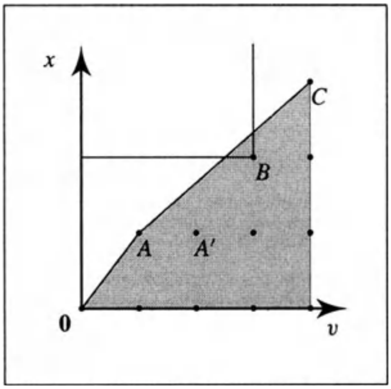

Figure II.29: Efficiency with respect to the convex hull of the graph GR

If points 0, A, B, and C in the figure opposite illustrate efficient activities, then there is no possible activity lying left above the points concerned (see point B). Moreover, A' is an output efficient but not an input efficient activity; the output x cannot be increased, whereas the input v can be reduced by one unit. At the same time point B illustrates that efficient activities, i.e. activities being output as well as input efficient, can lie in the interior of the convex hull of the graph GR without any problems. For this reason we define further:

Definition II.11[105] *An activity* $(\mathbf{v}, \mathbf{x}) \in \mathrm{GR}$ *is called* ***technically efficient with respect to*** conv GR *if the production technology satisfies*

$$(-\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \geq (-\mathbf{v}, \mathbf{x}) \iff (\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \notin \operatorname{conv} \mathrm{GR} .$$

Proposition II.8 *Under* [L3] *(Disposability of Inputs) and* [L4] *(Disposability of Outputs) all of the extreme points* $(\mathbf{v}^e, \mathbf{x}^e)$ *of* conv GR *with* $\mathbf{x}^e > \mathbf{0}$ *are efficient with respect to* conv GR.[106]

Proof: The assumption that the extreme point $(\mathbf{v}^e, \mathbf{x}^e)$ would not be efficient with respect to conv GR causes a contradiction, as it implies the existence of a point $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \in \operatorname{conv} \mathrm{GR}$ with $(-\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \geq (-\mathbf{v}^e, \mathbf{x}^e)$. Putting $\Delta\mathbf{v} := \mathbf{v}^e - \tilde{\mathbf{v}} \geqq \mathbf{0}$ and $\Delta\mathbf{x} := \tilde{\mathbf{x}} - \mathbf{x}^e \geqq \mathbf{0}$, it follows for $\lambda > 0$

$$\begin{aligned}(-\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) = (-\mathbf{v}^e, \mathbf{x}^e) + (\Delta\mathbf{v}, \Delta\mathbf{x}) &\geq (-\mathbf{v}^e, \mathbf{x}^e) \\ &\geq (-\mathbf{v}^e, \mathbf{x}^e) - \lambda(\Delta\mathbf{v}, \Delta\mathbf{x}) =: (-\mathbf{v}', \mathbf{x}').\end{aligned}$$

Because of $\mathbf{x}^e > \mathbf{0}$ the inequality $\mathbf{x}^e - \lambda\Delta\mathbf{x} = \mathbf{x}' \geqq \mathbf{0}$ is fulfilled for sufficiently small $\lambda > 0$. As it has been shown in the proof of Proposition II.5 under Axioms [L3] and [L4], we get $(\mathbf{v}', \mathbf{x}') \in \operatorname{conv} \mathrm{GR}$. Thus, $(\mathbf{v}^e, \mathbf{x}^e)$ can be expressed

[105]Cf. FRANK (1969, p. 43).

[106]Extreme points $(\mathbf{v}^e, \mathbf{x}^e)$ not satisfying $\mathbf{x}^e > \mathbf{0}$ can also be efficient with respect to conv GR. However, in this case there are also counterexamples. For instance, both activities $(\mathbf{v}, x_1, 0)$ and $(\mathbf{v}, 0, x_2)$ in Figure II.11, p. 26, can never be efficient with respect to conv GR.

as a convex combination of the two points $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}})$ and $(\mathbf{v}', \mathbf{x}')$ contradicting the assumption that $(\mathbf{v}^e, \mathbf{x}^e)$ is an extreme point of conv GR. ■

Proposition II.9 *No activity* $(\mathbf{v}, \mathbf{x}) \in \mathrm{GR}$ *obeying increasing returns to scale can be efficient with respect to* conv GR.

Proof: Supposing $(\mathbf{v}, \mathbf{x}) \in \mathrm{GR}$ obeys increasing returns to scale, there is a possible increase of the production level $\tilde{\lambda} > 1$ and an activity $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \in \mathrm{GR}$ such that $(-\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \geq \tilde{\lambda}(-\mathbf{v}, \mathbf{x})$. Together with the possibility of inaction $(\mathbf{0}, \mathbf{0}) \in \mathrm{GR}$ we get the convex combination

$$(1 - {}^1\!/_{\tilde{\lambda}})(-\mathbf{0}, \mathbf{0}) + {}^1\!/_{\tilde{\lambda}}(-\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \geq (-\mathbf{v}, \mathbf{x}).$$

Since the left hand side is an element of conv GR, the process $(\mathbf{v}, \mathbf{x})$ cannot be efficient with respect to conv GR. ■

The initial point for the discussion of technically efficient production was the unnecessary waste of scarce resources. Having discussed what is meant by input efficient or output efficient production and how to measure the degree of efficiency, the question remains as to what incentives exist for a firm to produce efficiently.

The answer is evident. Each firm not producing input efficiently is punished by avoidable factor costs provided the factor prices $\mathbf{q}$ are positive. Conversely, the firm fails to realize revenues from sale at positive commodity prices $\mathbf{p}$ provided it does not produce output efficiently. Both forms of punishment prevent it from realizing a profit maximum. The categories of technical efficiency are now completed by economic terms of efficiency referring to cost and revenue.

For instance, a firm producing a single good x by using the factors $\mathbf{v}$ may realize a set of data of the form $(\tilde{x}, \tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{q}})$, where the commodity price $\tilde{p}$ and the factor prices $\tilde{\mathbf{q}}$ are established by the market. Hence, the realized costs $\tilde{c} = \tilde{\mathbf{q}}^{\mathsf{T}}\tilde{\mathbf{v}}$ face the minimal costs $c(\tilde{\mathbf{q}}, \tilde{x})$ resulting at given factor prices $\tilde{\mathbf{q}}$ (and given technology) from the production of the amount $\tilde{x}$. While FARRELL's input efficiency measure generates a point $\mathbf{v}' = \mathcal{F}_I(\tilde{\mathbf{v}}, \tilde{x}) \cdot \tilde{\mathbf{v}}$ so that the degree of efficiency can be interpreted as distance ratio $\mathcal{F}_I(\tilde{\mathbf{v}}, \tilde{x}) = \|\mathbf{v}'\| / \|\tilde{\mathbf{v}}\|$, the ratio $c(\tilde{\mathbf{q}}, \tilde{x})/\tilde{c}$ gives a measure of the **cost efficiency** of an activity $(\tilde{\mathbf{v}}, \tilde{x})$.[107]

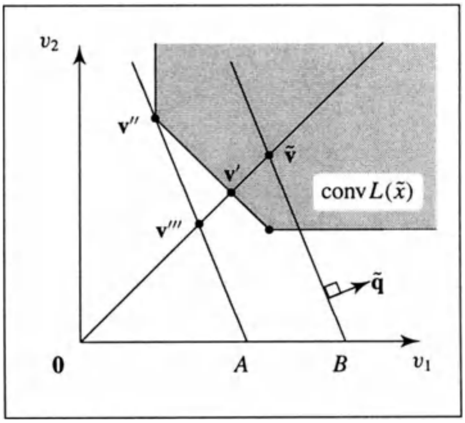

Figure II.30: Efficiency measures

[107]Cf. AFRIAT (1972, p. 582). The idea of using the ratio $c(\tilde{\mathbf{q}}, \tilde{x})/\tilde{c}$ as a measure of cost efficiency becomes clearer when we remember that the measure of input efficiency $\tilde{x}/f(\tilde{\mathbf{v}})$ was defined at the beginning, where f denotes the production function. Take into consideration that, similar to FARRELL's input efficiency measure (II.23), the measure of cost efficiency gives no information to the output efficiency of the carried out activity $(\mathbf{v}, \mathbf{x})$.

As shown by Figure II.30, both efficiency measures usually imply different results, i.e.

$$\mathcal{F}_I(\tilde{\mathbf{v}}, \tilde{x}) = \frac{\|\mathbf{v}'\|}{\|\tilde{\mathbf{v}}\|} \neq \frac{\|\mathbf{v}'''\|}{\|\tilde{\mathbf{v}}\|} = \frac{\overline{0A}}{\overline{0B}} = \frac{c(\tilde{\mathbf{q}}, \tilde{x})}{\tilde{\mathbf{q}}^{\mathsf{T}}\tilde{\mathbf{v}}} .$$

In actual fact, not all of the input efficient activities like $(\mathbf{v}', \tilde{x})$ are cost efficient at given factor prices $\tilde{\mathbf{q}}$ at the same time. However, input efficiency is a necessary condition for an activity $(\mathbf{v}'', \tilde{x})$ to be cost efficient. For instance, $c(\tilde{\mathbf{q}}, \tilde{x})/\tilde{\mathbf{q}}^{\mathsf{T}}\mathbf{v}'' = 1$ with $(\mathbf{v}'', \tilde{x}) \in \mathrm{GR}$ implies the condition $\mathcal{F}_I(\mathbf{v}'', \tilde{x}) = 1$. Analogously, output efficiency and technical efficiency can be compared to both **revenue efficiency** and **profit efficiency**.

The concluding expositions deal with an important special case, where a slightly modified form of FARRELL's input efficiency measure equals the measure of cost efficiency. First of all, the function $\mathcal{F}_I$ defined by (II.23) is modified to

$$\text{(II.25)} \qquad \mathcal{F}_I^{co}(\mathbf{v}, \mathbf{x}) := \begin{cases} \min\{\lambda \geqq 0 \mid \lambda\mathbf{v} \in \operatorname{conv}L(\mathbf{x})\} & \text{for } \mathbf{v} \in L(\mathbf{x}) \\ +\infty & \text{for } \mathbf{v} \notin L(\mathbf{x}). \end{cases}$$

This radial measure also scales each admissible input vector $\mathbf{v} \in L(\mathbf{x})$ along the ray through the origin and $\mathbf{v}$ whereby now a boundary point of the convex hull $\operatorname{conv}L(\mathbf{x})$ is generated. According to (II.13), p. 28 each of these boundary points can be expressed as a convex combination of extreme points and extreme directions of $\operatorname{conv}L(\mathbf{x})$. Furthermore, the extreme points of $\operatorname{conv}L(\mathbf{x})$ are admissible input vectors in $L(\mathbf{x})$. Thus, the admissible input vector $\mathbf{v}$ is put into relation to a convex combination of input vectors which are also admissible.

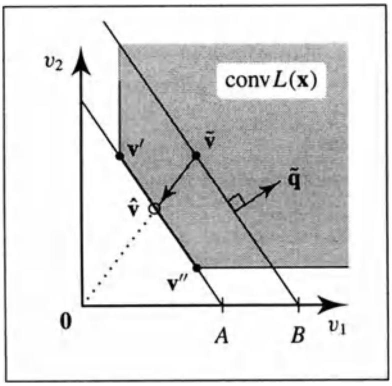

Figure II.31: The modified FARRELL's efficiency measure

Figure II.31 illustrates the example of an input requirement set containing the admissible input vectors $\mathbf{v}'$, $\mathbf{v}''$, and $\tilde{\mathbf{v}}$, where $\mathbf{v}'$ and $\mathbf{v}''$ denote two extreme points of the set $\operatorname{conv}L(\mathbf{x})$. Both inputs v_1 and v_2 are indivisible.

The inadmissible point $\hat{\mathbf{v}}$ in the boundary of $\operatorname{conv}L(\mathbf{x})$ results from $\hat{\mathbf{v}} = \mathcal{F}_I^{co}(\tilde{\mathbf{v}}, \mathbf{x})\,\tilde{\mathbf{v}}$. The adjoined movement takes place along the marked ray through the origin.

Now if a factor price vector $\tilde{\mathbf{q}}$ normal to $\operatorname{conv}\{\mathbf{v}', \mathbf{v}''\}$ is given for the measure of cost efficiency $c(\tilde{\mathbf{q}}, \mathbf{x})/\tilde{\mathbf{q}}^{\mathsf{T}}\tilde{\mathbf{v}}$, then both measures yield the same result with respect to the efficiency of the input vector $\tilde{\mathbf{v}}$:

$$\mathcal{F}_I^{co}(\tilde{\mathbf{v}}, \mathbf{x}) = \frac{\|\hat{\mathbf{v}}\|}{\|\tilde{\mathbf{v}}\|} = \frac{\overline{0A}}{\overline{0B}} = \frac{c(\tilde{\mathbf{q}}, \mathbf{x})}{\tilde{\mathbf{q}}^{\mathsf{T}}\tilde{\mathbf{v}}} .$$

Whereas the original measure $\mathcal{F}_I$ assesses the input vector $\tilde{\mathbf{v}}$ to be input efficient, the newly defined measure $\mathcal{F}_I^{co}$ says that the vector concerned is not efficient with respect to $\text{conv}L(\mathbf{x})$. Thereby the degree of efficiency adjusts the admissible input vectors $\mathbf{v}'$ and $\mathbf{v}''$. Moreover, if $\hat{\mathbf{v}}$ were admissible, then it would be efficient with respect to $\text{conv}L(\mathbf{x})$, but it is merely cost efficient for a price vector of the form $\lambda\tilde{\mathbf{q}}$ $(\lambda > 0)$. An exact analysis of the relationship between the cost function $c(\mathbf{q}, \mathbf{x})$ and the input distance function $t_I(\mathbf{v}, \mathbf{x})$ analogous to $\mathcal{F}_I(\mathbf{v}, \mathbf{x})$ or $\mathcal{F}_I^{co}(\mathbf{v}, \mathbf{x})$ will be discussed later in Section III.2.[108]

The next section deals with the description of optimal activities in the sense of profit maximization and serves essentially as transition to Chapter III of this book. At the same time technical efficiency (in its different aspects) is proven to be a necessary condition for a profit maximum. Moreover, an activity $(\mathbf{v}, \mathbf{x}) \in \text{GR}$ must be technically efficient with respect to conv GR to be capable of guaranteeing a profit maximum. Furthermore, those activities which exhibit increasing returns to scale (Proposition II.9) are ruled out as potential candidates for a profit maximum.

2.4.2 Determination of Optimal Activities

(a) Representation of Convex Sets by Functions Before the problem of profit maximization can be discussed we have to make some preparations. All of the preliminary remarks apply to an arbitrary set $C \subset \mathbb{R}^n$, which will be substituted later, for instance, by input requirement sets. Apart from special regularized sets, which are closely related to the set C, we present functions serving for the representation of the set C. Fundamental outcomes of optimization problems with the feasible region C are given by Theorems II.1 to II.3.

Besides the affine hull of the set C

$$\text{aff}C = \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{x}^i \mid \mathbf{x}^i \in C,\ \sum_{i=1}^{m} \lambda_i = 1,\ m = 1, 2, \ldots \right\}$$

and the convex hull

$$\text{conv}C = \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{x}^i \mid \mathbf{x}^i \in C,\ \lambda_i \geqq 0\ (i = 1, \ldots, m),\ \sum_{i=1}^{m} \lambda_i = 1,\ m = 1, 2, \ldots \right\}$$

three other not necessarily convex sets are important for the analysis. These are the aureoled hull aur C and the star-shaped hull star C introduced by (II.16) and (II.15) on p. 28 respectively, and the **projection cone**[109] defined by

$$\text{(II.26)} \qquad \text{cone}\, C := \{\lambda \mathbf{x} \mid \mathbf{x} \in C,\ \lambda \geqq 0\}\,.$$

[108] See definition (III.81), p. 152, where

$$t_I(\mathbf{v}, \mathbf{x}) \equiv \psi(\mathbf{v}|\text{conv}L(\mathbf{x})) = \max\{\lambda > 0 \mid \mathbf{v}/\lambda \in \text{conv}L(\mathbf{x})\}$$

is set for an admissible commodity bundle $\mathbf{x} \in X$.

[109] The projection cone, cone C, is also called the cone generated by the set C. This cone contains the vertex $\mathbf{0}$ and thus it has to be distinguished from the cone $K(C) := \text{cone}\, C \setminus \{\mathbf{0}\}$ which is also needed; see p. 153 ff. If C is a convex set not containing the origin, then $K(C) = \{\lambda\mathbf{x} \mid \mathbf{x} \in C,\ \lambda > 0\}$ denotes the smallest convex cone containing C; Cf. ROCKAFELLAR (1972, p. 14, Corollary 2.6.3).

A graphical representation of this cone results from laying Figures II.13 and II.14, p. 29 on top of each other, as $\operatorname{cone} C = \operatorname{star} C \cup \operatorname{aur} C$.

Apart from the projection cone we have already defined the **recession cone** by

(II.11) $$0^+C = \left\{ \mathbf{y} \in \mathbb{R}^n \mid \mathbf{x} + \lambda \mathbf{y} \in C \ \forall \lambda \geqq 0, \ \forall \mathbf{x} \in C \right\}.$$

This cone contains all directions $\mathbf{y}$ such that a ray with one of these directions starting at any point of C is contained entirely in C.[110]

After introducing some regularizations of the set C, we are now faced with the problem of describing the set C and the inferred sets by functions. Furthermore, we present implications resulting for sets with certain properties. The convexity of a set will be of crucial importance in this.

In fact the easiest representation of a set $C \subset \mathbb{R}^n$ results from the **indicator function** $\delta(\cdot|C)\colon \mathbb{R}^n \to [0, +\infty]$ with[111]

$$\delta(\mathbf{x}|C) := \begin{cases} 0 & \text{for } \mathbf{x} \in C \\ +\infty & \text{for } \mathbf{x} \notin C. \end{cases}$$

This function distinguishes points in $\mathbb{R}^n$ only by the criterion as to whether the respective point is contained in C or not.

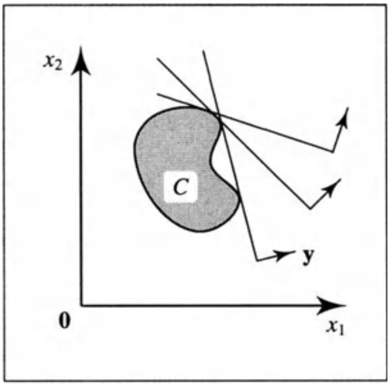

Figure II.32: Illustration of the support function of a set C

The **support function** $\sigma(\cdot|C)\colon \mathbb{R}^n \to [-\infty, +\infty]$ of the set C is defined by[112]

$$\sigma(\mathbf{y}|C) := \sup \left\{ \mathbf{y}^\mathsf{T}\mathbf{x} \mid \mathbf{x} \in C \right\}.$$

Using this function, we try to describe the set C by supporting hyperplanes, where $\mathbf{y}$ is normal to the respective plain. As shown by the figure opposite, this intension only partially succeeds for nonconvex sets. In the case of a closed convex set C the support function generates a system of half-spaces that permits a complete "outer representation" of the examined set according to the remarks of Figure II.10, p. 25, and in contrast to the "inner representation" by the indicator function.

The support function as a solution to a maximization problem can immediately be viewed in the light of the problem of revenue maximization as revenue function with given inputs, where C must be identified with the production possibility set.

[110]Two important cases have been mentioned. If all of the goods and factors are divisible, then on the one hand it is $0^+P(\mathbf{v}) = \{\mathbf{0}\}$ for a nonempty closed convex production possibility set. On the other hand the recession cone of an input requirement set is $0^+L(\mathbf{x}) = \mathbb{R}^m_+$.

[111]We have to distinguish between the indicator function $\delta(\cdot|C)$ and the characteristic function $\chi(\cdot|C)$.

$$\chi(\mathbf{x}|C) := \begin{cases} 1 & \text{for } \mathbf{x} \in C \\ 0 & \text{for } \mathbf{x} \notin C. \end{cases}$$

[112]For the properties of support functions, see Appendix D.2.1.

Similarly, the cost function is discussed as a reciprocal support function of an input requirement set, where the addition "reciprocal" now indicates that a minimization problem is to be investigated.

Like FARRELL's efficiency measures, the **gauge** $\gamma(\cdot|C)\colon \mathbb{R}^n \to [0, +\infty]$ of a nonempty set C is defined by[113]

$$\gamma(\mathbf{x}|C) := \inf\{\lambda \geqq 0 |\ \mathbf{x} \in \lambda C\}.$$

The idea of this function is to move a point $\tilde{\mathbf{x}} \in C$ along the ray through the origin and $\tilde{\mathbf{x}}$ until achieving a boundary point $\hat{\mathbf{x}} \in \partial C$; see the figure opposite.

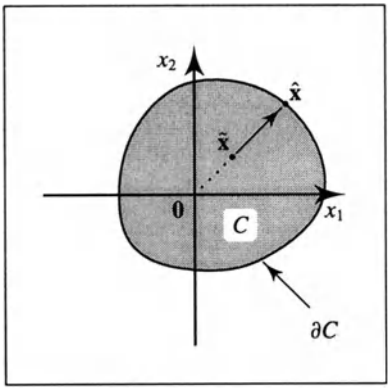

Figure II.33: Representation of the input distance function of a set C

Even input distance functions are merely partially suitable for the complete representation of a set. Nevertheless, we can show that the gauge $\gamma(\cdot|C)$ generates a unique representation of the boundary ∂C of the set C if the examined set is closed and star-shaped and if it has the origin in its interior, $\mathbf{0} \in \operatorname{int} C$:

$$\partial C = \{\mathbf{x}|\ \gamma(\mathbf{x}|C) = 1\}.$$

Since none of the relevant production possibility sets or input requirement sets satisfies the above conditions, we can make only little use of this representation of the boundary of a set.[114]

As shown by the duality theory, the two problems underlying the support function and the gauge function and, therefore, their solutions are inseparably related to each other. For example, the input efficiency of an activity turns out to be a necessary condition for this activity to realize a cost minimum.

To avoid linguistic confusion, we have to stress the difference between gauges and distance functions. The concept of the (German) "Distanzfunktion" goes back to MINKOWSKI and is therefore frequently called MINKOWSKI function.[115] As usual in mathematical bibliography, ROCKAFELLAR calls this function a *gauge*. However, the economic literature on this subject mostly follows SHEPHARD (1953), who speaks of the *distance* or *deflation function*.[116] This usage does not take into consideration that mathematicians have already reserved the term *distance function* as a synonym for the metric.[117]

The **(Euclidean) distance** $d(\cdot, C)\colon \mathbb{R}^n \to \mathbb{R}_+$ is defined by

$$d(\tilde{\mathbf{x}}, C) := \inf\{\|\tilde{\mathbf{x}} - \mathbf{x}\| \mid \mathbf{x} \in C\};$$

[113]For the properties of gauges, see Appendix D.2.2.
[114]See the remarks on Figures II.15 and II.16, p. 30.
[115]Cf. e.g. WETS (1976, p. 22).
[116]SHEPHARD (1953, p. 13) refers to the German genuine sources.
[117]See for example ROCKAFELLAR (1972, p. 28).

it will be of minor importance in the analysis. For instance, Chapter IV presents an ε-equilibrium deriving points which do not exceed a given distance ε from the demand set; see Proposition IV.13, p. 240.

No matter whether a minimum or a maximum is required the treated programs can be characterized as follows:[118] given the extended real-valued **objective function** $f\colon X \to [-\infty, +\infty]$ with the domain $X \subset \mathbb{R}^n$, the **minimum** of f over the **feasible region** $C \subset X$ is to be determined. Thus, we seek for a point $\hat{\mathbf{x}}$ such that

$$\text{(II.27)} \qquad \begin{aligned} &\hat{\mathbf{x}} \in C, \quad f(\hat{\mathbf{x}}) \leqq f(\mathbf{x}) \quad \forall \mathbf{x} \in C \\ \text{or} \quad &\hat{\mathbf{x}} \in C, \quad f(\hat{\mathbf{x}}) = \inf\{f(\mathbf{x}) \mid \mathbf{x} \in C\}. \end{aligned}$$

In short this problem is written

$$\text{(II.28)} \qquad \min\{f(\mathbf{x}) \mid \mathbf{x} \in C\} \quad \text{or} \quad \min_{\mathbf{x} \in C} f(\mathbf{x}).$$

Each point in the feasible region $\mathbf{x} \in C$ is said to be **feasible**. A point $\hat{\mathbf{x}}$ satisfying condition (II.27) is called an **optimal solution** or solution to the program. If a solution exists at all, then the solution need not necessarily be unique. Let $\hat{\mathbf{x}}$ be a solution to the program, then $\hat{f} \equiv f(\hat{\mathbf{x}})$ is called the **optimal value** of the program.
Since it cannot be guaranteed that a point $\hat{\mathbf{x}} \in C$ exists which satisfies the condition (II.27) or that the optimal value $f(\hat{\mathbf{x}})$ is finite, the problem (II.28) can be formulated more generally as

$$\text{(II.29)} \qquad \inf\{f(\mathbf{x}) \mid \mathbf{x} \in C\} \quad \text{or} \quad \inf_{\mathbf{x} \in C} f(\mathbf{x}).$$

When looking for the greatest lower bound – that is a value $\hat{f}$ fulfilling $\hat{f} \leqq f(\mathbf{x})$ for all $\mathbf{x} \in C$ – four cases may appear.

1. For $C = \emptyset$ we declare $\inf\{f(\mathbf{x}) \mid \mathbf{x} \in C\} = +\infty$. If the set of feasible solutions is empty, then the "impossibility of a smallest value of the objective function" is indicated by associating the smallest value of the objective function with the value $+\infty$. This case may especially occur when C is implicitly given.

2. If $C \neq \emptyset$ and if the finite infimum $\inf\{f(\mathbf{x}) \mid \mathbf{x} \in C\}$ is attained at a point $\hat{\mathbf{x}} \in C$, then we write

   $$\hat{f} = f(\hat{\mathbf{x}}) = \min\{f(\mathbf{x}) \mid \mathbf{x} \in C\} = \inf\{f(\mathbf{x}) \mid \mathbf{x} \in C\},$$

   and $\hat{\mathbf{x}} = \arg\min\{f(\mathbf{x}) \mid \mathbf{x} \in C\}$ is an optimal solution to the programs. A program with this property is said to be **feasible**.

3. If $C \neq \emptyset$ and if the finite infimum $\inf\{f(\mathbf{x}) \mid \mathbf{x} \in C\}$ is attained at *no* point $\mathbf{x} \in C$, then there is no optimal solution to the program and we call $\hat{f} = \inf\{f(\mathbf{x}) \mid \mathbf{x} \in C\}$ alternatively the optimal value to the program (II.29).

[118] Cf. BLUM, ÖTTLI (1975, p. 1 f.).

4. If $C \neq \emptyset$ and if the objective function is not bounded below in the feasible region C, $\inf\{f(\mathbf{x}) \mid \mathbf{x} \in C\} = -\infty$, then there is no optimal solution.

As far as possible optimal solutions to the treated programs are marked by a hat – for instance $\hat{\mathbf{x}}$. The following theorems prove the existence of optimal solutions and their properties.

Theorem II.1 (WEIERSTRASS)[119] *Provided the function* $f\colon X \to \mathbb{R}$ *is continuous on its compact domain* $X \subset \mathbb{R}^n$, *then* f *is bounded in* X *and*

$$\min\{f(\mathbf{x}) \mid \mathbf{x} \in X\} \quad \textit{and} \quad \max\{f(\mathbf{x}) \mid \mathbf{x} \in X\}$$

exist.

Thus, if the set C in Figure II.32 is bounded and closed, i.e. compact, then the support function is $\sigma(\mathbf{y} \mid C) = \max\left\{\mathbf{y}^{\mathsf{T}}\mathbf{x} \mid \mathbf{x} \in C\right\}$.

The next theorem confirms the supposition of Figure II.32 that the optimal value to a program is independent of the fact whether it is based on the feasible region C or its convex hull conv$C$. Take into consideration that extended real-valued convex functions are continuous in the relative interior of its effective domain so that WEIERSTRASS's Theorem can be applied if the effective domain is a subset in the feasible region C.

Theorem II.2[120] *Let* $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ *be a convex function and let* $C \subset \mathbb{R}^n$ *be an arbitrary set of points. Then*

$$\sup\{f(\mathbf{x}) \mid \mathbf{x} \in \operatorname{conv}C\} = \sup\{f(\mathbf{x}) \mid \mathbf{x} \in C\},$$

where the first supremum is achieved only if the second more restrictive supremum is achieved. For a concave function $g\colon X \to [-\infty, +\infty]$ *we have by analogy*

$$\inf\{g(\mathbf{x}) \mid \mathbf{x} \in \operatorname{conv}C\} = \inf\{g(\mathbf{x}) \mid \mathbf{x} \in C\}.$$

Whereas WEIERSTRASS's Theorem makes a statement on the existence of an optimal value, the two following theorems give information at which points the corresponding optimal solutions can be found.

Theorem II.3[121] *Let* $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ *be a convex function and let* C *be a closed convex set in the effective domain* Dom f *containing no line.*[122] *If the supremum of* f *is achieved relative to* C, *then it is achieved at an extreme point of* C.

[119] Cf. DALLMANN, ELSTER (1991a, p. 263, Theorem 15.16).
[120] Cf. ROCKAFELLAR (1972, p. 343, Theorem 32.2).
[121] Cf. ROCKAFELLAR (1972, p. 344, Corollary 32.3.1).
[122] A nonempty closed convex set containing no line has at least one extreme point; see ROCKAFELLAR (1972, p. 167, Corollary 18.5.3).

The relationship to Theorem II.2 becomes apparent when we take (II.8), p. 25, into account. By the theorem of KREIN-MILMAN the convex hull of the extreme points of a set C equals conv$C$. The issue of Theorem II.3 may be stated more precisely with respect to the level of the optimal value as follows:

Theorem II.4[123] *Let* $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ *be a convex function and let C be a compact convex set in the relative interior of the effective domain* rint(Dom f). *Then the supremum of f over C is finite and it is attained at an extreme point of C.*

Summarizing, the statements of Theorems II.1 to II.4 can be revealed by the example of Figure II.32. Since the function $\mathbf{y}^{\mathsf{T}}\mathbf{x}$ with the effective domain $\mathbb{R}^n$ is continuous in each set $C \subset \mathbb{R}^n$ for given $\mathbf{y}$, an optimal value $\max\{\mathbf{y}^{\mathsf{T}}\mathbf{x} \mid \mathbf{x} \in C\}$ exists according to Theorem II.1 (WEIERSTRASS). In accordance with Theorem II.2 the same outcome results when the set C is replaced with its convex hull conv$C$. If conv$C$ is closed, then the maximum is achieved by Theorem II.3 at an extreme point of conv$C$. Moreover, the optimal solution is finite by Theorem II.4 since conv$C$ is bounded.

(b) Profit Maximization In contrast to goods and factors the prices are not subject to any restrictions with respect to phenomena of indivisibility. The setting of the two spaces of commodity prices and factor prices

$$P_{\mathbf{p}} = \mathbb{R}^n_+ \quad \text{and} \quad Q = \mathbb{R}^m_+$$

thus considers only nonnegativity constraints.[124] Under the assumption of competitive markets all commodity prices $\mathbf{p} \in P_{\mathbf{p}}$ as well as all factor prices $\mathbf{q} \in Q$ are given from the point of view of an individual firm. Thus, the examined firm has no power to influence any price. If the firm as quantity adjuster pursues the goal of profit maximization $\mathbf{p}^{\mathsf{T}}\mathbf{x} - \mathbf{q}^{\mathsf{T}}\mathbf{v}$, then it must solve the problem[125]

$$\text{(II.30)} \qquad \pi(\mathbf{p}, \mathbf{q}) := \sup\{\mathbf{p}^{\mathsf{T}}\mathbf{x} - \mathbf{q}^{\mathsf{T}}\mathbf{v} \mid (\mathbf{v}, \mathbf{x}) \in \mathrm{GR}\},$$

where only possible activities $(\mathbf{v}, \mathbf{x}) \in \mathrm{GR}$ are allowed to be optimal solutions. Without specifying the properties of the profit function π in more detail each optimal solution to the problem, if it exists at all, is characterized by

Proposition II.10[126] *If* $\mathbf{p}^{\mathsf{T}}\mathbf{x} - \mathbf{q}^{\mathsf{T}}\mathbf{v} \leqq \mathbf{p}^{\mathsf{T}}\tilde{\mathbf{x}} - \mathbf{q}^{\mathsf{T}}\tilde{\mathbf{v}}$ *holds for all possible activities* $(\mathbf{v}, \mathbf{x}) \in \mathrm{GR}$ *with a given commodity price vector* $\mathbf{p} > \mathbf{0}$ *and a given factor price vector* $\mathbf{q} > \mathbf{0}$, *then* $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \in \mathrm{GR}$ *is a technically efficient activity.*

[123] Cf. ROCKAFELLAR (1972, p. 344, Corollary 32.3.2).

[124] We dispense with aspects of a smallest unit of money or negative prices, for example, for produced garbage.

[125] Cf. DIEWERT (1982, p. 537).
We go later into the cost minimization and the revenue maximization as necessarily to be solved subproblems of profit maximization.

[126] Cf. TAKAYAMA (1990, p. 56).

Proof: Suppose the process $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}})$ is not efficient, then there is a possible activity $(\mathbf{v}, \mathbf{x})$ with $(-\mathbf{v}, \mathbf{x}) \geq (-\tilde{\mathbf{v}}, \tilde{\mathbf{x}})$ such that the strict inequality $\mathbf{p}^{\mathsf{T}}\mathbf{x} - \mathbf{q}^{\mathsf{T}}\mathbf{v} > \mathbf{p}^{\mathsf{T}}\tilde{\mathbf{x}} - \mathbf{q}^{\mathsf{T}}\tilde{\mathbf{v}}$ results for positive prices. ■

If the problem of profit maximization has an optimal solution, then the optimal solution must be a technically efficient activity even when no assumptions on the production technology are made.

Afterwards two crucial questions must be answered: (1) When is the existence of a profit maximizing activity $(\hat{\mathbf{v}}, \hat{\mathbf{x}}) \in \mathrm{GR}$ guaranteed and what properties do these activities have? The answer is especially examined with respect to given prices such that the firm behaves as a quantity adjuster. (2) As by Proposition II.10 only technically efficient processes are of importance for the realization of a profit maximum, the question arises conversely as to which of the technically efficient activities yield a profit maximum at suitable prices $(\mathbf{q}, \mathbf{p})$.[127] If the production technology is characterized by nonconvexities, then we can identify technically efficient activities which are not profit maximizing for any price vector $(\mathbf{q}, \mathbf{p})$.

Proposition II.11 *Efficient activities* $(\mathbf{v}, \mathbf{x}) \in \mathrm{GR}$ *which are not efficient with respect to the convex hull of the graph* conv GR *can never lead to a profit maximum at positive prices.*[128]

Proof: If the activity $(\mathbf{v}, \mathbf{x}) \in \mathrm{GR}$ is not efficient with respect to conv GR, then there is a point $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \in \mathrm{conv\,GR}$ with $(-\tilde{\mathbf{v}}, \tilde{\mathbf{x}}) \geq (-\mathbf{v}, \mathbf{x})$ such that

$$\mathbf{p}^{\mathsf{T}}\mathbf{x} - \mathbf{q}^{\mathsf{T}}\mathbf{v} < \mathbf{p}^{\mathsf{T}}\tilde{\mathbf{x}} - \mathbf{q}^{\mathsf{T}}\tilde{\mathbf{v}}$$

holds for positive prices $\mathbf{q} > \mathbf{0}$ and $\mathbf{p} > \mathbf{0}$. By Theorem B.4 (CARATHÉODORY), p. 293, $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}})$ can be expressed as a convex combination of no more than $n + m + 1$ possible activities $(\mathbf{v}^i, \mathbf{x}^i) \in \mathrm{GR}$. Thus, according to Corollary B.4.3, at least one of these activities must satisfy

$$\mathbf{p}^{\mathsf{T}}\tilde{\mathbf{x}} - \mathbf{q}^{\mathsf{T}}\tilde{\mathbf{v}} \leqq \mathbf{p}^{\mathsf{T}}\mathbf{x}^i - \mathbf{q}^{\mathsf{T}}\mathbf{v}^i$$

such that $(\mathbf{v}, \mathbf{x})$ cannot be profit maximizing. ■

Since by Proposition II.9 no activity with increasing returns to scale is efficient with respect to conv GR, Proposition II.11 eliminates these activities to be a potential candidate for realizing a profit maximum. Note that this result only refers to the production technology as a restriction. In spite of increasing returns to scale, a profit maximum may be realized if we have to consider additional factor constraints or constraints regarding markets.

[127] SCARF (1981a) describes the starting point for this problem as follows: "The primary consequences of the convexity assumption is the existence of a vector of prices which supports an arbitrary efficient production plan." FRANK (1969, p. 70) discusses assumptions which also assure the existence of such a price vector for indivisible goods and factors. If, for example, indivisible factors can be substituted by divisible factors, then excess capacities caused by indivisibilities can be avoided by appropriate adjustment processes.

[128] Such an activity is illustrated by point *B* in Figure II.29, p. 56.

The analysis of profit maximization begins with the objective function $\varpi\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ which is at given prices

$$\varpi(\mathbf{x}, \mathbf{v}) := \mathbf{p}^{\mathsf{T}}\mathbf{x} - \mathbf{q}^{\mathsf{T}}\mathbf{v}\,.$$

This linear function is a hyperplane in $\mathbb{R}^{m+n}$ despite the fact that the quantities of outputs as well as inputs can possibly be subject to integer constraints. Therefore, the objective function ϖ is both convex and concave.[129] Theorem II.2 yields for this function regarding the set of points $\mathrm{GR} \subset \mathbb{R}^m \times \mathbb{R}^n$ the relation

$$\sup\left\{\mathbf{p}^{\mathsf{T}}\mathbf{x} - \mathbf{q}^{\mathsf{T}}\mathbf{v} \mid (\mathbf{v}, \mathbf{x}) \in \mathrm{GR}\right\} = \sup\left\{\mathbf{p}^{\mathsf{T}}\mathbf{x} - \mathbf{q}^{\mathsf{T}}\mathbf{v} \mid (\mathbf{v}, \mathbf{x}) \in \operatorname{conv}\mathrm{GR}\right\},$$

where the second supremum is attained only when the first more restrictive supremum is attained. Due to $\mathrm{GR} \subset V \times X \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+$, $\operatorname{conv}\mathrm{GR}$ cannot contain a line. If $\operatorname{conv}\mathrm{GR}$ is closed, then Theorem II.3 says: if the supremum of ϖ is attained relative to $\operatorname{conv}\mathrm{GR}$ at all,[130] then it is attained at an extreme point of $\operatorname{conv}\mathrm{GR}$. As gathered from the explanations on the convex hull of a set[131], every extreme point $(\hat{\mathbf{v}}, \hat{\mathbf{x}}) \in \operatorname{conv}\mathrm{GR}$ is a possible activity $(\hat{\mathbf{v}}, \hat{\mathbf{x}}) \in \mathrm{GR}$. Furthermore, every profit maximizing extreme point $(\hat{\mathbf{v}}, \hat{\mathbf{x}})$ with positive prices $(\mathbf{q}, \mathbf{p})$ must be efficient with respect to $\operatorname{conv}\mathrm{GR}$ (Proposition II.11).[132]

As the profit maximization takes place subject to the production technology GR, the preceding remarks suggest the examination of two subproblems of profit maximization. Accordingly the profit maximization regarding outputs adapts to the production technology $\big(L(\mathbf{x}) \mid \mathbf{x} \in X\big)$. The profit maximization regarding inputs, however, refers to the equivalent representation of the production technology $\big(P(\mathbf{v}) \mid \mathbf{v} \in V\big)$. While the first procedure yields the cost function as an interim result, the second procedure yields the revenue function in particular.

(c) Subproblems of Profit Maximization

(aa) Cost Minimization Under [L2] (*Attainability of Each Production*) the set of feasible solutions for a given commodity bundle $\tilde{\mathbf{x}} \in X$ is

$$\begin{aligned} M(\tilde{\mathbf{x}}) &:= \{(\mathbf{v}, \mathbf{x}) \in \mathrm{GR} \mid \mathbf{x} = \tilde{\mathbf{x}}\} \\ &= \{(\mathbf{v}, \mathbf{x}) \in V \times X \mid \mathbf{v} \in L(\tilde{\mathbf{x}})\}\,. \end{aligned}$$

Thus, the problem of profit maximization (II.30) can be rewritten as

$$\begin{aligned} \pi_O(\mathbf{p}, \mathbf{q}, \tilde{\mathbf{x}}) &= \sup\left\{\mathbf{p}^{\mathsf{T}}\tilde{\mathbf{x}} - \mathbf{q}^{\mathsf{T}}\mathbf{v} \mid (\mathbf{v}, \mathbf{x}) \in M(\tilde{\mathbf{x}})\right\} \\ &= \mathbf{p}^{\mathsf{T}}\tilde{\mathbf{x}} - \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in L(\tilde{\mathbf{x}})\right\}. \end{aligned}$$

[129] See Appendix C.1, Definition C.2.

[130] A priori does not need to be a finite profit maximum for any given commodity bundle $\mathbf{x}$. This case emerges for example for a substitutionable COBB-DOUGLAS production function when a factor price is set to zero. But if we suppose that the set of efficient input vectors is bounded (see footnote 88), then the profit maximum must be finite. Thus, in Figure II.29, p. 56, points 0, A, and C are possible candidates for a profit maximum.

[131] See Section 2.2 (a).

[132] As shown by Proposition II.9, this requirement holds at least for all extreme points $(\mathbf{v}^e, \mathbf{x}^e)$ with $\mathbf{x}^e > \mathbf{0}$.

In the literature on this point the function π_O, indicating the maximal profit at a given vector $\tilde{\mathbf{x}}$ and given prices, is frequently called the **restricted profit function** or **output conditional profit function**).[133] The problem is to find those input vectors $\mathbf{v} \in L(\tilde{\mathbf{x}})$ minimizing the factor costs for a given commodity bundle $\tilde{\mathbf{x}}$. Thus, the restricted problem of profit maximization can be converted into the following equivalent problem of cost minimization:[134]

$$c(\mathbf{q}, \tilde{\mathbf{x}}) = \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in L(\tilde{\mathbf{x}})\right\}.$$

The cost function c for a given commodity bundle $\tilde{\mathbf{x}}$ is called the factor price minimal cost function and each cost minimizing input vector $\hat{\mathbf{v}}$ is at the same time an optimal solution to the restricted problem of profit maximization. Using $c(\mathbf{q}, \tilde{\mathbf{x}})/\mathbf{q}^{\mathsf{T}}\mathbf{v}$ as a measure for cost efficiency of the activity $(\mathbf{v}, \tilde{\mathbf{x}})$, each input vector $\hat{\mathbf{v}}$ solving the problem of cost minimization is cost efficient, i.e. $c(\mathbf{q}, \tilde{\mathbf{x}})/\mathbf{q}^{\mathsf{T}}\hat{\mathbf{v}} = 1$. More properties of cost minimizing input vectors are discussed and compared in Section III.2. Intuitively, even at this point it can be stressed that each cost efficient activity $(\hat{\mathbf{v}}, \tilde{\mathbf{x}})$ must be input efficient in the sense of FARRELL, $\mathcal{F}_I(\hat{\mathbf{v}}, \tilde{\mathbf{x}}) = 1$; see the remarks on Figure II.31, p. 58.

Applying again Theorem II.2 yields

$$\inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in L(\tilde{\mathbf{x}})\right\} = \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in \operatorname{conv}L(\tilde{\mathbf{x}})\right\}.$$

Provided an optimal solution exists, then there is also an integer optimal solution $\hat{\mathbf{v}}$. For if the concave objective function $\varpi_O(\mathbf{v}) = \mathbf{q}^{\mathsf{T}}\mathbf{v}$ attains its infimum, then the infimum is attained at an (integer) extreme point of $\operatorname{conv}L(\tilde{\mathbf{x}})$. Remember that $\operatorname{conv}L(\tilde{\mathbf{x}})$ is a closed convex subset in the domain $\mathbb{R}^m$ of ϖ_O and that it does not contain a line (because $\operatorname{conv}L(\tilde{\mathbf{x}}) \subset \mathbb{R}^m_+$) so that the requirements of Theorem II.3 are satisfied.

Analogous to Proposition II.10, an input vector $\mathbf{v} \in L(\tilde{\mathbf{x}})$ only realizes minimal costs at positive factor prices $\mathbf{q} > \mathbf{0}$ if $\mathbf{v}$ is not only input efficient but also efficient with respect to $\operatorname{conv}L(\tilde{\mathbf{x}})$.[135] A firm violating this necessary condition of cost minimization is punished by avoidable costs at positive factor prices $\mathbf{q} > \mathbf{0}$. For each input vector $\mathbf{v}$ which is not efficient with respect to $\operatorname{conv}L(\tilde{\mathbf{x}})$ there is a vector $\tilde{\mathbf{v}} \in \operatorname{conv}L(\tilde{\mathbf{x}})$ with $\tilde{\mathbf{v}} \le \mathbf{v}$ and $\mathbf{q}^{\mathsf{T}}\tilde{\mathbf{v}} < \mathbf{q}^{\mathsf{T}}\mathbf{v}$. Analogous to the proof of Proposition II.10, apart from the vector $\tilde{\mathbf{v}}$ there must be an input vector $\hat{\mathbf{v}} \in L(\tilde{\mathbf{x}})$ which is associated with lower costs than $\mathbf{v}$. Although point A in Figure II.28, p. 55, is input efficient, there are activities – such as point B – incurring lower costs.

[133] Cf. VARIAN (1992, p. 26) or FÄRE (1988, p. 99).

[134] The structure of the problem does not change very much when the firm is subject to additional factor constraints $\mathcal{V} = \{\mathbf{v} \in V \mid \mathbf{v} \leqq \mathbf{b}\}$. In this case we look for the cost minimum of the set of possible activities.

$$\inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in L(\tilde{\mathbf{x}}),\ \mathbf{v} \in \mathcal{V}\right\} = \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in L(\tilde{\mathbf{x}}) \cap \mathcal{V}\right\}$$

[135] As mentioned above, the support function $\sigma(\cdot|\operatorname{conv}L(\mathbf{x}))$ corresponds to an "outer representation" of the set $\operatorname{conv}L(\mathbf{x})$ by supporting hyperplanes. The resulting set of optimal solutions reflects the boundary of the closed set $\operatorname{conv}L(\mathbf{x})$ and it contains especially those input vectors which are said to be efficient with respect to $\operatorname{conv}L(\mathbf{x})$.

This outcome will be of central importance in Chapter III. In particular, indivisible production factors imply nonconvex input requirement sets $L(\mathbf{x})$, which are regularly replaced with their convex hull conv$L(\mathbf{x})$. Within the framework of duality theory we can establish operations which are equivalent to the operation $L(\mathbf{x}) \to \text{conv}L(\mathbf{x})$ and which imply far-reaching properties of optimal activities.

Among other things we derive a result which can be outlined as follows: if the factor prices may vary arbitrarily, then under the introduced assumptions on the production technology there is a factor price vector $\tilde{\mathbf{q}}$ for each input vector $\mathbf{v}$ which is efficient with respect to conv$L(\mathbf{x})$ such that $\mathbf{v}$ solves the corresponding problem of cost minimization. In this case we are quite right to presuppose in the explanations of Figure II.31, p. 58, a vector of factor prices $\tilde{\mathbf{q}}$ normal to conv$\{\mathbf{v}', \mathbf{v}''\}$.

Finally, among all cost minimizing activities we have to filter out those activities which guarantee the maximal profit for a varying commodity bundle $\mathbf{x} \in X$.

$$\pi(\mathbf{p}, \mathbf{q}) = \sup \left\{ \mathbf{p}^{\mathsf{T}}\mathbf{x} - c(\mathbf{q}, \mathbf{x}) \mid \mathbf{x} \in X \right\}$$

Functions of this type will be discussed later under the concept of convex conjugate functions.[136]

Considering indivisible goods and factors the function of this optimization problem does not need to be continuous in the outputs $\mathbf{x}$. Hence, difficulties arise which can be illustrated by a simple example. Figure II.34[137] supposes a single-product firm producing the output x by a perfectly divisible factor v_1 and an indivisible factor v_2. To keep the example easy, we assume a price demand function such that the drawn parabola reflects the firm's revenue r. The costs of the first factor depending on the output quantity corresponds to the ray through origin $q_1 v_1$.

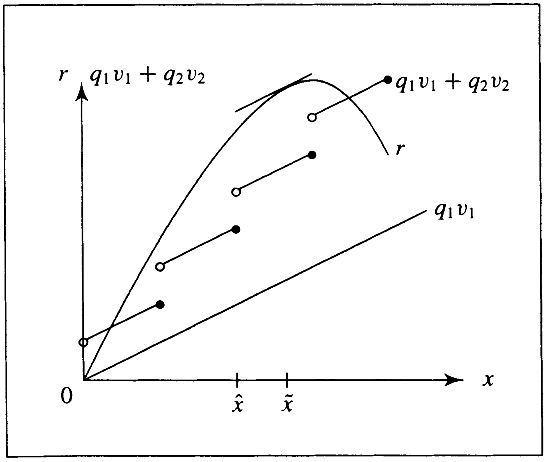

Figure II.34: Profit maximization

Considering the second indivisible factor, the step function indicates total costs $q_1 v_1 + q_2 v_2$ in producing the output x. From the point of view of determining profit maximizing outputs we usually investigate the necessary condition that marginal revenue must equal marginal cost. The result implies the output $\tilde{x}$. But the maximal profit is realized at the output $\hat{x}$ where the total costs make a jump.

(bb) Revenue Maximization In what follows, the problem of revenue maximization can be treated completely analogous to cost minimization. Given an input vector

[136] If the cost function $c(\mathbf{q}, \cdot)$ is proper, convex, and closed in $\mathbf{x}$, then

$$c(\mathbf{q}, \mathbf{x}) = \sup \left\{ \mathbf{p}^{\mathsf{T}}\mathbf{x} - \pi(\mathbf{p}, \mathbf{q}) \mid \mathbf{p} \in P_{\mathrm{p}} \right\}.$$

[137] A similar example is given by BREMS (1952, p. 583).

$\tilde{\mathbf{v}} \in V$, we get the subsequent set of feasible solutions:

$$M(\tilde{\mathbf{v}}) := \{(\mathbf{v}, \mathbf{x}) \in \mathrm{GR} \mid \mathbf{v} = \tilde{\mathbf{v}}\} = \{(\mathbf{v}, \mathbf{x}) \in V \times X \mid \mathbf{x} \in P(\tilde{\mathbf{v}})\}.$$

With that the problem of profit maximization (II.30) can be rewritten as

$$\pi_I(\mathbf{p}, \mathbf{q}, \tilde{\mathbf{v}}) = \sup\left\{\mathbf{p}^\mathrm{T}\mathbf{x} - \mathbf{q}^\mathrm{T}\tilde{\mathbf{v}} \mid (\mathbf{v}, \mathbf{x}) \in M(\tilde{\mathbf{v}})\right\} = \sup\left\{\mathbf{p}^\mathrm{T}\mathbf{x} \mid \mathbf{x} \in P(\tilde{\mathbf{v}})\right\} - \mathbf{q}^\mathrm{T}\tilde{\mathbf{v}}.$$

As before, the function π_I indicating the maximal profit at a given vector $\tilde{\mathbf{v}}$ and given prices $(\mathbf{q}, \mathbf{p})$ is also called **restricted profit function** or **input conditional profit function**). The equivalent problem of revenue maximization is now[138]

$$r(\mathbf{p}, \mathbf{v}) = \sup\left\{\mathbf{p}^\mathrm{T}\mathbf{x} \mid \mathbf{x} \in P(\mathbf{v})\right\},$$

and in this case Theorem II.2 yields

$$\sup\left\{\mathbf{p}^\mathrm{T}\mathbf{x} \mid \mathbf{x} \in P(\mathbf{v})\right\} = \sup\left\{\mathbf{p}^\mathrm{T}\mathbf{x} \mid \mathbf{x} \in \mathrm{conv}P(\mathbf{v})\right\}.$$

Since $\mathrm{conv}P(\mathbf{v})$ is compact, Theorem II.1 (WEIERSTRASS) can be applied.

$$r(\mathbf{p}, \mathbf{v}) = \max\left\{\mathbf{p}^\mathrm{T}\mathbf{x} \mid \mathbf{x} \in \mathrm{conv}P(\mathbf{v})\right\}$$

According to Theorem II.3, the maximum is attained at an extreme point $\hat{\mathbf{x}}$ of $\mathrm{conv}P(\mathbf{v})$ such that $\hat{\mathbf{x}} \in P(\mathbf{v})$ is satisfied. Again, the maximal profit results from a parametric variation of the input vector $\mathbf{v}$.

$$\pi(\mathbf{p}, \mathbf{q}) = \sup\left\{r(\mathbf{p}, \mathbf{v}) - \mathbf{q}^\mathrm{T}\mathbf{v} \mid \mathbf{v} \in V\right\}$$

At this point we refrain from aspects of the output efficiency. If $(\tilde{\mathbf{v}}, \tilde{\mathbf{x}}, \tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ denotes observed data, then the measure of revenue efficiency $r(\tilde{\mathbf{p}}, \tilde{\mathbf{v}})/\tilde{\mathbf{p}}^\mathrm{T}\tilde{\mathbf{x}}$ does not need any further explanations. Finally it should be mentioned that, for the case of a single-product firm with a given price of the good p, the problems of revenue maximization and output maximization are equivalent at given inputs $\mathbf{v}$. The optimal solutions to both problems are equal.

$$r(p, \mathbf{v}) = pf(\mathbf{v}) = p \sup\{x \mid x \in \mathrm{conv}P(\mathbf{v})\}$$

The next section deals with the properties of production functions which, in this case, are denoted by f.

2.4.3 Optimal Activities in the Production of One Good

This section draws the attention to firms producing a particular good x. In this case the commodity space is

$$X \subset \mathbb{R}_+.$$

[138]Cf. EICHHORN, SHEPHARD, STEHLING (1979, p. 344).

If the examined commodity is perfectly divisible, then $X = \mathbb{R}_+$. In the case of an indivisible good we put $X = \mathbb{Z}_+$. As already mentioned, the production function indicates the maximal output of a single-product firm when using the inputs $\mathbf{v}$ for production. The question when such a production *function* exists can be answered as follows:

Proposition II.12 *For each output correspondence* $P\colon V \to \mathfrak{P}(X)$ *of a single-product firm fulfilling Axioms* [P1a] *(Possibility of Inaction),* [P5] *(Boundedness) and* [P6] *(Closedness) a real-valued function* $f\colon V \to X$ *exists with*

$$f(\mathbf{v}) := \max\{x \mid x \in P(\mathbf{v})\},$$

which is called a ***production function****.*

Proof: Under Axioms [P1a], [P5] and [P6] the production possibility sets are nonempty and compact for all $\mathbf{v} \in V$. By Theorem II.1 (WEIERSTRASS) the continuous objective function ϕ with $\phi(x) = x$ attains its maximum over $P(\mathbf{v})$ for every $\mathbf{v} \in V$. As $P(\mathbf{v}) \subset X$, we have $f(\mathbf{v}) \geqq 0$ for all $\mathbf{v} \in V$. ■

Alternatively, the production function can be deduced from the inverse input correspondence L. Because of $x \in P(\mathbf{v}) \iff \mathbf{v} \in L(x)$, we have

$$f(\mathbf{v}) = \max\{x \mid \mathbf{v} \in L(x)\}.$$

Moreover, Theorem II.2 yields for the convex objective function ϕ

$$f(\mathbf{v}) = \max\{x \mid x \in P(\mathbf{v})\} = \max\{x \mid x \in \operatorname{conv} P(\mathbf{v})\} \quad \text{with} \quad \mathbf{v} \in V.$$

By Theorem II.4, the maximum is achieved at an extreme point of $\operatorname{conv} P(\tilde{\mathbf{v}})$ since $\operatorname{conv} P(\mathbf{v}) \subset \mathbb{R}_+$ is a nonempty compact convex set for each input vector $\mathbf{v} \in V$.

The generalized output correspondence[139] $P\colon \overline{V} \to \mathfrak{P}(X)$ with $P(\mathbf{v}) = \emptyset$ for all $\mathbf{v} \notin V$ induces a generalized production function $f\colon \overline{V} \to [-\infty, +\infty]$ with

$$f(\mathbf{v}) = \sup\{x \mid x \in P(\mathbf{v})\}.$$

In this definition we especially set $f(\mathbf{v}) = -\infty$ if $\mathbf{v}$ is no admissible input vector, i.e. $\mathbf{v} \notin V$. If the production function f yields $x = f(\mathbf{v})$, then the activity $(\mathbf{v}, x)$ is output efficient. Given the output x, the set of all output efficient activities is indicated by the isoquant $\operatorname{Isoq}(x) = \{\mathbf{v} \mid f(\mathbf{v}) = x\}$. Considering all inefficient activities with $f(\mathbf{v}) > x$, the input requirement set $L(x) = \{\mathbf{v} \mid f(\mathbf{v}) \geqq x\}$ arises. Corresponding to the expositions, this implicitly contains the condition that the input requirement sets are composed of admissible input vectors $\mathbf{v} \in V$.

Analogous to the production possibility sets inducing a production function, the production function f conversely induces **level sets** of the form $P_f(\mathbf{v}) =$

[139]The generalized output correspondence with $\overline{V} = \mathbb{R}^m$ has been introduced in Section 2.2.

$\{x \in X|\ f(\mathbf{v}) \geqq x\}$ for all $\mathbf{v} \in V$. From the conclusions for an arbitrary input vector $\mathbf{v} \in V$

(a) $\tilde{x} \in P_f(\mathbf{v}) \iff \tilde{x} \leqq f(\mathbf{v}) :\iff \tilde{x} \leqq \max\{x|\ x \in P(\mathbf{v})\} \overset{[P4]}{\iff} \tilde{x} \in P(\mathbf{v})$

(b) $\tilde{x} \in P(\mathbf{v}) \implies \max\{x|\ x \in P(\mathbf{v})\} \geqq \tilde{x} :\iff f(\mathbf{v}) \geqq \tilde{x} \implies \tilde{x} \in P_f(\mathbf{v})$

the statement

(II.31) $$P_f(\mathbf{v}) = P(\mathbf{v}) \qquad \forall \mathbf{v} \in V$$

results for the level sets and the production possibility sets. Therefore, we gain by Axioms [P1a], [P4], [P5], and [P6] the important relation[140]

(II.32) $$\forall x \in X,\ \forall \mathbf{v} \in V: \quad x \in P(\mathbf{v}) \iff x \leqq f(\mathbf{v}).$$

Even for an indivisible good the production function $f\colon V \to X$ is equivalent to the output correspondence $P\colon V \to \mathfrak{P}(X)$ and the inverse input correspondence $L\colon X \to \mathfrak{P}(V)$ respectively.

Apart from the proof of the existence of a production function the subsequent proposition on the properties of production functions can be noted, where V and $X \subset \mathbb{R}_+$ are the factor space and the commodity space respectively.

Proposition II.13 *An output correspondence* $P\colon V \to \mathfrak{P}(X)$ *with the level sets* $P(\mathbf{v}) = \{x|\ x \leqq f(\mathbf{v}),\ x \in X\}$ *satisfies Axioms* [P1]–[P6] *if and only if the production function* f *with* $f(\mathbf{v}) = \max\{x|\ x \in P(\mathbf{v})\}$ *satisfies the following five conditions:*

[$F1$] $f(\mathbf{0}) = 0$

[$F2$] *$f(\mathbf{v})$ is finite for every $\mathbf{v} \in V$ ($f\colon V \to X$ is a real-valued function.)*

[$F3$] $\forall \tilde{\mathbf{v}}, \mathbf{v} \in V: \left[\tilde{\mathbf{v}} \geq \mathbf{v} \iff f(\tilde{\mathbf{v}}) \geqq f(\mathbf{v})\right]$ *(f increases monotonically.)*

[$F4$] *f is upper semi-continuous on V.*

[$F5$] $\exists \{\mathbf{v}^\nu\} \subset V: \ f(\mathbf{v}^\nu) \to \infty$.

Proof: The proposition can be proved by five equivalence relations.
[F1] $\iff$ [P1] (*Possibility of Inaction* and *No Land of Cockaigne*)
For $\mathbf{v} = \mathbf{0}$ we obtain

$$f(\mathbf{0}) = \max\{x|\ x \in P(\mathbf{0})\} = 0 \iff P(\mathbf{0}) = \left[0, f(\mathbf{0})\right] = \{0\}.$$

[F2] $\iff$ [P5] (*Boundedness*)
If $P(\mathbf{v})$ is bounded [P5] for all $\mathbf{v} \in V$, then $f(\mathbf{v}) = \max\{x|\ x \in P(\mathbf{v})\}$ must be finite [F2]. If $f(\mathbf{v})$ is finite for all $\mathbf{v} \in V$, then $P(\mathbf{v}) = \{x|\ x \leqq f(\mathbf{v}),\ x \in X\}$ is not only bounded above but also bounded below because of $P(\mathbf{v}) \subset X$.
[F3] $\iff$ [P3] (*Disposability of Inputs*)

[140] Cf. FÄRE (1988, p. 24) or SHEPHARD (1953, p. 21).

For the input vectors $\tilde{\mathbf{v}}, \mathbf{v} \in V$ with $\tilde{\mathbf{v}} \geqq \mathbf{v}$ we have by [P3] $P(\mathbf{v}) \subset P(\tilde{\mathbf{v}})$ and, therefore, $f(\tilde{\mathbf{v}}) \geqq f(\mathbf{v})$ or [F3]. The reverse conclusion immediately results from the definition of $P(\mathbf{v})$ for $f(\tilde{\mathbf{v}}) \geqq f(\mathbf{v})$.

$$\begin{aligned} P(\mathbf{v}) &= \{x \mid x \leqq f(\mathbf{v}),\ x \in X\} \\ &\subset \{x \mid x \leqq f(\tilde{\mathbf{v}}),\ x \in X\} = P(\tilde{\mathbf{v}}) \end{aligned}$$

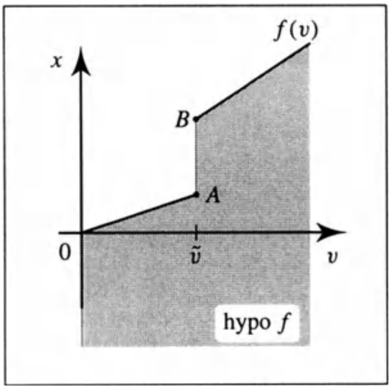

Figure II.35: Upper semi-continuity of a production function

[F4] $\Longleftrightarrow$ [P6] (*Closedness*)
By Theorem C.2, p. 301, the production function f is upper semi-continuous if and only if the hypograph

$$\text{hypo}\, f := \left\{(\mathbf{v}, \mu) \in \mathbb{R}^n \times \mathbb{R} \mid \mu \leqq f(\mathbf{v})\right\}$$

is closed in $\mathbb{R}^{n+1}$. Note that the closedness of the hypograph, shown as a gray area in Figure II.35, is satisfied, provided at point $\tilde{v}$ where f jumps all points between A and B are elements of the hypograph. This condition is fulfilled if the functional value $f(\tilde{v})$ equals the upper point B.

By [F2] $\mathbb{R}^n$ can be replaced in the definition with the effective domain $\text{Dom}\, f = V$ so that $\text{hypo}\, f := \{(\mathbf{v}, \mu) \in V \times \mathbb{R} \mid \mu \leqq f(\mathbf{v})\}$.

However, the output correspondence P is closed if its graph

$$\text{GR} = \{(\mathbf{v}, x) \mid \mathbf{v} \in V,\ x \in P(\mathbf{v})\}$$

is closed. By (II.32) we can write equivalently $\text{GR} = \{(\mathbf{v}, x) \in V \times X \mid x \leqq f(\mathbf{v})\}$. Consequently, the closedness of the graph GR at once results according to Theorem A.1, p. 283 (from the closedness of the hypograph hypo f) since

$$\text{hypo}\, f \cap (V \times X) = \text{GR} .$$

The reverse conclusion is more complicated. The closedness of the output correspondence P is equivalent to the closedness of the input correspondence L. Thus, the input requirement sets $L(x)$ are closed for all $x \in X$. However, the closedness of the production function f is given if and only if the sets $\{\mathbf{v} \in \mathbb{R}^n \mid \mu \leqq f(\mathbf{v})\}$ are closed for all $\mu \in \mathbb{R}$ (Theorem C.2, p. 301). Provided a divisible commodity exists,

$$\left\{\mathbf{v} \in \mathbb{R}^n \mid \mu \leqq f(\mathbf{v})\right\} = \{\mathbf{v} \in V \mid \mu \leqq f(\mathbf{v})\} = \begin{cases} V & \text{for } \mu \leqq 0 \\ L(\mu) & \text{for } \mu > 0 \end{cases}$$

completes the proof since the closedness of the input requirement sets $L(\mu)$ passes on to the sets $\{\mathbf{v} \in \mathbb{R}^n \mid \mu \leqq f(\mathbf{v})\}$. An indivisible good $x \in X = \mathbb{Z}_+$ however implies the relation $L(\mu) = \emptyset$ for $\mu \notin X$. If $\lceil\mu\rceil$ again denotes the smallest integer not smaller than μ, then the following equation holds for $\mu > 0$ because of $\lceil\mu\rceil \in X$

$$\{\mathbf{v} \in V \mid \mu \leqq f(\mathbf{v})\} = \{\mathbf{v} \in V \mid \lceil\mu\rceil \leqq f(\mathbf{v})\} = L(\lceil\mu\rceil).$$

So we can apply the case of divisible outputs; see Figure II.36, left hand part.
[F5] $\Longleftrightarrow$ [P2] (*Attainability of Each Production*)[141]
If no sequence of input vectors $\{\mathbf{v}^\nu\}$ exists such that $f(\mathbf{v}^\nu) \to +\infty$, then in contradiction to [P2] there is a quantity x which is not producible, $x > f(\mathbf{v})$ for all $\mathbf{v} \in V$. Regarding [P4] (*Disposability of Outputs*) the conclusion from [F5] to [P2] ensues: if the output correspondence P is defined by $P(\mathbf{v}) = \{x \mid x \leqq f(\mathbf{v}),\ x \in X\}$, then $\tilde{x}, x \in P(\mathbf{v})$ for all $\tilde{x}, x \in X$ with $\tilde{x} \leqq x \leqq f(\mathbf{v})$. ■

Figure II.36 depicts the graphs of two production functions $x = f_1(v)$ and $x = f_2(v)$ for an indivisible good x satisfying conditions [F1]–[F5]. The left hand part assumes a perfectly divisible production factor v, whereas the right hand part assumes an indivisible production factor. In particular, the points of jumps in the left hand graph can be used to get an idea of the upper semi-continuity of a production function.

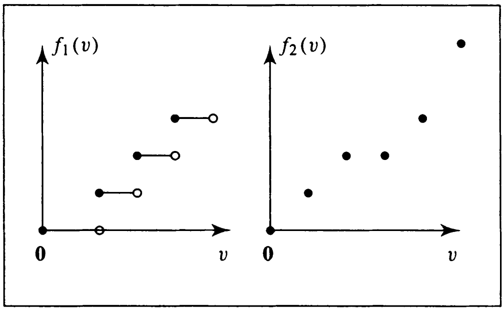

Figure II.36: Graph of special production functions

As the three following properties are based on the fundamental assumption of divisible production factors, we dispense with a detailed discussion.

1. The production function f is quasi-concave if and only if the output correspondence P is quasi-concave; see Theorem II.5 in the mathematical appendix of this section.

2. The assumption that the output correspondence P is concave in the factor space V is equivalent to the assumption that the production function f is concave in V.[142]

3. If the output correspondence $P\colon V \to \mathbb{R}$ is not only closed but also

[141]Since $f(\mathbf{v}^\nu)$ is bounded for all $\mathbf{v}^\nu \in V$, the limit $f(\mathbf{v}^\nu) \to +\infty$ can be fulfilled only for $\|\mathbf{v}^\nu\| \to +\infty$.
[142]Theorem C.1, p. 300, contains a criterion for the concavity of the production function f. Similarly, the output correspondence P is said to be concave in V if it satisfies the criterion $\lambda P(\mathbf{v}^1) + (1 - \lambda)P(\mathbf{v}^2) \subset P(\lambda\mathbf{v}^1 + (1-\lambda)\mathbf{v}^2)$ for all $\mathbf{v}^1, \mathbf{v}^2 \in V$ and for all $\lambda \in [0, 1]$.

continuous[143] with $P(\mathbf{v}) \neq \emptyset$ for all $\mathbf{v} \in V$, then a continuous production function f exists such that $f(\mathbf{v}) \in P(\mathbf{v})$ is satisfied for every $\mathbf{v}$.[144]

Finally, the problem of profit maximization is given.[145]

$$\pi(p, \mathbf{q}) = \sup\left\{px - \mathbf{q}^{\mathrm{T}}\mathbf{v} \mid x \leqq f(\mathbf{v})\right\}$$

The corresponding problem of cost minimization for a held fixed quantity of good x is

$$c(\mathbf{q}, x) = \inf\left\{\mathbf{q}^{\mathrm{T}}\mathbf{v} \mid x \leqq f(\mathbf{v})\right\}.$$

As already mentioned, the problem of revenue maximization and the problem of output maximization have the same optimal solutions at given inputs $\mathbf{v}$ and a given positive commodity price $p > 0$.

$$\begin{aligned} & r(p, \mathbf{v}) = \sup\{px \mid x \in \operatorname{conv} P(\mathbf{v})\} \\ \Longleftrightarrow \quad & pf(\mathbf{v}) = p \sup\{x \mid x \in \operatorname{conv} P(\mathbf{v})\} \end{aligned}$$

The problem of profit maximization with regard to the output

$$\sup\{px - c(\mathbf{q}, x) \mid x \in X\}$$

faces the equivalent problem of profit maximization with regard to inputs

$$\sup\left\{pf(\mathbf{v}) - \mathbf{q}^{\mathrm{T}}\mathbf{v} \mid \mathbf{v} \in V\right\}.$$

[143] As noted above, each continuous correspondence is upper semi-continuous (Definition C.5, p. 307) and each upper semi-continuous correspondence is itself closed (Theorem C.7, p. 309).

[144] This statement is a product of the Maximum Theorem (Theorem C.14, p. 311), where we have to put $x = \phi(x)$, $f(\mathbf{v}) = \psi(\mathbf{v})$, and $\Psi(\mathbf{v}) = \{x \in P(\mathbf{v}) \mid x = f(\mathbf{v})\}$.

[145] A detailed discussion of the given optimization problems may be found, for example, in BLACKORBY, PRIMONT, RUSSEL (1978) or DIEWERT (1982).

3 Summary

A household, trying to satisfy its needs as well as possible, adapts its preferences for the choice of commodity bundles which it can buy. These preferences are represented by preference orderings satisfying Axioms [$\mathscr{P}1$]–[$\mathscr{P}3$] and possibly [$\mathscr{P}4$]–[$\mathscr{P}6$]. The preference orderings in turn determine families of preference sets $\big(\mathscr{P}(\mathbf{x}) \mid \mathbf{x} \in X\big)$, which are called preference structures. Since each household may have different preferences, each household must be associated with a separate preference structure.

In the theory of the firm the production depends on its respective technology. This production technology can be described by a family of sets $\big(P(\mathbf{v}) \mid \mathbf{v} \in V\big)$, where this family is called a production structure if each of its members fulfills certain conditions, namely Axioms [P1]–[P6]. Each member of the family is called a production possibility set; it contains all of the commodity bundles which are producible by a given input vector. An input-output combination included by one member of the family is, therefore, technically feasible and is called a possible activity. Technically, the production technology is represented by a multi-valued mapping, namely the output correspondence.

The inverse mapping is called an input correspondence. As before, the production technology is represented by a family of sets $\big(L(\mathbf{x}) \mid \mathbf{x} \in X\big)$, although now, the members of the family collect all input vectors allowing the production of the respective commodity bundle. This family of input requirement sets is equivalent to the family of production possibility sets and is subject to Axioms [L1]–[L6], which are equivalent to [P1]–[P6] (Proposition II.2).

In order to keep a simple structure for the examined sets even under consideration of indivisible goods and factors, we introduce the surrogate of the convex hull after the description of production technologies. Each of the examined sets of points is associated with the smallest convex set containing the respective set. Among the various possibilities to represent convex hulls one particular form stands out. Using CARATHÉODORY's Theorem it is possible to express each point of the convex hull as a convex combination of a small number of points contained in the original set. In particular, we can take advantage of the fact that the extreme points of the convex hull belong to the genuine set. The convex hull gets further structural properties by the assumption of integer convexity. Under this assumption there is an one-to-one relationship between the generating set and its convex hull. Finally, it is shown how the "relevant" boundary points of the convex set concerned at least are determined by rays through the origin.

In the class of special production technologies the technologies with certain properties of homogeneity stand out. They give an answer to the question as to how the outcome of production varies when the production level changes. Although an activity with indivisible goods and factors cannot be multiplied by an arbitrary scalar, the idea of increasing, constant, or decreasing returns to scale remains the same. Apart from the total factor variation along a ray through the origin we can furthermore analyze the question concerning what outcomes result when different activities are combined additively. In particular, the joint use of indivisible factors as well as the joint production of indivisible goods hold the potential of increasing returns to scale. Finally, we deduce outcomes which result from the consideration of factor constraints in addition to the pure production technology.

Having presented the instruments for the representation of production technologies, we now seek for those activities satisfying certain optimality conditions. The criterion of technically efficient production indicates those activities which do not unnecessarily waste resources either as inputs or outputs. To describe the extent of the waste, different measures of technical efficiency of an activity are introduced. Beginning with the notion that boundary points in particular satisfy the criterion of technical efficiency, rays through the origin serve again as an initial point. The meaning of the efficiency measures is reduced by the transition to the convex hulls, but they do not lose all of their importance so that we keep on working with them.

The introduction of prices enables the assessment of a possible activity $(\mathbf{v}, \mathbf{x}) \in$ GR with respect to its cost or revenue efficiency. For example, the degree of cost efficiency of an activity $(\mathbf{v}, \mathbf{x})$ is determined by a comparison to a convex combination of activities which are input efficient as well as cost efficient with respect to the same output $\mathbf{x}$. With this in mind we can now discuss the goal of profit maximization concerning the restriction of a given production technology, remembering that a profit maximizing firm necessarily also pursues two subgoals. A profit maximum can be realized only if the corresponding commodity bundle is produced at minimal costs. The latter goal has not been successful if the firm wastes inputs, where the punishment is an avoidable cost. Again no profit maximum can be achieved if the firm wastes outputs by not selling these quantities. The shortcoming of the goal of revenue maximization at given inputs excludes a profit maximum. The concluding section is dedicated to a special case where the inspected firm produces a particular commodity.

4 Appendix

4.1 The Concept of Quasi-Concavity

Definition II.12[146] *The function* $f\colon X \to \mathbb{R}$ *is said to be* ***quasi-concave*** *in a convex subset X of* $\mathbb{R}^n$ *if for arbitrary* $\mathbf{x}, \tilde{\mathbf{x}} \in X$

$$f\big((1-\lambda)\mathbf{x} + \lambda\tilde{\mathbf{x}}\big) \geqq \min\{f(\mathbf{x}), f(\tilde{\mathbf{x}})\} \qquad \forall \lambda \in [0, 1].$$

The function $g\colon X \to \mathbb{R}$ *is called* ***quasi-convex*** *in X if* $-g$ *is quasi-concave in X, i.e. if for arbitrary* $\mathbf{x}, \tilde{\mathbf{x}} \in X$

$$g\big((1-\lambda)\mathbf{x} + \lambda\tilde{\mathbf{x}}\big) \leqq \max\{g(\mathbf{x}), g(\tilde{\mathbf{x}})\} \qquad \forall \lambda \in [0, 1].$$

The importance of quasi-concavity of a function results from the following:

Theorem II.5[147] *The level sets* $\Gamma_+(y) = \{\mathbf{x} \in X \mid f(\mathbf{x}) \geqq y\}$ *are convex for all* $y \in \mathbb{R}$ *if and only if the function f is quasi-concave on X.*
The level sets $\Gamma_-(y) = \{\mathbf{x} \in X \mid g(\mathbf{x}) \leqq y\}$ *are convex for all* $y \in \mathbb{R}$ *if and only if the function g is quasi-convex in X.*

Proof: The necessary part requires that the level sets $\Gamma_+(y)$ are convex for all $y \in \mathbb{R}$. If we set $y = \min\{f(\mathbf{x}), f(\tilde{\mathbf{x}})\}$ for arbitrary $\mathbf{x}, \tilde{\mathbf{x}} \in X$, then $\mathbf{x}, \tilde{\mathbf{x}} \in \Gamma_+(y)$. On the basis of the supposed convexity of $\Gamma_+(y)$ it follows $\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}} \in \Gamma_+(y)$ for all $\lambda \in [0, 1]$. Thus,

$$f\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big) \geqq y = \min\{f(\mathbf{x}), f(\tilde{\mathbf{x}})\}$$

must hold such that f is quasi-concave in X.
The sufficient part goes on the principle that f is quasi-concave in X. For an arbitrary $y \in \mathbb{R}$ we get $f(\mathbf{x}) \geqq y$ and $f(\tilde{\mathbf{x}}) \geqq y$ for all $\mathbf{x}, \tilde{\mathbf{x}} \in \Gamma_+(y)$. From the quasi-concavity of f it is found that $f\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big) \geqq \min\{f(\mathbf{x}), f(\tilde{\mathbf{x}})\} \geqq y$ for all $\lambda \in [0, 1]$. On the basis of the definition of $\Gamma_+(y)$ it is found that $\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}} \in \Gamma_+(y)$ for all $\lambda \in [0, 1]$ and, therefore, $\Gamma_+(y)$ is convex for all $y \in \mathbb{R}$.
The transference of the proof to convex level sets $\Gamma_-(y)$ and a quasi-convex function g becomes superfluous if we take into account that $-g$ is quasi-concave in X. ■

Definition II.13 *Let X be a convex subset in* $\mathbb{R}^n$. *The correspondence* $F\colon X \to \mathfrak{P}(Y)$ *with* $Y \subset \mathbb{R}^m$ *is said to be* ***quasi-concave***[148] *if for arbitrary* $\mathbf{x}, \tilde{\mathbf{x}} \in X$

$$F(\mathbf{x}) \cap F(\tilde{\mathbf{x}}) \subset F\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big) \qquad \forall \lambda \in [0, 1].$$

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[146] Cf. TAKAYAMA (1990, p. 113).

[147] Cf. CHIANG (1984, p. 391) and SHEPHARD (1953, p. 297), who, however, has a mistake in the second part of the theorem. The following proof is taken from SHEPHARD. Remember that f can only be quasi-concave in X if X is a convex subset in $\mathbb{R}^n$. Thus, for instance, the case $X = \mathbb{Z}_+ \times \mathbb{R}_+ \subset \mathbb{R}^2$ is ruled out.

[148] The criterion of a quasi-concave correspondence F should not be confused with the concavity. The correspondence F is called concave in X if it satisfies the criterion $\lambda F(\mathbf{x}^1) + (1-\lambda)F(\mathbf{x}^2) \subset F\big(\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2\big)$ for all $\mathbf{x}^1, \mathbf{x}^2 \in X$ and for all $\lambda \in [0, 1]$.

Defining the inverse correspondence F of Γ_+ of Theorem II.5 by using a function $f\colon X \to \mathbb{R}$, where $F(\mathbf{x}) := \{y \in \mathbb{R} \mid f(\mathbf{x}) \geqq y\}$, the quasi-concavity of the function f is equivalent to the quasi-concavity of the correspondence F. We obtain

$$\begin{aligned} & F(\mathbf{x}) \cap F(\tilde{\mathbf{x}}) \subset F\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big) \\ \iff\ & \{y \mid f(\mathbf{x}) \geqq y \text{ and } f(\tilde{\mathbf{x}}) \geqq y\} \subset \big\{y \mid f\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big) \geqq y\big\} \\ \iff\ & \min\{f(\mathbf{x}), f(\tilde{\mathbf{x}})\} \leqq f\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big) \end{aligned}$$

for all $\lambda \in [0, 1]$. The relation between the inverse correspondences Γ_+ and F induced by Theorem II.5 can be generalized to the following:

Theorem II.6 *Let F^{-1} be the inverse correspondence of $F\colon X \to \mathfrak{P}(Y)$. Then the correspondence F is quasi-concave in X if and only if $F^{-1}(\mathbf{y})$ is convex for all $\mathbf{y} \in Y$.*

Proof: The correspondences F and F^{-1} are said to be inverse to each other if they satisfy the following equivalence relation:

$$\mathbf{y} \in F(\mathbf{x}) \iff \mathbf{x} \in F^{-1}(\mathbf{y}).$$

If F is quasi-concave in X, then for arbitrary $\mathbf{x}, \tilde{\mathbf{x}} \in X$ and for all $\lambda \in [0, 1]$ it ensues

$$\begin{aligned} &\text{[a]} && \big[F(\mathbf{x}) \cap F(\tilde{\mathbf{x}}) \subset F\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big)\big] \\ &\text{[b]} && \iff \big[\mathbf{y} \in F(\mathbf{x}),\ \mathbf{y} \in F(\tilde{\mathbf{x}}) \implies \mathbf{y} \in F\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big)\big] \\ &\text{[c]} && \iff \big[\mathbf{x} \in F^{-1}(\mathbf{y}),\ \tilde{\mathbf{x}} \in F^{-1}(\mathbf{y}) \implies \lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}} \in F^{-1}(\mathbf{y})\big]. \end{aligned}$$

The level sets $F^{-1}(\mathbf{y})$ are convex at least at the points $\mathbf{y}$ examined in [b]. Suppose another point $\tilde{\mathbf{y}} \in Y$ exists at which $F^{-1}(\tilde{\mathbf{y}})$ is not convex. Then there must be two vectors $\mathbf{x} \in F^{-1}(\tilde{\mathbf{y}})$ and $\tilde{\mathbf{x}} \in F^{-1}(\tilde{\mathbf{y}})$ and a $\lambda \in [0, 1]$ such that $\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}} \notin F^{-1}(\tilde{\mathbf{y}})$. Due to $\tilde{\mathbf{y}} \in F(\mathbf{x})$ and $\tilde{\mathbf{y}} \in F(\tilde{\mathbf{x}})$, a contradiction of the presumed quasi-concavity of F results by $\tilde{\mathbf{y}} \notin F\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big)$.

The quasi-concavity of F yields convex level sets $F^{-1}(\mathbf{y})$ for all $\mathbf{y} \in Y$.

Supposing that $F^{-1}(\mathbf{y})$ is convex for all $\mathbf{y} \in Y$, we have as before [c] $\iff$ [b] $\iff$ [a] for arbitrary $\lambda \in [0, 1]$. The proof is complete if we can show that there is no pair of points $\mathbf{x}, \tilde{\mathbf{x}} \in X$ which is not included by [c] and which leads to $F(\mathbf{x}) \cap F(\tilde{\mathbf{x}}) \not\subset F\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big)$.

In the case that there are two of these points $\mathbf{x}, \tilde{\mathbf{x}} \in X$ and a $\lambda \in [0, 1]$, then

$$\begin{aligned} & && F(\mathbf{x}) \cap F(\tilde{\mathbf{x}}) \not\subset F\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big) \\ \implies\ & \exists\, \mathbf{y} \in Y: && \big[\mathbf{y} \in F(\mathbf{x}),\ \mathbf{y} \in F(\tilde{\mathbf{x}}) \implies \mathbf{y} \notin F\big(\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}}\big)\big] \\ \iff\ & \exists\, \mathbf{y} \in Y: && \big[\mathbf{x} \in F^{-1}(\mathbf{y}),\ \tilde{\mathbf{x}} \in F^{-1}(\mathbf{y}) \implies \lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}} \notin F^{-1}(\mathbf{y})\big] \end{aligned}$$

implies a contradiction of the assumed convexity of $F^{-1}(\mathbf{y})$ for all $\mathbf{y} \in Y$. ∎

4.2 Closedness of the Convex Hull of Input Requirement Sets

Proposition II.14[149] *If positive real numbers* k_i $(i = 1, \ldots, m)$ *exist for each input vector* $\mathbf{v}$ *in an input requirement set* $L(\mathbf{x}) \subset V$ *such that* $\tilde{\mathbf{v}} := \mathbf{v} + k_i \mathbf{e}^i$ *is also an element in* $L(\mathbf{x})$*, then the convex hull* $\operatorname{conv} L(\mathbf{x}) \subset \mathbb{R}^m_+$ *is closed under* [L6] *(Closedness), too.*

Before going into the proof, it is worthwhile making some remarks on the significance of this proposition.

1. The required property of the input correspondence L is particularly given when [L3] (*Disposability of Inputs*) holds good.
2. The transference of the proposition to a closed [$\mathcal{P}$4] preference set $\mathcal{P}(\mathbf{x})$ causes no problems. In particular, the property required by Proposition II.14 is satisfied with respect to the preference structure $\mathcal{P}$ when [$\mathcal{P}$5] (*Monotonicity*) holds.
3. In the examined optimization problems each feasible region is usually substituted by its convex hull. Since optimal solutions usually lie in the boundary of this convex hull, the closedness of the convex hull is fundamental to the existence of a feasible optimal solution.[150]
4. The convex hull of a bounded closed (i.e. compact) set $C \subset \mathbb{R}^m$ is also compact, but the convex hull of an unbounded closed set C does not need to be closed.[151] The special structure of the examined sets permits us of closing this gap.

By Theorem A.5 the convex hull of the input requirement set $\operatorname{conv} L(\mathbf{x}) \subset \mathbb{R}^m$ is closed if we can show that the limits of all convergent sequences of points in $\operatorname{conv} L(\mathbf{x})$ are again contained in $\operatorname{conv} L(\mathbf{x})$.

Proof: Assuming a convergent sequence of points $\{\mathbf{v}^\nu\} \subset \operatorname{conv} L(\mathbf{x})$ with limit $\mathbf{v}^0$, we have to show that this implies $\mathbf{v}^0 \in \operatorname{conv} L(\mathbf{x})$. By Theorem B.4 (CARATHÉODORY), p. 293, each element $\mathbf{v}^\nu$ can be expressed as a convex combination of no more than $m+1$ points in $L(\mathbf{x})$. Without requiring the $\mathbf{a}^\nu_j$ to be distinct, we obtain

$$\mathbf{v}^\nu = \sum_{j=0}^{m} \lambda^\nu_j \mathbf{a}^\nu_j \quad \text{with} \quad \mathbf{a}^\nu_j \in L(\mathbf{x}), \;\; \boldsymbol{\lambda}^\nu \in \Lambda^{m+1}.$$

Because of $\boldsymbol{\lambda}^\nu \in \Lambda^{m+1}$, it follows that $\|\boldsymbol{\lambda}^\nu\| \leqq 1$, i.e. the sequence of points $\{\boldsymbol{\lambda}^\nu\} = \left\{(\lambda^\nu_0, \ldots, \lambda^\nu_m)^{\mathsf{T}}\right\}$ is bounded. Thus, by Theorem A.4 (BOLZANO-WEIERSTRASS), p. 286, a convergent subsequence $\{\boldsymbol{\lambda}^{\nu_{k_1}}\}$ exists whose sequences of components satisfy $\lambda_j^{\nu_{k_1}} \to \lambda^0_j$ $(j = 0, \ldots, m)$.

[149] Proposition and a sketch of the proof can be found in STARR (1969).
[150] See Theorems II.2 and II.3.
[151] See the remarks on Figure B.1.

Each point $\boldsymbol{\lambda}^{\nu_{k_1}}$ is associated with a tuple $\left(\mathbf{a}_0^{\nu_{k_1}}, \ldots, \mathbf{a}_m^{\nu_{k_1}}\right)$, where the sequences of points $\left\{\mathbf{a}_j^{\nu_{k_1}}\right\}$ $(j = 0, \ldots, m)$ may be bounded or not. For the further procedure the subscripts of the bounded and unbounded sequences are collected in J_1 and J_2 respectively.

Because subsequences of convergent sequences have the same limit, by recursive procedure we can determine convergent subsequences $\{\boldsymbol{\lambda}^{\nu_{k_2}}\}$ and $\left\{\mathbf{a}_j^{\nu_{k_2}}\right\}$ with $j \in J_1$.

The other bounded sequences $\left\{\mathbf{a}_j^{\nu_{k_1}}\right\}$ with $j \in J_2$ are marked by the fact that there is a point $\mathbf{a}_j^{\nu_{k_1}}$ for every $c > 0$ such that $\|\mathbf{a}_j^{\nu_{k_1}}\| \geqq c$. Thus, the sequence of numbers $\left\{\|\mathbf{a}_j^{\nu_{k_1}}\|\right\}$ has a divergent subsequence $\left\{\|\mathbf{a}_j^{\nu_{k_2}}\|\right\}$ with limit $+\infty$, that is $\|\mathbf{a}_j^{\nu_{k_2}}\| \to \infty$.

By renaming the sequences $\{\boldsymbol{\lambda}^{\nu_{k_2}}\}$, $\left\{\mathbf{a}_j^{\nu_{k_2}}\right\}$ $(j = 0, \ldots, m)$ we gain

$$\begin{aligned} \lambda_j^\nu &\to \lambda_j^0 && j = 0, \ldots, m, \\ \mathbf{a}_j^\nu &\to \mathbf{a}_j^0 && j \in J_1, \\ \|\mathbf{a}_j^\nu\| &\to \infty && j \in J_2. \end{aligned}$$

Since $L(\mathbf{x})$ is closed by [L6], $\mathbf{a}_j^0 \in L(\mathbf{x})$ must be satisfied for all $j \in J_1$. Since the sequence of points $\{\mathbf{v}^\nu\} = \left\{\sum_{j=1}^m \lambda_j^\nu \mathbf{a}_j^\nu\right\}$ converges by assumption, $\lambda_j^0 = 0$ must hold for $j \in J_2$.[152] Now, in the limit case $\sum_{j\in J_1} \lambda_j^0 = 1$ follows from $\sum_{j=0}^m \lambda_j^0 = 1$.

Considering now $L(\mathbf{x}) \subset \mathbb{R}_+^m$, each point of the sequence $\{\mathbf{v}^\nu\}$ can be expressed as

$$\mathbf{v}^\nu = \sum_{j\in J_1} \lambda_j^\nu \mathbf{a}_j^\nu + \sum_{j\in J_2} \lambda_j^\nu \mathbf{a}_j^\nu \geqq \sum_{j\in J_1} \lambda_j^\nu \mathbf{a}_j^\nu.$$

The limit on the left has been denoted by $\mathbf{v}^0$. If we put $\tilde{\mathbf{v}} := \sum_{j\in J_1} \lambda_j^0 \mathbf{a}_j^0$ for the limit on the right, then $\mathbf{v}^0 \geqq \tilde{\mathbf{v}}$ where $\tilde{\mathbf{v}} \in \operatorname{conv} L(\mathbf{x})$.[153] The proof is completed if it is possible to replace $\mathbf{a}_j^0$ $(j \in J_1)$ with point $\mathbf{b}_j$ such that the residuum

$$\lim_{\nu\to\infty} \sum_{j\in J_2} \lambda_j^\nu \mathbf{a}_j^\nu = \mathbf{v}^0 - \tilde{\mathbf{v}} \geq \mathbf{0}$$

vanishes. Put $\mathbf{b}_j := \mathbf{a}_j^0 + \mathbf{v}^0 - \tilde{\mathbf{v}}$ $(j \in J_1)$ with $\mathbf{b}_j \geqq \mathbf{a}_j^0$, then

$$\mathbf{v}^0 = \sum_{j\in J_1} \lambda_j^0 \mathbf{b}_j, \tag{II.33}$$

where $\mathbf{v}^0 \in \operatorname{conv} L(\mathbf{x})$ provided the $\mathbf{b}_j$ can be expressed as convex combinations of points in $L(\mathbf{x})$.

[152] If $\lambda_j^\nu \to 0$ and $a_j^\nu \to +\infty$, then nothing is said about the limit $\lim_{\nu\to\infty} \lambda_j^\nu a_j^\nu$. Take $\lambda_j^\nu = 1/\nu$ and $a_j^\nu = \nu^2 \ln(1+1/\nu)$ as an illustration. Using L'HÔPITAL's rule it can be shown that $\lambda_j^\nu \to 0$, $a_j^\nu \to +\infty$, and $\lambda_j^\nu a_j^\nu \to 1$.

[153] The proof would have been completed at this point if $\lim_{\nu\to\infty} \sum_{j\in J_2} \lambda_j^\nu \mathbf{a}_j^\nu = \mathbf{0}$ or, equivalently, $\tilde{\mathbf{v}} = \mathbf{v}^0$ could be guaranteed.

In the case of $\mathbf{v}^0 = \tilde{\mathbf{v}}$ with $\mathbf{b}_j = \mathbf{a}_j^0$ there is nothing more to say since $\mathbf{a}_j^0 \in L(\mathbf{x})$ is already satisfied. In the case of $\mathbf{v}^0 \geq \tilde{\mathbf{v}}$ with $\mathbf{b}_j \geq \mathbf{a}_j^0$ we can make use of the assumption that there are positive numbers k_i $(i = 1, \ldots, m)$ for each $\mathbf{a}_j^0 \in L(\mathbf{x})$ such that $\mathbf{a}_j^0 + k_i \mathbf{e}^i \in L(\mathbf{x})$. Thus, from the k_i we can recursively derive arbitrarily large numbers μ_i such that $\mu_i > m(b_{ij} - a_{ij}^0)$ $(i = 1, \ldots, m)$; see the figure opposite. This results in

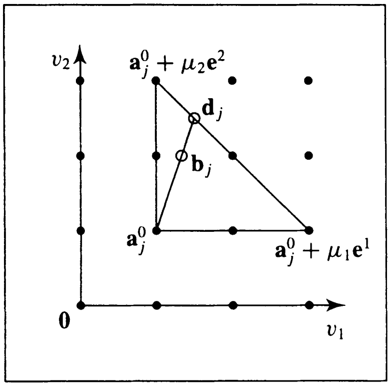

Figure II.37: Proof of Proposition II.14

$$\mathbf{b}_j = \mathbf{a}_j^0 + \sum_{i=1}^{m} \alpha_i \mu_i \mathbf{e}^i$$

$$\text{with} \quad \alpha_i := \frac{b_{ij} - a_{ij}^0}{\mu_i} < 1/m\,.$$

Considering $\mathbf{a}_j^0 + \mu_i \mathbf{e}^i \in L(\mathbf{x})$, the next convex combination for each $j \in J_1$

$$\text{(II.34)} \qquad \mathbf{d}_j := \sum_{i=1}^{m} \frac{\alpha_i}{\hat{\alpha}} (\mathbf{a}_j^0 + \mu_i \mathbf{e}^i) \quad \text{with} \quad 0 < \hat{\alpha} := \sum_{i=1}^{m} \alpha_i < 1$$

yields a point $\mathbf{d}_j \in \mathrm{conv}L(\mathbf{x})$. It follows by rearranging (II.34)

$$\begin{aligned} &\mathbf{d}_j = \mathbf{a}_j^0 + 1/\hat{\alpha}\, (\mathbf{b}_j - \mathbf{a}_j^0) \\ \Longleftrightarrow \quad &\mathbf{b}_j = \hat{\alpha}\mathbf{d}_j + (1 - \hat{\alpha})\mathbf{a}_j^0\,. \end{aligned}$$

Thus, the $\mathbf{b}_j$ can also be expressed as a convex combination of points of the input requirement set $L(\mathbf{x})$, $\mathbf{b}_j \in \mathrm{conv}L(\mathbf{x})$, and the proof is complete. ■

Chapter III.

Microeconomic Theory of Individual Agents

1 The Cost Structure of a Firm

1.1 Dual Statements in the Theory of a Firm

The notion of duality is used differently and is often misleading.[1] Usually, the attention is drawn to certain symmetry properties of statements or optimization problems. For example, we can formulate in accordance with mathematical logic the subsequent duality principle for the statements A and B which are either "true" or "false": if the word "and" is exchanged everywhere with the word "or" in the two semantically equivalent statements H_1 and H_2 which are only composed of the functional words "and", "or", and "not", then the resulting dual statements H_1^* and H_2^* are semantically equivalent, too.[2] When the text stresses this dual view of different statements, then such a substitution principle is intended bearing the symmetry properties in mind.[3]

The duality theory of mathematical programming deals with pairs of programs. An explicit rule always assigns a minimum problem P to a maximum problem P^* (or a maximum problem to a minimum problem). If the dual program P^* can be associated with a further dual program P^{**} such that P^{**} equals the primal program P, then we speak of a symmetric pair (P, P^*) of dual programs.[4] Whereas the variables of the primal program are taken from a certain space, the dual variables

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[1] See NEWMAN (1987b, p. 925), for more expositions.

[2] To illustrate the duality principle – from $H_1 \iff H_2$ it ensues $H_1^* \iff H_2^*$ – we can make use of DE MORGAN's laws:

$$\text{not } (A \text{ and } B) \iff (\text{not } A) \text{ or } (\text{not } B)$$
$$\text{not } (A \text{ or } B) \iff (\text{not } A) \text{ and } (\text{not } B)$$

[3] In particular, Theorem D.5, p. 320, emphasizes the dual view of different statements.

[4] Cf. BLUM, ÖTTLI (1975, pp. 113–114).

are assigned to the corresponding dual space. The intent of duality theory is to prove two statements for pairs of dual programs, which are then called duality theorems.[5] (1) If one of the two problems has an optimal solution, then the other one also has an optimal solution and the optimal values of both objective functions are the same. (2) A necessary and sufficient condition for both programs to have a solution is that both programs have feasible points.

The presented analysis directs the attention to the (dual) matching of quantities and prices. In doing so the introduced duality schemes should not be mixed up with pairs of dual programs. The duality schemes stress dual aspects of associated functions, whereas the underlying optimization problems are not dual programs.[6] The three presented duality schemes can be distinguished as follows:

(a) The first approach uses support functions and may be interpreted as a special case of the approaches (b) and (c). In doing so the treated functions only depend on dual variables. For a given commodity bundle $\tilde{\mathbf{x}}$ an input requirement set $L(\tilde{\mathbf{x}})$ corresponds to a subset in the primal factor space V, i.e. it embraces input vectors $\mathbf{v}$, whereas the cost function $c(\mathbf{q}, \tilde{\mathbf{x}})$ depends on factor prices or dual variables $\mathbf{q}$.

(b) Regarding the duality of conjugate functions, ROCKAFELLAR has introduced the concept of the FENCHEL transform into the field of convex analysis whereby a function f is related to a so called conjugate function g. Without exactly defining conjugate functions at this point, the examined pairs of conjugate functions correspond to the "best" pairs of functions (f, g) satisfying the following YOUNG-FENCHEL inequality:[7]

$$f(\mathbf{x}) + g(\mathbf{y}) \geqq \mathbf{x}^{\mathsf{T}}\mathbf{y} \qquad \forall \mathbf{x}, \ \forall \mathbf{y}.$$

Transferring this to economic theory, $f(\mathbf{v}^{-r}|L(\tilde{\mathbf{x}}))$ indicates the smallest quantity v_r of factor r such that v_r together with the other inputs $\mathbf{v}^{-r}$ allows the production of a given commodity bundle $\tilde{\mathbf{x}}$; see the right hand part of Figure III.1 with $r = 2$. The conjugate function $\tilde{c}(\cdot, \tilde{\mathbf{x}})$ describes, however, the minimal factor costs in units of factor r.[8] The normalized cost function $\tilde{c}(\cdot, \tilde{\mathbf{x}})$ depends on the dual variables $\mathbf{q}^{-r}$ where $\mathbf{q}^{-r}$ is a vector of normalized factor prices without the normalized price $q_r = 1$ of input r. For the representation of conjugate duality we especially have to emphasize the works of JORGENSON, LAU (1974) and NEWMAN (1987b).[9]

(c) The duality of polar gauges has been introduced into economic theory by SHEPHARD (1953). In contrast to (b), the theory of polar functions deals with "best" pairs (f, g) fulfilling MAHLER's inequality[10], i.e. an inequality of the form

$$f(\mathbf{x}) \cdot g(\mathbf{y}) \geqq \mathbf{x}^{\mathsf{T}}\mathbf{y} \qquad \forall \mathbf{x}, \ \forall \mathbf{y}.$$

[5] Both statements adjust the duality theorems to dual linear programs.

[6] The presented properties virtually serve as requirements for the duality theory of mathematical programming.

[7] For the YOUNG-FENCHEL inequalities, see (D.2a) and (D.2b), p. 317.

[8] The corresponding YOUNG-FENCHEL inequality is

$$-f(\mathbf{v}^{-r}|L(\tilde{\mathbf{x}})) + \tilde{c}(\mathbf{q}^{-r}, \tilde{\mathbf{x}}) \leqq (\mathbf{q}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) \qquad \forall \mathbf{v}^{-r}, \ \forall \mathbf{q}^{-r}.$$

[9] LAU (1974) contains further implications on econometric applications and further applications of this duality theory.

[10] See inequality (D.18), p. 335.

In this statement the role of the function f is adopted by the input distance function $t_I(\cdot, \tilde{\mathbf{x}})$, which depends on the inputs $\mathbf{v}$. The input distance function faces the cost function $c(\cdot, \tilde{\mathbf{x}})$; it depends on factor prices $\mathbf{q}$ and corresponds to the (reciprocally) polar gauge function g.[11] For a given commodity bundle $\tilde{\mathbf{x}}$ the cost function $c(\cdot, \tilde{\mathbf{x}})$ is associated with the input distance function $t_I(\cdot, \tilde{\mathbf{x}})$, which is a suitable measure for the efficiency of an input vector $\mathbf{v}$. A more detailed treatment of polar gauges is given in SHEPHARD (1953), who concentrates on single-product firms, and JACOBSEN (1970), who investigates multi-product firms. A lucid representation of SHEPHARD's duality theorem can be taken from JACOBSEN (1972).

The fundamental source for the basics of the duality of conjugate and polar functions is ROCKAFELLAR (1972). At the same time the transference of the results of convex analysis to microeconomic theory is determined by several aspects:

(a) The two theories of convex analysis conform to convex functions. However, economic theory often deals with concave functions. The change of sign corresponding to this observation is awkward, but it is accepted to assure, for example, nonnegative prices.

(b) The theory of polar sets and functions is founded on convex sets containing the origin, so they are star-shaped. The firm's production structure $\big(L(\mathbf{x}) \mid \mathbf{x} \in X\big)$ however consists of (convex) input requirement sets which – except for the special case $\mathbf{0} \in L(\mathbf{0})$ – do not contain the origin. The necessary switch-over to (aureoled) input requirement sets involves greater problems than one would suspect.

(c) While the functions used in convex analysis usually refer to an unique convex set $C \subset \mathbb{R}^m$, economic theory deals with (convex) level sets like $L(\mathbf{x})$ themselves depending on the parameters $\mathbf{x}$. The question arises how the cost function reacts to a variation of the outputs $\mathbf{x}$ and, therefore, to a change of the input requirement sets $L(\mathbf{x})$.

(d) Finally, we have to answer the crucial question as to what results are affected or not when the underlying sets are not convex. In dealing with indivisible production factors with nonconvex input requirement sets $L(\mathbf{x})$ we use the already presented surrogate of the convex hull $\operatorname{conv}L(\mathbf{x})$. This causes a certain loss of information, but many of the main results do not alter. Whereas the results on efficiency measuring using a (one-dimensional) ray through the origin may change dramatically by the transition from $L(\mathbf{x})$ to $\operatorname{conv}L(\mathbf{x})$, we do not observe a similar effect regarding the cost function with the $m-1$-dimensional objective function $\mathbf{q}^{\mathsf{T}}\mathbf{v}$.

The following sections deal with diverse operations regarding sets and functions. For instance, the operation $C \to \operatorname{conv}C$ assigns the convex hull $\operatorname{conv}C$ to the set C. In this sense the symbols "$\circ$" and "$*$" are reserved for indicating certain operations. Both symbols must be strictly distinguished with regard to whether it appears as a superscript (e.g. $C \to C^\circ$) or as a subscript (e.g. $C \to C_\circ$). All notations in dealing with functional symbols (e.g. $f = g$ or $c(\cdot, \mathbf{x})$) are taken from Appendix C.1.

[11] By Corollary III.18.1, p. 164, the inequality $t_I(\mathbf{v}, \tilde{\mathbf{x}}) \cdot c(\mathbf{q}, \tilde{\mathbf{x}}) \leqq \mathbf{q}^{\mathsf{T}}\mathbf{v}$ is only fulfilled for certain pairs of points $(\mathbf{q}, \mathbf{v})$.

Before going into alternative derivations of the firm's cost structure, it is worthwhile contrasting the working of the duality schemes used in Sections III.1 and III.2. Section III.1 refers to the usage of conjugate functions, while Section III.2 concentrates on dealing with polar gauges.

A comparison of both methods is given in Figure III.1;[12] it refers to various possibilities for describing a (convex) input requirement set $L(\mathbf{x})$ by functions. The left hand part of the figure is based on the input distance function

(III.1) $$t_I(\mathbf{v},\mathbf{x}) = \sup\{\lambda \geqq 0 \mid \mathbf{v} \in \lambda L(\mathbf{x})\},$$

which has already been introduced in the discussion of efficiency measuring. As shown by this figure, $t_I(\mathbf{v},\mathbf{x})$ determines the greatest λ such that the given commodity bundle $\mathbf{x}$ is producible by the inputs $\tilde{\mathbf{v}} = \mathbf{v}/\lambda$.

In the theory of conjugate functions this form of the total factor variation (along a ray through the origin) is compared to a partial factor variation, shown by the right hand part of the figure. The boundary of the input requirement set $L(\mathbf{x})$ is now indicated by the solutions to the problem

(III.2) $$\inf\left\{v_r \mid \mathbf{v} = (v_1,\ldots,v_m)^{\mathrm{T}} \in L(\mathbf{x})\right\}.$$

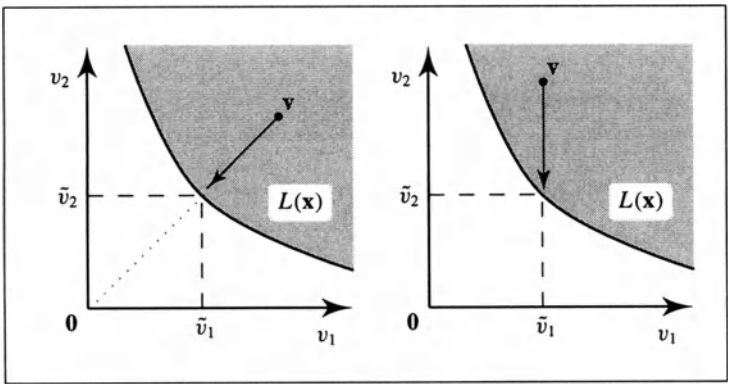

Figure III.1: Total versus partial factor variation

The left hand part of Figure III.1 shows an input vector $\binom{\tilde{v}_1}{\tilde{v}_2}$ such that $\lambda = 1$ solves (III.1). For $\lambda > 1$ each input vector $\binom{\tilde{v}_1}{\tilde{v}_2}/\lambda$ is not included in the input requirement set $L(\mathbf{x})$. The same input vector can be determined in the right hand part of Figure III.1 for the given input $\tilde{v}_1$ such that $\tilde{v}_2$ solves the problem (III.2). Not each vector $\binom{\tilde{v}_1}{v_2}$ with $v_2 < \tilde{v}_2$ belongs to $L(\mathbf{x})$.

Both methods are especially suitable for the representation of convex input requirement sets but they get into difficulties when certain factors are only available as integer units. Two problems are shown in Figure III.2 for the partial factor variation. Again, the two figures illustrate the input vector $\binom{\tilde{v}_1}{\tilde{v}_2}$. Certainly, this

[12]This comparison is picked up again in Figure III.36, p. 181.

vector cannot be input efficient in the left hand representation. A similar problem has already been discussed in view of FARRELL's input efficiency measure.

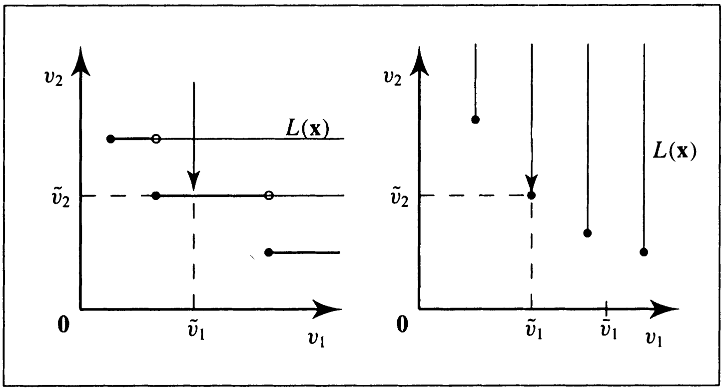

Figure III.2: The case of an indivisible production factor

The right hand graph includes admissible values for v_1 – see for instance $\bar{v}_1$ – such that there are no amounts of the second input satisfying $\binom{\bar{v}_1}{v_2} \in L(\mathbf{x})$. In these cases we put $\inf\{v_r|\ \mathbf{v} \in L(\mathbf{x})\} = +\infty$.

As long as convex input requirement sets are presupposed, we can understand the next theorem as a partial factor variation.

Theorem III.1[13] *Let C be a convex subset in* $\mathbb{R}^{n+1}$. *Then the function f, defined by*

(III.3) $$f(\mathbf{x}|C) := \inf\left\{\mu \mid \binom{\mathbf{x}}{\mu} \in C\right\},$$

is convex in $\mathbb{R}^n$.

A comparison to problem (III.2) shows how to describe the boundary of a convex input requirement set $L(\mathbf{x}) \subset \mathbb{R}^m$ by a convex function $f(\cdot|L(\mathbf{x}))$.[14] Take into account that the $m-1$ variables, which the function $f(\cdot|L(\mathbf{x}))$ depends on, are collected by $\mathbf{v}^{-r}$, i.e. the r-th component must be removed from the input vector $\mathbf{v}$.

If because of indivisible factors the input requirement set is not convex, then the function resulting from (III.2) will not be convex, either. Until now it has been suggested replacing the input requirement set $L(\mathbf{x})$ by its convex hull conv$L(\mathbf{x})$. The question arises as to what the effects of this procedure have on the boundary function which is to be determined.

The convex hull of a nonconvex function f (with a nonconvex epigraph epi$f$) is denoted by conv$f$ and it is given by Theorem III.1 for $C = \text{conv}(\text{epi}f)$.

---

[13]Cf. ROCKAFELLAR (1972, p. 33).

[14]The case of the single-product firm with $f(\cdot|L(x))$ corresponds to the isoquant at an output level x.

Therefore, $\operatorname{conv} f$ equals the greatest convex function satisfying[15]

$$\text{(III.4)} \qquad \operatorname{conv} f \leqq f\,.$$

An example for this operation is shown in Figure III.3. The underlying input requirement set $L(\mathbf{x})$ is given by the left hand part of Figure III.2. After the corresponding function $f(\cdot|L(\mathbf{x}))$ has been determined, the gray shadowed epigraph $\operatorname{epi} f(\cdot|L(\mathbf{x}))$ can also be marked. The function $\operatorname{conv} f(\cdot|L(\mathbf{x}))$ results from the convex hull of the epigraph $\operatorname{conv}\big(\operatorname{epi} f(\cdot|L(\mathbf{x}))\big)$ with the help of (III.3).

Given an arbitrary set $C \subset \mathbb{R}^{n+1}$, two functions $\operatorname{conv} f(\cdot|C)$ and $f(\cdot|\operatorname{conv} C)$ must be distinguished a priori. As shown by the following proposition, this differentiation is superfluous.

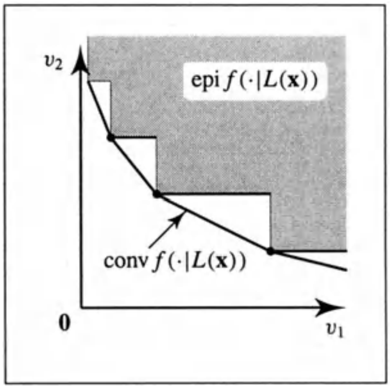

Figure III.3: Derivation of the function $\operatorname{conv} f(\cdot|L(\mathbf{x}))$

Proposition III.1 *For each set* $C \subset \mathbb{R}^{n+1}$ *with* $f(\cdot|C) > -\infty$ *we have*

$$\operatorname{conv} f(\cdot|C) = f(\cdot|\operatorname{conv} C).$$

Proof: In order to avoid terms of the form $-\infty + \infty$ in the subsequent calculations, we assume a set C with $f(\cdot|C) > -\infty$. Thus, the two appearing summations are unambiguous. In particular this assumption does not restrict the analysis for input requirement sets with $L(\mathbf{x}) \subset \mathbb{R}^m_+$.

By Theorem B.4 (CARATHÉODORY) each point $\binom{\mathbf{x}}{\mu} \in \operatorname{conv}\big(\operatorname{epi} f(\cdot|C)\big)$ can be expressed as a convex combination of no more than $n+2$ points in $\operatorname{epi} f(\cdot|C)$. Thus, the unit simplex Λ^{n+2} includes a vector $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_{n+1})^{\mathsf{T}}$ such that

$$\begin{pmatrix}\mathbf{x}\\ \mu\end{pmatrix} = \sum_{i=0}^{n+1} \lambda_i \begin{pmatrix}\mathbf{x}^i\\ \mu_i\end{pmatrix} = \begin{pmatrix}\sum \lambda_i \mathbf{x}^i\\ \sum \lambda_i \mu_i\end{pmatrix} \qquad \text{with} \quad \begin{pmatrix}\mathbf{x}^i\\ \mu_i\end{pmatrix} \in \operatorname{epi} f(\cdot|C).$$

On the basis of the definition of the epigraph of a function the relation $\binom{\mathbf{x}^i}{\mu_i} \in \operatorname{epi} f(\cdot|C)$ is satisfied if and only if $\mu_i \geqq f(\mathbf{x}^i|C)$. With respect to the index $i = 0, \dots, n+1$ we get

$$\begin{aligned}
&\operatorname{conv} f(\mathbf{x}|C)\\
&\quad = \inf\Big\{\mu \,\Big|\, \tbinom{\mathbf{x}}{\mu} \in \operatorname{conv}\big(\operatorname{epi} f(\cdot|C)\big)\Big\}\\
&\quad = \inf\left\{\mu \,\middle|\, \begin{pmatrix}\sum \lambda_i \mathbf{x}^i\\ \sum \lambda_i \mu_i\end{pmatrix} = \begin{pmatrix}\mathbf{x}\\ \mu\end{pmatrix},\ \begin{pmatrix}\mathbf{x}^i\\ \mu_i\end{pmatrix} \in \operatorname{epi} f(\cdot|C)\right\} \quad \text{for one } \boldsymbol{\lambda} \in \Lambda^{n+2}\\
&\quad = \inf\left\{\textstyle\sum \lambda_i \mu_i \,\middle|\, \sum \lambda_i \mathbf{x}^i = \mathbf{x},\ \mu_i \geqq f(\mathbf{x}^i|C)\right\}\\
&\quad = \inf\left\{\textstyle\sum \lambda_i f(\mathbf{x}^i|C) \,\middle|\, \sum \lambda_i \mathbf{x}^i = \mathbf{x}\right\}.
\end{aligned}$$

[15] Cf. also FRENK, DIAS, GROMICHO (1994, p. 154 ff.).

As each point $\binom{\mathbf{x}}{\mu} \in \operatorname{conv}C$ can also be expressed as a convex combination of no more than $n+2$ points in C, it ensues analogously

$$\begin{aligned} f(\mathbf{x}|\operatorname{conv}C) &= \inf\left\{\mu | \binom{\mathbf{x}}{\mu} \in \operatorname{conv}C\right\} \\ &= \inf\left\{\mu | \begin{pmatrix}\sum\lambda_i\mathbf{x}^i \\ \sum\lambda_i\mu_i\end{pmatrix} = \begin{pmatrix}\mathbf{x}\\ \mu\end{pmatrix}, \begin{pmatrix}\mathbf{x}^i\\ \mu_i\end{pmatrix} \in C\right\} \qquad \text{for one } \lambda \in \Lambda^{n+2} \\ &= \inf\left\{\sum\lambda_i\mu_i | \sum\lambda_i\mathbf{x}^i = \mathbf{x}, \begin{pmatrix}\mathbf{x}^i\\ \mu_i\end{pmatrix} \in C\right\} \\ &= \inf\left\{\sum\lambda_i f(\mathbf{x}^i|C) | \sum\lambda_i\mathbf{x}^i = \mathbf{x}\right\}. \end{aligned}$$

The proof closes with a comparison of the results. ■

Technically, Proposition III.1 offers a possibility of regularizing nonconvex functions. In the case of a (nonconvex) input requirement set $L(\mathbf{x})$ the function $f(\cdot|L(\mathbf{x}))$ is superseded by the greatest convex function $\operatorname{conv}f(\cdot|L(\mathbf{x})) = f(\cdot|\operatorname{conv}L(\mathbf{x}))$ which satisfies the inequality $\operatorname{conv}f(\cdot|L(\mathbf{x})) \leqq f(\cdot|L(\mathbf{x}))$. Note that the difficulties, described in Figure III.2, are not removed by the regularization.

Two properties of the input requirement sets make it easier to handle the (extended real-valued) function defined by (III.2). On the one hand all input requirement sets are subsets in $\mathbb{R}^m_+$ such that

$$\inf\{v_r | \mathbf{v} \in L(\mathbf{x})\} \geqq 0\,.$$

On the other hand the infimum is attained under [L6] (*Closedness*) only if the infimum $\tilde{v}_r$ is finite, $0 \leqq \tilde{v}_r < +\infty$. In this case the examined input vector satisfies the relation $\mathbf{v} \in L(\mathbf{x})$, where the r-th component of the input vector corresponds to the calculated infimum, $v_r = \tilde{v}_r$.

1.2 Determination of the Normalized Cost Function

In what follows, the spaces of primal (quantity) variables and of dual (price) variables are denoted by $\mathcal{V}$ and $\mathcal{Q}$ respectively. Both spaces can be specified by

$$\mathcal{V} = \mathbb{R}^{m-1} = \mathcal{Q}\,,$$

as according to the preliminary remarks we are dealing with, for example, input vectors which are denoted by $\mathbf{v}^{-r}$ and whose r-th component are missing. The analysis is founded on an input correspondence $L\colon X \to \mathfrak{P}(V)$, where an input requirement set $L(\mathbf{x}) \subset V$ denotes the set of all input vectors $\mathbf{v} \in V$ allowing the production of the commodity bundle $\mathbf{x}$. In contrast to $\mathcal{V} = \mathbb{R}^{m-1}$, we call $V \subset \mathbb{R}^m_+$ the factor space, where the integer constraints are already taken into account.

$$V = \mathbb{R}^{m_d}_+ \times \mathbb{Z}^{m-m_d}_+$$

The commodity space is denoted by $X \subset \mathbb{R}^n_+$, which contains n_d divisible outputs analogous to factor space.

$$X = \mathbb{R}^{n_d}_+ \times \mathbb{Z}^{n-n_d}_+$$

When a production technology satisfies Axioms [L1]–[L8], then certain properties of the function defined by (III.5) result. Since these properties form the basis of the subsequent analysis, we now give an examination of the central facts.

Regardless of the divisibility or indivisibility of inputs we define analogous to Theorem III.1 the following nonnegative function $f(\cdot|L(\mathbf{x})): \mathcal{V} \to [0, +\infty]$ with

$$\text{(III.5)} \qquad f(\mathbf{v}^{-r}|L(\mathbf{x})) := \inf\{v_r|\ \mathbf{v} \in L(\mathbf{x})\},$$

where $\mathbf{v}^{-r}$ is the input vector $\mathbf{v} = (v_1, \ldots, v_m)^\mathsf{T}$ without the r-th component. Given the other factor quantities, we seek for the smallest factor quantity v_r permitting us for producing $\mathbf{x}$. Setting without loss of generality $r = m$, then $\mathbf{v} = \binom{\mathbf{v}^{-r}}{v_r}$.

(III.5) cannot guarantee the existence of a minimum. If there is no v_r which permits the production of $\mathbf{x}$ at the given remaining inputs, then by definition

$$\text{(III.6)} \qquad f(\mathbf{v}^{-r}|L(\mathbf{x})) = \inf \emptyset = +\infty .$$

This convention symbolically indicates that not all of the activities $(\mathbf{v}, \mathbf{x})$ are possible. Despite an infinitely large quantity of factor r the commodity bundle $\mathbf{x}$ is not producible by using the other held fixed inputs $\mathbf{v}^{-r}$. The case described in (III.6) emerges especially for the following constellations:

- If at least one component of $\mathbf{v}^{-r}$ is negative or if $\mathbf{v}^{-r}$ violates integer constraints, then no $v_r \in \mathbb{R}$ exists such that $\mathbf{v} \in V$ would be feasible. The impossibility of producing the given commodity bundle $\mathbf{x} \in X$ is reflected by the functional value $f(\mathbf{v}^{-r}|L(\mathbf{x})) = +\infty$.

- If independent of v_r the inputs $\mathbf{v}^{-r}$ are not enough to produce the given commodity bundle $\mathbf{x} \in X$, then no $v_r \in \mathbb{R}$ exists such that $\mathbf{v} \in L(\mathbf{x})$ would be feasible. Again the functional value is $f(\mathbf{v}^{-r}|L(\mathbf{x})) = +\infty$.

- Letting $\mathbf{x}$ be no admissible output vector, $\mathbf{x} \notin X$, then $L(\mathbf{x}) = \emptyset$ and $f(\cdot|\emptyset) \equiv +\infty$. In this case an *improper* function $f(\cdot|\emptyset)$ results.

The nonnegative function $f(\cdot|L(\mathbf{x})) \geqq 0$ is thus defined on the entire range $\mathcal{V}$ and, above all, nowhere in $\mathcal{V}$ does it achieve the value $-\infty$. Moreover, if $\mathbf{x}$ is an admissible commodity bundle, then under [L2] (*Attainability of Each Production*) the function $f(\cdot|L(\mathbf{x}))$ achieves a finite value for at least one point, i.e., the function is **proper**[16]. The reason results from [L2]

$$\mathbf{x} \in X \implies L(\mathbf{x}) \neq \emptyset$$

[16]For the criterion of proper, n-proper, and improper functions, see Appendix C.1.

so that there is at least one input vector $\binom{\mathbf{v}^{-r}}{v_r} \in L(\mathbf{x}) \subset V$ having a finite quantity of factor r. Thus, the inspected (nonnegative) function has at least one finite functional value $f(\mathbf{v}^{-r}|L(\mathbf{x})) \leqq v_r$.

To grasp the economic meaning of the function $f(\cdot|L(\mathbf{x}))$, we suppose for a given admissible commodity bundle $\mathbf{x} \in X$ that the nonempty [L2] input requirement set $L(\mathbf{x})$ is a closed subset in $\mathbb{R}^m_+$ by [L6]. Then each finite functional value $f(\mathbf{v}^{-r}|L(\mathbf{x}))$ yields an input vector of the form[17]

$$\text{(III.7)} \qquad \begin{pmatrix} \mathbf{v}^{-r} \\ f(\mathbf{v}^{-r}|L(\mathbf{x})) \end{pmatrix} \in L(\mathbf{x}).$$

As shown in the right hand part of Figure III.1 for perfectly divisible factors, $f(\cdot|L(\mathbf{x}))$ describes the boundary of the input requirement set $L(\mathbf{x})$. This result will be stated more precisely at a later stage by Proposition III.3 since under [L3] (*Disposability of Inputs*) it is

$$L(\mathbf{x}) = \operatorname{epi} f(\cdot|L(\mathbf{x})).$$

For indivisible factors we get analogous to Figure III.3

$$\operatorname{conv} L(\mathbf{x}) = \operatorname{epi} f(\cdot|\operatorname{conv} L(\mathbf{x})).$$

In this case the graph of $f(\cdot|L(\mathbf{x}))$ indicates the (nonvertical) boundary of $L(\mathbf{x})$ or $\operatorname{conv} L(\mathbf{x})$. The functional values of $f(\cdot|L(\mathbf{x}))$ are measured in units of factor r.

The subsequent example graphically illustrates the correlation between an input requirement set $L(\mathbf{x})$, the function $f(\cdot|L(\mathbf{x}))$, and the epigraph $\operatorname{epi} f(\cdot|L(\mathbf{x}))$; see Figure III.4. Later we go into the convex biconjugate function $f^{**}(\cdot|L(\mathbf{x}))$. Note that in the case of the example the supposed input requirement set contradicts Axiom [L3] (*Disposability of Inputs*).[18]

Example: *The starting point is an input requirement set for the admissible output level $\tilde{x}$*

$$\text{(III.8)} \qquad L(\tilde{x}) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{Z}^2_+ \middle| \begin{array}{ll} v_2 \geqq v_2' & \text{for } v_1 = v_1' \text{ or } v_1 = v_1''' \\ v_2 \geqq v_2'' & \text{for } v_1 = v_1'' \end{array} \right\},$$

where by Figure III.4 it is supposed that $v_2' > v_2'' > 0$ and $v_1'' > v_1''' > v_1' > 0$. The naming of the factor quantities reflects the well-founded supposition that the quantity v_1''' will be irrelevant for the later determination of the cost function. Hence, in accordance with (III.5) the function $f(\cdot|L(\tilde{x})) \colon \mathcal{V} \to [0, +\infty]$ with $\mathcal{V} = \{v_1', v_1''', v_1''\}$ is

$$\text{(III.9)} \qquad f(v_1|L(\tilde{x})) = \inf\left\{v_2 \middle| \tbinom{v_1}{v_2} \in L(\tilde{x})\right\} = \begin{cases} v_2' & \text{for } v_1 = v_1' \text{ or } v_1 = v_1''' \\ v_2'' & \text{for } v_1 = v_1'' \\ +\infty & \text{otherwise.} \end{cases}$$

[17] No finite functional value $f(\mathbf{v}^{-r}|L(\mathbf{x}))$ can contradict any integer constraint even when the factor v_r is indivisible.

[18] Thus, we cannot rule out afterwards the case of a negative factor price $q_1 < 0$.

In addition to this function Figure III.4 represents its epigraph

$$\operatorname{epi} f(\cdot|L(\tilde{x})) = \left\{ \binom{v_1}{v_2} \in \mathcal{V} \times \mathbb{R} \middle| \, v_2 \geqq f(v_1|L(\tilde{x})) \right\}.$$

Finally, we get for the effective domain of the function $f(\cdot|L(\tilde{x}))$

$$\operatorname{Dom} f(v_1|L(\tilde{x})) = \left\{ v_1', v_1''', v_1'' \right\} = \mathcal{V}.$$

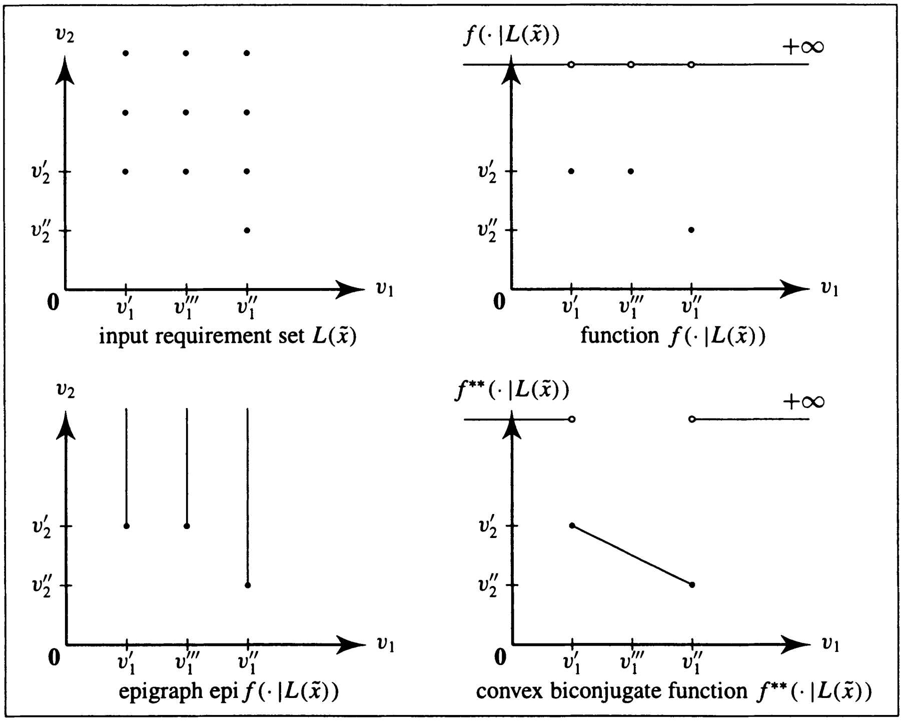

Figure III.4: Graphical representation of the example

In order to determine the factor costs incurred by the production of the commodity bundle $\mathbf{x}$, we fix a vector of nominal factor prices $\mathbf{q} \in Q$, where

$$Q = \mathbb{R}_+^m$$

denotes the space of the (nonnegative) factor price vectors in contrast to $\mathcal{Q} = \mathbb{R}^{m-1}$. By normalizing[19] the r-th factor price to unity ($\mathbf{q}/q_r$) a vector of relative factor prices $\mathbf{q}^{-r} \in \mathcal{Q}$ results. For $r = m$ we get $\mathbf{q}/q_r = \binom{\mathbf{q}^{-r}}{1}$. Thus, with respect to a price vector, the notation $\mathbf{q}^{-r}$ indicates *two* operations: the r-th

[19]Remember at this point that only the functional for determining the minimal costs implies a normalization of the factor price concerned.

component of the price vector $\mathbf{q}$ is not removed until it is normalized to unity. Consequently, the remaining components hold $q_i^{-r} = q_i/q_r$ with $i \neq r$.

The minimal costs are measured in units of factor r; they are given by[20]

$$\inf\left\{(\mathbf{q}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) + f(\mathbf{v}^{-r}|L(\mathbf{x}))|\ \mathbf{v}^{-r} \in \mathcal{V}\right\}.$$

With that the theory of conjugate functions in Section D.1 offers two starting points which differ with respect to the signs of the included parameters. First of all, the **convex FENCHEL transform**

$$f^*(\tilde{\mathbf{q}}^{-r}|L(\mathbf{x})) := \sup\left\{(\tilde{\mathbf{q}}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) - f(\mathbf{v}^{-r}|L(\mathbf{x}))|\ \mathbf{v}^{-r} \in \mathcal{V}\right\}$$

can be examined, where in the course of further calculations nonpositive vectors $\tilde{\mathbf{q}}^{-r} = -\mathbf{q}^{-r} \leqq \mathbf{0}$ arise. This (mathematically unproblematic) case can still be avoided by choosing the second alternative approach so that $\mathbf{q}^{-r}$ can further be interpreted as a vector of nonnegative factor prices. For this purpose the function[21] $g(\cdot|L(\mathbf{x})) = -f(\cdot|L(\mathbf{x}))$ is examined instead of $f(\cdot|L(\mathbf{x}))$. Since $f(\cdot|L(\mathbf{x}))$ is proper for each commodity bundle $\mathbf{x} \in X$, the function $g(\cdot|L(\mathbf{x}))$ must be *n-proper* for each commodity bundle, i.e. for all $\mathbf{x} \in X$

$$\text{(III.10)} \qquad \begin{aligned} &g(\cdot|L(\mathbf{x})) < +\infty \\ \text{and}\quad &g(\mathbf{v}^{-r}|L(\mathbf{x})) \text{ is finite for } \mathbf{v}^{-r} \in \mathcal{V}. \end{aligned}$$

With the above function g the **concave FENCHEL transform** now generates a function $g_*(\cdot|L(\mathbf{x})): \mathcal{Q} \to [-\infty, +\infty]$ with

$$\text{(III.11)} \qquad g_*(\mathbf{q}^{-r}|L(\mathbf{x})) := \inf\left\{(\mathbf{q}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) - g(\mathbf{v}^{-r}|L(\mathbf{x}))|\ \mathbf{v}^{-r} \in \mathcal{V}\right\},$$

where $g_*(\cdot|L(\mathbf{x}))$ is the minimal cost of producing the commodity bundle $\mathbf{x}$ measured in units of factor r. Finally, the outcome

$$\text{(III.12)} \qquad g_*(\mathbf{q}^{-r}|L(\mathbf{x})) = -f^*(-\mathbf{q}^{-r}|L(\mathbf{x})) \qquad \forall \mathbf{q}^{-r} \in \mathcal{Q}$$

will be important at a later stage.

Hence, for a given production technology represented by the input correspondence L or the production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$ we write for the (normalized) **cost function**

$$\text{(III.13)} \qquad g_*(\cdot|L(\mathbf{x})) \equiv \tilde{c}(\cdot, \mathbf{x}).$$

In correspondence with Theorem D.2, p. 316, the cost function $\tilde{c}(\cdot, \mathbf{x})$ as a concave conjugate function of $g(\cdot|L(\mathbf{x}))$ is n-proper if and only if $g(\cdot|L(\mathbf{x}))$ is n-proper according to (III.10). In this case

$$\text{(III.14)} \qquad \begin{aligned} &\tilde{c}(\cdot, \mathbf{x}) < +\infty \\ \text{and}\quad &\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) \text{ is finite for } \mathbf{q}^{-r} \in \mathcal{Q} \end{aligned}$$

[20] While the calculation of the functional value $f(\mathbf{v}^{-r}|L(\mathbf{x}))$ is carried out at given $\mathbf{v}^{-r}$ – that is in the sense of a partial factor variation – we now vary $\mathbf{v}^{-r}$ parametrically, where the adjoined optimal quantity of factor r – i.e. $f(\mathbf{v}^{-r}|L(\mathbf{x}))$ – must be taken into account.

[21] By Theorem III.1 the proper function $f(\cdot|L(\mathbf{x}))$ is convex if $L(\mathbf{x})$ is a nonempty convex subset in the factor space V. In this case $g(\cdot|L(\mathbf{x})) = -f(\cdot|L(\mathbf{x})) < +\infty$ is an n-proper concave function.

holds for all $\mathbf{x} \in X$. Moreover, Theorem D.2 states that the cost function $\tilde{c}(\cdot, \mathbf{x})$ is a closed[22] concave function. In the relevant case of an n-proper cost function we thus have by Theorem C.2, p. 301,

$$\text{(III.15)} \quad \mathrm{cl}\left(\mathrm{hypo}\,\tilde{c}(\cdot, \mathbf{x})\right) = \mathrm{hypo}\,\tilde{c}(\cdot, \mathbf{x}) \quad \text{or, equivalently,} \quad \mathrm{cl}\,\tilde{c}(\cdot, \mathbf{x}) = \tilde{c}(\cdot, \mathbf{x}).$$

The criterion of concavity is analogously defined by

$$\text{(III.16)} \qquad \mathrm{conv}\left(\mathrm{hypo}\,\tilde{c}(\cdot, \mathbf{x})\right) = \mathrm{hypo}\,\tilde{c}(\cdot, \mathbf{x}).$$

Before giving an example, it has to be noted that the results do not depend on the assumption as to whether the underlying input requirement set $L(\mathbf{x})$ is convex or not.

Proposition III.2 *Under the conditions of Proposition II.14 the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$, viewed as the concave conjugate function $g_*(\cdot|L(\mathbf{x}))$, equals the concave conjugate function $g_*(\cdot|\mathrm{conv}L(\mathbf{x}))$.*

$$\tilde{c}(\cdot, \mathbf{x}) \equiv g_*(\cdot|L(\mathbf{x})) = g_*(\cdot|\mathrm{conv}L(\mathbf{x}))$$

Proof: For an inadmissible vector $\mathbf{x} \notin X$ it is $L(\mathbf{x}) = \mathrm{conv}L(\mathbf{x}) = \emptyset$ and nothing remains to be shown. The cost function is improper;

$$\tilde{c}(\cdot, \mathbf{x}) \equiv g_*(\cdot|\emptyset) \equiv +\infty.$$

In order to reduce the notation for the rest of the proof, we disregard the input requirement set $L(\mathbf{x}) \subset \mathbb{R}^m$. Instead we inspect analogous to Theorem III.1 a nonempty set $C \subset \mathbb{R}^n_+$, whose convex hull $\mathrm{conv}C$ is closed. Furthermore, the effort can be reduced by carrying out the proof for the convex FENCHEL transform.[23] Instead of the pair of n-proper functions $g(\cdot|L(\mathbf{x}))$ and $g(\cdot|\mathrm{conv}L(\mathbf{x}))$ the following pair of proper functions is examined:

$$f_1(\mathbf{x}) = \inf\left\{\mu|\ \binom{\mathbf{x}}{\mu} \in C\right\} \geqq 0,$$
$$f_2(\mathbf{x}) = \inf\left\{\mu|\ \binom{\mathbf{x}}{\mu} \in \mathrm{conv}C\right\} \geqq 0.$$

In the sense of Proposition III.2 it is now proved that the adjoined convex conjugate functions

$$\text{(III.17a)} \qquad f_1^*(\mathbf{y}) = \sup\left\{\mathbf{y}^\mathsf{T}\mathbf{x} - f_1(\mathbf{x})|\ \mathbf{x} \in \mathbb{R}^{n-1}\right\},$$
$$\text{(III.17b)} \qquad f_2^*(\mathbf{y}) = \sup\left\{\mathbf{y}^\mathsf{T}\mathbf{x} - f_2(\mathbf{x})|\ \mathbf{x} \in \mathbb{R}^{n-1}\right\}$$

[22] For the criterion of a closed function, see Appendix C.1. As stressed on p. 304, we need not distinguish between the criterion of upper semi-continuity and the criterion of closedness for an n-proper concave functions.

[23] If the set C is identified with an input requirement set, then the switch-over to the concave FENCHEL transform is tediously long since we have set $-f(\cdot|L(\mathbf{x})) = g(\cdot|L(\mathbf{x}))$. In view of (III.12) we can avoid this effort.

satisfy $f_1^* = f_2^*$.

First of all, remember that $C \subset \operatorname{conv} C$ implies the inequality $f_1 \geqq f_2$. Transferred to convex conjugate functions we obtain $f_1^* \leqq f_2^*$. Finally, it must be shown that the strict inequality

$$f_1^*(\tilde{\mathbf{y}}) < f_2^*(\tilde{\mathbf{y}}) \tag{III.18}$$

is *not* satisfied at any point $\tilde{\mathbf{y}}$. At point $\mathbf{y} = \tilde{\mathbf{y}}$ the problem (III.17a) yields

$$f_1^*(\tilde{\mathbf{y}}) \geqq \tilde{\mathbf{y}}^{\mathsf{T}}\mathbf{x} - f_1(\mathbf{x}) \geqq \tilde{\mathbf{y}}^{\mathsf{T}}\mathbf{x} - \mu \qquad \forall \binom{\mathbf{x}}{\mu} \in C. \tag{III.19}$$

This inequality also holds for all $\binom{\mathbf{x}}{\mu} \in \operatorname{conv} C$ since by Theorem B.4 (CARATHÉODORY) each $\binom{\mathbf{x}}{\mu} \in \operatorname{conv} C$ can be expressed as

$$\binom{\mathbf{x}}{\mu} = \sum_{i=0}^{n} \lambda_i \binom{\mathbf{x}^i}{\mu_i} \qquad \text{with} \quad \binom{\mathbf{x}^i}{\mu_i} \in C, \ \lambda \in \Lambda^{n+1}.$$

From (III.19)

$$f_1^*(\tilde{\mathbf{y}}) \geqq \binom{\tilde{\mathbf{y}}}{-1}^{\mathsf{T}} \binom{\mathbf{x}^i}{\mu_i} = \tilde{\mathbf{y}}^{\mathsf{T}}\mathbf{x}^i - \mu_i \qquad (i = 0, \ldots, n)$$

ensues for each of the $\binom{\mathbf{x}^i}{\mu_i}$ such that

$$f_1^*(\tilde{\mathbf{y}}) = \sum_{i=0}^{n} \lambda_i f_1^*(\tilde{\mathbf{y}}) \geqq \sum_{i=0}^{n} \lambda_i (\tilde{\mathbf{y}}^{\mathsf{T}}\mathbf{x}^i - \mu_i) = \tilde{\mathbf{y}}^{\mathsf{T}}\mathbf{x} - \mu. \tag{III.20}$$

holds for each $\binom{\mathbf{x}}{\mu} \in \operatorname{conv} C$. Analogous to (III.19), we obtain from (III.17b)

$$f_2^*(\tilde{\mathbf{y}}) \geqq \tilde{\mathbf{y}}^{\mathsf{T}}\mathbf{x} - f_2(\mathbf{x}) \geqq \tilde{\mathbf{y}}^{\mathsf{T}}\mathbf{x} - \mu \qquad \forall \binom{\mathbf{x}}{\mu} \in \operatorname{conv} C.$$

The strict inequality (III.18) implies analogously

$$f_2^*(\tilde{\mathbf{y}}) > f_1^*(\tilde{\mathbf{y}}) \geqq \tilde{\mathbf{y}}^{\mathsf{T}}\mathbf{x} - \mu \qquad \forall \binom{\mathbf{x}}{\mu} \in \operatorname{conv} C.$$

This inequality can only be satisfied when the infimum f_2 is finite for at least one point. Otherwise we have $f_1 \equiv f_2 \equiv +\infty$ and therefore

$$f_2 \equiv +\infty \implies f_2^* \equiv -\infty \implies f_1^* \equiv -\infty \quad (\text{because of } f_1^* \leqq f_2^*).$$

Because of $f_2 \geqq 0$ the case $f_2(\mathbf{x}) = -\infty$ does not need to be inspected.[24] Assuming a finite infimum at point $\hat{\mathbf{x}}$ there must be an optimal $\hat{\mu}$ with respect to the

[24] If the set C allows the functional value $f_2(\mathbf{x}) = -\infty$, then

$$f_2(\mathbf{x}) = -\infty \implies f_2^* \equiv +\infty \text{ and } f_1^* \equiv +\infty \qquad (\text{because of (III.20) for } \mu \to -\infty).$$

closedness of conv$C$ so that the infimum is achieved, $\binom{\hat{\mathbf{x}}}{\hat{\mu}} \in \text{conv}C$ with $f_2(\hat{\mathbf{x}}) = \hat{\mu}$. The contradiction

$$\tilde{\mathbf{y}}^\mathsf{T}\hat{\mathbf{x}} - \hat{\mu} = f_2^*(\tilde{\mathbf{y}}) > f_1^*(\tilde{\mathbf{y}}) \geqq \tilde{\mathbf{y}}^\mathsf{T}\hat{\mathbf{x}} - \hat{\mu}$$

rules out the case $f_1^*(\tilde{\mathbf{y}}) < f_2^*(\tilde{\mathbf{y}})$ for all $\tilde{\mathbf{y}} \in \mathbb{R}^{n-1}$.

The proof is complete when we take into account that the convex hull of an input requirement set conv$L(\mathbf{x})$ is closed under the assumptions of Proposition II.14. ∎

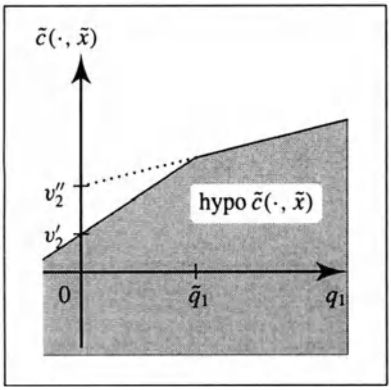

Figure III.5: Normalized cost function $\tilde{c}(\cdot, \tilde{x})$

The example shown in Figure III.4 can now be continued with respect to Proposition III.2 without knowing the convex hull of the input requirement set.

Example: *To avoid confusion, the normalized price of the first production factor q_1^{-r} with $r = 2$ is denoted by q_1 within the example. The price of factor 2 is set to one, $q_2 = 1$.*
For the input requirement set $L(\tilde{x})$ according to (III.8), the following cost function results corresponding to the concave conjugate function $g_*(\cdot|L(\tilde{x}))$:

$$\begin{aligned}\tilde{c}(q_1, \tilde{x}) &\equiv g_*(q_1|L(\tilde{x})) \\ &= \inf\{q_1 v_1 - [-f(v_1|L(\tilde{x}))]|\ v_1 \in \mathbb{R}\}.\end{aligned}$$

Considering (III.9) we obtain

$$\tilde{c}(q_1, \tilde{x}) = \inf \left\{ \begin{array}{ll} q_1 v_1 + v_2' & \text{for } v_1 = v_1' \text{ or } v_1 = v_1''' \\ q_1 v_1 + v_2'' & \text{for } v_1 = v_1'' \\ +\infty & \text{otherwise} \end{array} \right\}.$$

Due to $v_1''' > v_1'$, the input v_1''' is ruled out as a solution to the problem of cost minimization.

$$\text{(III.21)} \qquad \tilde{c}(q_1, \tilde{x}) = \begin{cases} q_1 v_1' + v_2' & \text{for } q_1 \geqq \tilde{q}_1 \\ q_1 v_1'' + v_2'' & \text{for } q_1 \leqq \tilde{q}_1 \end{cases} \qquad \text{with} \quad \tilde{q}_1 := \frac{v_2' - v_2''}{v_1'' - v_1'}$$

results in accordance with Figure III.5.

Thus, the graph of the normalized cost function $\tilde{c}(\cdot, \tilde{x})$ consists of a kink point and two half-lines. Finally, we have to note that the gray emphasized hypograph hypo $\tilde{c}(\cdot, \tilde{x})$ *is closed and convex by the relations (III.15) and (III.16) respectively.*

The normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ at once yields some basic statements.

- As mentioned above, each inadmissible vector $\mathbf{x} \notin X$ with the empty input requirement set $L(\mathbf{x}) = \emptyset$ and hence $g(\cdot|\emptyset) = -f(\cdot|\emptyset) \equiv -\infty$ implies an improper function $g_*(\cdot|\emptyset) \equiv \tilde{c}(\cdot, \mathbf{x}) \equiv +\infty$.

If an activity $(\mathbf{v}, \mathbf{x})$ with given inputs $\mathbf{v}^{-r}$ and a given commodity bundle $\mathbf{x} \in X$ is impossible in the sense of the production technology independent of the used quantity of factor r, $f(\mathbf{v}^{-r}\,|L(\mathbf{x})) = +\infty$, then $\mathbf{v}^{-r}$ with $g(\mathbf{v}^{-r}|L(\mathbf{x})) = -\infty$ is ruled out as solution to the problem of cost minimization. Infinitely high costs are in no case optimal.

- Given an admissible commodity bundle $\mathbf{x} \in X$ the following can be stated:
  - By Theorem D.2 the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ is n-proper since by [L2] (*Attainability of Each Production*) the genuine function $g(\cdot\,|L(\mathbf{x}))$ is n-proper for all $\mathbf{x} \in X$; see (III.14).
  - Because of $g(\cdot|L(\mathbf{x})) \leqq 0$ and $\mathbf{v}^{-r} \geqq \mathbf{0}$ we get nonnegative factor costs $\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) \geqq 0$, provided only nonnegative factor prices $\mathbf{q}^{-r} \geqq \mathbf{0}$ are admissible.
  - From a negative factor price, i.e. $\mathbf{q}^{-r} \ngeqq \mathbf{0}$, the functional value $\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) = -\infty$ results since by [L3] (*Disposability of Inputs*) the costs can be reduced permanently by enlarging more and more the factor quantity with the negative factor price.[25]
  - Summarizing the last two points gives for the effective domain of the normalized cost function

    (III.22) $$\text{n-Dom}\,\tilde{c}(\cdot, \mathbf{x}) = \mathbb{R}_+^{m-1}\,.$$

1.3 Reconstruction of the Production Structure

The previous explanations may be summarized as follows: beginning with the production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$, (III.5) introduces a function $f(\cdot|L(\mathbf{x}))$ serving for the representation of certain boundary points of an input requirement set $L(\mathbf{x})$. In doing so it turns out to be helpful to continue with the function $g(\cdot|L(\mathbf{x})) = -f(\cdot|L(\mathbf{x}))$. Finally, the function $g(\cdot|L(\mathbf{x}))$ has been assigned by the concave FENCHEL transform to a concave conjugate function $g_*(\cdot|L(\mathbf{x}))$, which indicates the minimal factor costs incurred by the production of the commodity bundle $\mathbf{x}$ in units of factor r. Thus, we write for the normalized cost function $\tilde{c}(\cdot, \mathbf{x}) \equiv g_*(\cdot|L(\mathbf{x}))$.

$$\begin{tikzcd}
L(\mathbf{x}) \arrow[d, shift right, "?"'] \arrow[dr] & \\
\operatorname{epi} f(\cdot|L(\mathbf{x})) \arrow[u, shift right] & -f(\cdot|L(\mathbf{x})) \equiv g(\cdot|L(\mathbf{x})) \arrow[l] \arrow[d] \\
& \tilde{c}(\cdot, \mathbf{x}) \equiv g_*(\cdot|L(\mathbf{x})) \arrow[d] \\
& g_{**}(\cdot|L(\mathbf{x})) \arrow[uu, bend right, "?"']
\end{tikzcd}$$

Figure III.6: Reconstruction of the production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$

Now we can give an idea of the following expositions. If the cost function $\tilde{c}(\cdot, \mathbf{x})$ can be deduced from the (boundary) function $f(\cdot|L(\mathbf{x}))$, in what way is it then possible to infer $f(\cdot|L(\mathbf{x}))$ from $\tilde{c}(\cdot, \mathbf{x})$? As shown by the outline opposite,

[25] The apparent contradiction to Figure III.5 results from the supposed input requirement set $L(\tilde{x})$ by (III.8). The quantity of factor 1 cannot arbitrarily be augmented for a negative factor price q_1.

for this purpose we apply the same operation to the function $g_*(\cdot|L(\mathbf{x}))$ as we did before to the function $g(\cdot|L(\mathbf{x}))$. By (III.23) the result $g_{**}(\cdot|L(\mathbf{x}))$ is closely related to the genuine function $g(\cdot|L(\mathbf{x}))$, where the degree of deviation can be assessed by means of the adjoined hypograph. If the result is $-f(\cdot|L(\mathbf{x})) = g(\cdot|L(\mathbf{x})) = g_{**}(\cdot|L(\mathbf{x}))$, then under certain assumptions the production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$ can be completely reconstructed from the cost function $\tilde{c}(\cdot,\mathbf{x}) \equiv g_*(\cdot|L(\mathbf{x}))$, i.e. $\operatorname{epi} f(\cdot|L(\mathbf{x})) = L(\mathbf{x})$; see Proposition III.3, p. 99.

The next step examines the **second concave FENCHEL transform**. This operation generates an n-proper closed concave function which is called the **concave biconjugate function** $g_{**}(\cdot|L(\mathbf{x}))\colon \mathcal{V} \to [-\infty,+\infty]$.

$$g_{**}(\mathbf{v}^{-r}|L(\mathbf{x})) := \inf\left\{(\mathbf{q}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) - g_*(\mathbf{q}^{-r}|L(\mathbf{x}))|\ \mathbf{q}^{-r} \in \mathcal{Q}\right\} \tag{III.23}$$

The YOUNG-FENCHEL inequality (D.1b), p. 317, yields analogously to (D.2b)

$$g_{**}(\cdot|L(\mathbf{x})) \geqq g(\cdot|L(\mathbf{x})).$$

The second concave FENCHEL transform associates an arbitrary function $g(\cdot|L(\mathbf{x}))$ with the smallest concave function $g_{**}(\cdot|L(\mathbf{x}))$ so that

$$\operatorname{hypo} g_{**}(\cdot|L(\mathbf{x})) = \operatorname{cl}\left\{\operatorname{conv}\big[\operatorname{hypo} g(\cdot|L(\mathbf{x}))\big]\right\}, \tag{III.24a}$$

where the hypograph of the function $g(\cdot|L(\mathbf{x}))$ is defined by

$$\operatorname{hypo} g(\cdot|L(\mathbf{x})) := \left\{\binom{\mathbf{v}^{-r}}{v_r} \in \mathbb{R}^m\Big|\ v_r \leqq g(\mathbf{v}^{-r}|L(\mathbf{x}))\right\}.$$

In view of $g(\cdot|L(\mathbf{x})) = -f(\cdot|L(\mathbf{x}))$ this statement is graphically illustrated by the convex biconjugate function $f^{**}(\cdot|L(\mathbf{x}))$ in Figure III.4, i.e.

$$\operatorname{epi} f^{**}(\cdot|L(\mathbf{x})) = \operatorname{cl}\left\{\operatorname{conv}\big[\operatorname{epi} f(\cdot|L(\mathbf{x}))\big]\right\}, \tag{III.24b}$$

where the epigraph of the function $f(\cdot|L(\mathbf{x}))$ is defined by

$$\operatorname{epi} f(\cdot|L(\mathbf{x})) := \left\{\binom{\mathbf{v}^{-r}}{v_r} \in \mathbb{R}^m\Big|\ v_r \geqq f(\mathbf{v}^{-r}|L(\mathbf{x}))\right\}.$$

Concerning duality theory, the question arises as to when $g_{**}(\cdot|L(\mathbf{x}))$ and $g(\cdot|L(\mathbf{x}))$ describe the same functional relation. If both functions contain the same information, then it is irrelevant whether the analysis underlies the function $g(\cdot|L(\mathbf{x}))$ or the concave conjugate function $g_*(\cdot|L(\mathbf{x}))$. In this case each function can be derived from the other one without any loss of information. If the functions $g_{**}(\cdot|L(\mathbf{x}))$ and $g(\cdot|L(\mathbf{x}))$ differ, then we have to reformulate the question; the pair $g_*(\cdot|L(\mathbf{x}))$ and $g_{**}(\cdot|L(\mathbf{x}))$ describes the same information, but the original function $g(\cdot|L(\mathbf{x}))$ cannot be reconstructed. Now we must find out what information is lost and under what assumptions this information can be ignored.

Since each function is unambiguously associated with a hypograph,[26] it seems reasonable to suspect that

$$g(\cdot|L(\mathbf{x})) = g_{**}(\cdot|L(\mathbf{x})) \tag{III.25}$$

[26] An analogous argument holds for the epigraph with respect to (III.24b).

holds if by (III.24a) the hypographs of both functions are equal. In order to guarantee that both hypographs are nonempty, it suffices by Theorem D.2, p. 316, to suppose an n-proper function $g(\cdot|L(\mathbf{x}))$.[27] In view of (III.24a) the hypographs are equal if the forming of the convex hull and the closure operation keep the hypograph $\operatorname{hypo} g(\cdot|L(\mathbf{x}))$ unchanged.

The forming of the convex hull may be ignored if $\operatorname{hypo} g(\cdot|L(\mathbf{x}))$ is convex and this case is given by definition when the function $g(\cdot|L(\mathbf{x}))$ is concave. This conclusion is confirmed by Theorem D.2, i.e.

$$g_{**}(\cdot|L(\mathbf{x})) = \operatorname{cl} g(\cdot|L(\mathbf{x}))$$

holds for an arbitrary concave function $g(\cdot|L(\mathbf{x}))$.[28] Note that the replacement of an input requirement set $L(\mathbf{x})$ by its convex hull $\operatorname{conv}L(\mathbf{x})$ guarantees according to Theorem III.1 the concavity of the function $g(\cdot|\operatorname{conv}L(\mathbf{x}))$ (or the convexity of the function $f(\cdot|\operatorname{conv}L(\mathbf{x}))$).

Moreover, the closure operation may be neglected in (III.24a) if the set $\operatorname{conv}\big(\operatorname{hypo} g(\cdot|L(\mathbf{x}))\big)$ concerned is closed. This case is given when the concave function $g(\cdot|L(\mathbf{x}))$ is closed; see Appendix C.1. Thus, if we suppose an n-proper closed concave function $g(\cdot|L(\mathbf{x}))$, then (III.24a) is satisfied. Moreover, Corollary D.3.1 (Fenchel, Moreau), p. 318, confirms that (III.25) is fulfilled if and only if $g(\cdot|L(\mathbf{x}))$ satisfies the three properties mentioned.

The justification for supposing an n-proper closed (and furthermore concave) function $g(\cdot|L(\mathbf{x}))$ results from the corresponding assumptions on a production technology. For an admissible commodity bundle $\mathbf{x} \in X$ [L2] (*Attainability of Each Production*) implies a nonempty input requirement set $L(\mathbf{x})$. Thus, the function

(III.5) $$g(\mathbf{v}^{-r}|L(\mathbf{x})) = -\inf\{v_r|\ \mathbf{v} \in L(\mathbf{x})\}$$

is n-proper since by [L2] at least one input vector $\mathbf{v} = \binom{\mathbf{v}^{-r}}{v_r} \in L(\mathbf{x})$ exists containing a (finite) quantity of factor v_r such that $(\mathbf{v}, \mathbf{x})$ is a possible activity.
By Proposition III.3 the closedness of the function $g(\cdot|L(\mathbf{x})) = -f(\cdot|L(\mathbf{x}))$ results mainly from [L6], i.e. the closedness of the input requirement set $L(\mathbf{x})$. Analogously, the concavity of the function $g(\cdot|L(\mathbf{x}))$ requires a convex input requirement set $L(\mathbf{x})$.

Concerning (III.26a), the equation (III.24b) can be put into concrete form by considering the relationship between the input requirement set $L(\mathbf{x})$, the describing function $f(\cdot|L(\mathbf{x}))$, and its epigraph $\operatorname{epi} f(\cdot|L(\mathbf{x}))$, i.e.

$$L(\mathbf{x}) \to f(\cdot|L(\mathbf{x})) \to \operatorname{epi} f(\cdot|L(\mathbf{x})) \overset{!}{\to} L(\mathbf{x}).$$

[27] By Theorem D.2 $g_*(\cdot|L(\mathbf{x}))$ is n-proper if and only if $g(\cdot|L(\mathbf{x}))$ has this property. The same argument can be transferred to $g_{**}(\cdot|L(\mathbf{x}))$ and $g_*(\cdot|L(\mathbf{x}))$. In accordance with (III.6) for $\mathbf{x} \notin X$ we have $g(\cdot|L(\mathbf{x})) \equiv -\infty$ where $\operatorname{hypo} g(\cdot|L(\mathbf{x})) = \emptyset$ and $\operatorname{cl}(\operatorname{conv}\emptyset) = \emptyset$.

[28] Theorem D.2 holds for improper functions, too. This case is given when $\mathbf{x}$ is not an admissible commodity bundle $\mathbf{x} \notin X$ with $L(\mathbf{x}) = \emptyset$. We obtain $g_{**}(\cdot|L(\mathbf{x})) \equiv +\infty \equiv \operatorname{cl} g(\cdot|L(\mathbf{x}))$. The relevant case of an admissible commodity bundle $\mathbf{x} \in X$ with $\operatorname{cl} g(\cdot|L(\mathbf{x})) < +\infty$ or more exactly $\operatorname{cl} g(\cdot|L(\mathbf{x})) \leqq 0$ is described by (III.5).

Proposition III.3 *Let* $\mathbf{x} \in X$ *be an admissible commodity bundle. If the inputs are perfectly divisible,*[29] *then under* [L2] (*Attainability of Each Production*), [L3] (*Disposability of Inputs*) *and* [L6] (*Closedness*) *we obtain*

$$L(\mathbf{x}) = \operatorname{epi} f(\cdot | L(\mathbf{x})).$$

Proof: According to [L2] the input requirement set $L(\mathbf{x})$ is not empty for all $\mathbf{x} \in X$. Each input vector $\mathbf{v} \in V$ can be expressed as $\mathbf{v} = \binom{\mathbf{v}^{-r}}{v_r}$. If $\hat{v}_r$ solves the problem $\inf\{v_r \geqq 0 | \, \mathbf{v} \in L(\mathbf{x})\}$ for a given vector $\mathbf{v}^{-r}$, then by definition of the epigraph $\binom{\mathbf{v}^{-r}}{v_r} \in \operatorname{epi} f(\cdot | L(\mathbf{x}))$ holds for all input vectors of the form $\binom{\mathbf{v}^{-r}}{v_r}$ with $v_r \geqq \hat{v}_r$. By [L6] closedness of the input requirement set $L(\mathbf{x})$ implies $\binom{\mathbf{v}^{-r}}{\hat{v}_r} \in L(\mathbf{x})$. Hence, under [L3] (*Disposability of Inputs*) $\binom{\mathbf{v}^{-r}}{v_r} \in L(\mathbf{x})$ is also fulfilled, provided that $v_r \geqq \hat{v}_r$ and that factor r is divisible.[30] Thus, each vector $\mathbf{v}$ which belongs to one and only one of the sets $L(\mathbf{x})$ and $\operatorname{epi} f(\cdot | L(\mathbf{x}))$ implies a contradiction. ■

Under the assumptions of Proposition III.3 the closedness of the describing function $f(\cdot | L(\mathbf{x}))$ at once results from the closedness of the epigraph $\operatorname{epi} f(\cdot | L(\mathbf{x})) = L(\mathbf{x})$ since $f(\cdot | L(\mathbf{x}))$ is proper for each commodity bundle $\mathbf{x} \in X$ and therefore[31]

$$\operatorname{cl} f(\cdot | L(\mathbf{x})) = f(\cdot | L(\mathbf{x})) \iff \operatorname{cl}\left[\operatorname{epi} f(\cdot | L(\mathbf{x}))\right] = \operatorname{epi} f(\cdot | L(\mathbf{x})).$$

Although this equivalence relation will not usually occur for indivisible factors – see Figures III.2 and III.3 – Proposition III.3 can be modified as follows: (Note that under [L7] (*Convexity*) the equation $L(\mathbf{x}) = \operatorname{conv} L(\mathbf{x})$ is satisfied.)

Corollary III.3.1 *Provided* $\mathbf{x} \in X$ *is an admissible commodity bundle. Then under* [L2] (*Attainability of Each Production*), [L3] (*Disposability of Inputs*) *and* [L6] (*Closedness*) *we have*

$$\operatorname{conv} L(\mathbf{x}) = \operatorname{epi} f(\cdot | \operatorname{conv} L(\mathbf{x})).$$

Proof: Because of [L2] the input requirement sets $L(\mathbf{x})$ and, therefore, their convex hulls $\operatorname{conv} L(\mathbf{x})$ are not empty for every $\mathbf{x} \in X$. By Proposition II.14 it is known that $\operatorname{conv} L(\mathbf{x})$ is closed under [L6] (*Closedness*) and [L3] (*Disposability of Inputs*). Moreover, considering Proposition II.4, [L3] yields

$$\forall \mathbf{v}, \tilde{\mathbf{v}} \in \mathbb{R}^m_+, \ \tilde{\mathbf{v}} \geqq \mathbf{v} : \mathbf{v} \in \operatorname{conv} L(\mathbf{x}) \implies \tilde{\mathbf{v}} \in \operatorname{conv} L(\mathbf{x})$$

for the convex hull of $L(\mathbf{x})$. Thus, the proof can be completed analogous to Proposition III.3. ■

[29] For the proof it is enough to assume one perfectly divisible factor r.

[30] Comparing the two subgraphs of Figure III.2 for $r = 2$, the condition that at least factor r must be divisible becomes evident. The condition is satisfied in the right hand part, whereas the left hand graph violates it.

[31] See p. 303. The same argument can be transferred to the function $g(\cdot | L(\mathbf{x}))$ as $g(\cdot | L(\mathbf{x}))$ is n-proper for each commodity bundle $\mathbf{x} \in X$ and therefore

$$\operatorname{cl} g(\cdot | L(\mathbf{x})) = g(\cdot | L(\mathbf{x})) \iff \operatorname{cl}[\operatorname{hypo} g(\cdot | L(\mathbf{x}))] = \operatorname{hypo} g(\cdot | L(\mathbf{x})).$$

Corollary III.3.1 includes the case of nonconvex input requirement sets and especially the case of indivisible production factors. Furthermore, the limit case of convex input requirement sets is considered, where we have to presume perfect divisibility of all factors.

First of all, we suppose convex input requirement sets by [L7] so that reverting to Theorem III.1 a convex function $f(\cdot|L(\mathbf{x}))$ (or a concave function $g(\cdot|L(\mathbf{x}))$) ensues. Moreover, Corollary D.3.1 (FENCHEL, MOREAU), p. 318, implies

$$\begin{aligned} f(\cdot|L(\mathbf{x})) = f^{**}(\cdot|L(\mathbf{x})) = -g(\cdot|L(\mathbf{x})) = -g_{**}(\cdot|L(\mathbf{x})) \\ \operatorname{epi} f(\cdot|L(\mathbf{x})) = \operatorname{epi} f^{**}(\cdot|L(\mathbf{x})) = L(\mathbf{x}) \end{aligned} \tag{III.26a}$$

for a proper function $f(\cdot|L(\mathbf{x}))$ (or an n-proper function $g(\cdot|L(\mathbf{x}))$). For indivisible factors the requirement of convexity of the inspected function f (or of the concavity of the corresponding function g) is assured by replacing the input requirement sets $L(\mathbf{x})$ in accordance with Corollary III.3.1 by their convex hulls $\operatorname{conv}L(\mathbf{x})$. Since $g_*(\cdot|L(\mathbf{x})) = g_*(\cdot|\operatorname{conv}L(\mathbf{x}))$ is satisfied by Proposition III.2, it is well-founded to suppose $g_{**}(\cdot|L(\mathbf{x})) = g_{**}(\cdot|\operatorname{conv}L(\mathbf{x}))$ and we have

$$\begin{aligned} f(\cdot|\operatorname{conv}L(\mathbf{x})) = f^{**}(\cdot|\operatorname{conv}L(\mathbf{x})) = -g(\cdot|\operatorname{conv}L(\mathbf{x})) = -g_{**}(\cdot|L(\mathbf{x})) \\ \operatorname{epi} f(\cdot|\operatorname{conv}L(\mathbf{x})) = \operatorname{epi} f^{**}(\cdot|L(\mathbf{x})) = \operatorname{conv}L(\mathbf{x}). \end{aligned} \tag{III.26b}$$

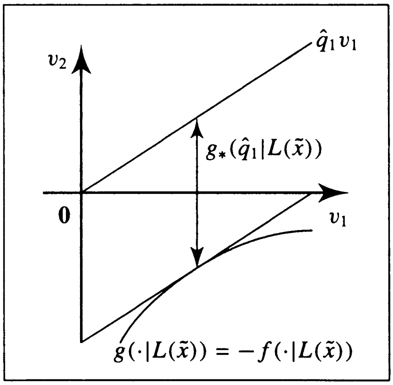

Figure III.7: Representation of the FENCHEL transform

Before going into the explicit derivation of the factor demand, the previous issues can be illustrated for the example of a simple production technology.

Example: *For the sake of clarity we omit an explicit consideration of indivisible goods as in the first example. Moreover, to avoid confusion within the calculation, the factor price q_1^{-r} with $r = 2$ is denoted within the example by q_1. The partial results are summarized by Figure III.8.*

For a fixed amount of the good $\tilde{x} > 0$ we suppose the input requirement set

$$L(\tilde{x}) = \left\{ \binom{v_1}{v_2} \in \mathbb{R}^2_+ \mid v_1 v_2 \geqq \tilde{x} \right\}$$

and by (III.5) the required function is

$$f(v_1|L(\tilde{x})) = \inf\left\{ v_2 \geqq 0 \mid \binom{v_1}{v_2} \in L(\tilde{x}) \right\} = \begin{cases} \tilde{x}/v_1 & \text{for } v_1 > 0 \\ +\infty & \text{for } v_1 \leqq 0. \end{cases}$$

Before the cost function can be calculated we have to carry out the change of sign as shown in Figure III.7, $g(\cdot|L(\tilde{x})) = -f(\cdot|L(\tilde{x}))$. This function is depicted as well in Quadrant IV of Figure III.8; here it describes the initial point A of the representation. The tangent drawn in Figure III.7 has been denoted in Figure III.8

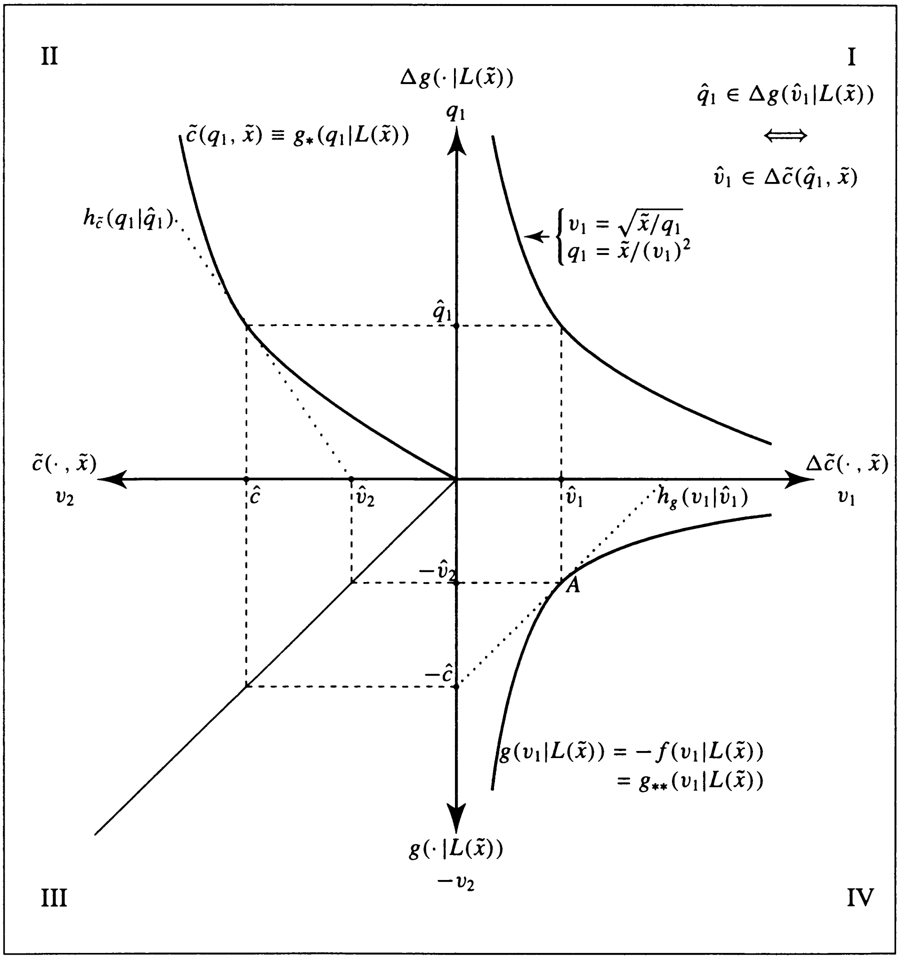

Figure III.8: A strictly convex input requirement set

by $h_g(v_1|\hat{v}_1)$. *Note that the downward directed ordinate shows not only* $g(\cdot|L(\tilde{x}))$ *but also* $-v_2$.

The determination of the normalized cost function $\tilde{c}(\cdot, \tilde{x})$ *with the concave* FENCHEL *transform*

$$\tilde{c}(q_1, \tilde{x}) \equiv g_*(q_1|L(\tilde{x})) = \inf\{q_1 v_1 - g(v_1|L(\tilde{x}))|\ v_1 \in \mathbb{R}\}$$

merely yields an optimal solution if $q_1 \geqq 0$ *is presupposed. Therefore, in Figure III.7 it is assumed* $\hat{q}_1 > 0$. *Since no* $v_1 \leqq 0$ *can be an optimal solution, the optimal amount of input* $\hat{v}_1$ *can be determined by differentiating* $q_1 v_1 + \tilde{x}/v_1$ *with* *respect to* v_1 *if we assume a positive factor price* $\hat{q}_1$.[32] *The result*

$$\hat{q}_1 - \tilde{x}/\hat{v}_1 = 0 \implies \hat{v}_1 = \sqrt{\tilde{x}/\hat{q}_1}$$

is depicted in Quadrant I of Figure III.8.

In this case the optimal value of the objective function is $\hat{q}_1\hat{v}_1 - \tilde{x}/\hat{v}_1 = 2\sqrt{\hat{q}_1\tilde{x}}$. *For* $q_1 < 0$ *the optimal amount of input* $\hat{v}_1 = +\infty$ *results, hence the optimal value of the objective function is* $q_1\hat{v}_1 - \tilde{x}/\hat{v}_1 = -\infty$. *Summarizing, we get the normalized cost function depicted in Quadrant II,*

$$\tilde{c}(q_1, \tilde{x}) \equiv g_*(q_1|L(\tilde{x})) = \begin{cases} 2\sqrt{q_1\tilde{x}} & \text{for } q_1 \geqq 0 \\ -\infty & \text{for } q_1 < 0\,. \end{cases}$$

As this function indicates the minimal factor costs in units of factor 2, the "ordinate" of Quadrant II depicts not only $\tilde{c}(\cdot, \tilde{x})$ *but also* v_2.

By the second concave FENCHEL *transform*

$$g_{**}(v_1|L(\tilde{x})) = \inf\{q_1v_1 - g_*(q_1|L(\tilde{x}))|\ q_1 \in \mathbb{R}\}$$

the original function $g(\cdot|L(\tilde{x}))$ *is reconstructed from the normalized cost function* $\tilde{c}(\cdot, \tilde{x}) \equiv g_*(\cdot|L(\tilde{x}))$. *Again* $q_1 < 0$ *cannot be an optimal solution. The minimization of* $q_1\hat{v}_1 + 2\sqrt{q_1\tilde{x}}$ *with respect to* q_1 *at a given* $\hat{v}_1 > 0$ *generates again the result described by Quadrant I:*

$$\hat{v}_1 - q_1^{-1/2}\tilde{x}^{1/2} = 0 \implies \hat{q}_1 = \tilde{x}/(\hat{v}_1)^2\,.$$

With that the optimal value of the objective function is $\hat{q}_1\hat{v}_1 - 2\hat{q}_1^{1/2}\tilde{x}^{1/2} = -\tilde{x}/\hat{v}_1$. *For* $v_1 \leqq 0$ *the optimal factor price is* $\hat{q}_1 = +\infty$ *since* $g(\cdot|L(\tilde{x}))$ *is bounded above. Summarizing, both results yield the initial function of Quadrant IV.*

$$g_{**}(v_1|L(\tilde{x})) = g(v_1|L(\tilde{x})) = -f(v_1|L(\tilde{x})) = \begin{cases} -\tilde{x}/v_1 & \text{for } v_1 > 0 \\ -\infty & \text{for } v_1 \leqq 0 \end{cases}$$

Before going into further aspects of Figure III.8, we have to concentrate on more analytical preparations. Furthermore, it must be taken into account that the example applies to a strictly convex input requirement set. However, the example of Figure III.12, p. 114, supposes indivisible factors.

1.4 Properties of Factor Demand

1.4.1 Convex Input Requirement Sets

An input vector $\mathbf{v} \in V$ is chosen if $\mathbf{v}$ is sufficient for the production of the given commodity bundle $\mathbf{x} \in X$ and if, at the same time, it minimizes the factor costs arising from the production of $\mathbf{x}$ at given factor prices. Denoting $f(\cdot|L(\mathbf{x}))$ the

[32] Second order optimum conditions for the determination of a minimum are omitted. For $q_1 = 0$ the minimum is not attained. As an alternative we may set $g_*(0|L(\tilde{x})) = 0$.

smallest quantity of factor r which is sufficient to produce the commodity bundle $\mathbf{x}$ with the given remaining inputs $\mathbf{v}^{-r}$, the quantities of factors $\mathbf{v}^{-r}$ are chosen if $\mathbf{v}^{-r}$ solves the problem of cost minimization

$$\text{(III.11)}\quad \tilde{c}(\mathbf{q}^{-r},\mathbf{x}) \equiv g_*(\mathbf{q}^{-r}|L(\mathbf{x})) = \inf\left\{(\mathbf{q}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) - g(\mathbf{v}^{-r}|L(\mathbf{x}))|\ \mathbf{v}^{-r} \in \mathcal{V}\right\}$$

at given (normalized) factor prices $\mathbf{q}^{-r}$. Since a solution to this problem does not need to be unique, we define the **factor demand correspondence**[33] $\tilde{D}(\cdot,\mathbf{x})\colon \mathcal{Q} \to \mathfrak{P}(\mathcal{V})$. The set $\tilde{D}(\mathbf{q}^{-r},\mathbf{x})$ contains all of the input vectors $\mathbf{v}^{-r}$ which minimize the factor costs in (III.11) for a given technology, given the factor prices $\binom{\mathbf{q}^{-r}}{1}$ and the output $\mathbf{x}$.

The next section deals with the properties of the demand correspondence $\tilde{D}$. As gathered from Section 1.4.3, we derive, for instance, a generalization of SHEPHARD's Lemma. Even at this point it can be noted that independent of the factor prices $\mathbf{q}^{-r}$ the set of cost minimizing input vectors $\tilde{D}(\mathbf{q}^{-r},\mathbf{x})$ is empty, provided an inadmissible vector $\mathbf{x} \notin X$ is given. In contrast, it can be assumed for each commodity bundle $\mathbf{x} \in X$ that the set $\tilde{D}(\mathbf{q}^{-r},\mathbf{x})$ includes at least one cost minimizing input vector $\mathbf{v}^{-r}$ at positive factor prices $\mathbf{q}^{-r} > \mathbf{0}$. In addition to this supposition the analysis states more precisely the correlation between the normalized factor prices $\mathbf{q}^{-r}$ and the cost minimizing input vectors $\hat{\mathbf{v}}^{-r}$.

Again the function $f(\cdot|L(\mathbf{x}))$, which serves for the description of certain boundary points of an input requirement set $L(\mathbf{x})$, is used as an initial point. Provided an admissible commodity bundle $\mathbf{x} \in X$ is held fixed, the n-proper function $g(\cdot|L(\mathbf{x})) \equiv -f(\cdot|L(\mathbf{x}))$ results by the above discussed change of sign. If we can further assure that the function $g(\cdot|L(\mathbf{x}))$ is concave, then regarding Corollary D.5.2, p. 320, important properties result for each element of the set of cost minimizing input vectors $\tilde{D}(\mathbf{q}^{-r},\mathbf{x})$.

The concavity of $g(\cdot|L(\mathbf{x}))$ is given by Theorem III.1, p. 86, if either convex input requirement sets $L(\mathbf{x})$ are considered or if the input requirement sets, including indivisible factors, are substituted by their convex hulls $\operatorname{conv}L(\mathbf{x})$.[34] Moreover, under the assumptions of Proposition III.3, p. 99, (or Corollary III.3.1) we can presume the closedness of the function $g(\cdot|L(\mathbf{x}))$.

The statement that an input vector $\hat{\mathbf{v}}^{-r}$ solves the problem of cost minimization (III.11), $\hat{\mathbf{v}}^{-r} \in \tilde{D}(\hat{\mathbf{q}}^{-r},\mathbf{x})$, is reflected by the following three equivalent conditions (III.27a)–(III.27c). For a given factor price vector $\hat{\mathbf{q}}^{-r}$ and an n-proper concave function $g(\cdot|L(\mathbf{x}))$ Corollary D.5.2 says that

$$\text{(III.27a)}\qquad (\hat{\mathbf{q}}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) - g(\mathbf{v}^{-r}|L(\mathbf{x}))\ \text{achieves the infimum at}\ \ \mathbf{v}^{-r} = \hat{\mathbf{v}}^{-r}$$

holds if and only if

$$\text{(III.27b)}\qquad g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) + g_*(\hat{\mathbf{q}}^{-r}|L(\mathbf{x})) = (\hat{\mathbf{q}}^{-r})^{\mathsf{T}}(\hat{\mathbf{v}}^{-r}).$$

[33]In this section we refrain from calling a multi-valued mapping $\Gamma\colon X \to \mathfrak{P}(Y)$ a correspondence only if $\Gamma(x) \neq \emptyset$ is satisfied for all $x \in X$.

[34]Remember at this point that $L(\mathbf{x}) = \operatorname{conv}L(\mathbf{x})$ for a convex input requirement set.

Take into consideration that the two functional values of $g(\cdot|L(\mathbf{x}))$ and the costs $\tilde{c}(\cdot,\mathbf{x}) \equiv g_*(\cdot|L(\mathbf{x}))$ are measured in units of factor r. An exemplary representation of this equivalence relation can be taken from Figure III.9, point A. Both conditions are equivalent to the statement that $\hat{\mathbf{q}}^{-r}$ is a supergradient of the function $g(\cdot|L(\mathbf{x}))$ at point $\hat{\mathbf{v}}^{-r}$. According to Definition D.2, p. 318, the factor price vector $\hat{\mathbf{q}}^{-r}$ is called a **supergradient** of the concave function $g(\cdot|L(\mathbf{x}))$ at point $\hat{\mathbf{v}}^{-r}$ if it fulfills

$$g(\mathbf{v}^{-r}|L(\mathbf{x})) \leqq g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) + (\hat{\mathbf{q}}^{-r})^{\mathrm{T}}(\mathbf{v}^{-r} - \hat{\mathbf{v}}^{-r}) \qquad \forall\, \mathbf{v}^{-r} \in \mathcal{V}\,.$$

The relation to (III.27a) can be shown by rearranging this inequality

$$g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) + \left[(\hat{\mathbf{q}}^{-r})^{\mathrm{T}}(\hat{\mathbf{v}}^{-r}) - g(\mathbf{v}^{-r}|L(\mathbf{x}))\right] \geqq (\hat{\mathbf{q}}^{-r})^{\mathrm{T}}(\mathbf{v}^{-r}) \qquad \forall\, \mathbf{v}^{-r} \in \mathcal{V}\,.$$

If the term in brackets achieves its infimum at point $\hat{\mathbf{v}}^{-r}$, then by definition it equals $g_*(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))$ and it follows (III.27b).

Before continuing with the analysis, we give a hint for the classification of supergradients. Supposing the function $g(\cdot|L(\mathbf{x}))$ is concave and differentiable[35] at a point $\hat{\mathbf{v}}^{-r}$, then the unique supergradient $\hat{\mathbf{q}}^{-r}$ by Theorem D.4, p. 319, equals the **gradient**

$$\nabla g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) = \left(\frac{\partial g}{\partial v_i}(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))\right)_{\substack{i=1,\dots,m\\ i\neq r}}\,.$$

Conversely, we can conclude from the existence of a gradient at point $\hat{\mathbf{v}}^{-r}$ that the supergradient is unique.

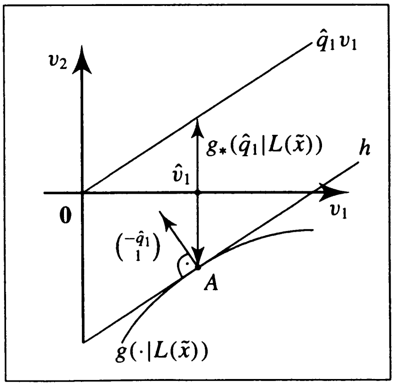

Figure III.9: Representation of a unique supergradient

Resuming Figure III.7, the graphical representation in Figure III.9 describes a further special case for a concave and differentiable function $g(\cdot|L(\mathbf{x}))$ since the number of production factors is restricted to two. Thus, $\mathbf{v}^{-r} = (v_1)$ holds for $r = 2$. In this case the unique supergradient $\hat{\mathbf{q}}^{-r} = (\hat{q}_1)$ at point $\hat{v}_1$ equals not only the gradient $\nabla g(\hat{v}_1|L(\tilde{x}))$ but also the derivative

$$\hat{q}_1 = \frac{\mathrm{d}g}{\mathrm{d}v_1}(\hat{v}_1|L(\tilde{x})).$$

As mentioned above, both statements (III.27a) and (III.27b) are equivalent to the condition that $\hat{q}_1$ is a supergradient of the function $g(\cdot|L(\tilde{x}))$ at point $\hat{v}_1$.

Before going into the construction of the vector $\binom{-\hat{q}_1}{1}$, it must be noted that a supergradient does not need to be unique at all. If the concave function $g(\cdot|L(\tilde{x}))$ has a kink point such as point A of Figure III.10, then each vector of the marked cone corresponds to a supergradient. In this case all of the supergradients of $g(\cdot|L(\mathbf{x}))$

[35]The assumption of differentiability of $g(\cdot|L(\mathbf{x}))$ and $\tilde{c}(\cdot,\mathbf{x})$ is discussed later in more detail.

at point $\hat{\mathbf{v}}^{-r}$ are collected in the (closed and convex) set $\Delta g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))$ and we call this set the **superdifferential** of $g(\cdot|L(\mathbf{x}))$ at point $\hat{\mathbf{v}}^{-r}$.

Now the statements (III.27a) and (III.27b) can be extended with regard to the superdifferential $\Delta g(\cdot|L(\mathbf{x}))$ by the equivalent condition

(III.27c) $$\hat{\mathbf{q}}^{-r} \in \Delta g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})).$$

To emphasize the used notation, we write

$$\Delta g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) = \left\{\nabla g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))\right\} = \left\{\hat{\mathbf{q}}^{-r}\right\}$$

in the case of a unique supergradient $\hat{\mathbf{q}}^{-r}$, which equals the gradient $\nabla g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))$.

To complete the construction principle of Figure III.9, scrutinize an optimal input vector $\hat{\mathbf{v}}^{-r}$. If we set for the infimum in (III.27a)

$$\alpha := g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) - (\hat{\mathbf{q}}^{-r})^{\mathsf{T}}(\hat{\mathbf{v}}^{-r}) = \text{const.},$$

then an affine function results,

$$h(\mathbf{v}^{-r}) = (\hat{\mathbf{q}}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) + \alpha\,,$$

which can be used in the inequality for the supergradient.

$$g(\mathbf{v}^{-r}|L(\mathbf{x})) \leqq h(\mathbf{v}^{-r}) \qquad \forall\, \mathbf{v}^{-r} \in \mathcal{V}$$

In particular, corresponding to point A in Figure III.9, the equation $g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) = h(\hat{\mathbf{v}}^{-r})$ holds at point $\hat{\mathbf{v}}^{-r}$. With regard to a graphical representation of the supergradient $\hat{\mathbf{q}}^{-r}$ the rearrangement

$$\begin{pmatrix} -\hat{\mathbf{q}}^{-r} \\ 1 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathbf{v}^{-r} \\ h(\mathbf{v}^{-r}) \end{pmatrix} = \alpha$$

turns out to be useful. Thus, the graph of the function h is a hyperplane H in $\mathbb{R}^m$ with the normal vector $\binom{-\hat{\mathbf{q}}^{-r}}{1}$. The equation $g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) = h(\hat{\mathbf{v}}^{-r})$ now states that the hyperplane H is tangent to the hypograph of the function $g(\cdot|L(\mathbf{x}))$ at point $\binom{\hat{\mathbf{v}}^{-r}}{g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))}$ so that

$$\begin{pmatrix} \hat{\mathbf{v}}^{-r} \\ g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) \end{pmatrix} \in \operatorname{hypo} g(\cdot|L(\mathbf{x})) \cap H\,.$$

In view of $\binom{\hat{\mathbf{v}}^{-r}}{g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))} = \binom{\hat{\mathbf{v}}^{-r}}{h(\hat{\mathbf{v}}^{-r})}$ we reach a graphical representation of the supergradient $\hat{\mathbf{q}}^{-r} \in \Delta g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))$ since the normal vector $\binom{-\hat{\mathbf{q}}^{-r}}{1}$ is especially normal at point $\binom{\hat{\mathbf{v}}^{-r}}{h(\hat{\mathbf{v}}^{-r})}$. While Figure III.9 has a unique supergradient $\hat{q}_1$ at point A, the superdifferential at point A of Figure III.10 corresponds to a convex cone.

The second part of Corollary D.5.2 states that dually to (III.27a)–(III.27c) three more conditions are equivalent to each other. These equivalence relations refer to the cost function $\tilde{c}(\cdot, \mathbf{x}) \equiv g_*(\cdot|L(\mathbf{x}))$ and to the concave conjugate function

$g_{**}(\cdot|L(\mathbf{x}))$ by (III.23), where the conditions which are sufficient for $g_{**}(\cdot|L(\mathbf{x})) = g(\cdot|L(\mathbf{x}))$ are picked up in (III.28).To assure that the conditions of an n-proper function $g(\cdot|L(\mathbf{x}))$ are satisfied, we have to assume at this point again an admissible commodity bundle $\mathbf{x} \in X$. [36]

(III.27d) $\quad (\mathbf{q}^{-r})^{\mathsf{T}}(\hat{\mathbf{v}}^{-r}) - g_*(\mathbf{q}^{-r}|L(\mathbf{x}))$ attains the infimum at $\mathbf{q}^{-r} = \hat{\mathbf{q}}^{-r}$

$$g_{**}(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) + g_*(\hat{\mathbf{q}}^{-r}|L(\mathbf{x})) = (\hat{\mathbf{q}}^{-r})^{\mathsf{T}}(\hat{\mathbf{v}}^{-r}) \tag{III.27e}$$

$$\hat{\mathbf{v}}^{-r} \in \Delta g_*(\hat{\mathbf{q}}^{-r}|L(\mathbf{x})) \tag{III.27f}$$

The assumed closedness of the function $g(\cdot|L(\mathbf{x}))$ implies especially the equation $\operatorname{cl} g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) = g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))$. Hence, by Corollary D.5.2, all of the six conditions (III.27a)–(III.27c) and (III.27d)–(III.27f) are equivalent. Each pair $(\hat{\mathbf{q}}^{-r}, \hat{\mathbf{v}}^{-r})$ satisfying the above six conditions is called a **pair of dual points**.

Finally, Corollary D.5.2 offers a statement collecting all pairs of dual points, where the attention is directed to the conditions (III.27c) and (III.27f). Since the supposed function $g(\cdot|L(\mathbf{x}))$ is n-proper, concave, and closed, the superdifferentials $\Delta g(\cdot|L(\mathbf{x})) : \mathcal{V} \to \mathfrak{P}(\mathcal{Q})$ and $\Delta g_*(\cdot|L(\mathbf{x})) : \mathcal{Q} \to \mathfrak{P}(\mathcal{V})$ are inverse to each other.

$$\mathbf{q}^{-r} \in \Delta g(\mathbf{v}^{-r}|L(\mathbf{x})) \iff \mathbf{v}^{-r} \in \Delta g_*(\mathbf{q}^{-r}|L(\mathbf{x}))$$

In particular, (III.27b) and (III.27e) reflect the symmetry of the FENCHEL transform by Corollary D.3.1 (FENCHEL, MOREAU), p. 318, with convex input requirement sets.

$$g(\cdot|L(\mathbf{x})) = g_{**}(\cdot|L(\mathbf{x})) \tag{III.28}$$

A comparison of (III.27a) and (III.27f) shows (*again for convex input requirement sets*), that for a given technology the superdifferential $\Delta g_*(\cdot|L(\mathbf{x}))$ is nothing else than the factor demand correspondence $\tilde{D}(\cdot, \mathbf{x})$.

$$\Delta g_*(\cdot|L(\mathbf{x})) \equiv \Delta\tilde{c}(\cdot, \mathbf{x}) = \tilde{D}(\cdot, \mathbf{x}) \tag{III.29}$$

In this way a factor vector $\mathbf{v}^{-r}$, solving the problem of cost minimization $\mathbf{v}^{-r} \in \tilde{D}(\mathbf{q}^{-r}, \mathbf{x})$, describes a supergradient of the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ at point $\mathbf{q}^{-r}$ and this relation is fulfilled if and only if the factor price vector $\mathbf{q}^{-r}$ is a supergradient of the (concave) function $g(\cdot|L(\mathbf{x}))$ at point $\mathbf{v}^{-r}$.

For the superdifferential $\Delta g(\cdot|L(\mathbf{x})) : \mathcal{V} \to \mathfrak{P}(\mathcal{Q})$ of an n-proper concave function $g(\cdot|L(\mathbf{x}))$ the following properties can be noted, which analogously hold for the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ and which are illustrated by the subsequent

[36] For this result of Corollary D.5.2 the findings of Theorem D.2, p. 316, are crucial. Accordingly, we may assume an n-proper closed concave function $g_*(\cdot|L(\mathbf{x}))$ for the commodity bundle $\mathbf{x} \in X$.

example.[37]

$$\begin{aligned}
\mathbf{v}^{-r} &\notin \text{n-Dom}\, g(\cdot|L(\mathbf{x})) &&\Longrightarrow \quad \Delta g(\mathbf{v}^{-r}|L(\mathbf{x})) = \emptyset\,;\\
\mathbf{v}^{-r} &\in \operatorname{rint}\big(\text{n-Dom}\, g(\cdot|L(\mathbf{x}))\big) &&\Longrightarrow \quad \Delta g(\mathbf{v}^{-r}|L(\mathbf{x})) \neq \emptyset\,;\\
\mathbf{v}^{-r} &\in \operatorname{int}\big(\text{n-Dom}\, g(\cdot|L(\mathbf{x}))\big) &&\Longleftrightarrow \quad \Delta g(\mathbf{v}^{-r}|L(\mathbf{x})) \neq \emptyset \quad \text{and bounded.}
\end{aligned}$$

As explained above, the concave function $g(\cdot|L(\mathbf{x}))$ has a uniquely determined supergradient $\Delta g(\mathbf{v}^{-r}|L(\mathbf{x})) = \{\mathbf{q}^{-r}\}$ at point $\mathbf{v}^{-r}$ *if and only if* the function is differentiable at the inspected point and has therefore the gradient $\nabla g(\mathbf{v}^{-r}|L(\mathbf{x})) = \mathbf{q}^{-r}$.[38]

Regarding the cost function $\tilde{c}(\cdot, \mathbf{x})$, which has by (III.22) the effective domain $\text{n-Dom}\,\tilde{c}(\cdot, \mathbf{x}) = \mathbb{R}_{+}^{m-1}$, we get

(III.30a) $\qquad \mathbf{q}^{-r} \notin \mathbb{R}_{+}^{m-1} \quad\Longrightarrow\quad \Delta\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) = \emptyset;$

(III.30b) $\qquad \mathbf{q}^{-r} \in \mathbb{R}_{++}^{m-1} \quad\Longleftrightarrow\quad \Delta\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) \neq \emptyset \quad \text{and bounded.}$

As long as $\mathbf{q}^{-r}$ contains a negative factor price, none of the input vectors is chosen, $\tilde{D}(\mathbf{q}^{-r}, \mathbf{x}) = \emptyset$.

Example: *Figure III.10 illustrates a superdifferential $\Delta g(\cdot|L(\tilde{x}))$ with a kinked isoquant $f(\cdot|L(\tilde{x}))$ for two factors, where the attention is directed to the later used convex hull of the input requirement set $L(\tilde{x})$ in (III.8). Considering the change of sign $f(\cdot|L(\tilde{x})) = -g(\cdot|L(\tilde{x}))$, the figure is based on the function*

$$g(v_1|L(\tilde{x})) = \begin{cases} a_1 + b_1 v_1 & \text{for } \underline{v}_1 < v_1 \leqq \tilde{v}_1 \\ a_2 + b_2 v_1 & \text{for } \tilde{v}_1 \leqq v_1 \leqq \overline{v}_1 \\ -\infty & \text{otherwise} \end{cases}$$

with the constants a_1, a_2, b_1, b_2 and $b_1 > b_2 > 0$ as well as $a_1 < a_2 < 0$. The effective domain

$$\text{n-Dom}\, g(\cdot|L(\tilde{x})) =]\underline{v}_1, \overline{v}_1]$$

clarifies the fact that the hypograph hypo $g(\cdot|L(\tilde{x}))$ *is not closed and, therefore, that the production technology does not fulfill Axiom* [L6] *(Closedness). Moreover,* [L3] *(Disposability of Inputs) is not satisfied. Ignoring the superscripts $-r$ with regard to the normalized factor price q_1^{-r} with $r = 2$ the following superdifferential emerges:*

$$\Delta g(v_1|L(\tilde{x})) = \begin{cases} \emptyset & \text{for } v_1 \leqq \underline{v}_1 \\ \{b_1\} & \text{for } \underline{v}_1 < v_1 < \tilde{v}_1 \\ \{q_1 \in \mathbb{R} |\, b_2 \leqq q_1 \leqq b_1\} & \text{for } v_1 = \tilde{v}_1 \\ \{b_2\} & \text{for } \tilde{v}_1 < v_1 < \overline{v}_1 \\ \{q_1 \in \mathbb{R} |\, q_1 < b_2\} & \text{for } v_1 = \overline{v}_1 \\ \emptyset & \text{for } v_1 > \overline{v}_1. \end{cases}$$

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[37] Cf. ROCKAFELLAR (1972, p. 217, Theorem 23.4).
For an $(m-1)$-dimensional convex set $\text{n-Dom}\, g(\cdot|L(\mathbf{x})) \subset \mathbb{R}^{m-1}$ we do not need to distinguish between the interior and the relative interior by (B.4).

$$\operatorname{rint}\big(\text{n-Dom}\, g(\cdot|L(\mathbf{x}))\big) = \operatorname{int}\big(\text{n-Dom}\, g(\cdot|L(\mathbf{x}))\big)$$

[38] See Theorem D.4, p. 319.

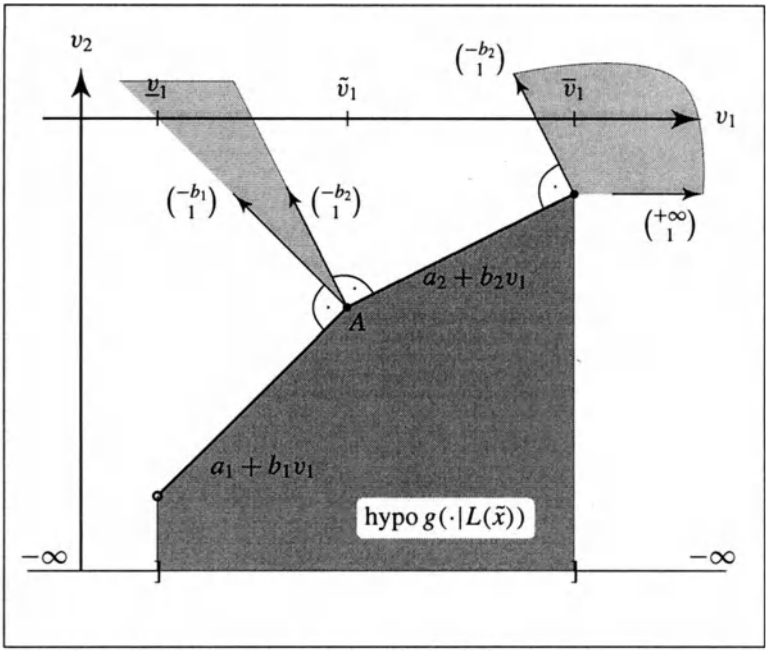

Figure III.10: Representation of the superdifferential $\Delta g(\cdot|L(\tilde{x}))$

Figure III.10 shows the nonempty bounded superdifferential at point $\tilde{v}_1$ *as a cone generated by the vectors* $\binom{-b_1}{1}$ *and* $\binom{-b_2}{1}$. *At point* $\underline{v}_1$ *there is no supergradient, i.e.* $\Delta g(\underline{v}_1|L(\tilde{x})) = \emptyset$. *At each other point* $\hat{v}_1$ *in the interval* $]\underline{v}_1, \overline{v}_1[$ *the function* $g(\cdot|L(\tilde{x}))$ *is differentiable. The corresponding superdifferentials consist of a unique element, namely the gradient or the derivative* $\mathrm{d}g(\hat{v}_1|L(\tilde{x}))/\mathrm{d}v_1$. *Because vertical hyperplanes with* $q_1 \rightarrow -\infty$ *cannot be expressed as affine functions, the nonempty, but unbounded superdifferential*[39] *at point* $\overline{v}_1$, *can only be indicated by the cone between* $\binom{-b_2}{1}$ *and* $\binom{+\infty}{1}$. *Furthermore, at this point it turns out that under* [L3] *(Disposability of Inputs)negative factor prices do not lead to any cost minimizing solution. In the example at hand each negative factor price* $\hat{q}_1 < 0$ *implies the optimal input* $\hat{v}_1 = \overline{v}_1$. *If the function* $g(\cdot|L(\tilde{x}))$, *which corresponds to the input requirement set* $L(\tilde{x})$, *satisfies Axiom [L3], then the factor* v_1 *could be arbitrarily enlarged and the problem*

$$\inf\left\{\hat{q}_1 v_1 - g(v_1|L(\tilde{x}))|\ v_1 \in \mathbb{R}\right\}$$

would have no solution for $\hat{q}_1 < 0$.

[39]Since $\overline{v}_1$ lies on the boundary of the effective domain n-Dom $g(\cdot|L(\tilde{x}))$, a priori nothing is known on the superdifferential $\Delta g(\overline{v}_1|L(\tilde{x}))$.

1.4.2 Consideration of Indivisible Production Factors

The results of the previous section are mainly based on the assumption of convex input requirement sets and, therefore, on divisible inputs. In particular, the identity of the superdifferential of the cost function and the factor demand correspondence, $\Delta\tilde{c}(\cdot, \mathbf{x}) \equiv \tilde{D}(\cdot, \mathbf{x})$, as given in (III.29), is no longer valid in view of indivisible factors. A vector $\mathbf{v}^{-r}$ solving the problem of cost minimization is not necessarily an integer; see point A in Figure III.11. Such a vector is ruled out for deriving the factor demand.

As before, we try to clarify the problem by replacing the input requirement sets $L(\mathbf{x})$ with their convex hulls. Consequently, $g(\cdot|\mathrm{conv}L(\mathbf{x}))$ is an n-proper closed concave function under the premises of Corollary III.3.1. By Corollary D.3.1 (FENCHEL, MOREAU), p. 318, we obtain

$$
\begin{aligned}
\text{(III.31)} \qquad -f(\cdot|\mathrm{conv}L(\mathbf{x})) &= g(\cdot|\mathrm{conv}L(\mathbf{x})) && \text{by definition} \\
&= g_{**}(\cdot|\mathrm{conv}L(\mathbf{x})) && \text{because of Corollary D.3.1} \\
&= g_{**}(\cdot|L(\mathbf{x})) && \text{because of Proposition III.2}
\end{aligned}
$$

with respect to Proposition III.2. Concerning Proposition III.2 for an admissible commodity bundle $\mathbf{x} \in X$, the six equivalent conditions are

(III.32a) $(\hat{\mathbf{q}}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) - g(\mathbf{v}^{-r}|\mathrm{conv}L(\mathbf{x}))$ achieves the infimum at $\mathbf{v}^{-r} = \hat{\mathbf{v}}^{-r}$;

(III.32b) $g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) + g_*(\hat{\mathbf{q}}^{-r}|L(\mathbf{x})) = (\hat{\mathbf{q}}^{-r})^{\mathsf{T}}(\hat{\mathbf{v}}^{-r})$;

(III.32c) $\hat{\mathbf{q}}^{-r} \in \Delta g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x}))$;

and their duals are

(III.32d) $(\mathbf{q}^{-r})^{\mathsf{T}}(\hat{\mathbf{v}}^{-r}) - g_*(\mathbf{q}^{-r}|L(\mathbf{x}))$ achieves the infimum at $\mathbf{q}^{-r} = \hat{\mathbf{q}}^{-r}$;

(III.32e) $g_{**}(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) + g_*(\hat{\mathbf{q}}^{-r}|L(\mathbf{x})) = (\hat{\mathbf{q}}^{-r})^{\mathsf{T}}(\hat{\mathbf{v}}^{-r})$;

(III.32f) $\hat{\mathbf{v}}^{-r} \in \Delta g_*(\hat{\mathbf{q}}^{-r}|L(\mathbf{x}))$.

As discussed above, contrary to (III.27a) not all of the vectors $\hat{\mathbf{v}}^{-r}$ can be interpreted as input vectors. The inverse superdifferentials

$$
\mathbf{q}^{-r} \in \Delta g(\mathbf{v}^{-r}|\mathrm{conv}L(\mathbf{x})) \iff \mathbf{v}^{-r} \in \Delta g_*(\mathbf{q}^{-r}|L(\mathbf{x})) \equiv \Delta\tilde{c}(\mathbf{q}^{-r}, \mathbf{x})
$$

also include those pairs of dual points $(\mathbf{q}^{-r}, \mathbf{v}^{-r})$, where $\mathbf{v}^{-r}$ does not satisfy the required integer constraints; see point A in Figure III.11.

When the operation $L(\mathbf{x}) \to \mathrm{conv}L(\mathbf{x})$ generates pairs of dual points $(\mathbf{q}^{-r}, \mathbf{v}^{-r})$ so that the corresponding vectors $\mathbf{v}^{-r}$ cannot be interpreted as input vectors, then the following question must be asked: what conditions are sufficient to filter out the set of optimal *input* vectors of the set of optimal vectors $\Delta\tilde{c}(\mathbf{q}^{-r}, \mathbf{x})$? See points B_1 and B_2 in Figure III.11.

First of all, we can note the following property of an optimal vector $\hat{\mathbf{v}}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$. The (nonpositive) functional value $g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x}))$ is finite[40] so that

$$\begin{pmatrix} \hat{\mathbf{v}}^{-r} \\ -g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) \end{pmatrix} \in \mathrm{conv}L(\mathbf{x})$$

holds for a closed set $\mathrm{conv}L(\mathbf{x})$. However, an optimal vector $\hat{\mathbf{v}}^{-r}$ is an admissible input vector only if $\hat{\mathbf{v}}^{-r}$, completed by the component $-g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x}))$, is an element in the factor space V.

$$\begin{pmatrix} \hat{\mathbf{v}}^{-r} \\ -g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) \end{pmatrix} \in V$$

Under [L7a] (*Integer Convexity*) both conditions suggest

$$\mathrm{conv}L(\mathbf{x}) \cap V = L(\mathbf{x})$$

so that the factor demand correspondence $\tilde{D}(\cdot, \mathbf{x})$ can be filtered out of the superdifferential $\Delta\tilde{c}(\cdot, \mathbf{x})$. The results of the following analysis can be taken from (III.35) and from the summary on p. 113.

First of all, keep in mind that

$$\text{(III.33)} \qquad \tilde{D}(\cdot, \mathbf{x}) \subset \Delta\tilde{c}(\cdot, \mathbf{x})$$

has to be satisfied since the problem of cost minimization is independent of the fact whether it is based on the input requirement set $L(\mathbf{x})$ or its convex hull $\mathrm{conv}L(\mathbf{x})$. Conversely, an optimal vector $\hat{\mathbf{v}}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$ implies a vector of factor quantities demanded $\hat{\mathbf{v}}^{-r} \in \tilde{D}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$ if

$$\begin{pmatrix} \hat{\mathbf{v}}^{-r} \\ -g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) \end{pmatrix} \in \mathrm{conv}L(\mathbf{x}) \cap V.$$

In this relation Axiom [L7a] (*Integer Convexity*) guarantees that all of these vectors can indeed be interpreted as an *input* vector, i.e.

$$\text{(III.34)} \qquad \begin{pmatrix} \hat{\mathbf{v}}^{-r} \\ -g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) \end{pmatrix} \in L(\mathbf{x}).$$

This relation can only be satisfied if $\hat{\mathbf{v}}^{-r}$ does not violate any integer constraints. Let V^{-r} denote the subspace in the factor space V containing the input vectors $\mathbf{v}^{-r}$, then the above relation

$$\hat{\mathbf{v}}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}) \cap V^{-r}$$

yields all of the vectors fulfilling (III.34).

---

[40]The relations (III.32b) and (III.32e) yield $g_{**}(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) = g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x}))$, where $g_{**}(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x}))$ is finite by (III.32d). Furthermore, $g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) = -\infty$ would contradict (III.32c).

In view of Proposition III.1

$$-g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) = f(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) = \mathrm{conv}f(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))$$

results in a statement equivalent to (III.34).

$$\begin{pmatrix} \hat{\mathbf{v}}^{-r} \\ \mathrm{conv}f(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) \end{pmatrix} \in L(\mathbf{x})$$

Because of (III.7) and (III.4), i.e.

$$\mathrm{conv}f(\cdot|L(\mathbf{x})) \leqq f(\cdot|L(\mathbf{x})),$$

the preceding relation is satisfied by the definition of $f(\cdot|L(\mathbf{x}))$ if and only if the relation $\mathrm{conv}f(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) = f(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))$ holds at point $\hat{\mathbf{v}}^{-r}$. When there is a cost minimizing *input* vector, then it is subject to the following equivalence relation:[41]

$$\text{(III.35)} \qquad \hat{\mathbf{v}}^{-r} \in \tilde{D}(\hat{\mathbf{q}}^{-r}, \mathbf{x}) \iff \left\{ \begin{matrix} \hat{\mathbf{v}}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}) \cap V^{-r} \\ f(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) = \mathrm{conv}f(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) \end{matrix} \right\}$$

Finally, we have to show whether (III.35) describes at least one cost minimizing input vector $\mathbf{v}^{-r}$ for each commodity bundle $\mathbf{x} \in X$ and for each price vector $\mathbf{q}^{-r} > \mathbf{0}$. In order to prove $\tilde{D}(\mathbf{q}^{-r}, \mathbf{x}) \neq \emptyset$, first of all

$$\text{(III.36)} \qquad \Delta\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) \cap V^{-r} \neq \emptyset$$

must hold, where it can be supposed that the closed convex superdifferential $\Delta\tilde{c}(\mathbf{q}^{-r}, \mathbf{x})$ is not empty for $\mathbf{q}^{-r} > \mathbf{0}$ according to (III.30b).

The idea of proving (III.36) is that the vectors $\mathbf{v}^{-r} \in \Delta\tilde{c}(\mathbf{q}^{-r}, \mathbf{x})$ determine all the points $\begin{pmatrix} \mathbf{v}^{-r} \\ f(\mathbf{v}^{-r}|\mathrm{conv}L(\mathbf{x})) \end{pmatrix}$ of an exposed face[42] of the set $\mathrm{conv}L(\mathbf{x})$. The following proposition, which states that each face of $\mathrm{conv}L(\mathbf{x})$ and in particular each exposed face of $\mathrm{conv}L(\mathbf{x})$ contains at least one point $\mathbf{v} \in L(\mathbf{x})$, must be taken into account. Thus, as described by Figure III.11 for two factors with $r = 2$, a nontrivial supporting hyperplane H of $\mathrm{conv}L(\mathbf{x})$ is needed such that

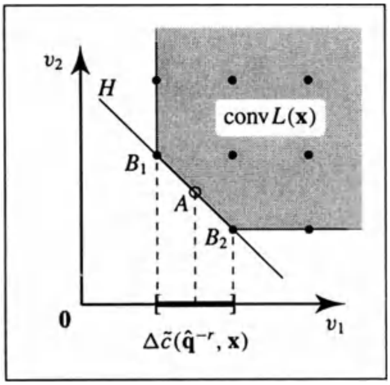

Figure III.11: Representation of an exposed face of $\mathrm{conv}L(\mathbf{x})$

(III.37)

$$\mathbf{v}^{-r} \in \Delta\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) \iff \begin{pmatrix} \mathbf{v}^{-r} \\ f(\mathbf{v}^{-r}|\mathrm{conv}L(\mathbf{x})) \end{pmatrix} \in H \cap \mathrm{conv}L(\mathbf{x}).$$

[41]For perfectly divisible inputs the right hand side of the equivalence relation reduces to $\hat{\mathbf{v}}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$; see (III.29).

[42]Each exposed face of a convex set C is a face of C. Without defining the term **face** of C it is enough to know the following statement: **exposed faces** of C are exactly those sets which can be expressed in the form $C \cap H$, where H is a nontrivial supporting hyperplane of C. Cf. ROCKAFELLAR (1972, p. 163).

The next proposition now follows for each (exposed) face:

Proposition III.4[43] *Let* $L(\mathbf{x})$ *be a set of points and let* C' *be a nonempty face of the convex hull* $\operatorname{conv} L(\mathbf{x})$. *Then* $C' = \operatorname{conv} L(\mathbf{x})'$, *where* $L(\mathbf{x})'$ *consists of those points in* $L(\mathbf{x})$ *which belong to* C'.

Since C' is nonempty, $L(\mathbf{x})' \subset L(\mathbf{x})$ must contain at least one point, too. According to the derivation of the equivalence relation (III.35), each of these points with

$$\text{(III.38)} \qquad \begin{pmatrix} \mathbf{v}^{-r} \\ f(\mathbf{v}^{-r}|\operatorname{conv} L(\mathbf{x})) \end{pmatrix} \in L(\mathbf{x})' \subset C'$$

satisfies $f(\mathbf{v}^{-r}|\operatorname{conv} L(\mathbf{x})) = f(\mathbf{v}^{-r}|L(\mathbf{x}))$ and yields (III.36). Now the required hyperplane H can be determined as follows: the equivalence of (III.32b) and (III.32f) states that

$$\text{(III.39)} \qquad \tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}) = \begin{pmatrix} \hat{\mathbf{q}}^{-r} \\ 1 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathbf{v}^{-r} \\ f(\mathbf{v}^{-r}|\operatorname{conv} L(\mathbf{x})) \end{pmatrix}$$

holds for all $\mathbf{v}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$. The nonempty convex set $\Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$ yields

$$\mathbf{v}_i^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}),\ \ \lambda_i \geqq 0,\ \sum \lambda_i = 1 \quad \Longrightarrow \quad \sum \lambda_i \mathbf{v}_i^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}).$$

Thus, (III.39) must hold in the form

$$\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}) = \begin{pmatrix} \hat{\mathbf{q}}^{-r} \\ 1 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \sum \lambda_i \mathbf{v}_i^{-r} \\ f(\sum \lambda_i \mathbf{v}_i^{-r}|\operatorname{conv} L(\mathbf{x})) \end{pmatrix}.$$

Furthermore, each single point $\mathbf{v}_i^{-r}$ satisfies the equation (III.39) so that

$$\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}) = \begin{pmatrix} \hat{\mathbf{q}}^{-r} \\ 1 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \sum \lambda_i \mathbf{v}_i^{-r} \\ \sum \lambda_i f(\mathbf{v}_i^{-r}|\operatorname{conv} L(\mathbf{x})) \end{pmatrix}.$$

The resulting equation

$$\text{(III.40)} \qquad f(\textstyle\sum \lambda_i \mathbf{v}_i^{-r}|\operatorname{conv} L(\mathbf{x})) = \sum \lambda_i f(\mathbf{v}_i^{-r}|\operatorname{conv} L(\mathbf{x}))$$
$$\text{with} \quad \mathbf{v}_i^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}),\ \ \lambda_i \geqq 0,\ \sum \lambda_i = 1$$

says that $f(\cdot|\operatorname{conv} L(\mathbf{x}))$ is a (partially) affine function in the set $\Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$. If we substitute $f(\cdot|\operatorname{conv} L(\mathbf{x}))$ by an affine function h which equals $f(\cdot|\operatorname{conv} L(\mathbf{x}))$ in

[43] Cf. ROCKAFELLAR (1972, p. 165, Theorem 18.3).

$\Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$, then[44]

$$\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}) = \begin{pmatrix} \hat{\mathbf{q}}^{-r} \\ 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \mathbf{v}^{-r} \\ h(\mathbf{v}^{-r}) \end{pmatrix}$$

represents a hyperplane H in $\mathbb{R}^m$. This hyperplane is a (nontrivial[45]) supporting hyperplane of the set $\mathrm{conv}L(\mathbf{x})$ since firstly the convexity of $f(\cdot|\mathrm{conv}L(\mathbf{x}))$ implies

$$\begin{pmatrix} \hat{\mathbf{q}}^{-r} \\ 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \mathbf{v}^{-r} \\ h(\mathbf{v}^{-r}) \end{pmatrix} \leqq \begin{pmatrix} \hat{\mathbf{q}}^{-r} \\ 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \mathbf{v}^{-r} \\ f(\mathbf{v}^{-r}|\mathrm{conv}L(\mathbf{x})) \end{pmatrix}$$

and secondly each point $\binom{\mathbf{v}^{-r}}{f(\mathbf{v}^{-r}|\mathrm{conv}L(\mathbf{x}))}$ with $\mathbf{v}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$ is a common point of H and $\mathrm{conv}L(\mathbf{x})$; see (III.37).
Conversely, each point $\binom{\mathbf{v}^{-r}}{h(\mathbf{v}^{-r})} \in H \cap \mathrm{conv}L(\mathbf{x})$ satisfies (III.39) so that $\mathbf{v}^{-r}$ determines a point in $\Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$. With

$$H \cap \mathrm{conv}L(\mathbf{x}) = \left\{ \begin{pmatrix} \mathbf{v}^{-r} \\ f(\mathbf{v}^{-r}|\mathrm{conv}L(\mathbf{x})) \end{pmatrix} \middle| \; \mathbf{v}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}) \right\}$$

the supporting hyperplane H, required for (III.37), is found. Thus, at least one point satisfies (III.38), which moreover assures that (III.36) is valid.

Summary: For each *output* vector $\mathbf{x} \in X$ and for each factor price vector $\hat{\mathbf{q}}^{-r} > \mathbf{0}$ an *input* vector $\hat{\mathbf{v}}^{-r} \in V^{-r}$ exists solving the problem of cost minimization; see points B_1 and B_2 in Figure III.11. The permissible and cost minimizing activity is

$$\left(\begin{pmatrix} \hat{\mathbf{v}}^{-r} \\ f(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) \end{pmatrix}, \mathbf{x} \right).$$

Accordingly, the quantities of inputs demanded can be deduced without the convexity of the input requirement set $L(\mathbf{x})$ from a given (concave) cost function $\tilde{c}(\cdot, \mathbf{x})$ provided $L(\mathbf{x})$ is integer convex. Without [L7a] (*Integer Convexity*) the existence of a cost minimizing input vector $\hat{\mathbf{v}} \in L(\mathbf{x})$ is not guaranteed but not all of the points $\binom{\mathbf{v}^{-r}}{f(\mathbf{v}^{-r}|L(\mathbf{x}))} \in \mathrm{conv}L(\mathbf{x}) \cap V$ with $\mathbf{v}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$ have to be possible activities $\mathbf{v} \in L(\mathbf{x})$. Such a case is illustrated by point B in the right hand part of Figure II.18, p. 32.

[44] Each vector $\hat{\mathbf{v}}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$ determines a pair of dual points $(\hat{\mathbf{q}}^{-r}, \hat{\mathbf{v}}^{-r})$ such that furthermore $\hat{\mathbf{q}}^{-r} \in \Delta g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x}))$ is satisfied. At the same time the supergradient $\hat{\mathbf{q}}^{-r}$ denotes a subgradient of the function $f(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x}))$, i.e. $-\hat{\mathbf{q}}^{-r} \in \partial f(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x}))$. Thus, by (D.3) we have the same support function

$$h(\mathbf{v}^{-r}, \hat{\mathbf{v}}^{-r}) = f(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) - (\mathbf{q}^{-r})^{\mathrm{T}}(\mathbf{v}^{-r} - \hat{\mathbf{v}}^{-r})$$

for each $\hat{\mathbf{v}}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$, where $\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}) = (\hat{\mathbf{q}}^{-r})^{\mathrm{T}}(\hat{\mathbf{v}}^{-r}) + f(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x}))$ must be taken into account.

[45] A supporting hyperplane H of $\mathrm{conv}L(\mathbf{x})$ is said to be nontrivial if $\mathrm{conv}L(\mathbf{x}) \not\subset H$. This relation is fulfilled in every case since $\mathrm{conv}L(\mathbf{x})$ is m-dimensional and H is an $m-1$-dimensional hyperplane.

Example: *Finally, we again fall back upon the example in Figure III.4, where the input requirement set*

$$L(\tilde{x}) = \left\{ \binom{v_1}{v_2} \in \mathbb{Z}_+^2 \mid \binom{v_1}{v_2} \geqq \binom{v_1'}{v_2'} \text{ or } \binom{v_1}{v_2} \geqq \binom{v_1''}{v_2''} \right\} \tag{III.41}$$

has been modified so that it suffices Axiom [L2] (*Attainability of Each Production*)*; see Quadrant IV in Figure III.12.*[46]

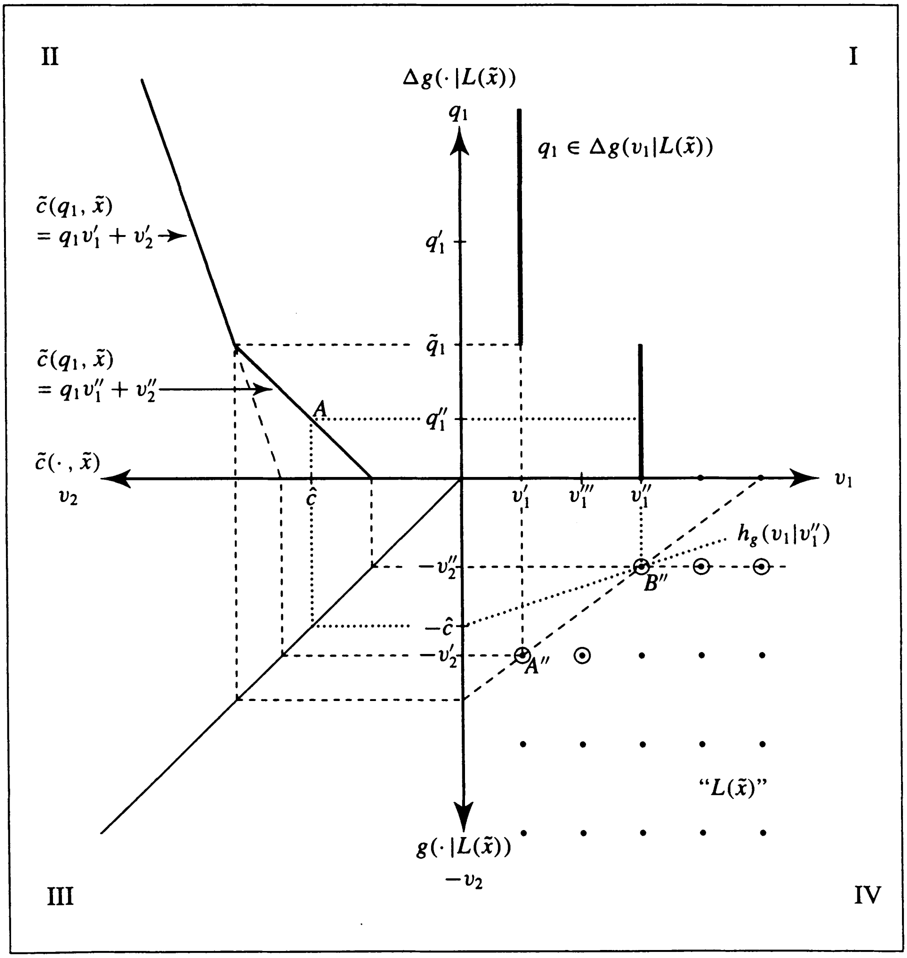

Figure III.12: Derivation of the cost function $\tilde{c}(\cdot\,, \tilde{x})$ from the function $g(\cdot\,|L(\tilde{x}))$

Again the normalized price q_1^{-r} of the first input (with $r = 2$) is denoted by q_1 within the example. Now we have to determine the smallest amount of factor 2

[46]Since $v_2 \leqq 0$ in Quadrant IV corresponds to the downward reflected input requirement set, $L(\tilde{x})$ is put in quotation marks.

which suffices for the production of the output $\tilde{x}$ at the given amount of input 1.[47] *The concluding change of signs yields*

$$\text{(III.42)} \qquad g(v_1|L(\tilde{x})) = -\inf\left\{v_2 \middle| \tbinom{v_1}{v_2} \in L(\tilde{x})\right\}$$
$$= \begin{cases} -v_2' & \text{for } v_1 = v_1' \text{ or } v_1 = v_1''' \\ -v_2'' & \text{for } v_1 \geqq v_1'' \text{ and } v_1 \in \mathbb{Z}_+ \\ -\infty & \text{otherwise,} \end{cases}$$

with the graph of this function being depicted in Quadrant IV by points of the form $\odot$.

Now the question can be answered what factor price must prevail such that a certain quantity of factor 1 is chosen. According to the inequality for a supergradient on p. 104, the input v_1' is chosen if the appropriately chosen factor price q_1 satisfies the inequality

$$q_1 v_1' - g(v_1'|L(\tilde{x})) \leqq q_1 v_1 - g(v_1|L(\tilde{x})) \qquad \forall\, v_1 \in \mathbb{Z}_{++}.$$

In view of (III.42) we have $g(v_1'|L(\tilde{x})) = -v_2'$ so that two inequalities result:

$$\begin{aligned} q_1 v_1' + v_2' &\leqq q_1 v_1 + v_2' && \text{for } v_1 = v_1' \text{ or } v_1 = v_1''' ; \\ q_1 v_1' + v_2' &\leqq q_1 v_1 + v_2'' && \text{for } v_1 \geqq v_1'' \text{ and } v_1 \in \mathbb{Z}_+. \end{aligned}$$

The first inequality only requires the factor price q_1 to be nonnegative. The second inequality induces the strongest restriction on $v_1 = v_1''$. Hence, the quantity v_1' is chosen when the factor price q_1 holds

$$\text{(III.43)} \qquad q_1 \geqq \frac{v_2' - v_2''}{v_1'' - v_1'} =: \tilde{q}_1 .$$

This property is symbolically indicated by $\Delta g(v_1'|L(\tilde{x})) = \{q_1 |\, q_1 \geqq \tilde{q}_1\}$ although in the strict sense the superdifferential $\Delta g(\cdot|L(\tilde{x}))$ is only defined for a convex function $g(\cdot|L(\tilde{x}))$; see Quadrant I of Figure III.12.

At point v_1''' the analogous inequalities

$$\begin{aligned} q_1 v_1''' + v_2' &\leqq q_1 v_1 + v_2' && \text{for } v_1 = v_1' \text{ or } v_1 = v_1''' \\ q_1 v_1''' + v_2' &\leqq q_1 v_1 + v_2'' && \text{for } v_1 \geqq v_1'' \text{ and } v_1 \in \mathbb{Z}_+ \end{aligned}$$

yield a contradiction.[48] *Thus, there is no price q_1 at which the quantity v_1''' is chosen, $\Delta g(v_1'''|L(\tilde{x})) = \emptyset$. Finally, one has to examine at which factor prices the quantity v_1'' is chosen. The necessary condition is now*

$$q_1 v_1'' - g(v_1''|L(\tilde{x})) \leqq q_1 v_1 - g(v_1|L(\tilde{x})) \qquad \forall\, v_1 \in \mathbb{Z}_{++}.$$

---

[47] See (III.9), p. 90.

[48] Because of $v_1' < v_1''$, the first inequality implies a negative factor price $q_1 < 0$. Hence the second inequality cannot be satisfied for $v_1 = v_1''$.

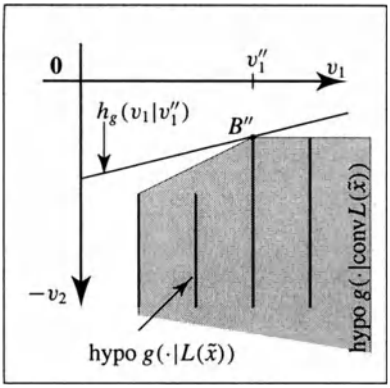

Figure III.13: Hypograph of the function $g(\cdot|L(\tilde{x}))$

Without investigating the resulting inequalities in more detail, for the factor price q_1'' the rearrangement

$$g(v_1|L(\tilde{x})) \leqq g(v_1''|L(\tilde{x})) - q_1''v_1'' + q_1''v_1 =: h_g(v_1|v_1'')$$

yields an affine function $h_g(\cdot|v_1'')$ touching the hypograph of the function $g(\cdot|L(\tilde{x}))$ as depicted in Figure III.13 at point v_1'' or more precisely at point B''. The remarks on Figure III.9, p. 104, have shown that the implied price vector $\binom{-q_1''}{1}$ is normal to the graph of the function $h_g(\cdot|v_1'')$.

Ignoring the relevant points $v_1 \geqq v_1''$, we get the correspondence depicted in Quadrant I of Figure III.12,[49]

$$\text{(III.44)} \qquad \Delta g(v_1|L(\tilde{x})) = \begin{cases} \{q_1|\, q_1 \geqq \tilde{q}_1\} & \text{for } v_1 = v_1' \\ \{q_1|\, 0 \leqq q_1 \leqq \tilde{q}_1\} & \text{for } v_1 = v_1'' \\ \{0\} & \text{for } v_1 > v_1'' \text{ and } v_1 \in \mathbb{Z}_+ \\ \emptyset & \text{otherwise.} \end{cases}$$

The connection between the four quadrants in Figure III.12 is explained afterwards in more detail.

Starting with the modified cost function $\tilde{c}(\cdot, \tilde{x})$ in accordance with (III.21)

$$\text{(III.45)} \qquad \tilde{c}(q_1, \tilde{x}) = \begin{cases} q_1v_1' + v_2' & \text{for } \tilde{q}_1 \leqq q_1 \\ q_1v_1'' + v_2'' & \text{for } 0 \leqq q_1 \leqq \tilde{q}_1 \\ -\infty & \text{for } \quad q_1 < 0\,, \end{cases}$$

we now have to calculate the superdifferential $\Delta\tilde{c}(\cdot, \tilde{x})$. By (III.33) this superdifferential includes all quantities of factor 1 solving the problem of cost minimization for a given factor price q_1. Analogous to the previous procedure, the representation is now given by Figure III.14. This figure is also explained in more detail later.

Figure III.5 suggests inspecting the factor price $q_1 = \tilde{q}_1$ given by (III.43) in more detail. Each factor quantity $\tilde{v}_1$ denotes a supergradient at point $\tilde{q}_1$ if $\tilde{v}_1$ satisfies

$$\tilde{c}(q_1, \tilde{x}) \leqq \tilde{c}(\tilde{q}_1, \tilde{x}) + \tilde{v}_1(q_1 - \tilde{q}_1) \qquad \forall q_1 \in \mathbb{R}.$$

Considering the cost function (III.45) as depicted in Quadrant II of Figures III.12 and III.14 respectively, it ensues

$$\left\{\begin{matrix} q_1v_1' + v_2' & \text{for} & \tilde{q}_1 \leqq q_1 \\ q_1v_1'' + v_2'' & \text{for} & 0 \leqq q_1 \leqq \tilde{q}_1 \\ -\infty & \text{for} & q_1 < 0 \end{matrix}\right\} \leqq \tilde{q}_1v_1' + v_2' + \tilde{v}_1(q_1 - \tilde{q}_1).$$

[49] An alternative possibility of representation has already been presented in Figure III.10.

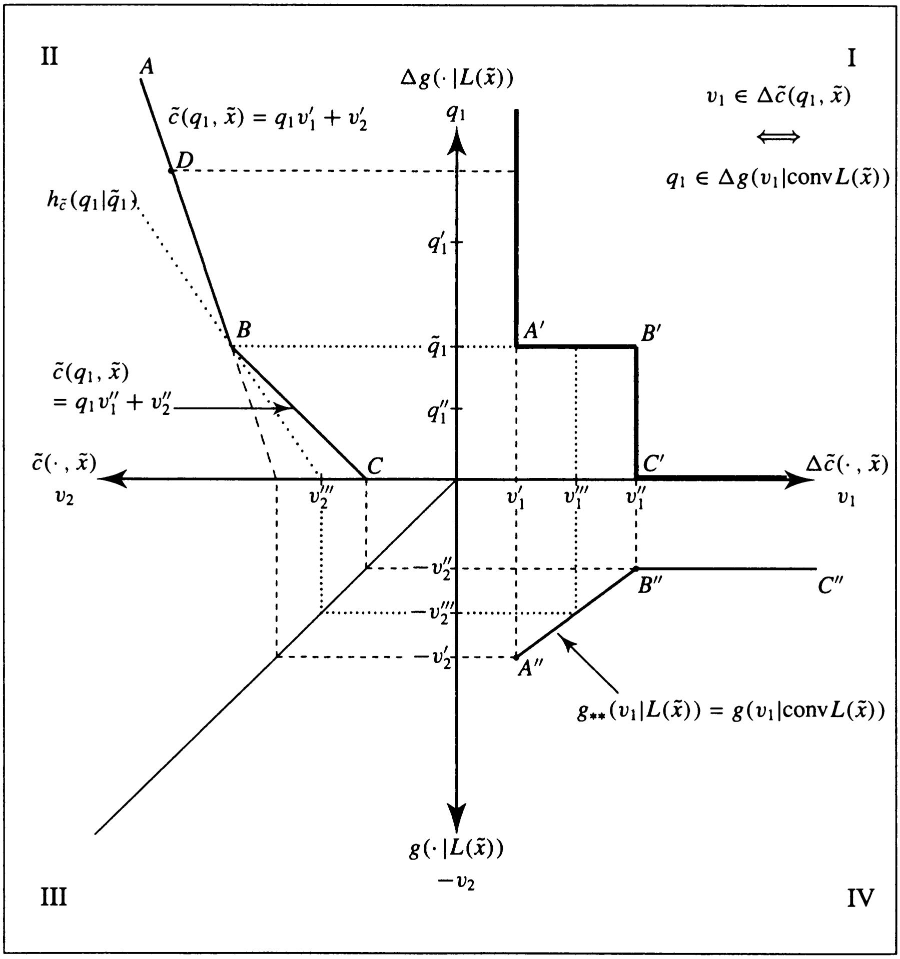

Figure III.14: Reconstruction of the concave biconjugate function $g_{**}(\cdot\,|L(\tilde{x}))$ from the cost function $\tilde{c}(\cdot\,, \tilde{x})$

In the case of $\tilde{q}_1 \leqq q_1$ *we get*

(III.46) $$q_1(v_1' - \tilde{v}_1) \leqq \tilde{q}_1(v_1' - \tilde{v}_1)$$

and this inequality is satisfied for all $\tilde{v}_1 \geqq v_1'$. *If we consider* $\tilde{q}_1 v_1' + v_2' = \tilde{q}_1 v_1'' + v_2''$ *according to (III.21), then it follows analogously for the second case with* $0 \leqq q_1 \leqq \tilde{q}_1$

$$q_1(v_1'' - \tilde{v}_1) \leqq \tilde{q}_1(v_1'' - \tilde{v}_1).$$

This inequality is satisfied for all $\tilde{v}_1 \leqq v_1''$. *Since a negative factor price* $q_1 < 0$

generates no further restrictions with respect to a supergradient, we obtain

$$\Delta\tilde{c}(\tilde{q}_1, \tilde{x}) = \left\{\tilde{v}_1 \mid v_1' \leqq \tilde{v}_1 \leqq v_1''\right\}$$

by summarizing the superdifferential. A representation of the superdifferential analogous to Figure III.10 is omitted at this point.

For a factor price $q_1' > \tilde{q}_1$ *a unique supergradient* $\tilde{v}_1 = v_1'$ *results directly from the above inequality (III.46). Thus, the cost function (III.21) is differentiable at this point (Theorem D.4).*

$$\frac{\mathrm{d}\tilde{c}(q_1', \tilde{x})}{\mathrm{d}q_1} = v_1'$$

Accordingly, $\mathrm{d}\tilde{c}(q_1'', \tilde{x})/\mathrm{d}q_1 = v_1''$ *holds for* $0 < q_1'' < \tilde{q}_1$. *The derived superdifferential*

$$\Delta\tilde{c}(q_1, \tilde{x}) = \begin{cases} \{v_1'\} & \text{for } q_1 > \tilde{q}_1 \\ \{v_1 \mid v_1' \leqq v_1 \leqq v_1''\} & \text{for } q_1 = \tilde{q}_1 \\ \{v_1''\} & \text{for } \tilde{q}_1 > q_1 > 0 \\ \{v_1 \mid v_1 \geqq v_1''\} & \text{for } q_1 = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

is again depicted in Quadrant I (Figure III.14).

Supposing the input requirement set $L(\tilde{x})$ *is integer convex with respect to the factor space* $V = \mathbb{Z}_+^2$, *then by (III.35)*

$$\Delta\tilde{c}(\tilde{q}_1, \tilde{x}) \cap V^{-2} = \left\{v_1', v_1''', v_1''\right\} \qquad \text{with} \quad V^{-2} = \mathbb{Z}_+ .$$

The demand set $\tilde{D}(\tilde{q}_1, \tilde{x})$ *consists of two elements* v_1' *and* v_1'' *since*

$$\begin{pmatrix} v_1' \\ \operatorname{conv} f(v_1' \mid L(\tilde{x})) \end{pmatrix} = \begin{pmatrix} v_1' \\ f(v_1' \mid L(\tilde{x})) \end{pmatrix} \quad \text{is chosen;}$$

$$\begin{pmatrix} v_1''' \\ \operatorname{conv} f(v_1''' \mid L(\tilde{x})) \end{pmatrix} \neq \begin{pmatrix} v_1''' \\ f(v_1''' \mid L(\tilde{x})) \end{pmatrix} \quad \text{is not chosen;}$$

$$\begin{pmatrix} v_1'' \\ \operatorname{conv} f(v_1'' \mid L(\tilde{x})) \end{pmatrix} = \begin{pmatrix} v_1'' \\ f(v_1'' \mid L(\tilde{x})) \end{pmatrix} \quad \text{is chosen.}$$

see points A'' *and* B'' *in Figures III.12 or III.14.*

Finally, we derive the concave biconjugate function $g_{**}(\cdot \mid L(\tilde{x}))$ *from the cost function* $\tilde{c}(\cdot, \tilde{x})$ *by (III.45) as illustrated in Quadrant IV of Figure III.14. This function denotes the smallest concave function with* $g_{**}(\cdot \mid L(\tilde{x})) \geqq g(\cdot \mid L(\tilde{x}))$. *Moreover,* $g_{**}(\cdot \mid L(\tilde{x})) = g(\cdot \mid \operatorname{conv} L(\tilde{x}))$ *holds. Thus,* $-g_{**}(\cdot \mid L(\tilde{x}))$ *is the* *nonvertical boundary of the convex hull of the input requirement set* $L(\tilde{x})$.

(III.47) $$g_{**}(v_1|L(\tilde{x})) = \inf\{q_1 v_1 - \tilde{c}(q_1,\tilde{x})|\ q_1 \in \mathbb{R}\}$$

[a] $$= \inf_{q_1\in\mathbb{R}} \left\{ q_1 v_1 - \left\{ \begin{array}{ll} q_1 v_1' + v_2' & \text{for } \tilde{q}_1 \leqq q_1 \\ q_1 v_1'' + v_2'' & \text{for } 0 \leqq q_1 \leqq \tilde{q}_1 \\ -\infty & \text{for } \quad q_1 < 0 \end{array} \right\} \right\}$$

[b] $$= \inf_{q_1\in\mathbb{R}} \left\{ \begin{array}{ll} q_1(v_1 - v_1') - v_2' & \text{for } \tilde{q}_1 \leqq q_1 \\ q_1(v_1 - v_1'') - v_2'' & \text{for } 0 \leqq q_1 \leqq \tilde{q}_1 \end{array} \right\}$$

[c] $$= \begin{cases} -\infty & \text{for } \quad v_1 < v_1' \\ \tilde{q}_1(v_1 - v_1') - v_2' & \text{for } v_1' \leqq v_1 \leqq v_1'' \\ -v_2'' & \text{for } v_1'' < v_1 \end{cases}$$

By the transition from [a] to [b] it is considered that no $q_1 < 0$ *can solve the problem. All three cases in [c] result from the following observations:*
For $v_1 < v_1'$ *the upper part of [b] is unbounded below* $(q_1 \to +\infty)$.
For $v_1 > v_1''$ *both parts of [b] are bounded. Each of the optimal values for* q_1 *give* $\tilde{q}_1(v_1 - v_1') - v_2'$ *for the upper part of [b] and* $-v_2''$ *for the lower part. Regarding the definition of* $\tilde{q}_1$, $q_1 = 0$ *turns out to be an optimal solution with the functional value* $-v_2''$.
Even for $v_1' \leqq v_1 \leqq v_1''$ *both parts of [b] are finite. Due to* $v_1' \leqq v_1$, *we choose the smallest possible value of* q_1 *in the upper part of [b]. Analogously, as* $v_1'' \geqq v_1$ *we choose the highest possible value of* q_1 *in the lower part of [b]. Calculation shows that the optimal solution* $\tilde{q}_1$ *gives the same value of the objective function in both cases.*

$$\tilde{q}_1(v_1 - v_1') - v_2' = \tilde{q}_1(v_1 - v_1'') - v_2''$$

In particular, we have

$$\begin{array}{rl} \textit{for } v_1 = v_1': & g_{**}(v_1'|L(\tilde{x})) = -v_2' \\ \textit{and for } v_1 = v_1'': & g_{**}(v_1''|L(\tilde{x})) = -v_2''. \end{array}$$

1.4.3 The Results under the Assumption of Differentiability

The following expositions begin with the inverse superdifferentials.

(III.48) $$\hat{\mathbf{q}}^{-r} \in \Delta g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) \iff \hat{\mathbf{v}}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r},\mathbf{x})$$

The relation on the right hand holds good when the vector $\hat{\mathbf{v}}^{-r}$ solves the problem of cost minimization at given normalized factor prices $\hat{\mathbf{q}}^{-r}$. Moreover, if $\hat{\mathbf{v}}^{-r}$ satisfies the integer constraints, then it is a cost minimizing *input* vector. For convex input requirement sets with $L(\mathbf{x}) = \mathrm{conv}L(\mathbf{x})$ the superdifferential of the cost function equals the factor demand correspondence:

$$\hat{\mathbf{v}}^{-r} \in \Delta\tilde{c}(\hat{\mathbf{q}}^{-r},\mathbf{x}) \iff \hat{\mathbf{v}}^{-r} \in \tilde{D}(\hat{\mathbf{q}}^{-r},\mathbf{x}).$$

The *super*differential on the left of (III.48) notes the inverse relation: the given vector $\hat{\mathbf{v}}^{-r}$ yields a cost minimum if the factor prices $\hat{\mathbf{q}}^{-r}$ prevail. In view of $g(\cdot|\mathrm{conv}L(\mathbf{x})) = -f(\cdot|\mathrm{conv}L(\mathbf{x}))$ an equivalent *sub*differential can be established:

$$\hat{\mathbf{q}}^{-r} \in \Delta g(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})) \iff -\hat{\mathbf{q}}^{-r} \in \partial f(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x})).$$

Analogous to the preceding graphical representations, the price vector $\binom{\hat{\mathbf{q}}^{-r}}{1}$ is normal to the hyperplane H tangent to the convex hull of the input requirement set $L(\mathbf{x})$ at the point $\binom{\hat{\mathbf{v}}^{-r}}{f(\hat{\mathbf{v}}^{-r}|\mathrm{conv}L(\mathbf{x}))}$.

For each pair of dual points $(\hat{\mathbf{q}}^{-r}, \hat{\mathbf{v}}^{-r})$ which satisfies the equivalence relation (III.48) Theorem D.4, p. 319, states: If the function $f(\cdot|L(\mathbf{x}))$ is differentiable at point $\hat{\mathbf{v}}^{-r}$ and if the cost function $\tilde{c}(\cdot, \mathbf{x})$ is differentiable at point $\hat{\mathbf{q}}^{-r}$, then

$$\text{(III.49)} \qquad -\hat{\mathbf{q}}^{-r} = \nabla f(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) \iff \hat{\mathbf{v}}^{-r} = \nabla \tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x}).$$

The right hand equation of this equivalence relation is known as **SHEPHARD's Lemma**: the partial derivative of the normed cost function with respect to the price of factor i yields the quantity of this factor demanded.

$$\text{(III.50a)} \qquad \hat{v}_i = \frac{\partial \tilde{c}}{\partial q_i^{-r}}(\hat{\mathbf{q}}^{-r}, \mathbf{x}) \qquad i = 1, \ldots, m;\ i \neq r$$

The left hand side of (III.49) is the so called **HICKS-ALLEN relation**. The relative price $\hat{q}_i^{-r}$ of factor i or the relation of nominal prices $\hat{q}_i/\hat{q}_r$ $(i = 1, \ldots, m;\ i \neq r)$ equals the marginal rate of (technical) substitution at the optimum.

$$\text{(III.50b)} \qquad \frac{\hat{q}_i}{\hat{q}_r} = -\frac{\partial f}{\partial v_i}(\hat{\mathbf{v}}^{-r}|L(\mathbf{x})) \qquad i = 1, \ldots, m;\ i \neq r$$

At the same time the (strict) convexity of $f(\cdot|L(\mathbf{x}))$ assures that each marginal rate of substitution $\partial f(\cdot|L(\mathbf{x}))/\partial v_i$ is a nondecreasing (increasing) function in v_i.

Provided the production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$ consists exclusively of closed convex input requirement sets, then the generalized HICKS-ALLEN relation $\hat{\mathbf{q}}^{-r} \in \Delta g(\hat{\mathbf{v}}^{-r}|L(\mathbf{x}))$ and the generalized HOTELLING-SHEPHARD Lemma of the form $\hat{\mathbf{v}}^{-r} \in \tilde{D}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$ are logically equivalent.

A comparison to SHEPHARD's Theorem (Proposition III.19, p. 164) shows that (III.50a) can again be found in the demand functions (III.100a). But the HICKS-ALLEN condition (III.50b) does *not* correspond to SHEPHARD's dual price demand functions (III.100b). This reflects the main difference between the duality schemes presented in the introduction of this section. While (III.50a) and (III.50b) or (III.49) result from the duality of conjugate functions, (III.100a) and (III.100b) are implications of the duality of polar gauges.

To answer the question when the requirements of equivalence relation (III.49) are fulfilled we revert to the concept of an exposed point.[50] A point $\mathbf{x}$ in the convex set $C \subset \mathbb{R}^n$ has been called an **exposed point of C** when a nontrivial supporting hyperplane[51] H exists such that $C \cap H = \{\mathbf{x}\}$.

[50] For the distinction between extreme points and exposed points, see Figure II.11, p. 26.

[51] See Appendix B. If a set $C \subset \mathbb{R}^3$ consists of a solitary point, then this point is not said to be an exposed point.

Now, an implication of Theorem D.4 is noted by

Corollary D.4.1[52] *Let* $k\colon \mathbb{R}^n \to [-\infty, +\infty]$ *be an n-proper concave function. A point* $\binom{\mathbf{y}}{\mu}$ *is an exposed point of the (closed convex) set* $\operatorname{hypo} k_* \subset \mathbb{R}^{n+1}$ *if and only if a point* $\mathbf{x}$ *exists at which the function k is differentiable with* $\nabla k(\mathbf{x}) = \mathbf{y}$. *Each of these exposed points is a point of the form* $\binom{\mathbf{y}}{k_*(\mathbf{y})}$.

The assumptions of this corollary hold especially for the function $g(\cdot|\operatorname{conv}L(\mathbf{x}))$, the cost function $\tilde{c}(\cdot, \mathbf{x}) \equiv g_*(\cdot|L(\mathbf{x}))$, and the function $g_{**}(\cdot|L(\mathbf{x}))$. As an example we now discuss the relation between exposed points of the convex hull of a nonempty input requirement set $\operatorname{conv}L(\mathbf{x})$ and the differentiability of the cost function $\tilde{c}(\cdot, \mathbf{x})$. In view of the corollary we have to replace the function k with the cost function $\tilde{c}(\cdot, \mathbf{x}) \equiv g_*(\cdot|L(\mathbf{x}))$ and the hypograph $\operatorname{hypo} k_*$ by $\operatorname{hypo} g_{**}(\cdot|L(\mathbf{x}))$. Regarding $g_{**}(\cdot|L(\mathbf{x})) = g(\cdot|\operatorname{conv}L(\mathbf{x}))$, we may refer to point B'' in Figure III.13 as an example for an exposed point of the set $\operatorname{hypo} g_{**}(\cdot|L(\mathbf{x}))$.

As already mentioned, the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ corresponds to an n-proper concave function on $\mathcal{Q} = \mathbb{R}^{m-1}$. Thus, the corollary says that a point $\binom{\mathbf{v}^{-r}}{\mu}$ is an exposed point of the hypograph $\operatorname{hypo} g_{**}(\cdot|L(\mathbf{x}))$ if and only if a factor price vector $\mathbf{q}^{-r}$ exists at which the cost function $\tilde{c}(\cdot, \mathbf{x})$ is differentiable, i.e.

$$\nabla\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) = \mathbf{v}^{-r}.$$

Like point B'' in Figure III.13 each of these exposed points is of the special form $\binom{\mathbf{v}^{-r}}{g_{**}(\mathbf{v}^{-r}|L(\mathbf{x}))}$ which by (III.31) can be rewritten as

$$\begin{pmatrix} \mathbf{v}^{-r} \\ f(\mathbf{v}^{-r}|\operatorname{conv}L(\mathbf{x})) \end{pmatrix} \in \operatorname{conv}L(\mathbf{x}).$$

Since each exposed point of $\operatorname{conv}L(\mathbf{x})$ is at the same time an extreme point of this convex hull, we have furthermore

$$\begin{pmatrix} \mathbf{v}^{-r} \\ f(\mathbf{v}^{-r}|\operatorname{conv}L(\mathbf{x})) \end{pmatrix} \in L(\mathbf{x}).$$

As shown in the derivation of (III.35), this relation can only be satisfied under the condition $f(\mathbf{v}^{-r}|L(\mathbf{x})) = f(\mathbf{v}^{-r}|\operatorname{conv}L(\mathbf{x}))$. Hence, we conclude

$$\nabla\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) = \mathbf{v}^{-r} \quad \Longrightarrow \quad \begin{pmatrix} \mathbf{v}^{-r} \\ f(\mathbf{v}^{-r}|L(\mathbf{x})) \end{pmatrix} \in L(\mathbf{x}).$$

If the cost function is differentiable at a point $\mathbf{q}^{-r}$, then the gradient $\mathbf{v}^{-r}$ yields an admissible input vector in the input requirement set $L(\mathbf{x})$ at this point together with $f(\mathbf{v}^{-r}|L(\mathbf{x}))$. For the case of indivisible production factors a graphical illustration of this statement ensues from the two exposed points A'' and B'' in Quadrant IV of Figure III.14. Point B'' with the coordinates $\binom{v_1''}{f(v_1''|L(\bar{x}))}$ is also explicitly marked in Figure III.13.

[52] Cf. ROCKAFELLAR (1972, p. 243, Corollary 25.1.2). The lemma has been transferred to n-proper concave functions.

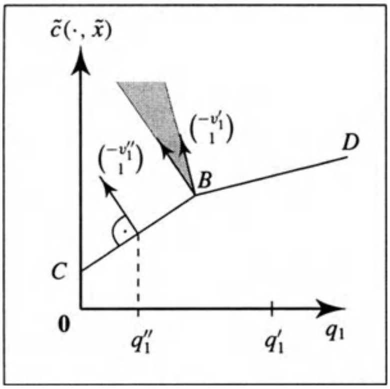

Figure III.15: Differentiability of the cost function $\tilde{c}(\cdot,\tilde{x})$

Similarly, Figure III.15 includes the cost function $\tilde{c}(\cdot,\tilde{x})$ of Quadrant II. The construction of the vector $\binom{-v_1''}{1}$ by the derivative of the cost function at point q_1'', i.e. $d\tilde{c}(q_1'',\tilde{x})/dq_1 = v_1''$, is known from Figure III.9, p. 104.

For strictly convex (and closed) input requirement sets each point of the form $\binom{\mathbf{v}^{-r}}{f(\mathbf{v}^{-r}|L(\mathbf{x}))} \in L(\mathbf{x})$ is an exposed point of the input requirement set $L(\mathbf{x})$. In this case we can show that the cost function $\tilde{c}(\cdot,\mathbf{x})$ is differentiable in the entire interior of its effective domain $\operatorname{int}\big(\text{n-Dom}\,\tilde{c}(\cdot,\mathbf{x})\big)$ – i.e. for every price vector $\mathbf{q}^{-r} > \mathbf{0}$. Thus, each superdifferential

$$\Delta\tilde{c}(\mathbf{q}^{-r},\mathbf{x}) \quad \text{with} \quad \mathbf{q}^{-r} \in \operatorname{int}\big(\text{n-Dom}\,\tilde{c}(\cdot,\mathbf{x})\big) = \mathbb{R}^{m-1}_{++}$$

is not empty and contains no more than one – i.e. one – point.

1.5 Summary

1.5.1 Graphical Representation of the Results

This section explains the theoretical results and their connection using four figures. Again we distinguish between three cases with respect to the input requirement sets.

1. Figure III.8, p. 101, illustrates the case of a strictly convex input requirement set, provided the differentiability of the two conjugate functions.
2. The convex input requirement set in Figure III.16, p. 125, has a sectional linear boundary with kink points.
3. The case of indivisible production factors is shown in Figures III.12 and III.14.

All of the figures illustrate the same facts in the respective quadrants.

1. Quadrant IV serves as an initial point and illustrates the given input requirement set $L(\mathbf{x})$. Remember that

$$g(\mathbf{v}^{-r}|L(\mathbf{x})) = -f(\mathbf{v}^{-r}|L(\mathbf{x})) = -\inf\{v_r|\mathbf{v} \in L(\mathbf{x})\}$$

so $g(\cdot|L(\mathbf{x}))$ is measured in units of factor r.

2. Similarly, Quadrant II contains the graph of the corresponding normalized cost function

$$\tilde{c}(\mathbf{q}^{-r},\mathbf{x}) \equiv g_*(\mathbf{q}^{-r}|L(\mathbf{x})) = \inf\left\{(\mathbf{q}^{-r})^{\mathrm{T}}(\mathbf{v}^{-r}) - g(\mathbf{v}^{-r}|L(\mathbf{x}))|\ \mathbf{v}^{-r} \in \mathcal{V}\right\}.$$

Remember again, that $\tilde{c}(\cdot,\mathbf{x})$ is measured in units of factor r.

3. Quadrant I presents the linkage between the concave conjugate functions and illustrates the superdifferentials of both functions.

$$\mathbf{q}^{-r} \in \Delta g(\mathbf{v}^{-r} | \mathrm{conv} L(\mathbf{x})) \iff \mathbf{v}^{-r} \in \Delta \tilde{c}(\mathbf{q}^{-r}, \mathbf{x})$$

4. Quadrant III merely serves for the reflection, where the levels are reflected absolutely.

Examples: *Since it is only necessary to distinguish between two inputs v_1 and v_2, the normalized price q_1^{-r} of the first factor is denoted by q_1 in the mentioned figures.*

1. Strictly convex input requirement sets: *Given two inputs with $r = 2$ and the output quantity $\tilde{x}$, the six equivalent relations (III.27a)–(III.27f) for the pair of dual points $(\hat{q}_1, \hat{v}_1)$ are*

(III.51a) $\hat{q}_1 v_1 - g(v_1 | L(\tilde{x}))$ *achieves the infimum at* $v_1 = \hat{v}_1$;

(III.51b) $g(\hat{v}_1 | L(\tilde{x})) + \tilde{c}(\hat{q}_1, \tilde{x}) = \hat{q}_1 \hat{v}_1$;

(III.51c) $\hat{q}_1 = \nabla g(\hat{v}_1 | L(\tilde{x}))$;

(III.51d) $\hat{v}_1 q_1 - \tilde{c}(q_1, \tilde{x})$ *achieves the infimum at* $q_1 = \hat{q}_1$;

(III.51e) $g_{**}(\hat{v}_1 | L(\tilde{x})) + \tilde{c}(\hat{q}_1, \tilde{x}) = \hat{q}_1 \hat{v}_1$;

(III.51f) $\hat{v}_1 = \nabla \tilde{c}(\hat{q}_1, \tilde{x})$.

The first three conditions are reflected by the tangent point in Quadrant IV of Figure III.8, where (III.51a) is explicitly shown in Figure III.7, p. 100. For the cost minimizing input $\hat{v}_1$ (III.51a) evokes by definition $\tilde{c}(\hat{q}_1, \tilde{x}) = \hat{q}_1 \hat{v}_1 - g(\hat{v}_1 | L(\tilde{x}))$ and, therefore, (III.51b). If $\hat{v}_2$ denotes the smallest amount of factor 2 which suffices for the production of the output $\tilde{x}$ at the given input $\hat{v}_1$, then $-\hat{v}_2 = g(\hat{v}_1 | L(\tilde{x}))$. Now we can linearize[53] the function $g(\cdot | L(\tilde{x}))$ at point $\hat{v}_1$,

$$h_g(v_1 | \hat{v}_1) := -\hat{v}_2 + \hat{q}_1 (v_1 - \hat{v}_1),$$

where the gradient $\hat{q}_1$ follows from (III.51c). Due to (III.51b) we have at the same time

$$-\hat{c} := h_g(0 | \hat{v}_1) = -\hat{v}_2 - \hat{q}_1 \hat{v}_1 = -\tilde{c}(\hat{q}_1, \tilde{x}).$$

That concludes the discussion of the point of tangency in Quadrant IV of Figure III.8 and we can turn to the equivalent tangent point in Quadrant II, which is essentially characterized by the conditions (III.51d)–(III.51f).

Analogously, the cost function $\tilde{c}(\cdot, \tilde{x})$ is linearized at point $\hat{q}_1$. Considering $\hat{c} = \tilde{c}(\hat{q}_1, \tilde{x})$, the gradient $\hat{v}_1$ in (III.51f) yields the function

$$h_{\tilde{c}}(q_1 | \hat{q}_1) := \hat{c} + \hat{v}_1 (q_1 - \hat{q}_1).$$

[53]The linearization of the function $g(\cdot | L(\tilde{x}))$ at point $\hat{v}_1$ is given in detail by (D.3) $h_g(v_1 | \hat{v}_1) = g(\hat{v}_1 | L(\tilde{x})) + \nabla g(\hat{v}_1 | L(\tilde{x}))(v_1 - \hat{v}_1)$.

For $q_1 = 0$ *(III.51e) implies*

$$h_{\tilde{c}}(0|\hat{q}_1) = \hat{c} - \hat{v}_1\hat{q}_1 = g_{**}(\hat{v}_1|L(\tilde{x})) = g(\hat{v}_1|L(\tilde{x})) = \hat{v}_2 .$$

The inverse superdifferentials $\Delta g(\cdot|L(\tilde{x}))$ *and* $\Delta\tilde{c}(\cdot, \tilde{x})$ *reflect the inverse* ***functions*** *by (III.51c) and (III.51f)*

(III.52) $$\hat{v}_1 = \sqrt{\tilde{x}/\hat{q}_1} \iff \hat{q}_1 = \tilde{x}/(\hat{v}_1)^2;$$

see Quadrant I of Figure III.8.

Given the n*-proper concave closed function* $g(\cdot|L(\tilde{x}))$*, the symmetry of the* FENCHEL *transform (*FENCHEL, MOREAU*) is expressed as*

$$g_{**}(\cdot|L(\tilde{x})) = g(\cdot|L(\tilde{x})).$$

Finally, it has to be mentioned that the point $\binom{\hat{v}_1}{g(\hat{v}_1|L(\tilde{x}))}$ *is an exposed point of the hypograph* hypo $g(\cdot|L(\tilde{x}))$ *if and only if the cost function* $\tilde{c}(\cdot, \tilde{x})$ *is differentiable at* $\hat{q}_1$*. The supposed differentiability of both functions over the interior of its effective domain is reflected by the fact that the demand function in (III.52) is invertible.*

2. Convex input requirement sets: *Figure III.16 abandons the assumption of a strictly convex input requirement set. The set* $L(\tilde{x})$ *now corresponds to a polehydral convex set. The concave conjugate cost function of the resulting polehydral function* $g(\cdot|L(\tilde{x})) = -f(\cdot|L(\tilde{x}))$ *is again a polehydral function.*[54]

As before, the six equivalent conditions for a pair of dual points are illustrated by $(\hat{q}_1, \hat{v}_1)$*. The (unique) linearization of the function* $g(\cdot|L(\tilde{x}))$ *at point* $\hat{v}_1$ *is again denoted by* $h_g(v_1|\hat{v}_1)$*. Similarly, a (nonunique) support function* $h_{\tilde{c}}(q_1|\hat{q}_1)$ *of the cost function* $\tilde{c}(\cdot, \tilde{x})$ *at point* $\hat{q}_1$ *can be marked in Quadrant II.*

In contrast to the expositions on a strictly convex input requirement set, the cost function is not differentiable at point $\hat{q}_1$*. The superdifferential* $\Delta\tilde{c}(\hat{q}_1, \tilde{x})$ *includes all factor quantities* v_1 *lying between the points* A *and* B*. At the same time it becomes evident that the point* $\binom{\hat{v}_1}{g(\hat{v}_1|L(\tilde{x}))}$ *cannot be an exposed point of* $L(\tilde{x})$*.*

Conversely, the differentiability of the function $g(\cdot|L(\tilde{x}))$ *at point* $\hat{v}_1$ *with*

$$\Delta g(\hat{v}_1|L(\tilde{x})) = \{\nabla g(\hat{v}_1|L(\tilde{x}))\} \quad \textit{and} \quad \nabla g(\hat{v}_1|L(\tilde{x})) = \frac{\mathrm{d}g}{\mathrm{d}v_1}(\hat{v}_1|L(\tilde{x})) = \hat{q}_1 ,$$

implies that $\binom{\hat{q}_1}{\tilde{c}(\hat{q}_1, \tilde{x})}$ *is an exposed point of the hypograph* hypo $\tilde{c}(\cdot, \tilde{x})$*. The statement that a point of the form* $\binom{q_1}{\tilde{c}(q_1, \tilde{x})}$ *is an exposed point of* hypo $\tilde{c}(\cdot, \tilde{x})$ *if and only if there is a factor quantity* v_1 *at which the function* $g(\cdot|L(\tilde{x}))$ *is differentiable (et vice versa) is even more clearly emphasized in the case of indivisible factors.*

Finally, it has to be mentioned that

$$g(\cdot|L(\tilde{x})) = g_{**}(\cdot|L(\tilde{x}))$$

[54]The statement is valid for all polehydral concave functions, see ROCKAFELLAR (1972, p. 173, Theorem 19.2).

holds again for the n-proper closed concave function $g(\cdot|L(\tilde{x}))$. *The inverse superdifferentials*

$$q_1 \in \Delta g(v_1|L(\tilde{x})) \iff v_1 \in \Delta\tilde{c}(q_1, \tilde{x})$$

can be taken from Quadrant I. In particular, the vertical line segment clarifies that the factor demand correspondence $\tilde{D}(\cdot, \tilde{x}) \equiv \Delta c(\cdot, \tilde{x})$ *yields no demand function. For different factor prices* q_1 *the same input* v_1 *can be optimal in the sense of cost minimization.*

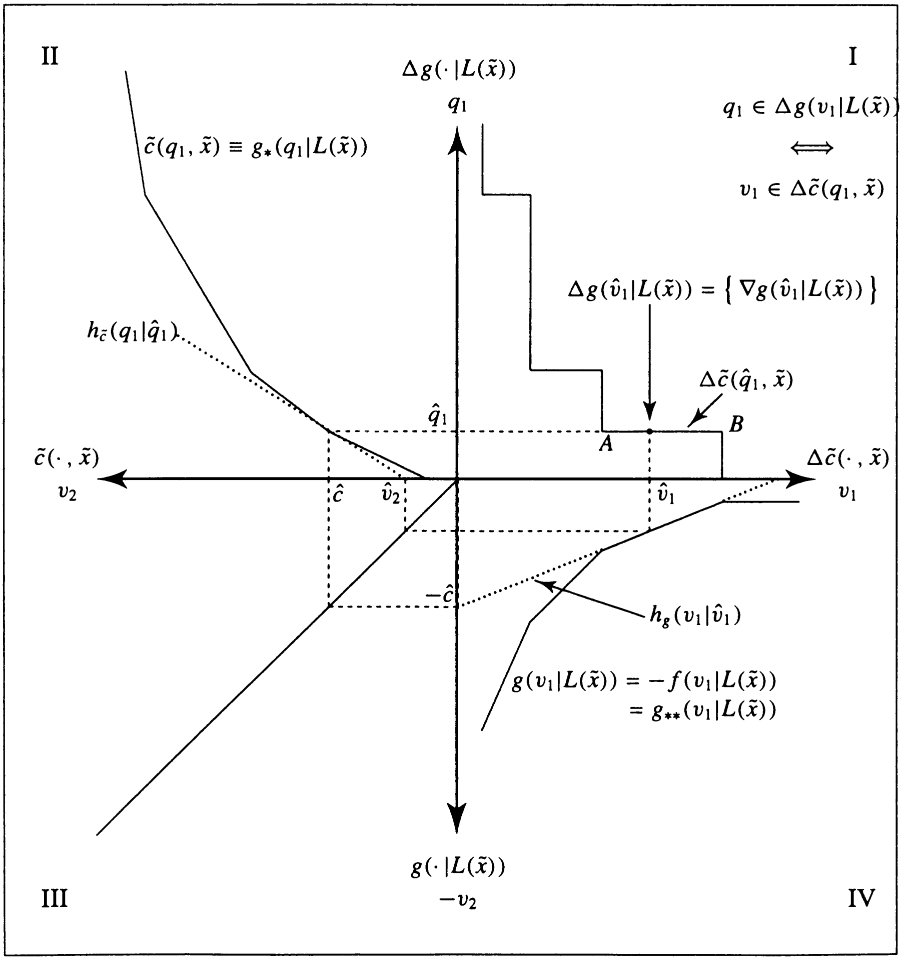

Figure III.16: Convex input requirement set

3. Indivisible production factors: *Because the function $g(\cdot|L(\tilde{x}))$ is not convex for indivisible inputs, it is*

$$g(\cdot|L(\tilde{x})) \neq g_{**}(\cdot|L(\tilde{x}))$$

so that the graphical representation is split into the two Figures III.12 on p. 114 and III.14 on p. 117. First of all, the normalized cost function is constructed by the function $g(\cdot|L(\tilde{x}))$, where Quadrant IV of Figure III.12 shows the functional values of (III.42) by points of the form $\odot$. The corresponding superdifferential (III.44) is shown by Quadrant I. The construction of the affine function

$$h_g(v_1|v_1'') = -\hat{c} + q_1''(v_1 - v_1''),$$

where $\hat{c} = g(v_1''|L(\tilde{x}))$, has already been discussed in Figure III.13; it touches the hypograph hypo $g(\cdot|L(\tilde{x}))$ *at point B'' and generates the pair of dual points (q_1'', v_1''). The corresponding point A on the cost function has the coordinates $\binom{q_1''}{\tilde{c}(q_1'', \tilde{x})}$, where the construction principle can be followed by the dotted lines. Varying the nonnegative factor price $q_1 \geqq 0$, each point of the cost function (III.45) can be derived geometrically in an analogous way.*

The inverse derivation of the function $g_{**}(\cdot|L(\tilde{x}))$ by the normalized cost function $\tilde{c}(\cdot, \tilde{x})$ is illustrated by Figure III.14. Now the support function

$$h_{\tilde{c}}(q_1|\tilde{q}_1) = v_2''' + v_1'''(q_1 - \tilde{q}_1)$$

corresponds to the pair of dual points $(\tilde{q}_1, v_1''')$. The resulting function $g_{**}(\cdot|L(\tilde{x}))$ is described by the line segments $A''B''C''$. We get the n-proper concave closed function

$$g_{**}(\cdot|L(\tilde{x})) = g(\cdot|\text{conv}L(\tilde{x})),$$

denoting the smallest concave function with $g_{**}(\cdot|L(\tilde{x})) \geqq g(\cdot|L(\tilde{x}))$. The inverse superdifferentials (Quadrant I of Figure III.14)

$$v_1 \in \Delta\tilde{c}(q_1, \tilde{x}) \iff q_1 \in \Delta g(v_1|\text{conv}L(\tilde{x}))$$

again form the link between both concave conjugate functions.

The cost function is differentiable except at points C and B for all factor prices $q_1 > 0$. Each point in the line $\overline{BC}$ is assigned to the exposed point B'' in conv$L(\tilde{x})$. *Conversely, the exposed point B'' implies the existence of a factor price q_1'' at which the cost function is differentiable; see line $\overline{B'C'}$.*

Analogously, the half-line $\overrightarrow{BC}$ and the exposed point A'' are assigned to each other by the vertical line starting in A'; see point D.

Dually, each point in the line $\overline{A''B''}$, at which the function $g_{**}(\cdot|L(\tilde{x}))$ is differentiable, leads to $\overline{A'B'}$ to the exposed point B of the hypograph hypo $\tilde{c}(\cdot, \tilde{x})$ *via the supergradients according. Similarly, point C and half-line $\overrightarrow{B''C''}$ are linked.*

Given the factor price $\tilde{q}_1$ with $\Delta\tilde{c}(\tilde{q}_1, \tilde{x}) \triangleq \overline{A'B'}$, we have as mentioned above

$$\Delta\tilde{c}(\tilde{q}_1, \tilde{x}) \cap \mathbb{Z}_+ = \{v_1', v_1''', v_1''\}.$$

Consequently, the input v_1''' is ruled out for the factor demand since a comparison of the corresponding functional values for v_1''' in Figures III.12 and III.14 shows

$$-\hat{v}_2 = g(v_1'''|\text{conv}L(\tilde{x})) > g(v_1'''|L(\tilde{x})) = -v_2'.$$

The equality, however, results for the inputs v_1' and v_1'' according to points A'' and B'' respectively. The demand set is

$$\tilde{D}(\hat{q}_1, \tilde{x}) = \{v_1', v_1''\}.$$

1.5.2 Results with Respect to the Output Correspondence

The previous expositions suppose the production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$ where the input requirement set $L(\mathbf{x})$ is described by a function

$$f(\mathbf{v}^{-r}|L(\mathbf{x})) = \inf\{v_r|\ \mathbf{v} \in L(\mathbf{x})\}.$$

According to the concave FENCHEL transform this function induces the normalized cost function

$$\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) = \inf\big\{(\mathbf{q}^{-r})^{\mathrm{T}}(\mathbf{v}^{-r}) + f(\mathbf{v}^{-r}|L(\mathbf{x}))|\ \mathbf{v}^{-r} \in \mathcal{V}\big\}.$$

Furthermore, various properties of factor demand have been presented.

We get similar statements when scrutinizing the production structure $\big(P(\mathbf{v})|\ \mathbf{v} \in V\big)$ instead of the equivalent production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$, where now each member of the family $P(\mathbf{v})$ denotes a production possibility set. Assuming n goods the spaces of primal (quantity) variables and dual (price) variables are now denoted by $\mathcal{X}$ and $\mathcal{R}_{\mathbf{p}}$ respectively:

$$\mathcal{X} = \mathbb{R}^{n-1} = \mathcal{R}_{\mathbf{p}}.$$

Supposing an admissible input vector $\mathbf{v} \in V$ and a vector of normalized commodity prices $\binom{\mathbf{p}^{-k}}{1}$, the n-proper (concave) function $g_P(\cdot|P(\mathbf{v}))\colon \mathcal{X} \to [0, +\infty[\cup \{-\infty\}$ with[55]

$$g_P(\mathbf{x}^{-k}|P(\mathbf{v})) := \sup\{x_k|\ \mathbf{x} \in P(\mathbf{v})\}$$

generates a proper (and convex) function $f_P(\cdot|P(\mathbf{v})) = -g_P(\cdot|P(\mathbf{v}))$. Moreover, the convex FENCHEL transform generates the convex conjugate function

$$f_P^*(\mathbf{p}^{-k}|P(\mathbf{v})) = \sup\big\{(\mathbf{p}^{-k})^{\mathrm{T}}(\mathbf{x}^{-k}) - f_P(\mathbf{x}^{-k}|P(\mathbf{v}))|\ \mathbf{x}^{-k} \in \mathcal{X}\big\},$$

indicating the the **normalized revenue** $\tilde{r}$ in units of good k,

$$\tilde{r}(\cdot, \mathbf{v}) \equiv f_P^*(\cdot|P(\mathbf{v})).$$

Analogous to the factor demand correspondence $\tilde{D}(\cdot, \mathbf{x})$, the **supply correspondence** $\tilde{S}(\cdot, \mathbf{v})\colon \mathcal{R}_{\mathbf{p}} \to \mathfrak{P}(\mathcal{X})$ includes all of the commodity bundles

[55] For the unusual domain $\sup\emptyset = -\infty$ must be taken into account.

solving the problem of revenue maximization depending on the normalized commodity prices $\mathbf{p}^{-k}$ at given factor quantities $\mathbf{v}$.

Assuming that all goods are divisible and that the production possibility sets are subject to Axiom [P7] (*Convexity*), then the supply correspondence $\tilde{S}(\cdot, \mathbf{v})$ and the subdifferential of the revenue function $\partial\tilde{r}(\cdot, \mathbf{v})$ are equivalent. If the boundary function $f_P(\cdot | P(\mathbf{v}))$ is not only proper and convex but also closed, then the relationship to the revenue function $\tilde{r}(\cdot, \mathbf{v})$ by Theorem D.5, p. 320, is indicated by six equivalent statements:

$$\begin{gathered}
(\hat{\mathbf{p}}^{-k})^{\mathsf{T}}(\mathbf{x}^{-k}) - f_P(\mathbf{x}^{-k}|P(\mathbf{v})) \ \text{ achieves the revenue maximum at } \ \mathbf{x}^{-k} = \hat{\mathbf{x}}^{-k};\\
f(\hat{\mathbf{x}}^{-k}|P(\mathbf{v})) + \tilde{r}(\hat{\mathbf{p}}^{-k}, \mathbf{v}) = (\hat{\mathbf{p}}^{-k})^{\mathsf{T}}(\hat{\mathbf{x}}^{-k});\\
\hat{\mathbf{q}}^{-k} \in \partial f_P(\hat{\mathbf{x}}^{-k}|P(\mathbf{v}));\\
(\mathbf{p}^{-k})^{\mathsf{T}}(\hat{\mathbf{x}}^{-k}) - \tilde{r}(\mathbf{p}^{-k}, \mathbf{v}) \ \text{ achieves the supremum at } \ \mathbf{p}^{-k} = \hat{\mathbf{p}}^{-k};\\
f_P^{**}(\hat{\mathbf{x}}^{-k}|P(\mathbf{v})) + \tilde{r}(\hat{\mathbf{p}}^{-k}, \mathbf{v}) = (\hat{\mathbf{p}}^{-k})^{\mathsf{T}}(\hat{\mathbf{x}}^{-k});\\
\hat{\mathbf{x}}^{-k} \in \tilde{S}(\hat{\mathbf{p}}^{-k}, \mathbf{v}).
\end{gathered}$$

In particular, the convex biconjugate function $f_P^{**}(\cdot | P(\mathbf{v}))$ satisfies

$$f_P(\mathbf{x}^{-k}|P(\mathbf{v})) = f_P^{**}(\mathbf{x}^{-k}|P(\mathbf{v})) := \sup\left\{(\mathbf{p}^{-k})^{\mathsf{T}}(\mathbf{x}^{-k}) - \tilde{r}(\mathbf{p}^{-k}, \mathbf{v}) | \ \mathbf{p}^{-k} \in \mathscr{P}_{\mathbf{p}}\right\}.$$

Each pair of points $(\hat{\mathbf{p}}^{-k}, \hat{\mathbf{x}}^{-k})$ satisfying the six mentioned conditions is called a **pair of dual points**, where Corollary D.5.1, p. 320, emphasizes the following inverse relation:

$$\hat{\mathbf{x}}^{-k} \in \tilde{S}(\hat{\mathbf{p}}^{-k}, \mathbf{v}) \quad \Longleftrightarrow \quad \hat{\mathbf{p}}^{-k} \in \partial f_P(\hat{\mathbf{x}}^{-k}|P(\mathbf{v})).$$

If the boundary function $f_P(\cdot|P(\mathbf{v}))$ and the revenue function $\tilde{r}(\cdot, \mathbf{v})$ are differentiable at points $\hat{\mathbf{x}}^{-k}$ and $\hat{\mathbf{p}}^{-k}$ respectively, then Corollary D.5.1 says with respect to the gradients:[56]

$$\hat{\mathbf{x}}^{-k} = \nabla\tilde{r}(\hat{\mathbf{p}}^{-k}, \mathbf{v}) \quad \Longleftrightarrow \quad \hat{\mathbf{p}}^{-k} = \nabla f_P(\hat{\mathbf{x}}^{-k}|P(\mathbf{v})).$$

The right hand equation reflects the HICKS-ALLEN condition since for a pair of dual points $(\hat{\mathbf{p}}^{-k}, \hat{\mathbf{x}}^{-k})$ the commodity price ratios $\hat{p}_j/\hat{p}_k = \hat{p}_j^{-k}$ equal the respective negative marginal rate of transformation $-\partial g_P(\hat{\mathbf{x}}^{-k}|P(\mathbf{v}))/\partial x_j^{-k}$. In contrast, the left hand equation corresponds to the HOTELLING-SHEPHARD Lemma. A commodity bundle $\hat{\mathbf{x}}^{-k}$ is supplied if it equals the gradient of the revenue function at point $\hat{\mathbf{p}}^{-k}$. Corollary D.5.1 states that both relations are equivalent for a pair of dual points $(\hat{\mathbf{p}}^{-k}, \hat{\mathbf{x}}^{-k})$. Thus, the commodity bundle $\hat{\mathbf{x}}^{-k}$ maximizes the revenue if and only if the commodity price vector $\hat{\mathbf{p}}^{-k}$ corresponds to the negative marginal rate of transformation $-\nabla g_P(\hat{\mathbf{x}}^{-k}|P(\mathbf{v}))$. The presented relations can be extended to

[56]The differentiability of the revenue function $\tilde{r}(\cdot, \mathbf{v})$ at point $\mathbf{p}^{-k}$ again implies a unique subgradient, $\tilde{S}(\mathbf{p}^{-k}, \mathbf{v}) = \partial r(\mathbf{p}^{-k}, \mathbf{v}) = \left\{\nabla\tilde{r}(\mathbf{p}^{-k}, \mathbf{v})\right\}$.

arbitrary goods j and r with $j, r \neq k$ by dividing the adjoined equations by each other.

$$\frac{\hat{x}_j}{\hat{x}_r} = \frac{\dfrac{\partial \tilde{r}}{\partial p_j}(\hat{\mathbf{p}}^{-k}, \mathbf{v})}{\dfrac{\partial \tilde{r}}{\partial p_r}(\hat{\mathbf{p}}^{-k}, \mathbf{v})} \quad \Longleftrightarrow \quad \frac{\hat{p}_j}{\hat{p}_r} = \frac{\dfrac{\partial g_P}{\partial x_j}(\hat{\mathbf{x}}^{-k}|P(\mathbf{v}))}{\dfrac{\partial g_P}{\partial x_r}(\hat{\mathbf{x}}^{-k}|P(\mathbf{v}))}$$

JORGENSON, LAU (1974) present analogous results with respect to the graph GR of a production technology. There the subdifferential of the profit function faces the inverse correspondence of profit maximizing production plans.

Technically, the convex conjugate function of the cost function $\tilde{c}$ by (III.13), p. 92 yields the profit function π_1 with

$$\pi_1(\mathbf{q}^{-r}, \mathbf{p}) \equiv \tilde{c}^*(\mathbf{q}^{-r}, \mathbf{p}) := \sup\left\{\mathbf{p}^{\mathsf{T}}\mathbf{x} - \tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) |\, \mathbf{x} \in X\right\}.$$

Analogously, the revenue function $\tilde{r}$ faces the profit function π_2 in the sense of a concave conjugate function:

$$-\pi_2(\mathbf{q}, \mathbf{p}^{-k}) \equiv -\tilde{r}_*(\mathbf{q}, \mathbf{p}^{-k}) := -\inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} - \tilde{r}(\mathbf{p}^{-k}, \mathbf{v}) |\, \mathbf{v} \in V\right\}.$$

The only difference between the two profit functions is that in π_1 the price of the r-th input and in π_2 the price of the k-th outputs is set to one. The technical properties of these two profit functions are omitted at this point.

2 Alternative Representation of the Firm's Cost Structure

2.1 The Cost Function

2.1.1 Properties of the Cost Function

The former expositions of conjugate functions are now followed by the analysis of polar functions. As already noted by Figure III.1, p. 85, the analysis differs by way of factor variation. Whereas the previously treated theory of conjugate functions is based on a partial factor variation, now a total factor variation is examined. With the alternative representation of input requirement sets further aspects result for the representation of a production structure and the appropriate cost structure of a firm.

The object of the next sections is to derive the relations shown in Figure III.29, p. 172. As a wealth of analytical results will be derived, it is recommended keeping this figure at hand as it virtually serves as a "map" for the further proceedings. The corresponding graphical representation can be found in Figure III.30, p. 174.[57] Beginning with the easiest representation of an input requirement set by an indicator function $\varrho(\cdot|L(\mathbf{x}))$, all of the main properties of the corresponding cost function

[57] For aspects of single-product firms, see Figure III.25, p. 167.

$c(\cdot, \mathbf{x})$ as the support function of the input requirement set can be derived (Proposition III.7) using the knowledge on concave conjugate functions from above. If the cost function is known, then for each member of the production structure $\big(L(\mathbf{x}) \mid \mathbf{x} \in X\big)$ the reverse conclusion indicates an approximate input requirement set which contains all of the essential properties of the genuine input requirement set.

The expositions in Section 2.1 on the relation between the cost function $c(\cdot, \mathbf{x})$ and the indicator function $\varrho(\cdot \mid L(\mathbf{x}))$ are followed by the proof that the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ of Section 1.2 equals the cost function $c(\cdot, \mathbf{x})$ despite of the different ways of construction. The ensuing Section 2.2 introduces the input distance function t_I with respect to measuring the technical efficiency of activities. This function has properties similar to the cost function (Proposition III.13).

Furthermore, by definition of polar input requirement sets $L_\circ(\mathbf{x})$ (composed of factor price vectors) the production structure $\big(L(\mathbf{x}) \mid \mathbf{x} \in X\big)$ is compared to a family $\big(L_\circ(\mathbf{x}) \mid \mathbf{x} \in X\big)$, which can be interpreted as a cost structure. As an input requirement set each polar input requirement set can be represented by the cost function and the input distance function, where both functions exchange their roles (Corollary III.16.1). This dual representation of the production structure as cost structure is gathered from Section 2.3, where SHEPHARD's Theorem (Proposition III.19) notes the direct relation between the cost function and the input distance function as a crucial issue.

For the subsequent analysis it is useful to replace the factor space $V = \mathbb{R}_+^{m_d} \times \mathbb{Z}_+^{m-m_d}$ as well as the space of the factor prices $Q = \mathbb{R}_+^m$ with

$$\overline{V} = \mathbb{R}^m = \overline{Q}$$

On the one hand this makes it easier to handle convex subsets in $\mathbb{R}^m$. On the other hand the analytical effort for the explicit consideration of nonnegativity constraints is reduced. As already mentioned, a vector $\mathbf{v} \in \overline{V}$ is said to be an *input* vector if, moreover, $\mathbf{v} \in V$ holds. Remember that input requirement sets consist only of input vectors, $L(\mathbf{x}) \subset V$. Also, for the sake of simplification, the commodity space $X = \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}$ and the space of commodity prices $P_\mathbf{p} = \mathbb{R}_+^n$ are superseded by

$$\overline{X} = \mathbb{R}^n = \overline{P_\mathbf{p}}\,.$$

Again $\mathbf{x} \in \overline{X}$ denotes a commodity bundle if and only if $\mathbf{x} \in X$ is satisfied. Each vector $\mathbf{x} \notin X$ evokes an empty input requirement set, $L(\mathbf{x}) = \emptyset$.

The analysis begins with the perhaps easiest representation of an input requirement set $L(\mathbf{x})$ by the **(reciprocal) indicator function**[58] $\varrho\colon \overline{V} \times \overline{X} \to [-\infty, 0]$ with

$$\varrho(\mathbf{v} \mid L(\mathbf{x})) := \begin{cases} 0 & \text{for } \mathbf{v} \in L(\mathbf{x}) \\ -\infty & \text{for } \mathbf{v} \notin L(\mathbf{x}). \end{cases}$$

[58] The indicator function $\varrho(\cdot \mid C)$, the support function $\sigma(\cdot \mid C)$, and the gauge $\gamma(\cdot \mid C)$ in Appendix D.2 face in the following sections the reciprocal indicator function $\varrho(\cdot \mid C)$, the reciprocal support function $\sigma(\cdot \mid C)$ and the reciprocal gauge $\gamma(\cdot \mid C)$ respectively. More appealing we omit the addition *reciprocal* unless ambiguity would result.

Accordingly, the indicator function ϱ distinguishes the vectors $\mathbf{v}$ only by the criterion whether they belong to the given input requirement set $L(\mathbf{x})$ or not.[59] In this way the whole production structure of a firm $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$ can be described by the indicator function ϱ.

Without going into more details on Axioms [L3] (*Disposability of Inputs*) and [L5] (*Boundedness*) the indicator function can be characterized as follows:

Proposition III.5 *The indicator function ϱ of an input requirement set $L(\mathbf{x})$ has the following properties:*

1. *Given the commodity bundle* $\mathbf{x} = \mathbf{0}$ *or the input vector* $\mathbf{v} = \mathbf{0}$ *we have*

$$\varrho(\mathbf{v}|L(\mathbf{0})) = 0 \quad \forall\, \mathbf{v} \in V \qquad \text{see [L1a];}$$
$$\varrho(\mathbf{0}|L(\mathbf{x})) = 0 \quad \forall\, \mathbf{x} \in X \setminus \{\mathbf{0}\} \qquad \text{see [L1b].}$$

2. *The function $\varrho(\cdot|L(\mathbf{x}))$ is n-proper if and only if $\mathbf{x}$ is a commodity bundle;* see [L2] (*Attainability of Each Production*) with $\mathbf{x} \in X \iff L(\mathbf{x}) \neq \emptyset$.
The impossibility of producing outputs $\mathbf{x} \notin X$ *is reflected by* $L(\mathbf{x}) = \emptyset$ *or, equivalently, by* $\varrho(\cdot|L(\mathbf{x})) = -\infty$.
3. *If $\varrho(\cdot|L(\mathbf{x}))$ is n-proper, then* $\varrho_{**}(\cdot|L(\mathbf{x})) = \varrho(\cdot|\,\mathrm{cl}(\mathrm{conv}L(\mathbf{x})))$; see Theorem D.2.
4. *The function $\varrho(\cdot|L(\mathbf{x}))$ is closed if and only if the inspected input requirement set $L(\mathbf{x})$ is closed;* see [L6] (*Closedness*).
5. *The function $\varrho(\cdot|L(\mathbf{x}))$ is convex if and only if the inspected input requirement set $L(\mathbf{x})$ is convex;* see [L7] (*Convexity*).
6. *If $\varrho(\cdot|L(\mathbf{x}))$ is n-proper, closed, and convex, then* $\varrho_{**}(\cdot|L(\mathbf{x})) = \varrho(\cdot|L(\mathbf{x}))$; see Theorem D.3 (FENCHEL, MOREAU).

Apart from the representation of a firm's production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$ by the indicator function ϱ we now seek for a function which represents the corresponding cost structure of the examined firm. For this purpose we define the **(reciprocal) support function**[60] $\varphi(\cdot|L(\mathbf{x}))\colon \overline{Q} \to [-\infty, +\infty]$ with

$$\text{(III.53)} \qquad \varphi(\mathbf{q}|L(\mathbf{x})) := \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v}|\ \mathbf{v} \in L(\mathbf{x})\right\}.$$

This function indicates the minimal factor costs incurred by producing the given vector $\mathbf{x}$. In the case of $\mathbf{x} \in X$ under [L2] (*Attainability of Each Production*) we speak of a commodity bundle with a nonempty input requirement set $L(\mathbf{x})$. As before, the impossibility of producing outputs $\mathbf{x} \notin X$ is reflected by an empty input requirement set $L(\mathbf{x}) = \emptyset$ or, equivalently, by infinitely large costs $\varphi(\cdot|\emptyset) \equiv +\infty$. For a given production technology L and a given $\mathbf{x}$

$$c(\cdot, \mathbf{x}) \equiv \varphi(\cdot|L(\mathbf{x}))$$

[59] A graphical representation of the modified indicator function $\varrho(\cdot|L(\mathbf{x})) + 1$ is given by Figure III.23, p. 156.

[60] See Definition D.3, p. 323.

is called the **factor price minimal cost function**. It faces the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ by (III.13) or (III.11), p. 92. As shown by a later comparison, both functions can be transformed into each other.

For a given production technology L the variation of $\mathbf{x}$ and, therefore, the variation of the input requirement set $L(\mathbf{x})$ generates an extended real-valued function $c\colon \overline{Q} \times \overline{X} \to [-\infty, +\infty]$ which is called the **minimal cost function** of the input correspondence L. The adjoined **factor demand correspondence** $D\colon Q \times X \to \mathfrak{P}(V)$ indicates all input vectors solving the problem of cost minimization at given factor prices $\mathbf{q} \in Q$ and given quantities $\mathbf{x} \in X$. [61]

$$D(\mathbf{q}, \mathbf{x}) = \left\{\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x}),\ \mathbf{q}^{\mathsf{T}}\mathbf{v} = c(\mathbf{q}, \mathbf{x})\right\} \tag{III.54}$$

Since the linear objective function $\mathbf{q}^{\mathsf{T}}\mathbf{v}$ is both convex and concave, Theorem II.2 yields

$$c(\cdot, \mathbf{x}) \equiv \varphi(\cdot \mid L(\mathbf{x})) = \varphi(\cdot \mid \mathrm{conv} L(\mathbf{x})). \tag{III.55}$$

It makes no difference whether the cost function is determined by a nonconvex input requirement set or its convex hull.[62] This result does not transmit to the factor demand correspondence. However, Theorem II.3 states that the infimum is attained at an extreme point of $\mathrm{conv}L(\mathbf{x})$ if it is attained at all. At least these extreme points belong to $L(\mathbf{x})$ and, therefore, to the set of input vectors demanded $D(\mathbf{q}, \mathbf{x})$.[63] The next proposition notes not only the relation between the indicator function of an input requirement set $\varrho(\cdot \mid L(\mathbf{x}))$ and the corresponding cost function $c(\cdot, \mathbf{x}) \equiv \varphi(\cdot \mid L(\mathbf{x}))$ but it also determines a series of properties of the cost function.

Proposition III.6 *The indicator function $\varrho(\cdot \mid C)$ and the support function $\varphi(\cdot \mid C)$ of a closed convex set $C \subset \overline{V}$ are concave conjugate to each other.*

$$\varphi(\cdot \mid C) = \varrho_*(\cdot \mid C) \quad \textit{and} \quad \varphi_*(\cdot \mid C) = \varrho(\cdot \mid C)$$

Proof: The definitions of the indicator function ϱ as well as the support function φ immediately give

$$\begin{aligned} \varrho_*(\mathbf{q} \mid C) &= \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} - \varrho(\mathbf{v} \mid C) \mid \mathbf{v} \in \overline{V}\right\} \\ &= \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in C\right\} \\ &= \varphi(\mathbf{q} \mid C) \end{aligned}$$

[61] As will be shown, $D(\mathbf{q}, \mathbf{x}) = \emptyset$ holds for $\mathbf{q} \notin Q$ or $\mathbf{x} \notin X$; see (III.58).

[62] The cost function even remains the same when the inspected input requirement set $L(\mathbf{x})$ is not closed. By the second separation theorem (Theorem B.10, p. 296), we obtain

$$c(\cdot, \mathbf{x}) \equiv \varphi(\cdot \mid L(\mathbf{x})) = \varphi(\cdot \mid \mathrm{cl}(\mathrm{conv}L(\mathbf{x}))) = \varphi(\cdot \mid \mathrm{rint}(\mathrm{conv}L(\mathbf{x}))).$$

[63] The definition of the correspondence $D(\cdot, \mathbf{x})$ allows a more precise formulation of this statement.

$$D(\mathbf{q}, \mathbf{x}) = \left\{\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x}),\ \mathbf{q}^{\mathsf{T}}\mathbf{v} = c(\mathbf{q}, \mathbf{x})\right\} = \left\{\mathbf{v} \mid \mathbf{v} \in \mathrm{conv}L(\mathbf{x}),\ \mathbf{q}^{\mathsf{T}}\mathbf{v} = c(\mathbf{q}, \mathbf{x})\right\} \cap L(\mathbf{x})$$

for each $\mathbf{q} \in \overline{Q}$. Analogous to Theorem D.8, p. 324, the second part of the proof results from

$$\begin{aligned} \varphi_*(\cdot|C) &= \varrho_{**}(\cdot|C) && \text{because of } \varphi(\cdot|C) = \varrho_*(\cdot|C) \\ &= \operatorname{cl}\varrho(\cdot|C) && \text{convexity of } C \text{ and Theorem D.2} \\ &= \varrho(\cdot|\operatorname{cl}C) && \\ &= \varrho(\cdot|C) && \text{closedness of } C. \quad \blacksquare \end{aligned}$$

If the premises of Proposition II.14 are satisfied, then the convex hull of an input requirement set $\operatorname{conv}L(\mathbf{x})$ fulfills the requirements of Proposition III.6. In view of (III.55) we get

$$\text{(III.56)} \qquad c(\cdot,\mathbf{x}) = \varrho_*(\cdot|\operatorname{conv}L(\mathbf{x})) \quad \text{and} \quad c_*(\cdot,\mathbf{x}) = \varrho(\cdot|\operatorname{conv}L(\mathbf{x}))$$

with respect to the input requirement set $L(\mathbf{x})$. No matter whether single goods or factors are indivisible, many characteristics of a cost function are not affected.

Proposition III.7 *The cost function c essentially has the following properties with regard to factor prices* $\mathbf{q}$*:*
For a given $\mathbf{x} \in \overline{X}$ *the cost function* $c(\cdot,\mathbf{x})$ *is 1. linearly homogeneous, 2. closed (and, therefore, lower semi-continuous), 3. concave, 4. superadditive, i.e.*

$$\text{(III.57)} \qquad c(\mathbf{q}+\tilde{\mathbf{q}},\mathbf{x}) \geqq c(\mathbf{q},\mathbf{x}) + c(\tilde{\mathbf{q}},\mathbf{x}) \qquad \forall\, \mathbf{q},\tilde{\mathbf{q}} \in \overline{Q},$$

and 5. nondecreasing in factor prices $\mathbf{q}$*. Moreover, the cost function* $c(\cdot,\mathbf{x})$ *is 6. n-proper if and only if* $\mathbf{x}$ *is a commodity bundle,* $\mathbf{x} \in X$*; and it is finally 7. continuous in* $\operatorname{rint}\left(\text{n-Dom}\, c(\cdot,\mathbf{x})\right) = \mathbb{R}^m_{++}$ *for each commodity bundle* $\mathbf{x} \in X$.

Proof: By (III.56) the cost function $c(\cdot,\mathbf{x})$, understood as a linearly homogeneous support function of the input requirement set $L(\mathbf{x})$, corresponds to the concave conjugate function of the indicator function $\varrho(\cdot|\operatorname{conv}L(\mathbf{x}))$. Thus, by Theorem D.2, p. 316, the cost function is closed and concave for each $\mathbf{x} \in \overline{X}$. Moreover, the same theorem states that $c(\cdot,\mathbf{x})$ is n-proper if and only if $\varrho(\cdot|L(\mathbf{x}))$ is n-proper. The latter property is given if and only if a commodity bundle $\mathbf{x} \in X$ or equivalently $L(\mathbf{x}) \neq \emptyset$ is presumed (Proposition III.5). The concavity of the cost function directly implies three more properties. The linearly homogeneous cost function is by Theorem C.4, p. 304, concave in $\mathbf{q}$ if and only if it is superadditive in $\mathbf{q}$. Interpreting $\tilde{\mathbf{q}}$ in (III.57) as change $\Delta\mathbf{q} \geqq \mathbf{0}$, then $c(\cdot,\mathbf{x})$ is nondecreasing in $\mathbf{q}$. Finally, every concave function is continuous in the relative interior of its effective domain n-Dom $c(\cdot,\mathbf{x})$. With respect to the functional values of the extended real-valued cost function it can be noted that

$$\text{(III.58)} \qquad \begin{aligned} c(\cdot,\mathbf{x}) &\equiv +\infty && \text{for } \mathbf{x} \notin X; \\ c(\mathbf{q},\mathbf{x}) &= -\infty && \text{for } \mathbf{x} \in X \text{ and } \mathbf{q} \ngeqq \mathbf{0}; \\ c(\mathbf{q},\mathbf{x}) &\geqq 0 && \text{for } \mathbf{x} \in X \text{ and } \mathbf{q} \geqq \mathbf{0}. \end{aligned}$$

Thus, the effective domain of the cost function is

$$\text{n-Dom}\, c(\cdot, \mathbf{x}) = Q \quad \text{for a given commodity bundle} \quad \mathbf{x} \in X. \qquad \blacksquare$$

From the linear homogeneity of the cost function the immediate result is that the factor demand correspondence is the homogeneity of degree 0.

$$\begin{aligned} D(\lambda\mathbf{q}, \mathbf{x}) &= \left\{\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x}),\ (\lambda\mathbf{q})^{\mathsf{T}}\mathbf{v} = c(\lambda\mathbf{q}, \mathbf{x})\right\} \\ &= \left\{\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x}),\ \lambda\mathbf{q}^{\mathsf{T}}\mathbf{v} = \lambda c(\mathbf{q}, \mathbf{x})\right\} \\ &= D(\mathbf{q}, \mathbf{x}) \end{aligned}$$

Already at this point we can note further properties of factor demand.

Proposition III.8 *Let $L(\mathbf{x})$ be a nonempty closed convex set and $\mathbf{q}$ be a point in $\overline{Q}$. Then at point $\mathbf{q}$ the superdifferential of the cost function $\Delta c(\cdot, \mathbf{x})$ consists of points at which the the linear function $\mathbf{q}^{\mathsf{T}}\mathbf{v}$ achieves its minimum over $L(\mathbf{x})$.*

$$\text{(III.59)} \qquad \hat{\mathbf{v}} \in \Delta c(\mathbf{q}, \mathbf{x}) \iff \hat{\mathbf{v}} = \arg\min \left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x})\right\}$$

Proof: Plug $g = \varrho(\cdot|L(\mathbf{x}))$ in Corollary D.5.2, p. 320. This leads to $g_* = \varrho_*(\cdot|L(\mathbf{x})) = \varphi(\cdot|L(\mathbf{x}))$ (Proposition III.6). As the support function $\varphi(\cdot|L(\mathbf{x})) \equiv c(\cdot, \mathbf{x})$ is closed (Proposition III.7), (III.59) ensues from Corollary D.5.2. ■

If the input requirement set examined in Proposition III.8 is not convex, then $L(\mathbf{x})$ has to be replaced with its (closed) convex hull. In this case the factor demand correspondence $D(\cdot, \mathbf{x})$ equals no longer the superdifferential of the cost function $\Delta c(\cdot, \mathbf{x})$. On the contrary we have at point $\mathbf{q}$

$$D(\mathbf{q}, \mathbf{x}) = \Delta c(\mathbf{q}, \mathbf{x}) \cap L(\mathbf{x}).$$

Provided the cost function is differentiable at $\mathbf{q}$, the superdifferential $\Delta c(\mathbf{q}, \mathbf{x})$ has precisely one supergradient[64] corresponding to the gradient

$$\nabla c(\mathbf{q}, \mathbf{x}) = \left(\frac{\partial c}{\partial q_i}(\mathbf{q}, \mathbf{x})\right)_{i=1,\dots,m}.$$

Thus, the vector $\hat{\mathbf{v}}$ solves the problem of cost minimization if and only if

$$\hat{\mathbf{v}} = \nabla c(\mathbf{q}, \mathbf{x}).$$

This property is called **SHEPHARD's Lemma**.[65]

Looking back at Corollary D.4.1, p. 121, keep in mind that indivisible factors are consistent with the differentiability of the cost function at least over some regions. By using the two concave conjugate functions $\varrho(\cdot|\text{conv}L(\mathbf{x}))$ and $c(\cdot, \mathbf{x})$ the corollary states that $\binom{\mathbf{v}}{\mu}$ is an exposed point of the hypograph

$$\text{hypo}\,\varrho(\cdot|\text{conv}L(\mathbf{x})) = \left\{\binom{\mathbf{v}}{\mu} \in \mathbb{R}^{m+1} \mid \mu \leqq \varrho(\mathbf{v}|\text{conv}L(\mathbf{x}))\right\}$$

[64] Cf. FÄRE, PRIMONT (1986, 1990).

[65] See Proposition III.19.

if and only if there is a factor price vector $\hat{\mathbf{q}}$ at which the cost function $c(\cdot, \mathbf{x})$ is differentiable – see points A and B in Figure III.17. All of these exposed points are then a point of the form

$$\begin{pmatrix} \hat{\mathbf{v}} \\ \varrho(\hat{\mathbf{v}}|\mathrm{conv}L(\mathbf{x})) \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{v}} \\ 0 \end{pmatrix}.$$

Accordingly, factor prices $\binom{\hat{q}_1}{\hat{q}_2}$ exist for the (only exposed) points A and B of the above figure, at which the cost function $c(\cdot, \mathbf{x})$ is differentiable. Conversely, the differentiability of the cost function at a point $\binom{\hat{q}_1}{\hat{q}_2}$ is associated with one of the two points A or B.

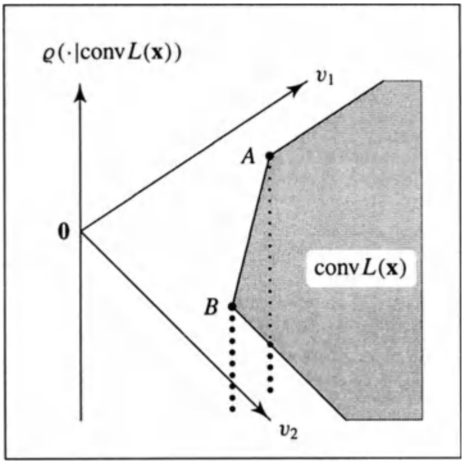

Figure III.17: The criterion of differentiability of a cost function

The properties of the cost function regarding a possible variation of the goods $\mathbf{x}$ can be characterized as follows, where in particular the first three implications are not affected considering indivisible goods.

Proposition III.9 *The cost function c has the following properties with respect to a variation of outputs* $\mathbf{x}$.

1. *For two commodity bundles* $\mathbf{x}, \tilde{\mathbf{x}} \in X$ *with* $\mathbf{x} \geqq \tilde{\mathbf{x}}$ *we have* $c(\cdot, \mathbf{x}) \geqq c(\cdot, \tilde{\mathbf{x}})$.
2. *For each sequence* $\{\mathbf{x}^\nu\} \subset X$ *with* $\|\mathbf{x}^\nu\| \to +\infty$ *and* $\mathbf{q} > \mathbf{0}$ *we have* $c(\mathbf{q}, \mathbf{x}^\nu) \to +\infty$.
3. *By (III.62b) the cost function c it subadditive in* $\mathbf{x}$ *if and only if the input correspondence L is superadditive.*
4. *The cost function c is convex in* $\mathbf{x}$ *for all* $\mathbf{q} > \mathbf{0}$ *if according to [T1] the graph of the technology is convex.*
5. *If in accordance with* [L6] *the input correspondence L is closed, then the cost function c is lower semi-continuous in X for a given* $\mathbf{q} > \mathbf{0}$.

Proof: Given any two commodity bundles $\mathbf{x}, \tilde{\mathbf{x}} \in X$ with $\mathbf{x} \geqq \tilde{\mathbf{x}}$, it is[66]

$$\begin{aligned} &\text{(III.60)} \qquad && L(\mathbf{x}) \subset L(\tilde{\mathbf{x}}) && \text{[L4] (\textit{Disposability of Outputs})} \\ &\iff && \varphi(\cdot|L(\mathbf{x})) \geqq \varphi(\cdot|L(\tilde{\mathbf{x}})) && \\ &\iff && c(\cdot,\mathbf{x}) \geqq c(\cdot,\tilde{\mathbf{x}}). && \end{aligned}$$

The second property results from a proof of contradiction. Let us examine a sequence $\{\mathbf{x}^\nu\} \subset X$ with $\|\mathbf{x}^\nu\| \to +\infty$. Supposing $\liminf_{\nu\to+\infty} c(\mathbf{q}, \mathbf{x}^\nu) < +\infty$,

[66]The technical proof of this proposition will be adduced below by (III.68).

then a subsequence $\{\mathbf{x}^{\nu_k}\}$ with $\|\mathbf{x}^{\nu_k}\| \to +\infty$ and $\lim\limits_{\nu \to +\infty} c(\mathbf{q}, \mathbf{x}^{\nu_k}) = c^0 < +\infty$ exists. Thus, the corresponding sequence $\{\hat{\mathbf{v}}^{\nu_k}\}$ of cost minimizing input vectors is bounded because of $\mathbf{q} > \mathbf{0}$ so that there is an input vector $\tilde{\mathbf{v}}$ with $\tilde{\mathbf{v}} \geqq \hat{\mathbf{v}}^{\nu_k}$ and therefore $\tilde{\mathbf{v}} \in L(\mathbf{x}^{\nu_k})$ for all ν_k; see [L3] (*Disposability of Inputs*). The contradiction to [L5] (*Boundedness*) completes the proof.

Referring to the third property, the following equation for all $\mathbf{x}, \tilde{\mathbf{x}} \in X$ is premised

$$\text{(III.61)} \qquad \begin{aligned} \varphi(\cdot|L(\mathbf{x}) + L(\tilde{\mathbf{x}})) &= \varphi(\cdot|L(\mathbf{x})) + \varphi(\cdot|L(\tilde{\mathbf{x}})) \\ &= c(\cdot, \mathbf{x}) + c(\cdot, \tilde{\mathbf{x}}). \end{aligned}$$

Thus, a superadditive input correspondence[67] satisfies for all commodity bundles $\mathbf{x}, \tilde{\mathbf{x}} \in X$ the following equivalence relation.

(III.62a)	$L(\mathbf{x}) + L(\tilde{\mathbf{x}}) \subset L(\mathbf{x} + \tilde{\mathbf{x}})$	**superadditivity**
$\Longleftrightarrow$	$\varphi(\cdot\vert L(\mathbf{x})) + \varphi(\cdot\vert L(\tilde{\mathbf{x}})) \geqq \varphi(\cdot\vert L(\mathbf{x} + \tilde{\mathbf{x}}))$	see (III.60) and (III.61)
(III.62b)		
$\Longleftrightarrow$	$c(\cdot, \mathbf{x}) + c(\cdot, \tilde{\mathbf{x}}) \geqq c(\cdot, \mathbf{x} + \tilde{\mathbf{x}})$	**subadditivity**

We can show by [T1], p. 22, that the convexity of graph GR implies a convex input correspondence[68] with the property

$$\text{(III.63)} \qquad \lambda L(\mathbf{x}) + (1 - \lambda) L(\tilde{\mathbf{x}}) \subset L\big(\lambda \mathbf{x} + (1 - \lambda)\tilde{\mathbf{x}}\big) \qquad \lambda \in [0, 1],$$

Therefore, the proof of the convexity of the cost function c is omitted, since inputs as well as outputs must be divisible. For a given factor price vector $\mathbf{q} > \mathbf{0}$ we obtain

$$\text{(III.64)} \qquad \lambda c(\mathbf{q}, \mathbf{x}) + (1 - \lambda) c(\mathbf{q}, \tilde{\mathbf{x}}) \geqq c\big(\mathbf{q}, \lambda \mathbf{x} + (1 - \lambda)\tilde{\mathbf{x}}\big).$$

Given a $\mathbf{q} > \mathbf{0}$, the lower semi-continuity of the cost function $c(\mathbf{q}, \cdot)$ holds if the subsequent criterion is met.

$$\liminf_{\mathbf{x}^\nu \to \mathbf{x}^0} c(\mathbf{q}, \mathbf{x}^\nu) = c(\mathbf{q}, \mathbf{x}^0)$$

As shown by SHEPHARD,[69] this condition can be derived from the closedness of the input correspondence, i.e.

$$\mathbf{v}^\nu \to \mathbf{v}^0,\ \mathbf{x}^\nu \to \mathbf{x}^0,\ \mathbf{v}^\nu \in L(\mathbf{x}^\nu) \quad \Longrightarrow \quad \mathbf{v}^0 \in L(\mathbf{x}^0).$$

At this point we can dispense with the proof. ■

[67] See Definition II.8, p. 44.

[68] For the proof, see SHEPHARD (1953, p. 91 f.). An input correspondence L with property (III.63) is said to be convex. *Convex* correspondences have to be distinguished from *convex-valued* correspondences. The later has only convex level sets. See Section II.4.1.

[69] See SHEPHARD (1953, p. 88 ff.).

To classify the lower semi-continuity of a cost function, two aspects are helpful. On the one hand each proper lower semi-continuous function achieves its minimum over a compact subset in its effective domain.[70] On the other hand it seems reasonable to suspect that stronger properties of the input correspondence are connected with stronger properties of the cost function. For example, if the inverse output correspondence P of L is continuous, then the Maximum Theorem[71] yields a revenue function $r(\mathbf{p}, \mathbf{v}) = \sup\{\mathbf{p}^{\mathsf{T}}\mathbf{x} | \mathbf{x} \in P(\mathbf{v})\}$, which is continuous at $\mathbf{v}$. In this the correspondence P is said to be continuous on V if it is both lower and upper semi-continuous and compact-valued. Moreover, each upper semi-continuous correspondence is closed by Theorem C.7, p. 309; see [P6] (*Closedness*).

Since the input correspondence L has no compact level sets $L(\mathbf{x})$, it cannot be *continuous on X*. Consequently, the Maximum Theorem cannot be applied with regard to the cost function $c(\mathbf{q}, \cdot)$. The Maximum Theorem is not even enough to derive the upper semi-continuity of the demand correspondence $D(\mathbf{q}, \cdot)$ by (III.54). Proving the existence of competitive equilibria, this property of demand correspondences will be of major importance in Chapter IV.

Proposition III.9 is followed by further remarks referring to the third and fourth property. While the subadditivity of the cost function is closely related to economies of scale and economies of scope, which might be caused by indivisibilities, the convexity of the cost function more likely reflects diseconomies of scale. The transference of the criterion of convexity to an indivisible commodity x results from (III.67) and is illustrated by Figure III.19.

With respect to the subadditivity of a cost function by (III.62b) there are two special cases to be stressed. On the one hand cost advantages may result by expanding the firm's product range (economies of scope). On the other hand the phenomenon of the natural monopoly can now be included.

Starting with the firm's product range $\mathbf{x} = (x_1, \ldots, x_r, 0, \ldots, 0)^{\mathsf{T}}$ the vector $\tilde{\mathbf{x}} = (0, \ldots, 0, \tilde{x}_{r+1}, \ldots, \tilde{x}_n)^{\mathsf{T}}$ denotes an extension of this range. If for a given technology it is more expensive to produce the commodity bundles $\mathbf{x}$ and $\tilde{\mathbf{x}}$ by two separate processes instead of joining them, then we speak of **economies of scope**.[72]

$$c(\cdot, \mathbf{x}) + c(\cdot, \tilde{\mathbf{x}}) \geqq c(\cdot, \mathbf{x} + \tilde{\mathbf{x}})$$

Thus, the easiest condition for the existence of economies of scope is given for two goods, $c(\cdot, x_1, 0) + c(\cdot, 0, \tilde{x}_2) \geqq c(\cdot, x_1, \tilde{x}_2)$. Looking back at Proposition II.7, indivisible goods and factors can especially be indicated as the origin of these cost advantages. The mutual provision of capacities not used by the single activities permits a joint production at lower costs.[73]

[70] Cf. BERGE (1963, p. 76, Theorem 2).

[71] See Theorem C.14, p. 311.

[72] A more general formulation of this phenomenon may be found in BAUMOL, PANZAR, WILLIG (1982, p. 71 f.), to which also FÄRE (1986) refers.

[73] As before, the effects can be clarified by the example of a building contractor. For example, each building project can only be carried out with the aid of a crane. The capacity of a single crane, however, suffices to serve for three projects. Given suitable excess capacities, two cranes could already be enough to undertake seven building projects.

Given the strict superadditivity[74] of the input correspondence, a stronger form of the third property in Proposition III.9 can be established, which reflects the case of a natural monopoly. The existence of an input vector $\bar{\mathbf{v}} \in L(\mathbf{x}+\tilde{\mathbf{x}})$ with $\bar{\mathbf{v}} \leq \mathbf{v}$ for all $\mathbf{v} \in L(\mathbf{x}) + L(\tilde{\mathbf{x}})$ implies

$$\begin{aligned} & \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x})\right\} + \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in L(\tilde{\mathbf{x}})\right\} \\ & \qquad > \mathbf{q}^{\mathsf{T}}\bar{\mathbf{v}} \geqq \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x}+\tilde{\mathbf{x}})\right\} \\ \Longleftrightarrow \quad & c(\mathbf{q},\mathbf{x}) + c(\mathbf{q},\tilde{\mathbf{x}}) > c(\mathbf{q},\mathbf{x}+\tilde{\mathbf{x}}) \end{aligned}$$

for all factor price vectors $\mathbf{q} > \mathbf{0}$. The cost function is said to be **strictly subadditive** in the outputs $\mathbf{x}$ in the interior of the factor price space Q, i.e. $\operatorname{int} Q = \mathbb{R}^m_{++}$. Interpreting $\mathbf{x}^i$ as output vector of a firm in an industry, then this industry is called a **natural monopoly** if the cost function of the examined industry is subadditive in the entire region.[75]

$$\sum_i c(\cdot,\mathbf{x}^i) > c(\cdot,\mathbf{x}) \qquad \text{with} \quad \mathbf{x} = \sum_i \mathbf{x}^i$$

The condition for a natural monopoly is not only closely related to the existence of economies of scope but also to the economies of scale a firm may enjoy.[76] First of all, the analysis of economies of scope refers to the notion of total factor variation. Since the multiplication of an activity $(\mathbf{v},\mathbf{x})$ by a scalar $\lambda > 0$ causes no problems for indivisible goods and factors, Proposition III.10 serves as the initial point for the following analysis:

Proposition III.10 *Given perfectly divisible goods and factors, we have three implications for a linearly homogeneous, superhomogeneous, or subhomogeneous*[77] *input correspondence L with closed convex input requirement sets.*

$$\lambda L(\mathbf{x}) = L(\lambda\mathbf{x}) \ \forall \lambda > 0 \implies \lambda c(\cdot,\mathbf{x}) = c(\cdot,\lambda\mathbf{x}) \ \forall \lambda > 0 \tag{III.65a}$$

$$\lambda L(\mathbf{x}) \subset L(\lambda\mathbf{x}) \ \forall \lambda \geqq 1 \implies \lambda c(\cdot,\mathbf{x}) \geqq c(\cdot,\lambda\mathbf{x}) \ \forall \lambda \geqq 1 \tag{III.65b}$$

$$\lambda L(\mathbf{x}) \supseteq L(\lambda\mathbf{x}) \ \forall \lambda \geqq 1 \implies \lambda c(\cdot,\mathbf{x}) \leqq c(\cdot,\lambda\mathbf{x}) \ \forall \lambda \geqq 1 \tag{III.65c}$$

[74] See again Definition II.8, p. 44.

[75] Cf. BAUMOL, PANZAR, WILLIG (1982, p. 17).

[76] Besides economies to scale and economies of scope BAUMOL (1987) stresses fixed and sunk cost as implications of indivisibilities. In Chapter V there are some bibliographical references which take sunk cost as a reason for barriers of market entry into account.

[77] See Definition II.6, p. 39.
The proposition holds similarly for an input correspondence which is homogeneous of degree r. In this case we can show that the cost function is homogeneous of degree r in the outputs $\mathbf{x}$, too, where $r < 1$ implies increasing returns to scale. Applying EULER's Theorem gives

$$S(\mathbf{x}) = 1/r = c(\mathbf{q},\mathbf{x}) \Big/ \sum_{j=1}^{n} \frac{\partial c(\mathbf{q},\mathbf{x})}{\partial x_j} x_j$$

a plausible measure for the degree of scale economies if the cost function is differentiable. Further expositions can be taken from BAUMOL, PANZAR, WILLIG (1982, p. 50) as well as FÄRE, GROSSKOPF, LOVELL (1986, p. 178).

Proof: First of all, it has to be noted that the linearly homogeneous reciprocal support function (and therefore the cost function) has the following property for all $\lambda > 0, \mathbf{x} \in \overline{X}$ and $\mathbf{q} \in \overline{Q}$.

$$\text{(III.66)} \qquad \begin{aligned} \lambda\varphi(\mathbf{q}|L(\mathbf{x})) &= \varphi(\lambda\mathbf{q}\,|L(\mathbf{x})) \\ &= \inf\left\{(\lambda\mathbf{q})^{\mathsf{T}}\mathbf{v}|\ \mathbf{v} \in L(\mathbf{x})\right\} \\ &= \inf\left\{\mathbf{q}^{\mathsf{T}}\tilde{\mathbf{v}}|\ \tilde{\mathbf{v}}/\lambda \in L(\mathbf{x})\right\} \\ &= \varphi(\mathbf{q}\,|\lambda L(\mathbf{x})) \end{aligned}$$

For example, we have with respect to (III.65b)

$$\lambda L(\mathbf{x}) \subset L(\lambda\mathbf{x}) \quad\Longrightarrow\quad \lambda\varphi(\mathbf{q}\,|L(\mathbf{x})) = \varphi(\mathbf{q}\,|\lambda L(\mathbf{x})) \geqq \varphi(\mathbf{q}\,|L(\lambda\mathbf{x})). \qquad \blacksquare$$

Regarding a single-product firm strict economies of scale in (III.65b) correspond to falling average costs along the ray through the origin λx.

$$\frac{c(\cdot, x)}{x} > \frac{c(\cdot, \lambda x)}{\lambda x} \qquad \forall\,\lambda \geqq 1$$

Similarly, we speak in (III.65c) (i.e. increasing average costs) of strict diseconomies of scale. The criterion for economies of scope derived from a total factor variation now faces a criterion, that is a priori not connected with any special form of factor variation.

An example illustrates that despite integer returns to scale by Definition II.7 it is quite useful to speak at least locally of economies of scale.[78]

Example: *Assuming a divisible output quantity x, a* LEONTIEF *production function* $x = \min\{v_1/a_1, v_2/a_2\}$ *is scrutinized, where it is also assumed that the first factor is divisible and that the second factor is only available at integer units. At the same time the constant input coefficients a_1 and a_2 describe the capacity of each factor, i.e. v_2/a_2 is the maximal output using the input v_2. If, for instance,* $x = v_1/a_1 < v_2/a_2$, *then the indivisible second factor has excess capacities. Full employment of both factors results in constant average costs.*[79]

$$\frac{c(\cdot, x)}{x} = \frac{c(\cdot, \lambda x)}{\lambda x} \qquad \forall\,\lambda \in \mathbb{Z}_{++}$$

The rest of the (technically efficient) activities (v_1, v_2, x) not exploiting the

[78]The question whether it is tautological to put economies to scale down to the indivisibility of production factors is disregarded at this point; see CHAMBERLIN (1947/48, p. 236 f.) as well as MCLEOD, HAHN (1949). In the above discussion, CHAMBERLIN supports the thesis that the divisibility of all production factors rules out both economies and diseconomies of scale. In particular, the conclusion this would at once result in an economy without firms assuming perfect competition (CHAMBERLIN (1947/48, p. 229)) seems to be of doubtful value without consideration of factor constraints.

[79]Remember that marginal costs $\partial c(q_1, q_2, x)/\partial x$ are not defined at the points of jumps. In WILLIAMSON (1966) these points are approximated by vertical lines. Finally, within a graphical analysis the determination of the welfare maximizing capacity follows – that is v_2 – where the demand for the good concerned is held fixed. Although the supposed production technology generates integer constant returns to scale, it is shown that the firm will usually operate with losses in the long-term welfare optimum. KUMAGAI (1962) also refers to welfare theoretical arguments to assess indivisible investments.

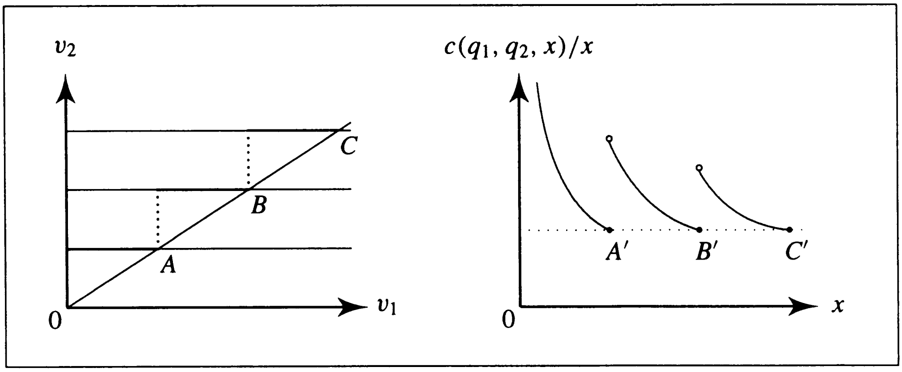

Figure III.18: Average costs with an indivisible input

capacities of the indivisible factors generates decreasing average costs over the interval $\lfloor v_2 \rfloor \leqq v_2 \leqq \lceil v_2 \rceil$. [80] *The left hand part of Figure III.18 illustrates the examined form of factor variation. Accordingly, the right hand part shows the appropriate development of average costs. Points A, B, and C reflect full employment of both factors and correspond to points A′, B′, and C′ with constant average costs.*

The presented phenomenon can be generalized for more than two production factors as the so called **harmonic law**.[81] For a LEONTIEF production function

$$x = \min\{v_i/a_i | \, i = 1, \ldots, m\}$$

all of the production factors are used up to their capacity provided $x = v_i/a_i$ is satisfied for all $i = 1, \ldots, m$. The average cost achieves its minimum whenever the output is a common multiple of the input coefficients of all indivisible factors. If the input coefficients of the indivisible inputs are $a_2 = 3$, $a_3 = 4$ and $a_4 = 6$ for a divisible factor 1, then the minimal average costs are attained at the "harmonic" output quantities 12, 24, 36 … . All the other output quantities are associated with excess capacities.[82] At this point we refrain from a graphical representation of the average costs. Similar to Figure III.18 the average cost curve jumps when at least one of the indivisible factors has to be augmented by one unit. The extent of the jumps is not least determined by what factor (2, 3 or 4) is raised and how many inputs are to be raised; at $x = 6$ the amounts v_2, and v_4 are augmented by one unit. Moreover, the jumps abate with rising outputs as shown in Figure III.18.[83]

[80] A similar example is presented in BREMS (1963/64). MATTHES (1996) describes an analogous phenomenon of indivisibility with respect to the calculation of telecommunication network charges. There the indivisibility is artificially introduced by checking telephone calls at discrete time intervals. As in WILSON (1993, especially Section 2.4) this problem is usually ignored.

[81] Cf. BREMS (1952, p. 580). The idea of the harmonic law has already been described in ROBINSON (1931, p. 33) and SCHNEIDER (1934, p. 83 ff.).

[82] SCAZZIERI (1993, p. 120 ff.) describes the analogous problem in time coordination of indivisible working steps.

[83] If the price of the good at hand is a small amount above minimal average cost, then the maximal

Besides the subadditivity of the cost function, Proposition III.9 yields a condition for the convexity of the cost function in commodities **x**, where perfectly divisible goods and factors are presumed. Supposing a firm produces a particular indivisible good $x \in \mathbb{Z}_+$, then the following criterion of an "integer convex" cost function at positive factor prices $\mathbf{q} > \mathbf{0}$ can be established:[84]

$$\text{(III.67)}\qquad \begin{array}{ll} c(\mathbf{q}, 1) > c(\mathbf{q}, 0) \quad \text{and} & \\ c(\mathbf{q}, x+2) - c(\mathbf{q}, x+1) \geqq c(\mathbf{q}, x+1) - c(\mathbf{q}, x) & \forall x \in \mathbb{Z}_+ \end{array}$$

The left hand part of Figure III.19 illustrates this assumption for $c(\mathbf{q}, 0) = 0$. Following (III.64) for $\lambda = 1/2$ the rearrangement of the second inequality leads to

$$1/2\, c(\mathbf{q}, \tilde{x}+2) + 1/2\, c(\mathbf{q}, \tilde{x}) \geqq c(\mathbf{q}, \tilde{x}+1) = c\big(\mathbf{q}, 1/2\,(\tilde{x}+2) + 1/2\,\tilde{x}\big).$$

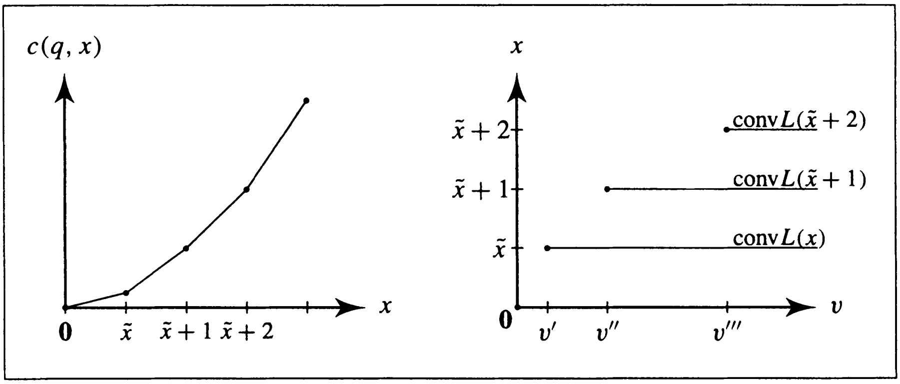

Figure III.19: Criterion of an integer convex cost function

Considering various interim findings, we obtain similar to (III.62a)

$$\begin{aligned} &1/2\, c(\mathbf{q}, \tilde{x}+2) + 1/2\, c(\mathbf{q}, \tilde{x}) \\ &\quad = 1/2\, \varphi\big(\mathbf{q} | \mathrm{conv} L(\tilde{x}+2)\big) + 1/2\, \varphi\big(\mathbf{q} | \mathrm{conv} L(\tilde{x})\big) && \text{see (III.55)} \\ &\quad = \varphi\big(\mathbf{q} | 1/2\, \mathrm{conv} L(\tilde{x}+2)\big) + \varphi\big(\mathbf{q} | 1/2\, \mathrm{conv} L(\tilde{x})\big) && \text{see (III.66)} \\ &\quad = \varphi\big(\mathbf{q} | 1/2\, \mathrm{conv} L(\tilde{x}+2) + 1/2\, \mathrm{conv} L(\tilde{x})\big) && \text{see (III.61)} \\ &\quad \geqq \varphi\big(\mathbf{q} | \mathrm{conv} L(\tilde{x}+1)\big) && \text{by assumption} \\ &\quad = c(\mathbf{q}, \tilde{x}+1). \end{aligned}$$

Thus, according to (III.60)

$$1/2\, \mathrm{conv} L(x+2) + 1/2\, \mathrm{conv} L(x) \subset \mathrm{conv} L(x+1) \qquad \forall x \in \mathbb{Z}_+$$

average profit is attained respectively with a harmonic output quantity. In the intervals between losses as well as profits per unit may occur.

[84] Cf. e.g. KANEKO, YAMAMOTO (1986, p. 122).

provides a necessary and sufficient criterion that the cost function c is integer convex. This condition is illustrated in the right hand part of Figure III.19 supposing a solitary input v. Referring to the marked quantities v', v'' and v''', the implication

$$\tfrac{1}{2}\,v''' + \tfrac{1}{2}\,v' \geqq v'' \quad\Longrightarrow\quad \tfrac{1}{2}\,v''' + \tfrac{1}{2}\,v' \in \operatorname{conv}L(\tilde{x}+1)$$

has already been noted by Proposition II.4, p. 31.

2.1.2 Reconstruction of the Production Structure

If the firm's cost function is known, then the question arises as to what information on the production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$ is included in the cost function. The subsequent proposition describes the reconstruction of an element of the examined family of input requirement sets. A graphical representation of the construction principle is shown by Figure II.32, p. 60.

Proposition III.11[85] *Each closed convex input requirement set $L(\mathbf{x})$ can be expressed as a system of inequalities given by the cost function.*

$$\begin{aligned} L(\mathbf{x}) &= \bigcap_{\mathbf{q}\in\overline{Q}} \big\{\mathbf{v}|\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq c(\mathbf{q},\mathbf{x})\big\} \\ &= \big\{\mathbf{v}|\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq c(\mathbf{q},\mathbf{x})\ \ \forall\mathbf{q} \in \overline{Q}\big\} \end{aligned}$$

Proof: Provided $L(\mathbf{x})$ is a closed convex input requirement set, then the system of inequalities in Proposition III.11 entails

$$\begin{aligned} &\big\{\mathbf{v}|\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq c(\mathbf{q},\mathbf{x})\ \ \forall\mathbf{q} \in \overline{Q}\big\} && \\ &= \big\{\mathbf{v}|\ \inf\big\{\mathbf{q}^{\mathsf{T}}\mathbf{v} - c(\mathbf{q},\mathbf{x})|\ \mathbf{q} \in \overline{Q}\big\} \geqq 0\big\} && \\ &= \{\mathbf{v}|\ c_*(\mathbf{v},\mathbf{x}) \geqq 0\} && \text{by definition} \\ &= \{\mathbf{v}|\ \varrho(\mathbf{v}|L(\mathbf{x})) \geqq 0\} && \text{see (III.56)} \\ & && \text{with}\ \ L(\mathbf{x}) = \operatorname{cl}(\operatorname{conv}L(\mathbf{x})) \\ &= L(\mathbf{x}) && \text{by definition} \end{aligned}$$

Recall at this point the favorable choice of functional values for the cost function by (III.58). ■

By Proposition III.11 the following equivalence relation immediately results for two closed convex input requirement sets[86]

$$c(\cdot,\mathbf{x}) \geqq c(\cdot,\tilde{\mathbf{x}}) \iff L(\mathbf{x}) \subset L(\tilde{\mathbf{x}}) \tag{III.68}$$

[85] The proposition corresponds to Corollary D.7.1, p. 323.

[86] See also (D.6), p. 324. Generally, two closed and convex sets C and D fulfill the equivalence relation

$$C \subset D \iff \varphi(\cdot|C) \geqq \varphi(\cdot|D).$$

Moreover, [L4] (*Disposability of Outputs*) states $\mathbf{x} \geqq \tilde{\mathbf{x}} \implies L(\mathbf{x}) \subset L(\tilde{\mathbf{x}})$.

since the left hand inequality implies

$$\{\mathbf{v}|\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq c(\mathbf{q},\mathbf{x})\ \ \forall\,\mathbf{q} \in \overline{Q}\} \subset \{\mathbf{v}|\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq c(\mathbf{q},\tilde{\mathbf{x}})\ \ \forall\,\mathbf{q} \in \overline{Q}\} \quad \text{(necessary part)},$$

and from the right hand inclusion ensues

$$\inf\{\mathbf{q}^{\mathsf{T}}\mathbf{v}|\ \mathbf{v} \in L(\mathbf{x})\} \geqq \inf\{\mathbf{q}^{\mathsf{T}}\mathbf{v}|\ \mathbf{v} \in L(\tilde{\mathbf{x}})\} \qquad \text{(sufficient part)}.$$

If the input requirement set $L(\mathbf{x})$ is not convex, then

$$L_{\diamond}(\mathbf{x}) := \{\mathbf{v}|\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq c(\mathbf{q},\mathbf{x})\ \ \forall\,\mathbf{q} \in \overline{Q}\}$$

is called an outer approximation of the input requirement set.[87] Observing Theorem D.2

$$\varrho_{**}(\cdot|\mathrm{conv}L(\mathbf{x})) = \mathrm{cl}\,\varrho(\cdot|\mathrm{conv}L(\mathbf{x})) = \varrho(\cdot|\,\mathrm{cl}(\mathrm{conv}L(\mathbf{x})))$$

we obtain

$$\begin{aligned}
L_{\diamond}(\mathbf{x}) &= \{\mathbf{v}|\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq \varphi(\mathbf{q}|L(\mathbf{x}))\ \ \forall\,\mathbf{q} \in \overline{Q}\} && \text{with } c(\cdot,\mathbf{x}) \equiv \varphi(\cdot|L(\mathbf{x}))\\
&= \{\mathbf{v}|\ \inf\{\mathbf{q}^{\mathsf{T}}\mathbf{v} - \varphi(\mathbf{q}|\mathrm{conv}L(\mathbf{x}))|\ \mathbf{q} \in \overline{Q}\} \geqq 0\} && \text{by (III.55)}\\
&= \{\mathbf{v}|\ \varphi_{*}(\mathbf{v}|\mathrm{conv}L(\mathbf{x})) \geqq 0\} && \text{by definition}\\
&= \{\mathbf{v}|\ \varrho_{**}(\mathbf{v}|\mathrm{conv}L(\mathbf{x})) \geqq 0\} && \text{by } \varphi(\cdot|\mathrm{conv}L(\mathbf{x}))\\
& && \quad = \varrho_{*}(\cdot|\mathrm{conv}L(\mathbf{x}))\\
&= \{\mathbf{v}|\ \varrho(\mathbf{v}|\,\mathrm{cl}(\mathrm{conv}L(\mathbf{x}))) \geqq 0\} && \text{Theorem D.2.}
\end{aligned}$$

Thus, the outer approximation $L_{\diamond}(\mathbf{x})$ denotes nothing more than the closure of the convex hull of the input requirement set concerned,[88]

$$L_{\diamond}(\mathbf{x}) = \mathrm{cl}\,(\mathrm{conv}L(\mathbf{x})).$$

Under the assumptions of Proposition II.14 the convex hull of an input requirement set is closed such that

$$L_{\diamond}(\mathbf{x}) = \mathrm{conv}L(\mathbf{x}).$$

Stipulating for the integer convexity of the input requirement sets by [L7a], the knowledge of the outer approximation $L_{\diamond}(\mathbf{x})$ is quite enough since the only observable market results are points which belong to $L_{\diamond}(\mathbf{x})$ as well as to $L(\mathbf{x}) = L_{\diamond}(\mathbf{x}) \cap V$. With regard to the cost function (III.55)[89]

$$\left.\begin{aligned} c(\cdot,\mathbf{x}) &\equiv \varphi(\cdot|L(\mathbf{x}))\\ c_{\diamond}(\cdot,\mathbf{x}) &\equiv \varphi(\cdot|L_{\diamond}(\mathbf{x}))\end{aligned}\right\} \Longrightarrow c(\cdot,\mathbf{x}) = c_{\diamond}(\cdot,\mathbf{x})$$

[87] Cf. DIEWERT (1982, p. 544) withholding the subsequent correlation.
[88] Cf. MAS-COLELL, WHINSTON, GREEN (1995, p. 78).
[89] Cf. MCFADDEN (1978, p. 22).

follows. Supposing a firm to produce a solitary good x, then a similar result can be offered with respect to the production function[90]

$$f(\mathbf{v}) = \max\{x|\ \mathbf{v} \in L(x)\}.$$

The adjoined cost function is[91]

$$\begin{aligned} c(\mathbf{q}, x) &= \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v}|\ \mathbf{v} \in L(x)\right\} \\ &= \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v}|\ f(\mathbf{v}) \geqq x\right\} \end{aligned}$$

and, therefore, it satisfies the premises of Propositions III.7 and III.9. If the cost function is known, then an approximate production function $f_\diamond$ can be established.

$$\begin{aligned} f_\diamond(\mathbf{v}) &:= \sup\left\{x|\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq c(\mathbf{q}, x)\ \ \forall \mathbf{q} \in \overline{Q}\right\} \\ &= \sup\{x|\ \mathbf{v} \in L_\diamond(x)\} \end{aligned}$$

Thus, $f = f_\diamond$ holds provided the input requirement set is closed and convex, $L(x) = L_\diamond(x)$. [92] Even if this assumption is not valid as in the case of an indivisible production factor, the observable market data $(\mathbf{v}, x, \mathbf{q})$ will not allow the differentiation between the two functions f and $f_\diamond$.[93] If the profit maximizing firm knows its particular production technology, then we have at least to presume $f(\mathbf{v}) = f_\diamond(\mathbf{v})$ for the factor quantities $\mathbf{v}$ demanded.

2.1.3 Comparison of the Derived Cost Functions

In the preceding sections two cost functions were derived from apparently independent concepts. On the one hand the concave FENCHEL transform generates by (III.13) the normalized cost function $\tilde{c}(\cdot, \mathbf{x}) : \mathcal{Q} \to [-\infty, +\infty]$ with

$$\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) \equiv g_*(\mathbf{q}^{-r}|L(\mathbf{x})) = \inf\left\{(\mathbf{q}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) - g(\mathbf{v}^{-r}|L(\mathbf{x}))|\ \mathbf{v}^{-r} \in \mathcal{V}\right\}.$$

On the other hand this definition faces the usual cost function $c(\cdot, \mathbf{x}) : \overline{Q} \to [-\infty, +\infty]$ in (III.55).

$$c(\mathbf{q}, \mathbf{x}) \equiv \varphi(\mathbf{q}|L(\mathbf{x})) = \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v}|\ \mathbf{v} \in L(\mathbf{x})\right\}$$

The difference between these two functions comes about by the diverse spaces of factor price vectors, $\mathcal{Q} = \mathbb{R}^{m-1}$ and $\overline{Q} = \mathbb{R}^m$. Both cost functions have the common property that their derivation does not depend on the fact whether it is based on the input requirement set $L(\mathbf{x})$ or on its convex hull $\mathrm{conv}L(\mathbf{x})$; see Proposition III.2 and Theorem II.2, p. 63. Moreover, the following section proves that both cost functions are equivalent.[94]

[90] For the definition of a production function, see Proposition II.12.

[91] This representation has been introduced in SHEPHARD (1953) and picked up in UZAWA (1964).

[92] Further statements of this form may be found in BLACKORBY, PRIMONT, RUSSEL (1978).

[93] Cf. DIEWERT (1982, p. 544).

[94] A similar proposition with respect to convex profit functions may be found in JORGENSON, LAU (1974, p. 193).

As shown by Proposition III.12, p. 147, the cost function $c(\cdot, \mathbf{x})$ denotes the smallest linearly homogeneous concave function generated by the modified normalized cost function $\tilde{c}(\cdot, \mathbf{x}) + \varrho(q_r|1)$.[95] Since Proposition III.14 comprehends a similar relation between the input distance function introduced below and the indicator function of an input requirement set, the expositions begin with an arbitrary concave function g, which, for the sake of a better distinction from the outputs $\mathbf{x}$, depends on the variables $\mathbf{y}$. The convex hypograph $\operatorname{hypo} g = \left\{\binom{\mathbf{y}}{\mu} \in \mathbb{R}^n \times \mathbb{R} \mid \mu \leqq g(\mathbf{y})\right\}$ of a concave function g can be associated with a unique smallest convex cone K containing the vertex $\mathbf{0} \in \mathbb{R}^{n+1}$ which is defined by $K = \{\lambda\xi \mid \lambda \geqq 0,\ \xi \in \operatorname{hypo} g\}$ with $\xi = \binom{\mathbf{y}}{\mu}$. Supposing now analogous to (III.3), p. 86,

$$k(\mathbf{y}) = \sup\left\{\mu \mid \tbinom{\mathbf{y}}{\mu} \in K\right\}, \tag{III.69}$$

k is called the **smallest linearly homogeneous concave function generated by the (concave) function g** since k is the smallest of all linearly homogeneous functions h fulfilling $h(\mathbf{0}) \geqq 0$ and $h \geqq g$.

Figure III.20 shows this construction principle for a concave function g. The hypograph $\operatorname{hypo} g$ corresponds to the set below g, whereas the cone K is enclosed by the rays through the origin, which are marked by k.

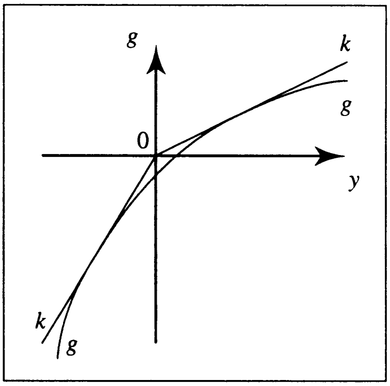

Figure III.20: The smallest linearly homogeneous concave function k generated by the function g.

Recursive substitution yields

$$k(\mathbf{y}) = \sup\left\{\mu \mid \tbinom{\mathbf{y}}{\mu} \in \{\lambda\xi \mid \lambda \geqq 0,\ \xi \in \operatorname{hypo} g\}\right\}.$$

In this function we have to distinguish the case $\lambda = 0$ from the case $\lambda > 0$. For an n-proper function g (with $\operatorname{hypo} g \neq \emptyset$) $\lambda = 0$ results in $k_1(\mathbf{y}) := \sup\left\{\mu \mid \binom{\mathbf{y}}{\mu} \in \mathbf{0}\right\}$ and therefore

$$k_1(\mathbf{y}) = \varrho(\mathbf{y}|\mathbf{0}) := \begin{cases} 0 & \text{for } \mathbf{y} = \mathbf{0} \\ -\infty & \text{for } \mathbf{y} \neq \mathbf{0}. \end{cases} \tag{III.70a}$$

If $\lambda > 0$, then by $\mathbf{y} = \lambda\tilde{\mathbf{y}}$ and $\mu = \lambda\tilde{\mu}$ it ensues

$$\begin{aligned} k_2(\mathbf{y}) &:= \sup\left\{\mu \mid \tbinom{\mathbf{y}}{\mu} \in \left\{\lambda\tbinom{\tilde{\mathbf{y}}}{\tilde{\mu}} \mid \lambda > 0,\ \tilde{\mu} \leqq g(\tilde{\mathbf{y}})\right\}\right\} \\ &= \sup\{\mu \mid \lambda > 0,\ \mu/\lambda \leqq g(\mathbf{y}/\lambda)\} \\ &= \sup\{\lambda g(\mathbf{y}/\lambda) \mid \lambda > 0\}. \end{aligned} \tag{III.70b}$$

[95] Appendix D.2.1 introduces the concept of the greatest linearly homogeneous function k generated by the (convex) function f. Similarly, we define at this point the smallest linearly homogeneous concave function generated by the (concave) function g.

By $k(\mathbf{y}) = \sup\{k_1(\mathbf{y}), k_2(\mathbf{y})\}$ both relations (III.70a) and (III.70b) evoke the smallest linearly homogeneous concave function k generated by g, i.e.

(III.71) $$k(\mathbf{y}) = \sup_{\lambda \geqq 0} \begin{Bmatrix} \lambda g(\mathbf{y}/\lambda) & \text{for } \lambda > 0 \\ \varrho(\mathbf{y}|\mathbf{0}) & \text{for } \lambda = 0 \end{Bmatrix}.$$

If $g(\mathbf{0}) < 0$ holds at point $\mathbf{y} = \mathbf{0}$, then the optimal λ in (III.70b) is $\lambda = 0$ so that (III.70a) must be applied instead of (III.70b). According to Figure III.20 that is $k(\mathbf{0}) = 0$. Actually for $g(\mathbf{0}) = 0$ we have $k(\mathbf{0}) = 0$. Nevertheless, $g(\mathbf{0}) > 0$ implies the relation $k(\mathbf{0}) = +\infty$. [96]

Equation (III.71) is again picked up in Proposition III.14. There, Figure III.23 also provides a graphical representation of the transferred results. If we derive the smallest linearly homogeneous concave function generated by the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$, then it turns out that the presented concept is not very helpful at this point. As shown by Quadrant II in Figure III.12, p. 114, the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ intersects the ordinate as a positive value provided factor r is used in the production at a positive amount, $\tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) > 0$ for $\mathbf{q}^{-r} = \mathbf{0}$.

But if we define the concave function

(III.72) $$\tilde{g}(\mathbf{q}^{-r}, \lambda) := \tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) + \varrho(\lambda|1) = \begin{cases} \tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) & \text{for } \lambda = 1 \\ -\infty & \text{for } \lambda \neq 1, \end{cases}$$

with $\mathbf{x}$ denoting an admissible commodity bundle with respect to an n-proper cost function $\tilde{c}(\cdot, \mathbf{x})$, then the function $\tilde{g}$ generates the following smallest linearly homogeneous concave function k.[97]

(III.73) $$k(\mathbf{q}^{-r}, \lambda) := \begin{cases} \lambda \tilde{c}(\mathbf{q}^{-r}/\lambda, \mathbf{x}) & \text{for } \lambda > 0 \\ \varrho(\mathbf{q}^{-r}|\mathbf{0}) & \text{for } \lambda = 0 \\ -\infty & \text{for } \lambda < 0 \end{cases}$$

The additional variable λ will adopt the role of the nominal factor price q_r afterwards. The purpose of the following explanations is to prove the close relationship between the function k defined by (III.73) and the cost function $c(\cdot, \mathbf{x})$. In doing this we fall back upon the construction principle of the function $c(\cdot, \mathbf{x})$ as a support function of the input requirement set $L(\mathbf{x})$.

Before we go into the concave form of Corollary D.9.1 using Proposition III.12 it has to be noted that according to (III.73) the closure of the concave function k is given by[98]

(III.74) $$\operatorname{cl} k(\mathbf{q}^{-r}, \lambda) = \begin{cases} \lambda \tilde{c}(\mathbf{q}^{-r}/\lambda, \mathbf{x}) & \text{for } \lambda > 0 \\ \lim\limits_{\lambda \downarrow 0} \lambda \tilde{c}(\mathbf{q}^{-r}/\lambda, \mathbf{x}) & \text{for } \lambda = 0 \\ -\infty & \text{for } \lambda < 0, \end{cases}$$

provided $\tilde{c}(\cdot, \mathbf{x})$ is an n-proper closed concave function with $\mathbf{0} \in \text{n-Dom}\, \tilde{c}(\cdot, \mathbf{x})$.

[96] In the last case the definition of the smallest linearly homogeneous concave function k generated by g remains without any economic meaning.

[97] Take into consideration that $\lambda \tilde{g}(\mathbf{q}^{-r}/\lambda, 1) = \lambda \tilde{c}(\mathbf{q}^{-r}/\lambda, \mathbf{x})$.

[98] The proof of this statement may be found in ROCKAFELLAR (1972, p. 67, Corollary 8.5.2).

Proposition III.12 *Letting* $\tilde{c}(\cdot, \mathbf{x})\colon \mathcal{Q} \to [-\infty, +\infty]$ *be the normalized cost function for a commodity bundle* $\mathbf{x} \in X$, *then the function* $c(\cdot, \mathbf{x})\colon \overline{Q} \to [-\infty, +\infty]$ *with*

$$\text{(III.75)} \qquad c\left(\binom{\mathbf{q}^{-r}}{\lambda}, \mathbf{x}\right) := \begin{cases} \lambda\tilde{c}(\mathbf{q}^{-r}/\lambda, \mathbf{x}) & \text{for } \lambda > 0 \\ \lim\limits_{\lambda \downarrow 0} \lambda\tilde{c}(\mathbf{q}^{-r}/\lambda, \mathbf{x}) & \text{for } \lambda = 0 \\ -\infty & \text{for } \lambda < 0\,. \end{cases}$$

is equivalent to the support function of the set $C \subset \overline{V}$ *with*

$$C = \left\{\binom{\mathbf{v}^{-r}}{\mu} \mid \mu \geqq -\tilde{c}_*(\mathbf{v}^{-r}, \mathbf{x})\right\} = \operatorname{epi}\left(-\tilde{c}_*(\cdot, \mathbf{x})\right).$$

Due to $C = \operatorname{conv} L(\mathbf{x})$, $c(\cdot, \mathbf{x})$ *can be understood as the corresponding cost function.*

Proof: First of all, it is shown that the function defined by (III.75) is the support function of the set C concerned. The ensuing expositions prove that the set C equals the convex hull of the input requirement set $L(\mathbf{x})$ so that (III.75) determines the cost function $c(\cdot, \mathbf{x})$ indeed.[99]

Analogous to (III.72), we define the function $\tilde{g}(\mathbf{q}^{-r}, \lambda) := \tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) + \varrho(\lambda | 1)$. Because by Proposition III.7 the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ is n-proper, closed, and concave for each commodity bundle $\mathbf{x} \in X$ and as $\varrho(\lambda | 1)$ has these properties as well, the function $\tilde{g}$ must be n-proper, closed, and concave, too. Thus, $\tilde{g}$ meets all requirements of the concave version of Theorem D.9.[100] At the same time by (III.73) the function k denotes the smallest linearly homogeneous concave function generated by $\tilde{g}$. For the closure $\operatorname{cl} k$ a comparison of (III.74) and (III.75) implies $\operatorname{cl} k = c(\cdot, \mathbf{x})$. [101] The concave version of Theorem D.9 says that $\operatorname{cl} k$ is the support function of the set $\left\{\binom{\mathbf{v}^{-r}}{\mu} \mid g_*(\mathbf{v}^{-r}, \mu) \geqq 0\right\}$. By rearranging

$$\begin{aligned} \tilde{g}_*(\mathbf{v}^{-r}, \mu) &= \inf\left\{(\mathbf{q}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) + \lambda\mu - \tilde{g}(\mathbf{q}^{-r}, \lambda) \mid \mathbf{q}^{-r} \in \mathcal{Q}\,,\ \lambda \in \mathbb{R}\right\} \\ &= \inf\left\{(\mathbf{q}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) + \lambda\mu - \tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) - \varrho(\lambda|1) \mid \mathbf{q}^{-r} \in \mathcal{Q}\,,\ \lambda \in \mathbb{R}\right\} \\ &= \inf\left\{(\mathbf{q}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) + \mu - \tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) \mid \mathbf{q}^{-r} \in \mathcal{Q}\right\} \\ &= \mu + \tilde{c}_*(\mathbf{v}^{-r}, \mathbf{x}) \end{aligned}$$

the first part of the proof is completed since

$$\left\{\binom{\mathbf{v}^{-r}}{\mu} \mid \tilde{g}_*(\mathbf{v}^{-r}, \mu) \geqq 0\right\} = \left\{\binom{\mathbf{v}^{-r}}{\mu} \mid \mu \geqq -\tilde{c}_*(\mathbf{v}^{-r}, \mathbf{x})\right\} =: C,$$

If the function $c(\cdot, \mathbf{x})$ defined by (III.75) is identical to the support function φ of the set C, $c(\cdot, \mathbf{x}) \equiv \varphi(\cdot | C)$, then (III.55) permits us for calling this

[99] In Figure III.36, p. 181, Proposition III.12 is again picked up and interpreted geometrically.

[100] Theorem D.9 and its concave version may be found on p. 327.
For $\mathbf{x} \notin X$ we get $\tilde{c}(\cdot, \mathbf{x}) \equiv +\infty$. In this economic irrelevant case the cost function $c(\cdot, \mathbf{x}) \equiv +\infty$ does not coincide with the function defined by (III.75).

[101] Corresponding to this relation [C4] says that the cost function $c(\cdot, \mathbf{x})$ is closed.

function a cost function if we can show that $C = \operatorname{conv} L(\mathbf{x})$. Considering $f(\cdot|L(\mathbf{x})) = -g(\cdot|L(\mathbf{x}))$ defined by (III.5), p. 89, we get $\tilde{c}_*(\cdot, \mathbf{x}) = g_{**}(\cdot|L(\mathbf{x}))$ in accordance with (III.13). Using Corollary III.3.1, p. 99, yields the required relation

$$\begin{aligned}
\operatorname{conv} L(\mathbf{x}) &= \operatorname{epi} f(\cdot|\operatorname{conv} L(\mathbf{x})) \\
&= \operatorname{epi} f^{**}(\cdot|L(\mathbf{x})) && \text{because of (III.26b)} \\
&= \left\{ \binom{\mathbf{v}^{-r}}{\mu} \middle| \; \mu \geqq f^{**}(\mathbf{v}^{-r}|L(\mathbf{x})) \right\} && \text{by definition} \\
&= \left\{ \binom{\mathbf{v}^{-r}}{\mu} \middle| \; \mu \geqq -g_{**}(\mathbf{v}^{-r}|L(\mathbf{x})) \right\} \\
&= \left\{ \binom{\mathbf{v}^{-r}}{\mu} \middle| \; \mu \geqq -\tilde{c}_*(\mathbf{v}^{-r}, \mathbf{x}) \right\} = C. && \blacksquare
\end{aligned}$$

After the cost function $c(\cdot, \mathbf{x})$ has been deduced from the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ by Proposition III.12, the question arises concerning the dimensions of the used variables. First of all, it is noticeable that both reciprocal support functions

$$\begin{aligned}
\text{(III.76a)} \quad c\left(\binom{\mathbf{q}^{-r}}{\lambda}, \mathbf{x}\right) &= \varphi\left(\binom{\mathbf{q}^{-r}}{\lambda} \middle| C\right) \\
&= \inf \left\{ (\mathbf{q}^{-r})^{\mathsf{T}}(\mathbf{v}^{-r}) + \lambda\mu \;\middle|\; \binom{\mathbf{v}^{-r}}{\mu} \in C \right\} && [\text{units of } v_r] \\
\text{(III.76b)} \quad c(\mathbf{q}, \mathbf{x}) &\equiv \varphi(\mathbf{q}|\operatorname{conv} L(\mathbf{x})) \\
&= \inf \left\{ \mathbf{q}^{\mathsf{T}}\mathbf{v} \;\middle|\; \mathbf{v} \in \operatorname{conv} L(\mathbf{x}) \right\} && [\$]
\end{aligned}$$

suggest the relation $\mathbf{q} = \binom{\mathbf{q}^{-r}}{\lambda}$. At the same time the new variable λ introduced in (III.75) must be dimensionless. The resulting contradiction can be eliminated by multiplying the linearly homogeneous function k of the proof by a factor price q_r of dimension $[\$/\text{units of } v_r]$ at point $\lambda = 1$.

$$q_r k(\mathbf{q}^{-r}, 1) = k(q_r \mathbf{q}^{-r}, q_r)$$

With that the m nominal factor prices $\mathbf{q} = \binom{q_r \mathbf{q}^{-r}}{q_r}$ with $[\$/\text{units of } v_i]$ result from the $m-1$ relative factor prices $\mathbf{q}^{-r}$ with $[\text{units of } v_r/\text{units of } v_i]$. Thus, regarding the cost functions we have at point $\lambda = 1$

$$c(\mathbf{q}, \mathbf{x}) = k(q_r \mathbf{q}^{-r}, q_r) = q_r \tilde{c}(\mathbf{q}^{-r}, \mathbf{x}) .$$

Alternatively, it can be supposed that the quantity of factor r is measured in money units [$]. Now for nominal factor prices $\mathbf{q}^{-r}$ the normalized cost function (III.11) is also measured in money units. After introducing the dimensionless variable λ with $\mathbf{q} = \binom{\mathbf{q}^{-r}}{\lambda}$ we obtain

$$c\left(\binom{\mathbf{q}^{-r}}{\lambda}, \mathbf{x}\right) = k(\mathbf{q}^{-r}, \lambda) = \lambda \tilde{c}(\mathbf{q}^{-r}/\lambda, \mathbf{x})$$

for all $\lambda > 0$. Because of economic aspects the first variant is subsequently preferred.

Finally, with regard to factor demand the deduced results can be compared. Specifically for differentiability of the two cost functions, the equivalence of

SHEPHARD's Lemma arises in the form of (III.50a) and (III.100a)[102] as follows: for a positive nominal price $q_r > 0$ of factor r and the relative prices of the other factors $q_i^{-r} = q_i/q_r$ $(i = 1,\dots,m,\ i \neq r)$ we gain the chosen quantity of factor i with $i \neq r$ from

$$\text{(III.77)} \qquad \hat{v}_i = \frac{\partial c}{\partial q_i}(\mathbf{q},\mathbf{x}) = q_r \cdot \frac{\partial \tilde{c}}{\partial q_i^{-r}}(\mathbf{q}^{-r},\mathbf{x}) \cdot \frac{\partial q_i^{-r}}{\partial q_i} = \frac{\partial \tilde{c}}{\partial q_i^{-r}}(\mathbf{q}^{-r},\mathbf{x}).$$

The quantity of factor r demanded results from

$$\begin{aligned}
\hat{v}_r = \frac{\partial c}{\partial q_r}(\mathbf{q},\mathbf{x}) &= \tilde{c}(\mathbf{q}^{-r},\mathbf{x}) + q_r \sum_{i \neq r} \frac{\partial \tilde{c}}{\partial q_i^{-r}}(\mathbf{q}^{-r},\mathbf{x}) \cdot \frac{\partial q_i^{-r}}{\partial q_r} && \text{note (III.77)}\\
&= \tilde{c}(\mathbf{q}^{-r},\mathbf{x}) - \sum_{i \neq r} \hat{v}_i q_i^{-r} && \text{since } q_i^{-r} = q_i/q_r\\
&= g_*(\mathbf{q}^{-r} | L(\mathbf{x})) - (\mathbf{q}^{-r})^{\mathsf{T}}(\hat{\mathbf{v}}^{-r}) && \text{by definition}\\
&= f(\hat{\mathbf{v}}^{-r} | L(\mathbf{x})) && \text{since (III.27b)}
\end{aligned}$$

As expected, by SHEPHARD's Lemma the two cost functions lead to the same factor demand.

Example: *To conclude, the example on p. 114 ff., where the explanations are based on the input requirement set*[103]

$$\text{(III.41)} \qquad L(\tilde{x}) = \left\{ \binom{v_1}{v_2} \in \mathbb{Z}_+^2 \,\middle|\, \binom{v_1}{v_2} \geqq \binom{v_1'}{v_2'} \text{ or } \binom{v_1}{v_2} \geqq \binom{v_1''}{v_2''} \right\}$$

is again picked up. According to (III.45), the normalized cost function $\tilde{c}(\cdot,\tilde{x})$ is now

$$\tilde{c}(q_1^{-r},\tilde{x}) = \begin{cases} q_1^{-r} v_1' + v_2' & \text{for } \quad q_1^{-r} \geqq \tilde{q}_1^{-r} \\ q_1^{-r} v_1'' + v_2'' & \text{for } 0 \leqq q_1^{-r} \leqq \tilde{q}_1^{-r} \\ -\infty & \text{for } \quad q_1^{-r} < 0. \end{cases} \qquad \text{with } \tilde{q}_1^{-r} := \frac{v_2' - v_2''}{v_1'' - v_1'},$$

where the superscript $-r$ (with $r = 2$) stresses that it is a normalized price. From now on we have to distinguish explicitly the relative factor price $q_1^{-r} \equiv q_1^{-2} := q_1/q_2$ from the nominal factor price q_1. Consequently, the price defined by (III.43) is denoted $\tilde{q}_1^{-r}$.

To derive the cost function $c(\cdot,\tilde{x})$, we have to calculate the function cl *k according to (III.74) using Proposition III.12:*

[102] See p. 120 as well as p. 164.

[103] A graphical discussion of the following relations is given in Figure III.34, p. 179.

(III.78)

$$
\begin{aligned}
\operatorname{cl} k(q_1^{-r}, \lambda) &= \begin{cases} \lambda \tilde{c}\left(q_1^{-r}/\lambda\,,\, \tilde{x}\right) & \text{for } \lambda > 0 \\ \lim\limits_{\lambda \downarrow 0} \lambda \tilde{c}\left(q_1^{-r}/\lambda\,,\, \tilde{x}\right) & \text{for } \lambda = 0 \\ -\infty & \text{for } \lambda < 0 \end{cases} \\
&= \begin{cases} \lambda\left((q_1^{-r}/\lambda)\cdot v_1' + v_2'\right) & \text{for } \lambda > 0 \text{ and } q_1^{-r}/\lambda \geqq \tilde{q}_1^{-r} \\ \lambda\left((q_1^{-r}/\lambda)\cdot v_1'' + v_2''\right) & \text{for } \lambda > 0 \text{ and } 0 < q_1^{-r}/\lambda \leqq \tilde{q}_1^{-r} \\ \lim\limits_{\lambda \downarrow 0} \lambda\left((q_1^{-r}/\lambda)\cdot v_1' + v_2'\right) & \text{for } \lambda = 0 \text{ and } q_1^{-r} > 0 \\ \lim\limits_{\lambda \downarrow 0} \lambda\left((0/\lambda)\cdot v_1'' + v_2''\right) & \text{for } \lambda = 0 \text{ and } q_1^{-r} = 0 \\ -\infty & \text{for } \lambda < 0 \text{ or } q_1^{-r} < 0 \end{cases} \\
&= \begin{cases} q_1^{-r} v_1' + \lambda v_2' & \text{for } \lambda > 0 \text{ and } q_1^{-r}/\lambda \geqq \tilde{q}_1^{-r} \\ q_1^{-r} v_1'' + \lambda v_2'' & \text{for } \lambda > 0 \text{ and } 0 < q_1^{-r}/\lambda \leqq \tilde{q}_1^{-r} \\ q_1^{-r} v_1' & \text{for } \lambda = 0 \text{ and } q_1^{-r} > 0 \\ 0 & \text{for } \lambda = 0 \text{ and } q_1^{-r} = 0 \\ -\infty & \text{for } \lambda < 0 \text{ or } q_1^{-r} < 0. \end{cases}
\end{aligned}
$$

Proposition III.12 states that this function equals the cost function $c(\cdot, \tilde{x})$ namely $c(\cdot, \tilde{x}) = \operatorname{cl} k$. If we calculate the cost function for the underlying input requirement set $L(\tilde{x})$ using (III.53), then

(III.79)

$$
\begin{aligned}
c(q_1, q_2, \tilde{x}) &\equiv \varphi(q_1, q_2 | L(\tilde{x})) \\
&= \inf\left\{ q_1 v_1 + q_2 v_2 \,\middle|\, \binom{v_1}{v_2} \in L(\tilde{x}) \right\} \\
&= \begin{cases} q_1 v_1' + q_2 v_2' & \text{for } q_2 > 0,\ q_1 > 0 \text{ and } q_1/q_2 \geqq \tilde{q}_1^{-r} > 0 \\ q_1 v_1'' + q_2 v_2'' & \text{for } q_2 > 0,\ q_1 > 0 \text{ and } q_1/q_2 \leqq \tilde{q}_1^{-r} \\ q_2 v_2'' & \text{for } q_2 > 0,\ q_1 = 0 \\ q_1 v_1' & \text{for } q_2 = 0,\ q_1 > 0 \\ 0 & \text{for } q_2 = 0,\ q_1 = 0 \\ -\infty & \text{for } q_2 < 0 \text{ or } q_1 < 0. \end{cases}
\end{aligned}
$$

Proposition III.12 is confirmed by the example when comparing (III.78) and (III.79). For $q_1^{-r}/\lambda = q_1$ and $q_2 = \lambda$ the equation $c(q_1, q_2, \tilde{x}) = \operatorname{cl} k(q_1, q_2)$ ensues.

The previous expositions introduce the relationship between the cost function $c(\cdot, \mathbf{x})$ and two more functions. On the one hand the cost function $c(\cdot, \mathbf{x})$ as the support function of the input requirement set $L(\mathbf{x})$ equals the concave conjugate function of the indicator function $\varrho(\cdot|\operatorname{conv} L(\mathbf{x}))$ (Proposition III.6). On the other hand the cost function $c(\cdot, \mathbf{x})$ corresponds to the closure of the smallest linearly homogeneous concave function k generated by the (modified) normalized cost function $\tilde{c}(\cdot, \mathbf{x}) + \varrho(\lambda|1)$ (Proposition III.12). With that the previous analysis is mainly characterized by the duality relations of *conjugate functions.*

The next section compares the cost function $c(\cdot, \mathbf{x})$ with a "gauge", where the relationship between the two functions is called the *polarity of gauge functions.* Before going into this property, we have to introduce the concept of gauges. In doing this the input distance function can be interpreted under certain assumptions as a measure for the input efficiency.

2.2 The Input Distance Function

Before going into the case of indivisible factors it turns out to be helpful first of all to suppose convex input requirement sets with divisible factors. However, each of the considered commodity bundles $\mathbf{x} \in X$ may involve indivisible outputs.

The **(reciprocal) gauge function** $\psi(\cdot|L(\mathbf{x}))\colon \overline{V} \to [0, +\infty] \cup \{-\infty\}$ of a nonempty convex input requirement set is defined by[104]

$$\text{(III.80)} \qquad \psi(\mathbf{v}|L(\mathbf{x})) := \sup\{\lambda \geqq 0 |\ \mathbf{v} \in \lambda L(\mathbf{x})\}.$$

For a given production technology L we also write

$$t_I(\cdot, \mathbf{x}) \equiv \psi(\cdot|L(\mathbf{x})),$$

where the function $t_I(\cdot, \mathbf{x})$ with a given commodity bundle $\mathbf{x} \in X$ is called the **restricted input distance function** (factor input minimal cost function or deflation function). Varying the vector $\mathbf{x}$ and, therefore, the input requirement set $L(\mathbf{x})$ results in a function $t_I\colon \overline{V} \times X \to [0, +\infty] \cup \{-\infty\}$, which is called the **input distance function** of the input correspondence L.

Intuitively, the concept of the input distance function can be understood when stipulating that there is a positive optimal solution $\hat{\lambda} > 0$ for (III.80). Under this assumption, (III.80) can be rewritten as

$$t_I(\mathbf{v}, \mathbf{x}) = \max\{\lambda > 0 |\ \mathbf{v}/\lambda \in L(\mathbf{x})\}.$$

Thus, given the commodity bundle $\mathbf{x}$, the input distance function provides a measure of how much the input vector $\mathbf{v}$ must be shrunk along a ray through the origin and $\mathbf{v}$ so that the resulting input vector is efficient with respect to the input requirement set $L(\mathbf{x})$.

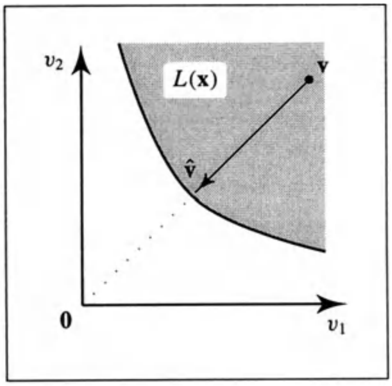

Figure III.21: Input distance function

For $t_I(\mathbf{v}, \mathbf{x}) > 1$ there is a smaller input vector which also permits the production of the commodity bundle $\mathbf{x}$. A functional value $t_I(\mathbf{v}, \mathbf{x}) < 1$ means that the input vector $\mathbf{v}$ is insufficient for the production of the commodity bundle $\mathbf{x}$. In accordance with (II.23) a comparison to FARRELL's

[104] See Definition D.4, p. 329. ZIESCHANG (1983) discusses how to transfer this concept of total factor variation to a partial factor variation.

input efficiency measure yields[105]

$$t_I(\mathbf{v},\mathbf{x}) = \frac{1}{\mathcal{F}_I(\mathbf{v},\mathbf{x})} \quad \text{for } \mathbf{v} \in L(\mathbf{x}).$$

According to Figure III.21 the term distance function can be justified if we express the input efficient vector $\hat{\mathbf{v}}$ as $\hat{\mathbf{v}} = \mathbf{v}/t_I(\mathbf{v},\mathbf{x})$ so that the value of the input distance function denotes the distance ratio,[106]

$$t_I(\mathbf{v},\mathbf{x}) = \|\mathbf{v}\| / \|\hat{\mathbf{v}}\| .$$

This form of the gauge follows the original definition of MINKOWSKI and it is used by SHEPHARD, who stipulates the existence of a vector $\hat{\mathbf{v}}$.[107] If $t_I(\mathbf{v},\mathbf{x}) = 1$, the input vector lies in the boundary of the input requirement set, $\mathbf{v} \in \partial L(\mathbf{x})$. For perfectly divisible inputs as well as for convex and closed input requirement sets this criterion establishes a necessary and sufficient condition for an efficient utilization of factors if the boundary of the input requirement set equals the set of all input vectors which are input efficient, $\partial L(\mathbf{x}) = \operatorname{Eff} L(\mathbf{x})$. An alternative representation of the input distance function is shown in Figure III.23 which is more suitable for a direct interpretation of (III.80) since the input requirement set $L(\mathbf{x})$ can be scaled directly.

Having pointed out the economic meaning of the input distance function as a measure for the input efficiency of an activity $(\mathbf{v},\mathbf{x})$, the question now arises as to how to overcome the case of indivisible production factors. Again we try to replace nonconvex input requirement sets with their convex hull $\operatorname{conv}L(\mathbf{x})$. Note that usually $\psi(\mathbf{v}|L(\mathbf{x})) \neq \psi(\mathbf{v}|\operatorname{conv}L(\mathbf{x}))$ will hold. But if we define the input distance function t_I with respect to the convex hull of an input requirement set, then we can write furthermore

$$\text{(III.81)} \qquad t_I(\cdot,\mathbf{x}) \equiv \psi(\cdot|\operatorname{conv}L(\mathbf{x})),$$

where $\operatorname{conv}L(\mathbf{x}) = L(\mathbf{x})$ has to be taken into account for convex input requirement sets.

Therefore, we run into difficulties which have been discussed in detail in Section 2.4.1. It can be ruled out that an efficient input vector becomes an efficient input vector by the operation $L(\mathbf{x}) \to \operatorname{conv}L(\mathbf{x})$, but not every efficient input vector $\tilde{\mathbf{v}}$ yields $t_I(\tilde{\mathbf{v}},\mathbf{x}) = 1$ for the value of the input distance function; see point A in Figure II.28, p. 55. Nevertheless, it turns out that such a vector always lies in the interior of the convex hull $\operatorname{conv}L(\mathbf{x})$. This results in two implications. First of all, there must be input efficient vectors corresponding to the extreme points of $\operatorname{conv}L(\mathbf{x})$. Hence, secondly it follows that not all of these extreme points can lead to higher costs than $\tilde{\mathbf{v}}$. Thus, efficient input vectors exist which permit a cheaper production of the commodity bundle $\mathbf{x}$ in comparison with $\tilde{\mathbf{v}}$ so that efficient input vectors as $\tilde{\mathbf{v}}$ may be disregarded without losing too much information.

A second problem has been treated, too. Not all vectors $\mathbf{v}$ with $t_I(\mathbf{v},\mathbf{x}) = 1$ can be interpreted as input vector $\mathbf{v} \in V$. Thus, the criterion $t_I(\hat{\mathbf{v}},\mathbf{x}) = 1$

[105] In contrast to FARRELL's measure of input efficiency, the input distance function also includes the cases $\mathbf{v} \notin L(\mathbf{x})$.

[106] In DEBREU (1951) the ratio $\|\hat{\mathbf{v}}\|/\|\mathbf{v}\|$ is called the coefficient of resource utilization.

[107] Cf. SHEPHARD (1953, p. 65 ff.).

is neither necessary nor sufficient for an efficient use of inputs. Not all efficient input vectors $\mathbf{v}$ lie in the boundary of the (closed) convex hull $\operatorname{conv}L(\mathbf{x})$ and not all points $\mathbf{v} \in \partial(\operatorname{conv}L(\mathbf{x}))$ describe an efficient input vector. However, [L7a] (*Integer Convexity*) permits the conclusion from $t_I(\mathbf{v}, \mathbf{x}) = 1$ and $\mathbf{v} \in V$ to an efficient input vector $\mathbf{v} \in L(\mathbf{x})$ provided $\operatorname{conv}L(\mathbf{x})$ is a closed set. In particular, the extreme points of $\operatorname{conv}L(\mathbf{x})$ denote efficient input vectors, at which the input distance function takes the value 1.

The main properties of the input distance function[108] are noted by the next proposition. Remember that the input vector $\mathbf{v} = \mathbf{0}$ is not included in the corresponding input requirement set $L(\mathbf{x})$ for any commodity bundle $\mathbf{x} \in X \setminus \{\mathbf{0}\}$; see Axiom [L1b].

Proposition III.13 *Let $L(\mathbf{x})$ be a nonempty subset in $\overline{V}$ with $\mathbf{0} \notin \operatorname{cl} L(\mathbf{x})$. Then the input distance function $t_I(\cdot, \mathbf{x})$ has the following properties:*

1. *The function $t_I(\cdot, \mathbf{x})$ is n-proper, $-\infty \leqq t_I(\cdot, \mathbf{x}) < +\infty \ \forall \mathbf{v} \in \overline{V}$.*
2. *The function $t_I(\cdot, \mathbf{x})$ is linearly homogeneous,*[109] *$\lambda t_I(\mathbf{v}, \mathbf{x}) = t_I(\lambda \mathbf{v}, \mathbf{x})$ $\forall \lambda > 0$. Furthermore, it is $t_I(\cdot, \mathbf{x}) \equiv \psi(\cdot | L(\mathbf{x})) = \lambda \psi(\cdot | \lambda L(\mathbf{x})) \ \forall \lambda > 0$.*
3. *We have $t_I(\mathbf{v}, \mathbf{x}) = 0$ if and only if $\mathbf{v} = \mathbf{0}$.*
   The inequality $t_I(\mathbf{v}, \mathbf{x}) > 0$ is satisfied if and only if $\mathbf{v} \in \operatorname{cone} L(\mathbf{x}) \setminus \{\mathbf{0}\}$.
   Thus, the effective domain is $\text{n-Dom}\, t_I(\cdot, \mathbf{x}) = \operatorname{cone} L(\mathbf{x})$.
4. *If $L(\mathbf{x})$ is closed and if $t_I(\mathbf{v}, \mathbf{x}) \geqq 0$, then the supremum is attained, $t_I(\mathbf{v}, \mathbf{x}) = \max \{\lambda \geqq 0 | \ \mathbf{v} \in \lambda L(\mathbf{x})\}$.*
5. *If $L(\mathbf{x})$ is closed, then $t_I(\cdot, \mathbf{x})$ is lower semi-continuous on* $\operatorname{cone} L(\mathbf{x})$.
6. *For convex $L(\mathbf{x})$ the function $t_I(\cdot, \mathbf{x})$ is concave and therefore superadditive.*
$$t_I(\mathbf{v} + \tilde{\mathbf{v}}, \mathbf{x}) \geqq t_I(\mathbf{v}, \mathbf{x}) + t_I(\tilde{\mathbf{v}}, \mathbf{x}) \qquad \forall \mathbf{v}, \tilde{\mathbf{v}} \in \overline{V}$$
7. *If $\operatorname{int} L(\mathbf{x}) \neq \emptyset$, then $t_I(\cdot, \mathbf{x})$ is continuous on* $\operatorname{cone}\left(\operatorname{int} L(\mathbf{x})\right)$.

The third property describes the effective domain of the input distance function by the projection cone $\operatorname{cone} L(\mathbf{x})$. This set denotes according to (II.26), p. 59, the cone generated by the input requirement set $L(\mathbf{x})$:

$$\operatorname{cone} L(\mathbf{x}) := \{\lambda \mathbf{v} | \ \mathbf{v} \in L(\mathbf{x}), \ \lambda \geqq 0\} \subset \mathbb{R}^m_+.$$

Under [L3] (*Disposability of Inputs*) together with the recession cone $0^+L(\mathbf{x}) = \mathbb{R}^m_+$ introduced in (II.11), p. 27, the closure results from[110]

$$\operatorname{cl}\left(\operatorname{cone} L(\mathbf{x})\right) = \operatorname{cone} L(\mathbf{x}) \cup 0^+L(\mathbf{x}) = \mathbb{R}^m_+.$$

[108] See Theorem D.11, p. 330.

[109] An output correspondence P is homogeneous of degree $r > 0$ if and only if the input distance function t_I is homogeneous of degree $-1/r$ in $\mathbf{x}$. Cf. JACOBSEN (1970, p. 762).

[110] Cf. ROCKAFELLAR (1972, p. 78, Theorem 9.6).

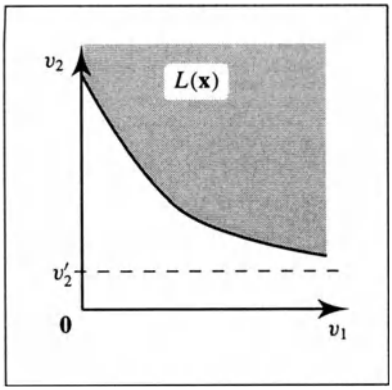

Figure III.22: The effective domain of the input distance function

Since the two sets $\operatorname{cone} L(\mathbf{x})$ and $\operatorname{cl}\big(\operatorname{cone} L(\mathbf{x})\big)$ can differ at the most by their relative boundary points,[111] the input distance function is at least defined for all positive input vectors $\mathbf{v} > \mathbf{0}$, i.e. in $\operatorname{rint}\big(\operatorname{cone} L(\mathbf{x})\big)$. But if an input vector $\tilde{\mathbf{v}}$ lies in the boundary of $\mathbb{R}^m_+$ and if it has at least one zero component, then it may occur that the input distance function $t_I(\tilde{\mathbf{v}}, \mathbf{x})$ takes the value $-\infty$. While the input distance function takes a finite value for all positive input vectors and for all input vectors in the vertical line in Figure III.22, each input vector in the horizontal line (except the origin) is mapped to the value $-\infty$. The reason for this is the minimum amount v_2' of the second factor required for the production of the output $\mathbf{x}$.[112] Not only the LEONTIEF production function but also the COBB-DOUGLAS production function may be cited as examples.

To exclude in future the special case $t_I(\mathbf{0}, \mathbf{x}) = 0$, it is easier to define the cone

$$K(L(\mathbf{x})) := \operatorname{cone} L(\mathbf{x}) \setminus \{\mathbf{0}\}.$$

Accordingly, $K(L(\mathbf{x})) = \{\lambda \mathbf{v} | \mathbf{v} \in L(\mathbf{x}),\ \lambda > 0\}$ holds true, if the input requirement set $L(\mathbf{x})$ does not contain the origin $\mathbf{v} = \mathbf{0}$. Thus, all commodity bundles are permissible except $\mathbf{x} = \mathbf{0}$.

In view of the closure of the n-proper concave input distance function $t_I(\cdot, \mathbf{x})$ which will be required later now

$$\text{(III.82)} \qquad t_I(\mathbf{v}, \mathbf{x}) = \operatorname{cl} t_I(\mathbf{v}, \mathbf{x}) > 0 \qquad \forall \mathbf{v} \in \mathbb{R}^m_{++}.$$

follows for each commodity bundle $\mathbf{x} \in X \setminus \{\mathbf{0}\}$:[113] Consequently, the region $\mathbb{R}^m_{++}$ results from the observation[114]

$$\operatorname{rint}\big(\text{n-Dom}\, t_I(\cdot, \mathbf{x})\big) = \operatorname{rint}\big(\operatorname{cone} L(\mathbf{x})\big) = \operatorname{rint}\big(\operatorname{cl}(\operatorname{cone} L(\mathbf{x}))\big) = \mathbb{R}^m_{++}.$$

Like the cost function $c(\cdot, \mathbf{x})$, the input distance function $t_I(\cdot, \mathbf{x})$ is also closely related to the indicator function $\varrho(\cdot | L(\mathbf{x}))$. This relationship is reflected by Proposition III.14 and afterwards it will be explained graphically by Figure III.23.

Proposition III.14 *The input distance function $t_I(\cdot, \mathbf{x})$ of a nonempty convex input requirement set $L(\mathbf{x})$ is the smallest linearly homogeneous concave function generated by the modified indicator function $\tilde{g} = \varrho(\cdot | L(\mathbf{x})) + 1$.*

[111] See Equation (B.3), p. 291.
[112] Thus, the second factor is called **essential** for the production.
[113] See Theorem C.3, p. 304.
[114] Equation (B.3), p. 291, especially means $\operatorname{rint}(\operatorname{cl} C) = \operatorname{rint} C$.

Proof: For each commodity bundle $\mathbf{x} \in X$ a nonempty input requirement set $L(\mathbf{x})$ exists, i.e. $\varrho(\cdot|L(\mathbf{x})) \not\equiv -\infty$. We have to substitute $L(\mathbf{x})$ by its convex hull $\operatorname{conv} L(\mathbf{x})$ for a nonconvex input requirement set. Using

$$\lambda\varrho(\mathbf{v}/\lambda|L(\mathbf{x})) = \varrho(\mathbf{v}/\lambda|L(\mathbf{x})) = \varrho(\mathbf{v}|\lambda L(\mathbf{x})) \quad \text{for } \lambda > 0$$

the smallest linearly homogeneous concave function generated by $\tilde{g} = \varrho(\cdot|L(\mathbf{x}))+1$ according to (III.71) is[115]

$$\begin{aligned} k(\mathbf{v}) &= \sup_{\lambda \geqq 0} \begin{Bmatrix} \varrho(\mathbf{v}|\lambda L(\mathbf{x})) + \lambda & \text{for } \lambda > 0 \\ \varrho(\mathbf{v}|\mathbf{0}) & \text{for } \lambda = 0 \end{Bmatrix} \\ &= \sup\{\varrho(\mathbf{v}|\lambda L(\mathbf{x})) + \lambda |\ \lambda \geqq 0\} \\ &= \sup\{\lambda |\ \lambda \geqq 0,\ \varrho(\mathbf{v}|\lambda L(\mathbf{x})) = 0\} \qquad \text{with} \quad \varrho(\cdot|L(\mathbf{x})) \not\equiv -\infty \\ &= \sup\{\lambda \geqq 0 |\ \mathbf{v} \in \lambda L(\mathbf{x})\}. \end{aligned}$$

Finally, we obtain

$$k(\mathbf{v}) = \psi(\mathbf{v}|L(\mathbf{x})) \equiv t_I(\mathbf{v}, \mathbf{x}),$$

where all of the $\mathbf{x} \in X$ and $\mathbf{v} \in \overline{V}$ are admissible. ■

Before continuing the examination of the input distance function, we have to stress a special case. The input requirement set $L(\mathbf{x})$ contains the origin $\mathbf{v} = \mathbf{0}$ solely for the commodity bundle $\mathbf{x} = \mathbf{0}$ so that $\tilde{g}(\mathbf{0}) = \varrho(\mathbf{0}|L(\mathbf{0})) + 1 > 0$. This case leads to $t_I(\mathbf{0}, \mathbf{0}) = +\infty$ or more generally[116]

$$t_I(\mathbf{v}, \mathbf{0}) = \begin{cases} +\infty & \text{for } \mathbf{v} \in \mathbb{R}^m_+ \\ -\infty & \text{for } \mathbf{v} \notin \mathbb{R}^m_+. \end{cases}$$

Finally, we give a graphical representation of the input distance function where the construction conforms to Proposition III.14.[117] While the left hand part of Figure III.23 illustrates the input distance function of an input requirement set $L(\mathbf{x})$ for a commodity bundle $\mathbf{x} \in X \setminus \{\mathbf{0}\}$, the right hand part illustrates the limit case for the commodity bundle $\mathbf{x} = \mathbf{0}$. Moreover, the modified indicator function $\varrho(\cdot|L(\mathbf{x})) + 1$ can be taken from the subgraphs. In particular, the left hand part is useful to illustrate the described scaling of the input requirement set $\lambda L(\mathbf{x})$ by $\lambda\varrho(\cdot|L(\mathbf{x})) + \lambda$.

With respect to the concave version of Theorem D.9, p. 327, we can now introduce the concept of a **polar input requirement set** $L_\circ(\mathbf{x})$. The set $L_\circ(\mathbf{x})$ defined by Proposition III.15 is also called the factor price requirement set. This set includes those factor price vectors $\mathbf{q}$ such that the factor costs $\mathbf{q}^T\mathbf{v}$ do not fall short

[115] At this point it has to be noted that for a commodity bundle $\mathbf{x} \neq \mathbf{0}$ and a given input vector $\mathbf{v}$ the case $\lambda = +\infty$ cannot result at any time.

[116] The smallest linearly homogeneous concave function k generated by a concave function g has been designed especially for $g(\mathbf{0}) \leqq 0$.

[117] Cf. also Figure D.4, p. 337.

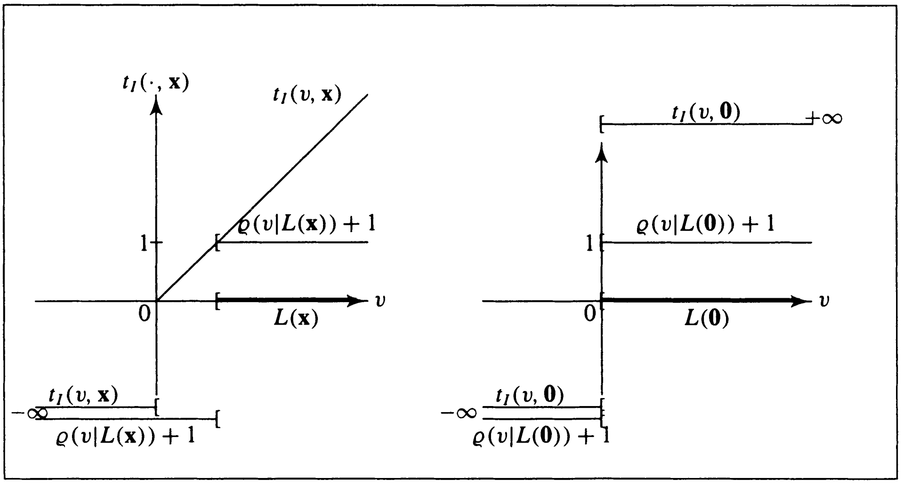

Figure III.23: Alternative graphical representation of the input distance function

of unity for any input vector $\mathbf{v}$ permitting the production of the given commodity bundle $\mathbf{x}$. As clarified by the graphical expositions, the set $L_\circ(\mathbf{x})$ plays the same role with respect to the cost structure as the input requirement set $L(\mathbf{x})$ with respect to the production structure.

The support function of the set $L_\circ(\mathbf{x})$ is sometimes called a price minimal cost function and in accordance with (III.53) it is defined by

(III.83) $$\varphi(\mathbf{v}|L_\circ(\mathbf{x})) := \inf\left\{\mathbf{q}^\mathsf{T}\mathbf{v}|\ \mathbf{q} \in L_\circ(\mathbf{x})\right\}.$$

Equivalent to (III.80) the gauge of the set $L_\circ(\mathbf{x})$ is given by

(III.84) $$\psi(\mathbf{q}|L_\circ(\mathbf{x})) := \sup\left\{\lambda \geqq 0|\ \mathbf{q} \in \lambda L_\circ(\mathbf{x})\right\}.$$

Proposition III.15 *For every nonempty convex set $L(\mathbf{x})$ the closure of the input distance function $t_I(\cdot, \mathbf{x})$ equals the support function $\varphi(\cdot|L_\circ(\mathbf{x}))$ of the set*

(III.85) $$L_\circ(\mathbf{x}) := \left\{\mathbf{q}|\ \mathbf{q}^\mathsf{T}\mathbf{v} \geqq 1\ \ \forall\, \mathbf{v} \in L(\mathbf{x})\right\},$$

where the closed and convex set $L_\circ(\mathbf{x})$ is called the ***(reciprocally)***[118] ***polar input requirement set.***

Proof: If the input distance function $t_I(\cdot, \mathbf{x})$ of a nonempty convex input requirement set $L(\mathbf{x})$ is the smallest linearly homogeneous concave function generated by $\tilde{g} = \varrho(\cdot|L(\mathbf{x})) + 1$, then the concave version of Theorem D.9, p. 327, states that the closure of the input distance function $\operatorname{cl} t_I(\cdot, \mathbf{x})$ equals the

[118] Provided there is no risk for confusion, we speak of polar input requirement sets.

support function of the set $\{\mathbf{q}|\ \tilde{g}_*(\mathbf{q}) \geqq 0\}$. From $\tilde{g}_* = \varrho_*(\cdot|L(\mathbf{x})) - 1$ it follows

$$
\begin{aligned}
\{\mathbf{q}|\ \tilde{g}_*(\mathbf{q}) \geqq 0\} &= \{\mathbf{q}|\ \varrho_*(\mathbf{q}|L(\mathbf{x})) \geqq 1\} \\
&= \left\{\mathbf{q}|\ \inf\left\{\mathbf{q}^\mathsf{T}\mathbf{v} - \varrho(\mathbf{v}|L(\mathbf{x}))|\ \mathbf{v} \in \overline{V}\right\} \geqq 1\right\} \\
&= \left\{\mathbf{q}|\ \inf\left\{\mathbf{q}^\mathsf{T}\mathbf{v}|\ \mathbf{v} \in L(\mathbf{x})\right\} \geqq 1\right\} \\
&= \left\{\mathbf{q}|\ \mathbf{q}^\mathsf{T}\mathbf{v} \geqq 1\ \ \forall\, \mathbf{v} \in L(\mathbf{x})\right\} = L_\circ(\mathbf{x})
\end{aligned}
\tag{III.86}
$$

or, equivalently,[119]

$$
\operatorname{cl} t_I(\cdot, \mathbf{x}) \equiv \operatorname{cl} \psi(\cdot|L(\mathbf{x})) = \varphi(\cdot|L_\circ(\mathbf{x})). \tag{III.87}
$$

Suppose an arbitrary set $C \subset \mathbb{R}^m$ in Proposition III.15. Then (D.11) yields the equivalence relation $\mathbf{0} \notin \operatorname{cl}(\operatorname{conv} C) \iff C_\circ \neq \emptyset$. If $\mathbf{0} \in \operatorname{cl}(\operatorname{conv} C)$, we obtain $\psi(\mathbf{0}|C) = +\infty$ and, therefore, $\operatorname{cl}\psi(\cdot|C) \equiv +\infty$.[120] Because of $\operatorname{cl}\psi(\cdot|C) = \varphi(\cdot|\emptyset) \equiv +\infty$ this case is perfectly consistent, but it can be eliminated *by allowing only for commodity bundles with* $\mathbf{x} \in X \setminus \{\mathbf{0}\}$ *so that* $\mathbf{0} \notin \operatorname{cl}(\operatorname{conv} L(\mathbf{x}))$ *and* $L_\circ(\mathbf{x}) \neq \emptyset$. ■

According to (III.82), Proposition III.15 implies that the input distance function $t_I(\cdot, \mathbf{x})$ equals the support function φ of the polar input requirement set $L_\circ(\mathbf{x})$ for every commodity bundle $\mathbf{x} \in X \setminus \{\mathbf{0}\}$:

$$
t_I(\cdot, \mathbf{x}) = \varphi(\cdot|L_\circ(\mathbf{x})) \qquad \forall\, \mathbf{v} \in \mathbb{R}^m_{++}.
$$

If all of the vectors $\mathbf{v} \in \overline{V}$ are admissible, then the support function $\varphi(\cdot|L_\circ(\mathbf{x})) = \operatorname{cl} t_I(\cdot, \mathbf{x})$ yields a system of inequalities which permits analogous to Proposition III.11 a representation of the polar input requirement set equivalent to (III.85).

$$
\begin{aligned}
L_\circ(\mathbf{x}) &= \left\{\mathbf{q}|\ \mathbf{q}^\mathsf{T}\mathbf{v} \geqq \varphi(\mathbf{v}|L_\circ(\mathbf{x}))\ \ \forall\, \mathbf{v} \in \overline{V}\right\} \\
&= \bigcap_{\mathbf{v} \in \overline{V}} \left\{\mathbf{q}|\ \mathbf{q}^\mathsf{T}\mathbf{v} \geqq \operatorname{cl} t_I(\mathbf{v}, \mathbf{x})\right\}
\end{aligned}
\tag{III.88}
$$

Hence, the family $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$ of input requirement sets, which is also called the production structure, faces a family $\big(L_\circ(\mathbf{x})|\ \mathbf{x} \in X\big)$. While an input requirement set $L(\mathbf{x})$ contains all of the input vectors $\mathbf{v}$ permitting the production of the output $\mathbf{x}$, the polar input requirement set $L_\circ(\mathbf{x})$ is a collection of factor price vectors $\mathbf{q}$ which at least imply the costs $\mathbf{q}^\mathsf{T}\mathbf{v} = 1$ at the production of $\mathbf{x}$. Thus, we are quite right to call the family $\big(L_\circ(\mathbf{x})|\ \mathbf{x} \in X\big)$ a **cost structure.**[121]

2.3 Dual Representation of the Production Structure as Cost Structure

The symmetry between the production structure and the corresponding cost structure can be explained by holding a commodity bundle $\mathbf{x} \in X$ fixed. Now

[119] Based on SHEPHARD (1953) this result is frequently called SHEPHARD's duality theorem – see for example JACOBSEN (1974) – although this is emphatically rejected by SHEPHARD (1974a).

[120] See the remarks on Definition C.4, p. 302.

[121] For a discussion of the properties of this cost structure, see SHEPHARD (1953, p. 96 ff.).

an input requirement set $L(\mathbf{x})$ faces its polar set $L_\circ(\mathbf{x})$. The two sets are associated with a support function φ and a gauge ψ respectively. The next section discusses conditions so that the symmetry of the presented relations may be summarized in the following figure.[122] The economic meaning of the marked gauge and support function will be stressed in Figure III.29.

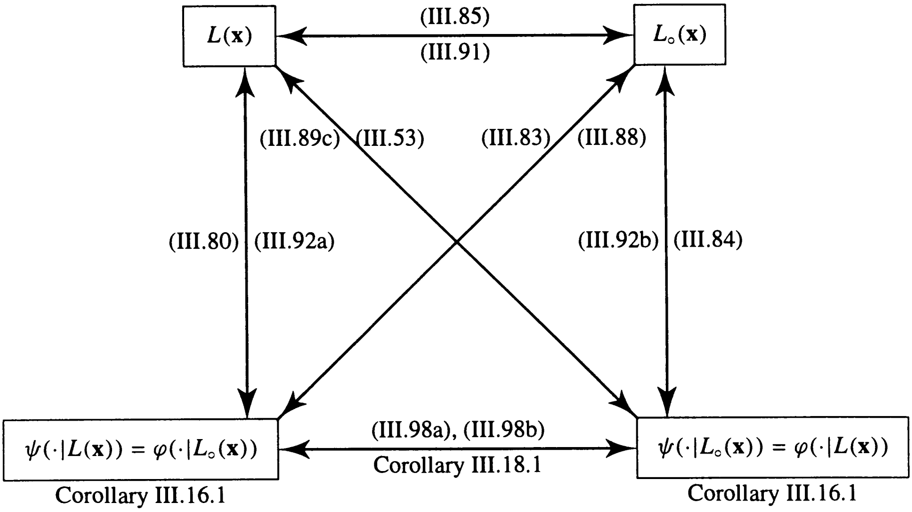

Figure III.24: Duality relations in the sense of SHEPHARD (1)

First of all, the input requirement set $L(\mathbf{x})$ in Proposition III.15 is substituted by $L_\circ(\mathbf{x})$. The resulting **bipolar set** consists of input vectors and is defined by[123]

(III.89a)	$L_{\circ\circ}(\mathbf{x}) := \{\mathbf{v} \mid \mathbf{q}^\mathsf{T}\mathbf{v} \geqq 1 \;\; \forall \mathbf{q} \in L_\circ(\mathbf{x})\}$	like (III.85)
	$= \{\mathbf{v} \mid \varrho_*(\mathbf{v}\mid L_\circ(\mathbf{x})) \geqq 1\}$	analogous to (III.86)
	$= \{\mathbf{v} \mid \varphi(\mathbf{v}\mid L_\circ(\mathbf{x})) \geqq 1\}$	Proposition III.6
(III.89b)	$= \{\mathbf{v} \mid \operatorname{cl} \psi(\mathbf{v}\mid L(\mathbf{x})) \geqq 1\}$	by (III.87).

This set yields a relation equivalent to (III.87), $\operatorname{cl}\psi(\cdot|L_\circ(\mathbf{x})) = \varphi(\cdot|L_{\circ\circ}(\mathbf{x}))$, and analogous to (III.88) it is now $L_{\circ\circ}(\mathbf{x}) = \{\mathbf{v} \mid \mathbf{q}^\mathsf{T}\mathbf{v} \geqq \varphi(\mathbf{v}|L_{\circ\circ}(\mathbf{x})) \;\; \forall \mathbf{q} \in \overline{Q}\}$. On the basis of the second separation theorem[124] we may set $\varphi(\cdot|L_{\circ\circ}(\mathbf{x})) = \varphi(\cdot|L(\mathbf{x}))$

[122] See also Figure D.3, p. 336. The symmetry of dual relations is stressed in particular by HANOCH (1978), who deals with the case of single-product firms.

[123] The remarks on Theorem D.14, p. 333, yield $L(\mathbf{x}) \subset L_{\circ\circ}(\mathbf{x})$.

[124] The Bipolar Theorem (Theorem D.14, p. 333) states $L(\mathbf{x}) \subset L_{\circ\circ}(\mathbf{x})$ such that $\varphi(\cdot|L(\mathbf{x})) \geqq \varphi(\cdot|L_{\circ\circ}(\mathbf{x}))$. Assuming that $\varphi(\cdot|L(\mathbf{x})) > \varphi(\cdot|L_{\circ\circ}(\mathbf{x}))$ holds at a point, then there must be a $\mathbf{v} \in L(\mathbf{x})$ with $\mathbf{q}^\mathsf{T}\mathbf{v} > \varphi(\cdot|L_{\circ\circ}(\mathbf{x}))$. Thus, by the second separation theorem (Theorem B.10, p. 296) there is a hyperplane separating point $\mathbf{v}$ and set $L_{\circ\circ}(\mathbf{x})$ properly. The contradiction to $L(\mathbf{x}) \subset L_{\circ\circ}(\mathbf{x})$ completes the proof.

with regard to the support function. Thus, it not only follows

(III.89c) $$L_{\circ\circ}(\mathbf{x}) = \left\{\mathbf{v} \mid \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq \varphi(\mathbf{v} \mid L(\mathbf{x}))\ \forall \mathbf{q} \in \overline{Q}\right\},$$

but also

(III.90) $$\operatorname{cl} \psi(\cdot \mid L_{\circ}(\mathbf{x})) = \varphi(\cdot \mid L(\mathbf{x})).$$

As the bipolar set $L_{\circ\circ}(\mathbf{x})$ consists of input vectors, it seems reasonable to examine its relation to the input requirement set $L(\mathbf{x})$. A major outcome follows from the assumption that $L(\mathbf{x})$ is a closed convex set not including the origin $\mathbf{v} = \mathbf{0}$. Moreover, if according to Figure II.14, p. 29, the set $L(\mathbf{x})$ is aureoled[125] $L(\mathbf{x}) = \operatorname{aur} L(\mathbf{x})$, then by Corollary D.14.1, p. 333,

(III.91) $$L_{\circ\circ}(\mathbf{x}) = L(\mathbf{x}).$$

In this case (III.89a) and (III.89c) describe the input requirement set $L(\mathbf{x})$ by a system of tangent hyperplanes, where the construction principle is given by Figure II.32, p. 60. However, (III.89b) conforms to rays through the origin. If the boundary point $\hat{\mathbf{v}}$ in Figure III.21 is known, then $\psi(\hat{\mathbf{v}} \mid L(\mathbf{x})) = 1$. Thus, each point on the ray through the origin and $\hat{\mathbf{v}}$, which is further away from the origin than $\hat{\mathbf{v}}$, is an element of the input requirement set $L(\mathbf{x})$.

Before making use of (III.91) we offer further results.

Proposition III.16[126] *Given an arbitrary set* $L(\mathbf{x}) \subset \overline{V}$, *we have*

$$\psi(\mathbf{q} \mid L_{\circ}(\mathbf{x})) = \varphi(\mathbf{q} \mid L(\mathbf{x})) \qquad \forall \mathbf{q} \in K(L_{\circ}(\mathbf{x}));$$
$$\psi(\mathbf{v} \mid L_{\circ\circ}(\mathbf{x})) = \varphi(\mathbf{v} \mid L_{\circ}(\mathbf{x})) \qquad \forall \mathbf{v} \in K(L_{\circ\circ}(\mathbf{x})).$$

Proof: For every given $\mathbf{q} \in K(L_{\circ}(\mathbf{x}))$ we may presume a positive λ in the subsequent rearrangement (Proposition III.13).

$$\begin{aligned}
\psi(\mathbf{q} \mid L_{\circ}(\mathbf{x})) &= \sup\{\lambda \geqq 0 \mid \mathbf{q}/\lambda \in L_{\circ}(\mathbf{x})\} \\
&= \sup\left\{\lambda \geqq 0 \mid (\mathbf{q}/\lambda)^{\mathsf{T}}\mathbf{v} \geqq 1\ \forall \mathbf{v} \in L(\mathbf{x})\right\} \\
&= \sup\left\{\lambda \geqq 0 \mid \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq \lambda\ \forall \mathbf{v} \in L(\mathbf{x})\right\} \\
&= \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x})\right\} \\
&= \varphi(\mathbf{q} \mid L(\mathbf{x}))
\end{aligned}$$

---

[125] As has been mentioned, this condition corresponds to Axiom [L3] (*Disposability of Inputs*) for divisible goods.
Under the above conditions the outer approximation of the input requirement set satisfies furthermore $L_{\diamond}(\mathbf{x}) = L(\mathbf{x})$; see p. 143.

[126] Cf. NEWMAN (1987c, p. 486, Proposition 5). Although NEWMAN explicitly rules out $L_{\circ}(\mathbf{x}) = \emptyset$ with $L_{\circ\circ}(\mathbf{x}) = \mathbb{R}^m$ regarding the second implication no contradiction results, $\psi(\cdot \mid \mathbb{R}^m) = \varphi(\cdot \mid \emptyset) \equiv +\infty$.
NEWMAN points out furthermore that Proposition III.16 can be extended to all $\mathbf{q} \in \overline{Q}$ (and analogously to all $\mathbf{v} \in \overline{V}$) if none of the supporting hyperplanes of $L(\mathbf{x})$ contains the origin, i.e. $\mathbf{q} \neq \mathbf{0} \implies \varphi(\mathbf{q} \mid L(\mathbf{x})) \equiv c(\mathbf{q}, \mathbf{x}) \neq 0$. Since in the actual case both sets $L(\mathbf{x})$ and $L_{\circ}(\mathbf{x})$ will never fulfill this condition at the same time, we can dispense with a more detailed discussion of the implication. See the remarks on Figure III.33.

The same reckoning can be done with regard to $\psi(\mathbf{v}|L_{\circ\circ}(\mathbf{x}))$ for all $\mathbf{v} \in K(L_{\circ\circ}(\mathbf{x}))$.
■

The following corollary of Proposition III.16 stipulates the equivalence $L_{\circ\circ}(\mathbf{x}) = L(\mathbf{x})$. It characterizes the cost function $c(\cdot, \mathbf{x})$ as well as the input distance function $t_I(\cdot, \mathbf{x})$ as support functions and gauges respectively.

Corollary III.16.1[127] *Let $L(\mathbf{x})$ be a nonempty input requirement set satisfying the premises of (III.91). Then the relation between the gauge ψ of the set $L_\circ(\mathbf{x})$ and the support function φ of $L(\mathbf{x})$ – i.e. the cost function $c(\cdot, \mathbf{x})$ – is given by*

$$\psi(\cdot|L_\circ(\mathbf{x})) = \varphi(\cdot|L(\mathbf{x})) \equiv c(\cdot, \mathbf{x}) \qquad \forall \mathbf{q} \in K(L_\circ(\mathbf{x})).$$

Dually, the support function φ of the polar input requirement set $L_\circ(\mathbf{x})$ equals the gauge ψ of the input requirement set $L(\mathbf{x})$ – i.e. the input distance function $t_I(\cdot, \mathbf{x})$.

$$\varphi(\cdot|L_\circ(\mathbf{x})) = \psi(\cdot|L(\mathbf{x})) \equiv t_I(\cdot, \mathbf{x}) \qquad \forall \mathbf{v} \in K(L(\mathbf{x}))$$

Proof: Supposing a commodity bundle $\mathbf{x} \in X \setminus \{\mathbf{0}\}$, the input requirement set $L(\mathbf{x})$ is not empty [L2] and closed [L6]. If needed, $L(\mathbf{x})$ must be replaced with its convex hull, where [L3] (*Disposability of Inputs*) assures the closedness of $\operatorname{conv} L(\mathbf{x})$. Further, Axiom [L3] implies $L(\mathbf{x}) = \operatorname{aur} L(\mathbf{x})$. We also have $\mathbf{0} \notin L(\mathbf{x})$ [L1b] and, therefore, $L_\circ(\mathbf{x}) \neq \emptyset$. In view of (III.91), we get $L(\mathbf{x}) = L_{\circ\circ}(\mathbf{x}) \neq \emptyset$. The rest follows from Proposition III.16. ■

The statements III.15 and III.16 can be summarized with regard to (III.91).

Proposition III.17 *Suppose the input requirement sets $L(\mathbf{x})$ satisfy (III.91). Then each member of the production structure $\big(L(\mathbf{x}) |\, \mathbf{x} \in X \setminus \{\mathbf{0}\}\big)$ faces an equivalent member of the cost structure $\big(L_\circ(\mathbf{x}) |\, \mathbf{x} \in X \setminus \{\mathbf{0}\}\big)$.*

$$L(\mathbf{x}) = \{\mathbf{v} |\; t_I(\mathbf{v}, \mathbf{x}) \geqq 1\} \tag{III.92a}$$

$$L_\circ(\mathbf{x}) = \{\mathbf{q} |\; c(\mathbf{q}, \mathbf{x}) \geqq 1\} \tag{III.92b}$$

Proof: The representation of the polar input requirement set $L_\circ(\mathbf{x})$ has already been proved implicitly by Proposition III.15. The rearrangement (III.86) only requires the substitution $c(\cdot, \mathbf{x}) = \varrho_*(\cdot|L(\mathbf{x}))$ by (III.56), where $L(\mathbf{x})$ must be convex according to (III.91).[128]

If $L_{\circ\circ}(\mathbf{x}) = L(\mathbf{x})$, (III.89b) induces an analogous representation of the input requirement set, $L(\mathbf{x}) = \{\mathbf{v} |\; \operatorname{cl} \psi(\mathbf{v}, \mathbf{x}) \geqq 1\}$. Therefore, it remains to be shown

[127] See Corollary D.15.1.

[128] JACOBSEN (1972, p. 461) emphasizes the relation

$$\varrho(\mathbf{q}|L_\circ(\mathbf{x})) = \varphi_*(\mathbf{q}|L_\circ(\mathbf{x})) = \begin{cases} 0 & \text{for } c(\mathbf{q}, \mathbf{x}) \geqq 1 \\ -\infty & \text{for } c(\mathbf{q}, \mathbf{x}) < 1. \end{cases}$$

Without comprehending the proof in JACOBSEN this result follows from Proposition III.6.

that cl $\psi(\mathbf{v}, \mathbf{x})$ can be substituted by $\psi(\mathbf{v}, \mathbf{x})$ in the case at hand. Considering the above presented results, we have for all $\mathbf{v} \in K(L(\mathbf{x}))$

$$\begin{aligned} \psi(\mathbf{v}|L(\mathbf{x})) &= \varphi(\mathbf{v}|L_\circ(\mathbf{x})) && \text{because of Proposition III.16} \\ &= \text{cl}\,\psi(\mathbf{v}|L(\mathbf{x})) && \text{because of (III.87).} \end{aligned}$$

Due to $L(\mathbf{x}) \subset K(L(\mathbf{x}))$ this transformation is especially valid for all $\mathbf{v} \in L(\mathbf{x})$ and it ensues $L(\mathbf{x}) = \{\mathbf{v}|\ \psi(\mathbf{v}, \mathbf{x}) \geqq 1\}$ or (III.92a). ■

Up until now the analysis has concentrated on the various interpretations of the cost function and the input distance function. For instance, Corollary III.16.1 stresses that the cost function $c(\cdot, \mathbf{x})$ corresponds not only to the support function φ of the input requirement set $L(\mathbf{x})$ but also to the gauge ψ of the polar input requirement set $L_\circ(\mathbf{x})$. Hence, the presented results yield the "vertical" relations in Figure III.24.

The next expositions serve for the derivation of a direct relation between both gauges ψ of the input requirement set $L(\mathbf{x}) = L_{\circ\circ}(\mathbf{x})$ and of the polar input requirement set $L_\circ(\mathbf{x})$. Transferred to the cost function $c(\cdot, \mathbf{x}) = \psi(\cdot|L_\circ(\mathbf{x}))$ and to the input distance function $t_I(\cdot, \mathbf{x}) = \psi(\cdot|L(\mathbf{x}))$ the result is noted by Corollary III.18.1. In order to analyze the relationship between the two mentioned gauges $\psi(\cdot|L(\mathbf{x}))$ and $\psi(\cdot|L_\circ(\mathbf{x}))$, first of all, we introduce the concept of a (reciprocally) polar gauge.

The definition (III.93) can be justified with respect to (III.85) by fixing a $\mu > 0$ such that the polar sets $L(\mathbf{x})$ and $L_\circ(\mathbf{x})$ hold the following equivalence relation:

$$\mathbf{q}^\mathsf{T}\mathbf{v} \geqq \mu \quad \forall\, \mathbf{v} \in L(\mathbf{x}) \iff \mathbf{q}/\mu \in L_\circ(\mathbf{x}).$$

The left hand side may be multiplied by an arbitrary $\lambda > 0$.

$$\begin{aligned} & \lambda(\mathbf{q}/\mu)^\mathsf{T}\mathbf{v} \geqq \lambda && \forall\, \mathbf{v} \in L(\mathbf{x}),\ \forall\, \lambda > 0 && (\text{put } \tilde{\mathbf{v}} = \lambda\mathbf{v}) \\ \iff\ & (\mathbf{q}/\mu)^\mathsf{T}\tilde{\mathbf{v}} \geqq \lambda && \forall\, \tilde{\mathbf{v}} \in L(\mathbf{x}),\ \forall\, \lambda > 0 \end{aligned}$$

If we substitute λ by the greatest positive possible λ, i.e.[129]

$$\psi(\tilde{\mathbf{v}}|L(\mathbf{x})) = \sup\{\lambda \geqq 0|\ \tilde{\mathbf{v}} \in \lambda L(\mathbf{x})\} \qquad \forall\, \tilde{\mathbf{v}} \in \text{n-Dom}\,\psi(\tilde{\mathbf{v}}|L(\mathbf{x})) \setminus \{\mathbf{0}\},$$

then

$$\mathbf{q}^\mathsf{T}\mathbf{v} \geqq \mu\psi(\tilde{\mathbf{v}}|L(\mathbf{x})) \qquad \forall\, \tilde{\mathbf{v}} \in \text{n-Dom}\,\psi(\tilde{\mathbf{v}}|L(\mathbf{x})) \setminus \{\mathbf{0}\}.$$

Similar to Definition D.6, p. 334, we can now switch to the dual view. If we assume an n-proper (reciprocal) gauge $k = \psi(\cdot|L(\mathbf{x}))$ instead of μ[130] then the **(reciprocally) polar gauge** $k_\circ$ denotes the greatest nonnegative admissible μ depending on $\mathbf{q}$.

$$\text{(III.93)} \qquad k_\circ(\mathbf{q}) := \sup\left\{\mu \geqq 0|\ \mathbf{q}^\mathsf{T}\mathbf{v} \geqq \mu k(\mathbf{v})\ \forall\, \mathbf{v} \in \text{n-Dom}\,k \setminus \{\mathbf{0}\}\right\}$$

[129] The unwanted case $\lambda = \psi(\tilde{\mathbf{v}}|L(\mathbf{x})) = 0$ cannot occur by Proposition III.13 or Theorem D.11 for $\tilde{\mathbf{v}} \neq \mathbf{0}$.

[130] First of all, it turns out to be useful to scrutinize the function k instead of the gauge $\psi(\cdot|L(\mathbf{x}))$.

As shown by Proposition III.18, $k_\circ$ is also a gauge so that the bipolar gauge is given by

$$k_{\circ\circ}(\mathbf{v}) := \sup\left\{\lambda \geqq 0 \mid \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq \lambda k_\circ(\mathbf{q}) \;\; \forall\, \mathbf{q} \in \text{n-Dom}\, k_\circ \setminus \{\mathbf{0}\}\right\}.$$

The properties of these functions require some remarks.

1. According to Proposition III.13, an n-proper gauge k with the effective domain $\text{n-Dom}\, k \neq \emptyset$ is supposed.
2. To avoid polar gauges $k_\circ$ which are not n-proper,[131] we have to impose the effective domain n-Dom k without the origin $\mathbf{v} = \mathbf{0}$ instead of $\mathbf{v} \in \overline{V}$.
3. If $0 > \mathbf{q}^{\mathsf{T}}\mathbf{v}$ for all $\mathbf{v} \in \text{n-Dom}\, k \setminus \{\mathbf{0}\}$, then $k_\circ(\mathbf{q}) = -\infty$.

Analogous to (D.15) MAHLER's inequality remains the same.

$$\mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq k(\mathbf{v})\, k_\circ(\mathbf{q}) \qquad \forall\, \mathbf{v} \in \text{n-Dom}\, k \setminus \{\mathbf{0}\},\;\; \forall\, \mathbf{q} \in \text{n-Dom}\, k_\circ \setminus \{\mathbf{0}\}$$

Bear in mind according to Proposition III.13 that the effective domain of the gauge $\psi(\cdot|L(\mathbf{x}))$ is given by $\text{n-Dom}\, \psi(\cdot|L(\mathbf{x})) = \text{cone}\, L(\mathbf{x})$. [132]

Proposition III.18 *Let $L(\mathbf{x})$ be a nonempty convex set with the gauge $\psi(\cdot|L(\mathbf{x}))$. Then the polar set $L_\circ(\mathbf{x})$ of $L(\mathbf{x})$ yields the polar gauge*

$$\text{(III.94a)} \qquad \psi_\circ(\mathbf{q}|L(\mathbf{x})) = \psi(\mathbf{q}|L_\circ(\mathbf{x})) \qquad \forall\, \mathbf{q} \in K(L_\circ(\mathbf{x})).$$

The bipolar gauge can be determined by the bipolar set $L_{\circ\circ}(\mathbf{x})$.

$$\text{(III.94b)} \qquad \psi_{\circ\circ}(\mathbf{v}|L(\mathbf{x})) = \psi(\mathbf{v}|L_{\circ\circ}(\mathbf{x})) \qquad \forall\, \mathbf{v} \in K(L_{\circ\circ}(\mathbf{x})).$$

Proof: If the set $L(\mathbf{x})$ is nonempty, then $\psi(\mathbf{v}|L(\mathbf{x})) > 0$ for at least one $\mathbf{v}$. From $\mathbf{q} = \mathbf{0}$ it now ensues $\psi_\circ(\mathbf{0}|L(\mathbf{x})) = 0$. For $\mu > 0$ (with $\mathbf{q} \neq \mathbf{0}$) the constraint in (III.93) can be rewritten as follows, where $K(L(\mathbf{x})) = \{\lambda\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x}),\ \lambda > 0\}$ must be taken into account:

$$\begin{aligned}
& & \mathbf{q}^{\mathsf{T}}\tilde{\mathbf{v}} &\geqq \mu\psi(\tilde{\mathbf{v}}|L(\mathbf{x})) & & \forall\, \tilde{\mathbf{v}} \in K(L(\mathbf{x})) \\
&\Longleftrightarrow & (\mathbf{q}/\mu)^{\mathsf{T}}(\lambda\mathbf{v}) &\geqq \psi(\lambda\mathbf{v}|L(\mathbf{x})) & & \forall\, \mathbf{v} \in L(\mathbf{x}),\ \forall\, \lambda > 0 \\
&\Longleftrightarrow & (\mathbf{q}/\mu)^{\mathsf{T}}\mathbf{v} &\geqq \psi(\mathbf{v}|L(\mathbf{x})) & & \forall\, \mathbf{v} \in L(\mathbf{x}) \text{ (linear homogeneity)}
\end{aligned}$$

Furthermore, due to $\mathbf{v} \in L(\mathbf{x}) \implies \psi(\mathbf{v}|L(\mathbf{x})) \geqq 1$ we get

$$\Longleftrightarrow \qquad (\mathbf{q}/\mu)^{\mathsf{T}}\mathbf{v} \geqq 1 \qquad \forall\, \mathbf{v} \in L(\mathbf{x}).$$

[131] Each $\mathbf{v} \notin \text{n-Dom}\, k$ with $k(\mathbf{v}) = -\infty$ would induce an improper function $k_\circ(\mathbf{q}) = +\infty$. Analogously, $k(\mathbf{0}) = 0$ implies the polar gauge $k_\circ \equiv +\infty$.

[132] Thus, we have $K(L(\mathbf{x})) = \text{cone}\, L(\mathbf{x}) \setminus \{\mathbf{0}\} = \text{n-Dom}\, \psi(\cdot|L(\mathbf{x})) \setminus \{\mathbf{0}\}$.

The last condition is equivalent to $\mathbf{q}/\mu \in L_\circ(\mathbf{x})$ or $\mathbf{q} \in \mu L_\circ(\mathbf{x})$. Thus, (III.93) ensues

$$\begin{aligned} \psi_\circ(\mathbf{q}|L(\mathbf{x})) &= \sup\{\mu \geqq 0 |\ \mathbf{q} \in \mu L_\circ(\mathbf{x})\} \\ &= \psi(\mathbf{q}|L_\circ(\mathbf{x})) \end{aligned}$$

for $\mu > 0$ with respect to (III.84). Because of $\mu > 0$ for all $\mathbf{q} \in K(L_\circ(\mathbf{x}))$ (Proposition III.13) the proof is complete. The second part of Proposition III.18 merely consists of applying the first part to the polar set

$$\begin{aligned} \psi_\circ(\mathbf{v}|L_\circ(\mathbf{x})) &= \sup\left\{\mu \geqq 0 |\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq \mu\psi(\mathbf{q}|L_\circ(\mathbf{x})) \quad \forall \mathbf{q} \in K(L_\circ(\mathbf{x}))\right\} \\ &= \sup\left\{\mu \geqq 0 |\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq \mu\psi_\circ(\mathbf{q}|L(\mathbf{x})) \quad \forall \mathbf{q} \in K(L_\circ(\mathbf{x}))\right\} \\ &=: \psi_{\circ\circ}(\mathbf{v}|L(\mathbf{x})). \end{aligned}$$

Replacing $L(\mathbf{x})$ by $L_\circ(\mathbf{x})$ in (III.94a) implies (III.94b).

$$\psi_{\circ\circ}(\mathbf{v}|L(\mathbf{x})) = \psi_\circ(\mathbf{v}|L_\circ(\mathbf{x})) = \psi(\mathbf{v}|L_{\circ\circ}(\mathbf{x})) \qquad \forall \mathbf{v} \in K(L_{\circ\circ}(\mathbf{x})) \quad \blacksquare$$

Given a nonempty input requirement set $L(\mathbf{x})$ fulfilling (III.91), the following symmetry between the gauges $k = \psi(\cdot|L(\mathbf{x}))$ and $h = \psi_\circ(\cdot|L(\mathbf{x}))$ holds.[133]

$$\begin{aligned} h_\circ(\mathbf{v}) = k_{\circ\circ}(\mathbf{v}) = k(\mathbf{v}) \qquad & \forall \mathbf{v} \in K(L(\mathbf{x})), \\ k_\circ(\mathbf{q}) = h(\mathbf{q}) = h_{\circ\circ}(\mathbf{q}) \qquad & \forall \mathbf{q} \in K(L_\circ(\mathbf{x})). \end{aligned}$$

Under the same condition – i.e. $L(\mathbf{x}) = L_{\circ\circ}(\mathbf{x})$ – Proposition III.18 gives

(III.95) $$\psi_{\circ\circ}(\mathbf{v}|L(\mathbf{x})) = \psi(\mathbf{v}|L(\mathbf{x})) \qquad \forall \mathbf{v} \in K(L(\mathbf{x})).$$

Now Proposition III.16 can be rewritten as

$$\begin{aligned} \psi_\circ(\mathbf{q}|L(\mathbf{x})) = \varphi(\mathbf{q}|L(\mathbf{x})) \quad & \forall \mathbf{q} \in K(L_\circ(\mathbf{x})) \qquad \text{by (III.94a)} \\ \psi(\mathbf{v}|L(\mathbf{x})) = \varphi(\mathbf{v}|L_\circ(\mathbf{x})) \quad & \forall \mathbf{v} \in K(L(\mathbf{x})) \qquad \text{by (III.94b) and (III.95).} \end{aligned}$$

At the same time the duality of the input distance function $\psi(\cdot|L(\mathbf{x})) \equiv t_I(\cdot, \mathbf{x})$ and the cost function $\psi_\circ(\cdot|L(\mathbf{x})) \equiv c(\cdot, \mathbf{x})$ with[134]

(III.96a)
$$c(\mathbf{q}, \mathbf{x}) = \sup\left\{\lambda \geqq 0 |\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq \lambda t_I(\mathbf{v}, \mathbf{x}) \ \forall \mathbf{v} \in K(L(\mathbf{x}))\right\} \qquad \forall \mathbf{q} \in K(L_\circ(\mathbf{x}))$$

(III.96b)
$$t_I(\mathbf{v}, \mathbf{x}) = \sup\left\{\mu \geqq 0 |\ \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq \mu c(\mathbf{q}, \mathbf{x}) \ \forall \mathbf{q} \in K(L_\circ(\mathbf{x}))\right\} \qquad \forall \mathbf{v} \in K(L(\mathbf{x}))$$

may be summarized as follows:[135]

[133] See Corollary D.16.1 (Gauge Duality), p. 335.

[134] See McFadden (1978, p. 28). Furthermore, we have $c(\mathbf{0}, \mathbf{x}) = 0$ and $t_I(\mathbf{0}, \mathbf{x}) = 0$. Under the above conditions two equivalent systems of equations face each other.

$$\begin{aligned} c(\mathbf{q}, \mathbf{x}) \equiv \varphi(\mathbf{q}|L(\mathbf{x})) &= \varphi_\circ(\mathbf{q}|L_\circ(\mathbf{x})) = \psi_\circ(\mathbf{q}|L(\mathbf{x})) = \psi(\mathbf{q}|L_\circ(\mathbf{x})) \\ t_I(\mathbf{v}, \mathbf{x}) \equiv \psi(\mathbf{v}|L(\mathbf{x})) &= \psi_\circ(\mathbf{v}|L_\circ(\mathbf{x})) = \varphi(\mathbf{v}|L_\circ(\mathbf{x})) = \varphi_\circ(\mathbf{v}|L(\mathbf{x})) \end{aligned}$$

[135] See Corollary D.16.2, p. 335.

Corollary III.18.1 *Let $L(\mathbf{x})$ be a nonempty input requirement set satisfying the premises of (III.91). Then the input distance function $t_I(\cdot, \mathbf{x})$ and the cost function $c(\cdot, \mathbf{x})$ are polar to each other and fulfill* MAHLER*'s inequality*

$$\text{(III.97)} \qquad \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq c(\mathbf{q}, \mathbf{x})\, t_I(\mathbf{v}, \mathbf{x}) \qquad \forall \mathbf{q} \in K(L_\circ(\mathbf{x})), \;\; \forall \mathbf{v} \in K(L(\mathbf{x})).$$

If the vectors $\mathbf{v}$ and $\mathbf{q}$ fulfill the inequality (III.97) for a given commodity bundle $\mathbf{x} \in X \setminus \{\mathbf{0}\}$, then $(\mathbf{q}, \mathbf{v})$ is called a **pair of polar points**.

Whereas MCFADDEN discusses (III.96a) and (III.96b), it turns out to be more favorable regarding Proposition III.19 to express the two equations as follows:[136]

$$\text{(III.98a)} \qquad c(\mathbf{q}, \mathbf{x}) = \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid t_I(\mathbf{v}, \mathbf{x}) \geqq 1\right\} \qquad \forall \mathbf{q} \in K(L_\circ(\mathbf{x}))$$

$$\text{(III.98b)} \qquad t_I(\mathbf{v}, \mathbf{x}) = \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid c(\mathbf{q}, \mathbf{x}) \geqq 1\right\} \qquad \forall \mathbf{v} \in K(L(\mathbf{x}))$$

Both pairs of equations (III.96a), (III.96b) and (III.98a), (III.98b) are equivalent. For if the nonempty input requirement set $L(\mathbf{x})$ satisfies condition (III.91), then, considering Proposition III.17, it is

$$\begin{aligned} c(\mathbf{q}, \mathbf{x}) &= \psi_\circ(\mathbf{q} \mid L(\mathbf{x})) && \text{yields (III.96a) by } t_I(\cdot, \mathbf{x}) \equiv \psi(\cdot \mid L(\mathbf{x})) \\ &= \varphi(\mathbf{q} \mid L(\mathbf{x})) \\ &= \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x})\right\} && \text{yields (III.98a) by (III.92a).} \end{aligned}$$

The same argument with respect to the input distance function $t_I(\cdot, \mathbf{x})$ relates (III.96b) and (III.98b), where now (III.92b) must be taken into account.

Proposition III.19 (SHEPHARD's Theorem)[137] *For a pair of polar points $(\hat{\mathbf{q}}, \hat{\mathbf{v}})$, to satisfy (III.97) for a given commodity bundle $\mathbf{x} \in X \setminus \{\mathbf{0}\}$ as an equation*[138]

$$\text{(III.99)} \qquad c(\hat{\mathbf{q}}, \mathbf{x})\, t_I(\hat{\mathbf{v}}, \mathbf{x}) = \hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}},$$

it is necessary and sufficient that the input vector $\hat{\mathbf{v}}$ solves the problem of cost minimization for a given factor price vector $\hat{\mathbf{q}} \in K(L_\circ(\mathbf{x}))$ or dually that $\hat{\mathbf{q}}$ is an optimal solution to (III.83) for a given vector $\hat{\mathbf{v}} \in K(L(\mathbf{x}))$.

Provided the functions are differentiable, the KUHN-TUCKER *conditions yield two systems of equations which are dual to each other for an optimal pair of polar points $(\hat{\mathbf{q}}, \hat{\mathbf{v}})$ with $\hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}} = 1$:*

$$\text{(III.100a)} \qquad \frac{\partial c}{\partial q_i}(\hat{\mathbf{q}}, \mathbf{x}) = \hat{v}_i \qquad i = 1, \ldots, m$$

$$\text{(III.100b)} \qquad \frac{\partial t_I}{\partial v_i}(\hat{\mathbf{v}}, \mathbf{x}) = \hat{q}_i \qquad i = 1, \ldots, m.$$

---

[136]This form of representation is taken from SHEPHARD (1953, p. 159). The duality of (III.98a) and (III.98b) does not depend on assumptions on properties of homogeneity of the production technology as suggested by DIEWERT (1974). Cf. SHEPHARD (1974a).

[137]Cf. SHEPHARD (1953, p. 171).

[138]In DEATON (1979) this equation serves as a starting point for the derivation of indices to measure the change of the utility level. See AHLHEIM, ROSE (1992, Chapter 9), for further remarks.

Whereas (III.100a) indicates the system of factor demand functions, (III.100b) is the system of inverse factor demand functions.

Proof: If the nonempty input requirement set $L(\mathbf{x})$ is convex, then for a given factor price vector $\hat{\mathbf{q}} \in K(L_\circ(\mathbf{x}))$ (III.98a) is a convex program[139]

$$\text{(P)} \qquad c(\hat{\mathbf{q}}, \mathbf{x}) = \inf\left\{\hat{\mathbf{q}}^\mathsf{T}\mathbf{v} \mid 1 - t_I(\mathbf{v}, \mathbf{x}) \leqq 0\right\},$$

where the constraint $f_1 = 1 - t_I(\cdot, \mathbf{x})$ is defined on the nonempty convex set $\operatorname{Dom} f_1 = \operatorname{cone} L(\mathbf{x})$.[140] Here we have to rule out the commodity bundle $\mathbf{x} = \mathbf{0}$ since $t_I(\mathbf{v}, \mathbf{0}) \equiv +\infty$. Note that cone $L(\mathbf{x})$ has at least one relatively interior point $\tilde{\mathbf{v}}$ satisfying $f_1(\tilde{\mathbf{v}}) < 0$ or $t_I(\tilde{\mathbf{v}}, \mathbf{x}) > 1$. Thus, $\tilde{\mathbf{v}}$ fulfills SLATER's conditions[141] so that the following KUHN-TUCKER conditions are necessary and sufficient for $(\hat{\mathbf{v}}, \hat{\lambda})$ to be a saddlepoint of the LAGRANGEan Φ or, equivalently, for $\hat{\mathbf{v}}$ to solve problem (P) or problem (III.98a) for $\mathbf{q} = \hat{\mathbf{q}}$.

The LAGRANGEan function

$$\Phi_1(\mathbf{v}, \lambda) = \hat{\mathbf{q}}^\mathsf{T}\mathbf{v} + \lambda\big(1 - t_I(\mathbf{v}, \mathbf{x})\big)$$

with the LAGRANGEan multiplier λ implies the following KUHN-TUCKER conditions:[142]

$$\text{[a]} \qquad \hat{\lambda} \geqq 0, \quad 1 - t_I(\hat{\mathbf{v}}, \mathbf{x}) \leqq 0, \quad \hat{\lambda}\big(1 - t_I(\hat{\mathbf{v}}, \mathbf{x})\big) = 0,$$

$$\text{[b]} \qquad \Phi_1(\hat{\mathbf{v}}, \hat{\lambda}) \leqq \Phi_1(\mathbf{v}, \hat{\lambda}) \qquad \forall \mathbf{v} \in \operatorname{cone} L(\mathbf{x}),$$

where in [b] the following property of the subgradient must be taken into account:

$$\Phi_1(\hat{\mathbf{v}}, \hat{\lambda}) = \min\left\{\Phi_1(\mathbf{v}, \hat{\lambda}) \mid \mathbf{v} \in \operatorname{cone} L(\mathbf{x})\right\} \iff \mathbf{0} \in \partial_\mathbf{v}\Phi_1(\hat{\mathbf{v}}, \hat{\lambda}).$$

Thus, the LAGRANGEan function $\Phi_1(\cdot, \hat{\lambda})$ attains its minimum at point $\hat{\mathbf{v}}$ if and only if $\mathbf{0}$ is a subgradient of the convex function $\Phi_1(\cdot, \hat{\lambda})$ at point $\hat{\mathbf{v}}$.

If we take into account that the objective function $f_0(\mathbf{v}) = \hat{\mathbf{q}}^\mathsf{T}\mathbf{v}$ is defined on $\mathbb{R}^m$, then

$$\operatorname{rint}(\operatorname{Dom} f_0) \cap \operatorname{rint}(\operatorname{Dom} f_1) = \mathbb{R}^m \cap \operatorname{rint}\big(\operatorname{cone} L(\mathbf{x})\big) \neq \emptyset$$

and, therefore,[143]

$$\mathbf{0} \in \partial_\mathbf{v}\Phi_1(\hat{\mathbf{v}}, \hat{\lambda}) \iff \mathbf{0} \in \left[\partial(\hat{\mathbf{q}}^\mathsf{T}\hat{\mathbf{v}}) + \hat{\lambda}\,\partial(1 - t_I(\hat{\mathbf{v}}, \mathbf{x}))\right].$$

[139] At the same time $\hat{\mathbf{q}} \in K(L_\circ(\mathbf{x})) \equiv \operatorname{cone} L_\circ(\mathbf{x}) \setminus \{\mathbf{0}\}$ rules out $c(\hat{\mathbf{q}}, \mathbf{x}) = 0$ contradicting (III.99).

[140] By Proposition III.13 the effective domain of the (n-proper) input distance function $t_I(\cdot, \mathbf{x})$ satisfies n-Dom $t_I(\cdot, \mathbf{x}) = \operatorname{cone} L(\mathbf{x})$. Thus, the (proper) function f_1 holds $\operatorname{Dom} f_1 = \operatorname{cone} L(\mathbf{x})$.

[141] Cf. BLUM, ÖTTLI (1975, p. 63, p. 69 (Theorem 8), and p. 71 (Theorem 12)).

[142] Cf. BLUM, ÖTTLI (1975, p. 63) and also ROCKAFELLAR (1972, p. 281, Theorem 28.3).

[143] With the aid of ROCKAFELLAR (1972, p. 223, Theorem 23.8) we obtain the following equation for the subdifferential of the LAGRANGEan function $\Phi(\mathbf{v}, \boldsymbol{\lambda}) = f_0(\mathbf{v}) + \lambda_1 f_1(\mathbf{v}) + \cdots + \lambda_n f_n(\mathbf{v})$,

$$\partial_\mathbf{v}\Phi(\mathbf{v}, \boldsymbol{\lambda}) = \partial f_0(\mathbf{v}) + \lambda_1 \partial f_1(\mathbf{v}) + \cdots + \lambda_n \partial f_n(\mathbf{v}) \qquad \forall \mathbf{v},$$

provided the functions f_0 and f_i are proper and convex and the convex sets $\operatorname{rint}(\operatorname{Dom} f_i)$ $(i = 0, \ldots, n)$ have a point in common. MITITELU (1994, p. 217) presents KUHN-TUCKER conditions already taking this result into account.

In the last term[144] $\partial(\hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}}) = \{\hat{\mathbf{q}}\}$ can immediately be substituted so that[145]

$$\mathbf{0} \in \left[\{\hat{\mathbf{q}}\} + \hat{\lambda}\,\partial(-t_I(\hat{\mathbf{v}}, \mathbf{x}))\right] \iff \hat{\mathbf{q}} \in \hat{\lambda}\,\Delta t_I(\hat{\mathbf{v}}, \mathbf{x}).$$

Due to $\hat{\mathbf{q}} \neq \mathbf{0}$ it follows $\hat{\lambda} > 0$ and therefore $t_I(\hat{\mathbf{v}}, \mathbf{x}) = 1$; see [a]. Thus, $\hat{\mathbf{q}}$ denotes an optimal solution to

$$1 = t_I(\hat{\mathbf{v}}, \mathbf{x}) \equiv \varphi(\hat{\mathbf{v}}|L_\circ(\mathbf{x})) = \inf\left\{\mathbf{q}^{\mathsf{T}}\hat{\mathbf{v}}|\ \mathbf{q} \in L_\circ(\mathbf{x})\right\} \tag{III.83}$$

since each alternative price vector $\mathbf{q}$ with $t_I(\hat{\mathbf{v}}, \mathbf{x}) < 1$ is inadmissible by (III.85), $\mathbf{q} \notin L_\circ(\mathbf{x})$. At the same time it is $\Phi_1(\hat{\mathbf{v}}, \hat{\lambda}) = c(\hat{\mathbf{q}}, \mathbf{x}) = \hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}}$, i.e. the linear function $\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{v}$ achieves its minimum over $L(\mathbf{x})$ at $\hat{\mathbf{v}}$. Hence, by Proposition III.8 each cost minimizing input vector $\hat{\mathbf{v}}$ provides a supergradient of the cost function

$$\hat{\mathbf{v}} \in \Delta c(\hat{\mathbf{q}}, \mathbf{x})$$

or a gradient $\hat{\mathbf{v}} = \nabla c(\hat{\mathbf{q}}, \mathbf{x})$ according to (III.100a) provided the cost function is differentiable.[146] The results $t_I(\hat{\mathbf{v}}, \mathbf{x}) = 1$ and $c(\hat{\mathbf{q}}, \mathbf{x}) = \hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}}$ commonly imply (III.99). Dually, (III.99) can also be derived by the convex program

$$t_I(\hat{\mathbf{v}}, \mathbf{x}) = \inf\left\{\mathbf{q}^{\mathsf{T}}\hat{\mathbf{v}}|\ 1 - c(\mathbf{q}, \mathbf{x}) \leqq 0\right\}.$$

The LAGRANGEan function $\Phi_2(\mathbf{q}, \mu) = \mathbf{q}^{\mathsf{T}}\hat{\mathbf{v}} + \mu\big(1 - c(\mathbf{q}, \mathbf{x})\big)$ then gives $c(\hat{\mathbf{q}}, \mathbf{x}) = 1$ and $t_I(\hat{\mathbf{v}}, \mathbf{x}) = \hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}}$ as well as $\hat{\mathbf{q}} \in \Delta t_I(\hat{\mathbf{v}}, \mathbf{x})$.

Presuming differentiability of the input distance function at point $\hat{\mathbf{v}}$ and of the cost function at point $\hat{\mathbf{q}}$, the two mentioned LAGRANGEan functions satisfy at the optimum

$$\begin{aligned}
\Phi_1(\hat{\mathbf{v}}, \hat{\lambda}) &= c(\hat{\mathbf{q}}, \mathbf{x}) = \hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}} = \hat{\lambda}\big(\nabla t_I(\hat{\mathbf{v}}, \mathbf{x})\big)^{\mathsf{T}}\hat{\mathbf{v}} && \text{with} \quad \hat{\mathbf{q}} = \hat{\lambda}\,\nabla t_I(\hat{\mathbf{v}}, \mathbf{x}),\\
\Phi_2(\hat{\mathbf{q}}, \hat{\mu}) &= t_I(\hat{\mathbf{v}}, \mathbf{x}) = \hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}} = \hat{\mu}\big(\nabla c(\hat{\mathbf{q}}, \mathbf{x})\big)^{\mathsf{T}}\hat{\mathbf{q}} && \text{with} \quad \hat{\mathbf{v}} = \hat{\mu}\,\nabla c(\hat{\mathbf{q}}, \mathbf{x}).
\end{aligned}$$

Now with respect to an interpretation of the LAGRANGE multipliers λ and μ we can make use of EULER's Theorem. The linear homogeneity of the input distance function (and analogously the cost function) implies $t_I(\hat{\mathbf{v}}, \mathbf{x}) = 1 = \big(\nabla t_I(\hat{\mathbf{v}}, \mathbf{x})\big)^{\mathsf{T}}\hat{\mathbf{v}}$. With that the results

$$c(\hat{\mathbf{q}}, \mathbf{x}) = \hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}} = \hat{\lambda}, \quad t_I(\hat{\mathbf{v}}, \mathbf{x}) = 1, \quad \hat{\mathbf{v}} = \nabla c(\hat{\mathbf{q}}, \mathbf{x}) \quad \text{with} \quad \hat{\mathbf{q}} \in K(L_\circ(\mathbf{x}))$$

are reflected by the following equivalent results.

$$t_I(\hat{\mathbf{v}}, \mathbf{x}) = \hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}} = \hat{\mu}, \quad c(\hat{\mathbf{q}}, \mathbf{x}) = 1, \quad \hat{\mathbf{q}} = \nabla t_I(\hat{\mathbf{v}}, \mathbf{x}) \quad \text{with} \quad \hat{\mathbf{v}} \in K(L(\mathbf{x})). \quad \blacksquare$$

[144] Given the linear function $f_0(\mathbf{v}) = \hat{\mathbf{q}}^{\mathsf{T}}\mathbf{v}$, the subdifferential $\partial f_0(\mathbf{v}) = \{\hat{\mathbf{q}}\}$ is equivalent to the gradient $\nabla f_0(\mathbf{v}) = \hat{\mathbf{q}}$.

[145] The subdifferential $\partial f(\mathbf{x})$ of a convex function f at point $\mathbf{x}$ and the superdifferential $\Delta(-f(\mathbf{x}))$ of the concave function $-f$ at point $\mathbf{x}$ satisfy the relation $-\partial f(\mathbf{x}) = \Delta(-f(\mathbf{x}))$.

[146] In economic bibliography the outcome of Proposition III.8 is frequently described as a result of the more general envelope theorem. Apart from the more general objective functions, the envelope theorem takes into account additional parameter variations regarding the restrictions. Cf. e.g. TAKAYAMA (1990, p. 138).

Following BLACKORBY, PRIMONT, RUSSEL (1978), we now give a schematic representation of the duality theory with regard to a firm producing a solitary good x.[147] The construction of the graph adjusts Figure III.24, p. 158, where a transference of the technical correlations to the case of a single-product firm is omitted. With that we only have to concern ourselves about the meaning of the function z.

First of all, the **polar production function** z is defined analogous to a **production function** f.

$$f(\mathbf{v}) := \sup\{x \in X \mid \mathbf{v} \in L(x)\}$$
$$z(\mathbf{q}) := \inf\{x \in X \mid \mathbf{q} \in L_{\circ}(x)\}$$

While $f(\mathbf{v})$ is the greatest output x producible by inputs $\mathbf{v}$, $z(\mathbf{q})$ denotes the smallest output leading to factor costs not lower than unity, $\mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq 1$, at fixed factor prices $\mathbf{q}$. Given the impositions of (III.92b), we have $\mathbf{q} \in L_{\circ}(x) \iff c(\mathbf{q}, x) \geqq 1$ so that the polar production function can be rewritten as

$$\begin{tikzcd}
f(\mathbf{v}) = x \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & z(\mathbf{q}) = x \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
t_I(\mathbf{v}, x) = 1 \arrow[r, leftrightarrow] & c(\mathbf{q}, x) = 1
\end{tikzcd}$$

Figure III.25: Duality relations in a single-product firm

$$z(\mathbf{q}) = \inf\{x \in X \mid c(\mathbf{q}, x) \geqq 1\}.$$

Thus, the cost function $c(\mathbf{q}, x)$ must take at least the value 1. For a comparison to the usual definition of the indirect production function $\tilde{z}$ it is convenient to refer to a further representation using the definition of the polar input requirement set $L_{\circ}(\mathbf{x})$.

(III.101) $$z(\mathbf{q}) = \inf\left\{x \in X \mid \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq 1 \;\; \forall \mathbf{v} \in L(x)\right\}$$

The polar production function faces the **indirect production function** $\tilde{z}$.[148]

(III.102) $$\begin{aligned} \tilde{z}(\mathbf{q}, \lambda) &:= \sup\left\{f(\mathbf{v}) \mid \mathbf{q}^{\mathsf{T}}\mathbf{v} \leqq \lambda\right\} \\ &= \sup\left\{x \in X \mid \mathbf{q}^{\mathsf{T}}\mathbf{v} \leqq \lambda, \;\; \mathbf{v} \in L(x)\right\} \end{aligned}$$

The indirect production function indicates the maximal admissible output x which is producible at given factor prices $\mathbf{q}$ such that the factor costs $\mathbf{q}^{\mathsf{T}}\mathbf{v}$ do not pass the value λ. Considering the cost function, this constraint is satisfied if $c(\mathbf{q}, x) \leqq \lambda$. Thus, in contrast to the polar production function z the indirect production function $\tilde{z}$ can be written as

$$\tilde{z}(\mathbf{q}, \lambda) = \sup\{x \in X \mid c(\mathbf{q}, x) \leqq \lambda\}.$$

[147] DIEWERT (1974) and BLACKORBY, PRIMONT, RUSSEL (1978, p. 39) discuss results regarding the household's preferences.

[148] The transference of the indirect production function to indirect output correspondences for multi-product firms is presented in SHEPHARD (1974b) and is discussed in full by FÄRE (1988). The indirect output correspondence $\mathbb{P}\colon Q \times \mathbb{R}_+ \to \mathfrak{P}(X)$ is defined by

$$\mathbb{P}(\mathbf{q}, \lambda) := \{\mathbf{x} \mid c(\mathbf{q}, \mathbf{x}) \leqq \lambda\}.$$

The dual view of both problems (III.101) and (III.102) is illustrated in the left hand part of Figure III.26, where the construction of both problems at once gives $\tilde{z}(\cdot, 1) \leqq z$. Whereas the orientation in direction to the origin corresponds to the problem (III.101), (III.102) reflects a movement which is directed away from the origin.

The right hand graph in Figure III.26 shows that (III.101) and (III.102) may lead to different results. The example is constructed for two indivisible production factors such that point A with $x = 1$ is assigned to problem (III.102) for $\lambda = 1$. At the same time point B with $x = 2$ corresponds to the result in (III.101).

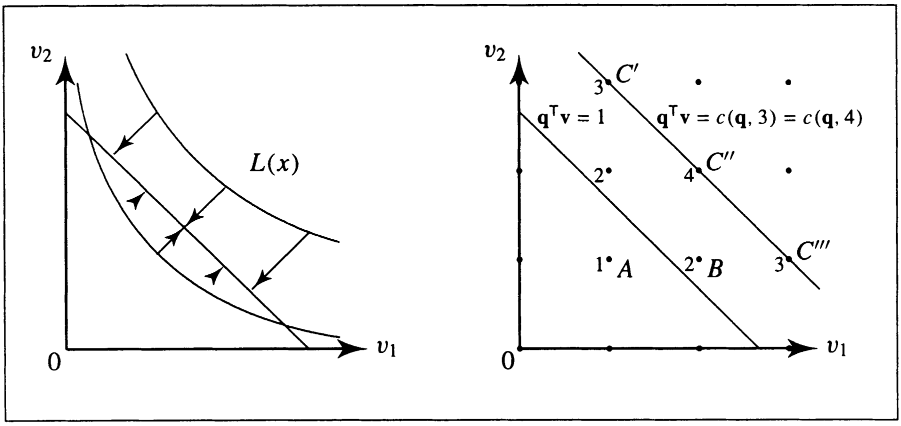

Figure III.26: Dual representation of optimal input vectors

The question as to when the problems (III.101) and (III.102) have the same result, can be answered intuitively as follows: both problems must be associated with the same minimal costs. Therefore as shown by points A and B in Figure III.26, the "right" λ must be fixed in (III.102). But even such a λ does not guarantee that the results (III.101) and (III.102) are equal. As shown by the upper isocost line in the right hand part of Figure III.26, (III.101) with $x = 3$ would give one of the points C' or C'''. However, (III.102) yields point C'' with $x = 4$. For perfectly divisible goods and factors this case can be ruled out by stipulating that the two diverse isoquants may not touch each other. Transferred to the case of indivisible goods and factors it is required that the convex hulls of two diverse input requirement sets have no boundary points in common.[149]

Proposition III.20 *Suppose the outputs x and $\tilde{x}$ satisfy*[150]

$$x > \tilde{x} \implies \operatorname{conv}L(x) \subset \operatorname{int}\big(\operatorname{conv}L(\tilde{x})\big).$$

[149]For this reason BLACKORBY, PRIMONT, RUSSEL (1978, p. 15) require the strict quasi-concavity of their underlying utility function u, i.e. for arbitrary $\lambda \in [0, 1]$

$$u(\mathbf{x}) > u(\tilde{\mathbf{x}}) \implies u\big(\lambda\mathbf{x} + (1 - \lambda)\tilde{\mathbf{x}}\big) > u(\tilde{\mathbf{x}}).$$

[150]More severely, Proposition III.9 requires $c(\cdot, x) > c(\cdot, \tilde{x})$ for all $x > \tilde{x}$; see also Theorem D.7.

Then, for the values of the polar production function and the indirect production function to be equal, $z(\mathbf{q}) = \tilde{z}(\mathbf{q}, 1) = \hat{x}$, *it is necessary and sufficient that the factor prices* $\mathbf{q}$ *are normalized such that* $c(\mathbf{q}, \hat{x}) = 1$. [151]

Proof: The necessary part directly follows from the definitions of the functions z and $\tilde{z}$ because $z(\mathbf{q}) = \tilde{z}(\mathbf{q}, 1) = \hat{x}$ means $\mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq 1$ for all $\mathbf{v} \in L(\hat{x})$ and $\mathbf{q}^{\mathsf{T}}\hat{\mathbf{v}} \leqq 1$ for at least one $\hat{\mathbf{v}} \in L(\hat{x})$. Thus, $c(\mathbf{q}, \hat{x}) = \mathbf{q}^{\mathsf{T}}\hat{\mathbf{v}} = 1$.

In the sufficient part we stipulate $c(\mathbf{q}, \hat{x}) = 1$. Hence, $H(\mathbf{q}, 1) = \{\mathbf{v} | \mathbf{q}^{\mathsf{T}}\mathbf{v} = 1\}$ denotes a (nontrivial) supporting hyperplane of the set $\mathrm{conv}L(\hat{x})$, i.e.

[a] $\quad \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq 1 \quad \forall\, \mathbf{v} \in \mathrm{conv}L(\hat{x})$

[b] $\quad \mathbf{q}^{\mathsf{T}}\hat{\mathbf{v}} = 1 \quad$ for one $\hat{\mathbf{v}} \in \mathrm{conv}L(\hat{x})$.

Because the hyperplane $H(\mathbf{q}, 1)$ separates each point $\hat{\mathbf{v}}$ fulfilling condition [b] from the set $\mathrm{conv}L(\hat{x})$ properly, $\hat{\mathbf{v}}$ may not lie in the interior of the inspected (m-dimensional) set, $\hat{\mathbf{v}} \notin \mathrm{int}(\mathrm{conv}L(\hat{x}))$. [152] By assumption, for such an input vector $\hat{\mathbf{v}}$ the condition $x > \hat{x}$ implies $\hat{\mathbf{v}} \notin \mathrm{conv}L(x)$. Thus, $\hat{x}$ is the smallest of all outputs x fulfilling [a] and [b]. In particular, [b] yields $z(\mathbf{q}) = \hat{x}$.
If the input vector lies in the boundary of $\mathrm{conv}L(\hat{x})$, then by imposition $x < \hat{x}$ implies $\hat{\mathbf{v}} \in \mathrm{int}(\mathrm{conv}L(x))$. Contrary to [a] $H(\mathbf{q}, 1)$ cannot be a supporting hyperplane of the set $\mathrm{conv}L(x)$. With that $\hat{x}$ is the largest of all outputs x satisfying [a] and [b]. In particular, [a] leads to $\tilde{z}(\mathbf{q}, 1) = \hat{x}$. ■

Proposition III.20 can also be viewed in reverse order for a given factor price vector $\mathbf{q}$. If the cost function $c(\mathbf{q}, \cdot)$ is continuous at x and if $c(\mathbf{q}, \cdot)$ takes all values between 0 and $C > 1$, then there must be an output level $\hat{x}$ with $c(\mathbf{q}, \hat{x}) = 1$. In this case the pair $(\mathbf{q}, \hat{x})$ fulfills[153]

$$\inf\{x \in X | \, c(\mathbf{q}, x) \geqq 1\} = z(\mathbf{q}) = \hat{x} = \tilde{z}(\mathbf{q}, 1) = \sup\{x \in X | \, c(\mathbf{q}, x) \leqq 1\}.$$

2.4 Summary

2.4.1 Schematic Construction of Duality Theory

The three following figures are constructed similarly to Figure III.24 each emphasizing different aspects. Figure III.27 draws the attention to a nonempty closed convex input requirement set $L(\mathbf{x})$ not containing the origin $\mathbf{v} = \mathbf{0}$. Hence, a commodity bundle $\mathbf{x} \in X \setminus \{\mathbf{0}\}$ must be presumed. If the examined input requirement set is not convex, then the basic statements for the convex hull $\mathrm{conv}L(\mathbf{x})$ as a substitute for $L(\mathbf{x})$ remain unchanged, particularly as this set is closed under the assumptions of Proposition II.14.

[151] Considering the homogeneity of degree 0 of the indirect production function $\tilde{z}$ in $\mathbf{q}$ and λ, the mentioned equivalence relation can be written in the form $z(\mathbf{q}/\lambda) = \tilde{z}(\mathbf{q}, \lambda) = \hat{x} \iff c(\mathbf{q}, \hat{x}) = \lambda$.

[152] Cf. ROCKAFELLAR (1972, p. 100, Theorem 11.6).

[153] This outcome is confirmed by BLACKORBY, PRIMONT, RUSSEL (1978, p. 379, Theorem A.10).

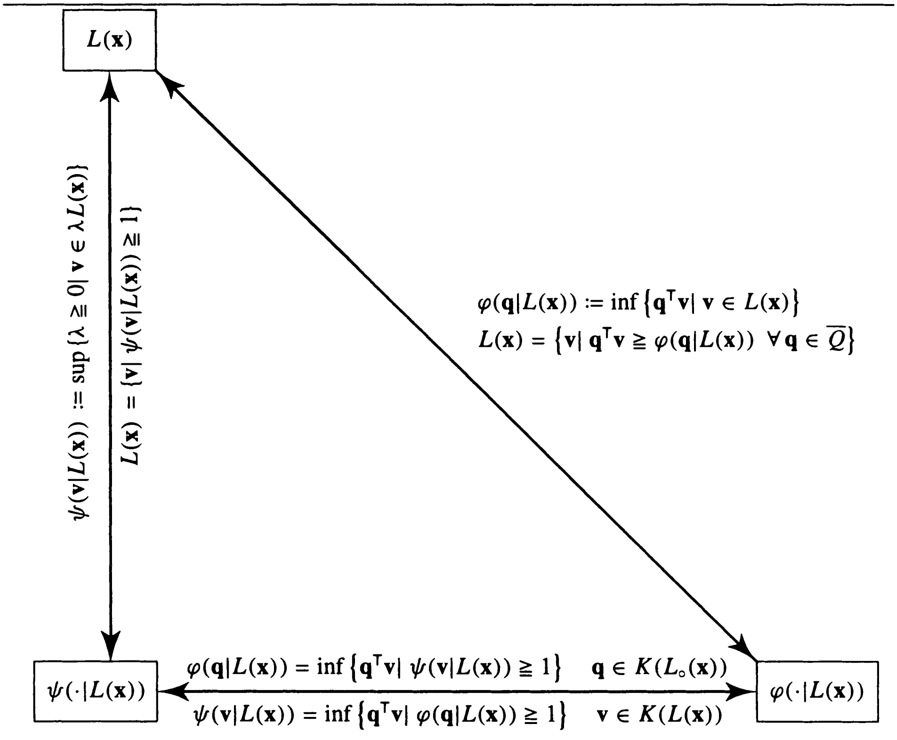

Figure III.27: Gauge and support function of the input requirement set $L(\mathbf{x})$

The input requirement set is associated with two functions. The support function $\varphi(\cdot|L(\mathbf{x}))$ depends on the factor prices $\mathbf{q}$ and indicates the minimal factor costs occurring at the production of the given commodity bundle $\mathbf{x}$; therefore, it is called the cost function. However, the gauge $\psi(\cdot|L(\mathbf{x}))$ serves as a measure for the efficiency of an input vector $\mathbf{v}$. If $\psi(\mathbf{v}|L(\mathbf{x})) = 1$, then no alternative input vector lies in the ray through the origin and $\mathbf{v}$ being closer to the origin. Thus, $\mathbf{v}$ lies in the boundary of the input requirement set $L(\mathbf{x})$ and there is a hyperplane $H(\mathbf{q}, \alpha)$ separating the point $\mathbf{v}$ and the set $L(\mathbf{x})$ properly. At the same time $\varphi(\mathbf{q}|L(\mathbf{x})) = \alpha$ denotes the distance between the origin $\mathbf{v} = \mathbf{0}$ and this hyperplane. Scaling the factor prices $\mathbf{q}$ suitably, the relation between the support function $\varphi(\cdot|L(\mathbf{x}))$ and the gauge $\psi(\cdot|L(\mathbf{x}))$ can now be justified by normalizing α to unity, i.e. $\varphi(\mathbf{q}/\lambda|L(\mathbf{x})) = \alpha/\lambda = 1$. However, this procedure is only admissible if the corresponding hyperplane $H(\mathbf{q}, \alpha)$ does not contain the origin, i.e. $\alpha > 0$. All of these supporting hyperplanes of an input requirement set $L(\mathbf{x})$ are determined by a perpendicular vector $\mathbf{q}$ of factor prices so that it seems to be reasonable to define the polar input requirement set $L_{\circ}(\mathbf{x})$ as follows:

$$L_{\circ}(\mathbf{x}) = \left\{\mathbf{q}|\ \mathbf{q}^{\mathrm{T}}\mathbf{v} \geqq 1\ \ \forall \mathbf{v} \in L(\mathbf{x})\right\}.$$

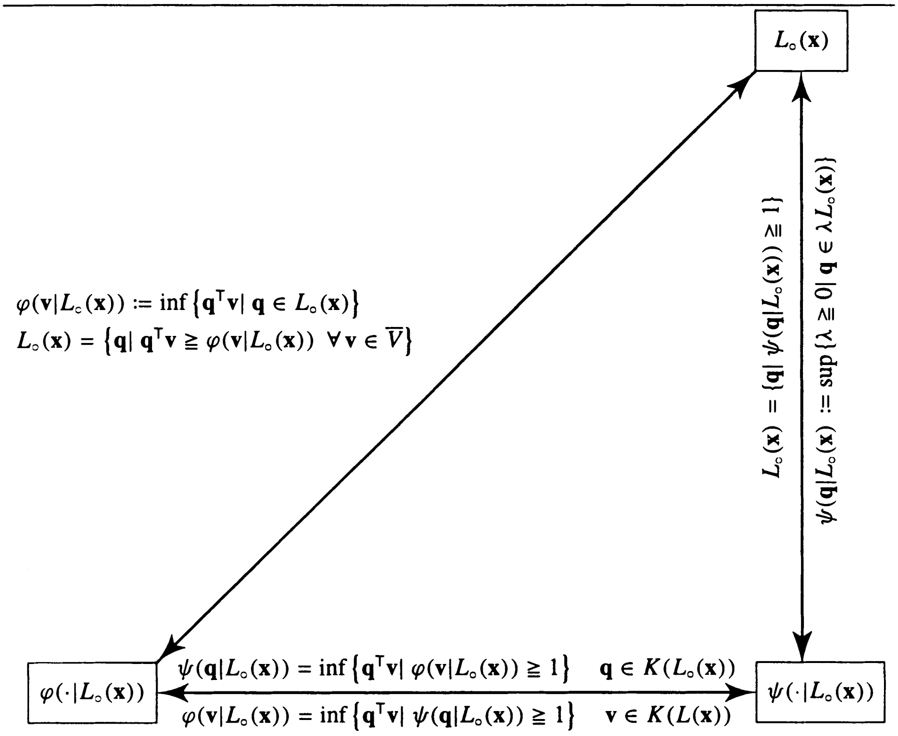

Figure III.28: Gauge and support function of the polar input requirement set $L_\circ(\mathbf{x})$

This set collects all factor price vectors which guarantee that the factor costs $\mathbf{q}^\mathsf{T}\mathbf{v}$ do not fall below unity in the production of the commodity bundle $\mathbf{x}$. Each input requirement set can be assigned to such a polar input requirement set so that the production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$ faces a family $\big(L_\circ(\mathbf{x})|\ \mathbf{x} \in X\big)$, which is called the cost structure. For each commodity bundle $\mathbf{x} \in X \setminus \{\mathbf{0}\}$ it can be shown that the set $L_\circ(\mathbf{x})$ as well as the input requirement set itself are nonempty, closed, and convex and do not contain the origin $\mathbf{q} = \mathbf{0}$. Thus, it seems reasonable to associate the polar input requirement set $L_\circ(\mathbf{x})$ with the support function $\varphi(\cdot|L_\circ(\mathbf{x}))$ and the gauge $\psi(\cdot|L_\circ(\mathbf{x}))$. Figure III.28 differs from Figure III.27 only by the fact that the input requirement set contains input vectors, $L(\mathbf{x}) \subset \overline{V}$, and that the polar input requirement set consists of factor price vectors, $L_\circ(\mathbf{x}) \subset \overline{Q}$.

The close relation between Figures III.27 and III.28 results in several facts. On the basis of the subsequently given relations both figures may be placed on top of each other so that Figure III.29 ensues, which is equivalent to Figure III.24, p. 158.

To stress the symmetry $L(\mathbf{x}) \rightleftarrows L_\circ(\mathbf{x})$, bear in mind that the bipolar input requirement set

$$L_{\circ\circ}(\mathbf{x}) = \{\mathbf{v}|\ \mathbf{q}^\mathsf{T}\mathbf{v} \geqq 1\ \ \forall\mathbf{q} \in L_\circ(\mathbf{x})\}$$

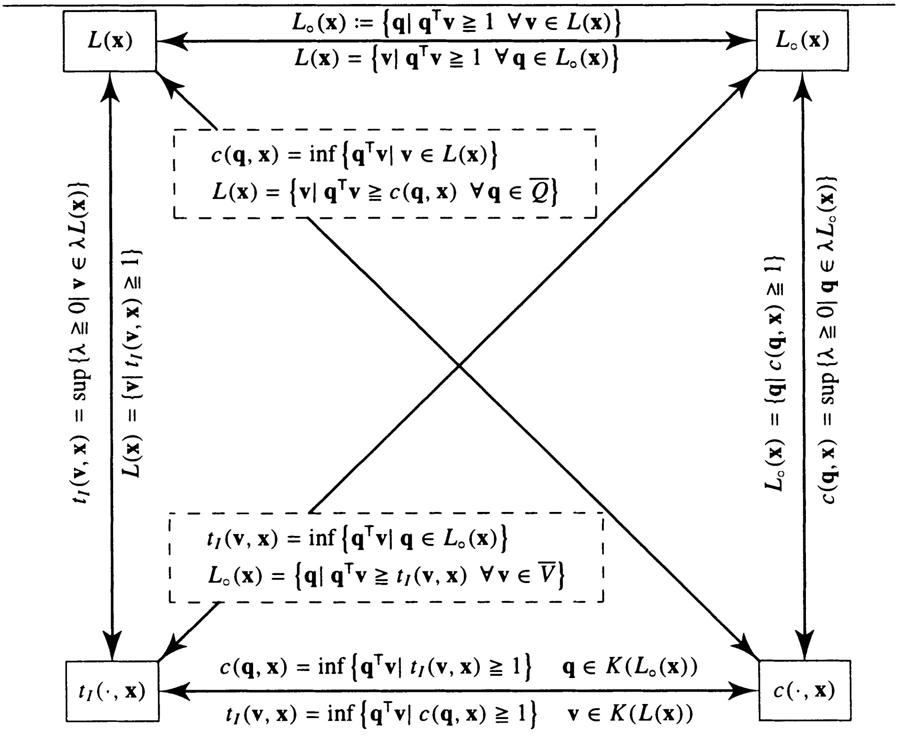

Figure III.29: Duality relations in the sense of SHEPHARD (2)

corresponds to the set which has been suggested as a replacement for the input requirement set assuming that $L(\mathbf{x})$ is not convex, i.e. $L_{\circ\circ}(\mathbf{x}) = \mathrm{cl}\left(\mathrm{conv}L(\mathbf{x})\right)$. In particular, a nonempty closed convex input requirement set $L(\mathbf{x})$ not containing the origin fulfills

$$L_{\circ\circ}(\mathbf{x}) = L(\mathbf{x})$$

provided [L3] (*Disposability of Inputs*) holds. While the cost function $c(\cdot, \mathbf{x})$ is defined in the sense of a support function φ of the input requirement set $L(\mathbf{x})$, this function can now be interpreted as the gauge ψ of the polar input requirement set $L_\circ(\mathbf{x})$,

$$c(\cdot, \mathbf{x}) \equiv \varphi(\cdot|L(\mathbf{x})) = \psi(\cdot|L_\circ(\mathbf{x})).$$

If $\psi(\mathbf{q}|L_\circ(\mathbf{x})) = 1$, then the (normalized) factor price vector $\mathbf{q}$ lies in the boundary $\partial L_\circ(\mathbf{x})$ of the polar input requirement set $L_\circ(\mathbf{x})$.[154] For all input vectors $\mathbf{v} \in L(\mathbf{x})$ we have $\mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq 1$ and for at least one input vector $\hat{\mathbf{v}}$ the minimal costs $c(\mathbf{q}, \mathbf{x}) = \mathbf{q}^{\mathsf{T}}\hat{\mathbf{v}} = 1$ are attained.

[154] MCFADDEN calls the boundary of the polar input requirement set $L_\circ(\mathbf{x})$ the factor price frontier.

Analogously, the input distance function $t_I(\cdot, \mathbf{x})$, which is defined as the gauge ψ of the set $L(\mathbf{x})$, can be interpreted as a support function φ of the polar set $L_\circ(\mathbf{x})$,

$$t_I(\cdot, \mathbf{x}) \equiv \psi(\cdot|L(\mathbf{x})) = \varphi(\cdot|L_\circ(\mathbf{x})).$$

Finally, the direct relationship $c(\cdot, \mathbf{x}) \rightleftarrows t_I(\cdot, \mathbf{x})$ between the cost function and the input distance function is characterized by MAHLER's inequality.

$$c(\mathbf{q}, \mathbf{x})\, t_I(\mathbf{v}|L(\mathbf{x})) \leqq \mathbf{q}^{\mathsf{T}}\mathbf{v} \qquad \forall \mathbf{q} \in K(L_\circ(\mathbf{x})),\ \forall \mathbf{v} \in K(L(\mathbf{x}))$$

If for a pair of polar points $(\hat{\mathbf{q}}, \hat{\mathbf{v}})$ the equation $c(\hat{\mathbf{q}}, \mathbf{x})\, t_I(\hat{\mathbf{v}}|L(\mathbf{x})) = \hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}}$ holds, then $\hat{\mathbf{q}}$ lies in the boundary of the polar input requirement set, $\hat{\mathbf{q}} \in \partial L_\circ(\mathbf{x})$, and $\hat{\mathbf{v}}$ lies in the boundary of the input requirement set, $\hat{\mathbf{v}} \in \partial L(\mathbf{x})$. Note that the equation requires a pair of polar points $(\hat{\mathbf{q}}, \hat{\mathbf{v}})$ at which neither the cost function nor the input distance function takes the value zero. This restriction is satisfied if $\hat{\mathbf{q}}$ and $\hat{\mathbf{v}}$ are contained in the respective given cones $K(L_\circ(\mathbf{x}))$ and $K(L(\mathbf{x}))$. These cones are explained in more detail in the following graphical discussion.

2.4.2 Graphical Representation of the Results

(a) Convex Input Requirement Sets First of all, analogous to Section 1.5.1, the graphical comparison of an input requirement set $L(\mathbf{x})$ and its polar set $L_\circ(\mathbf{x})$ takes place for the case of two divisible production factors v_1 and v_2. Whereas this case presumes a strictly convex input requirement set without loss of generality, the consideration of indivisible production factors implies a nonconvex input requirement set. This case is inspected separately and conforms to the example underlying the Figures III.12 (p. 114) and III.14 (p. 117). Moreover, Figure III.36 (p. 181) demonstrates how the results of Figures III.12 or III.14 are connected with the results of Figures III.34 (p. 179) or III.35 (p. 180). In this case the factor demand correspondences will serve as a link.

Figure III.30 illustrates the geometrical derivation[155] of the polar input requirement set $L_\circ(\mathbf{x})$ (Quadrant III) from the given input requirement set $L(\mathbf{x})$ (Quadrant I). First of all, an arbitrary point lying on the boundary of the set $L(\mathbf{x})$ is picked out, $\hat{\mathbf{v}} \in \partial L(\mathbf{x})$. Thus, a hyperplane $H(\tilde{\mathbf{q}}, \alpha)$ exists which separates the input vector $\hat{\mathbf{v}}$ and the input requirement set properly; see $\overline{AA'}$. At the same time the distance between the origin $\mathbf{v} = \mathbf{0}$ and these hyperplanes denotes the minimal costs $c(\tilde{\mathbf{q}}, \mathbf{x}) = \tilde{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}} = \alpha$. Using the linear homogeneity of the cost function in the factor prices, α can be normalized to one, $c(\hat{\mathbf{q}}, \mathbf{x}) = 1$, where $\hat{\mathbf{q}}$ is set to $\tilde{\mathbf{q}}/\alpha$. If an isocost curve like $\overline{AA'}$ intersects the ordinate (point A), then $c(\mathbf{q}) = q_2 v_2 = 1$. The corresponding hyperbola is depicted in Quadrant II and particularly contains point B. Analogously, we get the relation $c(\mathbf{q}) = q_1 v_1 = 1$ for each point in the horizontal line (point A'). Quadrant IV shows the adjoined hyperbola, where point B' can be found. With the points B and B' both factor prices, which determine

[155] Similar representations may be found for example in DARROUGH, SOUTHEY (1977) or SGRO (1986). The relation to SHEPHARD's Theorem (Proposition III.19) is emphasized in FÄRE (1984).

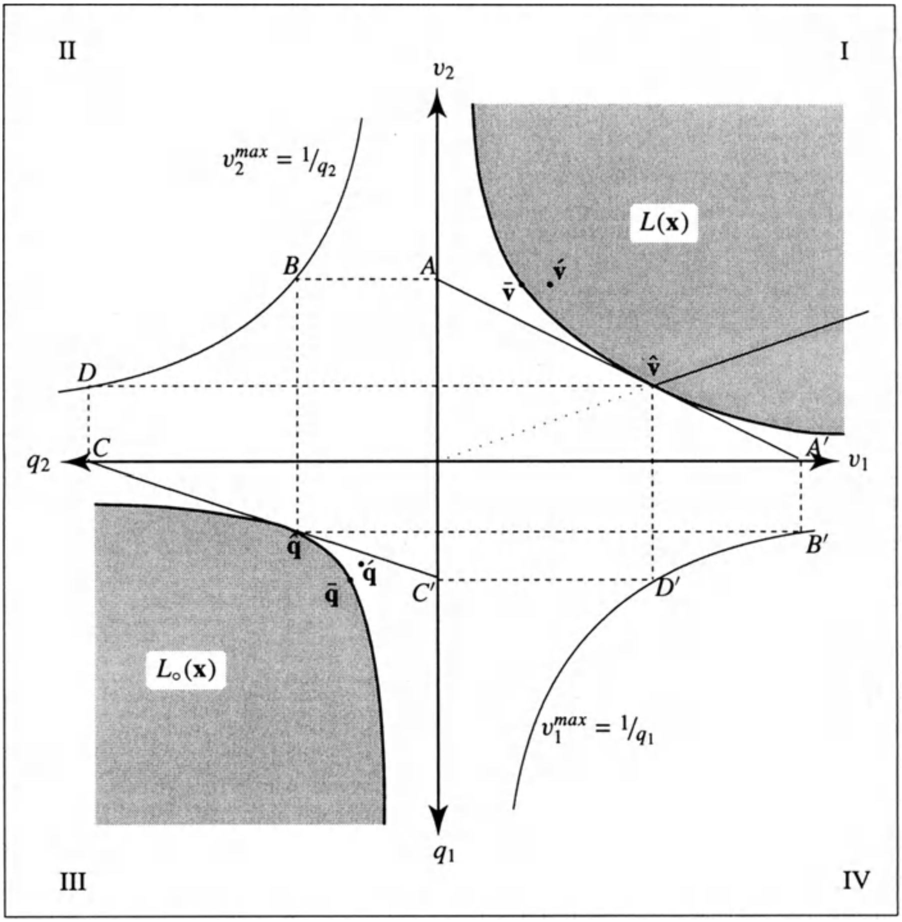

Figure III.30: Geometrical derivation of the polar input requirement set $L_\circ(\mathbf{x})$

the hyperplane $H(\hat{\mathbf{q}}, 1)$, are known. In virtue of this, the first point of the polar set $L_\circ(\mathbf{x})$ is found.

Since the input vector $\hat{\mathbf{v}}$ solves the corresponding problem of cost minimization $\inf\left\{\hat{\mathbf{q}}^\mathsf{T}\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x})\right\}$, it ensues $t_I(\hat{\mathbf{v}}, \mathbf{x}) = 1$; see SHEPHARD's Theorem (Proposition III.19). Thus, $\hat{\mathbf{q}}$ lies in the boundary of the polar input requirement set, $\hat{\mathbf{q}} \in \partial L_\circ(\mathbf{x})$. Each alternative boundary point of the set $L(\mathbf{x})$ generates a further boundary point of the set $L_\circ(\mathbf{x})$, until eventually $L_\circ(\mathbf{x})$ is completely determined; see for instance $\bar{\mathbf{v}}$ and $\bar{\mathbf{q}}$. The reconstruction of the set $L(\mathbf{x}) = L_{\circ\circ}(\mathbf{x})$ is carried out in the same way. The supporting hyperplane $\overline{CC'}$ implies points D and D', which both yield the initial point $\hat{\mathbf{v}}$.

In particular, the various forms of describing an input requirement set $L(\mathbf{x})$

are now plausible.[156] The presented construction of a single point corresponds to expression (III.89a), p. 158,

$$L(\mathbf{x}) = \left\{\mathbf{v} \mid \mathbf{q}^{\mathrm{T}}\mathbf{v} \geqq 1 \;\; \forall \mathbf{q} \in L_{\circ}(\mathbf{x})\right\}.$$

The hyperplane $\overline{AA'}$ yields one of the inequalities in

$$L(\mathbf{x}) = \bigcap_{\mathbf{q} \in \overline{Q}} \left\{\mathbf{v} \mid \mathbf{q}^{\mathrm{T}}\mathbf{v} \geqq c(\mathbf{q}, \mathbf{x})\right\}.$$

Finally, the ray starting in $\hat{\mathbf{v}}$ can be associated with the following equivalent form of representation in (III.92a), p. 160.

$$L(\mathbf{x}) = \{\mathbf{v} \mid t_I(\mathbf{v}, \mathbf{x}) \geqq 1\}$$

The construction principle presented by Figure III.30 immediately yields further properties indicating the relationship between the sets $L(\mathbf{x})$ and $L_{\circ}(\mathbf{x})$.

(1) If two commodity bundles $\mathbf{x}$ and $\tilde{\mathbf{x}}$ with $\mathbf{x} \geq \tilde{\mathbf{x}}$ are examined, then by [L4] (*Disposability of Outputs*) $L(\mathbf{x}) \subset L(\tilde{\mathbf{x}})$. From that follows the relation[157] shown by Figure III.31, $L_{\circ}(\mathbf{x}) \supseteq L_{\circ}(\tilde{\mathbf{x}})$. On the one hand $L(\mathbf{x})$ is further away from the origin than $L(\tilde{\mathbf{x}})$, i.e. by (III.68) $c(\cdot, \mathbf{x}) \geqq c(\cdot, \tilde{\mathbf{x}})$. On the other hand $L_{\circ}(\mathbf{x})$ lies closer to the origin than $L_{\circ}(\tilde{\mathbf{x}})$ such that $t_I(\cdot, \mathbf{x}) \leqq t_I(\cdot, \tilde{\mathbf{x}})$, with respect to Proposition III.19, results. These outcomes are noted by the figure opposite without deriving them geometrically according to Figure III.30.

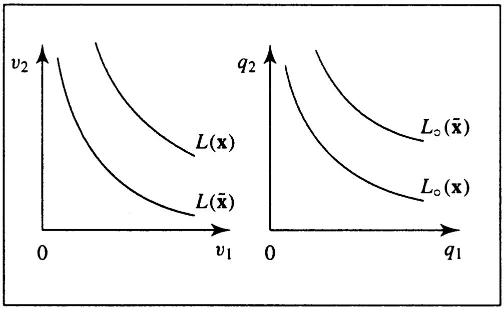

Figure III.31: Geometry of polar sets (1)

(2) The stronger the boundary $\partial L(\mathbf{x})$ is curved, the weaker the boundary $\partial L_{\circ}(\mathbf{x})$ is curved and vice versa. An extreme case is already visible in Figure III.30. If the line $\overline{AA'}$ is the (not curved) boundary of an input requirement set, then the construction principle implies a right-angled set $\partial L_{\circ}(\mathbf{x})$ with vertex $\hat{\mathbf{q}}$. Another example is illustrated by the input vector $\bar{\mathbf{v}}$ and the corresponding factor price vector $\bar{\mathbf{q}}$. If $\partial L(\mathbf{x})$ were curved more strongly, then at the same price ratio a cost minimizing input vector like $\check{\mathbf{v}}$ would result. The corresponding factor price vector $\check{\mathbf{q}}$ must lie in the same ray through the origin as $\bar{\mathbf{q}}$ – but closer to the origin. The stronger curvature of $\partial L(\mathbf{x})$ yields a weaker curvature of $\partial L_{\circ}(\mathbf{x})$.

To provide a technical argument for this behavior of curvature, we suppose that the boundary $\partial L(\mathbf{x})$ can be given by the implicit function $f(\mathbf{v}, \mathbf{x}) = 1$, which is differentiable at $\mathbf{v}$. If the input vector $\hat{\mathbf{v}}$ solves the problem of cost minimization, then the marginal rate of substitution MRS between the factors v_1 and v_2 equals the

[156] See Figure III.29. The polar input requirement set satisfies analogous relations.
[157] See Appendix D.2.3.

factor price ratio.

$$MRS(\hat{\mathbf{v}}) := \frac{\partial f(\hat{\mathbf{v}}, \mathbf{x})/\partial v_1}{\partial f(\hat{\mathbf{v}}, \mathbf{x})/\partial v_2} = -\left.\frac{\mathrm{d}v_2}{\mathrm{d}v_1}\right|_{\mathbf{v}=\hat{\mathbf{v}}} = \frac{q_1}{q_2}$$

Now with the (positive) elasticity of substitution[158]

$$\eta_{\mathbf{v}} := \frac{\frac{\mathrm{d}(v_2/v_1)}{v_2/v_1}}{\frac{\mathrm{d}MRS(\hat{\mathbf{v}})}{MRS(\hat{\mathbf{v}})}} = \frac{\mathrm{d}\ln(v_2/v_1)}{\mathrm{d}\ln(q_1/q_2)}$$

we can give a measure for the curvature of the implicit function $f(\cdot, \mathbf{x})$ at point $\hat{\mathbf{v}}$.

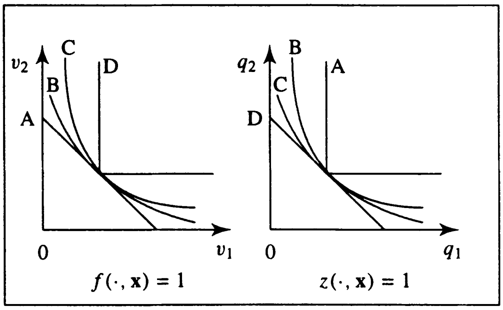

Figure III.32: Geometry of polar sets (2)

If an increase of the factor price ratios q_1/q_2 by one percent implies a relative large increase of the factor intensity v_2/v_1, then the function $f(\cdot, \mathbf{x})$ is not curved very strongly and vice versa; see the left hand part of the figure opposite. If the analogous function $z(\mathbf{q}, \mathbf{x}) = 1$ corresponds to the boundary of the set $L_\circ(\mathbf{x})$, then the elasticity

$$\eta_{\mathbf{q}} := \frac{\mathrm{d}\ln(q_1/q_2)}{\mathrm{d}\ln(v_2/v_1)}$$

yields a measure for the curvature of the function $z(\cdot, \mathbf{x})$. In view of Figure III.25 not only the notation but also the relation between the functions f and z gives rise to the suspicion that $\eta_{\mathbf{v}} = 1/\eta_{\mathbf{q}}$.[159] Without going into more detail at this point, we refer to Figure III.32 where the cases A, B, C and D are linked. In case A the corresponding elasticities are $\eta_{\mathbf{v}} = +\infty$ and $\eta_{\mathbf{q}} = 0$.

(3) In the discussion of polar gauges the two cones $K(L(\mathbf{x})) = \{\mu\mathbf{v} \mid \mathbf{v} \in L(\mathbf{x}), \mu > 0\}$ and $K(L_\circ(\mathbf{x})) = \{\lambda\mathbf{q} \mid \mathbf{q} \in L_\circ(\mathbf{x}), \lambda > 0\}$ were repeatedly established as feasible regions. The two cones have a geometric meaning[160] when examining the behavior of the two sets $L(\mathbf{x})$ and $L_\circ(\mathbf{x})$ in the boundary of $\mathbb{R}^m_+$.

As stressed by MCFADDEN (1978), we have to distinguish between three cases which have different effects on the cone concerned.

1. The set $L(\mathbf{x})$ touches an axis asymptotically if and only if the polar set $L_\circ(\mathbf{x})$ touches the other axis asymptotically; see Figure III.33. If $L(\mathbf{x})$ touches both axes asymptotically as in the case of the COBB-DOUGLAS production function, then $K(L(\mathbf{x})) = \mathbb{R}^2_{++}$ and $K(L_\circ(\mathbf{x})) = \mathbb{R}^2_{++}$.

[158] Cf. VARIAN (1992, p. 13).

[159] A detailed proof for the validity of this relation may be found in MCFADDEN (1978, p. 41 ff.).

[160] A similar discussion is given by DIEWERT (1971).

2. The set $L(\mathbf{x})$ is tangent to an axis if and only if the polar set $L_\circ(\mathbf{x})$ touches a parallel of the other axis asymptotically et vice versa; see Figure III.33. If $L(\mathbf{x})$ is tangent to both axes, then $K(L(\mathbf{x})) = \mathbb{R}^2_+ \setminus \{\mathbf{0}\}$ and $L_\circ(\mathbf{x})$ touch both axes asymptotically, i.e. $K(L_\circ(\mathbf{x})) = \mathbb{R}^2_{++}$.

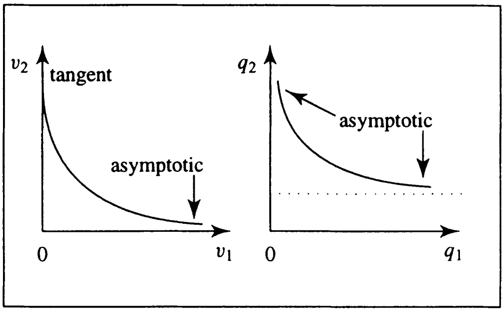

Figure III.33: Geometry of polar sets (3)

3. The set $L(\mathbf{x})$ touches an axis (not tangentially) if and only if the polar set $L_\circ(\mathbf{x})$ touches a parallel of the other axis (not tangentially) and falls short of it.
   The opposite case is given by Figure III.35. The set $L_\circ(\tilde{x})$ touches both axes and $L(\tilde{x})$ touches the vertical line through A or the horizontal line through B. The corresponding cone now satisfies $K(L(\mathbf{x})) = \mathbb{R}^2_{++}$ and $K(L_\circ(\mathbf{x})) = \mathbb{R}^2_+ \setminus \{\mathbf{0}\}$.

Whenever a supporting hyperplane $H(\mathbf{q}, \alpha)$ of the set $L(\mathbf{x})$ contains the origin, then $\lambda\mathbf{q} \notin K(L_\circ(\mathbf{x}))$ for all $\lambda > 0$. Actually, the same fact is valid with regard to the polar set $L_\circ(\mathbf{x})$. Both implications reflect the assumptions on the equation (III.99) in SHEPHARD's Theorem (Proposition III.19).

(b) Consideration of Indivisible Production Factors For the concluding graphical discussion of indivisible production factors it is convenient to pick up again the example of Section 2.1.3.[161]

Example: *The input requirement set*

(III.41) $$L(\tilde{x}) = \left\{ \binom{v_1}{v_2} \in \mathbb{Z}^2_+ \mid \binom{v_1}{v_2} \geqq \binom{v_1'}{v_2'} \text{ or } \binom{v_1}{v_2} \geqq \binom{v_1''}{v_2''} \right\}$$

is shown in Quadrant I of Figure III.34. Defining the factor price ratio given by points A and B

(III.43) $$\tilde{q}_1^{-r} := \frac{v_2' - v_2''}{v_1'' - v_1'} = \frac{\tilde{v}_2}{\tilde{v}_1},$$

the corresponding cost function is

[161] See p. 114 ff. and p. 149 ff.

(III.79)
$$c(q_1, q_2, \tilde{x}) = \begin{cases} q_1 v_1' + q_2 v_2' & \text{for } q_2 > 0,\ q_1 > 0 \text{ and } q_1/q_2 \geqq \tilde{q}_1^{-r} > 0 \\ q_1 v_1'' + q_2 v_2'' & \text{for } q_2 > 0,\ q_1 > 0 \text{ and } q_1/q_2 \leqq \tilde{q}_1^{-r} \\ q_2 v_2'' & \text{for } q_2 > 0,\ q_1 = 0 \\ q_1 v_1' & \text{for } q_2 = 0,\ q_1 > 0 \\ 0 & \text{for } q_2 = 0,\ q_1 = 0 \\ -\infty & \text{for } q_2 < 0 \text{ or } q_1 < 0. \end{cases}$$

Quadrant III contains the isocost curve $c(q_1, q_2, \tilde{x}) = 1$, *in which the points* A', B' *and* C' *can be found. At the same time these points are the extreme points of the polar set* $L_\circ(\mathbf{x}) = \{\mathbf{q} |\ c(q_1, q_2, \tilde{x}) \geqq 1\}$. *According to Figure III.30, Quadrants II and IV contain the hyperbolas* $v_1 = {}^1/_{q_1}$ *and* $v_2 = {}^1/_{q_2}$ *respectively. The presented geometrical construction principle yields point* B' *for each point in the line* $\overline{AB}$. *In this the marked distance ratios imply the relation* $\tilde{q}_1^{-r} = q_1/q_2 = \tilde{v}_2/\tilde{v}_1$ *for the ray through the origin and* B'. *Point* A *is, however, compatible with each price ratio in line* $\overline{B'C'}$. *Similarly, point* B *is mapped into the line* $\overline{A'B'}$. *Each point on the horizontal line through* B *is cost minimizing if and only if* $\mathbf{q}$ *lies on the* q_2*-axis to the left of* A', *i.e. especially* $q_1 = 0$.

After the polar input requirement set $L_\circ(\tilde{x})$ *has been deduced, now the bipolar set* $L_{\circ\circ}(\tilde{x}) = \text{conv}L(\tilde{x})$ *can be constructed by using the reverse conclusion. Take into consideration, according to Quadrant I in Figure III.35, that the approximate input requirement set* $\text{conv}L(\tilde{x})$ *is aureoled by* [L3] *(Disposability of Inputs) and closed by Proposition II.14.*

$$\text{conv}L(\tilde{x}) = \text{aur}\left(\text{conv}L(\tilde{x})\right) \quad \textit{and} \quad \text{conv}L(\tilde{x}) = \text{cl}\left(\text{conv}L(\tilde{x})\right)$$

The construction principle is now known and needs no further explanation. The line $1 = q_1 \acute{v}_1 + q_2 \acute{v}_2$ *supports the set* $L_\circ(\tilde{x})$ *at point* B' *and generates the boundary point* $(\acute{v}_1, \acute{v}_2)$ *of the bipolar set* $L_{\circ\circ}(\tilde{x}) = \text{conv}L(\tilde{x})$.

The representation of the convex hull $\text{conv}L(\tilde{x})$ *is effected by two functions. While the input distance function* $t_I\left(\binom{v_1}{v_2}, \tilde{x}\right) = 1$ *implicitly indicates the entire boundary of the set* $\text{conv}L(\tilde{x})$, *the vertical area, starting at point* A *on the function* $f(v_1 | \text{conv}L(\tilde{x})) = \inf\left\{v_2 |\ \binom{v_1}{v_2} \in \text{conv}L(\tilde{x})\right\}$, *is ignored.*[162] *Considering* $f(\cdot | \text{conv}L(\tilde{x})) = -g_{**}(\cdot | L(\tilde{x}))$, *we obtain*[163]

$$f(v_1 | \text{conv}L(\tilde{x})) = \begin{cases} +\infty & \text{for } v_1 < v_1' \\ v_2' - \tilde{q}_1^{-r}(v_1 - v_1') & \text{for } v_1' \leqq v_1 \leqq v_1'' \\ v_2'' & \text{for } v_1'' < v_1 \end{cases}$$

from the concave biconjugate function $g_{**}(\cdot | L(\tilde{x}))$ *in (III.47).*

This function indicates the nonvertical boundary of $\text{conv}L(\tilde{x})$ *and is depicted in Quadrant I of Figures III.35 and III.36. For* $v_1' \leqq v_1 \leqq v_1''$ *the function* $f(\cdot | \text{conv}L(\tilde{x}))$ *is a straight line between* $A = (v_1', v_2')$ *and* $B = (v_1'', v_2'')$.

[162] A comparison of both construction principles may be found in Figure III.1, p. 85.
[163] See (III.26b), p. 100.

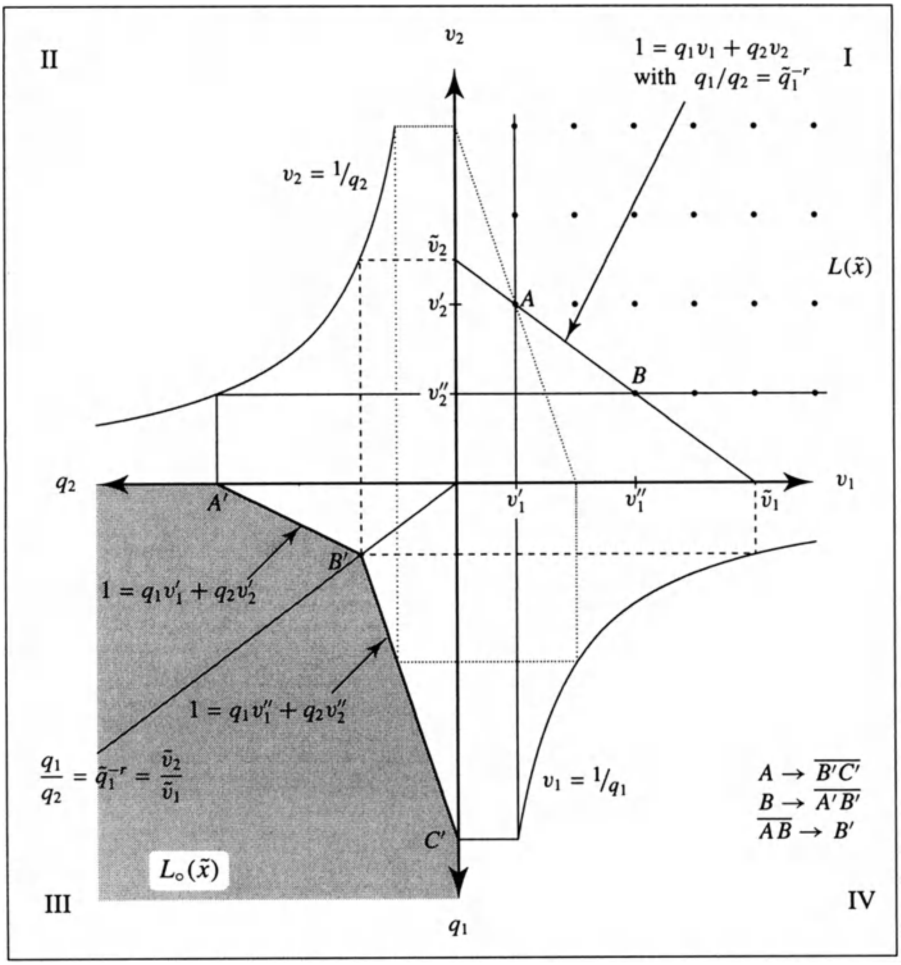

Figure III.34: Geometrical derivation of the isocost curve $c(q_1, q_2, \tilde{x}) = 1$

The example, underlying the Figures III.34 and III.35 as well as III.12 and III.14, can now be completed. According to Sections III.1 and III.2 the concluding Figure III.36 contains a mixed form of the presented duality schemes. Note that we have to distinguish the relative factor price $q_1^{-r} = q_1/q_2$ (with $r = 2$) from the nominal factor price q_1.

Analogous to Figure III.34, the representation of the isocost curve $c(q_1, q_2, \tilde{x}) = 1$ occurs in Quadrant III of Figure III.36 by the curve $A'B'C'$. The super-differential $\Delta\tilde{c}(\cdot, \tilde{x})$ of the normalized cost function $\tilde{c}(\cdot, \tilde{x})$, which includes the demand correspondence $D(\cdot, \tilde{x})$ regarding factor 1, is depicted in Quadrant IV.[164]

[164]A similar form of representation is chosen in EATON, LEMCHE (1992), where the attention is directed to the commodity supply of a multi-product firm.

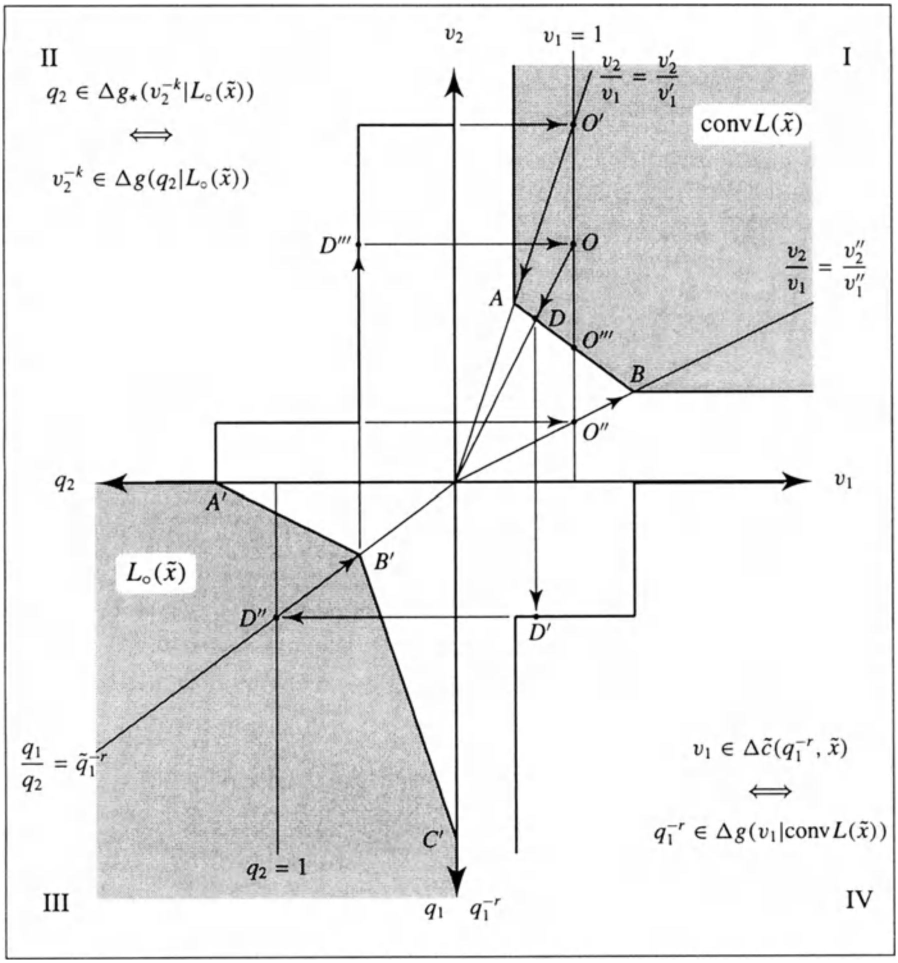

Figure III.35: Geometrical derivation of the approximate input requirement set

This correspondence has been derived geometrically in Figure III.14 and is depicted in Quadrant I.

Starting at point O (with given $v_1 = 1$) in Quadrant I, the geometric linkage of Quadrants I, III and IV can be expressed as follows: while the solution to the problem $\inf\{v_2| \binom{1}{v_2} \in \text{conv}L(\tilde{x})\}$ results in point O''', $\sup\{\lambda \geqq 0| \binom{1}{v_2} \in \lambda\text{conv}L(\tilde{x})\}$ yields point D by dividing the coordinates of point O by the optimal λ. Since both points are usually associated with different ratios of factor prices, it is supposed that both problems generate the same optimal point D. Thus, instead of the initial point O we presume alternative starting points such that the analysis can continue with point D.

As can be seen from the remarks on Figure III.12, p. 114, each point on the

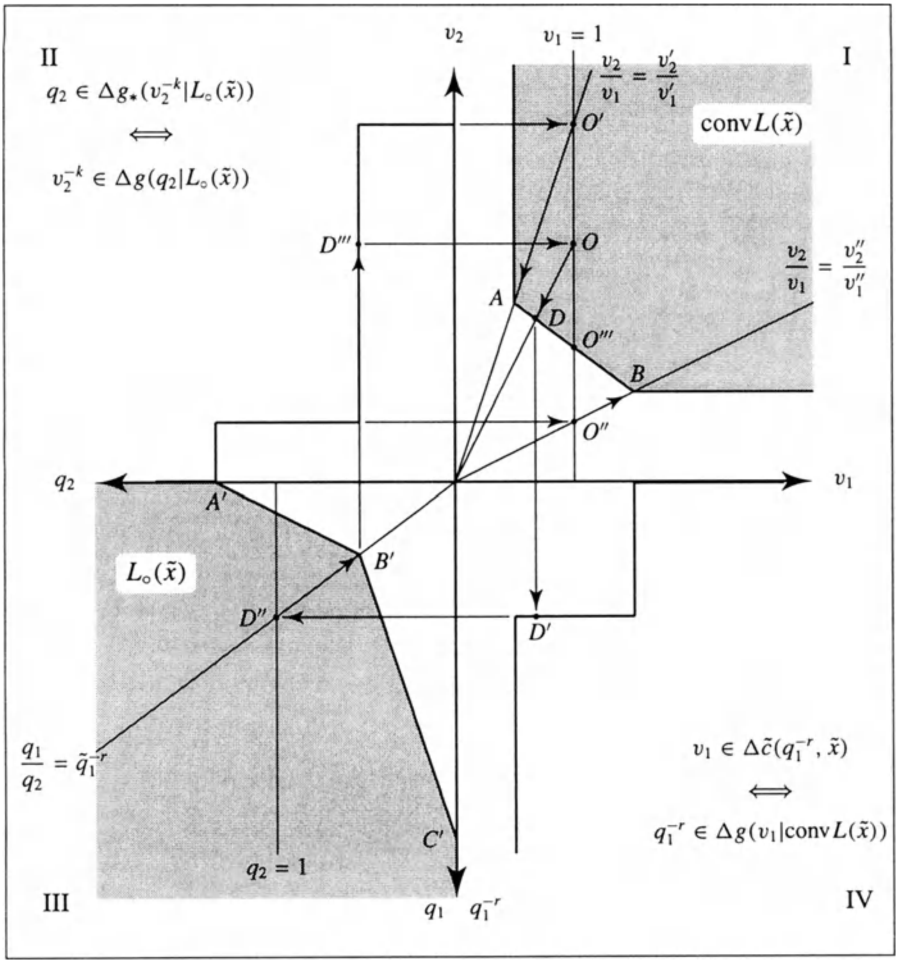

Figure III.36: Comparison of the duality schemes

line $\overline{AB}$ (i.e. in particular point D) is linked with point B' on the isocost curve $c(q_1, q_2, \tilde{x}) = 1$.

At the same time each point on the line $\overline{AB}$ is mapped into a point of the superdifferential $\Delta\tilde{c}(q_1^{-r}, \tilde{x})$ (i.e. in particular $D \to D'$), which, if it is applicable, indicates the demand for v_1 at a normalized price q_1^{-r} of this factor. Since the superdifferential depends on the normalized factor price q_1^{-r}, point D' cannot directly be transferred to Quadrant III. In spite of that, point D'' with $q_2 = 1$ results in $q_1 = q_1^{-r}$, that is $c(q_1, q_2, \tilde{x}) = \tilde{c}(q_1^{-r}, \tilde{x})$ in view of the values of the two cost functions.

Thus, point B' faces point D''. As shown in Proposition III.12, both points *satisfy the equation*

$$c(q_1^{-r}, \lambda, \tilde{x}) = \lambda\tilde{c}(q_1^{-r}/\lambda, \tilde{x}) \quad \text{or} \quad c(q_1, q_2, \tilde{x}) = q_2\tilde{c}(q_1/q_2, \tilde{x}).$$

The determination of the adjoined λ, *holding* $c(q_1, q_2, \tilde{x}) = 1$, *corresponds to the depicted movement on the ray through the origin and the points* B' *and* D''.

Since the described method can be followed in the reverse order, we get the association $\overline{AB} \leftrightarrow B'$. The associations $A \leftrightarrow \overline{A'B'}$ and $B \leftrightarrow \overline{B'C'}$ are *derived analogously.*

For the sake of completeness Quadrant II describes the counterpart of Quadrant IV. The difference is that the inverse superdifferentials for $k = 1$ *(instead of* $r = 2$*)*

$$q_2 \in \Delta g_*(v_2^{-k}|L_\circ(\tilde{x})) \iff v_2^{-k} \in \Delta g(q_2|L_\circ(\tilde{x}))$$

are depicted regarding the polar set $L_\circ(\tilde{x})$. *The normalization of the first input to* $v_1 = 1$ *should be understood as an interim finding. Now each point in the line* $\overline{B'C'}$ *is again mapped into point* A *via the intermediate point* O' *with* $v_1 = 1$. *Similarly,* O'' *serves as intermediate step for the association* $B \leftrightarrow \overline{A'B'}$. *Each point of the segment* $\overline{AB}$ *can be determined in the same way, for instance, by* $B' \leftrightarrow D''' \leftrightarrow O \leftrightarrow D$. *Note that the cost function* $c(\cdot, \tilde{x})$ *is differentiable at each point in the curve* $A'B'C'$ *except at the points* A', B', *and* C'. *Again the differentiability of the cost function* $c(\cdot, \tilde{x})$ *assures that an integer exposed point of* $\operatorname{conv} L(\tilde{x})$ *– i.e.* A *or* B *– is attained.*

2.4.3 Results with Respect to the Output Correspondence

The concluding remarks apply to the production structure $\big(P(\mathbf{v})|\ \mathbf{v} \in V\big)$, which is equivalent to $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$. The main difference in the form of the analysis becomes apparent as follows: whereas an (aureoled) input requirement set $L(\mathbf{x})$ does not contain the origin $\mathbf{v} = \mathbf{0}$ for any commodity bundle $\mathbf{x} \in X \setminus \{\mathbf{0}\}$, [P1a] (*Possibility of Inaction*) guarantees the origin $\mathbf{x} = \mathbf{0}$ to be an element of the (star-shaped) production possibility set $P(\mathbf{v})$. Moreover, [P5] (*Boundedness*) and [P6] (*Closedness*) state that the sets $P(\mathbf{v})$ are compact for all input vectors $\mathbf{v} \in V$.[165] Under these conditions the representation of a production possibility set $P(\mathbf{v})$ by its support function $\sigma(\cdot|P(\mathbf{v}))$ or its gauge $\gamma(\cdot|P(\mathbf{v}))$ is much easier than the characterization of an input requirement set $L(\mathbf{x})$ by the reciprocal support function $\varphi(\cdot|L(\mathbf{x}))$ or the reciprocal gauge $\psi(\cdot|L(\mathbf{x}))$. Thus, a more detailed explanation of the duality relations as shown in Figure III.37 can be omitted at this point.[166] Figure III.37 is nothing more than an adequate transference of the issues in Figure D.3, p. 336.

The support function $\sigma(\cdot|P(\mathbf{v}))$ indicates the maximal revenue which can be obtained at a given input vector $\mathbf{v} \in V$ and commodity prices[167] $\mathbf{p} \in \overline{P_\mathbf{p}}$. Thus,

[165]If $P(\mathbf{v})$ is compact, then the convex hull $\operatorname{conv} P(\mathbf{v})$ is compact, too.

[166]However, Figure III.37 requires a convex set $P(\mathbf{v})$.

[167]Economically, a vector $\mathbf{p} \in \overline{P_\mathbf{p}}$ with $\overline{P_\mathbf{p}} := \mathbb{R}^n$ should be called a commodity bundle only if it is nonnegative.

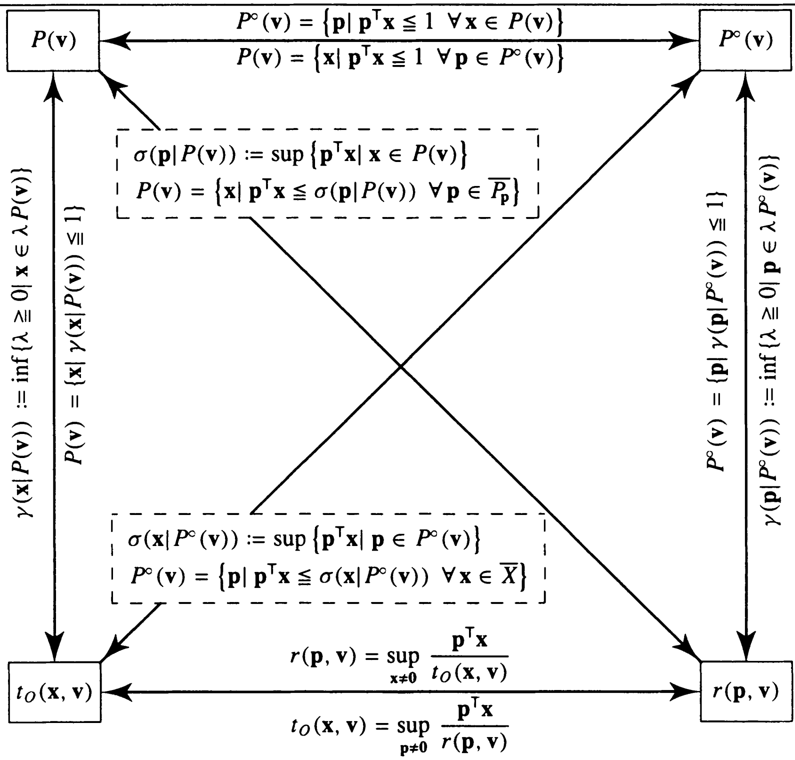

Figure III.37: Duality relations regarding revenues

$\sigma(\cdot \mid P(\mathbf{v}))$ is also called the **revenue function** $r(\cdot, \mathbf{v})$.

$$r(\mathbf{p}, \mathbf{v}) \equiv \sigma(\mathbf{p} \mid P(\mathbf{v})) = \sup\{\mathbf{p}^\mathsf{T}\mathbf{x} \mid \mathbf{x} \in P(\mathbf{v})\}$$

In the reverse conclusion the revenue function yields a system of inequalities which is suitable for the determination of the approximate production possibility set.

$$\operatorname{cl}\left(\operatorname{conv} P(\mathbf{v})\right) = \bigcap_{\mathbf{p} \in \overline{P_\mathbf{p}}} \{\mathbf{x} \mid \mathbf{p}^\mathsf{T}\mathbf{x} \leqq r(\mathbf{p}, \mathbf{v})\}$$

The gauge $\gamma(\cdot \mid P(\mathbf{v}))$ measures to what extent a commodity bundle $\mathbf{x} \in P(\mathbf{v})$ can be increased along a ray through the origin such that the resulting commodity bundle remains producible by the input vector $\mathbf{v} \in V$. The function $\gamma(\cdot \mid P(\mathbf{v}))$ is now called the **output distance function** $t_O(\cdot, \mathbf{v})$.

$$t_O(\mathbf{x}, \mathbf{v}) \equiv \gamma(\mathbf{x} \mid P(\mathbf{v})) := \inf\{\lambda \geqq 0 \mid \mathbf{x} \in \lambda P(\mathbf{v})\}$$

If the production possibility set is closed and convex, then the reverse conclusion yields

$$P(\mathbf{v}) = \{\mathbf{x} \mid t_O(\mathbf{x}, \mathbf{v}) \leqq 1\}.$$

The polar set $P^\circ(\mathbf{v})$ of $P(\mathbf{v})$ contains all the commodity price vectors $\mathbf{p}$ which assure that the revenues $\mathbf{p}^\mathsf{T}\mathbf{x}$ do not pass unity for any commodity bundle $\mathbf{x}$ producible by $\mathbf{v}$.

$$P^\circ(\mathbf{v}) = \left\{\mathbf{p} \mid \mathbf{p}^\mathsf{T}\mathbf{x} \leqq 1 \;\; \forall \mathbf{x} \in P(\mathbf{v})\right\}$$

As before, the family $\left(P^\circ(\mathbf{v}) \mid \mathbf{v} \in V\right)$ can be interpreted as a **revenue structure**. Since the set $P(\mathbf{v})$ contains the origin $\mathbf{x} = \mathbf{0}$, its convex hull is star-shaped, i.e. $\operatorname{conv} P(\mathbf{v}) = \operatorname{star}\left(\operatorname{conv} P(\mathbf{v})\right)$. With that the bipolar set is

$$P^{\circ\circ}(\mathbf{v}) = \operatorname{cl}\left(\operatorname{conv} P(\mathbf{v})\right) = \left\{\mathbf{x} \mid \mathbf{p}^\mathsf{T}\mathbf{x} \leqq 1 \;\; \forall \mathbf{p} \in P^\circ(\mathbf{v})\right\}.$$

Regarding the support function σ and the gauge γ of the bipolar set $P^{\circ\circ}$, it is[168]

$$\sigma(\cdot \mid P^{\circ\circ}(\mathbf{v})) = \gamma(\cdot \mid P^\circ(\mathbf{v})) \quad \text{and} \quad \gamma(\cdot \mid P^{\circ\circ}(\mathbf{v})) = \sigma(\cdot \mid P^\circ(\mathbf{v})).$$

If the production possibility set $P(\mathbf{v})$ is convex, then the revenue function $r(\cdot, \mathbf{v})$ can be interpreted as the gauge of the polar set $P^\circ(\mathbf{v})$. At the same time the output distance function $t_O(\cdot, \mathbf{v})$ may be viewed as the support function of the polar set.[169]

$$r(\cdot, \mathbf{v}) \equiv \sigma(\cdot \mid P(\mathbf{v})) = \gamma(\cdot \mid P^\circ(\mathbf{v}))$$
$$t_O(\cdot, \mathbf{v}) \equiv \gamma(\cdot \mid P(\mathbf{v})) = \sigma(\cdot \mid P^\circ(\mathbf{v}))$$

Therefore, $r(\cdot, \mathbf{v})$ and $t_O(\cdot, \mathbf{v})$ are polar gauges[170] fulfilling MAHLER's inequality (D.18), p. 335.

$$\mathbf{p}^\mathsf{T}\mathbf{x} \leqq r(\mathbf{p}, \mathbf{v})\, t_O(\mathbf{x}, \mathbf{v}) \qquad \forall \mathbf{p}, \;\; \forall \mathbf{x}$$

For a pair of polar points $(\hat{\mathbf{p}}, \hat{\mathbf{x}})$ this inequality is satisfied as an equation if and only if $\hat{\mathbf{x}}$ solves the problem of revenue maximization for a given price vector $\hat{\mathbf{p}}$ and if dually $\hat{\mathbf{p}}$ solves the problem $\sup\left\{\mathbf{p}^\mathsf{T}\hat{\mathbf{x}} \mid \mathbf{p} \in P^\circ(\mathbf{v})\right\}$. Under differentiability we obtain the supply of goods $\hat{\mathbf{x}} = \nabla r(\hat{\mathbf{p}}, \mathbf{v})$ or dually $\hat{\mathbf{p}} = \nabla t_O(\hat{\mathbf{x}}, \mathbf{v})$.

To sum up, four families of sets are available to describe different aspects of the representation of firms; see Figure III.38. First of all, Chapter II introduces the equivalent representation of a production technology by the production structures $\left(P(\mathbf{v}) \mid \mathbf{v} \in V\right)$ and $\left(L(\mathbf{x}) \mid \mathbf{x} \in X\right)$. While $P(\mathbf{v})$ indicates all commodity bundles $\mathbf{x}$ producible by inputs $\mathbf{v}$, the input requirement set $L(\mathbf{x})$ includes all input vectors $\mathbf{v}$ which are sufficient for the production of the outputs $\mathbf{x}$. The inverse relation between both production structures is reflected by the equivalence relation

$$\mathbf{x} \in P(\mathbf{v}) \iff \mathbf{v} \in L(\mathbf{x}).$$

[168] See Theorem D.15, p. 334.

[169] See Corollary D.15.1, p. 334.
FÄRE, PRIMONT (1994) present further duality relations. For example, we can define an indirect distance function

$$It_O(\mathbf{x}, \mathbf{q}/C) := \inf_{\mathbf{v}, \lambda} \left\{\lambda \mid \mathbf{x} \in \lambda P(\mathbf{v}),\; \mathbf{q}^\mathsf{T}\mathbf{v} \leqq C\right\},$$

where in contrast to the output distance function $t_O(\cdot, \mathbf{v})$ the factor prices $\mathbf{q}$ and the cost C but not the factor quantities $\mathbf{v}$ are given exogenously.

[170] See Theorem D.16, p. 335.

output oriented production structure | revenue structure

$$
\begin{tikzcd}
\big(P(\mathbf{v}) \mid \mathbf{v} \in V\big) \arrow[r, leftrightarrow, "\text{polar}"] \arrow[d, leftrightarrow, "\text{inverse}"] & \big(P^{\circ}(\mathbf{v}) \mid \mathbf{v} \in V\big) \\
\big(L(\mathbf{x}) \mid \mathbf{x} \in X\big) \arrow[r, leftrightarrow, "\text{reciprocally polar}"] & \big(L_{\circ}(\mathbf{x}) \mid \mathbf{x} \in X\big)
\end{tikzcd}
$$

input oriented production structure | cost structure

Figure III.38: Theory of the firm

In Chapter III we concentrate on the comparison of the production structure $\big(L(\mathbf{x}) \mid \mathbf{x} \in X\big)$ and the cost structure $\big(L_{\circ}(\mathbf{x}) \mid \mathbf{x} \in X\big)$. Each input requirement set $L(\mathbf{x})$ is associated with a polar set $L_{\circ}(\mathbf{x})$including all of the factor price vectors $\mathbf{q}$ which at least imply costs $\mathbf{q}^{\mathrm{T}}\mathbf{v} \geqq 1$ for the given commodity bundle $\mathbf{x}$. If we apply the same operation as in $L(\mathbf{x}) \to L_{\circ}(\mathbf{x})$ to the polar set, $L_{\circ}(\mathbf{x}) \to L_{\circ\circ}(\mathbf{x})$, then we get the bipolar set $L_{\circ\circ}(\mathbf{x})$. Under certain conditions this set equals the original input requirement set, $L(\mathbf{x}) = L_{\circ\circ}(\mathbf{x})$. But for indivisible inputs it merely yields the approximation $L_{\circ\circ}(\mathbf{x}) = \mathrm{cl}\big(\mathrm{conv} L(\mathbf{x})\big)$.

Afterwards we went briefly into the analogous comparison of the production structure $\big(P(\mathbf{v}) \mid \mathbf{v} \in V\big)$ and the revenue structure $\big(P^{\circ}(\mathbf{v}) \mid \mathbf{v} \in V\big)$. Now the set $P(\mathbf{v})$ faces a polar production possibility set $P^{\circ}(\mathbf{v})$. As before, the operation $P(\mathbf{v}) \to P^{\circ}(\mathbf{v})$ can also be applied to the polar set, $P^{\circ}(\mathbf{v}) \to P^{\circ\circ}(\mathbf{v})$. Again, in the case of indivisible outputs the bipolar set $P^{\circ\circ}(\mathbf{v})$ serves as an approximation of the genuine set $P^{\circ\circ}(\mathbf{v}) = \mathrm{cl}(\mathrm{conv} P(\mathbf{v}))$.

3 Optimal Activities in the Theory of the Household

3.1 Demand for Commodities

3.1.1 The Expenditure Structure of a Household

Having introduced the dual representation of the production structure $\big(L(\mathbf{x}) \mid \mathbf{x} \in X\big)$ by a cost structure $\big(L_\circ(\mathbf{x}) \mid \mathbf{x} \in X\big)$ into the theory of the firm, we have now to consider an appropriate representation of a household.

A comparison of the household's preference structure $\big(\mathcal{P}(\mathbf{x}) \mid \mathbf{x} \in X\big)$ to the firm's production structure $\big(L(\mathbf{x}) \mid \mathbf{x} \in X\big)$ suggests that the technical results of the cost theory can be transferred adequately to the theory of households. Each result, shown in Figure III.29, now faces a counterpart in expenditure theory, where $\big(\mathcal{P}_\circ(\mathbf{x}) \mid \mathbf{x} \in X\big)$ represents the expenditure structure of the household at hand.[171]

This supposition is confirmed when it is possible to derive the following statements from Axioms [$\mathcal{P}$1]–[$\mathcal{P}$6] which correspond to the Axioms [L1]–[L4] as well as [L6] and [L7] on an input correspondence L on p. 19.

1. No commodity bundle is worse than $\mathbf{x} = \mathbf{0}$, i.e. $\mathcal{P}(\mathbf{0}) = X$. Each commodity bundle $\mathbf{x} \geq \mathbf{0}$ is preferred to $\mathbf{x} = \mathbf{0}$.

$$\forall \mathbf{x} \in X \setminus \{\mathbf{0}\} : \quad \mathbf{0} \notin \mathcal{P}(\mathbf{x})$$

2. Each preference set $\mathcal{P}(\mathbf{x})$ contains a commodity bundle, namely $\mathbf{x}$.

$$\forall \mathbf{x} \in X : \quad \mathcal{P}(\mathbf{x}) \neq \emptyset$$

3. If a commodity bundle $\mathbf{x}'$ yields at least the utility level of $\mathbf{x}$, then this relation is true for each commodity bundle $\mathbf{x}''$ which is at least as large as $\mathbf{x}'$.

$$\forall \mathbf{x}', \mathbf{x}'' \in X,\ \mathbf{x}' \leqq \mathbf{x}'' : \quad \mathbf{x}' \in \mathcal{P}(\mathbf{x}) \implies \mathbf{x}'' \in \mathcal{P}(\mathbf{x})$$

4. If a commodity bundle $\mathbf{x}$ yields at least the utility level of $\mathbf{x}'$, then $\mathbf{x}$ is at least as good as each commodity bundle $\mathbf{x}''$, which is not greater than $\mathbf{x}'$.

$$\forall \mathbf{x}', \mathbf{x}'' \in X,\ \mathbf{x}' \geqq \mathbf{x}'' : \quad \mathbf{x} \in \mathcal{P}(\mathbf{x}') \implies \mathbf{x} \in \mathcal{P}(\mathbf{x}'')$$

5. Axiom [L5] has no direct counterpart in the theory of the household. This apparent deficit will be discussed later in more detail.

6. The preference sets $\mathcal{P}(\mathbf{x})$ are closed for every commodity bundle $\mathbf{x} \in X$

7. The preference sets $\mathcal{P}(\mathbf{x})$ are convex for every commodity bundle $\mathbf{x} \in X$.

Since the subsequent analysis does not direct the attention to the derivation of the household's expenditure structure according to Figure III.29, we can dispense with the proof of Propositions 1–7.

[171] Analogous to polar input requirement sets $L_\circ(\mathbf{x})$, the polar preference sets $\mathcal{P}_\circ(\mathbf{x})$ are defined by $\mathcal{P}_\circ(\mathbf{x}) := \big\{\mathbf{p} \mid \mathbf{p}^\mathsf{T}\tilde{\mathbf{x}} \geqq 1 \ \forall \tilde{\mathbf{x}} \in \mathcal{P}(\mathbf{x})\big\}$.

Merely point 5, needs mentioning. Note first of all that Axiom [L5]

$$\|\mathbf{x}^\nu\| \to \infty : \bigcap_{\nu=1}^{\infty} L(\mathbf{x}^\nu) = \emptyset$$

has no influence on the analysis of the firm's cost structure. Thus, the analogous assumption

$$\text{(III.103)} \quad \|\mathbf{x}^\nu\| \to \infty : \bigcap_{\nu=1}^{\infty} \mathcal{P}(\mathbf{x}^\nu) = \emptyset ,$$

would have no consequence for the household's expenditure structure, even if it is fulfilled.

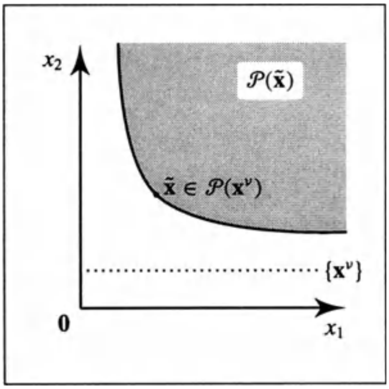

Figure III.39: Counterexample for Assumption (III.103)

To give a reason why (III.103) does not need to be satisfied, we define the inverse correspondence of $\mathcal{P}$ with the level sets $\{\mathbf{x} \in X \mid \mathbf{x}' \succcurlyeq \mathbf{x}\}$, i.e.

$$\mathbf{x}' \in \{\mathbf{x} \in X \mid \mathbf{x} \succcurlyeq \mathbf{x}''\} = \mathcal{P}(\mathbf{x}'') \iff \mathbf{x}'' \in \{\mathbf{x} \in X \mid \mathbf{x}' \succcurlyeq \mathbf{x}\} .$$

Looking back at Proposition II.2, p. 20[172], compact level sets $\{\mathbf{x} \in X \mid \mathbf{x}' \succcurlyeq \mathbf{x}\}$ result if (III.103) is valid. As it is not difficult to define preference orderings contradicting this implication, (III.103) does not need to be satisfied, too; see Figure III.39. The relation $\tilde{\mathbf{x}} \succcurlyeq \mathbf{x}^\nu$ holds for all ν such that the set $\{\mathbf{x} \in X \mid \tilde{\mathbf{x}} \succcurlyeq \mathbf{x}\}$ is bounded.

Although (III.103) is of no importance for the theory of the household, similar statements will be important. For instance, Proposition III.22 proves for a *compact* subset C in the commodity space X that $\bigcap_{\mathbf{x} \in C} [C \cap \mathcal{P}(\mathbf{x})] \neq \emptyset$, i.e. each such set C contains a commodity bundle demanded by the household. Whereas a compact set C considers only sequences of commodity bundles with $\|\mathbf{x}^\nu\| \leqq c < \infty$, Propositions III.28 and III.29 deal with a closed but *unbounded* set C. Proposition III.28 yields a relation of the form $\bigcap_{\mathbf{x} \in C} [C \cap \mathcal{P}(\mathbf{x})] = \emptyset$, i.e. the household does not demand any commodity bundle in C. Proposition III.29 notes an analogous implication of the form $\|\mathbf{x}^\nu\| \to \infty$, which also results from the transition from a bounded to an unbounded set C. That means that the quantity of goods demanded becomes infinitely large if the household is allowed to choose a commodity bundle in a bounded set C.

3.1.2 The Individual Demand for Goods

In the previously introduced theory of the firm we dispense with the question as to how to aggregate the factor demand or the supply of goods of individual firms to a market demand and a market supply respectively. In particular, how the firms' sector reacts regarding price changes on single markets was ignored.

[172] See there the equivalence relation [P5] $\iff$ [L5].

However, the properties of aggregate commodity demand or aggregate excess demand of the households' sector will be of crucial importance with respect to the discussion of competitive equilibria. In the preparatory analysis we establish the properties of the individual commodity demand, where the explicit analysis of the household's expenditure structure analogous to the firm's cost structure recedes into the background.

Having presented the main properties of the individual and the aggregate demand for goods of the households in the subsequent sections, Chapter IV deals with the answer to the central question as to

> what conditions suffice for the existence of a commodity price vector[173] $\mathbf{p} \in P_{\mathbf{p}}$ such that the aggregate excess demand vanishes on all of the n commodity markets.

The answer is founded basically on fixed-point theorems and refers in particular to the fixed-point theorems of BROUWER and of KAKUTANI.[174] With that the further procedure is characterized by the question as to what assumptions on the preferences of individual households have to be made such that the aggregate demand satisfies the requirement of the respective fixed-point theorem. But before going into the individual commodity demand, we have to establish the framework of the analysis.

The examined economy embraces a set A of persons, where each person $a \in A$ possesses his particular preference ordering $\succcurlyeq_a$. For the sake of simplicity it is assumed that the preference orderings of all persons are defined on the same commodity space

$$X = \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}.$$

Suppose again that only n_d out of the n goods are divisible. Moreover, the examined economy has a total commodity endowment $\mathbf{w}_A \in X$, where the analysis only considers those goods occurring as a positive quantity, $\mathbf{w}_A > \mathbf{0}$. The total endowment $\mathbf{w}_A$ is composed of the individual endowments $\mathbf{w}_a$ of all households.

$$\mathbf{w}_A := \sum_{a \in A} \mathbf{w}_a$$

To avoid an unnecessary restriction to the analysis, the endowments $\mathbf{w}_a \in \mathbb{R}_+^n$ need not necessarily be integer. This associates the notion that a person may by all means own fractions of an indivisible good. Conversely, it is assumed that each unit of an indivisible good can only be consumed by an individual person. If different persons own fractions of an indivisible commodity unit, then no household can benefit from these fractions. While the case $\mathbf{w}_a \notin X$ is admissible, each

[173] The commodity price space has already been defined by $P_{\mathbf{p}} = \mathbb{R}_+^m$.

[174] Appendix C.3 contains apart from the fixed-point theorems of BROUWER (Theorem C.15, p. 312) and of KAKUTANI (Theorem C.16, p. 313) a fixed-point theorem of DEBREU, GALE, NIKAIDO (Theorem C.17, p. 314). As gathered from UZAWA (1962) and HEUSER (1992), all of the theorems are equivalent.

household will choose only consumable (integer) commodity bundles $\mathbf{x} \in X$. [175] As in the theory of the firm before, the distinction between a point $\mathbf{x} \in \mathbb{R}^n_+$ and a commodity bundle $\mathbf{x} \in X$ will accompany the whole analysis.

If the persons are allowed to use their endowments $\mathbf{w}_a$ for the exchange of goods owned by other persons, then the result of the exchange processes can be described by an allocation. First of all, an **allocation** $\mathfrak{x}$ with

$$\mathfrak{x} := (\mathbf{x}_a)_{a \in A} \in \bigtimes_{a \in A} \mathbb{R}^n_+$$

assigns each person a to a *point* $\mathbf{x}_a \in \mathbb{R}^n_+$. With that we agree on the following notation for an alternative allocation $\tilde{\mathfrak{x}}$:

$$\tilde{\mathfrak{x}} = (\tilde{\mathbf{x}}_a)_{a \in A} \quad \text{with} \quad \tilde{\mathbf{x}}_a = (\tilde{x}_{1a}, \ldots, \tilde{x}_{na})^{\mathsf{T}} \quad \text{for all} \quad a \in A\,.$$

Given everybody's individual endowments $\mathbf{w}_a$, an allocation $\mathfrak{x}$ is called **feasible** if $\sum_{a \in A} \mathbf{x}_a \leqq \mathbf{w}_A$. Note that one feasible allocation is already known from the endowments of the households $(\mathbf{w}_a)_{a \in A}$. At the same time it becomes evident that even feasible allocations are not subject to the integer constraints $\mathbf{x}_a \in X$. This requirement will be integrated in the later defined WALRASian allocation.

Keeping these preliminary remarks in mind, the individual commodity demand D_a of a household a results from the subsequent calculus. Each person chooses with respect to his preferences the best of all commodity bundles he can buy. The person concerned is capable of buying each commodity bundle $\mathbf{x} \in X$ which does not exceed the value of the endowment $\mathbf{p}^{\mathsf{T}}\mathbf{w}_a$ for a given commodity price vector $\mathbf{p} \in P_{\mathbf{p}}$. To debar nobody from trading, it is supposed that each person owns a positive quantity of some good, $\mathbf{w}_a \geq \mathbf{0}$.

Before the budget set $B(\mathbf{p}, \mathbf{w}_a)$ is defined, it is helpful for further analysis to collect all points $\mathbf{x} \in \mathbb{R}^n_+$ which do not exceed the value of the endowment $\mathbf{w}_a$ at prices $\mathbf{p}$ in the set

$$G(\mathbf{p}, \mathbf{w}_a) := \left\{\mathbf{x} \in \mathbb{R}^n_+ \,|\, \mathbf{p}^{\mathsf{T}}\mathbf{x} \leqq \mathbf{p}^{\mathsf{T}}\mathbf{w}_a\right\}.$$

As long as we assume goods to be divisible, each vector $\mathbf{x} \in G(\mathbf{p}, \mathbf{w}_a)$ can be interpreted as an admissible commodity bundle $\mathbf{x} \in X$. For indivisible goods the set $G(\mathbf{p}, \mathbf{w}_a)$ even includes such points which do not belong to the commodity space X. See Figure III.40 showing the case of a divisible good 1 and an indivisible good 2.

The **budget set**

$$B(\mathbf{p}, \mathbf{w}_a) := G(\mathbf{p}, \mathbf{w}_a) \cap X$$

gets rid of this deficit. It contains all *commodity bundles* person a can buy with the initial endowment $\mathbf{w}_a$ at given prices $\mathbf{p}$. In Figure III.40 these commodity bundles are marked by bold sections.

[175] Because aspects of time are not taken into account, the household cannot use the acquired fractions of an indivisible good for future exchanges.

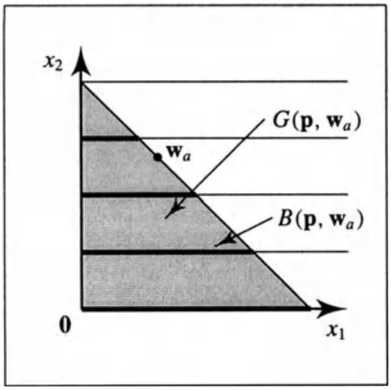

Figure III.40: The budget constraint of a household

For the commodity demand of person a with the preference ordering $\succcurlyeq_a$ merely those commodity bundles $\mathbf{x} \in X$ are considered whose utility is at least as large as that of each alternative commodity bundle $\tilde{\mathbf{x}} \in X$ the person can buy.[176] Hence, the set of commodity bundles $D_a(\mathbf{p}, \mathbf{w}_a)$, demanded by person a at prices $\mathbf{p}$, is

$$D_a(\mathbf{p}, \mathbf{w}_a) := \{\mathbf{x} \mid \mathbf{x} \in B(\mathbf{p}, \mathbf{w}_a),\ \mathbf{x} \succcurlyeq_a \tilde{\mathbf{x}}\ \forall \tilde{\mathbf{x}} \in B(\mathbf{p}, \mathbf{w}_a)\},$$

where the subscript a indicates that each person may possess different preferences. As will be shown, each demand set is not empty for positive price vectors $\mathbf{p} > \mathbf{0}$. Consequently, $D_a(\cdot, \mathbf{w}_a): \mathbb{R}^n_{++} \to \mathfrak{P}(X)$ may be called the **individual demand correspondence**[177] of person a having the preference ordering $\succcurlyeq_a$ and the initial endowment $\mathbf{w}_a$. This correspondence immediately results in

Proposition III.21 *The individual demand correspondence $D_a(\cdot, \mathbf{w}_a)$ is homogeneous of degree 0 in prices* $\mathbf{p}$.

Proof: If all prices are multiplied by the same factor $\lambda > 0$, then the budget set does not change.

$$B(\mathbf{p}, \mathbf{w}_a) = B(\lambda\mathbf{p}, \mathbf{w}_a) \qquad \forall \lambda > 0$$

Since the preference ordering does not change for varying λ – this fact is called the absence of money illusion – the examined person demands the same commodity bundles for each price vector $\lambda\mathbf{p}$ with $\lambda > 0$.

$$D_a(\mathbf{p}, \mathbf{w}_a) = D_a(\lambda\mathbf{p}, \mathbf{w}_a) \qquad \forall \lambda > 0 \quad \blacksquare$$

In the sense of a dimensionless scaling, the homogeneity of degree 0 of the demand correspondence $D_a(\cdot, \mathbf{w}_a)$ permits the division of each price $\tilde{p}_j$ $(j = 1,\ldots, n)$ of a price vector $\tilde{\mathbf{p}}$ (see Figure III.41) by the sum of the single prices without changing the demand set $D_a(\tilde{\mathbf{p}}, \mathbf{w}_a)$.

In this way the set of scrutinized price vectors $P_{\mathbf{p}} = \mathbb{R}^n_+$ can be restricted to the **price simplex**

$$\Delta := \left\{\mathbf{p} \in \mathbb{R}^n_+ \mid \sum_{j=1}^{n} p_j = 1\right\}$$

[176] Commodity bundles with this property have been defined as maximal elements.

[177] A correspondence $\Gamma: X \to \mathfrak{P}(Y)$ assigns each element of the set X to a nonempty subset in Y.

as it is depicted in Figure III.41 for the case of three goods. Since the analysis has to distinguish positive price vectors from price vectors containing at least one zero component, for the sake of brevity we use

$\operatorname{rint}\Delta = \Delta \cap \mathbb{R}^n_{++}$ (relative) interior of Δ

$\partial\Delta = \Delta \setminus \mathbb{R}^n_{++}$ (relative) boundary of Δ.

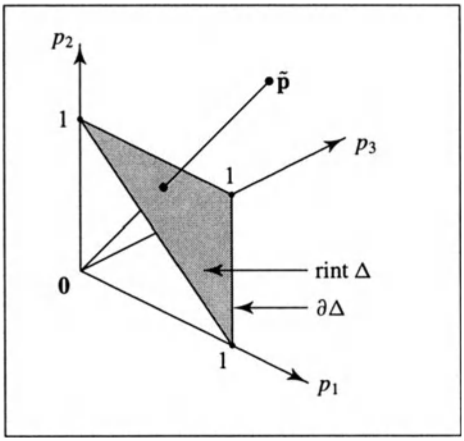

Figure III.41: Price simplex Δ

3.1.3 The Aggregate Excess Demand

The homogeneity of degree 0 of the individual demand correspondences passes directly on to the **aggregate demand correspondence**[178] $\widehat{D}\colon \operatorname{rint}\Delta \to \mathfrak{P}(X)$ with

$$\widehat{D}(\mathbf{p}) := \sum_{a \in A} D_a(\mathbf{p}, \mathbf{w}_a)\,.$$

Since the **aggregate excess demand correspondence** $Z\colon \operatorname{rint}\Delta \to \mathfrak{P}(\mathbb{R}^n)$ with

$$Z(\mathbf{p}) := \widehat{D}(\mathbf{p}) - \mathbf{w}_A$$

is only expanded by the constant total endowment $\mathbf{w}_A$, Z must also be homogeneous of degree 0 in prices.

At this point two aspects must be stressed, which refer to the emergence of the aggregate excess demand. On the one hand each person individually decides which commodity bundle he chooses. In this way all exchange processes are based on **decentralized** decisions. On the other hand the households maximize their utility without knowing the demand of other persons or contemplating their wishes. Thus, it is supposed that the individuals **do not cooperate**. The coordination of the exchange plans results from market prices. By valuating their initial endowments at market prices, the agents only react to changes of commodity prices instead of affecting each other by a direct exchange of goods.[179]

The coordination of particular exchange plans is successful when the commodity prices $\mathbf{p}^\circ$ provided by the market cause individual exchange plans, which themselves lead to a feasible allocation of goods $\mathfrak{x}^\circ$. We speak of a **WALRASian equilibrium** or

[178] For the addition of sets, see Appendix B.

[179] Cf. ALIPRANTIS, BROWN, BURKINSHAW (1989, p. 52).

a **competitive equilibrium** $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ **with free disposability of goods** (free disposal equilibrium) when first each person $a \in A$ is associated with an (integer) commodity bundle $\mathbf{x}_a^\circ \in X$ which he demands at commodity prices $\mathbf{p}^\circ \in \Delta$,

$$\mathbf{x}_a^\circ \in D_a(\mathbf{p}^\circ, \mathbf{w}_a) \text{ for each person } a \in A,$$

and second if the resulting **WALRASian allocation** $\mathfrak{x}^\circ \in \times_{a \in A} X$ is feasible,

$$\mathbf{z}^\circ = \sum_{a \in A} \mathbf{x}_a^\circ - \mathbf{w}_A \leqq \mathbf{0} \quad \text{with} \quad \mathbf{p}^\circ \in \Delta \quad \text{and} \quad \mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ = 0.$$

Accordingly, a WALRASian equilibrium requires that there is no excess demand for any good, $\mathbf{z}^\circ \in Z(\mathbf{p}^\circ)$ with $\mathbf{z}^\circ \leqq \mathbf{0}$.

The question of the existence of an exchange equilibrium can now be stated more precisely as follows:

> In the examined economy $\mathcal{E}$ each person $a \in A$ is characterized by a continuous preference ordering $\succcurlyeq_a \in \Pi$ and an initial endowment $\mathbf{w}_a \in \mathbb{R}_+^n$. What conditions must be fulfilled by this **exchange economy** – interpreted as mapping $\mathcal{E} : A \to \Pi \times \mathbb{R}_+^n$ – so that there is a vector of commodity prices $\mathbf{p}^\circ$ at which an equilibrium prevails. In doing so, the described WALRASian equilibrium in its strict sense requires on the one hand that each person may carry out his exchange plans and on the other hand that the aggregate excess demand must vanish on all commodity markets,[180] $\mathbf{0} \in Z(\mathbf{p}^\circ)$.

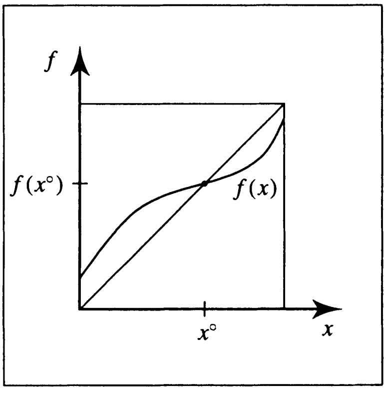

Figure III.42: Fixed-point

Some answers to this question are discussed in detail by Propositions IV.6 to IV.13. The proof of each proposition refers to Corollary C.15.1 and Theorem C.17, which themselves result from the already mentioned fixed-point theorems.

The easiest case of a fixed-point of the function f is shown in Figure III.42. The continuous function $f: C \to C$ must intersect the diagonal at at least one point. At this **fixed-point** we have $x^\circ = f(x^\circ)$. Thus, it immediately becomes clear that the existence of a fixed-point depends on whether the function f is continuous or jumps. With respect to the existence of a fixed-point of the form $\mathbf{0} \in Z(\mathbf{p}^\circ)$ the assumptions, which will be introduced later, especially intend to avoid unwanted jumps of the demand correspondence $D_a(\cdot, \mathbf{w}_a)$ and the aggregate excess demand Z. The concept of upper semi-continuity of the correspondence Z will play the role of the continuity of the function f.

In view of the premises of Theorem C.17 (DEBREU, GALE, NIKAIDO), p. 314 the structure of the further analysis up to the discussion of exchange equilibria can

[180]With that the value of the excess demand at once results in $\mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ = \mathbf{p}^{\circ\mathsf{T}}\mathbf{0} = 0$.

be justified in the following way. The theorem requires a closed convex subset S in the price simplex Δ. Ideally the set S would also include all of the price vectors $\mathbf{p}$, i.e. $S = \Delta$. Moreover, Theorem C.17 presumes a correspondence $\Psi\colon S \to \mathfrak{P}(\mathbb{R}^n)$ which to all intents and purposes is identified by the aggregate excess demand correspondence Z. The answer to the question as to whether a price vector with $\mathbf{0} \in Z(\mathbf{p})$ exists, stipulates four properties of the correspondence Ψ.[181]

1. The sets $\Psi(\mathbf{p})$ are nonempty and convex for all $\mathbf{p} \in S$;
2. The correspondence Ψ is closed;
3. The correspondence Ψ is bounded. Thus, there is a ball $K \subset \mathbb{R}^n$ such that $\Psi(\mathbf{p}) \subset K$ for each $\mathbf{p} \in S$;
4. $\mathbf{p}^{\mathsf{T}}\mathbf{z} \leqq 0$ holds good for each $\mathbf{p} \in S$ and each $\mathbf{z} \in \Psi(\mathbf{p})$.

Actually, the greatest trouble in transferring these properties to the correspondence Z result from point 1. As will be shown, $Z(\mathbf{p}) = \emptyset$ holds for each price vector $\mathbf{p}$ lying in the boundary of the price simplex Δ and therefore containing at least one zero component. Moreover, the sets $Z(\mathbf{p})$ are not necessarily convex when the analysis concerns indivisible goods.

First of all, the problem of nonconvex sets $Z(\mathbf{p})$ is not taken into account, whereas the examination is subdivided into positive price vectors $\mathbf{p} \in \operatorname{rint}\Delta$ and price vectors containing at least one zero component, $\mathbf{p} \in \partial\Delta$. Remember that the set $\operatorname{rint}\Delta$ is convex but not closed as required by Theorem C.17 for the set S. Further, remember that merely continuous preference orderings are assumed for positive price vectors $\mathbf{p} \in \operatorname{rint}\Delta$; see Section 3.2.1. The treatment of price vectors having zero components, $\mathbf{p} \in \partial\Delta$, however, stipulates the additional assumption of monotone preference orderings; see Sections 3.2.2.

Regarding the four premises the subsequently derived results can be classified as follows:

on 1. Proposition III.23: the sets $Z(\mathbf{p})$ are not empty for all $\mathbf{p} \in \operatorname{rint}\Delta$.
Proposition III.28: the sets $Z(\mathbf{p})$ are empty for all $\mathbf{p} \in \partial\Delta$.

on 2. Proposition III.27: the correspondence Z is upper semi-continuous and, therefore, closed in $\operatorname{rint}\Delta$ provided there is at least one divisible good.
The preliminary remarks are subdivided into two cases. While Corollary III.25.1 assumes exclusively divisible goods, Corollary III.25.3 assumes at least one perfectly divisible good.

on 3. Corollary III.29.1: if a sequence $\{\mathbf{p}^\nu\}$ of positive price vectors tends to a price vector $\tilde{\mathbf{p}} \in \partial\Delta$ containing a zero component, then the condition $\|\mathbf{z}^\nu\| \to \infty$ is valid for each sequence $\{\mathbf{z}^\nu\}$ with $\mathbf{z}^\nu \in Z(\mathbf{p}^\nu)$. Thus, the correspondence Z cannot be bounded.

[181] If point 3 is satisfied, then the closedness of the correspondence Ψ (point 2) is equivalent to the upper semi-continuity of this correspondence by Theorem C.8, p. 309. Moreover, Theorem C.10, p. 309, affords a criterion to prove the upper semi-continuity of the correspondence Ψ.

on 4. Since each person balances his budget constraint $\mathbf{p}^{\mathsf{T}}\mathbf{x}_a \leqq \mathbf{p}^{\mathsf{T}}\mathbf{w}_a$, the total economy satisfies

$$\text{(III.104)} \qquad \mathbf{p}^{\mathsf{T}}\mathbf{z} \leqq 0 \quad \text{for all} \quad \mathbf{p} \in \operatorname{rint}\Delta \quad \text{and for all} \quad \mathbf{z} \in Z(\mathbf{p}).$$

Obviously, the mentioned properties of the correspondence Z are not enough to apply Theorem C.17 (DEBREU, GALE, NIKAIDO), p. 314. Nevertheless, the main implications are established so that Chapter IV can immediately go into the existence of equilibrium commodity prices.[182]

3.2 Special Preference Orderings

3.2.1 Implications of the Continuity of Preference Orderings

(a) Perfectly Divisible Goods Regarding the mathematical proof of the existence of a competitive equilibrium, we require technical properties of the individual demand correspondence, which are also of economic importance.
Without going into the technically necessary properties, it is shown that each person demands some commodity bundle at positive commodity prices. Moreover, an arbitrary amount cannot be chosen at positive prices. If positive prices change marginally, then the behavior of agents does not make any arbitrary jumps. "Small" price variations usually imply "small" reactions of a person. If some goods have a zero price so that the use of these goods is not subject to any restrictions, then the agents' behavior changes suddenly.

Proposition III.22[183] *If $\succcurlyeq_a$ is a continuous preference ordering in the commodity space X, then each nonempty compact subset C in X has at least one best element. Moreover, the set of best elements $D_B(C, \succcurlyeq_a)$ is compact.*

Proof: With the aid of the preference sets $\mathcal{P}_a(\mathbf{x})$ for each $\mathbf{x} \in C$ the set of best elements $D_B(C, \succcurlyeq_a)$ is given by

$$D_B(C, \succcurlyeq_a) = \bigcap_{\mathbf{x} \in C} \left[C \cap \mathcal{P}_a(\mathbf{x}) \right].$$

On the basis of the assumption of a continuous preference ordering the preference sets are closed. Thus, the hypothesis of a compact set C implies a bounded and closed – i.e. compact – set of best elements, where $C \cap \mathcal{P}_a(\mathbf{x}) \subset C$.
Finally, we have to show that $D_B(C, \succcurlyeq_a)$ consists of at least one element. For this purpose we inspect an arbitrary but finite set of points $\{\mathbf{x}^1, \ldots, \mathbf{x}^\mu\}$ in C. The reflexivity of the preference ordering yields $\mathbf{x}^\nu \in \mathcal{P}_a(\mathbf{x}^\nu)$. The transitivity of the

[182]Readers who are not interested in the technical details of Sections 3.2.1 and 3.2.2 are recommended to skip these two sections. However, Figures III.44, III.47, and III.48 should be taken into account to get a feeling for the treatment of indivisible goods.

[183]Cf. HILDENBRAND, KIRMAN (1988, p. 62, Proposition 2.1).

preference ordering assures that each finite set of points in C has a best element; see Proposition II.1, p. 9. If $\mathbf{x}^\mu$ is a best element of the set $\{\mathbf{x}^1, \ldots, \mathbf{x}^\mu\}$, then

$$\mathbf{x}^\mu \in \bigcap_{\nu=1}^{\mu} \left[C \cap \mathcal{P}_a(\mathbf{x}^\nu)\right] \neq \emptyset.$$

Considering the results in Appendix A.4, it ensues $D_B(C, \succcurlyeq_a) \neq \emptyset$ from the "finite intersection property". ■

Proposition III.23 *The demand set $D_a(\mathbf{p}, \mathbf{w}_a)$ is not empty and compact for positive prices $\mathbf{p} > \mathbf{0}$ provided the preference ordering $\succcurlyeq_a$ is continuous.*

Proof: Even if a person holds no initial endowment, $\mathbf{w}_a = \mathbf{0}$, the budget set is not empty, $\mathbf{0} \in B(\mathbf{p}, \mathbf{w}_a)$. Furthermore, the budget set $B(\mathbf{p}, \mathbf{w}_a) = G(\mathbf{p}, \mathbf{w}_a) \cap X$ is closed and bounded – i.e. compact – for positive prices $\mathbf{p} > \mathbf{0}$.
In view of the next steps it is useful to rewrite the demand set as follows:

$$\begin{aligned} D_a(\mathbf{p}, \mathbf{w}_a) &= \{\mathbf{x} \mid \mathbf{x} \in B(\mathbf{p}, \mathbf{w}_a),\ \mathbf{x} \succcurlyeq_a \tilde{\mathbf{x}}\ \forall \tilde{\mathbf{x}} \in B(\mathbf{p}, \mathbf{w}_a)\} \\ &= \bigcap_{\tilde{\mathbf{x}} \in B(\mathbf{p}, \mathbf{w}_a)} \{\mathbf{x} \mid \mathbf{x} \in B(\mathbf{p}, \mathbf{w}_a),\ \mathbf{x} \succcurlyeq_a \tilde{\mathbf{x}}\} \\ &= \bigcap_{\tilde{\mathbf{x}} \in B(\mathbf{p}, \mathbf{w}_a)} \left[B(\mathbf{p}, \mathbf{w}_a) \cap \mathcal{P}_a(\tilde{\mathbf{x}})\right]. \end{aligned}$$

Thus, the demand set $D_a(\mathbf{p}, \mathbf{w}_a)$ equals the set of best elements $D_B\big(B(\mathbf{p}, \mathbf{w}_a), \succcurlyeq_a\big)$ with respect to the budget set $B(\mathbf{p}, \mathbf{w}_a)$.

- Proof of the compactness of $D_a(\mathbf{p}, \mathbf{w}_a)$ for $\mathbf{p} > \mathbf{0}$:

Because of the assumption of a continuous preference ordering the set

$$\mathcal{P}_a^s(\tilde{\mathbf{x}}) := B(\mathbf{p}, \mathbf{w}_a) \cap \mathcal{P}_a(\tilde{\mathbf{x}}) = X \cap G(\mathbf{p}, \mathbf{w}_a) \cap \mathcal{P}(\tilde{\mathbf{x}})$$

is closed. Moreover, $\mathcal{P}_a^s(\tilde{\mathbf{x}})$ is a bounded subset in the compact set $G(\mathbf{p}, \mathbf{w}_a)$. Thus, $D_a(\mathbf{p}, \mathbf{w}_a)$ is as well compact since $D_a(\mathbf{p}, \mathbf{w}_a)$ is the intersection of closed sets, all of which are contained in $G(\mathbf{p}, \mathbf{w}_a)$.

- Proof that $D_a(\mathbf{p}, \mathbf{w}_a)$ is not empty for $\mathbf{p} > \mathbf{0}$:

Since we seek for the best elements in a nonempty compact budget set $B(\mathbf{p}, \mathbf{w}_a)$ at positive prices $\mathbf{p} > \mathbf{0}$, $D_a(\mathbf{p}, \mathbf{w}_a)$ cannot be empty by Proposition III.22. ■

The individual demand *correspondence* $D_a(\cdot, \mathbf{w}_a)$: $\operatorname{rint} \Delta \to \mathfrak{P}(X)$ maps, therefore, each positive price vector $\mathbf{p} \in \operatorname{rint} \Delta$ into a nonempty (and compact) demand set, $D_a(\mathbf{p}, \mathbf{w}_a) \subset X$.

Because of the assumption of convex preferences the next proposition refers to economies in which all goods are divisible, $X = \mathbb{R}^n_+$. Continuity of the preference ordering is not required at this point.

Proposition III.24 *Given a convex preference ordering $\succcurlyeq_a$, all of the demand sets $D_a(\mathbf{p}, \mathbf{w}_a)$ are convex. Moreover, the aggregate demand correspondence $\widehat{D}$ and the aggregate excess demand correspondence Z are convex-valued, too.*

Proof: Under the assumption of convex preferences the set $D_a(\mathbf{p}, \mathbf{w}_a)$ results from the intersection of a family of convex sets and is, therefore, convex itself.
The correspondence $\widehat{D}$ is called convex-valued if $\widehat{D}(\mathbf{p})$ is convex for all $\mathbf{p}$. As the sum of sets is convex,[184] the convexity of $D_a(\mathbf{p}, \mathbf{w}_a)$ at once implies convex valued correspondences $\widehat{D}$ and Z. ∎

One important implication of continuity of a preference ordering is given by

Proposition III.25[185] *Let $\{\mathbf{p}^\nu\}$ be a sequence of price vectors whose limit $\mathbf{p}^0$ satisfies $\mathbf{p}^{0\mathsf{T}}\mathbf{w}_a > 0$. If the relation $\mathbf{x}^\nu \in D_a(\mathbf{p}^\nu, \mathbf{w}_a)$ holds for a sequence $\{\mathbf{x}^\nu\}$ of commodity vectors with limit $\mathbf{x}^0$, then $\mathbf{x}^0 \in D_a(\mathbf{p}^0, \mathbf{w}_a)$ provided the preference ordering is continuous and all of the goods are divisible.*

Proof: Without indivisible goods – that is for $n = n_d$ – the commodity space is $X = \mathbb{R}^n_+$ and, therefore, $G(\mathbf{p}, \mathbf{w}_a) = B(\mathbf{p}, \mathbf{w}_a)$. Considering the individual demand correspondence

$$D_a(\mathbf{p}, \mathbf{w}_a) = \bigcap_{\tilde{\mathbf{x}} \in G(\mathbf{p}, \mathbf{w}_a)} \left[G(\mathbf{p}, \mathbf{w}_a) \cap \mathscr{P}_a(\tilde{\mathbf{x}})\right],$$

we obtain $\mathbf{p}^{\nu\mathsf{T}}\mathbf{x}^\nu \leqq \mathbf{p}^{\nu\mathsf{T}}\mathbf{w}_a$ for each ν because of $\mathbf{x}^\nu \in G(\mathbf{p}^\nu, \mathbf{w}_a)$. Calculating the limit

$$\lim_{\nu\to\infty} \mathbf{p}^{\nu\mathsf{T}}\mathbf{x}^\nu = \mathbf{p}^{0\mathsf{T}}\mathbf{x}^0 \leqq \lim_{\nu\to\infty} \mathbf{p}^{\nu\mathsf{T}}\mathbf{w}_a = \mathbf{p}^{0\mathsf{T}}\mathbf{w}_a$$

entails $\mathbf{x}^0 \in G(\mathbf{p}^0, \mathbf{w}_a)$.

Thus, it remains to be shown that each $\tilde{\mathbf{x}} \in G(\mathbf{p}^0, \mathbf{w}_a)$ implies the relation $\mathbf{x}^0 \succcurlyeq_a \tilde{\mathbf{x}}$ so that $\mathbf{x}^0 \in D_a(\mathbf{p}^0, \mathbf{w}_a)$ is satisfied. For this purpose we have to distinguish the following cases:

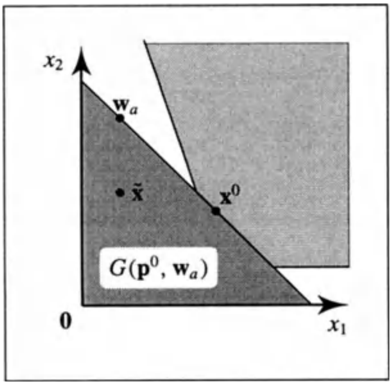

Figure III.43: Proof of Proposition III.25

(a) If $\mathbf{p}^{0\mathsf{T}}\tilde{\mathbf{x}} < \mathbf{p}^{0\mathsf{T}}\mathbf{w}_a$ and, therefore, $\tilde{\mathbf{x}} \in G(\mathbf{p}^0, \mathbf{w}_a)$, then the sequence $\{\tilde{\mathbf{x}}^\nu\}$ with $\tilde{\mathbf{x}}^\nu = \tilde{\mathbf{x}}$ and sufficiently large ν fulfills $\mathbf{p}^{\nu\mathsf{T}}\tilde{\mathbf{x}} = \mathbf{p}^{\nu\mathsf{T}}\tilde{\mathbf{x}}^\nu < \mathbf{p}^{\nu\mathsf{T}}\mathbf{w}_a$. Due to $\mathbf{x}^\nu \in D_a(\mathbf{p}^\nu, \mathbf{w}_a)$ the relation $\mathbf{x}^\nu \succcurlyeq_a \tilde{\mathbf{x}}^\nu$ arises for these ν. Now from the continuity of the preference ordering it ensues $\mathbf{x}^0 \succcurlyeq_a \tilde{\mathbf{x}}$. Summarizing, we get the implication illustrated in Figure III.43:

$$\mathbf{p}^{\nu\mathsf{T}}\tilde{\mathbf{x}} < \mathbf{p}^{\nu\mathsf{T}}\mathbf{w}_a \implies \mathbf{x}^0 \succcurlyeq_a \tilde{\mathbf{x}}.$$

(b) In contrast to (a), we now examine those points $\tilde{\mathbf{x}}$ satisfying $\mathbf{p}^{0\mathsf{T}}\tilde{\mathbf{x}} = \mathbf{p}^{0\mathsf{T}}\mathbf{w}_a$. We require a sequence $\{\tilde{\mathbf{x}}^\nu\}$ of commodity bundles converging to $\tilde{\mathbf{x}}$ and whose elements satisfy $\mathbf{p}^{0\mathsf{T}}\tilde{\mathbf{x}}^\nu < \mathbf{p}^{0\mathsf{T}}\mathbf{w}_a$. Considering (a), $\mathbf{x}^0 \succcurlyeq_a \tilde{\mathbf{x}}^\nu$ holds for each ν so that the continuity of the preference ordering yields the needed result $\mathbf{x}^0 \succcurlyeq_a \tilde{\mathbf{x}}$. Because

[184] See Appendix B.

[185] See HEUSER (1992, p. 626 f.).

of the assumption $\mathbf{p}^{0\mathsf{T}}\tilde{\mathbf{x}} = \mathbf{p}^{0\mathsf{T}}\mathbf{w}_a > 0$ we have $\tilde{\mathbf{x}} \geq \mathbf{0}$ such that the sequence we are looking for can be established without difficulties for perfectly divisible goods. Take, for example, $\left\{(1 - {}^1\!/_\nu)\tilde{\mathbf{x}}\right\}$. ■

Corollary III.25.1[186] *If a person's initial endowment satisfies* $\mathbf{w}_a \geq \mathbf{0}$, *the individual demand correspondence* $D_a(\cdot, \mathbf{w}_a)$ *is closed for every price vector* $\mathbf{p} > \mathbf{0}$.

Proof: For $\mathbf{w}_a \geq \mathbf{0}$ and $\mathbf{p} > \mathbf{0}$ we have $\mathbf{p}^\mathsf{T}\mathbf{w}_a > 0$. By Theorem C.6, p. 308, the closedness of $D_a(\cdot, \mathbf{w}_a)$ at point $\mathbf{p}$ immediately results from Proposition III.25. ■

(b) Indivisible Goods

Corollary III.25.2 *Proposition III.25 remains also valid under consideration of indivisible goods* ($n_d \leqq n$) *as long as at least one divisible good* ($n_d \geqq 1$) *exists which has a positive quantity within the vector* $\mathbf{x}^0$ *and whose price is positive at limit* $\mathbf{p}^0$.

Proof: The proof is analogous to Proposition III.25. Nevertheless, each element of the sequence of points $\{\mathbf{x}^\nu\}$ now fulfills $\mathbf{x}^\nu \in G(\mathbf{p}^\nu, \mathbf{w}_a) \cap X$, where the commodity space is given by $X = \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}$. The limit calculation now yields $\mathbf{x}^0 \in G(\mathbf{p}^0, \mathbf{w}_a)$ and $\mathbf{x}^0 \in X$. Regarding the two cases to be distinguished, $G(\mathbf{p}, \mathbf{w}_a)$ must be replaced with the budget set $B(\mathbf{p}, \mathbf{w}_a)$.
By Proposition III.25 at least one good is divisible and leaves its mark on the vector $\mathbf{x}^0$ in a positive amount. Furthermore, without loss of generality it can be supposed that the first good satisfies this assumption. In part (b) of the proof the needed sequence $\{\tilde{\mathbf{x}}^\nu\}$ can be specified by $\tilde{\mathbf{x}}^\nu = \left((1 - {}^1\!/_\nu)\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n\right)^\mathsf{T}$ where $\tilde{\mathbf{x}}^\nu \in X = \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}$ is taken into account. The inequality

$$\begin{aligned}\mathbf{p}^{0\mathsf{T}}\tilde{\mathbf{x}}^\nu &= p_1^0(1 - {}^1\!/_\nu)\tilde{x}_1 + p_2^0\tilde{x}_2 + \cdots + p_n^0\tilde{x}_n \\ &< p_1^0\tilde{x}_1 + p_2^0\tilde{x}_2 + \cdots + p_n^0\tilde{x}_n = \mathbf{p}^{0\mathsf{T}}\tilde{\mathbf{x}} = \mathbf{p}^{0\mathsf{T}}\mathbf{w}_a\end{aligned}$$

is always satisfied provided $p_1^0 > 0$ and $\tilde{x}_1 > 0$. ■

Corollary III.25.3 *If a person's initial endowment satisfies* $\mathbf{w}_a \geq \mathbf{0}$, *then the individual demand correspondence* $D_a(\cdot, \mathbf{w}_a)$ *is closed for every price vector* $\mathbf{p} > \mathbf{0}$ *even under consideration of indivisible goods* ($n_d \leqq n$) *provided at least one divisible good* ($n_d \geqq 1$) *exists which has a positive amount within the vector* $\mathbf{x}^0$.

[186] As stressed by BARTEN, BÖHM (1982, p. 397), empirical studies of consumers' behavior give reason to the conjecture that consumers possess no transitive preferences. Although Proposition III.23 is not based on this assumption, one can show that the result of Corollary III.25.1 are not affected for a continuous preference relation under relatively weak assumptions. See SONNENSCHEIN (1971).

The proof is analogous to Corollary III.25.1.

The two preceding corollaries are based one the assumption that at least one divisible good exists that is chosen as a positive quantity. As soon as this assumption is dropped, we can construct examples with the demand correspondence no longer being closed. Thus, without the existence of a divisible good ($n_d = 0$) the conclusion in Corollary III.25.2 breaks down.

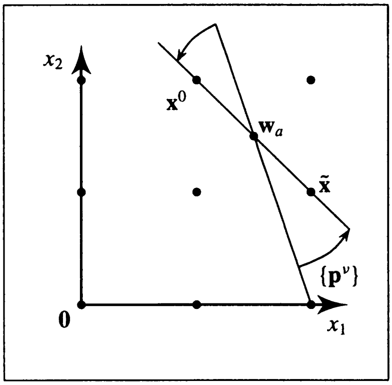

Figure III.44: Counterexample for Corollary III.25.3

Figure III.44 illustrates an example yielding $\mathbf{x}^0 \notin D_a(\mathbf{p}^0, \mathbf{w}_a)$ contrary to Corollary III.25.2. The case of of two indivisible goods starts at a noninteger initial endowment $\mathbf{w}_a$. Besides the sequence of output vectors $\{\mathbf{x}^\nu\}$ with $\mathbf{x}^\nu = \mathbf{x}^0$ there is a sequence of price vectors $\{\mathbf{p}^\nu\}$ with $\mathbf{p}^\nu \to \mathbf{p}^0$ by assumption. If $\tilde{\mathbf{x}}$ is preferred to the commodity bundle $\mathbf{x}^0$, $\tilde{\mathbf{x}} \succ_a \mathbf{x}^0$, then we obtain $D_a(\mathbf{p}^0, \mathbf{w}_a) = \{\tilde{\mathbf{x}}\}$ and $\mathbf{x}^0 \notin D_a(\mathbf{p}^0, \mathbf{w}_a)$ at the limit, although $\mathbf{x}^0 \in D_a(\mathbf{p}^\nu, \mathbf{w}_a)$ is satisfied for all ν. The demand jumps from $\mathbf{x}^0$ to $\tilde{\mathbf{x}}$ at point $\mathbf{p}^0$.

As these jumps contradict the premises of the used fixed-point theorems they are ruled out by the following assumption:

Assumption 1 (BROOME)[187] *The examined economy has at least one perfectly divisible good ($n_d \geqq 1$) desired by each person as a positive quantity.*

Note that this assumption can only be fulfilled if each person holds an initial endowment with positive value, $\mathbf{p}^\mathsf{T}\mathbf{w}_a > 0$.

The next propositions follow from Corollary III.25.2 and note *local* properties of the individual demand correspondence and the aggregate excess demand correspondence.

Proposition III.26 *Provided a continuous preference ordering, the individual demand correspondence $D_a(\cdot, \mathbf{w}_a)$: $\operatorname{rint}\Delta \to \mathfrak{P}(X)$ is upper semi-continuous at each point $\mathbf{p} > \mathbf{0}$ if Assumption 1 ($n_d \geqq 1$) holds.*

Proof: Since by Proposition III.23 the demand set $D_a(\mathbf{p}, \mathbf{w}_a)$ is compact for each $\mathbf{p} > \mathbf{0}$ we have to show by Theorem C.9, p. 309, that for each sequence of price vectors $\{\mathbf{p}^\nu\}$ with limit $\mathbf{p}^0 > \mathbf{0}$ and for each sequence $\{\mathbf{x}^\nu\}$ with $\mathbf{x}^\nu \in D_a(\mathbf{p}^\nu, \mathbf{w}_a)$ a convergent subsequence $\{\mathbf{x}^{\nu_k}\}$ exists whose limit $\mathbf{x}^0$ belongs to $D_a(\mathbf{p}^0, \mathbf{w}_a)$.
(a) For $\mathbf{w}_a = \mathbf{0}$ the sets are $G(\mathbf{p}^\nu, \mathbf{w}_a) = G(\mathbf{p}^0, \mathbf{w}_a) = \{\mathbf{0}\}$. Together with $\{\mathbf{0}\} \in X$ we also have $D_a(\mathbf{p}^\nu, \mathbf{w}_a) = D_a(\mathbf{p}^0, \mathbf{w}_a) = \{\mathbf{0}\}$. Although Assumption 1 ($n_d \geqq 1$) is not fulfilled, we can offer the sequence of commodity bundles $\{\mathbf{x}^\nu\}$ with

[187] Authors supposing a sole divisible good usually speak of money. Cf. e.g. QUINZII (1984) or SVENSSON (1991).

$\mathbf{x}^\nu = \mathbf{0}$ whose limit holds $\mathbf{0} \in D_a(\mathbf{p}^0, \mathbf{w}_a)$. [188]
(b) For $\mathbf{w}_a \geq \mathbf{0}$ it suffices to examine sequences of positive price vectors $\{\mathbf{p}^\nu\}$ so that the limit $\mathbf{p}^0$ is itself positive. At the same time $\mathbf{p}^{0\mathsf{T}}\mathbf{w}_a > 0$ arises, which is needed to use Proposition III.25 or Corollary III.25.2. Due to $\mathbf{x}^\nu \in D_a(\mathbf{p}^\nu, \mathbf{w}_a)$ the budget restriction $\mathbf{p}^{\nu\mathsf{T}}\mathbf{x}^\nu \leqq \mathbf{p}^{\nu\mathsf{T}}\mathbf{w}_a$ is satisfied for all elements of the sequence. If we choose a positive number α and an index ν^0 such that the inequality

$$0 < \alpha \leqq p_j^\nu \qquad \text{for all} \quad j = 1, \ldots, n \quad \text{and} \quad \nu \geqq \nu^0,$$

holds for each single price, then

$$\alpha x_1^\nu + \cdots + \alpha x_n^\nu \leqq p_1^\nu x_1^\nu + \cdots + p_n^\nu x_n^\nu \leqq \mathbf{p}^{\nu\mathsf{T}}\mathbf{w}_a$$
$$\text{or} \quad 0 \leqq x_j^\nu \leqq \frac{\mathbf{p}^{\nu\mathsf{T}}\mathbf{w}_a}{\alpha} \qquad (j = 1, \ldots, n)$$

for all $\nu \geqq \nu^0$. As the sequence $\{\mathbf{p}^{\nu\mathsf{T}}\mathbf{w}_a\}$ converges and is, therefore, bounded, each sequence of components $\{x_j^\nu\}$ $(j = 1, \ldots, n)$ and the sequence of points $\{\mathbf{x}^\nu\}$ itself must be bounded, too. Thus, the sequence of output vectors $\{\mathbf{x}^\nu\}$ includes a convergent subsequence $\{\mathbf{x}^{\nu_k}\}$ whose limit $\mathbf{x}^0$ belongs to $D_a(\mathbf{p}^0, \mathbf{w}_a)$ by Corollary III.25.2 since Assumption 1 $(n_d \geqq 1)$ holds and a continuous preference ordering has been supposed. The price of the divisible good is positive in all cases as $\mathbf{p}^0 > \mathbf{0}$.
■

Because of the following Proposition III.28 the set $D_a(\tilde{\mathbf{p}}, \mathbf{w}_a)$ is empty for a monotone preference ordering as long as the price vector $\tilde{\mathbf{p}}$ has a zero component. Therefore, the question on the upper semi-continuity of the individual demand correspondence $D_a(\cdot, \mathbf{w}_a)$ at point $\tilde{\mathbf{p}}$ does not need to be investigated.
The main local property for applying the fixed-point theorems is given by

Proposition III.27 *For continuous preference orderings of all persons the aggregate demand correspondence* $\widehat{D}$: $\operatorname{rint}\Delta \to \mathfrak{P}(X)$ *and the aggregate excess demand correspondence* Z: $\operatorname{rint}\Delta \to \mathfrak{P}(\mathbb{R}^n)$ *are upper semi-continuous and compact-valued for every* $\mathbf{p} > \mathbf{0}$, *if Assumption 1* $(n_d \geqq 1)$ *holds.*

Proof: Supposing continuous preference orderings, by Proposition III.23 the sets $D_a(\mathbf{p}, \mathbf{w}_a)$ are compact at $\mathbf{p} > \mathbf{0}$. Thus[189]

$$\widehat{D}(\mathbf{p}) = \sum_{a \in A} D_a(\mathbf{p}, \mathbf{w}_a)$$

is also compact for each $\mathbf{p} > \mathbf{0}$. The same argument holds for $Z(\mathbf{p})$.
In order to prove the upper semi-continuity of $\widehat{D}$ at each $\mathbf{p} > \mathbf{0}$ according to Theorem C.9, p. 309, it remains to be shown that for each sequence of price vectors $\{\mathbf{p}^\nu\}$ with limit $\mathbf{p}^0$ and for each sequence $\{\widehat{\mathbf{x}}^\nu\}$ with $\widehat{\mathbf{x}}^\nu \in \widehat{D}(\mathbf{p}^\nu)$ there is a convergent

[188] Thus, the jump from $\tilde{\mathbf{x}}$ to $\mathbf{x}^0$ described by Figure III.44 may *not* occur.
[189] See Appendix B.2, in particular p. 294.

subsequence $\{\widehat{\mathbf{x}}^{\nu_k}\}$ whose limit $\widehat{\mathbf{x}}^0$ belongs to $\widehat{D}(\mathbf{p}^0)$. Every sequence $\{\widehat{\mathbf{x}}^{\nu}\}$ can be expressed as

$$\widehat{\mathbf{x}}^{\nu} = \sum_{a \in A} \mathbf{x}_a^{\nu} \quad \text{with} \quad \widehat{\mathbf{x}}^{\nu} \in \widehat{D}(\mathbf{p}^{\nu})$$

$$\text{and} \quad \mathbf{x}_a^{\nu} \in D_a(\mathbf{p}^{\nu}, \mathbf{w}_a) \quad (\text{for all} \quad a \in A).$$

Using the convergent subsequences determined in the proof of Corollary III.25.2, a subsequence $\{\widehat{\mathbf{x}}^{\nu_k}\}$ results with

$$\widehat{\mathbf{x}}^{\nu_k} = \sum_{a \in A} \mathbf{x}_a^{\nu_k} .$$

For $\mathbf{w}_a \geq \mathbf{0}$ and $\mathbf{p} > \mathbf{0}$ the inequality $\mathbf{p}^{\mathsf{T}}\mathbf{w}_a > 0$ in Proposition III.25 holds for each person. In accordance with Corollary III.25.2 Assumption 1 ($n_d \geqq 1$) assures that the limit $\mathbf{x}_a^0 = \lim_{\nu_k \to \infty} \mathbf{x}_a^{\nu_k}$ belongs to $D_a(\mathbf{p}^0, \mathbf{w}_a)$. We obtain

$$\lim_{\nu_k \to \infty} \widehat{\mathbf{x}}^{\nu_k} = \lim_{\nu_k \to \infty} \sum_{a \in A} \mathbf{x}_a^{\nu_k} \quad \Longrightarrow \quad \widehat{\mathbf{x}}^0 = \sum_{a \in A} \mathbf{x}_a^0 \quad \text{with} \quad \widehat{\mathbf{x}}^0 \in \widehat{D}(\mathbf{p}^0).$$

With that the aggregate demand correspondence $\widehat{D}$ is upper semi-continuous and compact-valued for each $\mathbf{p} > \mathbf{0}$. The same argument is valid for the aggregate excess demand correspondence Z with $Z(\mathbf{p}) = \widehat{D}(\mathbf{p}) - \mathbf{w}_A$. ■

Finally, the subsequent *global* properties of the examined correspondences must be noted:[190]

- The individual demand correspondences $D_a(\cdot, \mathbf{w}_a)$ of all persons $a \in A$, the aggregate demand correspondence $\widehat{D}$, and the aggregate excess demand correspondence Z are upper semi-continuous in the relative interior of the price simplex rint Δ.
- For convex preference orderings the above mentioned correspondences are convex-valued.

(c) **Graphical Explanations** At this point it seems to be worthwhile to go into the concept of the upper semi-continuity of an individual demand correspondence in more detail.

In Figure III.46 we refer to the graphically representable case of a divisible good 1 and an indivisible good 2 for an idea of the upper semi-continuity of a demand correspondence. But, first of all, we discuss with the help of Figure III.45 the upper semi-continuity conforming to the technical Definition C.5, p. 307.

A correspondence $\Gamma\colon X \to \mathfrak{P}(Y)$ is said to be upper semi-continuous or upper semi-continuous at point $\mathbf{x}^0 \in X \subset \mathbb{R}^n$ if it makes no "explosive" jumps for small changes of its arguments $\mathbf{x}^0$. This notion of an "explosive" jump is illustrated by the expositions of Figure III.45.

[190]By Definition C.5, p. 307, a correspondence $\Gamma\colon X \to \mathfrak{P}(Y)$ is called **upper semi-continuous** in X if it is upper semi-continuous at $\mathbf{x}$ for all $\mathbf{x} \in X$ *and* if $\Gamma(\mathbf{x})$ is compact for all $\mathbf{x} \in X$.

Technically, the problem can be ruled out as follows:[191] for each open set V containing $\Gamma(\mathbf{x}^0)$ a neighborhood U of $\mathbf{x}^0$ must exist such that $\Gamma(\mathbf{x}) \subset V$ also holds for each $\mathbf{x}$ in the neighborhood U. Accordingly, there is an "explosive jump" if each neighborhood $U(\mathbf{x}^0)$ contains an $\mathbf{x}$ with $\Gamma(\mathbf{x}) \not\subset V$ but $\Gamma(\mathbf{x}^0) \subset V$. *The correspondence virtually jumps out of V.*

This case is shown in Figure III.45 at point $x^0 \in X \subset \mathbb{R}$. Although $\Gamma(x^0) \subset V$ is satisfied, each neighborhood of x^0 contains an $\tilde{x}$ with $\Gamma(\tilde{x}) \not\subset V$. Note that the criterion of upper semi-continuity allows "imploding" jumps as at point x^1.

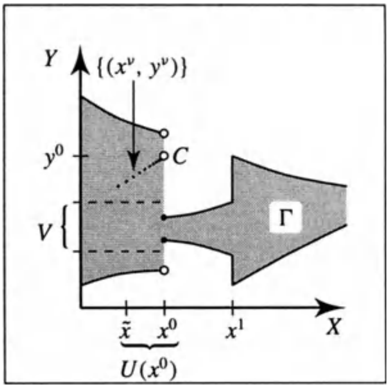

Figure III.45: On the criterion of upper semi-continuity of a correspondence Γ

Theorem C.9, p. 309, states that there is an equivalent criterion of upper semi-continuity at point $\mathbf{x}^0$ for a compact-valued correspondence. Accordingly, for a sequence $\{\mathbf{x}^\nu\}$ with $\mathbf{x}^\nu \to \mathbf{x}^0$ and for each sequence $\{\mathbf{y}^\nu\}$ with $\mathbf{y}^\nu \in \Gamma(\mathbf{x}^\nu)$ there must be a convergent subsequence $\{\mathbf{y}^{\nu_k}\}$ such that $\mathbf{y}^{\nu_k} \to \mathbf{y}^0 \in \Gamma(\mathbf{x}^0)$ holds. Figure III.45 illustrates this criterion by a sequence $\{x^\nu\}$ converging from the left hand side to x^0. A sequence $\{y^\nu\}$ with $y^\nu \in \Gamma(x^\nu)$ converging to y^0 now violates the criterion $y^0 \in \Gamma(x^0)$. See point $C = (x^0, y^0)$ with $y^0 \notin \Gamma(x^0)$.

With regard to economic aspects Figure III.46 serves as a reference situation. The two initial points $\mathbf{x}^1$ and $\mathbf{x}^2$ are supposed to be best elements with respect to the price vectors $\mathbf{p}^1$ and $\mathbf{p}^2$, where the price vectors are described by two budget lines containing the initial endowment $\mathbf{w}_a$. If $D_a(\mathbf{p}^1, \mathbf{w}_a) = \{\mathbf{x}^1\}$ and $D_a(\mathbf{p}^2, \mathbf{w}_a) = \{\mathbf{x}^2\}$ hold, then it remains to be settled what behavior of the demand correspondence may occur for continuous changes of the price vector from $\mathbf{p}^1$ to $\mathbf{p}^2$ without violating the requirements of upper semi-continuity.

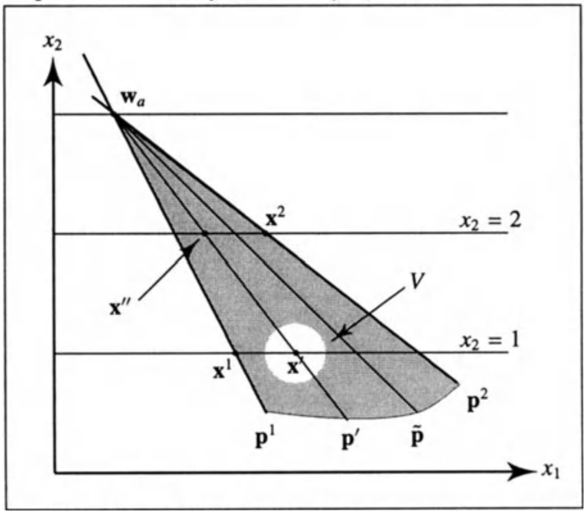

Figure III.46: Upper semi-continuity of a demand correspondence

First of all, it is striking that the demand correspondence must jump from $x_2 = 1$ to $x_2 = 2$ for at least one price vector (for instance, at $\mathbf{p}'$). Technically, it cannot

[191] This criterion is given in Definition C.5.

be ruled out that the correspondence concerned includes commodity bundles with $x_2 = 1$ and $x_2 = 2$ at diverse price vectors. Suppose in the inspected cone between $\mathbf{p}^1$ and $\mathbf{p}^2$ one and only one jump at point $\mathbf{p}'$ emerges, then a demand correspondence for alternative price vectors between $\mathbf{p}^1$ and $\mathbf{p}^2$ results, which is similar to the bold marked sections in Figure III.47. Furthermore, not only $\mathbf{x}'$ but also $\mathbf{x}''$ must be an element in $D_a(\mathbf{p}', \mathbf{w}_a)$ presuming the upper semi-continuity.

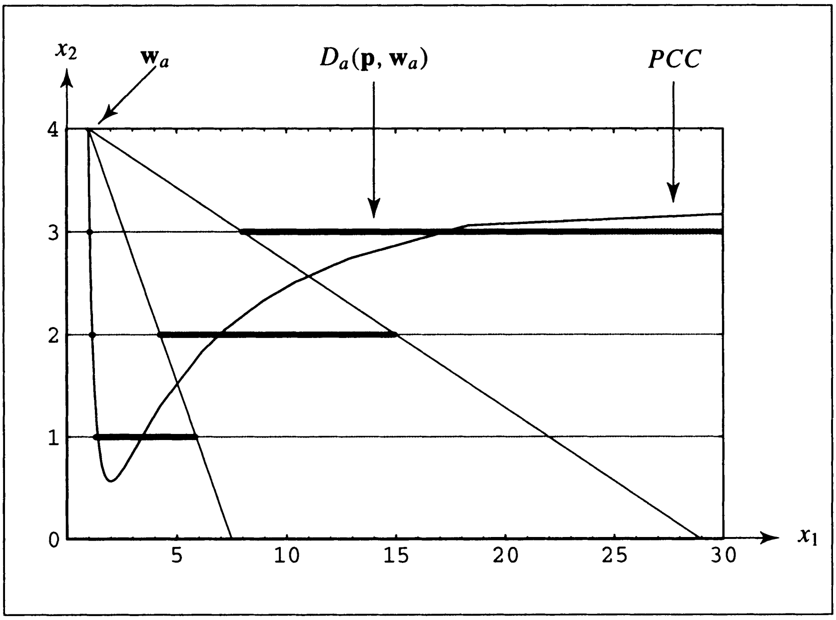

Figure III.47: Price consumption curve

The proof is offered assuming $D_a(\mathbf{p}', \mathbf{w}_a) = \{\mathbf{x}'\}$, i.e. $\mathbf{x}'' \notin D_a(\mathbf{p}', \mathbf{w}_a)$. Because each neighborhood $U(\mathbf{p}')$ contains a price vector $\tilde{\mathbf{p}}$ with $D_a(\tilde{\mathbf{p}}, \mathbf{w}_a) \not\subset V$, the demand correspondence cannot be upper semi-continuous. Similarly, we can show that the demand correspondence must not have any further "holes", i.e. for instance, no commodity bundle must be missing over the segment $\overline{\mathbf{x}^1\mathbf{x}'}$. Technically, this statement is included in Theorem C.7. Accordingly, each upper semi-continuous correspondence is closed and, therefore, it has a closed graph.

Figure III.47 illustrates the demand correspondence analogous to a price consumption curve. Beginning with an initial endowment $(w_{1a}, w_{2a}) = (1, 4)$ we have derived a price consumption curve PCC on the basis of a CES utility function $u(x_1, x_2) = (x_1^{-1} + x_2^{-1})^{-1}$. [192] If we suppose that x_2 can only appear at integer

[192] The result has been computed by MATHEMATICA®. Generally, the optimum conditions of the LAGRANGEan function $L(x_1, x_2, \lambda) = (x_1^{-1} + x_2^{-1})^{-1} + \lambda(p_1 w_{1a} + p_2 w_{2a} - p_1 x_1 - p_2 x_2)$ yield a price consumption curve of the implicit form $x_2^2/x_1^2 = -(x_2 - w_{2a})/(x_1 - w_{1a})$. Note that the symmetry of the price consumption curve only appears to vanish because of the chosen scaled part of the commodity space.

units, then the resulting demand correspondence $D_a(\cdot, \mathbf{w}_a)$ can be represented by the bold segments.

3.2.2 Implications of the Monotonicity of Preference Orderings

(a) Introduction of Additional Assumptions Before going into the proof of the existence of competitive equilibria, we have to investigate how the individual demand correspondences behave when the examined price vectors have zero components. To assure that the value of the initial endowment is positive at each price vector $\mathbf{p} \in \Delta$ for every person $a \in A$, we establish the following

Assumption 2 *Every person* $a \in A$ *has a positive initial endowment,* $\mathbf{w}_a > \mathbf{0}$.

As long as positive price vectors $\mathbf{p}$ are supposed, we can dispense with this very restrictive assumption. In this case the value of an initial endowment $\mathbf{w}_a \geq \mathbf{0}$ is always positive, $\mathbf{p}^{\mathsf{T}}\mathbf{w}_a > 0$.

If one of the examined price vectors $\tilde{\mathbf{p}}$ has a zero component, then we can construct examples based on the indivisibility of a commodity, which evoke a certain misbehavior of the demand correspondence. At point $\tilde{\mathbf{p}}$ "a jump to infinity" virtually emerges, which is possible as the budget set $B(\tilde{\mathbf{p}}, \mathbf{w}_a)$ is unbounded.

The problem is illustrated in Figure III.48 by two goods ($n = 2$) with the divisible good 1 ($n_d = 1$) and the indivisible good 2. If the sequence of price vectors $\{\mathbf{p}^\nu\} = \{(p_1^\nu, p_2^\nu)\}$ converges to a limit $(\tilde{p}_1, \tilde{p}_2) = (0, \tilde{p}_2)$, then a horizontal budget line results. The corresponding budget set is unbounded. At the same time it is assumed that the examined person a prefers his initial endowment $\mathbf{w}_a$ to every commodity bundle in the horizontal ray[193] starting at the commodity bundle $\tilde{\mathbf{x}}$, i.e.

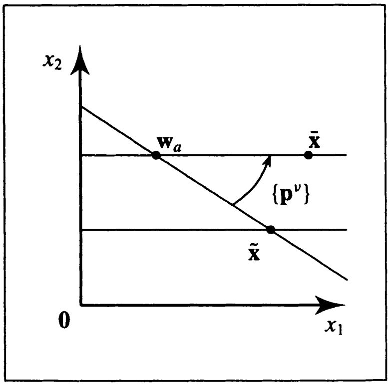

Figure III.48: Noncontinuity of the correspondence $D_a(\cdot, \mathbf{w}_a)$

$$\text{(III.105)} \qquad \mathbf{w}_a \succ_a \tilde{\mathbf{x}} + \lambda\mathbf{e}^1 \qquad \forall\, \lambda > 0\,.$$

For a strongly monotone preference ordering $\mathbf{w}_a$ cannot be a best element in the limit case since $\bar{\mathbf{x}} \succ_a \mathbf{w}_a$. Although the sequence $\{\mathbf{p}^\nu\}$ converges to $\tilde{\mathbf{p}}$, at which the limit is $\tilde{\mathbf{p}}^{\mathsf{T}}\mathbf{w}_a > 0$ and $\mathbf{w}_a \in D_a(\mathbf{p}^\nu, \mathbf{w}_a)$ holds for all ν, $\mathbf{w}_a \in D_a(\tilde{\mathbf{p}}, \mathbf{w}_a)$ does *not* result as suggested by Proposition III.25. The jump from $D_a(\mathbf{p}^\nu, \mathbf{w}_a) = \{\mathbf{w}_a\}$ to a point with an infinitely large quantity of the first good, $x_1 = +\infty$, becomes possible by the unboundedness of the budget set $B(\tilde{\mathbf{p}}, \mathbf{w}_a)$. Since no commodity bundle $\binom{x_1}{x_2}$ satisfies the relation $x_1 = +\infty$, none of them is chosen $D_a(\tilde{\mathbf{p}}, \mathbf{w}_a) = \emptyset$.

The reason for the "misbehavior" of the demand correspondence $D_a(\cdot, \mathbf{w}_a)$ results from the assumption (III.105). Thus, we have a reason for the subsequent assumption, which states that each person prefers a commodity bundle $\mathbf{x}$ to each

[193] As before, $\mathbf{e}^1$ denotes the unit vector with 1 as the first component.

commodity bundle $\tilde{\mathbf{x}}$ when the first commodity bundle is augmented by a sufficiently large amount of the divisible good x_1.

Assumption 3[194] *For all persons* $a \in A$ *and all commodity bundles* $\mathbf{x}, \tilde{\mathbf{x}} \in X$ *there is a positive number* λ *such that* $\mathbf{x} + \lambda \mathbf{e}^1 \succcurlyeq_a \tilde{\mathbf{x}}$.

Geometrically, this assumption excludes that the distance between two indifferent commodity bundles can become arbitrarily large when these commodity bundles lie in a line parallel to the axis of the divisible good. Accordingly, the boundary of the convex hull of an arbitrary preference set must cross the entire commodity space X.

(b) Price Vectors with Zero Components The subsequent propositions describe the behavior of persons of the examined economy when the price of a good falls to zero given strongly monotone preference orderings. The main result is that the aggregate excess demand tends to infinity for at least one good. Since no commodity bundle has an infinitely large component $x_j = +\infty$, the demand set is empty, $D_a(\tilde{\mathbf{p}}, \mathbf{w}_a) = \emptyset$.

Proposition III.28 *Provided that the continuous preference ordering of a person* $a \in A$ *is strongly monotone and that the price vector* $\tilde{\mathbf{p}}$ *has at least one zero component so that* $\tilde{\mathbf{p}} \in \partial\Delta$, *then the demand set is empty.*

$$D_a(\tilde{\mathbf{p}}, \mathbf{w}_a) = \emptyset \qquad \forall \tilde{\mathbf{p}} \in \partial\Delta$$

Proof: Suppose $D_a(\tilde{\mathbf{p}}, \mathbf{w}_a)$ contains the commodity bundle $\mathbf{x}$, where without loss of generality it is supposed that $\tilde{p}_1 = 0$. Then each commodity bundle $\tilde{\mathbf{x}}$ with $\tilde{x}_1 > x_1$ and $\tilde{x}_j = x_j$ $(j \neq 1)$ satisfies $\tilde{\mathbf{p}}^\mathsf{T}\mathbf{x} = \tilde{\mathbf{p}}^\mathsf{T}\tilde{\mathbf{x}} \leqq \tilde{\mathbf{p}}^\mathsf{T}\mathbf{w}_a$ or $\tilde{\mathbf{x}} \in B(\tilde{\mathbf{p}}, \mathbf{w}_a)$. Due to $\mathbf{x} \in D_a(\tilde{\mathbf{p}}, \mathbf{w}_a)$ we get $\mathbf{x} \succcurlyeq_a \tilde{\mathbf{x}}$ in contradiction to $\tilde{\mathbf{x}} \succ_a \mathbf{x}$ because of the strongly monotone preference ordering. ■

Proposition III.29[195] *Suppose the continuous preference ordering of person a is strongly monotone* $(\succcurlyeq_a \in \Pi_{smo})$ *and a sequence of price vectors* $\{\mathbf{p}^\nu\}$ *with* $\mathbf{p}^\nu > \mathbf{0}$ *converges to a price vector* $\tilde{\mathbf{p}} \in \partial\Delta$ *including at least one zero component. Under Assumptions 1* $(n_d \geqq 1)$ *and 2* $(\mathbf{w}_a > \mathbf{0})$ *each sequence of commodity bundles* $\{\mathbf{x}^\nu\}$ *with* $\mathbf{x}^\nu \in D_a(\mathbf{p}^\nu, \mathbf{w}_a)$ *satisfies the relation* $\|\mathbf{x}^\nu\| \to \infty$ *if the preference ordering fulfills Assumption 3.*

Proof: Assumption 2 $(\mathbf{w}_a > \mathbf{0})$ implies $\tilde{\mathbf{p}}^\mathsf{T}\mathbf{w}_a > 0$.

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[194] BROOME (1972, p. 228). Technically, this assumption holds for the origin $\mathbf{x} = \mathbf{0}$, too. We get $\lambda \mathbf{e}^1 \succcurlyeq_a \tilde{\mathbf{x}}$ when λ is chosen sufficiently large. Those implications can be avoided by alternative assumptions but these assumptions are associated with a considerably more extensive technical setting. SVENSSON (1991) describes a market with m houses and the divisible good money. If each person can own one house at the most then Assumption 3 means that the loss of each house may be compensated by a sufficiently large amount of money.

[195] Cf. HILDENBRAND, KIRMAN (1988, pp. 93 and 95).
As shown by the proof, Assumptions 1 and 3 are not required for convex preference orderings.

The proposition says that the sequence $\{\mathbf{x}^\nu\}$ satisfies the condition $\|\mathbf{x}^\nu\| \to \infty$ under certain premises.[196] Thus, if under the same conditions we suppose the existence of a bounded subsequence $\{\mathbf{x}^{\nu_k}\}$, then a contradiction must result since in this case a finite number $c > 0$ exists such that $\|\mathbf{x}^{\nu_k}\| < c$ is valid for all ν_k. In the sense of a proof of contradiction it is now supposed that an arbitrary sequence of points $\{\mathbf{x}^\nu\}$ with $\mathbf{x}^\nu \in D_a(\mathbf{p}^\nu, \mathbf{w}_a)$ contains a bounded subsequence $\{\mathbf{x}^{\nu_k}\}$.

(a) First of all, we assume that all goods are divisible ($n_d = n$): since each bounded sequence $\{\mathbf{x}^{\nu_k}\}$ contains a convergent subsequence $\{\mathbf{x}^{\nu_h}\}$ whose limit is denoted by $\tilde{\mathbf{x}}$, the relation $\tilde{\mathbf{x}} \in D_a(\tilde{\mathbf{p}}, \mathbf{w}_a)$ must be satisfied because of $\tilde{\mathbf{p}}^\mathsf{T}\mathbf{w}_a > 0$ by Proposition III.25. But this implication contradicts Proposition III.28. Thus, we have $\|\mathbf{x}^\nu\| \to \infty$.

(b) Considering indivisible goods with at least one divisible good (Assumption 1, $n_d \geqq 1$), we need a case distinction: **(aa)** if the price of the divisible good in $\tilde{\mathbf{p}}$ is positive, then the same result as in (a) follows from using Corollary III.25.2.

(bb) Without loss of generality it is supposed that the first good is divisible and that the sequence $\{\mathbf{p}^\nu\}$ converges to a limit $\tilde{\mathbf{p}}$ with $\tilde{p}_1 = 0$. If the subsequence $\{\mathbf{x}^{\nu_k}\}$ is bounded, then there is a convergent subsequence $\{\mathbf{x}^{\nu_h}\}$ with $\mathbf{x}^{\nu_h} \to \tilde{\mathbf{x}}$ and $\mathbf{p}^{\nu_h} \to \tilde{\mathbf{p}}$.

The procedure within the rest of the proof is illustrated by Figure III.49. For each commodity bundle $\mathbf{x} < \mathbf{w}_a$ with $\mathbf{w}_a > \mathbf{0}$ (Assumption 2) a positive λ^* exists such that $\mathbf{x} + \lambda^*\mathbf{e}^1 \succ_a \tilde{\mathbf{x}}$ (Assumption 3). Because of $\mathbf{x} \in G(\mathbf{p}, \mathbf{w}_a) \cap X$ for all $\mathbf{p}$ we get $\mathbf{x} + \lambda\mathbf{e}^1 \in G(\tilde{\mathbf{p}}, \mathbf{w}_a) \cap X$ for all $\lambda > 0$ so that $\mathbf{x}+\lambda^*\mathbf{e}^1 \in G(\mathbf{p}^{\nu_h}, \mathbf{w}_a) \cap X$ must be fulfilled for sufficiently large ν_h.

On the basis of the closedness of preference sets (assumption of a continuous preference ordering) there is an ε-neighborhood of the point $\tilde{\mathbf{x}}$ such that

$$U_\varepsilon(\tilde{\mathbf{x}}) \cap \mathcal{P}_a(\mathbf{x} + \lambda^*\mathbf{e}^1) = \emptyset .$$

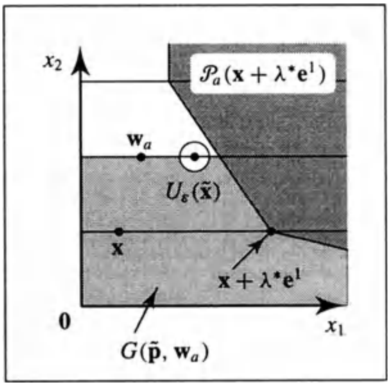

Figure III.49: Proof of Proposition III.29

For sufficiently high ν_h the relations $\mathbf{x}^{\nu_h} \in U_\varepsilon(\tilde{\mathbf{x}})$ and $\mathbf{x} + \lambda^*\mathbf{e}^1 \in G(\mathbf{p}^{\nu_h}, \mathbf{w}_a)$ must be satisfied. This contradicts the hypothesis $\mathbf{x}^{\nu_h} \in D_a(\mathbf{p}^{\nu_h}, \mathbf{w}_a)$. Again, we obtain $\|\mathbf{x}^\nu\| \to \infty$. ■

Corollary III.29.1[197] *Suppose the premises of Proposition III.29 hold for each person $a \in A$ and the sequence of price vectors $\{\mathbf{p}^\nu\}$ with $\mathbf{p}^\nu > \mathbf{0}$ converges to a price vector $\tilde{\mathbf{p}} \in \partial\Delta$. Then we have for each sequence of aggregate excess demand vectors $\{\mathbf{z}^\nu\}$ with $\mathbf{z}^\nu \in Z(\mathbf{p}^\nu)$ the relation $\|\mathbf{z}^\nu\| \to \infty$.*

[196] If a given sequence has $-\infty$ or $+\infty$ as sole cluster point, then it is called *divergent* with limit $-\infty$ or $+\infty$. See Appendix A.3.

[197] Cf. HILDENBRAND, KIRMAN (1988, p. 97).

Proof: Due to $Z(\mathbf{p}) = \widehat{D}(\mathbf{p}) - \mathbf{w}_A$ with $\widehat{D}(\mathbf{p}) = \sum_{a \in A} D_a(\mathbf{p}, \mathbf{w}_a)$ the proposition at once follows from Proposition III.29. ■

(c) WALRAS' Law

Proposition III.30 (WALRAS' Law) *Suppose all persons $a \in A$ possess strongly monotone continuous preference orderings and there is at least one divisible good. Then the value of aggregate excess demand is zero, $\mathbf{p}^{\mathsf{T}}(\mathbf{x}_A - \mathbf{w}_A) = \mathbf{p}^{\mathsf{T}}\mathbf{z} = 0$, for every price vector $\mathbf{p} > \mathbf{0}$ and every vector $\mathbf{x}_A \in \widehat{D}(\mathbf{p})$.*

Proof: If there is at least one divisible good, then each person $a \in A$ with a strongly monotone preference ordering satisfies the budget constraint

$$\mathbf{p}^{\mathsf{T}}\mathbf{x}_a = \mathbf{p}^{\mathsf{T}}\mathbf{w}_a \qquad \forall \mathbf{x}_a \in D_a(\mathbf{p}, \mathbf{w}_a)$$

because the divisibility of a good guarantees the existence of an $\tilde{\mathbf{x}} \geq \mathbf{x}$ for each $\mathbf{x}$ holding $\mathbf{p}^{\mathsf{T}}\mathbf{x} < \mathbf{p}^{\mathsf{T}}\mathbf{w}_a$, where $\tilde{\mathbf{x}}$ satisfies the budget constraint, too. On the basis of strongly monotone preference orderings it is $\tilde{\mathbf{x}} \succ_a \mathbf{x}$ so that $\mathbf{x}$ cannot be a best element.
The summation of individual budget constraints yields $\mathbf{p}^{\mathsf{T}}\mathbf{x}_A = \mathbf{p}^{\mathsf{T}}\mathbf{w}_A$ for all $\mathbf{x}_A \in \widehat{D}(\mathbf{p})$ or $\mathbf{p}^{\mathsf{T}}\mathbf{z} = 0$ for all $\mathbf{z} \in Z(\mathbf{p})$. ■

Corollary III.30.1 (Free Goods)[198] *Suppose each person possesses a strongly monotone preference ordering and there is at least one divisible good. If $\mathbf{p}^{\circ}$ is a WALRASian equilibrium and if there is an excess supply of good j, $z_j^{\circ} < 0$, then the price of this free good is zero, $p_j^{\circ} = 0$.*

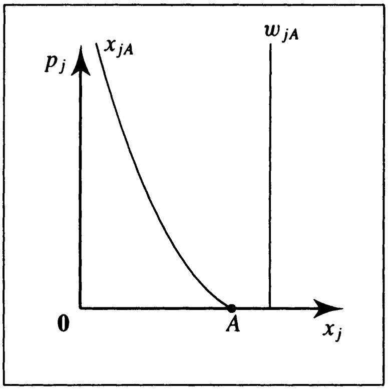

Figure III.50: Free good

Proof: Within a WALRASian equilibrium $(\mathbf{x}^{\circ}, \mathbf{p}^{\circ})$ with $\mathbf{z}^{\circ} = \sum_{a \in A} \mathbf{x}_a^{\circ} - \mathbf{w}_A \leqq \mathbf{0}$ and $z_j^{\circ} < 0$, each positive price p_j° of good j contradicts WALRAS' law, i.e. $\mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^{\circ} = 0$ for all $\mathbf{z}^{\circ} \in Z(\mathbf{p}^{\circ})$. ■

The case of a free good can be ruled out with respect to a WALRASian equilibrium when everyone of the examined exchange economy has strongly monotone preference orderings.
Hence, the situation illustrated by point A in Figure III.50 may not occur. Every excess supply of commodity j being however large is cleared if the positive price p_j is small enough.

Corollary III.30.2 *If $(\mathbf{x}^{\circ}, \mathbf{p}^{\circ})$ is a WALRASian equilibrium with strongly monotone preference orderings, then $\mathbf{p}^{\circ} > \mathbf{0}$ and $\mathbf{z}^{\circ} = \mathbf{0}$ with $\mathbf{z}^{\circ} \in Z(\mathbf{p}^{\circ})$ hold.*

198 VARIAN (1992, p. 318).

Proof: Because of the strongly monotone preference orderings each price $p_j^\circ = 0$ yields a positive excess demand for good j, $z_j^\circ > 0$, which is inconsistent with the assumption of a WALRASian equilibrium.

However, by Corollary III.30.1 an excess supply $z_j^\circ < 0$ yields $p_j^\circ = 0$. But at this price strong monotonicity of the preference orderings implies $z_j^\circ > 0$ contradicting the presumed excess supply. ■

If $(\mathbf{x}^\circ, \mathbf{p}^\circ)$ is a WALRASian equilibrium, then Corollary III.30.2 means that on all markets the aggregate demand curve x_{jA} must have a point in common with the aggregate supply curve w_{jA}; see Figure III.50. We have to take two aspects into account. On the one hand the aggregate demand w_{jA} always fulfills the integer constraint by assumption. On the other hand the curve of aggregate demand x_{jA} jumps when good j is indivisible. Although an aggregate demand x_{jA} can be determined for each price p_j, there does not need to be any price p_j° so that the supply w_{jA} meets the demand x_{jA}. However, each price vector $\mathbf{p}^\circ > \mathbf{0}$ describes a competitive equilibrium $(\mathbf{x}^\circ, \mathbf{p}^\circ)$ in the reverse conclusion of Corollary III.30.2 if there is a $\mathbf{z}^\circ = \mathbf{x}_A - \mathbf{w}_A = \mathbf{0}$ with $\mathbf{z}^\circ \in Z(\mathbf{p}^\circ)$.

3.3 Summary

Because the first two sections of Chapter III concentrate on the analysis of the firm's cost structure, in the third section the analysis of the household's expenditure structure may recede into the background. Instead of this we derive properties of the aggregate excess demand from the individual commodity demand, which are of crucial importance for the existence proof of exchange equilibria.

For this purpose we introduce an exchange economy which first of all consists of a set of persons. Each person pursues utility maximization without cooperating with other persons. After a person has valuated his initial endowment at market prices, he knows all of the consumption bundles he can buy with respect to a corresponding supply of his initial endowment. If each household individually chooses a utility maximizing commodity bundle that it can buy, then the following question is asked. When the individual exchange plans are made on the basis of market prices, what market prices assure that aggregate demand meets aggregate supply on all commodity markets? Before this question can be answered, it must be settled whether some market prices are able to remove the aggregate excess demand on all commodity markets at all. The analysis in Chapter IV will show that especially the consideration of indivisible goods can exclude the existence of equilibrium prices, but does not have to.

The premises for the existence of equilibrium commodity prices involve a series of properties of the aggregate excess demand implied by certain properties of the individual preference orderings. Under this aspect Sections 3.2.1 and 3.2.2 present implications of continuous as well as strongly monotone preference orderings.

The crucial problem considering indivisible goods is that the individual commodity demand and therefore the aggregate excess demand can jump when the market prices change marginally. To avoid these jumps being of arbitrary form two assumptions are introduced. Assumption 1 (BROOME) requires the existence of at least one divisible good desired by each person in a positive amount. This assumption suffices for positive prices to rule out explosive jumps of the individual commodity demand as described in Section 3.2.1(c). Assumption 3 prevents virtual "jumps to infinity". If the price of a good continually falls to zero, then we can construct examples as in Figure III.48, which imply the following phenomenon. For each positive price of the commodity at hand a certain finite quantity of this good is chosen. As soon as this price has fallen to zero, an infinitely large demand is the consequence. This jump to infinity should not be mixed up with a monotonic increase of a commodity demand with a falling commodity price.

Chapter IV.

Theory of Market Equilibria

1 The Problem of General Equilibria

1.1 Approaches to Treating Indivisible Goods

In every economy it is necessary to know who can dispose of what quantities of goods and for what purpose. The examination of these problems assumes that the property rights of goods are exerted personally by individual persons, i.e. common property is ruled out.

Given a property distribution of the commodity stocks among the persons of the economy, each economic agent is allowed to exchange his endowment against property rights of other amounts of goods, where at the same time no agent can be forced to exchange. The distinction between consumer goods and production factors indicates how each individual agent can put his initial endowment to its intended purpose of either consumption or production. A good is not only chosen to be consumed but also to be used in the production of new goods. Only the production of new goods permits future consumption.

Conversely, a supply of goods on a market merely happens because not all initial endowments satisfy the needs of the person concerned. To gain a "better" commodity bundle, one must offer a supply such that there is another economic agent with compatible wishes, who agrees to the supply.

With the described processes there are three fundamental questions:[1]

1. What will be the outcome of the implied decisions?
2. What outcomes are desirable?
3. What allocative mechanisms are appropriate for achieving desirable outcomes?

The presented text deals particularly with the question of the existence of general equilibria, where the main emphasis lies on the aspect of the indivisibility of goods.

[1] See VARIAN (1992, p. 314).

To avoid additional problems resulting from the indivisibility of some production factors, the production processes are not taken into consideration. The existence of an equilibrium serves not only as a precondition for the calculation of an equilibrium but it often gives hints of algorithms to calculate the equilibria.

The first approach to a pure exchange economy excludes the production of goods. This simplified abstraction of a production economy can be justified by avoiding the additional problems in describing production processes. Moreover, there can only be satisfactory answers regarding a production economy if suitable answers are available with respect to an exchange economy. In particular, in view of the indivisibility of consumption goods and production factors does seem useful to rule out problems of production for the moment.

The examination of equilibria in exchange economies has to distinguish between two concepts, the competitive equilibrium and the core. Both concepts can be described as follows: the exchange economy includes a set of persons each of whom holds a commodity bundle as an initial endowment. These initial endowments can be used for trade, where it is supposed that each person possesses a well defined preference ordering on the set of all commodity bundles. The result of the exchange is indicated by an allocation $\mathfrak{x}$, which describes the reallocation of the initial endowments between persons.

A **competitive equilibrium** $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ consists of two components, a system of commodity prices $\mathbf{p}^\circ$, at which the aggregate supply of each good meets the aggregate demand for this good, and a commodity allocation $\mathfrak{x}^\circ$ resulting from trade at the system of commodity prices. More precisely, the equilibrium allocation associates each person with a commodity bundle such that no person can buy a preferred commodity bundle with respect to the value of his initial endowment.

An allocation $\mathfrak{x}^c$ belongs to the **core of the exchange economy** if there is no coalition of market participants which is able to improve the allocation $\mathfrak{x}^c$ to their advantage. In other words, there must be no group of persons ignoring the other market participants so that each member of the group can realize a more favorable result than by the allocation $\mathfrak{x}^c$.

While the concept of a competitive equilibrium is only useful when supposing perfect competition, the concept of the core does not depend on this assumption. This difference is reflected among other things by the fact that supply and demand are coordinated in the competitive equilibrium by market prices, while the concept of the core ignores the price mechanism and relies upon the direct exchange between the market participants. In particular, the core can be investigated when the exchange economy only consists of a few persons. The relationship of competitive equilibria $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ to those allocations $\mathfrak{x}^c$ lying in the core can intuitively be comprehended by understanding money and prices as a help to make trade easier. In this case it is expected that each competitive equilibrium lies in the core. Conversely, not all of the allocations in the core can be transmuted by an appropriate price system into a competitive equilibrium.

The examination of the core is not the most important factor when proving the existence of competitive equilibria. But if it can be shown that the core is empty, then we can stop looking for an (exact) competitive equilibrium.

In the relevant literature on this point there are different approaches dealing with the existence of competitive equilibria.[2] The way chosen here concentrates on applying the fixed-point theorems of BROUWER and of KAKUTANI. However, DEBREU (1982) basically distinguishes between three ways.

(1) The simultaneous optimization approach examines a social system, where the m participants are collected in the set A.[3] Each agent $a \in A$ is associated with a set B_a^* of *a priori* admissible actions $\mathbf{x}_a$. If $m - 1$ agents have made their decisions, then $B_a \subset B_a^*$ indicates the resulting (nonempty) set of feasible actions. In this case it turns out to be more favorable to examine the set $B_a(\mathfrak{x})$ with $\mathfrak{x} := (\mathbf{x}_a)_{a \in A}$ instead of $B_a(\mathfrak{x}^{-a})$ with $\mathfrak{x}^{-a} := (\mathbf{x}_k)_{\substack{k \in A \\ k \neq a}}$. Agent a chooses a best element over $B_a(\mathfrak{x})$ with respect to his preferences which are represented by a utility function $u_a(\mathfrak{x}) \equiv u_a(\mathbf{x}_a, \mathfrak{x}^{-a})$. The set of best elements for agent a is denoted by $D_a(\mathfrak{x}) \subset B_a(\mathfrak{x})$. If each agent receives a best element $\mathbf{x}_a^\circ \in D_a(\mathfrak{x}^\circ)$, then the allocation $\mathfrak{x}^\circ = (\mathbf{x}_a^\circ)_{a \in A} \in \times_{a \in A} B_a^*$ is an equilibrium since no agent can improve his situation by choosing an alternative action. There is an equilibrium for the social system $(B_a^*, u_a, B_a)_{a \in A}$ if the mapping $D := \times_{a \in A} D_a$ has a fixed-point, i.e.

$$\mathfrak{x}^\circ \in D(\mathfrak{x}^\circ) = \underset{a \in A}{\times} D_a(\mathfrak{x}^\circ).$$

(2) Concentrating – as in the presented analysis of exchange economies – on the excess demand Z of the aggregate economy, then the question of the existence of an equilibrium can be reformulated as follows: each person owns an initial endowment which is valuated at alternative price vectors $\mathbf{p}$. Depending on the initial endowment and the respective price vector a set of commodity bundles results that the person concerned can buy. The person can choose an arbitrary commodity bundle within this budget set. He will only choose best commodity bundles with respect to his preferences. These utility maximizing commodity bundles generate an individual excess demand correspondence Z_a, which, corresponding to the discourse, depends on the commodity prices, the respective initial endowment, and the individual preferences. With that an equilibrium exists if there is a price vector $\mathbf{p}^\circ$ for the aggregate excess demand $Z := \sum_{a \in A} Z_a$ such that each excess demand vanishes. Thus, for given initial endowments and given individual preferences we have to prove the validity of the following relation:

$$\mathbf{0} \in Z(\mathbf{p}^\circ)$$

(3) In large economies with a very large number of persons, whose individual influence on the total economy can be ignored, a further procedure can be offered which refers to the use of fixed-point theorems. Without going into the instruments of measure theory in more detail the problem can be stated in the following way.[4]

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[2]See DEBREU (1982, p. 697 f.).

[3]A sketch of this approach is presented at the treatment of production economies (Section 3.1). There, the market is interpreted as agent choosing a price vector such that the value of excess demand is maximized.

[4]Cf. DEBREU (1982) and in particular KIRMAN (1981).

The nonempty set A is the set of economic agents. By μ we define an appropriate measure so that each (nonempty) coalition $A_j \subset A$ is associated with a number $\mu(A_j)$, which can be interpreted as a share of A_j of the whole set A.[5] Since the set A consists of an infinite number of persons, it is furthermore required that no coalition A_j consisting of finitely many persons has a positive weight, $\mu(A_j) = 0$. In particular, individual persons must not have any influence. If moreover $\mu(A) = 1$ holds, then μ is called an (atomless) probability measure.

A price vector $\mathbf{p}^\circ$ and the allocation $\mathfrak{x}^\circ = (\mathbf{x}_a^\circ)_{a \in A}$ are called an equilibrium when almost all agents receive a best element and when the "means of supply and demand" are equal. Here "almost all" means that the set of all persons who receive no best element with respect to their preferences, their initial endowment $\mathbf{w}_a$, and the price vector $\mathbf{p}^\circ$ is a set of measure zero. As the initial endowments of all persons determine the aggregate supply, $\sum_{a \in A} \mathbf{x}_a^\circ = \sum_{a \in A} \mathbf{w}_a$ must be valid in a WALRASian equilibrium. For finitely many agents this equivalence of aggregate demand and aggregate supply can be rewritten as

$$\frac{1}{\#A} \sum_{a \in A} \mathbf{x}_a^\circ = \frac{1}{\#A} \sum_{a \in A} \mathbf{w}_a .$$

Similarly, the mean values of aggregate demand and aggregate supply must be equal even for infinitely many persons.[6]

$$\int_A \mathbf{x}^\circ(a) \mathrm{d}\mu = \int_A \mathbf{w}(a) \mathrm{d}\mu$$

Again we seek for properties of the aggregate excess demand Z so that the existence of a fixed-point assures the existence of a price vector $\mathbf{p}^\circ$ at which $\mathbf{0} \in Z(\mathbf{p}^\circ)$ holds.

Further analytical approaches to prove the existence of general equilibria are omitted. However, a more recent direction of theory must not be neglected which has been developed in particular by the influence of SCARF.

SCARF (1967) and SHAPLEY, SCARF (1974) discuss a class of markets for an indivisible good – for instance houses – whose core is not empty. The examined market includes an infinite number of persons, each of whom has one unit of the indivisible good. Assuming that no person can utilize more than one unit of the good at hand and that there is no other exchange, then the only consequence by

[5]For finitely many agents we could use the measure $\mu(A_j) = \#A_j/\#A$ for a coalition $A_j \subset A$.

[6]The transition between the equations can be justified as follows: for an infinite number of persons both sides of the first equation can be estimated by partitioning A into n pairwise disjoint sets A_j with

$$\bigcup_{j=1}^{n} A_j = A \quad \text{and} \quad \sum_{j=1}^{n} \mu(A_j) = \mu(A).$$

Afterwards the approximated sum

$$S := \sum_{j=1}^{n} \mathbf{x}^\circ(a_j)\mu(A_j) \qquad \text{with} \quad a_j \in A_j$$

is calculated for each of these partitions. Intuitively, we can imagine the next step as a continually "refined" partition so that a sequence $\{S^\nu\}$ of approximated sums emerges whose limit S is denoted by $S := \int_A \mathbf{x}^\circ(a)\mathrm{d}\mu$ if it exists at all. Cf. DALLMANN, ELSTER (1991a, pp. 455–457).

market activities is a reallocation of the indivisible commodity units between the persons. Within the framework of game theory[7] it is now shown what assumptions suffice for the existence of a feasible market result that cannot be improved by any coalition of market participants. Moreover, a procedure is described as to how to calculate competitive prices so that the market result lies in the core of the market.

A main extension to the analysis of indivisible goods results by adding a divisible good to the exchange economy. Now the problem is to ascertain a reallocation of indivisible commodity units considering offsets with the divisible good so that no coalition of market participants can improve the result of the reallocation. As shown by QUINZII (1984), we can establish conditions assuring a nonempty core of this exchange economy.[8] Moreover, the assumptions under which the set of all allocations in the core of the exchange economy equals the set of all competitive equilibrium allocations is shown.[9] In this case for each allocation in the core there is a price vector, which jointly form a competitive equilibrium.

Alternative to QUINZII, the approach of SHAPLEY, SCARF is also extended by a divisible good in SVENSSON (1984). As before, every person consumes exactly one unit of the indivisible good. But in contrast to QUINZII it is assumed that each person can own fractions of the indivisible commodity units before trade, i.e. the property rights are supposed to be divisible. Again we can establish conditions so that there is a competitive equilibrium.[10]

The extension of the approach of SHAPLEY and SCARF has been continued with reference to further properties of allocations apart from the proof of the existence of a competitive equilibrium. For example, SVENSSON (1983) and MASKIN (1987) provide conditions so that there is a fair allocation. We speak of a fair allocation if the allocation is PARETO efficient and if no person envies another person his commodity bundle.[11] TADENUMA, THOMSON (1991) examine within the same model the consistency of an allocation. A fair allocation is said to be consistent if each coalition owns resources so that in turn this assignment constitutes a fair allocation within the examined coalition.[12]

If there is a competitive equilibrium, then there is the immediate question as to how the equilibrium prices are determined. Provided fixed-point theorems and theorems which guarantee the existence of equilibrium prices are equivalent to each other, then each numerical procedure for computing equilibrium prices must at the same time be an algorithm for computing fixed-points. A class of efficient algorithms of combinatory nature for determining approximate equilibria is given by SCARF (1982).

The calculation of equilibria is also discussed in KEHOE (1991). As discussed above, economic equilibria are usually solutions to fixed-point problems and not

[7]The examined market is modelled as balanced n-person game.

[8]Cf. QUINZII (1984, p. 44, Theorem 1).

[9]See QUINZII (1984, p. 54, Theorem 3). For existence and calculation of equilibria on the described market, see also KANEKO, YAMAMOTO (1986).

[10]Cf. SVENSSON (1984, p. 380, Theorem 2).

[11]For the concept of absence of envy and alternative approaches to fairness, see VARIAN (1974).

[12]ALKAN, DEMANGLE, GALE (1991) offer further aspects on fair allocations of indivisible goods. In particular, the assumption of an equal number of persons and objects is eliminated.

to problems of the convex programming, so two problems basically occur: on the one hand equilibria are often difficult to calculate and on the other hand there can be more than one equilibrium. These and further aspects are discussed by KEHOE without emphasizing explicitly the indivisibility of goods and factors.

The presented approach concerning the examination of equilibria in exchange economies with indivisible goods corresponds to that of BROOME (1972). For a finite number of both persons and goods it is proved that there is an allocation which must not arbitrarily differ from an exact equilibrium. The proof is founded mainly on the assumption that there is at least one divisible good. The derived allocation must be called a quasi-equilibrium for two reasons. (1) The allocation is only approximately feasible, i.e. the aggregate demand may exceed the total endowment of the economy even if not arbitrarily. (2) Not all agents prefer the allocated commodity bundle to all the others the person concerned could buy. Both errors may be ignored the more persons there are in the economy.

An alternative approach to the description of an exchange economy with exclusively indivisible goods is presented in DIERKER (1971) with respect to HENRY (1970). As before, it is shown that there is a price vector and a corresponding allocation such that the budget constraint is satisfied for each person. If each person chooses an optimal commodity bundle, then the aggregate excess demand of the derived *planned-price allocation* is bounded. Again asymptotic properties can be established for an increasing number of persons. The difference with BROOME (1972) is that we treat the examination without any divisible good within a modified analytical framework.

Finally, we draw attention to a theory investigating the existence of equilibria with reference to measure theory. The previous approaches of SCARF (1967), DIERKER (1971) and BROOME (1972) assume finitely many persons and finitely many goods. According to OSTROY[13] these so called *small-square economies* are opposite to *large-square economies* with an infinite number of persons and an infinite number of goods.[14] The idealized representation by infinitely many persons as continuum is based on AUMANN (1964). He proves conditions with respect to an exchange economy with a continuum of persons and (finitely many) divisible goods so that the core of the economy equals the set of equilibria. AUMANN (1966) gives conditions assuring a nonempty core. The transference of these outcomes to exchange economies with a continuum of persons concerning indivisible goods was made by MAS-COLELL.

MAS-COLELL (1975) distinguishes between two classes of goods. Apart from at least one homogeneous good he examines (infinitely many) differentiated goods which can only be consumed at integer units. Each differentiated good can be specified by characteristics. The infinitely many consumers possess continuous preferences with respect to these characteristics.[15] As before, AUMANN and MAS-COLELL show when the core of the examined exchange economy equals the set

[13] OSTROY (1984) investigates the existence of WALRASian equilibria in *large-square economies*.
[14] Cf. also ALIPRANTIS (1995, Chapter 3).
[15] The idea of identifying goods with their characteristics has already been picked up in ROSEN (1974).

of equilibria and furthermore when it is not empty. The explicit consideration of (finitely many) indivisible goods besides at least one divisible good may be found in MAS-COLELL (1977). Apart from the existence of equilibria (Theorem 1) in an exchange economy with a continuum of persons the attention is now directed to the determinateness of equilibria (Theorem 3). The author himself describes his result as follows: *"[...] i.e., the existence of a dense set of economies having a finite number of equilibria each one of which is 'stable' (i.e., not very sensitive) under perturbations of the economy."*[16]

KHAN, YAMAZAKI (1981) also examine exchange economies with a continuum of persons and finitely many indivisible goods beside a divisible good. Their assumptions establish a nonempty core of the underlying exchange economy, but they do not necessarily assure the existence of a competitive equilibrium.

DUNZ (1992) deals with an approach following closely MAS-COLELL (1975). However, the assumption of the existence of a homogeneous good is eliminated. The model with an infinite number of indivisible goods and an infinite number of consumers supposes that each person has one indivisible good, and preferences with respect to the indivisible goods. Like SHAPLEY, SCARF (1974) it is supposed that each person consumes exactly one indivisible good so that DUNZ's approach is so to speak the "*large-square*" version of SHAPLEY and SCARF. Although DUNZ is not able to prove an equilibrium distribution, he presents a result on the existence of a quasi-equilibrium distribution which is not basically weaker. An equilibrium distribution is characterized by three criteria. (1) The households' share of the total population that is endowed with a certain type of house before trade equals the share of households which want to own this type of house. (2) The households' share of the total population, living in a certain subregion, corresponds to the share of this region to the total region. (3) All but the most households obtain the most preferred house they can buy at the given price function. As opposed to an equilibrium distribution, the third point is weakened in a quasi-equilibrium as follows: everybody receives a house so that there is no preferred house that costs less.

ROSEN (1974), like MAS-COLELL (1975), also examines a market for a large number of differentiated goods but without referring to measure theory. The model of a production economy describes each good by finitely many characteristics, where these attributes determine an implicit or hedonistic price of the good. Moreover, each person possesses a preference ordering over the shape of the features of all goods. The indivisibility of goods is overcome by the assumption that there are sufficiently many goods so that the choice between different shapes of the characteristics can be varied continuously. Correspondingly, it is supposed that production technology allows a supply with continuously variable properties of the goods. Thus, the market for differentiated goods is in fact transformed into a market for "perfectly divisible characteristics", for which the question of the existence of a competitive equilibrium can be answered. To calculate the competitive equilibrium we can refer for example to a simulation model in LUK (1993). This model is based

[16]MAS-COLELL (1977, p. 444).

on a class of fixed-point algorithms in SCARF (1982) and supposes a continuum of persons.

1.2 Graphical Representation of Simple Exchange Economies

By an exchange economy we understand a set A of persons each holding an initial endowment $\mathbf{w}_a = (w_{1a}, \ldots, w_{na})^\mathsf{T}$ $(a \in A)$ exchangeable for other amounts of goods. Each individual agrees to an exchange of goods provided this exchange improves its utility, i.e. the person at hand prefers the new commodity bundle $\mathbf{x}_a = (x_{1a}, \ldots, x_{na})^\mathsf{T}$ to his initial endowment $\mathbf{w}_a$.
In order that trade takes place, the individual exchange plans of all agents must be compatible with each other. Commodities supplied by one individual must be chosen by the other one. Both agents must agree to the relation at which they exchange their goods.

Without defining the used concepts accurately at this point the problem to be examined can be illustrated graphically (Figure IV.1) for two persons and two goods by an **EDGEWORTH box**. Both persons own an initial endowment $\mathbf{w}_1 = (w_{11}, w_{21})^\mathsf{T}$ and $\mathbf{w}_2 = (w_{12}, w_{22})^\mathsf{T}$ respectively, which sum up to the total endowment $\mathbf{w}_A = (w_{1A}, w_{2A})^\mathsf{T}$ of the economy (initial point A^1). The first person – centered at the origin O^1 – prefers all commodity bundles lying in the **preference set** above the **indifference curve** I^1. Similarly, the second person – centered at

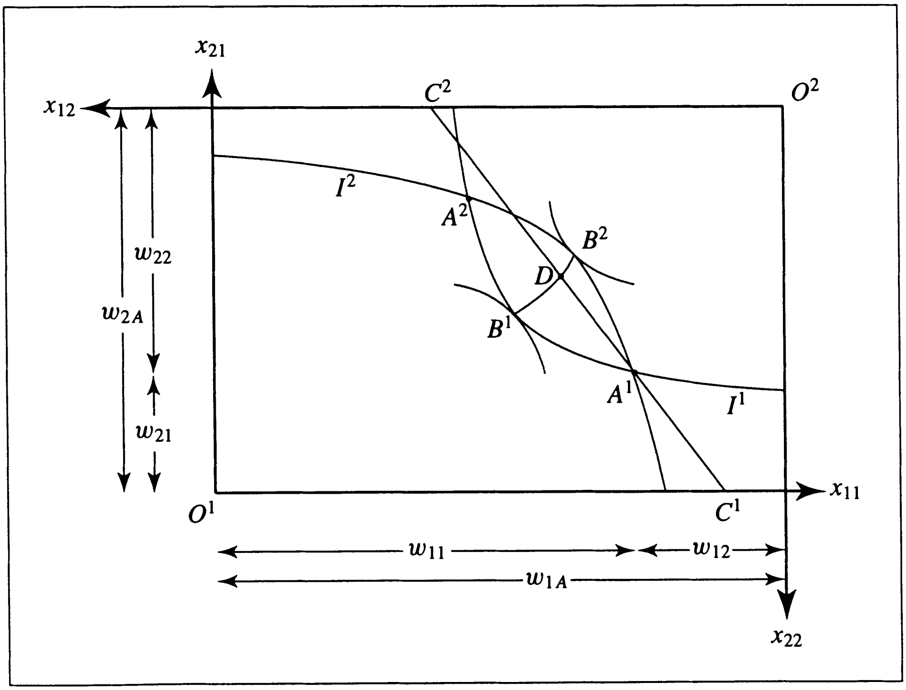

Figure IV.1: EDGEWORTH box

the origin O^2 – prefers all commodity bundles "above" the indifference curve I^2. Consequently, an exchange of goods results in a higher utility level for both persons, provided the new **allocation** lies within the lens (A^1, A^2). Each exchange within this **exchange lens** improves the welfare of both consumers, until the indifference curves of both persons are eventually tangent. After that there is no further exchange possibility improving the situation of one individual without worsening the situation of the other one. Allocations with this property are said to be **PARETO efficient**. The **core** of the examined economy is described by the locus (B^1, B^2) and indicates the set of all PARETO efficient points which are feasible from the initial situation A^1. Point D will be realized if both consumers agree on an exchange ratio corresponding to the line (C^1, C^2). Here, nothing is said about the slope of the line (C^1, C^2) and the slopes of the indifference curve at point D. In particular, for a small number of market participants the realized exchange ratio depends not least on the market power of each agent.

What is the "optimal" allocation resulting from the initial endowments and the preferences of both actors? To answer this question, it is worthwhile to dispense with the pure exchange of goods and to introduce a third agent – the market. The market "chooses" a commodity price vector $\mathbf{p} = (p_1, p_2)^\mathsf{T}$ maximizing the value of the excess demand $\mathbf{p}^\mathsf{T}\mathbf{z} = \mathbf{p}^\mathsf{T}(\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{w}_1 - \mathbf{w}_2)$. At the same time the relative price p_1/p_2 determines the exchange ratio x_2/x_1. A possible relative price is represented by the line (C^1, C^2).

The simultaneous optimization approach regarding these three agents can now be described as follows, presupposing the preferences of both consumers can be expressed as a utility function u_a $(a = 1, 2)$ respectively. Both consumers maximize their utility by valuating their initial endowment at market prices $\mathbf{p}$ and choosing the "best" commodity bundle they can buy.

$$\max \left\{u_a(\mathbf{x}_a) \mid \mathbf{p}^\mathsf{T}\mathbf{x}_a \leqq \mathbf{p}^\mathsf{T}\mathbf{w}_a\right\} \qquad a = 1, 2$$

These individual choices are made without knowing the demand of the other person or considering his wishes. Thus, we speak of decentralized and noncooperative decisions. Both agents behave as price takers and quantity adjusters and react only to the market instead of affecting each other as in the direct exchange of goods. The market simultaneously chooses a price vector $\mathbf{p}$ out of the price simplex Δ such that the value of the excess demand is maximized.

$$\max \left\{\mathbf{p}^\mathsf{T}\mathbf{z} \mid \mathbf{p} \in \Delta\right\}$$

In this sense the market behaves as price adjuster by taking the excess demand $\mathbf{z} = \mathbf{x}_1 + \mathbf{x}_2 - \mathbf{w}_A$ of both consumers as given.

Suppose the triple $(\mathbf{x}_1^\circ, \mathbf{x}_2^\circ, \mathbf{p}^\circ)$ denotes a simultaneous solution to the above three problems; that is, the households choose one of the utility maximizing commodity bundles $\mathbf{x}_1^\circ$ or $\mathbf{x}_2^\circ$ at given market prices $\mathbf{p}^\circ$ and the market chooses an optimal price vector $\mathbf{p}^\circ \in \Delta$ at the given vector $\mathbf{z}^\circ = \mathbf{x}_1^\circ + \mathbf{x}_2^\circ - \mathbf{w}_A$. As both households are restricted to their budget constraint $\mathbf{p}^{\circ\mathsf{T}}(\mathbf{x}_a^\circ - \mathbf{w}_a) \leqq 0$, it follows $\mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ = \mathbf{p}^{\circ\mathsf{T}}(\mathbf{x}_1^\circ - \mathbf{x}_2^\circ - \mathbf{w}_A) \leqq 0$. If the market maximizes the value of the excess demand, then $\mathbf{p}^T\mathbf{z}^\circ \leqq 0$ for all $\mathbf{p} \in \Delta$ and, therefore, $\mathbf{z}^\circ \leqq \mathbf{0}$. Hence, the state $(\mathbf{x}_1^\circ, \mathbf{x}_2^\circ, \mathbf{p}^\circ)$ is feasible because of $\mathbf{x}_1^\circ + \mathbf{x}_2^\circ \leqq \mathbf{w}_A$. If all households satisfy their budget constraints for appropriate assumptions on their preferences, then $\mathbf{p}^{\circ T}\mathbf{z}^\circ = 0$ results (WALRAS' law). Thus, $(\mathbf{x}_1^\circ, \mathbf{x}_2^\circ, \mathbf{p}^\circ)$ denotes a **WALRASian equilibrium.**

In Figure IV.2 point D marks a possible candidate for a WALRASian equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ at which the corresponding WALRASian allocation $\mathfrak{x}^\circ = (\mathbf{x}_1^\circ, \mathbf{x}_2^\circ)$ is feasible since $\mathbf{x}_1^\circ + \mathbf{x}_2^\circ = \mathbf{w}_A$. At the same time both markets are cleared, $\mathbf{z}^\circ = \mathbf{0}$. Because the price ratio p_1° / p_2° determines a budget line (C^1, C^2) containing points A^1 and D, the budgets of both households are balanced and the value of the excess demand satisfies $\mathbf{p}^{\circ T}\mathbf{z}^\circ = 0$. Point D therefore yields a WALRASian equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ if the allocation $\mathfrak{x}^\circ$ assigns utility maximizing commodity bundles to both persons at given commodity prices $\mathbf{p}^\circ$.

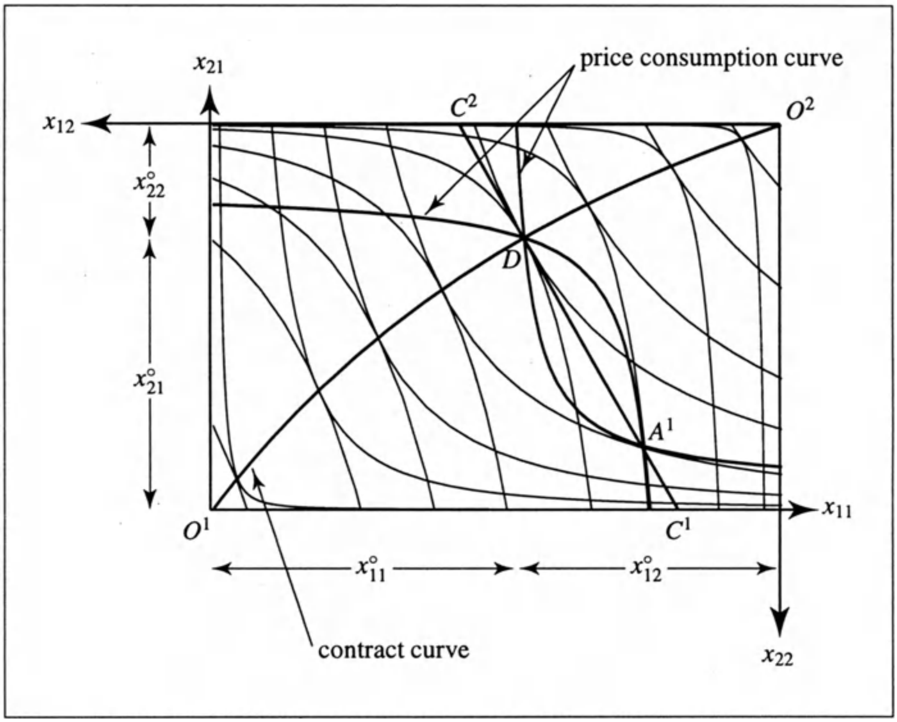

Figure IV.2: Exchange equilibrium

The last step can be explained by Figure IV.2[17], showing the same situation as Figure IV.1. If the budget line (C^1, C^2) corresponding to point D separates the preference sets of both persons,[18] then each commodity bundle preferred to $\mathbf{x}_1^\circ$ or $\mathbf{x}_2^\circ$

[17]The results have been derived on the basis of two COBB-DOUGLAS utility functions with the assistance of *MATHEMATICA*®. Further remarks on using *MATHEMATICA*® may be taken from NOGUCHI (1993) and BOBZIN, BUHR, CHRISTIAANS (1995).

[18]Given the differentiability of the utility functions, the price ratio p_1° / p_2° corresponds to the slopes of both indifference curves or the marginal rate of substitution for both persons.

costs more than the respective person can afford at the given initial endowment and the given commodity prices. Thus, each commodity bundle $\mathbf{x}_1^\circ$ and $\mathbf{x}_2^\circ$ maximizes utility.

At the same time the budget line (C^1, C^2), separating the corresponding preference sets of both persons, implies that the adjoined indifference curves are tangent to each other at point D. Thus, the allocation $(\mathbf{x}_1^\circ, \mathbf{x}_2^\circ)$ is PARETO efficient. All other points with this property are collected in **contract curve** (O^1, O^2) which is marked in bold. As a result Figure IV.2 suggests that WALRASian allocations are PARETO efficient and, moreover, that they lie in the core. This supposition is inspected and proved in Section 3.2.1.

An alternative representation of the WALRASian equilibrium follows from determining a common point of the two price consumption curves[19] (drawn in bold) through the initial point A^1. The price consumption curves indicate the locus of all commodity bundles demanded at alternative price ratios. Thus, the price consumption curves generate an aggregate excess demand $\mathbf{z}(\mathbf{p})$ so that all markets are cleared at the point of intersection, $\mathbf{z}(\mathbf{p}^\circ) = \mathbf{0}$ – i.e. at the WALRASian equilibrium. This aspect will be the central point of the following analysis.

The question as to what conditions are sufficient for the existence of a WALRASian equilibrium is illuminated under consideration of the fixed-point theorems of BROUWER and of KAKUTANI. Here a positive answer depends basically on the assumption of convex preference sets. As shown by Figure IV.3, we can construct examples for the case of nonconvex preference sets such that there is no price vector at which the corresponding budget line separates the preference sets.

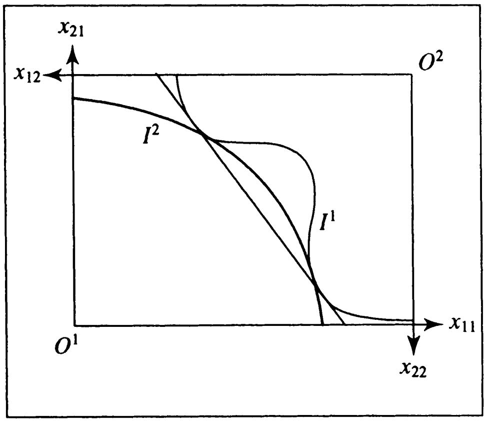

Figure IV.3: Nonconvex preference sets

Before going into the determination of a quasi-equilibrium with nonconvex preference sets, the problem of indivisible goods can be picked up. The subsequent Figure IV.4 illustrates the case of a divisible good 1 and an indivisible good 2. If the second good can appear only at integer units, then the total endowment w_{A2} of this good must be an integer. As before the initial endowments of both persons correspond to point A. It is not required that the initial endowments w_{21} and w_{22} must be integer for this point. Each person may initially own fractions of a good that can be consumed only at integer units.

Now the preference sets of both persons consist only of commodity bundles satisfying the integer constraints. The bold lines indicate two preference sets. The lower preference set contains all commodity bundles which the second person thinks to be not worse than $\mathbf{x}_2^d$. To get an optically better idea, the convex hull of this preference set is also marked. The first person's marked preference set includes all

[19]Cf. VARIAN (1992, p. 317).

commodity bundles which are not worse than $\hat{\mathbf{x}}_1^d$. As shown by the corresponding convex hull, not all points of this convex hull, satisfying at the same time the integer constraints, are elements of the preference set; see point $\mathbf{x}_2^d$ and the section to the right hand of this point. The diagonal (C^1, C^2) determines not only an exchange ratio or a relative price p_1° / p_2° but also at the same time separates the convex hulls from the above mentioned preference sets.

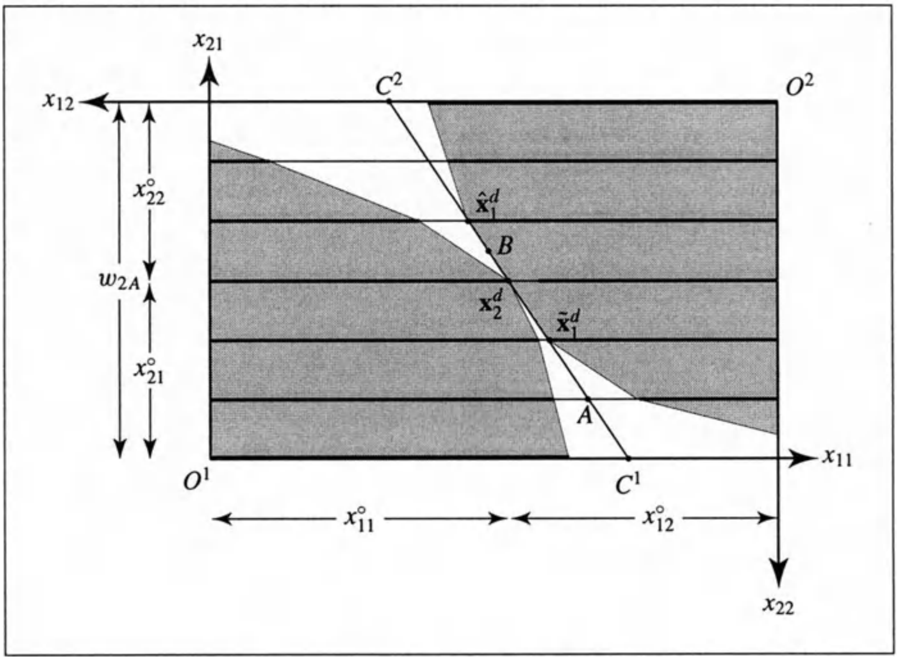

Figure IV.4: Nonconvex preferences with an indivisible good

The given price ratio implies two demand sets – $\{\hat{\mathbf{x}}_1^d, \tilde{\mathbf{x}}_1^d\}$ for the first person and $\{\mathbf{x}_2^d\}$ for the second person – which themselves do not allow any compatible exchange plan. Neither $\hat{\mathbf{x}}_1^d - \mathbf{w}_1$ nor $\tilde{\mathbf{x}}_1^d - \mathbf{w}_1$ equals $-(\mathbf{x}_2^d - \mathbf{w}_2)$ so that the price vector $\mathbf{p}^\circ$ cannot provide a WALRASian equilibrium. However, for the marked exchange ratio there is an allocation $(\mathbf{x}_1^\circ, \mathbf{x}_2^\circ) = (\mathbf{x}_1^\circ, \mathbf{x}_2^d)$ having the following properties:

- Both persons utilize their whole budgets.

$$p_1^\circ x_{11}^\circ + p_2^\circ x_{21}^\circ = p_1^\circ w_{11} + p_2^\circ w_{21} \quad \text{and} \quad p_1^\circ x_{12}^\circ + p_2^\circ x_{22}^\circ = p_1^\circ w_{12} + p_2^\circ w_{22}$$

- The allocation $\mathbf{x}^\circ = (\mathbf{x}_1^\circ, \mathbf{x}_2^\circ)$ is feasible since $\mathbf{x}_1^\circ + \mathbf{x}_2^\circ = \mathbf{w}_1 + \mathbf{w}_2$.

- The two commodity bundles $\mathbf{x}_1^\circ$ and $\mathbf{x}_2^\circ$ are contained in the convex hull of the demand set.

$$\mathbf{x}_1^\circ \in \operatorname{conv}\{\hat{\mathbf{x}}_1^d, \tilde{\mathbf{x}}_1^d\} = \{\lambda \hat{\mathbf{x}}_1^d + (1-\lambda)\tilde{\mathbf{x}}_1^d \mid \lambda \in [0,1]\} \quad \text{and} \quad \mathbf{x}_2^\circ \in \operatorname{conv}\{\mathbf{x}_2^d\}$$

- At least one of the two persons receives a commodity bundle he demands. For the given price ratio p_1° / p_2° the commodity bundle $\mathbf{x}_2^\circ = \mathbf{x}_2^d$ maximizes the utility of the second person. In other words, $\mathbf{x}_2^\circ$ denotes an element in the demand set, $\mathbf{x}_2^\circ \in \{\mathbf{x}_2^d\}$.
  However, the first person $\mathbf{x}_1^\circ$ does *not* demand, $\mathbf{x}_1^\circ \notin \{\hat{\mathbf{x}}_1^d, \tilde{\mathbf{x}}_1^d\}$. Although this commodity bundle belongs to the convex hull of the preference set with respect to $\hat{\mathbf{x}}_1^d$.

Starting with the marked exchange ratio and with an allocation $\mathfrak{x}^\circ = (\mathbf{x}_1^\circ, \mathbf{x}_2^\circ)$, the question is now asked as to what shortcoming results from $(\mathbf{x}_1^\circ, \mathbf{x}_2^\circ)$ and the allocation demanded – for instance, $(\hat{\mathbf{x}}_1^d, \mathbf{x}_2^d)$ – when a person does not demand the allocated commodity bundle. If one person at the most receives a commodity bundle he does not demand, then the distance d between the points C^1 and C^2 (with appropriate preferences) gives an upper bound for this shortcoming, $\|\mathbf{x}_1^\circ - \mathbf{x}_1^d\| < d(C^1, C^2)$.

Furthermore, we can ask whether there is an allocation $\mathfrak{x}^{\circ\circ} = (\mathbf{x}_1^{\circ\circ}, \mathbf{x}_2^{\circ\circ})$ alternative to $\mathfrak{x}^\circ = (\mathbf{x}_1^\circ, \mathbf{x}_2^\circ)$ so that the burden, like $\mathbf{x}_1^\circ - \hat{\mathbf{x}}_1^d$, is distributed "equally" among all persons and not borne by a particular person. Point B illustrates such an allocation. Neither of the two persons is more remote from his real demand than $\|\mathbf{x}_1^\circ - \hat{\mathbf{x}}_1^d\|/2$. For a sufficient large number of persons it turns out that this distance becomes negligibly small.

As suggested by the graphical introduction, the following Sections 2.1 to 2.3 are subdivided hierarchically with respect to the convexity of preferences.

(1) The case of strictly convex preferences conforms to VARIAN (1992). No indivisible goods can be included, but the more familiar dealing with demand *functions* instead of demand *correspondences* enables an easier lead-in to dealing with fixed-point theorems. A commodity price vector $\mathbf{p}^\circ$ is required so that the resulting aggregate excess demand vector vanishes, $\mathbf{z}(\mathbf{p}^\circ) = \mathbf{0}$. In view of the equilibrium price vector the WALRASian equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ is completely determined.

(2) For convex preferences we present the dealings with demand correspondences without considering the difficulties caused by indivisible goods. The procedure follows HILDENBRAND, KIRMAN (1988). The existence proof is given first of all by a procedure frequently used in economic theory, where the commodity space is truncated from above. The proof is completed when a price vector $\mathbf{p}^\circ$ is found which permits all markets to be cleared, $\mathbf{p}^\circ \in Z(\mathbf{p}^\circ)$. Thus, a WALRASian equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ exists.
Afterwards an alternative proof is presented as it is more suitable for the treatment of indivisible goods. This procedure is taken from HEUSER (1992) and does not need the restriction of the commodity space.

(3) In the case of indivisible goods no exact equilibria can be proved. However, there are pairs $(\mathfrak{x}, \mathbf{p})$ which do not arbitrarily differ from an exact WALRASian equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$. The ROTHENBERG equilibrium presented in Proposition IV.10, p. 233, and the ε-equilibria illuminate the extent of deviation from an exact equilibrium under various aspects.[20]

1.3 The Problem of Unbounded Demand for Goods

1.3.1 Convex Preference Orderings

In view of using the fixed-point theorems (Corollary C.15.1 and Theorem C.17) the results of Section 3.2.2(b) show that the aggregate excess demand correspondence $Z\colon \Delta \to \mathfrak{P}(\mathbb{R}^n)$ does not satisfy the required conditions.

Assuming convex preference orderings with divisible goods, $Z(\mathbf{p})$ is convex (Proposition III.24), but for continuous and strongly monotone preference orderings we have $Z(\mathbf{p}) = \emptyset$ for all price vectors $\mathbf{p} \in \partial\Delta$ containing a zero component; see Proposition III.28. At the same time Corollary III.29.1 implies that the correspondence Z cannot be bounded.

There are consequently two possibilities to apply Theorem C.17 (DEBREU, GALE, NIKAIDO), p. 314. Either the convex closed subset S to be chosen from the price simplex Δ may not contain price vectors with zero components,[21] i.e. $S \cap \partial\Delta = \emptyset$. Or we at once put $S = \Delta$ and modify the aggregate excess demand correspondence Z afterwards so that it holds the premises of Theorem C.17.

Usually, economic literature on this subject suggests the second way.[22] As a result the correspondence Z has to be modified so that the level sets of the new correspondence Z^s are nonempty, i.e. $Z^s(\mathbf{p}) \neq \emptyset$ for all $\mathbf{p} \in \Delta$. Moreover, we have to assure that the new correspondence Z^s maps into a compact set.

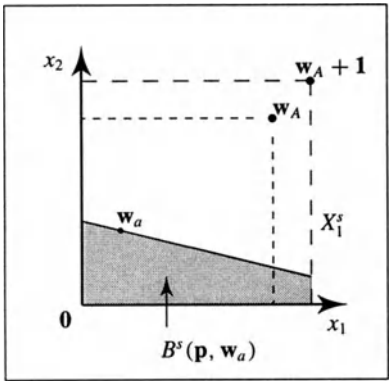

Figure IV.5: The restricted commodity space X

In accordance with Figure IV.5 the procedure starts with a truncation of the commodity space $X = \mathbb{R}^n_+$.

$$X_1^s := \{\mathbf{x} \in X \mid \mathbf{0} \leqq \mathbf{x} \leqq \mathbf{w}_A + \mathbf{1}\}.$$

If each person is only allowed to choose commodity bundles within the set X_1^s, then he can demand more of a good than is available in the economy. But corresponding to Figure IV.5 the level of excess demand is bounded for each person, $\mathbf{x}_a^d(\mathbf{p}, \mathbf{w}_a) - \mathbf{w}_A \leqq \mathbf{1}$.

For finitely many persons included in the economy this restricts the aggregate excess demand to $\#A \cdot \mathbf{1}$. As long as the level of the excess demand is of minor consequence, the previous analysis remains the same particularly as the case of a WALRASian equilibrium excludes the excess demand for a good.

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[20] The presented results follow mainly from the analysis of nonconvex preferences.

[21] This method is presented in Section 2.2.

[22] Keep in mind that the transition of the correspondence Z to a correspondence Z^s may not destroy the economic meaning of an aggregate excess demand correspondence.

In view of the introduced truncation

$$\begin{aligned} B^s(\mathbf{p}, \mathbf{w}_a) &:= B(\mathbf{p}, \mathbf{w}_a) \cap X_1^s \\ &= G(\mathbf{p}, \mathbf{w}_a) \cap X_1^s \qquad \text{(because of } X_1^s \subset X) \end{aligned}$$

person a's demand set at prices $\mathbf{p}$

$$D_a(\mathbf{p}, \mathbf{w}_a) = \{\mathbf{x} | \mathbf{x} \in B(\mathbf{p}, \mathbf{w}_a), \ \mathbf{x} \succcurlyeq_a \tilde{\mathbf{x}} \ \forall \tilde{\mathbf{x}} \in B(\mathbf{p}, \mathbf{w}_a)\}$$

has to be modified to

$$D_a^s(\mathbf{p}, \mathbf{w}_a) := \left\{\mathbf{x} | \mathbf{x} \in B^s(\mathbf{p}, \mathbf{w}_a), \ \mathbf{x} \succcurlyeq_a \tilde{\mathbf{x}} \ \forall \tilde{\mathbf{x}} \in B^s(\mathbf{p}, \mathbf{w}_a)\right\}.$$

Neither sets differ as long as $D_a(\mathbf{p}, \mathbf{w}_a)$ is already contained in X_1^s. Moreover, the newly defined correspondence $D_a^s(\cdot, \mathbf{w}_a)$ is homogeneous of degree 0 in prices. Furthermore, $D_a^s(\cdot, \mathbf{w}_a)$ has the following properties for all price vectors in the price simplex Δ.

Proposition IV.1 *The synthetic sets $D_a^s(\mathbf{p}, \mathbf{w}_a)$ are nonempty and compact for all price vectors $\mathbf{p} \in \Delta$ provided $\succcurlyeq_a$ is a continuous preference ordering.*

Proof: The set $G(\mathbf{p}, \mathbf{w}_a)$ is closed for every price vector $\mathbf{p} \in \Delta$. Furthermore, X_1^s is compact. Thus, the synthetic budget set $B^s(\mathbf{p}, \mathbf{w}_a) = G(\mathbf{p}, \mathbf{w}_a) \cap X_1^s$ is compact and not empty as it contains the origin $\mathbf{x} = \mathbf{0}$. Proposition III.22 now means that the set of best elements $D_a^s(\mathbf{p}, \mathbf{w}_a)$ in the set $B^s(\mathbf{p}, \mathbf{w}_a)$ is not empty and compact. ■

In particular, it follows

Proposition IV.2 *Provided the preference orderings are continuous and convex, then the synthetic demand correspondence $D_a^s(\cdot, \mathbf{w}_a) : \Delta \to \mathfrak{P}(X_1^s)$ is upper semi-continuous in the price simplex Δ if Assumption 2 ($\mathbf{w}_a > \mathbf{0}$) holds.*

Proof: By definition the correspondence $D_a^s(\cdot, \mathbf{w}_a)$ is upper semi-continuous in Δ if it is upper semi-continuous at every point $\mathbf{p} \in \Delta$ *and* if $D_a^s(\mathbf{p}, \mathbf{w}_a)$ is compact for all $\mathbf{p} \in \Delta$. Since the criterion of compactness is already fulfilled by Proposition IV.1, we have only to prove the upper semi-continuity at each $\mathbf{p} \in \Delta$.

As X_1^s is compact, by Theorem C.10, p. 309, it suffices to show that

$$\left[\mathbf{p}^\nu \to \mathbf{p}^0, \ \mathbf{x}^\nu \to \mathbf{x}^0, \ \mathbf{x}^\nu \in D_a^s(\mathbf{p}^\nu, \mathbf{w}_a)\right] \implies \mathbf{x}^0 \in D_a^s(\mathbf{p}^0, \mathbf{w}_a).$$

is valid for every sequence $\{\mathbf{p}^\nu\}$ of price vectors in Δ and for every sequence $\{\mathbf{x}^\nu\}$ of commodity bundles in X_1^s. Because of $\mathbf{x}^\nu \in D_a^s(\mathbf{p}^\nu, \mathbf{w}_a)$ both $\mathbf{x}^\nu \in G(\mathbf{p}^\nu, \mathbf{w}_a)$ and $\mathbf{x}^\nu \in X_1^s$ must hold since $D_a^s(\mathbf{p}, \mathbf{w}_a)$ can be expressed as

$$D_a^s(\mathbf{p}, \mathbf{w}_a) = \bigcap_{\tilde{\mathbf{x}} \in B^s(\mathbf{p}, \mathbf{w}_a)} G(\mathbf{p}, \mathbf{w}_a) \cap X_1^s \cap \mathcal{P}_a(\tilde{\mathbf{x}}).$$

For the closed set X_1^s we obtain $\mathbf{x}^\nu \to \mathbf{x}^0 \in X_1^s$. All that is left to show has been proved by Proposition III.25 in which $G(\mathbf{p}, \mathbf{w}_a)$ must be replaced with $B^s(\mathbf{p}, \mathbf{w}_a)$. Note that the condition $\mathbf{p}^{0\mathsf{T}}\mathbf{w}_a > 0$ (Proposition III.25) holds under Assumption 2 for all price vectors $\mathbf{p} \in \Delta$. ■

Again the actual properties of the new individual demand correspondences immediately transfer to the synthetic aggregate demand correspondence $\widehat{D}^s$ with

$$\widehat{D}^s(\mathbf{p}) := \sum_{a \in A} D_a^s(\mathbf{p}, \mathbf{w}_a)$$

and the synthetic aggregate excess demand correspondence Z^s with

$$Z^s(\mathbf{p}) := \widehat{D}^s(\mathbf{p}) - \mathbf{w}_A .$$

Summarizing with respect to Theorem C.17 (DEBREU, GALE, NIKAIDO) we have the following:

Proposition IV.3 *If each person $a \in A$ has a positive initial endowment $\mathbf{w}_a > \mathbf{0}$ (Assumption 2) and a continuous convex preference ordering, then the synthetic aggregate excess demand correspondence Z^s has the following properties:*

1. *$Z^s(\mathbf{p})$ is not empty and convex for all $\mathbf{p} \in \Delta$;*
2. *Z^s is upper semi-continuous and thus closed in the entire price simplex Δ;*
3. *Z^s is bounded, i.e. $Z^s(\mathbf{p}) \subset X_3^s := \{\mathbf{z} \mid -\mathbf{w}_A \leqq \mathbf{z} \leqq \#A \cdot \mathbf{1}\}$;*
4. *For every $\mathbf{p} \in \Delta$ and every $\mathbf{z} \in Z^s(\mathbf{p})$ we have $\mathbf{p}^\mathsf{T}\mathbf{z} \leqq 0$.*

Proof: The first property results from Propositions IV.1 and III.24. Regarding Theorem C.7, p. 309, Proposition IV.2 implies the second property. The third property ensues from the definitions of the synthetic correspondences $\widehat{D}^s$ and Z^s, i.e.

$$\begin{aligned} D_a^s(\cdot, \mathbf{w}_a)&: \Delta \to \mathfrak{P}(X_1^s) \quad \text{with} \quad X_1^s = \{\mathbf{x} \mid \mathbf{0} \leqq \mathbf{x} \leqq \mathbf{w}_A + \mathbf{1}\}, \\ \widehat{D}^s&: \Delta \to \mathfrak{P}(X_2^s) \quad \text{with} \quad X_2^s := \{\widehat{\mathbf{x}} \mid \mathbf{0} \leqq \widehat{\mathbf{x}} \leqq \mathbf{w}_A + \#A \cdot \mathbf{1}\}, \\ Z^s&: \Delta \to \mathfrak{P}(X_3^s) \quad \text{with} \quad X_3^s := \{\mathbf{z} \mid -\mathbf{w}_A \leqq \mathbf{z} \leqq \#A \cdot \mathbf{1}\}. \end{aligned}$$

Take into consideration that the set X_3^s is compact. Since each person is restricted to his budget constraint $\mathbf{p}^\mathsf{T}\mathbf{x}_a \leqq \mathbf{p}^\mathsf{T}\mathbf{w}_a$ for all price vectors $\mathbf{p} \in \Delta$ the fourth property holds for the aggregate economy. ■

1.3.2 Consideration of Indivisible Commodities

Allowing for indivisible goods with the commodity space $X = \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}$ or nonconvex preference orderings, the aggregate excess demand correspondence Z or the synthetic correspondence Z^s will usually not be convex-valued as required by Theorem C.17 (DEBREU, GALE, NIKAIDO).

BROOME (1972) gets over this problem mainly by superseding the preference sets $\mathcal{P}_a(\mathbf{x})$ by their convex hulls. Without repeating the procedure exactly,

$$\tilde{D}_a^s(\mathbf{p}, \mathbf{w}_a) = \bigcap_{\tilde{\mathbf{x}} \in B^s(\mathbf{p}, \mathbf{w}_a)} G(\mathbf{p}, \mathbf{w}_a) \cap \text{conv} X_1^s \cap \text{conv} \mathcal{P}_a(\tilde{\mathbf{x}})$$

offers an idea of its operating method. Accordingly, $\tilde{D}_a^s(\mathbf{p}, \mathbf{w}_a)$ results from an intersection of nonempty convex sets, where $B^s(\mathbf{p}, \mathbf{w}_a) = G(\mathbf{p}, \mathbf{w}_a) \cap X_1^s$ guarantees that only admissible commodity bundles $\tilde{\mathbf{x}} \in X_1^s$ are taken into account so that especially $\text{conv}\mathcal{P}_a(\tilde{\mathbf{x}}) \neq \emptyset$ holds.
Afterwards it is shown by a relatively extensive proof that the derived synthetic individual demand correspondences $\tilde{D}_a^s(\cdot, \mathbf{w}_a): \Delta \rightarrow \mathfrak{P}(\mathbb{R}^n)$ are convex-valued and upper semi-continuous in the entire price simplex Δ provided there is at least one divisible good.[23]
Thus, a synthetic aggregate excess demand correspondence $\tilde{Z}^s$ results satisfying the assumptions of the fixed-point theorems.

2 Existence of Competitive Equilibria

2.1 Strictly Convex Preference Orderings

In the case of strictly convex preference orderings the commodity space is $X = \mathbb{R}_+^n$. The existence of an exchange equilibrium is proved when it is possible to show the existence of a positive price vector $\mathbf{p}^\circ > \mathbf{0}$ at which the aggregate excess demand vanishes, $\mathbf{z}(\mathbf{p}^\circ) = \mathbf{0}$. Before this is proved with the help of Corollary C.15.1, p. 312 (applying BROUWER's fixed-point theorem), we have to note the following implications of strictly convex preference orderings.

Proposition IV.4 *Given strictly convex (and continuous) preference orderings, the demand set $D_a(\mathbf{p}, \mathbf{w}_a)$ contains precisely one element $\mathbf{x}_a^d(\mathbf{p}, \mathbf{w}_a)$ for every price vector* $\mathbf{p} > \mathbf{0}$.

Proof: Suppose that $D_a(\mathbf{p}, \mathbf{w}_a)$ includes two diverse commodity bundles $\mathbf{x}^1$ and $\mathbf{x}^2$. Then person a can also buy each commodity bundle $\mathbf{x} = \lambda\mathbf{x}^1 + (1 - \lambda)\mathbf{x}^2$. On the basis of strictly convex preference orderings the person at hand prefers the commodity bundle $\mathbf{x}$ to commodity bundle $\mathbf{x}^1$ so that $\mathbf{x}^1$ cannot be a best element in contradiction to the above assumption. Since $D_a(\mathbf{p}, \mathbf{w}_a)$ is not empty (Proposition III.23), the demand set must consist of one and only one element. ■

The transition from an individual demand *correspondence* $D_a(\cdot, \mathbf{w}_a)$: $\text{rint}\,\Delta \rightarrow \mathfrak{P}(X)$ to an individual demand *function* $\mathbf{x}_a^d(\cdot, \mathbf{w}_a)$: $\text{rint}\,\Delta \rightarrow X$ yields the next

Proposition IV.5 *Given a strictly convex preference ordering of person a, the individual demand function $\mathbf{x}_a^d(\cdot, \mathbf{w}_a)$ is continuous in the relative interior of the*

[23]Cf. BROOME (1972, p. 237, Lemma 4.7).

price simplex rint Δ. *If the preference orderings of all persons $a \in A$ are strictly convex, then the aggregate demand function $\widehat{\mathbf{x}}^d$ and the aggregate excess demand function $\mathbf{z}$ are continuous in* rint Δ *as well.*

Proof: For strictly convex preference orderings the demand set contains one and only one element, $D_a(\mathbf{p}, \mathbf{w}_a) = \{\mathbf{x}_a^d(\mathbf{p}, \mathbf{w}_a)\}$ for each price vector $\mathbf{p}$ (Proposition IV.4). For $D_a(\cdot, \mathbf{w}_a) = \{\mathbf{x}_a^d(\cdot, \mathbf{w}_a)\}$ the definitions of an upper semi-continuous[24] correspondence $D_a(\cdot, \mathbf{w}_a)$ equals the definition of a continuous function $\mathbf{x}_a^d(\cdot, \mathbf{w}_a)$.[25] With that $\widehat{\mathbf{x}}^d$, being the sum of continuous functions, is also continuous in rint Δ. The same argument holds for the aggregate excess demand function $\mathbf{z}$. ■

The restriction of the commodity space to X_1^s guarantees that the synthetic demand correspondences $D_a^s(\cdot, \mathbf{w}_a)$ is upper semi-continuous in the entire price simplex Δ; see Proposition IV.2. Analogous to Proposition IV.5, synthetic individual demand functions $\mathbf{x}_a^{ds}(\cdot, \mathbf{w}_a)\colon \Delta \rightarrow X_1^s$ result, which are continuous in the entire price simplex Δ. This evokes a synthetic aggregate excess demand function $\mathbf{z}^s$, which is continuous in Δ. Applying Corollary C.15.1 now guarantees the existence of a price vector $\mathbf{p}^\circ \in \Delta$ satisfying

$$\text{(IV.1)} \qquad \mathbf{z}^s(\mathbf{p}^\circ) = \sum_{a \in A} \mathbf{x}_a^{ds}(\mathbf{p}^\circ, \mathbf{w}_a) - \mathbf{w}_A \leqq \mathbf{0}.$$

Moreover, this inequality means that no individual person $a \in A$ demands more of a good than is available in the economy, $\mathbf{x}_a^{ds}(\mathbf{p}^\circ, \mathbf{w}_a) \leqq \mathbf{w}_A$. Thus, the additional constraint of a truncated commodity space X_1^s becomes obsolete for price vectors $\mathbf{p}^\circ$ with $\mathbf{z}^s(\mathbf{p}^\circ) \leqq \mathbf{0}$, i.e.[26]

$$\mathbf{z}^s(\mathbf{p}^\circ) \leqq \mathbf{0} \quad \Longrightarrow \quad \mathbf{x}_a^{ds}(\mathbf{p}^\circ, \mathbf{w}_a) = \mathbf{x}_a^d(\mathbf{p}^\circ, \mathbf{w}_a).$$

Since the restriction X_1^s is obsolete for each person, the restriction X_3^s also loses its significance for the total economy

$$\mathbf{z}^s(\mathbf{p}^\circ) \leqq \mathbf{0} \quad \Longrightarrow \quad \mathbf{z}^s(\mathbf{p}^\circ) = \mathbf{z}(\mathbf{p}^\circ)$$

so that the subsequent propositions in fact hold for the aggregate excess demand function $\mathbf{z}$.

Since $p_j^\circ = 0$ leads to a positive excess demand $z_j(\mathbf{p}^\circ) > 0$ for strongly monotone preference orderings, there must be a positive commodity price vector $\mathbf{p}^\circ > \mathbf{0}$ by (IV.1). Both inequalities $\mathbf{p}^\circ > \mathbf{0}$ and $\mathbf{z}(\mathbf{p}^\circ) \leqq \mathbf{0}$ are only compatible with WALRAS' law (Proposition III.30) $\mathbf{p}^{\circ\mathrm{T}}\mathbf{z}(\mathbf{p}^\circ) = 0$ if the excess demand vanishes on all commodity markets,[27] $\mathbf{z}(\mathbf{p}^\circ) = \mathbf{0}$. Summarizing, we obtain the next

[24]The upper semi-continuity of the individual demand correspondences has been proved in Proposition III.26.

[25]Cf. BERGE (1963, p. 109).

[26]If $\mathbf{z}^\circ = \mathbf{x}_A - \mathbf{w}_A \leqq \mathbf{0}$, then regarding Figure IV.5 point $\mathbf{x}_A$ must lie in the box with the dotted line corresponding to the initial endowment $\mathbf{w}_A$. Thus, each point $\mathbf{x}_a^{ds}(\mathbf{p}^\circ, \mathbf{w}_a)$ lies also in this box.

[27]Cf. VARIAN (1992, p. 318 f.).

Proposition IV.6 (Competitive Equilibrium)[28] *Under Assumption 2* ($\mathbf{w}_a > \mathbf{0}$) *in each exchange economy* $\mathcal{E} : A \to \Pi_{\substack{smo\\sco}} \times \mathbb{R}^n_+$ *an equilibrium price vector* $\mathbf{p}^\circ > \mathbf{0}$ *exists.*

If $\mathbf{p}^\circ > \mathbf{0}$ is an equilibrium price vector, i.e. $\mathbf{z}(\mathbf{p}^\circ) = \mathbf{0}$, then the corresponding WALRASian allocation $\mathfrak{x}^\circ$ is also known from the persons' demand functions $\mathbf{x}^\circ_a = \mathbf{x}^d_a(\mathbf{p}^\circ, \mathbf{w}_a)$.

Starting with an imbalanced price vector $\mathbf{p}$, we can imagine the following adjustment mechanism to a WALRASian equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$. Note that BROUWER's fixed-point theorem (Theorem C.15) and Corollary C.15.1 are equivalent. Hence, each numerical procedure for computing equilibrium prices has, at the same time, to be an algorithm for computing fixed-points of continuous mappings.[29]

Because a WALRASian equilibrium[30] only allows the occurrence of excess supplies, $\mathbf{z}(\mathbf{p}^\circ) \leqq \mathbf{0}$, we have merely to discuss the clearing of an excess demand. By an appropriate modification of the function $\mathbf{g}$ with the components

$$g_j(\mathbf{p}) = \frac{p_j + \max\{0, \psi_j(\mathbf{p})\}}{1 + \sum_{k=1}^n \max\{0, \psi_k(\mathbf{p})\}} \qquad j = 1, \ldots, n$$

defined in the proof of Corollary C.15.1 a function $\tilde{\mathbf{g}}$ results with

$$\tilde{g}_j(\mathbf{p}) = \frac{p_j + \max\{0, z^s_j(\mathbf{p})\}}{1 + \sum_{k=1}^n \max\{0, z^s_k(\mathbf{p})\}} \qquad j = 1, \ldots, n.$$

If there is an excess demand on one market, $z^s_j(\mathbf{p}) > 0$, then the relative price p_j of the examined good is raised until $z^s_j(\mathbf{p}^\circ) \leqq 0$ is satisfied. With that it is assumed that the excess demand for a good diminishes with an increasing price for this good. Finally, in the WALRASian equilibrium all excess demands are removed, $\mathbf{z}(\mathbf{p}^\circ) \leqq \mathbf{0}$. At the same time strongly monotone preferences exclude an excess supply (Corollary III.30.2). Thus, at an equilibrium, represented by BROUWER's fixed-point $\mathbf{p}^\circ = \tilde{\mathbf{g}}(\mathbf{p}^\circ)$, all markets are cleared, $\mathbf{z}^s(\mathbf{p}^\circ) = \mathbf{0}$.

Example: *To clarify the structure of the solution, imagine an economy in which all persons* $a \in A$ *possess a utility function* $u_a(x_1, x_2) = x_1^\alpha x_2^\beta$ *and an initial endowment* (w_{1a}, w_{2a}). *Calculating the excess demand functions* $z_{ja}(p_1, p_2)$ *for good* j *and for each person* a, *two functions for the aggregate excess demand result with the following structure:*

$$\begin{aligned} z_1(p_1, p_2) &= \frac{\alpha}{\alpha + \beta}\left(w_{1A} + \frac{p_2}{p_1} w_{2A}\right) - w_{1A} \\ z_2(p_1, p_2) &= \frac{\beta}{\alpha + \beta}\left(w_{2A} + \frac{p_1}{p_2} w_{1A}\right) - w_{2A}, \end{aligned}$$

[28] Cf. HILDENBRAND, KIRMAN (1988, p. 108).
Note that Assumption 2 is required for the proof of Proposition IV.2.

[29] Cf. SCARF (1982, p. 1014 f.).

[30] For the definition of a WALRASian equilibrium, see p. 192.

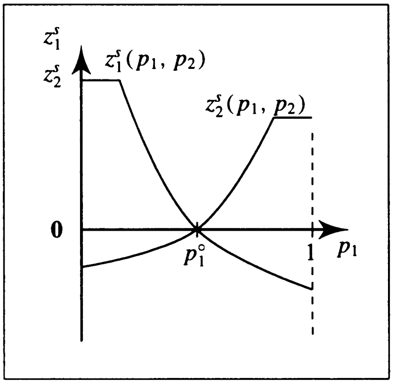

Figure IV.6: Representation of a competitive equilibrium with two goods

where (w_{1A}, w_{2A}) denotes the summed up initial endowments (w_{1a}, w_{2a}) of all persons. Since the price vector (p_1, p_2) denotes a point in the price simplex Δ it is furthermore $p_1 + p_2 = 1$. The restriction of the commodity space $X = \mathbb{R}_+^2$ to

$$X_1^s = \left\{ \mathbf{x} \in X \mid \mathbf{0} \leqq \mathbf{x} \leqq \begin{pmatrix} w_{1A} + 1 \\ w_{2A} + 1 \end{pmatrix} \right\}$$

yields the synthetic excess demand functions z_1^s and z_2^s according to Figure IV.6. This restriction is irrelevant for the fixed-point (p_1°, p_2°) with $p_2^\circ = 1 - p_1^\circ$ and $z_j(p_1^\circ, p_2^\circ) = 0 \quad (j = 1, 2)$.

2.2 Convex Preference Orderings

As before, convex preference orderings are only compatible with perfectly divisible goods. Thus, the commodity space can be identified with $X = \mathbb{R}_+^n$. For the proof of the existence of an exchange equilibrium it must be shown that there is a price vector $\mathbf{p}^\circ > \mathbf{0}$ satisfying $\mathbf{0} \in Z(\mathbf{p}^\circ)$. The proof ensues with the aid of Theorem C.17 (DEBREU, GALE, NIKAIDO), p. 314.

By Proposition IV.3 the synthetic aggregate excess demand correspondence $Z^s\colon \ \Delta \to \mathfrak{P}(X_3^s)$ derived in Section 1.3.1 satisfies all premises of Theorem C.17 if Assumption 2 ($\mathbf{w}_a > \mathbf{0}$) holds. This implies:

> *There is a $\mathbf{p}^\circ \in \Delta$ and a $\mathbf{z}^\circ \in Z^s(\mathbf{p}^\circ)$ such that $\mathbf{p}^\mathsf{T}\mathbf{z}^\circ \leqq 0$ holds for all $\mathbf{p} \in \Delta$.*

With that each vector $\mathbf{z}^\circ$ with a positive component $z_j^\circ > 0$ leads to a contradiction since the price vector[31] $\mathbf{p} = \mathbf{e}^j \in \Delta$ yields $\mathbf{p}^\mathsf{T}\mathbf{z}^\circ > 0$. Thus, $\mathbf{z}^\circ \leqq \mathbf{0}$. Again this inequality means that no person $a \in A$ demands more of a commodity than is available in the economy; see (IV.1). The additional restriction of a bounded commodity space becomes obsolete, i.e.[32]

$$\mathbf{z}^\circ \leqq \mathbf{0}, \ \mathbf{z}^\circ \in Z^s(\mathbf{p}^\circ) \implies \mathbf{z}^\circ \in Z(\mathbf{p}^\circ).$$

For monotone preference orderings each price $p_j^\circ = 0$ leads to a positive excess demand $z_j^\circ > 0$. Because of $\mathbf{z}^\circ \leqq \mathbf{0}$ it is thus well-founded to assume $\mathbf{p}^\circ > \mathbf{0}$. In view of WALRAS' law (Proposition III.30) $\mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ = 0$ must also be satisfied. From $\mathbf{p}^\circ > \mathbf{0}$ it at once ensues $\mathbf{z}^\circ = \mathbf{0}$ for the examined vector $\mathbf{z}^\circ \in Z(\mathbf{p}^\circ)$ so that the existence of a competitive equilibrium is proved.

[31] We denote the j-th unit vector by $\mathbf{e}^j$.

[32] It should not be forgotten at this point the hint of footnote 26.

Proposition IV.7 (Competitive Equilibrium) *Under Assumption 2* ($\mathbf{w}_a > \mathbf{0}$) *in each exchange economy* $\mathcal{E} : A \to \Pi_{\underset{co}{smo}} \times \mathbb{R}^n_+$ *there is an equilibrium price vector* $\mathbf{p}^\circ > \mathbf{0}$.

An alternative proof for the existence of exchange equilibria is given by HEUSER (1992, p. 632 f.). This proof does not need the truncation of the commodity space and is suitable to avoid analytic difficulties considering indivisible commodities as established in Section 1.3.2. The disadvantage is a more complex proof. Preparing for the treatment of indivisible goods in Section 2.3, we now give a representation of the proof. The idea of the proof refers to the already proved properties of the aggregate excess demand Z and is based on constructing a sequence of fixed-points of the form $(\mathbf{p}^\nu, \mathbf{z}^\nu)$.

If the correspondence $Z\colon \operatorname{rint}\Delta \to \mathfrak{P}(\mathbb{R}^n)$ is restricted to a compact set $S \subset \operatorname{rint}\Delta$, i.e.

$$\widehat{Z}\colon S \to \mathfrak{P}(\mathbb{R}^n) \quad \text{with} \quad \widehat{Z}(\mathbf{p}) = Z(\mathbf{p}) \quad \forall \mathbf{p} \in S,$$

then the resulting correspondence $\widehat{Z}$ has the following properties:

1. Given continuous convex preference orderings, the sets $\widehat{Z}(\mathbf{p})$ are not empty (Proposition III.23) and convex (Proposition III.24) for all $\mathbf{p} \in S$.
2. The correspondence $\widehat{Z}$ is upper semi-continuous (Proposition III.27) and, therefore, closed in S (Theorem C.7, p. 309).
3. If the upper semi-continuous and compact-valued correspondence Z is restricted to a compact set $S \subset \operatorname{rint}\Delta$, then the resulting correspondence $\widehat{Z}$ is bounded (Theorem C.5, p. 308).
4. For every $\mathbf{p} \in S$ and every $\mathbf{z} \in \widehat{Z}(\mathbf{p})$ we have $\mathbf{p}^\mathsf{T}\mathbf{z} \leqq 0$ (see (III.104), p. 194).

Thus, by Theorem C.17 (DEBREU, GALE, NIKAIDO) the correspondence $\widehat{Z}$ has a fixed-point $(\widehat{\mathbf{p}}, \widehat{\mathbf{z}})$ with $\widehat{\mathbf{p}} \in \widehat{S}$ and $\widehat{\mathbf{z}} \in \widehat{Z}(\widehat{\mathbf{p}})$ such that $\mathbf{p}^\mathsf{T}\widehat{\mathbf{z}} \leqq 0$ holds for all $\mathbf{p} \in S$.

With this outcome it seems reasonable to choose a suitable sequence $\{S^\nu\}$ of closed convex subsets in the price simplex Δ instead of the set S such that $S^\nu \to \Delta$ holds for the limit. The corresponding sequence of fixed-points $(\mathbf{p}^\nu, \mathbf{z}^\nu)$ is investigated afterwards.

Defining the sets

$$S^\nu := \left\{\mathbf{p} \in \Delta \mid p_j \geqq {}^1\!/_\nu \quad (j = 1, \ldots, n)\right\} \qquad \text{for } \nu \geqq n$$

leads to the following inclusion

$$\text{(IV.2)} \qquad S^n \subset S^{n+1} \subset S^{n+2} \cdots \quad \text{with} \quad S^\nu \to \Delta$$

and each price vector $\mathbf{p} \in \Delta$ is included at least at the limit.

Let Z^ν be the correspondence Z restricted to the domain S^ν. For $\nu \geqq n$ these correspondences fulfill the four above mentioned properties. Hence, by Theorem

C.17 (DEBREU, GALE, NIKAIDO) each of these correspondences has a fixed-point $(\mathbf{p}^\nu, \mathbf{z}^\nu)$ with $\mathbf{p}^\nu \in S^\nu$ and $\mathbf{z}^\nu \in Z^\nu(\mathbf{p}^\nu)$ so that

(IV.3) $$\mathbf{p}^\mathsf{T}\mathbf{z}^\nu \leqq 0 \qquad \forall\, \mathbf{p} \in S^\nu.$$

With the help of the sequence of fixed-points $(\mathbf{p}^\nu, \mathbf{z}^\nu)$ Proposition IV.7 can be proved in four steps.

Step 1: The sequence of price vectors $\{\mathbf{p}^\nu\} \subset \Delta$ is bounded. Accordingly, there is a convergent subsequence $\{\mathbf{p}^{\nu_k}\}$ with its limit $\mathbf{p}^\circ$ belonging to the (closed) price simplex Δ, $\mathbf{p}^{\nu_k} \to \mathbf{p}^\circ$, and $\mathbf{p}^\circ \in \Delta$.

Step 2: On the basis of the definition of an aggregate excess demand the sequence $\{\mathbf{z}^\nu\}$ is bounded below by $-\mathbf{w}_A \leqq \mathbf{z}^\nu$ for all $\nu \geqq n$.

Moreover, (IV.3) implies for an arbitrary[33] $\tilde{\mathbf{p}} \in S^n$ (with $\tilde{\mathbf{p}} > \mathbf{0}$) the inequality $\tilde{\mathbf{p}}^\mathsf{T}\mathbf{z}^n \leqq 0$. Considering the inclusion (IV.2), it follows furthermore $\tilde{\mathbf{p}}^\mathsf{T}\mathbf{z}^\nu \leqq 0$ for all $\nu \geqq n$. Because of $-\mathbf{w}_A \leqq \mathbf{z}^\nu$ the sequence of points $\{\mathbf{z}^\nu\}$ must also be bounded above. Both bounds yield the shadowed area in Figure IV.7. Thus, the sequence of points $\{\mathbf{z}^\nu\}$ is bounded and a convergent subsequence $\{\mathbf{z}^{\nu_k}\}$ exists with $\mathbf{z}^{\nu_k} \to \mathbf{z}^\circ$.

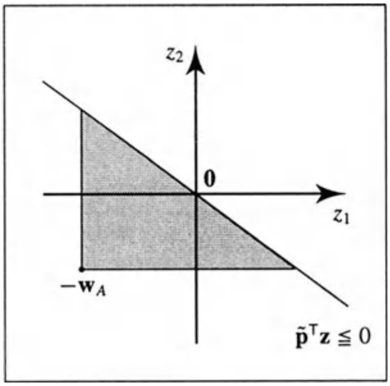

Figure IV.7: Boundedness of the sequence of points $\{\mathbf{z}^\nu\}$

Step 3: For $\mathbf{w}_A > \mathbf{0}$ (Assumption 2) and $\mathbf{p}^\circ \in \Delta$ it is $\mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_A > 0$. Hence, we must have a positive price vector $\mathbf{p}^\circ > \mathbf{0}$ because otherwise by Corollary III.29.1 it would be $\|\mathbf{z}^{\nu_k}\| \to \infty$ for strongly monotone preference orderings[34] contradicting the boundedness of the sequence $\{\mathbf{z}^\nu\}$; see Figure IV.7. Closedness of the correspondence Z for all $\mathbf{p} > \mathbf{0}$ assures that

$$\left[\mathbf{p}^{\nu_k} \to \mathbf{p}^\circ > \mathbf{0},\ \mathbf{z}^{\nu_k} \to \mathbf{z}^\circ,\ \mathbf{z}^{\nu_k} \in Z(\mathbf{p}^{\nu_k})\right] \Longrightarrow \mathbf{z}^\circ \in Z(\mathbf{p}^\circ).$$

Step 4: WALRAS' law (Proposition III.30) requires $\mathbf{p}^\mathsf{T}\mathbf{z} = 0$ for all $\mathbf{z} \in Z(\mathbf{p})$. In particular, $\mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ = 0$ follows from $\mathbf{z}^\circ \in Z(\mathbf{p}^\circ)$. The relation (IV.2) includes every positive price vector satisfying condition (IV.3) for sufficiently large ν. Thus, $\mathbf{p}^\mathsf{T}\mathbf{z}^\circ \leqq 0$ holds for all $\mathbf{p} > \mathbf{0}$ such that $\mathbf{z}^\circ \leqq \mathbf{0}$ must be satisfied. Relations $\mathbf{p}^\circ > \mathbf{0}$, $\mathbf{z}^\circ \leqq \mathbf{0}$ and $\mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ = 0$ imply finally $\mathbf{z}^\circ = \mathbf{0}$. Thus, the alternative proof of Proposition IV.7 is completed.

The concluding remarks intend to draw the attention to the major features of the aggregate excess demand correspondence Z. Only these properties guarantee the existence of a competitive equilibrium. All additional assumptions are made in consideration of the aspect that the correspondence Z fulfills the relevant premises.

Two conditions are given in advance of the subsequent proposition.[35]

[33] As $\mathbf{p} \in \Delta$ the set S^n contains in the presented case a single price vector $\tilde{\mathbf{p}} = (1/\nu, \ldots, 1/\nu)^\mathsf{T}$.

[34] Corollary III.29.1 is based on the same assumptions as Proposition III.29. Thus, for convex preference orderings we have merely to stipulate Assumption 2 ($\mathbf{w}_a > \mathbf{0}$); see Proposition IV.7.

[35] See Proposition III.30 (WALRAS' Law) and Corollary III.29.1.

WALRAS' Law: For all $\mathbf{p} > \mathbf{0}$ and for all $\mathbf{z} \in Z(\mathbf{p})$ it is $\mathbf{p}^{\mathsf{T}}\mathbf{z} = 0$.

Boundary Condition: Suppose the sequence of price vectors $\{\mathbf{p}^{\nu}\}$ with $\mathbf{p}^{\nu} > \mathbf{0}$ converges to a price vector $\tilde{\mathbf{p}}$ having a zero component, $\tilde{\mathbf{p}} \in \partial\Delta$. Then for each sequence of excess demand vectors $\{\mathbf{z}^{\nu}\}$ with $\mathbf{z}^{\nu} \in Z(\mathbf{p}^{\nu})$ the relation $\|\mathbf{z}^{\nu}\| \to +\infty$ is valid.

Proposition IV.8 (Competitive Equilibrium)[36] *Suppose that the aggregate excess demand correspondence Z is convex-valued, bounded below, and upper semi-continuous for all positive price vectors* $\mathbf{p} > \mathbf{0}$. *Then there is an equilibrium price vector* $\mathbf{p}^{\circ} > \mathbf{0}$ *such that* $\mathbf{0} \in Z(\mathbf{p}^{\circ})$ *if Z satisfies* WALRAS' *Law as well as the Boundary Condition.*

We refrain from repeating the above proof since merely slight modifications must be carried out.

The relationship with Proposition IV.7 becomes apparent when observing the additional assumptions. The correspondence Z satisfies all premises of Proposition IV.8 if we assume that each person has a positive initial endowment $\mathbf{w}_a > 0$ (Assumption 2) and a strongly monotone convex preference ordering. Proposition IV.8 implies, therefore, Proposition IV.7.

2.3 Consideration of Indivisible Commodities

From Proposition III.27, p. 199, it is known that the aggregate excess demand correspondence Z is upper semi-continuous and compact-valued for every $\mathbf{p} > \mathbf{0}$ if there is at least one divisible good demanded by each person in a positive quantity (Assumption 1).

Since the correspondence Z is usually not convex-valued for indivisible goods, the sets $Z(\mathbf{p})$ are substituted by their convex hull $Z_{co}(\mathbf{p}) := \operatorname{conv} Z(\mathbf{p})$ for all $\mathbf{p} > \mathbf{0}$. Theorem C.11, p. 310, assures that the new correspondence Z_{co} is upper semi-continuous and compact-valued for every $\mathbf{p} > \mathbf{0}$.

Let Z_{co}^{ν} be the correspondence Z_{co} restricted to the area S^{ν}. Each correspondence Z_{co}^{ν} is upper semi-continuous in S^{ν}, i.e. it is upper semi-continuous and compact valued for every $\mathbf{p} \in S^{\nu}$. All correspondences Z_{co}^{ν} satisfy the following conditions:

1. For continuous convex preference orderings the sets $Z_{co}^{\nu}(\mathbf{p})$ are not empty and convex for all $\mathbf{p} \in S$.

2. The correspondences Z_{co}^{ν} are upper semi-continuous and, therefore, closed in the respective set S^{ν}.[37]

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[36] Cf. DEBREU (1982, p. 722, Theorem 8), where an alternative proof is presented which goes back to NEUEFEIND (1977).

[37] Theorem C.11 states that Z_{co} is compact-valued and upper semi-continuous for every $\mathbf{p} > \mathbf{0}$. By Proposition III.27 this property passes on to the restricted correspondences Z_{co}^{ν}. Finally, the closedness of the correspondences Z_{co}^{ν} follows from Theorem C.7.

3. If the correspondence Z_{co} is restricted to a compact set $S^\nu \subset \operatorname{rint} \Delta$, then the resulting correspondence Z_{co}^ν is bounded.[38]

4. In view of Corollary B.4.3, p. 293, for every $\mathbf{p} \in S^\nu$ and for every $\mathbf{z} \in Z_{co}^\nu(\mathbf{p})$ the inequality $\mathbf{p}^\mathsf{T}\mathbf{z} \leqq 0$ must be satisfied.

As before, we obtain from Theorem C.17 (DEBREU, GALE, NIKAIDO), p. 314, for each correspondence Z_{co}^ν a fixed-point $(\mathbf{p}^\nu, \mathbf{z}^\nu)$ with $\mathbf{p}^\nu \in S^\nu$ and $\mathbf{z}^\nu \in Z_{co}^\nu(\mathbf{p}^\nu)$ so that

$$\mathbf{p}^\mathsf{T}\mathbf{z}^\nu \leqq 0 \qquad \forall \mathbf{p} \in S^\nu.$$

Analogous to Proposition IV.7 (competitive equilibrium with convex preferences), it follows now

Proposition IV.9 *Let Assumptions 1, 2, and 3 hold. Then in each exchange economy* $\mathcal{E} : A \to \Pi_{smo} \times \mathbb{R}^n_+$ *there is a price vector* $\mathbf{p}^\circ > \mathbf{0}$ *so that* $\mathbf{0} \in Z_{co}(\mathbf{p}^\circ)$.

Proof: Using the sequence of fixed-points $(\mathbf{p}^\nu, \mathbf{z}^\nu)$ for the correspondences Z_{co}^ν: $S^\nu \to \mathfrak{P}(\mathbb{R}^n)$ with $S^\nu \to \Delta$, Proposition IV.9 can be proved by the same four steps as before Proposition IV.7. Hence, only a few comments are necessary.

If we replace the correspondence Z with Z_{co} in step 3, p. 230, then to use Corollary III.29.1 we have to keep in mind that the analysis is now based on indivisible goods. Thus, all of the three Assumptions 1, 2, and 3 are required. Furthermore, the closedness of the correspondence Z_{co} has already been noted.

Take Corollary B.4.3, p. 293, into account for step 4. If $\mathbf{p}^\mathsf{T}\mathbf{z} = 0$ holds for all $\mathbf{z} \in Z(\mathbf{p})$, then the equation is satisfied for all $\mathbf{z} \in Z_{co}(\mathbf{p})$, too. ■

The main difference between Propositions IV.7 and IV.9 is that the first assures in contrast to the second one – apart from the existence of an equilibrium price vector $\mathbf{p}^\circ$ – the existence of a WALRASian allocation $\mathbf{x}^\circ \in \times_{a \in A} X$, i.e.

$$\mathbf{x}_a^\circ \in D_a(\mathbf{p}^\circ, \mathbf{w}_a) \qquad \text{for every person } a \in A$$

$$\text{and} \qquad \sum_{a \in A} \mathbf{x}_a^\circ = \mathbf{w}_A .$$

This implication can only partially be transferred to Proposition IV.9 because the aggregate excess demand correspondence Z has been substituted by the convex-valued correspondence Z_{co}. Thus, the conclusion from the aggregate excess demand $Z_{co}(\mathbf{p}^\circ)$ to the individual commodity demand $D_a(\mathbf{p}^\circ, \mathbf{w}_a)$ remains impossible. A feasible allocation $\mathbf{x}^\circ \in \times_{a \in A} \mathbb{R}^n_+$ exists but not all persons receive a commodity bundle they demand. Nevertheless, the issues which can be proved for convex preference orderings (Proposition IV.7) do not become completely obsolete.

[38] If the correspondence Z_{co}^ν is upper semi-continuous in S^ν, then it is upper semi-continuous and compact-valued for every $\mathbf{p} \in S^\nu$. Everything else results again from Theorem C.7, p. 309.

Considering Theorem B.5, p. 294,

$$Z_{co}(\mathbf{p}^\circ) = \operatorname{conv}\left(\sum_{a\in A} D_a(\mathbf{p}^\circ, \mathbf{w}_a) - \mathbf{w}_A\right) = \sum_{a\in A} D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a) - \mathbf{w}_A\,,$$

where we set again $D_a^{co}(\mathbf{p}, \mathbf{w}_a) := \operatorname{conv} D_a(\mathbf{p}, \mathbf{w}_a)$, a pair $(\mathbf{x}^\circ, \mathbf{p}^\circ)$ corresponding to Proposition IV.9 can be understood as follows:

Proposition IV.10 (ROTHENBERG Equilibrium)[39] *Suppose that Assumptions 1, 2, and 3 hold. Then in each exchange economy* $\mathcal{E} : A \to \Pi_{smo} \times \mathbb{R}^n_+$ *there is a price vector* $\mathbf{p}^\circ > \mathbf{0}$ *and a feasible allocation* $\mathbf{x}^\circ$ *so that*

(IV.4a) $\mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^\circ = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a$ *for each person* $a \in A$,

(IV.4b) $\mathbf{x}_a^\circ \in D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$ *for each person* $a \in A$ *and*

$\#\{a \in A \mid \mathbf{x}_a^\circ \notin D_a(\mathbf{p}^\circ, \mathbf{w}_a)\} \leqq n - 1$,

(IV.4c) $\sum_{a\in A} \mathbf{x}_a^\circ = \mathbf{w}_A$.

Some comments must first be made in advance of the proof. By (IV.4b) and (IV.4c) the above mentioned pair $(\mathbf{x}^\circ, \mathbf{p}^\circ)$ contains a feasible allocation $\mathbf{x}^\circ \in \times_{a\in A} \mathbb{R}^n_+$, where the elements $\mathbf{x}_a^\circ$ do not necessarily satisfy the integer constraints. The budgets of all persons are, however, balanced for the price vector $\mathbf{p}^\circ$; see (IV.4a). The statement (IV.4b) notes two properties of the pair $(\mathbf{x}^\circ, \mathbf{p}^\circ)$. On the one hand concerning n goods the allocation $\mathbf{x}^\circ$ assigns a vector $\mathbf{x}_a^\circ$, not required by these persons, at the most $n - 1$ out of $\#A$ persons. On the other hand the allocation $\mathbf{x}^\circ$ does not consist of arbitrary vectors. Each vector $\mathbf{x}_a^\circ$ belongs to the convex hull of those sets of commodity bundles which are chosen by the person a concerned at prices $\mathbf{p}^\circ$. This result has already been noted in Figure IV.4, p. 220, for two persons and two goods. If the demand set $D_a(\mathbf{p}^\circ, \mathbf{w}_a)$ is a singleton, then $D_a(\mathbf{p}^\circ, \mathbf{w}_a) = D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$. Thus, by (IV.4b) this person at least would receive a commodity bundle $\mathbf{x}_a^\circ \in X$ he demands.

Proof: As each person with a monotone preference ordering and at least one divisible good only demands those commodity bundles which make use of the whole budget, it ensues that

$$\mathbf{p}^\mathsf{T}\mathbf{x}_a^d = \mathbf{p}^\mathsf{T}\mathbf{w}_a \qquad \forall\, \mathbf{x}_a^d \in D_a(\mathbf{p}, \mathbf{w}_a).$$

Thus, if (IV.4b) holds, Corollary B.4.3, p. 293, entails (IV.4a).

$$\mathbf{p}^{\circ\mathsf{T}}\mathbf{x} = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a \qquad \forall\, \mathbf{x} \in D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$$

Defining the individual excess demand by $\mathbf{z}_a := \mathbf{x}_a^d - \mathbf{w}_a$ with $\mathbf{x}_a^d \in D_a(\mathbf{p}, \mathbf{w}_a)$ for each person $a \in A$, then the set of excess demand vectors for each given

[39] Cf. STARR (1969, p. 29).

price vector $\mathbf{p}^\circ > \mathbf{0}$ is a compact subset in the $(n-1)$-dimensional hyperplane $\mathbf{p}^{\circ\mathsf{T}}\mathbf{z} = 0$. By SHAPLEY and FOLKMAN[40] for $\#A > n-1$ and $\mathbf{0} \in Z_{co}(\mathbf{p}^\circ)$ there are vectors of the form $\mathbf{z}_a \in D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a) - \mathbf{w}_a$ or, equivalently, $\mathbf{x}_a^\circ := \mathbf{z}_a + \mathbf{w}_a \in D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$ fulfilling two conditions: first $\sum_{a\in A} \mathbf{z}_a = \mathbf{0}$ and second $\mathbf{z}_a \in D_a(\mathbf{p}^\circ, \mathbf{w}_a) - \mathbf{w}_a$ is satisfied with at the most $n-1$ exceptions. Therefore no further explanations of (IV.4b) and (IV.4c) are needed. ■

As previously shown, the ROTHENBERG equilibrium $(\boldsymbol{x}^\circ, \mathbf{p}^\circ)$ of Proposition IV.10 involves a feasible allocation $\boldsymbol{x}^\circ$, but not all vectors $\mathbf{x}_a^\circ$ correspond to a commodity bundle demanded by person a at prices $\mathbf{p}^\circ$. As every person individually decides on an optimal commodity bundle $\mathbf{x}_a^d \in X$ at given commodity prices $\mathbf{p}^\circ$, now the question arises as to what extent the feasible but not necessarily chosen allocation $\boldsymbol{x}^\circ$ with

$$(\mathbf{x}_a^\circ)_{a\in A} \in \times_{a\in A} \mathbb{R}_+^n$$

differs from the chosen but not necessarily feasible allocation $\boldsymbol{x}^d$ with

$$(\mathbf{x}_a^d)_{a\in A} \in \times_{a\in A} X$$

when both allocations do not differ arbitrarily according to Proposition IV.10.

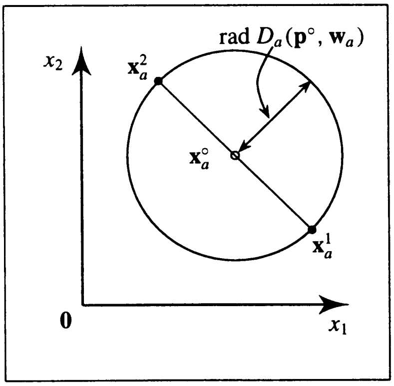

Figure IV.8: Radius of the demand set $D_a(\mathbf{p}^\circ, \mathbf{w}_a)$

To offer a measure for the degree of deviation between both allocations $\boldsymbol{x}^\circ$ and $\boldsymbol{x}^d$, we define the radius of the set $D_a(\mathbf{p}, \mathbf{w}_a)$:

$$\operatorname{rad} D_a(\mathbf{p}, \mathbf{w}_a) := \inf_{\tilde{\mathbf{x}}\in\mathbb{R}^n} \sup_{\mathbf{x}\in D_a(\mathbf{p},\mathbf{w}_a)} \|\tilde{\mathbf{x}} - \mathbf{x}\| .$$

This measure indicates the radius of the smallest closed ball containing the nonempty compact set $D_a(\mathbf{p}, \mathbf{w}_a)$.[41] Assuming that the burden borne by person a by a vector $\mathbf{x}_a^\circ \in D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$ rises with increasing distance to a chosen commodity bundle $\mathbf{x}_a^d \in D_a(\mathbf{p}^\circ, \mathbf{w}_a)$, then

$$\operatorname{rad} D_a(\mathbf{p}^\circ, \mathbf{w}_a) = \operatorname{rad} D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$$

indicates the worst possible case for the examined person.[42]

Figure IV.8 shows the demand set $D_a(\mathbf{p}^\circ, \mathbf{w}_a) = \{\mathbf{x}_a^1, \mathbf{x}_a^2\}$. Thus the greatest deviation of a point in $D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$ results from the commodity bundles $\mathbf{x}_a^1$ and $\mathbf{x}_a^2$ at $\mathbf{x}_a^\circ$. From the point of view of person a the point $\mathbf{x}_a^\circ \in D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$ is the worst possible case.

[40] See Theorem B.6, p. 295.

[41] Therefore BONNESEN, FENCHEL (1934, p. 54) call rad C the radius of the circumscribing ball of the set C.

[42] Note that upward and downward deviations from a feasible (integer) commodity bundle $\mathbf{x}_a^d$ are treated as being equal: "Thus, 11/10 and 9/10 of a car are equally bad."

With regard to each person the radius of a commodity demand set $D_a(\mathbf{p}^\circ, \mathbf{w}_a)$ gives a nonnegative upper bound for the deviation between both $\mathbf{x}_a^\circ \in D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$ and $\mathbf{x}_a^d \in D_a(\mathbf{p}^\circ, \mathbf{w}_a)$. Hence, there is an upper bound for the accumulated deviation between both allocations $\mathfrak{x}^\circ$ and $\mathfrak{x}^d$ with respect to the total economy.

Proposition IV.11 *Let Assumptions 1, 2, and 3 hold. Then in each exchange economy* $\mathcal{E} : A \to \Pi_{smo} \times \mathbb{R}_+^n$ *there is a price vector* $\mathbf{p}^\circ > \mathbf{0}$ *as well as a feasible allocation* $\mathfrak{x}^\circ$ *and an allocation* $\mathfrak{x}^d$ *chosen such that*

$$\text{(IV.5a)} \qquad \mathbf{x}_a^d \in D_a(\mathbf{p}^\circ, \mathbf{w}_a) \qquad \textit{for each} \quad a \in A,$$

$$\text{(IV.5b)} \qquad \Big\| \sum_{a \in A} (\mathbf{x}_a^d - \mathbf{x}_a^\circ) \Big\|^2 \leqq \sum_{a \in A} \operatorname{rad}^2 D_a(\mathbf{p}^\circ, \mathbf{w}_a).$$

Before proving this proposition, two comments are given in advance. The allocation $\mathfrak{x}^d$ is described by (IV.5a). Each person chooses a commodity bundle $\mathbf{x}_a^d \in X$ demanded at commodity prices $\mathbf{p}^\circ$. The inequality (IV.5b) provides a first upper bound for the accumulated deviation between the (feasible) allocation $\mathfrak{x}^\circ$ and the allocation $\mathfrak{x}^d$ chosen at prices $\mathbf{p}^\circ$.

The second comment notes the consistence of this outcome. If the demand sets $D_a(\mathbf{p}^\circ, \mathbf{w}_a)$ consist of one and only one element so that $\operatorname{rad} D_a(\mathbf{p}^\circ, \mathbf{w}_a) = 0$, then it ensues that $\| \sum_{a \in A} (\mathbf{x}_a^d - \mathbf{x}_a^\circ) \|^2 \leqq 0$. As found out before, this inequality holds if and only if each person receives a commodity bundle he demands, $\mathbf{x}_a^\circ = \mathbf{x}_a^d$ for all $a \in A$.

Proof: By Proposition IV.10 under the Assumptions 1, 2, and 3 there is a ROTHENBERG equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ in the examined economy $\mathcal{E}$ with

$$\sum_{a \in A} (\mathbf{x}_a^\circ - \mathbf{w}_a) = \mathbf{0} \in Z_{co}(\mathbf{p}^\circ).$$

With that and by Theorem B.7, p. 295, there is a commodity bundle $\mathbf{x}_a^d - \mathbf{w}_a \in D_a(\mathbf{p}^\circ, \mathbf{w}_a) - \mathbf{w}_a$ so that (IV.5a) is satisfied.[43] Furthermore, Theorem B.7 results in

$$\Big\| \sum_{a \in A} (\mathbf{x}_a^\circ - \mathbf{w}_a) - \sum_{a \in A} (\mathbf{x}_a^d - \mathbf{w}_a) \Big\|^2 \leqq \sum_{a \in A} \operatorname{rad}^2 \big[D_a(\mathbf{p}^\circ, \mathbf{w}_a) - \mathbf{w}_a \big].$$

Because of $\operatorname{rad} D_a(\mathbf{p}, \mathbf{w}_a) = \operatorname{rad} D_a(\mathbf{p}, \mathbf{w}_a) - \mathbf{w}_a$ this at once yields (IV.5b). ■

A weakness of the estimation (IV.5b) is that the right hand side of the inequality depends among other things on the number of persons, on the price vector $\mathbf{p}^\circ$, and on the form of preferences. Thus, the estimation will vary with each change of these parameters. This deficit can be sorted out by an additional assumption. Supposing an upper bound for the radius of the sets $D_a(\mathbf{p}, \mathbf{w}_a)$, asymptotic propositions about the extent of deviation between both allocations $\mathfrak{x}^\circ$ and $\mathfrak{x}^d$ can be made.

[43] A similar result may be found in BROOME (1972, p. 238 f., Lemma 4.9).

Assumption 4[44] *For the preference orderings of all persons there is a finite number D^{sup} with*

$$D^{sup} := \sup_{a \in A} \sup_{\mathbf{p} \in \text{int}\,\Delta} \text{rad}\, D_a(\mathbf{p}, \mathbf{w}_a)\,.$$

With that, two commodity bundles chosen under the same conditions cannot differ more than $2 \cdot D^{sup}$; see the commodity bundles $\bar{\mathbf{x}}_a^d$ and $\tilde{\mathbf{x}}_a^d$ demanded in Figure IV.9. Thus, the number D^{sup} is also used as measure for the degree of nonconvexity of preferences. Continuing Proposition IV.11 we now gain

Corollary IV.11.1 (Quasi-Equilibrium)[45] *Under Assumptions 1 to 4 in each exchange economy $\mathcal{E} : A \to \Pi_{smo} \times \mathbb{R}_+^n$ there is a price vector $\mathbf{p}^\circ > \mathbf{0}$ as well as a feasible allocation $\mathfrak{x}^\circ$ and an allocation $\mathfrak{x}^d$ chosen such that*

(IV.6a) $\quad \mathbf{x}_a^d \in D_a(\mathbf{p}^\circ, \mathbf{w}_a) \quad$ *for each person $a \in A$ and*

(IV.6b) $\quad \| \sum_{a \in A} (\mathbf{x}_a^d - \mathbf{x}_a^\circ) \| \leqq \sqrt{n-1}\, D^{sup}.$

Proof: Using the allocation $\mathfrak{x}^d$ and the price vector $\mathbf{p}^\circ$ of Proposition IV.11, we know that the sets $\{\mathbf{z}_a \mid \mathbf{z}_a = \mathbf{x}_a^d - \mathbf{w}_a,\ \mathbf{x}_a^d \in D_a(\mathbf{p}^\circ, \mathbf{w}_a)\}$ are not empty compact subsets in the $(n-1)$-dimensional hyperplane $\{\mathbf{z} \mid \mathbf{p}^{\circ\mathsf{T}}\mathbf{z} = 0\}$. Concerning

$$\mathbf{0} \in Z_{co}(\mathbf{p}^\circ) = \sum_{a \in A} D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a) - \sum_{a \in A} \mathbf{w}_a\,,$$

by Corollary B.7.1, p. 295, there are commodity bundles $\mathbf{x}_a^d \in D_a(\mathbf{p}^\circ, \mathbf{w}_a)$ so that

$$\|\mathbf{0} - \sum_{a \in A} \mathbf{z}_a\|^2 = \|\sum_{a \in A} (\mathbf{x}_a^d - \mathbf{w}_a)\|^2 \leqq R\,,$$

where R is defined as the sum of the $\min\{\#A, n-1\}$ greatest $\text{rad}^2\, D_a(\mathbf{p}^\circ, \mathbf{w}_a)$. Considering Assumption 4, Proposition IV.11, (IV.5b) implies

$$\|\sum_{a \in A} (\mathbf{x}_a^d - \mathbf{x}_a^\circ)\|^2 \leqq R \leqq (n-1)(D^{sup})^2$$

for the examined ROTHENBERG equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ and, therefore, (IV.6b). ■

The advantage of Corollary IV.11.1 over Proposition IV.11 is that the estimation (IV.6b) in contrast to (IV.5b) does not depend on the number of persons but on the positive number D^{sup}. With this fact the ROTHENBERG equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ of Proposition IV.10 can be viewed from an alternative point of view.

Proposition IV.10 proves the existence of a feasible allocation $\mathfrak{x}^\circ$ by which at the most $n-1$ persons receive a vector $\mathbf{x}_a^\circ$ they do not ask for. This result is not affected by increasing the number of persons.

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[44] The assumptions used until now were introduced on pages 198, 203, and 204.

[45] Cf. STARR (1969) and ARROW, HAHN (1971, Chapter 7).

Corollary IV.11.1, however, suggests that with an increasing number of persons the degree of deviation between both allocations $\mathfrak{x}^\circ$ and $\mathfrak{x}^d$ *per head* diminishes. Obviously, this view is merely useful when the occurring burdens are distributed equally among all persons instead of being born by at the most $n-1$ persons as in Proposition IV.10.

The next expositions serve for the equal distribution of the occurring burdens. The allocation $\mathfrak{x}^\circ$ known from Proposition IV.10 is modified to an alternative allocation $\mathfrak{x}^{\circ\circ}$ which must satisfy certain premises.

1. Proposition IV.12 presents conditions for a feasible allocation $\mathfrak{x}^{\circ\circ}$ such that no person a receives a commodity bundle $\mathbf{x}_a^{\circ\circ}$ which differs on average in more than ε from a chosen commodity bundle $\mathbf{x}_a^d$.

2. Proposition IV.13 has more restrictive conditions for a feasible allocation $\mathfrak{x}^{\circ\circ}$ such that none of the commodity bundles $\mathbf{x}_a^{\circ\circ}$ is further away from a chosen commodity bundle $\mathbf{x}_a^d$ than ε.

Starting with the ROTHENBERG equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ of Proposition IV.10, each person receives a vector $\mathbf{x}_a^\circ \in \mathbb{R}_+^n$ deviating by $\boldsymbol{\delta}_a := \mathbf{x}_a^d - \mathbf{x}_a^\circ$ from a commodity bundle $\mathbf{x}_a^d \in X$ demanded. Hence, for at the most $n-1$ persons $\boldsymbol{\delta}_a \neq \mathbf{0}$. If each person's vector $\mathbf{x}_a^\circ$ is expanded by the respective $\boldsymbol{\delta}_a$,[46] then the feasible allocation $\mathfrak{x}^\circ$ results in an allocation $\mathfrak{x}^d$ chosen, which differs by

$$\text{(IV.7)} \qquad \boldsymbol{\delta}_A := \sum_{a\in A} \boldsymbol{\delta}_a = \sum_{a\in A} (\mathbf{x}_a^d - \mathbf{x}_a^\circ) = \sum_{a\in A} \mathbf{x}_a^d - \mathbf{w}_A$$

from a feasible allocation. The task now is to partition $\boldsymbol{\delta}_A$ into $\boldsymbol{\varepsilon}_a$ such that the outcome $\mathbf{x}_a^{\circ\circ} := \mathbf{x}_a^d - \boldsymbol{\varepsilon}_a$ is a feasible allocation $\mathfrak{x}^{\circ\circ}$, where no commodity bundle of anyone differs on average in more than ε from a chosen commodity bundle $\mathbf{x}_a^d$.

The step described above may be clearly summarized in a table.

stage	allocation			person $a \in A$
1.	$\mathfrak{x}^\circ$	feasible	usually not chosen	$\mathbf{x}_a^\circ$
2.	$\mathfrak{x}^d$	usually feasible	chosen	$\mathbf{x}_a^d = \mathbf{x}_a^\circ + \boldsymbol{\delta}_a$
3.	$\mathfrak{x}^{\circ\circ}$	feasible	usually not chosen	$\mathbf{x}_a^{\circ\circ} = \mathbf{x}_a^d - \boldsymbol{\varepsilon}_a$

The problem is to find a feasible allocation $\mathfrak{x}^{\circ\circ}$ where the choice of the $\boldsymbol{\varepsilon}_a$ is subject to several restrictions which can be established by the following premises.[47]

1. "The reallocation should not cost anything."
   Because of (IV.4a) $\mathbf{p}^{\circ\mathsf{T}}\boldsymbol{\delta}_a = \mathbf{p}^{\circ\mathsf{T}}\boldsymbol{\varepsilon}_a = 0$ must hold for each person since

$$\mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^\circ = \mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^d = \mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^{\circ\circ} = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a.$$

[46] Remember that $\mathbf{p}^{\circ\mathsf{T}}\boldsymbol{\delta}_a = 0$.

[47] See Figure IV.9.

2. "The reallocation has to be feasible."
   Because of (IV.4c) the reallocation is restricted to $\sum_{a\in A}\delta_a = \sum_{a\in A}\varepsilon_a$ since

$$\sum_{a\in A}\mathbf{x}_a^{\circ\circ} = \sum_{a\in A}(\mathbf{x}_a^d - \varepsilon_a) = \sum_{a\in A}(\mathbf{x}_a^\circ + \delta_a - \varepsilon_a) = \mathbf{w}_A + \sum_{a\in A}\delta_a - \sum_{a\in A}\varepsilon_a.$$

3. "Nothing can be taken away from a person that it does not have."
   The choice of the ε_a must assure $\mathbf{x}_a^{\circ\circ} = \mathbf{x}_a^d - \varepsilon_a \geqq \mathbf{0}$; see $\bar{\mathbf{x}}_a^d - \varepsilon_a$ in Figure IV.9. Thus, $\varepsilon_a = \delta_A/\#A$ is ruled out since this reallocation will usually not satisfy the nonnegativity constraint.

4. Finally, the ε_a, and their respective components, must be chosen such that they have the same signs as the components of δ_A;[48] this condition is necessary to estimate (IV.8).

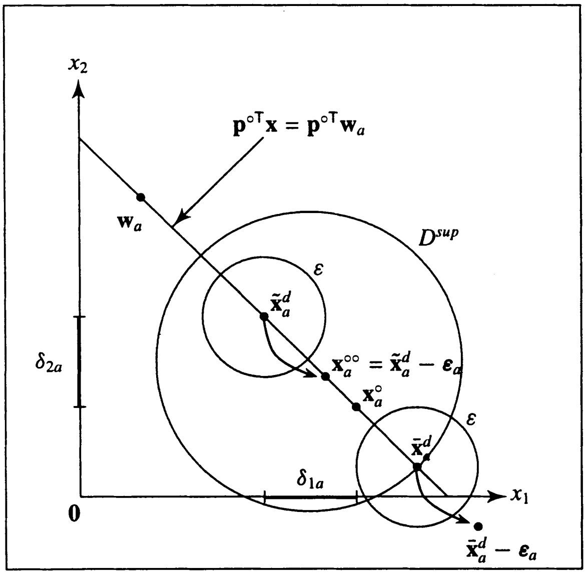


Figure IV.9: The problem of a suitable reallocation

Figure IV.9 describes the restrictions on a suitable reallocation $(\varepsilon_a)_{a\in A}$ for an individual person with the initial endowment $\mathbf{w}_a$. The person demands one of the two commodity bundles $\bar{\mathbf{x}}_a^d$ or $\tilde{\mathbf{x}}_a^d$ at prices $\mathbf{p}^\circ$, that is

$$D_a(\mathbf{p}^\circ, \mathbf{w}_a) = \left\{\bar{\mathbf{x}}_a^d, \tilde{\mathbf{x}}_a^d\right\}.$$

According to the ROTHENBERG equilibrium $(\boldsymbol{x}^\circ, \mathbf{p}^\circ)$, the vector $\mathbf{x}_a^\circ$ is allocated to the person concerned. The difference between this vector and the chosen commodity bundle $\tilde{\mathbf{x}}_a^d$ is $\delta_a = \tilde{\mathbf{x}}_a^d - \mathbf{x}_a^\circ$. Accumulating these differences $\delta_A = \sum_{a\in A}\delta_a$ and partitioning this δ_A into appropriate ε_a vectors, yields $\mathbf{x}_a^{\circ\circ} = \tilde{\mathbf{x}}_a^d - \varepsilon_a$ for person a. In the example shown by the figure we have $\|\tilde{\mathbf{x}}_a^d - \mathbf{x}_a^{\circ\circ}\| = \|\varepsilon_a\| > \varepsilon$ for person a, but anticipating (IV.9d) only $\sqrt{\sum_{a\in A}\|\varepsilon_a\|^2} < \#A\cdot\varepsilon$ is required for the economy.

[48]This property is not valid for the δ_a!

If a suitable reallocation $(\boldsymbol{\varepsilon}_a)_{a\in A}$ with the above properties exists, then

$$\begin{aligned} & \boldsymbol{\delta}_A = \sum_{a\in A} \boldsymbol{\delta}_a = \sum_{a\in A} \boldsymbol{\varepsilon}_a \\ \Longrightarrow\quad & \boldsymbol{\delta}_A^{\mathsf{T}}\boldsymbol{\delta}_A = \sum_{a\in A} \boldsymbol{\varepsilon}_a^{\mathsf{T}}\boldsymbol{\varepsilon}_a + \sum_{a\in A}\sum_{\substack{\tilde a\in A\\ \tilde a\neq a}} \boldsymbol{\varepsilon}_a^{\mathsf{T}}\boldsymbol{\varepsilon}_{\tilde a} \geqq \sum_{a\in A} \boldsymbol{\varepsilon}_a^{\mathsf{T}}\boldsymbol{\varepsilon}_a \\ \text{(IV.8)}\qquad \Longleftrightarrow\quad & \|\boldsymbol{\delta}_A\|^2 \geqq \sum_{a\in A} \|\boldsymbol{\varepsilon}_a\|^2 = \sum_{a\in A} \|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\|^2 \end{aligned}$$

holds and we gain

Proposition IV.12 (ε-Equilibrium)[49] *Suppose that we can prove a reallocation $(\boldsymbol{\varepsilon}_a)_{a\in A}$ with the above given properties. Let Assumptions 1 to 4 hold. Then in each exchange economy $\mathcal{E} : A \to \Pi_{smo} \times \mathbb{R}_+^n$ there are a price vector $\mathbf{p}^\circ > \mathbf{0}$, a feasible allocation $\mathbf{x}^{\circ\circ}$ and an allocation $\mathbf{x}^d$ chosen such that*

(IV.9a) $\quad \mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^{\circ\circ} = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a \qquad$ *for each person $a \in A$*

(IV.9b) $\quad \mathbf{x}_a^d \in D_a(\mathbf{p}^\circ, \mathbf{w}_a) \qquad$ *for each person $a \in A$*

(IV.9c) $\quad \sum_{a\in A} \mathbf{x}_a^{\circ\circ} = \mathbf{w}_A$

is satisfied. Moreover,

$$\text{(IV.9d)}\qquad \sqrt{\sum_{a\in A} \|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\|^2} < \#A \cdot \varepsilon$$

holds if the inequality $\#A > \sqrt{n-1}\, D^{sup}/\varepsilon$ is satisfied for sufficiently large $\#A$.

Before this proposition is proved, let us look at the main results. The allocation $\mathbf{x}^d \in \times_{a\in A} X$ chosen by (IV.9b) faces an allocation $\mathbf{x}^{\circ\circ} \times_{a\in A} \mathbb{R}_+^n$ being feasible by (IV.9c) so that each person satisfies his budget constraint at prices $\mathbf{p}^\circ$ by (IV.9a). The degree of deviation between these two allocations is indicated by (IV.9d) as an average deviation per head. If the economy $\mathcal{E}$ has a sufficiently large number of persons $\#A$, then the commodity bundle of no person is on average further away from a chosen commodity bundle $\mathbf{x}_a^d$ than ε. In this $\#A > \sqrt{n-1}\, D^{sup}/\varepsilon$ is fulfilled for an increasing number of persons $\#A$ if ε continually falls.[50] In the case of two goods – i.e. for $n = 2$ as in Figure IV.9 – (IV.9d) simplifies to $\#A > D^{sup}/\varepsilon$.

[49] Cf. STARR (1969, p. 31).
Analogous to the ROTHENBERG equilibrium $(\mathbf{x}^\circ, \mathbf{p}^\circ)$, the pair $(\mathbf{x}^{\circ\circ}, \mathbf{p}^\circ)$ is called an ε-equilibrium in the sense of STARR.

[50] MAS-COLELL (1985, p. 144 ff.) present similar results.

Proof: The initial point of the proof is the ROTHENBERG equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ of Proposition IV.10, p. 233. Setting $\mathbf{x}_a^{\circ\circ} = \mathbf{x}_a^d - \boldsymbol{\varepsilon}_a = \mathbf{x}_a^\circ + \boldsymbol{\delta}_a - \boldsymbol{\varepsilon}_a$ for each person $a \in A$ leads to

$$\#A \cdot \varepsilon > \sqrt{n-1}\, D^{sup} \geqq \Big\| \sum_{a \in A} (\mathbf{x}_a^d - \mathbf{x}_a^\circ) \Big\| \qquad \text{by Corollary IV.11.1}$$

$$= \|\boldsymbol{\delta}_A\| \geqq \sqrt{\sum_{a \in A} \|\boldsymbol{\varepsilon}_a\|^2} \qquad \text{is subject to (IV.8)}$$

$$= \sqrt{\sum_{a \in A} \|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\|^2}$$

for a given ε and a sufficiently large number of persons $\#A$. The proof of Proposition IV.12 is completed having found an appropriate reallocation $(\boldsymbol{\varepsilon}_a)_{a \in A}$. At this point we refrain from the determination of the $\boldsymbol{\varepsilon}_a$ vectors.[51] However, it is pointed out that the vectors, having the required properties, are determined within the proof of Proposition IV.13. In (IV.15a) these vectors are denoted by $\boldsymbol{\lambda}/\#A_k$. ■

The ε-equilibrium in the sense of STARR (Proposition IV.12) refers to the accumulated deviations $\|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\|$. But the inequality (IV.9d)

$$\frac{1}{\#A} \sqrt{\sum_{a \in A} \|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\|^2} < \varepsilon$$

can nevertheless be violated for individual persons. Subsequently we examine an ε-equilibrium in the sense of HILDENBRAND and KIRMAN, where $\|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\| \leqq \varepsilon$ is required for each person.[52] Thus, it is not surprising that the examined exchange economies are subject to further and more restrictive regularity constraints. As will be shown, the examined economies must be "large" with regard to different aspects. An interpretation of the parameters ($\widehat{w}$, ε, δ and $\bar{n}$) used in the following proposition will be given later. Similar to Proposition IV.12 we have

Proposition IV.13 (ε-Equilibrium)[53] *Suppose that Assumptions 1, 2 and 3 are satisfied and each person holds a bounded initial endowment, i.e.* $\|\mathbf{w}_a\| \leqq \widehat{w}$ *for all* $a \in A$. *Then an integer* $\bar{n}$ *exists for each* $\varepsilon > 0$ *and for each* $\delta > 0$ *such that in each exchange economy* $\mathcal{E} : A \to \Pi_{smo} \times \mathrm{IR}_+^n$ *there is a price vector* $\mathbf{p}^\circ > \mathbf{0}$, *a feasible allocation* $\mathfrak{x}^{\circ\circ}$, *and an allocation* $\mathfrak{x}^d$ *chosen with*

(IV.10a)	$\mathbf{p}^{\circ\mathrm{T}}\mathbf{x}_a^{\circ\circ} = \mathbf{p}^{\circ\mathrm{T}}\mathbf{w}_a$	*for each person* $a \in A$,
(IV.10b)	$\mathbf{x}_a^d \in D_a(\mathbf{p}^\circ, \mathbf{w}_a)$	*for each person* $a \in A$,
(IV.10c)	$\sum_{a \in A} \mathbf{x}_a^{\circ\circ} = \mathbf{w}_A$,	
(IV.10d)	$\|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\| \leqq \varepsilon$	*for each person* $a \in A$.

[51] An appropriate reallocation $(\boldsymbol{\varepsilon}_a)_{a \in A}$ is presented in STARR (1969, p. 31).
[52] The case illustrated in Figure IV.9 for $\tilde{\mathbf{x}}_a^d$ and $\mathbf{x}_a^{\circ\circ}$ should no longer occur.
[53] Cf. HILDENBRAND, SCHMEIDLER, ZAMIR (1973, p. 1160, Theorem 1).

Moreover,

(IV.10e) $$\#\{a \in A \mid \mathbf{x}_a^{\circ\circ} \notin D_a(\mathbf{p}^\circ, \mathbf{w}_a)\} < \bar{n}$$

holds when the two inequalities $\#A \geqq \bar{n}$ and $\mathbf{w}_A/\#A > \delta\mathbf{1}$ are satisfied.

Before going into the proof of this proposition, we first make some comments. Analogous to Proposition IV.12, the allocation $\mathfrak{x}^d \in \underset{a\in A}{\times} X$ demanded by (IV.10b) faces an allocation $\mathfrak{x}^{\circ\circ} \underset{a\in A}{\times} \mathbb{R}^n_+$ being feasible by (IV.10c) such that all persons satisfy their budget constraint for the price vector $\mathbf{p}^\circ$ by (IV.10a). According to (IV.10d), the vector $\mathbf{x}_a^{\circ\circ}$ of no person $a \in A$ is "far" away from a chosen commodity bundle $\mathbf{x}_a^d \in X$; the parameter ε serves as an upper bound. Obviously, this fact is only relevant for those persons who are associated with a vector $\mathbf{x}_a^{\circ\circ}$ outside of their demand sets $D_a(\mathbf{p}^\circ, \mathbf{w}_a)$. By (IV.10e) the number of these persons is smaller than $\bar{n}$. Moreover, the accumulated deviation is

$$\sum_{a\in A} \|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\| \leqq \varepsilon\bar{n} \leqq \varepsilon \cdot \#A\,,$$

where $\varepsilon\bar{n}$ does not depend on the number of persons.
After these preliminary comments it becomes clear why the examined exchange economy is said to be "large". On the one hand it is required that the economy $\mathcal{E}$ contains no less than $\bar{n}$ persons $(\#A \geqq \bar{n})$, on the other hand the total endowment $\mathbf{w}_A$ of the economy must be larger than $\#A\delta\mathbf{1}$. Interpreting the parameter $\delta > 0$ it is useful to rewrite the corresponding inequality as follows:

$$\mathbf{w}_A/\#A > \delta\mathbf{1} \quad \Longleftrightarrow \quad \min\{w_{1A},\ldots,w_{nA}\}/\#A > \delta\,.$$

Finally, a parameter $\widehat{w}$ is given which limits the influence of an individual person a. His initial endowment $\mathbf{w}_a$ is restricted to $\|\mathbf{w}_a\| \leqq \widehat{w}$.

Proof:[54] Again the ROTHENBERG equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ of Proposition IV.10 serves for the initial point of the explanations. If each person $a \in A$ asks for a commodity bundle $\mathbf{x}_a^d \in D_a(\mathbf{p}^\circ, \mathbf{w}_a)$, then we can set as before

(IV.7) $$\boldsymbol{\delta}_A = \sum_{a\in A} \boldsymbol{\delta}_a = \sum_{a\in A}(\mathbf{x}_a^d - \mathbf{x}_a^\circ) = \sum_{a\in A} \mathbf{x}_a^d - \mathbf{w}_A\,.$$

At the same time the budget balance $\mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^d = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a$ implies $\mathbf{p}^{\circ\mathsf{T}}\boldsymbol{\delta}_A = 0$ for each person. The problem is to distribute $\boldsymbol{\delta}_A$ among the persons of the economy such that the resulting allocation $\mathfrak{x}^{\circ\circ}$ satisfies the premises of Proposition IV.13, i.e. especially

$$\sum_{a\in A}(\mathbf{x}_a^{\circ\circ} - \mathbf{w}_a) = \mathbf{0}\,; \quad \mathbf{x}_a^{\circ\circ} \geqq \mathbf{0} \qquad \forall a \in A \quad \text{(feasible allocation } \mathfrak{x}^{\circ\circ})$$

$$\|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\| \leqq \varepsilon\,, \quad \mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^d = \mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^{\circ\circ} = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a \quad \forall a \in A\,.$$

[54] For the proof, see HILDENBRAND, SCHMEIDLER, ZAMIR (1973, p. 1161 ff.).

From person a's point of view the reallocation appears as follows:

$$\mathbf{x}_a^{\circ} \xrightarrow{\delta_a} \mathbf{x}_a^{d} \xrightarrow{\varepsilon_a} \mathbf{x}_a^{\circ\circ}.$$

As already mentioned above, the intuitive first approach $\boldsymbol{\delta}_A/\#A$ to the reallocation of $\boldsymbol{\delta}_A$ is ruled out because it usually violates the condition $\mathbf{x}_a^{\circ\circ} \geqq \mathbf{0}$. A positive quantity δ_{jA} can only be distributed equally among the members of a group $A_j \subset A$ if each member of the group demands more than $\delta_{jA}/\#A_j$:[55]

$$\text{(IV.11)} \qquad x_{ja}^{\circ\circ} = x_{ja}^{d} - \frac{\delta_{jA}}{\#A_j} > 0 \qquad \forall a \in A_j.$$

If we introduce a threshold ξ such that person a belongs to group A_j if and only if $x_{ja}^{d} > \xi$, then the following problem arises: for large ξ the number of the members of the group $\#A_j$ will be small so that $\xi \cdot \#A_j < \delta_{jA}$ may arise, i.e. the potential of redistribution $\xi \cdot \#A_j$ does not meet the need for redistribution δ_{jA}. Conversely, this case may even occur for a very small threshold ξ. Again $\xi \cdot \#A_j < \delta_{jA}$ holds despite large groups $\#A_j$. The proof requires a threshold $\xi < \delta$. In this $\xi = \delta/n$ turns out to be a favorable choice, see (IV.12) and (IV.14).

Because the redistribution has an upper bound by (IV.10d), the following estimation can be made:

$$\varepsilon \cdot \#A \geqq \sqrt{\sum_{a\in A} \|\mathbf{x}_a^{d} - \mathbf{x}_a^{\circ\circ}\|^2}.$$

Using (IV.7), the right hand side can be estimated by $\varepsilon \cdot \#A \geqq \|\boldsymbol{\delta}_A\|$. Indeed, this inequality holds true, if the relation $\varepsilon \cdot \#A_j \geqq \|\boldsymbol{\delta}_A\|$ is satisfied for each group $A_j \subset A$; see (IV.13). The relationship to (IV.11) becomes apparent, when $\|\boldsymbol{\delta}_A\| \geqq \delta_{jA}$ is taken into account. If each person belongs to exactly one group A_j, then two effects occur: first, (IV.11) guarantees a positive $x_{ja}^{\circ\circ}$ and at the same time $\mathbf{x}_a^{\circ\circ} \geqq \mathbf{0}$ holds since $\mathbf{x}_a^{d} \geqq \mathbf{0}$ and $\mathbf{x}_a^{\circ\circ}$ differ only by the j-th component. Second, the upper bound of the reallocation $\varepsilon \geqq \|\mathbf{x}_a^{d} - \mathbf{x}_a^{\circ\circ}\|$ is guaranteed by $\varepsilon \geqq \delta_{jA}/\#A_j = x_{ja}^{d} - x_{ja}^{\circ\circ} > 0$.[56] This is the reason for partitioning the set A with respect to (IV.12) and (IV.13).

The rest of the proof succeeds in two steps. First, it is supposed that the mentioned partition of A into suitable groups is possible. Second, it is shown how to carry out the grouping.

Step 1: Partition the set of all persons A into n disjoint groups $(A_j)_{j=1,\dots,n}$ (for each good one group) such that

$$\text{(IV.12)} \qquad x_{ja}^{d} > \delta/n \qquad \forall a \in A_j$$

$$\text{(IV.13)} \qquad \min\{\varepsilon, \delta/n\} \cdot \#A_j \geqq \|\boldsymbol{\delta}_A\|.$$

[55] At the same time it becomes evident that the inverse case with $\delta_{jA} \leqq 0$ is no real problem.

[56] If the both vectors $\mathbf{x}_a^{d}$ and $\mathbf{x}_a^{\circ\circ}$ differ at the most by the j-th component with $x_{ja}^{d} - x_{ja}^{\circ\circ} \geqq 0$, then $\|\mathbf{x}_a^{d} - \mathbf{x}_a^{\circ\circ}\| = x_{ja}^{d} - x_{ja}^{\circ\circ} \leqq \varepsilon$.

The two conditions can be summarized to

(IV.14) $$x_{ja}^{d} \cdot \#A_j > \|\boldsymbol{\delta}_A\| \geqq \delta_{jA} \, .$$

Given $\boldsymbol{\delta}_A$, we define two index sets

$$J^+ := \left\{ j \mid \delta_{jA} > 0,\ j = 1,\ldots,n \right\}$$
$$\text{and} \qquad J^- := \left\{ j \mid \delta_{jA} \leqq 0,\ j = 1,\ldots,n \right\} \, .$$

Because of $\mathbf{p}^{\circ\mathrm{T}}\boldsymbol{\delta}_A = 0$ the vector $\boldsymbol{\delta}_A$ must vanish for $J^+ = \emptyset$, i.e. $\boldsymbol{\delta}_A = \mathbf{0}$. With that a reallocation of $\boldsymbol{\delta}_A$ becomes superfluous. Thus, we suppose $k \in J^+$ so that $J^- \neq \emptyset$ is implied. Now a $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_n)^{\mathrm{T}}$ is determined such that

$$\begin{array}{rll} 0 \geqq \lambda_j \geqq \delta_{jA} & \text{for } j \in J^- & (\text{note } \delta_{jA} \leqq 0) \\ \lambda_j = \delta_{jA} & \text{for } j = k & (\text{note } \delta_{kA} > 0) \\ \lambda_j = 0 & \text{for } j \in J^+ \setminus \{k\} & \end{array}$$

and $\mathbf{p}^{\circ\mathrm{T}}\boldsymbol{\lambda} = p_k^\circ \delta_{kA} + \sum_{j \in J^-} p_j^\circ \lambda_j = 0$ are satisfied. In particular, $\|\boldsymbol{\lambda}\| \leqq \|\boldsymbol{\delta}_A\|$ can be presumed. Because of $\mathbf{p}^{\circ\mathrm{T}}\boldsymbol{\delta}_A = 0$ such a choice of $\boldsymbol{\lambda}$ is always possible since

$$\sum_{j \in J^+} p_j^\circ \delta_{jA} + \sum_{j \in J^-} p_j^\circ \delta_{jA} = 0 \quad \overset{k \in J^+}{\Longrightarrow} \quad p_k^\circ \delta_{kA} + \sum_{j \in J^-} p_j^\circ \delta_{jA} \leqq 0 \, .$$

Assuming $\mathbf{x}_a^d \in D_a(\mathbf{p}^\circ, \mathbf{w}_a)$ with $\mathbf{p}^{\circ\mathrm{T}}\mathbf{x}_a^d = \mathbf{p}^{\circ\mathrm{T}}\mathbf{w}_a$ for all $a \in A$, we now put

(IV.15a) $$\mathbf{x}_a^{\circ\circ} := \mathbf{x}_a^d - \boldsymbol{\lambda}/\#A_k \qquad \text{for } a \in A_k \, ,$$

(IV.15b) $$\mathbf{x}_a^{\circ\circ} := \mathbf{x}_a^d \qquad \text{for } a \notin A_k \, .$$

Thus, the reallocation of the positive quantity δ_{kA} takes place exclusively within the group A_k. The persons of this group must give away some of good k and they are offset in the sense of a balanced budget by some amounts of other goods with $\delta_{jA} < 0$. We get the following properties of the above redistribution:

1. Because of $\mathbf{p}^{\circ\mathrm{T}}(\boldsymbol{\lambda}/\#A_k) = (\mathbf{p}^{\circ\mathrm{T}}\boldsymbol{\lambda})/\#A_k = 0$ everyone's budget remains balanced; (IV.10a) follows from (IV.15a) and (IV.15b).

2. Each person receives an $\mathbf{x}_a^{\circ\circ} \geqq \mathbf{0}$. In particular a positive quantity of good k results from (IV.14),
$$x_{ka}^{\circ\circ} = x_{ka}^d - \delta_{kA}/\#A_k > 0 \, .$$

3. Each person gets an $\mathbf{x}_a^{\circ\circ}$ being not "far" away from $\mathbf{x}_a^d$: By (IV.13) $\varepsilon \cdot \#A_k \geqq \|\boldsymbol{\delta}_A\| \geqq \|\boldsymbol{\lambda}\|$ implies $\|\boldsymbol{\lambda}/\#A_k\| \leqq \varepsilon$. Thus, (IV.10d) is proved regarding
$$\begin{array}{lll} \|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\| = \|\boldsymbol{\lambda}/\#A_k\| \leqq \varepsilon & \forall a \in A_k & \text{see (IV.15a)}, \\ \|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\| = 0 & \forall a \notin A_k & \text{see (IV.15b)}. \end{array}$$

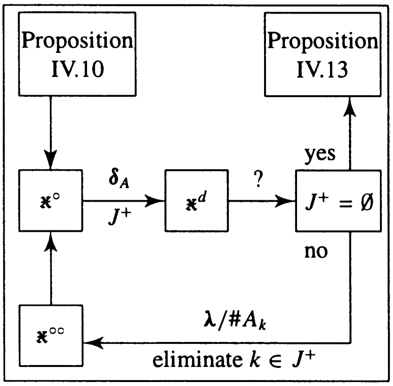


Figure IV.10: Proof of Proposition IV.13 (Step 1)

The resulting allocation $\mathbf{x}^{\circ\circ}$ constitutes a new $\boldsymbol{\delta}_A := \sum_{a\in A} \mathbf{x}_a^{\circ\circ} - \mathbf{w}_A$ as well as new index sets J^+ and J^-, where J^+ now has one element less. Repeating the above steps, it eventually ensues $J^+ = \emptyset$ and, therefore, $\boldsymbol{\delta}_A = \mathbf{0}$ or (IV.10c); see Figure IV.10.

Step 2: The two inequalities (IV.16) and (IV.17) are required for an appropriate partition of the set of persons A; they are deduced from the following steps (aa)–(ee).

(aa) For all $\mathbf{x} = (x_1, \ldots, x_n)^\mathsf{T} \geqq \mathbf{0}$ we have $\|\mathbf{x}\| \leqq \sum_{j=1}^n x_j$.[57] At the same time one gets $\|\mathbf{p}\| \leqq \sum_{j=1}^n p_j = 1$ for all $\mathbf{p} \in \Delta$.

(bb) Set $p^\circ_{min} := \min\{p^\circ_1, \ldots, p^\circ_n\}$. Because of $\mathbf{p}^\circ > \mathbf{0}$ the price p°_{min} must be positive, too. Thus, for all $\mathbf{x} \in \mathbb{R}^n_+$ with $\mathbf{p}^{\circ\mathsf{T}}\mathbf{x} = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a$ the inequality

$$p^\circ_{min}\mathbf{1}^\mathsf{T}\mathbf{x} \leqq \mathbf{p}^{\circ\mathsf{T}}\mathbf{x} = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a \iff \mathbf{1}^\mathsf{T}\mathbf{x} = \sum_{j=1}^n x_j \leqq \frac{\mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a}{p^\circ_{min}}$$

holds; see Figure IV.11 with $p^\circ_{min} = p^\circ_2 < p^\circ_1$.

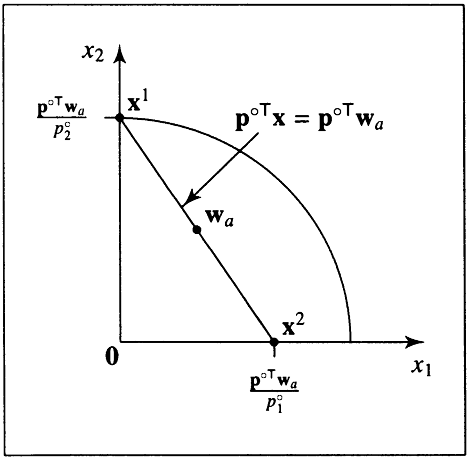


Figure IV.11: Proof of Proposition IV.13

(cc) The first two points imply

$$x_j \leqq \|\mathbf{x}\| \leqq \sum_{j=1}^n x_j \leqq \frac{\mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a}{p^\circ_{min}} \qquad \forall \mathbf{x} \in \mathbb{R}^n_+ \text{ with } \mathbf{p}^{\circ\mathsf{T}}\mathbf{x} = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a$$

for all $j = 1, \ldots, n$. Due to $\|\mathbf{p}^\circ\| \leqq 1$ and $\|\mathbf{w}_a\| \leqq \widehat{w}$ for each person $a \in A$, SCHWARZ's inequality (A.1), p. 282, $\mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a \leqq \|\mathbf{p}^\circ\|\cdot\|\mathbf{w}_a\| \leqq \widehat{w}$ yields

$$\text{(IV.16)} \quad x_j \leqq \|\mathbf{x}\| \leqq \frac{\mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a}{p^\circ_{min}} \leqq \frac{\widehat{w}}{p^\circ_{min}}$$

for each good j $(j = 1, \ldots, n)$.

[57] The conclusion for $\mathbf{x} \geqq \mathbf{0}$ results from

$$(\textstyle\sum_j x_j)^2 = \sum_j x_j^2 + \sum_j \sum_{k\neq j} x_j x_k \geqq \sum_j x_j^2 \iff \sum_j x_j \geqq \sqrt{\sum_j x_j^2} = \|\mathbf{x}\|$$

where $j, k = 1, \ldots, n$.

(dd) Thus, by the triangular inequality[58] it ensues

$$\|\mathbf{x}_a - \mathbf{x}_a^{\circ\circ}\| \leqq \|\mathbf{x}_a\| + \|\mathbf{x}_a^{\circ\circ}\| \leqq 2\,\frac{\mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a}{p^\circ_{min}} \leqq \frac{2\widehat{w}}{p^\circ_{min}}$$

for all $\mathbf{x}_a, \mathbf{x}_a^{\circ\circ} \in \mathbb{R}^n_+$ with $\mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a = \mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^{\circ\circ}$. Figure IV.11 illustrates this estimation of the distance between points $\mathbf{x}_a$ and $\mathbf{x}_a^{\circ\circ}$ for the case $\mathbf{x}^1 = \mathbf{x}_a$ and $\mathbf{x}^2 = \mathbf{x}_a^{\circ\circ}$.

(ee) For $\delta_A = \sum\limits_{a\in A}(\mathbf{x}_a^d - \mathbf{x}_a^\circ)$ the triangular inequality furthermore provides

$$\|\delta_A\| = \|\sum_{a\in A}(\mathbf{x}_a^d - \mathbf{x}_a^\circ)\| \leqq \sum_{a\in A}\|\mathbf{x}_a^d - \mathbf{x}_a^\circ\|\,.$$

Since $\mathbf{x}_a^d \neq \mathbf{x}_a^\circ$ holds for at the most $n-1$ persons by (IV.4b), it follows from (dd) with regard to $\mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^d = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a$ for each person $a \in A$

$$\|\delta_A\| \leqq (n-1)\cdot\frac{2\widehat{w}}{p^\circ_{min}}\,. \tag{IV.17}$$

Now all information on the explicit subdivision of the set of persons A into disjoint groups $(A_j)_{j=1,\dots,n}$ is available so that each group holds (IV.12) and (IV.13). The way taken can be sketched by the inclusion $A_j \subset \tilde{A}_j \subset \bar{\bar{A}}_j \subset \bar{A}_j$.

The starting points for the determination of the disjoint groups $(A_j)_{j=1,\dots,n}$ are the following groups which are not necessarily disjoint

$$\bar{A}_j := \{a \in A \mid x^\circ_{ja} > \delta/n\} \qquad (j = 1,\dots,n),$$

where the definition conforms to the remarks on (IV.11). The result of the subsequent estimation of group sizes is taken from (IV.19).
By the assumption $\mathbf{w}_A/\#A > \delta\mathbf{1}$ we obtain for each $j = 1,\dots,n$

$$\begin{aligned}
&& \#A\cdot\delta < w_{jA} &= \sum_{a\in\bar{A}_j} x^\circ_{ja} + \sum_{a\notin\bar{A}_j} x^\circ_{ja} && \\
&& &\overset{\text{(IV.16)}}{\leqq} \#\bar{A}_j\cdot\frac{\widehat{w}}{p^\circ_{min}} + (\#A - \#\bar{A}_j)\frac{\delta}{n} && \\
&\Longleftrightarrow & \#A\cdot\delta\frac{n-1}{n} &< \#\bar{A}_j\cdot\frac{\widehat{w}n - \delta p^\circ_{min}}{p^\circ_{min}n} && \\
&\Longrightarrow & \frac{\#A}{\#\bar{A}_j} &< \frac{\widehat{w}n - \delta p^\circ_{min}}{p^\circ_{min}\delta(n-1)} && (n \geqq 2) \\
\text{(IV.18)} &\Longrightarrow & \Omega := \frac{\widehat{w}}{\delta p^\circ_{min}} &> \frac{\#A}{\#\bar{A}_j} \geqq 1. &&
\end{aligned}$$

[58] The triangular inequality [No3], p. 282, denotes one property of the norm.

To estimate $\#\bar{A}_j > \#A/\Omega$, we choose a positive constant[59]

$$N := \frac{2(n-1)\widehat{w}}{p^{\circ}_{min}} \cdot \max\left\{{}^{1}\!/_{\varepsilon}, {}^{n}\!/_{\delta}\right\}$$

such that the smallest integer $\lceil N \rceil$ not smaller than N fulfills $\lceil N \rceil \geqq 1$. Besides N we have to fix an integer $\bar{n}$ satisfying

$$\bar{n} \geqq (n\lceil N \rceil + n - 1)\Omega \overset{(IV.18)}{>} n\lceil N \rceil + n - 1$$

If $\varepsilon < \delta/n$, then ε determines the value of N. A reduction of ε enlarges not only N but also $\bar{n}$. Provided that $\#A \geqq \bar{n}$, the number of persons $\#A$ in the economy concerned must be even larger the smaller ε is chosen. For too restrictive parameters ε and δ we have $\#A < \bar{n}$ so that ε and possibly δ must be relaxed, but the increase of δ has an upper bound by $\mathbf{w}_A > \#A \cdot \delta \mathbf{1}$. The assumption $\#A \geqq \bar{n}$ merely says that each group A_j contains at least

(IV.19) $$\#\bar{A}_j \overset{(IV.18)}{>} \frac{\#A}{\Omega} \geqq \frac{\bar{n}}{\Omega} \geqq n\lceil N \rceil + n - 1$$

persons. For subsets $\bar{\bar{A}}_j$ in $\bar{A}_j$ which are subject to the restrictions

$$\begin{array}{lll} \bar{\bar{A}}_j = \bar{A}_j & & \text{for } \#\bar{A}_j \leqq \bar{n} \\ \bar{\bar{A}}_j \subset \bar{A}_j & \text{with } \#\bar{\bar{A}}_j = \bar{n} & \text{for } \#\bar{A}_j > \bar{n} \end{array}$$

we get

$$\bar{n} \geqq \#\bar{\bar{A}}_j \geqq n\lceil N \rceil + n - 1 \qquad (j = 1, \ldots, n).$$

Because of (IV.4b) $\mathbf{x}^d_a \neq \mathbf{x}^{\circ}_a$ holds for at the most $n - 1$ persons and we can determine subsets by

$$\tilde{A}_j := \left\{a \in \bar{\bar{A}}_j \mid x^d_{ja} > \delta/n\right\} \qquad (j = 1, \ldots, n)$$

such that $\#\bar{\bar{A}}_j - \#\tilde{A}_j \leqq n - 1$ or

$$\bar{n} \geqq \#\tilde{A}_j \geqq n\lceil N \rceil \qquad (j = 1, \ldots, n).$$

Because each of the n sets $\tilde{A}_j$ contains more than $n\lceil N \rceil$ persons, we can now choose n disjoint sets $A_j \subset \tilde{A}_j$ with the group size $\#A_j = \lceil N \rceil$. As in the first step of the proof we only modify the commodity bundles of persons in $\bigcup_{j=1}^{n} A_j$ and because of $\#A \geqq \bar{n} > n\lceil N \rceil = n \cdot \#A_j$ there are less than $\bar{n}$ persons receiving a vector $\mathbf{x}^{\circ\circ}_a$ they do not ask for; see (IV.10e).

[59]The construction of the constant N can be justified with respect to (IV.17) by $\min\{\varepsilon, \delta/n\} \cdot N \geqq \|\delta_A\|$. An appropriate partition of the set A into groups A_j will satisfy (IV.13) if $\#A_j \geqq N$ holds.

Inequality (IV.12) is satisfied by the definition of $\tilde{A}_j$. From

$$\#A_j = \lceil N \rceil \geqq N = \frac{2(n-1)\widehat{w}}{p^\circ_{min}} \cdot \max\left\{{}^1\!/_\varepsilon, {}^n\!/_\delta\right\} \qquad \text{by definition}$$

$$\text{and} \qquad \frac{2(n-1)\widehat{w}}{p^\circ_{min}} \geqq \|\delta_A\| \qquad \text{by (IV.17)}$$

follows inequality (IV.13), $\min\{\varepsilon, \delta/n\} \cdot \#A_j \geqq \|\delta_A\|$. ■

2.4 Summary

Finally, we carry out a comparison of the discussed equilibria and their properties. First of all, remember that exact competitive equilibria have only been proved for convex preferences. Considering indivisible goods, the existence of an exact equilibrium is not assured.

In particular, the assumption of strictly convex preferences with perfectly divisible goods allows to presume demand functions in Section 2.1. The proof of an exact competitive equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ by Proposition IV.6 follows from BROUWER's fixed-point theorem. One of the requirements of this theorem is a bounded aggregate demand of the economy. This condition is fulfilled when the commodity space is restricted to an appropriate compact set. Since a positive commodity price vector $\mathbf{p}^\circ > \mathbf{0}$ is calculated, the above introduced restriction turns out to be superfluous to the outcome.

If we suppose only convex preferences instead of strictly convex preferences, then we have to consider the possibility that for each person there may be more than one best commodity bundle. The existence of a competitive equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ is guaranteed by Proposition IV.7, i.e. the price vector $\mathbf{p}^\circ > \mathbf{0}$ permits the clearing of all commodity markets, but in contrast with strictly convex preferences we can no longer assume the uniqueness of the allocation $\mathfrak{x}^\circ$.[60] To prove Proposition IV.7, we present two procedures both of which are based on Theorem C.17 (DEBREU, GALE, NIKAIDO), where this theorem follows from KAKUTANI's fixed-point theorem. As before, the first method gets over the problem of unbounded excess demand by an appropriate restriction of the commodity space. The second method avoids this procedure, which is associated with analytical difficulties with respect to the treatment of indivisible goods. Instead of this a sequence of fixed-points is constructed, which is used to prove Proposition IV.7.

The second method is now transferred to the case of indivisible goods by replacing the aggregate excess demand correspondence Z by an appropriate convex-valued correspondence Z_{co}. For this correspondence a fixed-point can be determined (Proposition IV.9), but the conclusion to the original correspondence and, therefore, the conclusion to the individual demand correspondences $D_a(\cdot, \mathbf{w}_a)$ remains essentially blocked.

[60]This observation is also reflected by the method of proof. Instead of BROUWER's fixed-point theorem referring to functions we now use KAKUTANI's fixed-point theorem allowing to deal with correspondences.

However, it is possible to prove feasible allocations which may not deviate arbitrarily from an equilibrium allocation. For the sake of clarity the presented allocations $\mathfrak{x}^\circ$, $\mathfrak{x}^{\circ\circ}$, and $\mathfrak{x}^d$ as well as their properties are summarized in the following table where $\#A^-$ indicates $\#\{a \in A \mid \mathbf{x}_a^* \notin D_a(\mathbf{p}^\circ, \mathbf{w}_a)\}$ with $\mathbf{x}_a^*$ belonging to the respective allocation.

ROTHENBERG equilibrium	ε-equilibrium	reference situation
	allocation	
$\mathfrak{x}^\circ$ by Proposition IV.10	$\mathfrak{x}^{\circ\circ}$ by Proposition IV.13	$\mathfrak{x}^d$
	budget constraint	
$\mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^\circ = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a$	$\mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^{\circ\circ} = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a$	$\mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^d = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a$
	deviation from individual demand $D_a(\mathbf{p}^\circ, \mathbf{w}_a)$	
$\mathbf{x}_a^\circ \in D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$ $\#A^- \leqq n-1$	$\Vert\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\Vert \leqq \varepsilon$ $\#A^- < \bar{n} \leqq \#A$	$\mathbf{x}_a^d \in D_a(\mathbf{p}^\circ, \mathbf{w}_a)$ $\#A^- = 0$
	feasibility of the allocation	
$\sum_{a\in A}\mathbf{x}_a^\circ = \mathbf{w}_A$	$\sum_{a\in A}\mathbf{x}_a^{\circ\circ} = \mathbf{w}_A$	usually $\sum_{a\in A}\mathbf{x}_a^d \neq \mathbf{w}_A$

In the ROTHENBERG equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ with n goods at the most $n-1$ out of $\#A$ persons receive a vector $\mathbf{x}_a^\circ$ they do not ask for. Without giving the distance to a chosen commodity bundle $\mathbf{x}_a^d$, for the moment it is only known that $\mathbf{x}_a^\circ$ can be expressed as a convex combination of demanded commodity bundles, $\mathbf{x}_a^\circ \in D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$.

Referring now to the smallest ball containing $D_a^{co}(\mathbf{p}^\circ, \mathbf{w}_a)$ and, therefore, $\mathbf{x}_a^\circ$,[61] the maximum distance between a point $\mathbf{x}_a^\circ$ and a chosen commodity bundle $\mathbf{x}_a^d$ can be limited by assuming that the radius of the examined ball cannot pass the value D^{sup} (Assumption 4).

Corollary IV.11.1 states that under this assumption there is a ROTHENBERG allocation $\mathfrak{x}^\circ$ and an allocation $\mathfrak{x}^d$ chosen whose deviation is restricted to

$$\Vert\sum_{a\in A}(\mathbf{x}_a^d - \mathbf{x}_a^\circ)\Vert \leqq \sqrt{n-1}\; D^{sup}.$$

As the inequality does not depend on the number of persons $\#A$ included in the economy the deviation per head can be arbitrarily reduced by an increasing $\#A$. Nevertheless, we have $\boldsymbol{\delta}_a = \mathbf{x}_a^d - \mathbf{x}_a^\circ \neq \mathbf{0}$ for at the most $n-1$ persons. The particular situation of these persons will not necessarily be improved by a rising number of persons $\#A$.

A first approach to get over this dilemma is presented in Proposition IV.12. Starting with a ROTHENBERG allocation $\mathfrak{x}^\circ$, we derive an alternative feasible allocation $\mathfrak{x}^{\circ\circ}$ by an appropriate reallocation. For a sufficient large number of

[61] Figure IV.9 shows the implications of the comments for an individual person.

persons $\#A$ we now have

$$\sqrt{\sum_{a \in A} \|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\|^2} < \#A \cdot \varepsilon$$

if $\#A > \sqrt{n-1}\, D^{sup}/\varepsilon$ holds. The advantage over the above estimation is that now all differences $\delta_a = \mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}$ are included by positive weights. Yet the problem of an individual treatment still exists. Although the deviation per head ε can be reduced by enlarging the number of persons $\#A$, $\|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\| > \varepsilon$ may, however, hold for an individual person. In addition the problem emerges that possibly considerably more than $n-1$ persons receive a vector $\mathbf{x}_a^{\circ\circ}$ they do not ask for.

Both problems are considered in Proposition IV.13. First of all, it is explicitly required that

$$\|\mathbf{x}_a^d - \mathbf{x}_a^{\circ\circ}\| \leqq \varepsilon$$

is satisfied for each person in the ε-equilibrium $(\boldsymbol{\varkappa}^{\circ\circ}, \mathbf{p}^{\circ})$ concerned. But this stronger result is based on more restrictive assumptions. For example, the examined economy must have a "large" (but finite) number of persons $\#A$ and a "large" total endowment $\mathbf{w}_A$. Finally, the number of persons who receive no commodity bundle demanded is bounded, $\bar{n} < \#A$. Nevertheless, the larger the number $\bar{n}$ is the more restrictive the parameter ε is given.

Analogous issues referring to economies with a large number of persons are given by KIRMAN (1981). A model that explicitly considers infinitely many indivisible goods and infinitely many consumers is presented in DUNZ (1992).

3 Generalization of the Model of an Exchange Economy

3.1 Equilibria in Production Economies

3.1.1 Description of a Production Economy

The preceding sections discuss the existence of competitive equilibria within the setting of an exchange economy. The question now arises as to how the results change if we consider not only the exchange of goods but also the production of goods.

For the proof for the existence of competitive equilibria in such production economies there are different approaches available, where the technical proof again relies upon KAKUTANI's fixed-point theorem. Whereas WALD (1933–34, 1934–35, 1936) presents a first exact but tedious proof, we find in KUHN (1956) or DORFMAN, SAMUELSON, SOLOW (1958, Chapter 13) more favorable procedures using the results of linear programming. Although these works present a relatively simple proof for the existence of competitive equilibria, the formulation of the problem involves some unsatisfactory aspects. For example, neither the behavior of the consumers nor the behavior of the producers is described.[62] The kernel of the matter is whether a certain set of equations has a solution or not.

A more modern formulation of the problem goes back to ARROW, DEBREU (1954). The question is now whether a set of optimization problems can be solved simultaneously, where the optimization problems are associated with individual economic agents facing different restrictions independent of each other.[63] As in the treatment of an exchange economy, the production economy is modelled as a noncooperative n-person game. First of all, a consumption set and a production set is specified for each household and each firm respectively. Moreover, we make assumptions on the economic behavior of the economic agents and define a competitive equilibrium. Finally, we try to prove the existence of a competitive equilibrium using the established assumptions.

The presented analysis conforms to ARROW, HAHN (1971, Chapter 7), where similar approaches may be found in MCKENZIE (1959, 1981) and DEBREU (1959). Further aspects can be taken from the more recent works of DEBREU (1982) and SMALE (1981).

While an exchange economy contains a finite set A of households, the examined production economy contains in addition a finite set B of producers. Suppose again that only n_d out of the n goods are perfectly divisible. Thus, the commodity space X or the nonnegative region of the commodity space X_+ is

$$X = \mathbb{R}^{n_d} \times \mathbb{Z}^{n-n_d} \quad \text{and} \quad X_+ = \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}$$

respectively. Each household $a \in A$ is indicated by a triple $(X_a, \succcurlyeq_a, \mathbf{w}_a)$. Here $X_a \subset X$ denotes the **consumption set** of the examined household. Up until now this set has been substituted by $X_a = X_+$, but alternative consumption sets can certainly be specified.[64] Each household has a continuous preference ordering $\succcurlyeq_a$[65] defined in the entire consumption set X_a, i.e. $\succcurlyeq_a \in X_a \times X_a$. Moreover, each household has an initial endowment $\mathbf{w}_a \in \mathbb{R}_+^n$. As before, the total endowment of the economy is denoted by $\mathbf{w}_A = \sum_{a \in A} \mathbf{w}_a$.

Each firm $b \in B$ is characterized by a **production set**

$$Y_b := \{\mathbf{y}_b \in X | \ \mathbf{y}_b \text{ is a possible activity for the firm } b\},$$

where the components of a **netput vector** $\mathbf{y}_b$ represent (demanded) quantities of inputs if $y_{jb} < 0$ or (supplied) quantities of output if $y_{jb} > 0$. According to the remarks on the production technology of a firm b, we suppose the following

[62] Cf. TAKAYAMA (1990, p. 261).

[63] Cf. KOOPMANS (1957, p. 60).

[64] NEWMAN (1987a, p. 616) delivers three arguments to associate each person a with his own consumption set X_a. (1) X_a denotes the subset in the commodity space X, in which the preferences are defined. (2) X_a contains only commodity bundles which guarantee the existence minimum of person a. (3) The household's maximal labor supply and its initial endowment form a natural upper bound for the consumption set X_a.

[65] See Axioms [$\mathscr{P}1$]–[$\mathscr{P}4$], p. 7 f.

properties of a production set Y_b:[66] (1) Each firm b has the possibility of inaction, $\mathbf{0} \in Y_b$. This relation corresponds to Axiom [P1a] with $\mathbf{0} \in P_b(\mathbf{0})$. (2) By [P1b] (*No Land of Cockaigne*) no netput vector $\mathbf{y}_b$ can be strictly positive, $Y_b \cap X_+ = \{\mathbf{0}\}$. (3) The production set Y_b is closed analogous to [P6]. (4) Similar to [T1] we suppose a convex production set Y_b. This assumption involves two problems. First, all goods which leave their mark on the production set Y_b must be divisible. Moreover, the assumption of a convex production set rules out increasing returns to scale.[67] (5) Free disposability of all goods by [P3] (*Disposability of Inputs*) and [P4] (*Disposability of Outputs*) implies $Y_b \supseteq (-X_+)$.

Concerning the union of all firms, the aggregate technology is denoted by

$$Y_B := \sum_{b \in B} Y_b.$$

Now the examined **production economy** $\mathcal{E}$ can be indicated by a tuple

$$\mathcal{E} = \big((X_a, \succcurlyeq_a, \mathbf{w}_a)_{a \in A}, (Y_b)_{b \in B}, (\theta_{ab})_{a \in A, b \in B}\big),$$

where $\theta_{ab} \geqq 0$ is the household a's share of firm b's profit. Provided all profits are completely apportioned to the households,[68]

$$\sum_{a \in A} \theta_{ab} = 1 \qquad \forall\, b \in B \tag{IV.20}$$

must be fulfilled. In the production economy an **allocation**

$$(\mathfrak{x}, \mathfrak{y}) := \big((\mathbf{x}_a)_{a \in A}, (\mathbf{y}_b)_{b \in B}\big) \in \mathop{\times}_{a \in A} X_a \times \mathop{\times}_{b \in B} Y_b$$

is said to be **attainable** if the aggregate endowments of the individual economic agents do not exceed the endowment of the economy.[69]

$$\sum_{a \in A} \mathbf{x}_a - \sum_{b \in B} \mathbf{y}_b - \mathbf{w}_A \leqq \mathbf{0} \tag{IV.21}$$

The following three points describe the individual and the aggregate behavior of the economic actors.

(a) Households: Each household $a \in A$ chooses a commodity bundle satisfying its needs best possible over the set of all commodity bundles it can buy. In this the budget set

$$B_a(\mathbf{p}) = \left\{\mathbf{x}_a \in X_a \mid \mathbf{p}^{\mathsf{T}}\mathbf{x}_a \leqq w_a(\mathbf{p})\right\}$$

consists of all commodity bundles the household a can buy in principle with wealth $w_a(\mathbf{p})$ and which belong to the consumption set X_a. The wealth corresponds to the

[66] See Figure IV.12, p. 253.

[67] Suppose that Y_b is a closed convex cone. Then constant returns to scale are supposed according to the expositions in Section II.2.3.1.

[68] Since all firms are owned by the households, we speak of a private ownership economy.

[69] Remember that an allocation is called feasible if it hold (IV.21).

value of the initial endowment $\mathbf{w}_a$ expanded by the profit share $\theta_{ab}\mathbf{p}^{\mathsf{T}}\mathbf{y}_b$ which is entitled to household a.

(IV.22)
$$w_a(\mathbf{p}) := \mathbf{p}^{\mathsf{T}}\mathbf{w}_a + \sum_{b\in B}\theta_{ab}\mathbf{p}^{\mathsf{T}}\mathbf{y}_b$$

As before, the individual commodity demand correspondence is defined by

$$D_a(\mathbf{p}) := \{\mathbf{x}_a \in B_a(\mathbf{p}) |\, \mathbf{x}_a \succcurlyeq_a \tilde{\mathbf{x}}_a \quad \forall\, \tilde{\mathbf{x}}_a \in B_a(\mathbf{p})\}.$$

(b) Firms: Each firm $b \in B$ maximizes the profit π_b with respect to its production set Y_b.

$$\pi_b(\mathbf{p}) := \sup\left\{\mathbf{p}^{\mathsf{T}}\mathbf{y}_b |\, \mathbf{y}_b \in Y_b\right\}$$

If each firm b has the possibility of inaction, $\mathbf{0} \in Y_b$, then the profit cannot be negative, $\pi_b(\mathbf{p}) \geqq 0$. The correspondence S_b with

$$S_b(\mathbf{p}) := \left\{\mathbf{y}_b \in Y_b |\, \mathbf{p}^{\mathsf{T}}\mathbf{y}_b \geqq \mathbf{p}^{\mathsf{T}}\tilde{\mathbf{y}}_b \quad \forall\, \tilde{\mathbf{y}}_b \in Y_b\right\}$$

includes all of firm b's profit maximizing activities $\mathbf{y}_b$; it describes the commodity supply $(y_{jb} > 0)$ as well as the factor demand $(y_{jb} \leqq 0)$.

The left hand part of Figure IV.12 shows a production set Y_b fulfilling the required properties. The commodity prices given by a' and b' yield the price ratios a and b, which is noted together with the supplied quantities of good 2 in the right hand part of the figure. An analogous representation of the demand for good 1 is omitted.

Given constant returns to scale, Y_b is a cone. In this case no positive profits may occur, hence $\pi_b(\mathbf{p}) = 0$. If a price vector leads to a positive profit, then the profit could be raised by each enlargement of the production level.[70] Thus, no finite maximal profit exists as well as no profit maximizing activity, i.e. $S_b(\mathbf{p}) = \emptyset$.

(c) Aggregate behavior: Summing up over all economic agents, we gain the aggregate excess demand correspondence Z with

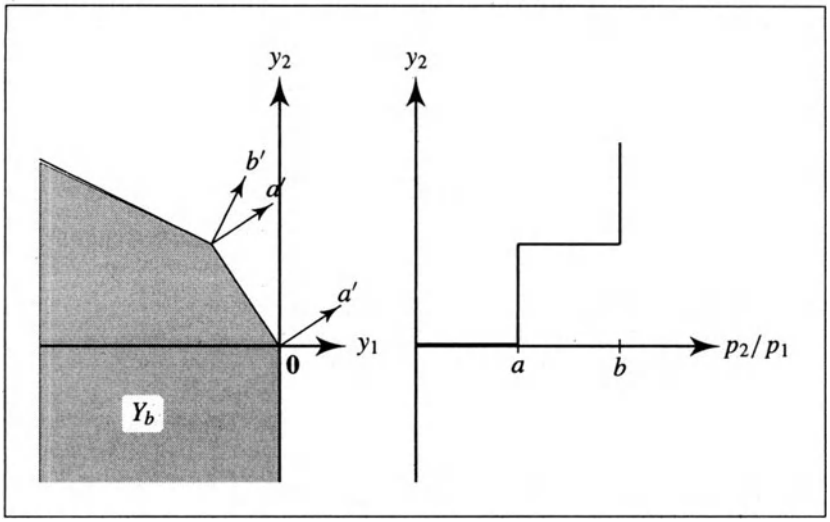
$$Z(\mathbf{p}) := \sum_{a\in A} D_a(\mathbf{p}) - \sum_{b\in B} S_b(\mathbf{p}) - \mathbf{w}_A.$$

As every firm realizes a nonnegative profit $\mathbf{p}^{\mathsf{T}}\mathbf{y}_b \geqq 0$, the aggregation of the individual budget constraints $\mathbf{p}^{\mathsf{T}}\left(\mathbf{x}_a - \mathbf{w}_a - \sum_{b\in B}\theta_{ab}\mathbf{y}_b\right) \leqq 0$ yields similar to (III.104), p. 194:

(IV.23)
$$\mathbf{p}^{\mathsf{T}}\mathbf{z} \leqq 0 \quad \text{for every} \quad \mathbf{p} \in \operatorname{Dom} Z \quad \text{and every} \quad \mathbf{z} \in Z(\mathbf{p})$$

On the basis of the homogeneity of degree 0 of all correspondences D_a and S_b at this point we need only consider price vectors $\mathbf{p}$ which belong to the price simplex Δ.

[70] If Y_b is a cone and if $\mathbf{p}^{\mathsf{T}}\mathbf{y}_b > 0$ for $\mathbf{y}_b \in Y_b$, then the profit can be increased arbitrarily for each production level $\lambda > 0$ because of $\lambda\mathbf{y}_b \in Y_b$.



Source: ARROW, HAHN (1971, p. 56).

Figure IV.12: Graphical representation of the supply correspondence

If a price vector $\tilde{\mathbf{p}} \in \Delta$ and an excess demand vector $\tilde{\mathbf{z}}$ hold $\tilde{\mathbf{z}} \in Z(\tilde{\mathbf{p}})$, then an allocation $(\tilde{\mathfrak{x}}, \tilde{\mathfrak{y}})$ exists such that

(IV.24a) $\tilde{\mathbf{x}}_a \in D_a(\tilde{\mathbf{p}})$ for all households $a \in A$

(IV.24b) $\tilde{\mathbf{y}}_b \in S_b(\tilde{\mathbf{p}})$ for all firms $b \in B$

(IV.24c) and $\tilde{\mathbf{z}} = \sum_{a \in A} \tilde{\mathbf{x}}_a - \sum_{b \in B} \tilde{\mathbf{y}}_b - \mathbf{w}_A$.

Furthermore, if $\tilde{\mathbf{z}} \leqq \mathbf{0}$, then the allocation $(\tilde{\mathfrak{x}}, \tilde{\mathfrak{y}})$ is feasible and

$$(\tilde{\mathfrak{x}}, \tilde{\mathfrak{y}}, \tilde{\mathbf{p}}) = \left((\tilde{\mathbf{x}}_a)_{a \in A}, (\tilde{\mathbf{y}}_b)_{b \in B}, \tilde{\mathbf{p}}\right)$$

is called an attainable state[71] of the production economy. Each firm chooses a profit maximizing activity $\tilde{\mathbf{y}}_b$ with respect to its technology and each household chooses a best element $\tilde{\mathbf{x}}_a$ with respect to its preferences. Moreover, by (IV.24a) the commodity bundles $\tilde{\mathbf{x}}_a$ belong to the respective consumption set, $\tilde{\mathbf{x}}_a \in X_a$. If the consumption set X_a contains no point of saturation, then for strongly monotone preferences a commodity bundle $\mathbf{x}''_a \geq \mathbf{x}'_a$ exists for every $\mathbf{x}'_a \in X_a$ with $\mathbf{x}''_a \succ_a \mathbf{x}'_a$. Thus, each household utilizes its whole budget provided the consumption set X_a includes at least one divisible good.[72]

$$\tilde{\mathbf{p}}^{\mathsf{T}}\tilde{\mathbf{x}}_a = \tilde{\mathbf{p}}^{\mathsf{T}}\mathbf{w}_a + \sum_{a \in A} \theta_{ab}\tilde{\mathbf{p}}^{\mathsf{T}}\tilde{\mathbf{y}}_b \qquad \text{for all households } a \in A$$

[71]Cf. DEBREU (1982, p. 705).

[72]This analogous argument has been used in the proof of WALRAS' law (Proposition III.30).

Considering (IV.20), the summation of all budget equations yields **WALRAS' law.**[73]

$$\tilde{\mathbf{p}}^{\mathsf{T}}\tilde{\mathbf{z}} = 0 \quad \text{for every} \quad \tilde{\mathbf{p}} \in \operatorname{Dom} Z \quad \text{and every} \quad \tilde{\mathbf{z}} \in Z(\tilde{\mathbf{p}}).$$

Note that neither (IV.23) nor WALRAS' law requires $\tilde{\mathbf{z}} \leqq \mathbf{0}$.

3.1.2 Existence of Competitive Equilibria

(a) Perfectly Divisible Goods Having introduced the examined production economy, we can now formulate the requirements of an equilibrium state. An attainable state $(\mathfrak{x}^\circ, \mathfrak{y}^\circ, \mathbf{p}^\circ)$ fulfilling apart from (IV.24a) and (IV.24b) the condition

$$\text{(IV.25)} \qquad \mathbf{z}^\circ = \sum_{a\in A} \mathbf{x}_a^\circ - \sum_{b\in B} \mathbf{y}_b^\circ - \mathbf{w}_A = \mathbf{0}$$

is called a **WALRASian equilibrium** or a **competitive equilibrium**. Moreover, $\mathbf{p}^\circ \in \Delta$ is called an equilibrium price vector.

Looking back at (IV.24a)–(IV.24c), the proof of the existence of a WALRASian equilibrium is completed if it can be shown that there is an equilibrium price vector $\mathbf{p}^\circ$ satisfying $\mathbf{0} \in Z(\mathbf{p}^\circ)$.

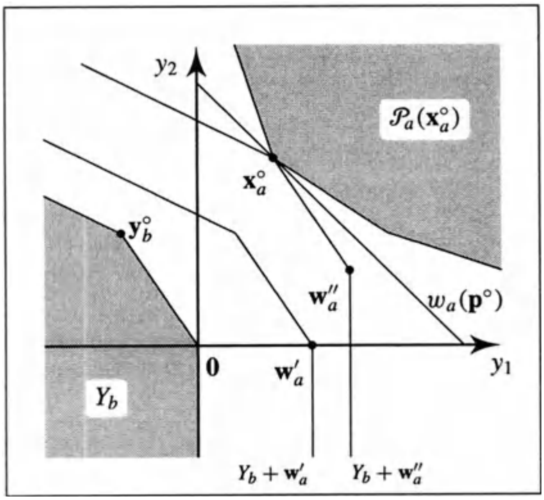


Figure IV.13: Illustration of a WALRASian equilibrium

The representation of a WALRASian equilibrium is given by Figure IV.13 for a particular firm with the production set Y_b and for a particular household with the consumption set $X_a = \mathbb{R}_+^2$. If the household owns the initial endowment $\mathbf{w}_a'$, then each quantity of good 2 the household wants to consume must be produced by the firm using the first good. For the initial endowment $\mathbf{w}_a''$ both of the commodities are available in a positive amount even without any production. Only for a relatively large price ratio p_2/p_1 the firm agrees to take up the production of the second good.[74] In the illustrated WALRASian equilibrium $(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$ with $\mathbf{y}_b^\circ = \mathbf{x}_a^\circ - \mathbf{w}_a''$ the firm earns a positive profit $\pi_b(\mathbf{p}^\circ) = \mathbf{p}^{\circ\mathsf{T}}\mathbf{y}_b^\circ$, which is certainly earned by the household as sole owner of the firm, $\theta_{ab} = 1$. In this way the household has the income $w(\mathbf{p}^\circ) = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a'' + \mathbf{p}^{\circ\mathsf{T}}\mathbf{y}_b^\circ$, which equals the consumption expenditures $w(\mathbf{p}^\circ) = \mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^\circ$.

[73]The excess demand $Z(\mathbf{p})$ does not need to be defined for all $\mathbf{p} \in \Delta$. This case may occur for example when the production technology of a firm obeys constant returns to scale. Depending on what commodity prices $\mathbf{p}$ are considered, the profit is unbounded above. Then both $S_b(\mathbf{p}) = \emptyset$ and $Z(\mathbf{p}) = \emptyset$ hold.

[74]See Figure IV.12.

Replacing (IV.25) by the weaker requirement[75]

(IV.26) $$\mathbf{z}^\circ = \sum_{a \in A} \mathbf{x}_a^\circ - \sum_{b \in B} \mathbf{y}_b^\circ - \mathbf{w}_A \leqq \mathbf{0} \quad \text{with} \quad \mathbf{p}^\circ \in \Delta \quad \text{and} \quad \mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ = 0\,,$$

there may be no excess demand on the market for good j, $z_j^\circ = 0$, as long as the price of this good is positive, $p_j^\circ > 0$. Conversely, an excess supply $z_j^\circ < 0$ implies a commodity price $p_j^\circ = 0$. Thus, we look for an attainable state $(\mathfrak{x}^\circ, \mathfrak{y}^\circ, \mathbf{p}^\circ)$ with $\mathbf{z}^\circ \leqq \mathbf{0}$ and $\mathbf{p}^\circ \in \Delta$ which satisfies WALRAS' law, i.e. $\mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ = 0$.

To isolate those problems resulting from the indivisibility of some goods, we present for the moment the case of divisible goods. Thus, the commodity space is $X = \mathbb{R}^n$ or $X_+ = \mathbb{R}^n_+$.

Proposition IV.14[76] *In each production economy $\mathcal{E}$ fulfilling the following assumptions there is a* WALRAS*ian equilibrium $(\mathfrak{x}^\circ, \mathfrak{y}^\circ, \mathbf{p}^\circ)$ satisfying the three conditions (IV.24a), (IV.24b), and (IV.26).*

1. *For each household $a \in A$ we have:*

   (a) *The consumption sets X_a are closed, convex, and bounded below, i.e. there is an $\mathbf{x}_a^{min}$ so that $\mathbf{x}_a^{min} \leqq \mathbf{x}_a$ for all $\mathbf{x}_a \in X_a$.*

   (b) *Every household owns an initial endowment $\mathbf{w}_a \in X_+$ so that there is a commodity bundle $\tilde{\mathbf{x}}_a \in X_a$ with $\tilde{\mathbf{x}}_a < \mathbf{w}_a$.*

   (c) *Every household possesses a continuous preference ordering satisfying $\lambda\mathbf{x} + (1-\lambda)\tilde{\mathbf{x}} \succ_a \tilde{\mathbf{x}}$, $\lambda \in]0, 1]$ for all $\mathbf{x}, \tilde{\mathbf{x}} \in X_a$ with $\mathbf{x} \succ_a \tilde{\mathbf{x}}$.*

   (d) *The consumption sets X_a have no point of saturation.*

   (e) *The firms are owned by the households. Thus, the relative profit shares θ_{ab} are subject to (IV.20); the households' wealth $w_a(\mathbf{p})$ follows from (IV.22).*

2. *For each firm $b \in B$ we assume:*

   (a) *Each firm has the possibility of inaction, $\mathbf{0} \in Y_b$; the production sets Y_b are closed and convex.*

   (b) *The aggregate technology holds $Y_B \cap (-Y_B) = \{\mathbf{0}\}$ and $Y_B \supseteq (-X_+)$.*

A proof of Proposition IV.14 is omitted since the method varies only slightly from the methods applied with respect to exchange economies. At this point it may be enough to present the meaning of the premises of Proposition IV.14 and to give an

[75]DEBREU (1982, p. 704) refers to this case as *(Walrasian) free disposal equilibrium*. This condition is important if and only if all of the goods are assumed to be disposable and, therefore, that they could be thrown away without causing costs. In this case the price vector $\mathbf{p}^\circ$ must be nonnegative. If one good has a negative price, then in contradiction to (IV.24b) a firm could increase its profit arbitrarily by expanding the quantity of the (free disposable) good.

[76]Cf. ARROW, DEBREU (1954).

outline of the proof. The method of the proof is discussed especially with respect to Figure IV.14, which itself is founded on Figure IV.13.

Assumptions 1.(a) and 1.(c) are needed to guarantee the existence of a utility maximizing commodity bundle for each household. Here point $\mathbf{x}_a^{min}$ illustrated in Figure IV.14 can be understood as an existence minimum.[77] Assumption 1.(b) assures that each household has an initial endowment $\mathbf{w}_a$ so that there is a commodity bundle $\tilde{\mathbf{x}}_a \in X_a$ which satisfies $\mathbf{x}_a^{min} \leqq \tilde{\mathbf{x}}_a < \mathbf{w}_a$. Hence, each household can survive without exchange. In view of the existence proof there is one more important implication: independent of the price vector $\mathbf{p} \in \Delta$ each household owns an initial endowment with positive value,[78] i.e. $w_a(\mathbf{p}) = \mathbf{p}^{\mathsf{T}}\mathbf{w}_a > 0$.

Furthermore, the wealth of each household has to be greater than the expenditures for the cheapest commodity bundle $\mathbf{x}_a \in X_a$. Accordingly, household a can buy in particular the existence minimum $\mathbf{x}_a^{min}$; the budget set $B_a(\mathbf{p})$ is not empty.

The condition in 1.(c) is satisfied if the preferences are strictly convex, and it implies itself convex preferences;[79] the preference set $\mathcal{P}_a(\mathbf{x}_a^\circ)$ in Figure IV.13 has not been transferred to Figure IV.14. If 1.(c) is expanded by 1.(d), then all households satisfy their budget constraint, $\mathbf{p}^{\mathsf{T}}\mathbf{x}_a^d = w_a(\mathbf{p})$, and WALRAS' law holds.

All assumptions on the households together assure that the demand correspondences D_a are upper semi-continuous and that the profits are distributed completely by 1.(e) among the households.

Because a convex production set Y_b excludes increasing returns to scale for the firm b, it is more generally presumed, that Y_B is convex.[80] As proved by NIKAIDO, the outcomes of Proposition IV.14 will still be the same.[81] MCKENZIE supposes further that the set Y_B is a closed convex cone.[82] Thus, he presumes constant returns to scale at least for the aggregate technology. Whereas Figure IV.13 includes a convex production set Y_b, Figure IV.14 illustrates the production set Y_b as a convex cone.

By Assumption 2.(b) it is supposed that the aggregate processes are irreversible, $Y_B \cap (-Y_B) = \{\mathbf{0}\}$. If $\mathbf{y}_B \neq \mathbf{0}$ is composed of possible activities $\mathbf{y}_b$ of the sole

[77]For a comparison of the assumptions to the similar conditions in MCKENZIE (1981, p. 821), we have to take into account that the consumption sets $\bar{X}_a$ in MCKENZIE describes the household a's excess demand, i.e. $\mathbf{x}_a \in X_a \implies \mathbf{x}_a - \mathbf{w}_a \in \bar{X}_a$.

[78]Under these conditions the equilibria discussed in ARROW, HAHN (1971), Theorem 5 (p. 119, *competitive equilibrium*) and Theorem 4 (p. 116, *compensated equilibrium*) are equivalent; see Theorem 2 (p. 109).
The condition that each person has a positive wealth for every price vector and that no person has a point of saturation (1.(b) and 1.(e)) may be substituted by alternative (local) conditions. For example, MCKENZIE (1959) introduces an assumption, which is called the *irreducibility of the (convex) market*; it assures that each person has a positive wealth in a neighborhood of the equilibrium prices and that there is at least locally nonsaturation.

[79]Cf. NIKAIDO (1968, p. 241, Lemma 15.5).

[80]For equilibria in production economies with increasing returns to scale, see VILLAR (1996, Chapter 4).

[81]Cf. NIKAIDO (1968, p. 256, Theorem 16.4).

[82]Cf. MCKENZIE (1981, p. 821).

firm, then the aggregate activity $-\mathbf{y}_B$, where the roles of all inputs and outputs are interchanged, is not viable. Given free disposability of all goods, $Y_B \supseteq (-X_+)$, the land of Cockaigne is ruled out by $Y_B \cap X_+ = \{\mathbf{0}\}$ since

$$\{\mathbf{0}\} = Y_B \cap (-Y_B) \supseteq Y_B \cap X_+ \supseteq \{\mathbf{0}\} \implies Y_B \cap X_+ = \{\mathbf{0}\}.$$

Astonishing, with respect to production 2.(d) is enough to prove that all activities $\mathbf{y}_b$ lie within a compact set for all production allocations $\mathfrak{y} = (\mathbf{y}_b)_{b \in B}$ which are possible by (IV.21).[83] This implication is important for the existence proof and is discussed in Figure IV.14 by the set $\widehat{Y}_b$.

In particular, if the restrictive assumption 1.(b) is relaxed, then there must be a prerequisite that there is an admissible allocation $(\mathfrak{x}, \mathfrak{y})$ associated with a positive excess supply for all goods,[84] $\sum_{b \in B} \mathbf{y}_b + \mathbf{w}_A > \sum_{a \in A} \mathbf{x}_a$. This implication is not explicitly required by Proposition IV.14, but it is implicitly involved by 1.(b) and 2.(a): if each firm chooses the possibility of inaction $\mathbf{y}_b = \mathbf{0}$, then 1.(b) yields $\mathbf{w}_A > \sum_{a \in A} \tilde{\mathbf{x}}_a$.

Sketch of the proof of Proposition IV.14: Looking at the proof for the existence of competitive equilibria in exchange economies suggests the conjecture that at this point an appropriate treatment of the aggregate excess demand Z should lead to the goal, too. This is an illusion. As mentioned above, the additional consideration of production implies that the excess demand need not be defined for each price vector $\mathbf{p} > \mathbf{0}$; see footnote 70.

With regard to KAKUTANI's fixed-point theorem[85] it is convenient to use the method presented in Section 1.3. We seek for a fixed-point of the correspondence Γ defined afterwards by (IV.28), which reflects the simultaneous optimization approach of all market participants. Before going into the relevant properties of the correspondence Γ, it must be proved that the examined correspondence is defined on a nonempty, *compact*, and convex set C and that it maps into this set, i.e. $\Gamma\colon C \to \mathfrak{P}(C)$. Regarding a production economy, the sets $\widehat{X}_a$ and $\widehat{Y}_b$ (shown in Figure IV.14) together with the price simplex Δ will adopt the role of C; see (IV.27) below.

Analogous to Figure IV.5, p. 222, Figure IV.14 sketches the procedure when dealing with a production economy. The consumption set X_a of the sole household faces the production set Y_b of the sole firm. The cone, denoted by Y_b, implies constant returns to scale and excludes a profit maximum for a sufficiently large price ratio p_2/p_1.

Conversely, the production of the second commodity good takes place only if the price ratio p_2/p_1 is not too low. The household has the initial endowment $\mathbf{w}_a > \mathbf{x}_a^{min}$. Thus, $\bar{\mathbf{y}}_b$ denotes the production activity with the highest of all possible production levels, i.e. $\bar{\mathbf{y}}_b + \mathbf{w}_a \geqq \mathbf{0}$. Despite this, it must be taken into account that

[83] Cf. SMALE (1981, p. 364 f., Lemma A.2) and ARROW, HAHN (1971, p. 66, Theorem 2).

[84] See ARROW, DEBREU (1954, p. 280).
MCKENZIE (1981, p. 821) supposes that the sets $X_A = \sum_{a \in A}(X_a - \mathbf{w}_a)$ and $Y_B = \sum_{b \in B} Y_b$ have a relatively interior point in common, $\operatorname{rint} X_A \cap \operatorname{rint} Y_B \neq \emptyset$.

[85] See Theorem C.16, p. 313.

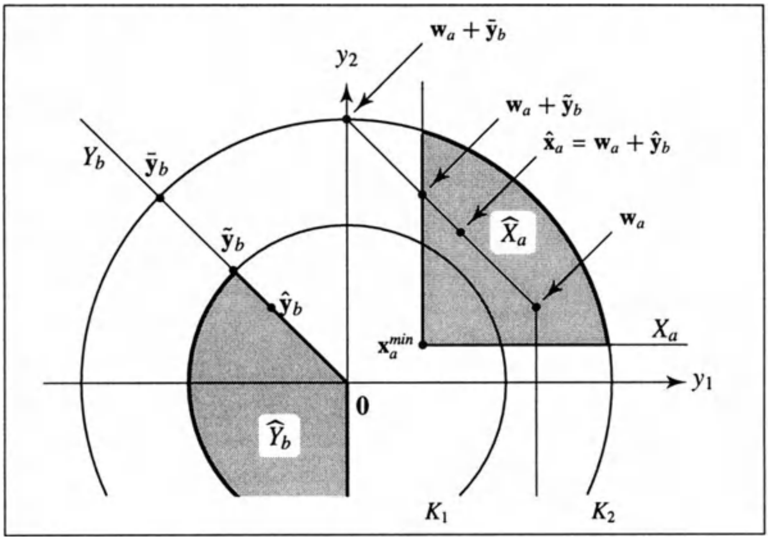


Figure IV.14: Proof of Proposition IV.14

the activity $\tilde{\mathbf{y}}_b$ is the highest production level consistent with the existence minimum $\mathbf{x}_a^{min}$, i.e. $\tilde{\mathbf{y}}_b + \mathbf{w}_a \geqq \mathbf{x}_a^{min}$.

With respect to the required compact and convex set C we now define a ball K_1 centered at the origin of radius $\|\tilde{\mathbf{y}}_b\|$. Thus, $\widehat{Y}_b := Y_b \cap K_1$ is a compact convex set containing all possible production activities $\mathbf{y}_b \in Y_b$ with $\|\mathbf{y}_b\| \leqq \|\tilde{\mathbf{y}}_b\|$. The restriction to the set $\widehat{Y}$ is irrelevant for an optimal solution $\hat{\mathbf{y}}_b$ to the problem of profit maximization $\max\left\{\mathbf{p}^{\mathrm{T}}\mathbf{y}_b \mid \mathbf{y}_b \in \widehat{Y}_b\right\}$ as long as $\|\hat{\mathbf{y}}_b\| < \|\tilde{\mathbf{y}}_b\|$ is satisfied. In this case $\hat{\mathbf{y}}_b$ also solves the genuine problem $\max\left\{\mathbf{p}^{\mathrm{T}}\mathbf{y}_b \mid \mathbf{y}_b \in Y_b\right\}$.

Similarly, a ball K_2 centered at the origin[86] can be determined such that the relation $\|\mathbf{x}_a\| \leqq \|\tilde{\mathbf{y}}_b + \mathbf{w}_a\|$ holds for all admissible commodity bundles $\mathbf{x}_a \in X_a$. As before, a compact and convex set $\widehat{X}_a = X_a \cap K_2$ results. If a commodity bundle $\hat{\mathbf{x}}_a \in \widehat{X}_a$ solves the problem of utility maximization, then $\hat{\mathbf{x}}_a$ is also an optimal solution to the original problem of utility maximization, provided $\|\hat{\mathbf{x}}_a\| < \|\tilde{\mathbf{y}}_b + \mathbf{w}_a\|$ holds good.

The above determined sets $\widehat{X}_a$ and $\widehat{Y}_b$ are shown in Figure IV.14 by gray areas and they can now be united with the price simplex Δ to the required set

$$C = \widehat{X}_a \times \widehat{Y}_b \times \Delta. \tag{IV.27}$$

Since all subsets are not empty, compact, and convex, C has these properties, too.

The concluding comments apply directly to KAKUTANI's fixed-point theorem,[87] where we look for a fixed-point of the correspondence $\Gamma\colon C \to \mathfrak{P}(C)$ defined by (IV.28). Under the premises of Proposition IV.14 it can be shown that household

[86] The ball K_2 chosen in Figure IV.14 suffices for the sake of demonstration.
[87] For a generalization to many households and firms, see DEBREU (1982).

a's preferences can be described by a continuous quasi-concave utility function u_a. When we introduce the market as agent choosing a price vector $\mathbf{p} \in \Delta$ so that the value of the excess demand on all markets is maximized, then the *continuous* objective functions of all market participants in the simultaneous optimization approach are given by

$$\begin{aligned}
\tilde{u}_a(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p}) &= u_a(\mathbf{x}_a) && \text{for the household } a,\\
\tilde{\pi}_b(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p}) &= \mathbf{p}^\mathsf{T}\mathbf{y}_b && \text{for the firm } b,\\
t(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p}) &= \mathbf{p}^\mathsf{T}\mathbf{z} && \text{for the market with } \mathbf{z} = \mathbf{x}_a - \mathbf{y}_b - \mathbf{w}_a.
\end{aligned}$$

The households' budget set is now $\tilde{B}_a(\mathbf{p}) = \{\mathbf{x}_a \in \widehat{X}_a | \, \mathbf{p}^\mathsf{T}\mathbf{x}_a \leqq \mathbf{p}^\mathsf{T}\mathbf{w}_a + \pi_b(\mathbf{p})\}$. If two out of the three scrutinized market participants have chosen an activity, then the third agent must consider these facts at the choice of his particular optimal activity. Now the set $T(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b)$ collects all of the price vectors which are optimal for the market, provided the household chooses the commodity bundle $\tilde{\mathbf{x}}_a$ and the firm chooses the activity $\tilde{\mathbf{y}}_b$. It turns out to be technically more pleasant to denote the set $T(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b)$ by $T(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b, \tilde{\mathbf{p}})$, where it must be taken into account that $T(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b, \tilde{\mathbf{p}})$ does not change with any variation of $\tilde{\mathbf{p}}$.

For a state of the economy $(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b, \tilde{\mathbf{p}}) \in \widehat{X}_a \times \widehat{Y}_b \times \Delta$ the following sets of optimal activities result regarding the three market participants – the household a, the firm b, and the market – respectively.[88]

$$\begin{aligned}
\tilde{D}_a(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b, \tilde{\mathbf{p}}) &= \{\hat{\mathbf{x}}_a \in \tilde{B}_a(\tilde{\mathbf{p}}) | \, u_a(\hat{\mathbf{x}}_a) = \max\{u_a(\mathbf{x}_a) | \, \mathbf{x}_a \in \tilde{B}_a(\tilde{\mathbf{p}})\}\}\\
\tilde{S}_b(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b, \tilde{\mathbf{p}}) &= \{\hat{\mathbf{y}}_b \in \widehat{Y}_b | \, \tilde{\mathbf{p}}^\mathsf{T}\hat{\mathbf{y}}_b = \max\{\tilde{\mathbf{p}}^\mathsf{T}\mathbf{y}_b | \, \mathbf{y}_b \in \widehat{Y}_b\}\}\\
T(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b, \tilde{\mathbf{p}}) &= \{\hat{\mathbf{p}} \in \Delta | \, \hat{\mathbf{p}}^\mathsf{T}\tilde{\mathbf{z}} = \max\{\mathbf{p}^\mathsf{T}\tilde{\mathbf{z}} | \, \mathbf{p} \in \Delta\}\}
\end{aligned}$$

Following KAKUTANI's fixed-point theorem, the purpose of the analysis can now be made aware of. Concerning the nonempty convex compact set $C = \widehat{X}_a \times \widehat{Y}_b \times \Delta$ we define the correspondence $\Gamma : C \to \mathfrak{P}(C)$ with[89]

$$\Gamma(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p}) := \tilde{D}_a(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p}) \times \tilde{S}_b(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p}) \times T(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p}) \tag{IV.28}$$

so that it must be shown that the correspondence Γ has a fixed-point

$$(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ) \in \Gamma(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ).$$

If the above state of the economy satisfies $(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b, \tilde{\mathbf{p}}) = (\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$, then the simultaneous optimization approach permits the choice $(\hat{\mathbf{x}}_a, \hat{\mathbf{y}}_b, \hat{\mathbf{p}}) = (\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$. Thus, the examined economy with the fixed-point $(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$ has a state at which each actor makes an optimal choice.

Since the sole household is restricted to the budget constraint $\mathbf{x}_a^\circ \in \tilde{B}_a(\mathbf{p}^\circ)$ the equation $\mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ = \mathbf{p}^{\circ\mathsf{T}}(\mathbf{x}_a^\circ - \mathbf{w}_a - \mathbf{y}_b^\circ) \leqq 0$ must hold. If the market maximizes

[88] At this point is has to be shown moreover that the sets $\tilde{D}_a(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p})$, $\tilde{S}_b(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p})$, and $T(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p})$ are not empty and convex for all states $(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b, \tilde{\mathbf{p}}) \in \widehat{X}_a \times \widehat{Y}_b \times \Delta$ of the economy.

[89] The Cartesian product of upper semi-continuous correspondences is also upper semi-continuous. At this point we refrain from proving the upper semi-continuity of the correspondences $\tilde{D}_a$, $\tilde{S}_b$, and T.

the value of the excess demand, then $\mathbf{p}^{\mathsf{T}}\mathbf{z}^\circ \leqq 0$ for all $\mathbf{p} \in \Delta$ and, therefore, $\mathbf{z}^\circ \leqq \mathbf{0}$. Thus, the state $(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$ is attainable. Assuming nonsaturation, the household will utilize its whole budget, where at the same time $\mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ = 0$ corresponds to WALRAS' law. We obtain a WALRASian equilibrium of the form (IV.26).

(b) Consideration of Indivisible Goods If some goods are only available at integer units, then the consumption set X_a and the production set Y_b are usually nonconvex subsets in the commodity space $X = \mathbb{R}^{n_d} \times \mathbb{Z}^{n-n_d}$. Thus, it seems to be reasonable to replace the consumption sets and the production sets with their convex hulls $\operatorname{conv} X_a$ and $\operatorname{conv} Y_b$ respectively. After substituting the budget sets by

$$B_a^{co}(\mathbf{p}) := \Big\{\mathbf{x}_a \in \operatorname{conv} X_a \,|\, \mathbf{p}^{\mathsf{T}}\mathbf{x}_a \leqq \mathbf{p}^{\mathsf{T}}\mathbf{w}_a + \sum_{b\in B} \theta_{ab}\mathbf{p}^{\mathsf{T}}\mathbf{y}_b\Big\}$$

we have a problem analogous to the one described in Figure IV.14. For households as well as for firms we have to establish nonempty compact convex sets embracing all possible activities of the respective economic agent. The adequate sets are now $\widehat{Y}_b^{co} := \operatorname{conv} Y_b \cap K_1$ and $\widehat{X}_a^{co} := \operatorname{conv} X_a \cap K_2$. As before, for a given state of the economy

$$(\tilde{\mathfrak{x}}, \tilde{\mathfrak{y}}, \tilde{\mathbf{p}}) \in C^{co} \quad \text{with} \quad C^{co} := \mathop{\times}_{a\in A} \widehat{X}_a^{co} \times \mathop{\times}_{b\in B} \widehat{Y}_b^{co} \times \Delta$$

the subsequent synthetic sets[90] of optimal activities for the respective market participants result – the households $a \in A$, the firms $b \in B$, and the market.

$$\begin{aligned}
\tilde{D}_a^{co}(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b, \tilde{\mathbf{p}}) &= \bigcap_{\mathbf{x}_a\in B_a(\tilde{\mathbf{p}})} B_a^{co}(\tilde{\mathbf{p}}) \cap \widehat{X}_a^{co} \cap \operatorname{conv}\mathcal{P}_a(\mathbf{x}_a) \\
\tilde{S}_b^{co}(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b, \tilde{\mathbf{p}}) &= \left\{\hat{\mathbf{y}}_b \in \widehat{Y}_b^{co} \,|\, \tilde{\mathbf{p}}^{\mathsf{T}}\hat{\mathbf{y}}_b = \max\left\{\tilde{\mathbf{p}}^{\mathsf{T}}\mathbf{y}_b \,|\, \mathbf{y}_b \in \widehat{Y}_b^{co}\right\}\right\} \\
T(\tilde{\mathbf{x}}_a, \tilde{\mathbf{y}}_b, \tilde{\mathbf{p}}) &= \left\{\hat{\mathbf{p}} \in \Delta \,|\, \hat{\mathbf{p}}^{\mathsf{T}}\tilde{\mathbf{z}} = \max\left\{\mathbf{p}^{\mathsf{T}}\tilde{\mathbf{z}} \,|\, \mathbf{p} \in \Delta\right\}\right\}
\end{aligned}$$

The difference to the previous analysis is particularly revealed by the synthetic demand correspondence $\tilde{D}_a^{co}$. Since the premises are not enough to guarantee the existence of a continuous and quasi-concave utility function u_a, we have to revert to an alternative derivation of the correspondence $\tilde{D}_a^{co}$. The surrogate $\tilde{D}_a^{co}$ must not arbitrarily differ from the real demand correspondence D_a and it must be upper semi-continuous in the entire price simplex Δ. Furthermore, KAKUTANI's fixed-point theorem requires the level sets $\tilde{D}_a^{co}(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p})$ to be not empty, compact, and convex. In the derivation of these sets especially the relation $\mathbf{x}_a \in B_a(\mathbf{p}) \subset X_a$ assures that $\operatorname{conv}\mathcal{P}_a(\mathbf{x}_a) \neq \emptyset$ holds.

Assume the suggested modifications suffice to prove a fixed-point of the correspondence $\Gamma^{co}\colon C^{co} \to \mathfrak{P}(C^{co})$ with

$$\Gamma^{co}(\mathfrak{x}, \mathfrak{y}, \mathbf{p}) = \mathop{\times}_{a\in A} \tilde{D}_a^{co}(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p}) \times \mathop{\times}_{b\in B} \tilde{S}_b^{co}(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p}) \times T(\mathbf{x}_a, \mathbf{y}_b, \mathbf{p}).$$

[90]The definition of the synthetic demand correspondence has already been suggested in Section 1.3.2 and is discussed in detail by BROOME (1972).

Then this fixed-point $(\mathfrak{x}^\circ, \mathfrak{y}^\circ, \mathbf{p}^\circ) \in \Gamma(\mathfrak{x}^\circ, \mathfrak{y}^\circ, \mathbf{p}^\circ)$ denotes a state of the "convex-valued" production economy $\mathcal{E}$ satisfying[91]

(IV.29a)	$\mathbf{x}_a^\circ \in \tilde{D}_a^{co}(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$	for all households $a \in A$,
(IV.29b)	$\mathbf{y}_b^\circ \in \tilde{S}_b^{co}(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$	for all firms $b \in B$,
(IV.29c)	$\mathbf{p}^\circ \in T(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$	for the market.

Moreover, if suitable assumptions assure – as in Proposition III.30 (WALRAS' law) – that each household makes use of its whole budget, $\mathbf{p}^{\circ\mathsf{T}}\mathbf{x}_a^\circ = \mathbf{p}^{\circ\mathsf{T}}\mathbf{w}_a + \sum_b \theta_{ab}\mathbf{p}^{\circ\mathsf{T}}\mathbf{y}_b^\circ$, then analogous to Proposition IV.14 it ensues

$$\text{(IV.26)} \quad \mathbf{z}^\circ = \sum_{a\in A}\mathbf{x}_a^\circ - \sum_{b\in B}\mathbf{y}_b^\circ - \mathbf{w}_A \leqq \mathbf{0} \quad \text{with} \quad \mathbf{p}^\circ \in \Delta \quad \text{and} \quad \mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ = 0\,.$$

With that we have reached a point for the presented production economy that has been noted by Proposition IV.9, p. 232, regarding exchange economies.

If the production technologies also satisfy Assumption 4, p. 236, i.e. $D^{sup} \geqq \operatorname{rad} S_b(\mathbf{y}_b)$, then, analogous to Corollary IV.11.1, we can estimate how far the allocation $(\mathfrak{x}^\circ, \mathfrak{y}^\circ)$ concerned can deviate at the most from an allocation $(\mathfrak{x}^d, \mathfrak{y}^d)$ with

$\mathbf{y}_b^d \in S_b(\mathbf{p}^\circ)$	for all firms $b \in B$,
$\mathbf{x}_a^d \in D_a(\mathbf{p}^\circ)$	for all households $a \in A$.

In this context ARROW and HAHN[92] present the subsequent estimation for nonconvex preferences and nonconvex production sets Y_b

$$\|(\sum_{a\in A}\mathbf{x}_a^\circ - \sum_{b\in B}\mathbf{y}_b^\circ) - (\sum_{a\in A}\mathbf{x}_a^d - \sum_{b\in B}\mathbf{y}_b^d)\| \leqq \sqrt{n}\,D^{sup}\,.$$

Inequalities of this form have been discussed in more detail in Section IV.2.3 and this need not be repeated at this point since we cannot expect that this analysis gives basically new insights. Nevertheless, the question now arises as to how far the production technologies hold $D^{sup} \geqq \operatorname{rad} S_b(\mathbf{y}_b)$ when considering indivisible goods. The number D^{sup} has been interpreted at its introduction as a measure for the degree of nonconvexity of preference sets. Referring to production this interpretation is lacking in content when relative to the examined economy large and significant indivisibilities appear which result in increasing economies to scale. As examples for this SCARF (1994) offers production lines, bridges, and networks which are only available at certain discrete sizes and whose utilization is merely useful at a high production scale. The results following from an approximation of the production set Y_b by its convex hull $\operatorname{conv} Y_b$ are useless in this case.

To give an idea of the described problem, Figure IV.15 offers a production set Y_b which has similar properties as in Figure II.23, p. 42.[93] Remember that the

[91] A similar statement is given by ARROW, HAHN (1971, p. 177) regarding nonconvex preferences.
[92] Cf. ARROW, HAHN (1971, p. 178, Theorem 1).
[93] A similar example may be found in MAS-COLELL, WHINSTON, GREEN (1995, p. 572).

production technology is indicated by netput vectors. After the convex hull of the production set Y_b has been deduced, the household a's initial endowment $\mathbf{w}_a$ can be added to the set $\operatorname{conv}Y_b$. Analogous to Figure IV.13, we get a graphical representation of the set $\operatorname{conv}Y_b + \mathbf{w}_a$. Starting with the netput vectors $\mathbf{y}_b^1, \mathbf{y}_b^2, \mathbf{0} \in Y_b$, then $\mathbf{x}_a^1 = \mathbf{y}_b^1 + \mathbf{w}_a$, $\mathbf{x}_a^2 = \mathbf{y}_b^2 + \mathbf{w}_a$, and $\mathbf{w}_a$ denote the relevant points of the set $\operatorname{conv}Y_b + \mathbf{w}_a$.

To grasp the problem of significant indivisibilities with increasing returns to scale, we now refer to point $\mathbf{x}_a^\circ$ which is assigned to the state $(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$ (with $\mathbf{x}_a^\circ = \mathbf{w}_a + \mathbf{y}_b^\circ$). If this point fulfills the conditions (IV.29a)–(IV.29c) and (IV.26), then it must be an integer commodity bundle $\mathbf{x}_a^\circ \in X$ since the correspondence $\tilde{D}_a^{co}$, used in (IV.29a), is based on the set $\operatorname{conv}\mathcal{P}_a(\mathbf{x}_a^\circ)$. In the case of $\mathbf{x}_a^\circ \notin X$ not only the preference set $\mathcal{P}_a(\mathbf{x}_a^\circ)$ but also its convex hull $\operatorname{conv}\mathcal{P}_a(\mathbf{x}_a^\circ)$ and, therefore, $\tilde{D}_a^{co}(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$ were empty contrary to the hypothesis $\mathbf{x}_a^\circ \in \tilde{D}_a^{co}(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$. Given an admissible commodity bundle $\mathbf{x}_a^\circ \in X$, the problem is limited to production, $\mathbf{y}_b^\circ \in \tilde{S}_b^{co}(\mathbf{x}_a^\circ, \mathbf{y}_b^\circ, \mathbf{p}^\circ)$ by (IV.29b). In the sense of important indivisibilities with increasing returns to scale it is now supposed that there is no possible netput vector[94] $\mathbf{y}_b^\circ \in Y_b$ such that $\mathbf{y}_b^\circ + \mathbf{w}_a \in \operatorname{conv}\{\mathbf{x}_a^1, \mathbf{x}_a^2, \mathbf{w}_a\}$ holds.[95] In this case $\mathbf{x}_a^\circ = \mathbf{y}_b^\circ + \mathbf{w}_a$ merely gives an approximation to the possible point $\mathbf{x}_a^1 = \mathbf{y}_b^1 + \mathbf{w}_a$. This approximation must particularly be considered as unsatisfactory if we lay great emphasis on large indivisibilities.

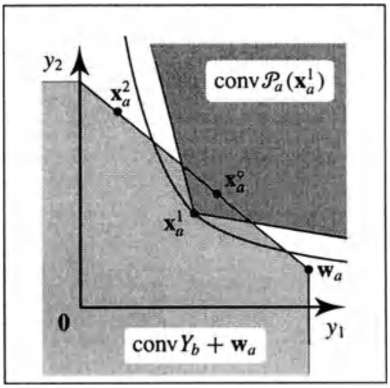


Figure IV.15: Increasing returns to scale

Moreover, Figure IV.15 illustrates another problem that has already been mentioned in a similar form by Figure IV.3, p. 219, concerning exchange economies. If the "boundary points" of the set $Y_b + \mathbf{w}_a$ are given by $\mathbf{w}_a$, $\mathbf{x}_a^1$, and $\mathbf{x}_a^2$ and if $\mathbf{x}_a^1 \succ_a \mathbf{w}_a \succ_a \mathbf{x}_a^2$ holds for the household a, then there is no price *line*, separating the sets $Y_b + \mathbf{w}_a$ and $\operatorname{conv}\mathcal{P}(\mathbf{x}_a^1)$. At the same time the existence of a WALRAS equilibrium is excluded.

With that the question arises as to whether it is in principle possible to construct a nonlinear pricing rule[96] superceding the price line and yielding $\mathbf{x}_a^1 = \mathbf{y}_a^1 + \mathbf{w}_a$ as a market result. Such a nonlinear pricing rule is sketched in Figure IV.15 by the curve through $\mathbf{x}_a^1$. In particular, $\mathbf{x}_a^2 = \mathbf{y}_b^2 + \mathbf{w}_a$ and $\mathbf{w}_a$ (with $\mathbf{y}_b = \mathbf{0}$) lie below this curve so that $\mathbf{y}_b^1 = \mathbf{x}_a^1 - \mathbf{w}_a$ corresponds to the profit maximizing activity of the firm.

[94]Since both $\mathbf{x}_a^\circ$ and $\mathbf{w}_a$ satisfy the integer constraints, $\mathbf{y}_b^\circ = \mathbf{x}_a^\circ - \mathbf{w}_a$ satisfies the conditions, too. Thus, in the case of an integer convex production set Y_b the relation $\mathbf{y}_b \in \operatorname{conv}Y_b$ would imply a possible netput vector $\mathbf{y}_b \in Y_b$. The example would be null and void.

[95]The previous remarks only imply $\mathbf{y}_b^\circ \in \operatorname{conv}Y_b$ and $\mathbf{x}_a^\circ = \mathbf{y}_b^\circ + \mathbf{w}_a \in \operatorname{conv}\{\mathbf{x}_a^1, \mathbf{x}_a^2, \mathbf{w}_a\}$.

[96]JOUINI (1992) examines general equilibria with respect to nonconvex production sets, where the firm sets the commodity prices in accordance with nonlinear pricing rules. One pricing rule determines a lower bound for the firm's losses, another rule fixes the commodity price corresponding to marginal costs. Last but not least WILSON (1993) provides a detailed discussion of nonlinear pricing.

Such pricing rules are examined in BROWN, HEAL (1980) and KAMIYA (1995) for a general equilibrium model.[97] Here the pricing rules should not primarily assure the existence of equilibria but for the moment PARETO optimal allocations. The following section is dedicated to this criterion of optimality.

3.2 Alternative Criteria for Optimal Market Results

3.2.1 The Core of an Exchange Economy

3.2.2 PARETO Optimal Allocations

The WALRASian equilibrium has been characterized by high requirements on market results. The class of all PARETO efficient allocations is less restrictive and requires a very weak property of economic efficiency. A feasible allocation $\mathfrak{x}$ is said to be **PARETO efficient** or **PARETO optimal** if there is no alternative feasible allocation $\mathfrak{x}'$ which puts none of the persons ($\mathbf{x}_a \succcurlyeq_a \mathbf{x}'_a$ for all $a \in A$) in a worse position and at least one of them ($\mathbf{x}_a \succ_a \mathbf{x}'_a$ for at least one $a \in A$) in a better position.

Letting $\mathfrak{X}$ be the set of all feasible allocations $\mathfrak{x} \in \underset{a \in A}{\times} X$, then we can define a binary relation $\succ_A$ on $\mathfrak{X}$ by putting

$$\mathfrak{x} \succ_A \mathfrak{x}' \iff \left[\mathbf{x}_a \succcurlyeq_a \mathbf{x}'_a \text{ for all } a \in A \quad \text{and} \quad \mathbf{x}_a \succ_a \mathbf{x}'_a \text{ for one } a \in A\right]$$

for all $\mathfrak{x},\ \mathfrak{x}' \in \mathfrak{X}$. Thus, a PARETO optimal allocation corresponds to a maximal element in $\mathfrak{X}$ with respect to relation $\succ_A$.

The following propositions refer to an economy in which exclusively indivisible goods occur. Thus, for a finite endowment $\mathbf{w}_A > \mathbf{0}$ it is guaranteed that $\mathfrak{X}$ consists of a finite number of feasible allocations. To prove the existence of a PARETO optimal allocation, we have now to show:

Proposition IV.15[98] *If each person $a \in A$ possesses a preference ordering $\succcurlyeq_a$, then $\succ_A$ is an irreflexive quasi-ordering, i.e. $\succ_A$ is irreflexive and transitive.*

Proof: The assumed preference orderings are reflexive [$\mathscr{P}$1] and transitive [$\mathscr{P}$3] by definition; see p. 6.

(a) For each allocation $\mathfrak{x} \in \mathfrak{X}$ the implication

$$\mathfrak{x} \succ_A \mathfrak{x} \implies \mathbf{x}_a \succ_a \mathbf{x}_a \quad \text{for one } a \in A$$

generates a contradiction of the assumed reflexivity of the preference orderings since $\mathbf{x}_a \succ_a \mathbf{x}_a$ implies $\neg(\mathbf{x}_a \succcurlyeq_a \mathbf{x}_a)$. Thus the relation $\succ_A$ is irreflexive because of $\neg(\mathfrak{x} \succ_A \mathfrak{x})$ for all $\mathfrak{x} \in \mathfrak{X}$.

(b) For three feasible allocations $\mathfrak{x}$, $\mathfrak{x}'$, and $\mathfrak{x}''$ with $\mathfrak{x} \succ_A \mathfrak{x}'$ and $\mathfrak{x}' \succ_A \mathfrak{x}''$ we obtain

[97] A survey on general equilibrium theory with increasing returns to scale is given by CORNET (1988), where the attention is drawn to "nonconvex economies" and not directly to indivisible goods. VILLAR (1996) examines increasing returns to scale.

[98] Cf. EMMERSON (1972, p. 180).

1. $\mathbf{x}_{\tilde{a}} \succ_{\tilde{a}} \mathbf{x}'_{\tilde{a}}$ for at least one person $\tilde{a} \in A$. Due to $\mathbf{x}_{\tilde{a}} \succ_{\tilde{a}} \mathbf{x}'_{\tilde{a}}$ and $\mathbf{x}'_{\tilde{a}} \succcurlyeq_{\tilde{a}} \mathbf{x}''_{\tilde{a}}$ it ensues $\mathbf{x}_{\tilde{a}} \succ_{\tilde{a}} \mathbf{x}''_{\tilde{a}}$ by transitivity.

2. for all of the remaining persons $a \in A$ $(a \neq \tilde{a})$ the implication $\mathbf{x}_a \succcurlyeq_a \mathbf{x}'_a$, $\mathbf{x}'_a \succcurlyeq_a \mathbf{x}''_a \implies \mathbf{x}_a \succcurlyeq_a \mathbf{x}''_a$ follows immediately from the supposed transitivity.

The union of the two arguments induces $\mathbf{x} \succ_A \mathbf{x}''$ and therefore the transitivity of the relation $\succ_A$. ■

Hence, considering Proposition II.1, p. 9, we can state:

> *For each economy with a finite number of indivisible goods and a finite endowment there is a* PARETO *optimal allocation if each of the finitely many persons has a preference ordering.*

In each exchange economy with a divisible good we can think of an infinite number of feasible allocations. Nevertheless, the existence of an individually rational PARETO optimum is assured if each consumer has a continuous preference ordering.[99] As the proof of this statement does not refer to the divisibility or indivisibility of goods, the proof is omitted.[100]

In Figure IV.2, p. 218, the set of all PARETO efficient allocations is indicated by the contract curve (O^1, O^2). Each PARETO efficient allocation corresponds to a point of tangency of two indifference curves. At the same time it becomes apparent that not all of the PARETO efficient allocations can lead to satisfying results. A person will not agree to any allocation $\mathbf{x}$ in which his position is worsened in comparison with his initial endowment $\mathbf{w}_a$ $(\mathbf{w}_a \succ_a \mathbf{x}_a)$.

In this sense HILDENBRAND, KIRMAN (1988) pick up the idea of an "individually rational PARETO optimum", where each person must accept the PARETO optimal $\mathbf{x}$ allocation $(\mathbf{x}_a \succcurlyeq_a \mathbf{w}_a$ for all $a \in A)$. If we ignore for a moment the criterion of PARETO efficiency, then we have to ask when an allocation can be "improved" at all. For this purpose we examine arbitrary nonempty subsets S in the set of all persons A which are called **coalitions**. A coalition S is able to improve the allocation $\mathbf{x}$ if an alternative allocation $\mathbf{x}'$ exists such that $\mathbf{x}'_a \succ_a \mathbf{x}_a$ for every person a of the allocation S and $\sum_{a\in S} \mathbf{x}'_a = \sum_{a\in S} \mathbf{w}_a$. Eventually, the **core** of an economy contains all allocations which cannot be improved by any coalition.[101] Figure IV.1, p. 216, with two divisible goods illustrates the core as a part of the contract curve determined by the section (B^1, B^2).

From this approach of game theory it follows that each allocation $\mathbf{x}$ in the core is individually rational $(\mathbf{x}_a \succcurlyeq_a \mathbf{w}_a$ for all $a \in A)$ and it is weakly PARETO efficient (there is no allocation $\mathbf{x}'$ with $\mathbf{x}'_a \succ_a \mathbf{x}_a$ for all $a \in A$).[102] For the following propositions it is important to know when the criteria of PARETO efficiency equals the weak PARETO efficiency, where the influence of indivisible goods must be taken into account.

[99] See [$\mathcal{P}$4] on p. 6

[100] See ALIPRANTIS, BROWN, BURKINSHAW (1989, p. 40, Theorem 1.5.3).

[101] See HILDENBRAND (1987).

[102] Cf. ALIPRANTIS, BROWN, BURKINSHAW (1989, pp. 39–42), in particular Theorem 1.5.5.

Proposition IV.16 *Suppose that all persons in an exchange economy possess continuous and strongly monotone preference orderings. If there is at least one divisible commodity, then a feasible allocation* $\mathbf{x}$ *is* PARETO *efficient if and only if it is weakly* PARETO *efficient.*

Proof: Each PARETO efficient allocation is immediately weakly PARETO efficient and we have merely to prove the reverse conclusion. Thus, no feasible allocation $\mathbf{x}'$ can imply a PARETO improvement to a weakly PARETO efficient allocation $\mathbf{x}$.[103] The starting point of the proof of contradiction is a weakly PARETO optimal allocation $\mathbf{x}$ and a feasible allocation $\mathbf{x}'$ being a PARETO improvement, i.e.

$$\begin{aligned} & \mathbf{x}'_a \succcurlyeq_a \mathbf{x}_a \quad \text{for each person} \quad a \in A \\ \text{and} \quad & \mathbf{x}'_{\tilde{a}} \succ_{\tilde{a}} \mathbf{x}_{\tilde{a}} \quad \text{for one person} \quad \tilde{a} \in A. \end{aligned}$$

Without loss of generality it is supposed that the first good is divisible. If we take away the positive quantity α of the first good from person $\tilde{a}$, then $\mathbf{x}'_{\tilde{a}} \succ_{\tilde{a}} \mathbf{x}'_{\tilde{a}} - \alpha \mathbf{e}^1$ for a strongly monotone preference ordering. The continuity of the preference ordering means that the preference set $\mathcal{P}_{\tilde{a}}(\mathbf{x}'_{\tilde{a}})$ is closed. Due to $\mathbf{x}_{\tilde{a}} \notin \mathcal{P}_{\tilde{a}}(\mathbf{x}'_{\tilde{a}})$ there must be an open ball K centered at $\mathbf{x}_{\tilde{a}}$ of radius α such that $K(\mathbf{x}_{\tilde{a}}, \alpha) \cap \mathcal{P}_{\tilde{a}}(\mathbf{x}'_{\tilde{a}}) = \emptyset$. Thus for a sufficiently small quantity α and strong monotonicity of the preference ordering we have

$$\mathbf{x}'_{\tilde{a}} \succ_{\tilde{a}} \mathbf{x}''_{\tilde{a}} := \mathbf{x}'_{\tilde{a}} - \alpha \mathbf{e}^1 \succ_{\tilde{a}} \mathbf{x}_{\tilde{a}} .$$

Now the quantity α can be distributed equally among all the remaining persons $a \in A, \ a \neq \tilde{a}$). With respect to strongly monotone preference orderings it follows

$$\mathbf{x}''_a := \mathbf{x}'_a + \frac{\alpha}{\#A - 1} \mathbf{e}^1 \succ_a \mathbf{x}_a .$$

Although the generated allocation $\mathbf{x}''$ is feasible, $\mathbf{x}''_a \succ_a \mathbf{x}_a$ holds for all $a \in A$ because of the strong monotonicity. Hence, $\mathbf{x}$ cannot be weakly PARETO optimal which is contrary to the supposition. If there is no allocation $\mathbf{x}'$ with the quoted properties, then the weakly PARETO optimal allocation $\mathbf{x}$ is also PARETO optimal. ■

3.2.3 First Theorem of Welfare Economics

Under the premises of Proposition IV.16 we can now prove that each WALRASian allocation in an exchange economy also lies in the core of this economy.[104] Supposing this statement were wrong, then there must be a WALRASian allocation $\mathbf{x}^\circ$ not lying in the core of the economy. In this case there is a coalition S and

[103] Cf. e.g. ALIPRANTIS, BROWN, BURKINSHAW (1989, p. 40, Theorem 1.5.2). The cited theorem does not explicitly presume the divisibility of goods; the proof requires, however, this assumption.

[104] Cf. DEBREU, SCARF (1963, p. 240, Theorem 1) or HILDENBRAND, KIRMAN (1988, p. 83, Proposition 2.4).

an allocation $\mathfrak{x}'$ satisfying two conditions: (i) $\mathbf{x}'_a \succ_a \mathbf{x}^\circ_a$ for all $a \in S$ and (ii) $\sum_{a \in S} \mathbf{x}'_a = \sum_{a \in S} \mathbf{w}_a$. For the WALRASian equilibrium $(\mathbf{p}^\circ, \mathfrak{x}^\circ)$ (i) implies in conjunction with the premises of Proposition IV.16

$$\mathbf{p}^{\circ\mathrm{T}}\mathbf{x}'_a > \mathbf{p}^{\circ\mathrm{T}}\mathbf{x}^\circ_a = \mathbf{p}^{\circ\mathrm{T}}\mathbf{w}_a \qquad \forall a \in S .$$

In contradiction to (ii) we get

$$\sum_{a \in S} \mathbf{p}^{\circ\mathrm{T}}\mathbf{x}'_a > \sum_{a \in S} \mathbf{p}^{\circ\mathrm{T}}\mathbf{w}_a .$$

Thus, the WALRASian allocation $\mathfrak{x}^\circ$ lies in the core of the economy. Conversely, a WALRASian allocation can only exist if the core of the economy is not empty. Accordingly, the search for a WALRASian equilibrium need not be continued if it has been shown that the core of the economy concerned is empty.

Finally, it remains to be noted that each WALRASian allocation must be PARETO efficient (**first theorem of welfare economics**) under the premises of Proposition IV.16 since all allocations in the core of the economy have this property.[105] This outcome is noteworthy if we take into account that the described WALRASian equilibria are the result of a simultaneous optimization approach in which the agents do not cooperate.

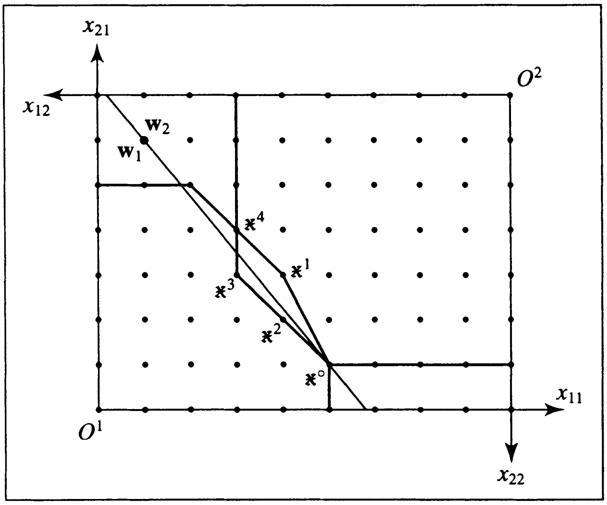


Figure IV.16: Non PARETO efficient WALRASian equilibrium

For exchange economies including only indivisible goods contrary to Proposition IV.16, the statement of the first theorem of welfare economics is no longer valid. As shown by the EDGEWORTH box in Figure IV.16 for two persons $a = 1, 2$, we can construct examples where not all equilibria are at the same time PARETO efficient.[106]

Starting with the initial endowments $\mathbf{w}_1$ and $\mathbf{w}_2$ of two persons in Figure IV.16, the feasible allocation $\mathfrak{x}^\circ = (\mathbf{x}^\circ_1, \mathbf{x}^\circ_2)$ implies the budget balance for both persons given the depicted price ratio: $\mathbf{p}^{\circ\mathrm{T}}\mathbf{w}_a = \mathbf{p}^{\circ\mathrm{T}}\mathbf{x}^\circ_a$ with $a = 1, 2$.

[105] KHAN, YAMAZAKI (1981) present similar results referring to economies with indivisible goods and a continuum of households. The authors discuss assumptions so that the core of the examined economies is not empty. Furthermore, it is shown that the set of WALRASian allocations equals the set of all allocations in the core, where the WALRASian allocations underly slightly weaker requirements.
QUINZII (1984) describes a market for m houses and the perfectly divisible good money. Again assumptions are presented under which the set of WALRASian allocations equals the set of all allocations in the core. Concerning this market, SVENSSON (1991) shows that each WALRASian equilibrium is furthermore equivalent to a (strong) NASH equilibrium.

[106] A similar example may be found in EMMERSON (1972, p. 184).

Thus, the allocation $\mathfrak{x}^\circ$ denotes a WALRASian equilibrium if $\mathbf{x}_1^\circ$ and $\mathbf{x}_2^\circ$ are best commodity bundles relative to the respective budget sets of both persons.[107] In this sense it is supposed for person 1 that the allocations $\mathfrak{x}^\circ$, $\mathfrak{x}^2$, and $\mathfrak{x}^3$ reflect the following indifference: $\mathbf{x}_1^\circ \sim_1 \mathbf{x}_1^2 \sim_1 \mathbf{x}_1^3$. Analogously, it is assumed that the allocations $\mathfrak{x}^\circ$ and $\mathfrak{x}^1$ evoke $\mathbf{x}_2^\circ \sim_2 \mathbf{x}_2^1$ for the second person.

Although $\mathfrak{x}^\circ$ corresponds to a WALRASian allocation, this allocation cannot be PARETO efficient for strongly monotone preferences. For instance, the allocation $\mathfrak{x}^1$ is a PARETO improvement to $\mathfrak{x}^\circ$ since we have $\mathbf{x}_1^1 \succ_1 \mathbf{x}_1^\circ$ for the first person besides $\mathbf{x}_2^1 \sim_2 \mathbf{x}_2^\circ$.

The equilibrium allocation $\mathfrak{x}^\circ$ does not even need to be weakly PARETO efficient. If both agents ignore the given price ratio and agree to the allocation $\mathfrak{x}^4$, then the result can be favorable for both actors. In this case the WALRASian allocation $\mathfrak{x}^\circ$ would not lie in the core of the economy since a coalition of both persons would block $\mathfrak{x}^\circ$ because they can realize a better result than $\mathfrak{x}^4$. [108]

3.2.4 Comments on the Second Theorem of Welfare Economics

After the expositions on the first theorem of welfare economics the question now arises as to what extent indivisible goods influence the validity of the second theorem of welfare economics.

Second Theorem of Welfare Economics[109] *Suppose $\mathfrak{x}^\circ$ is a* PARETO *efficient allocation with $\mathbf{x}_a^\circ > \mathbf{0}$ for each person $a \in A$. Provided the preference orderings are convex, continuous, and monotone, $\mathfrak{x}^\circ$ is a* WALRAS*ian equilibrium for the initial endowments $\mathbf{w}_a = \mathbf{x}_a^\circ$ $(a \in A)$.*

Thus, each of the mentioned PARETO efficient allocations can be converted into a WALRASian equilibrium by an appropriate reallocation of the initial endowments.

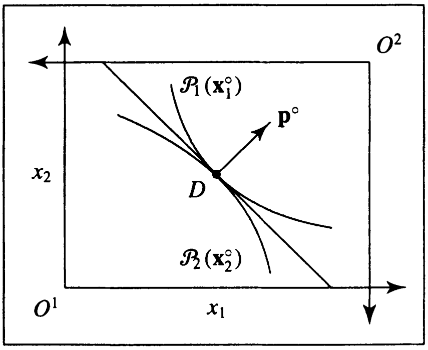


Figure IV.17: Second theorem of welfare economics

The idea of the proof can be justified for two persons and two goods with the help of the EDGEWORTH Box opposite. If point D is a PARETO efficient allocation $(\mathbf{x}_1^\circ, \mathbf{x}_2^\circ)$ with $\mathbf{x}_1^\circ = \mathbf{w}_1$ and $\mathbf{x}_2^\circ = \mathbf{w}_2$, then there must be a hyperplane separating both convex closed preference sets $\mathscr{P}_1(\mathbf{x}_1^\circ)$ and

[107]Technically $\mathbf{x}_a^\circ \in D_a(\mathbf{p}^\circ, \mathbf{w}_a)$ must hold for both persons $a = 1, 2$, where the demand correspondence $D_a(\cdot, \mathbf{w}_a)$ has been defined on p. 190.

[108]More examples with WALRASian allocations being not PARETO efficient may be found in GUESNERIE (1975, p. 23 ff.). The presented production economies presume divisible goods but nonconvex production technologies. The presented nonconvex technologies as a cause for PARETO inefficient WALRASian allocations cannot be transferred to the case of indivisible goods without further assumptions.

[109]VARIAN (1992, p. 326).

$\mathcal{P}_2(\mathbf{x}_2^\circ)$.[110] Now for strongly monotone preferences the corresponding price vector $\mathbf{p}^\circ$ yields the WALRASian equilibrium $(\mathbf{x}_1^\circ, \mathbf{x}_2^\circ, \mathbf{p}^\circ)$.

According to VARIAN (1992), the proof of such a hyperplane relies crucially on the convexity of both preference sets. Thus, it is not surprising that the statement of the second theorem of welfare economics is no longer valid under the consideration of indivisible goods. At this point it may be enough to note two statements offered without proof.

(1) Despite an indivisible good the second theorem of welfare economics remains valid if the following assumptions are made. The economy concerned embraces two persons and two goods. The first good is perfectly divisible and it is chosen by each person in a positive amount.[111] The preferences satisfy Assumption 3, i.e. for both persons and for all commodity bundles $\mathbf{x}, \mathbf{x}' \in X$ a positive number λ exists such that $\mathbf{x} + \lambda \mathbf{e}^1 \succcurlyeq_a \mathbf{x}'$ $(a = 1, 2)$. The sets $\{\mathbf{x} \in X | \, \mathbf{x} \succ_a \tilde{\mathbf{x}}\}$ $(a = 1, 2)$ are integer convex, i.e. for each commodity vector $\tilde{\mathbf{x}}$ in the commodity space $X = \mathbb{R}_+ \times \mathbb{Z}_+$ we have

$$\operatorname{conv}\{\mathbf{x} \in X | \, \mathbf{x} \succ_a \tilde{\mathbf{x}}\} \cap X = \{\mathbf{x} \in X | \, \mathbf{x} \succ_a \tilde{\mathbf{x}}\}.$$

Although the proof is omitted, we can make use of Figure IV.4, p. 220. Substituting the first person's preference set by an integer convex preference set, the marked allocation $(\mathbf{x}_1^\circ, \mathbf{x}_2^\circ)$ is PARETO efficient, and the price line $\overline{C^1 C^2}$ separates the convex hulls $\operatorname{conv}\{\mathbf{x} \in X | \, \mathbf{x} \succ_1 \mathbf{x}_1^\circ\}$ and $\operatorname{conv}\{\mathbf{x} \in X | \, \mathbf{x} \succ_2 \mathbf{x}_2^\circ\}$. The corresponding price vector $\mathbf{p}^\circ$ constitutes the WALRASian equilibrium $(\mathbf{x}_1^\circ, \mathbf{x}_2^\circ, \mathbf{p}^\circ)$.

(2) Although it seems intuitively reasonable to make a generalization for more than two goods, we can already construct PARETO efficient allocations for three goods (two indivisible goods and one divisible good) which do not determine a WALRASian equilibrium.[112]

3.3 Summary

Section IV.3 presents initial points which continue the results of the existence of exchange equilibria regarding two aspects. On the one hand the exchange economy is expanded by the ignored aspects of production and on the other hand the PARETO efficiency gives an alternative criterion for assessing an allocation.

The consideration of production causes no major problems for the moment. All assumptions on the production technology of the individual firm have already been introduced in Chapter I. As seen in the expositions of Proposition IV.14, the

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[110] Such a hyperplane implies that there is no allocation improving the position of one of both persons without worsening the position of the other person.

[111] This condition was introduced by Assumption 1 (BROOME).

[112] *MATHEMATICA*® is especially suitable for a geometrical construction of three-dimensional EDGEWORTH boxes with appropriate preference structures.
EMMERSON (1972, Theorem 2) asserts that monotonicity and integer convexity of preferences would suffice to prove that each PARETO optimal allocation describes at the same time a WALRASian allocation given an appropriate price vector. But the proof stipulates that the sum of integer convex sets is also integer convex; see EMMERSON (1972, p. 182). Even simple examples disprove this assumption.

existence of a WALRASian equilibrium with divisible goods can be proved with the same arguments as introduced in the treatment of exchange economies. The simultaneous optimization approach is merely expanded by firms as additional actors. Provided that it is possible to convert the described production economy regarding indivisible goods into a "convex-valued" economy, then the results cannot essentially differ from the results with respect to exchange economies. As before, we have to suppose that no exact equilibria need to exist, but at least for nonincreasing returns to scale the simultaneous optimization approach will end up in a state $(\mathfrak{x}^d, \mathfrak{y}^d, \mathbf{p}^\circ)$ of the economy which cannot arbitrarily differ from a determined fixed-point $(\mathfrak{x}^\circ, \mathfrak{y}^\circ, \mathbf{p}^\circ)$.

For significant indivisibilities implying increasing returns to scale the established estimations of the deviation between both allocations $(\mathfrak{x}^d, \mathfrak{y}^d)$ and $(\mathfrak{x}^\circ, \mathfrak{y}^\circ)$ ought to be useless. Keeping that in mind, we are faced with the question which ties in with one cause for the deviation of the preceding allocations. Although the maximal utility level of the representative household can be associated with a certain production level, no price vector needs to exist so that the corresponding production activity is profit maximizing. As no linear price plane supports the production set at the point of the utility maximizing production activity, we try to construct a nonlinear price plane, which furthermore transfers the utility maximizing activity into a profit maximum; see Figure IV.15.

To assess an attainable allocation, the criterion of a WALRASian equilibrium faces the criterion of PARETO efficiency. By the first theorem of welfare economics we can prove, under relatively weak assumptions, that each WALRASian allocation has to be PARETO efficient at the same time. The reverse conclusion of the second theorem of welfare economics is usually not admissible for given initial endowments. In this sense the PARETO efficiency is often understood as a weaker criterion of optimality. But if all of the goods are indivisible, then we can offer examples in which WALRASian allocations are not PARETO efficient.

Chapter V.

Critique

In treating indivisible goods and factors we follow a path which is based on well known approaches in microeconomic theory.

The analysis begins with the basic requirements regarding the preferences of a household and the production technology of a firm. Representing the household's preference structure by a family of preference sets $\big(\mathcal{P}(\mathbf{x})|\ \mathbf{x} \in X\big)$ it becomes evident that the properties of continuous preference orderings do not depend on the consideration of indivisible goods. Merely the assumption of convex preference sets contradicts the requirement that some goods are only available at integer amounts. Since the convexity of sets is associated with a series of analytical advantages, we look for a suitable substitute for nonconvex sets so that we can take advantage of convexity.

Before concerning ourselves with the household's preference structure, we examine the analogous problem with respect to the firm's production technology (Section II.1.2). Again the production structure $\big(P(\mathbf{v})|\ \mathbf{v} \in V\big)$, which consists of production possibility sets $P(\mathbf{v})$, has several properties which are not influenced by the existence of indivisible factors; see [P1]–[P6], p. 11 ff. These features face equivalent properties [L1]–[L6] of the inverse production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$, where $L(\mathbf{x})$ denotes an input requirement set (Proposition II.2).

In order to make use of the analytical advantages of convex sets, in the next step we introduce the concept of the convex hull (Section II.2.2.1). This set is the smallest of all convex sets containing the genuine set. In the case of an input requirement set the transition to the convex hull is indicated by $L(\mathbf{x}) \to \operatorname{conv} L(\mathbf{x})$. To represent convex hulls, two views are compared (Figure II.10). On the one hand each point in the convex hull can be expressed as a convex combination of points in the original set (inner representation). On the other hand a (closed) convex hull is the intersection of all (closed) half-spaces containing the original set (outer representation). These views of an input requirement set $L(\mathbf{x})$ are reflected by the cost function $c(\cdot, \mathbf{x})$ and the input distance function $t_I(\cdot, \mathbf{x})$. While the cost function is determined by a system of hyperplanes tangent to the convex hull of the input requirement set (Proposition III.11), the input distance function indicates how much a point in the convex hull can be deflated along a ray through the origin until it reaches the frontier of this set.

Since both functions refer in particular to boundary points of the convex hull of an input requirement set, the interrelation between the convex hull and certain boundary points is first analyzed. Not only extreme points but also exposed points turn out to be of major importance. For instance, it is shown that a cost minimum is always achieved at an extreme point of the convex hull $\text{conv}L(\mathbf{x})$ of the input requirement set. In order to prove that both extreme points and exposed points are admissible in principle, it is furthermore shown that the convex hull $\text{conv}L(\mathbf{x})$ is closed under the assumptions on the production technology (Proposition II.14).

The comparison of the cost function $c(\cdot, \mathbf{x})$ and the input distance function $t_I(\cdot, \mathbf{x})$ is determined by certain characteristics of the input requirement sets. Analogously, the revenue function $r(\cdot, \mathbf{v})$ and the output distance function $t_O(\cdot, \mathbf{v})$ are related to each other via the equivalent properties of the production possibility sets, which also result from the assumptions on the production technology. Whereas the convex hull $\text{conv}P(\mathbf{v})$ of a production possibility set contains the origin $\mathbf{x} = \mathbf{0}$ and is star-shaped (Figure II.15), the convex hull of an input requirement set $\text{conv}L(\mathbf{x})$ is an aureoled set, which does not contain the origin $\mathbf{v} = \mathbf{0}$ (Figure II.16). Using these properties the relevant part of the frontiers of the respective convex hull can be described by gauges which correspond to the economic distance functions. If the frontiers are known, then the assumption of integer convexity enables us to determine unequivocally all of the possible activities. In this case there is no loss of information if we go over from a set to its convex hull. In particular, the relevant integer boundary points of the convex hull are at the same time points of the genuine set (Section II.2.2.2).

No matter what the set is, this property holds for all of the extreme points of a convex hull, hence we can now give a plausible measure for the technical efficiency of activities. If we apply FARRELL's input efficiency measure to the convex hull of the input requirement set concerned, then the measure compares each input vector with a boundary point of the convex hull lying on the same ray through the origin. Since each boundary point can be expressed as a convex combination of extreme points of the convex hull and all of the extreme points are at the same time efficient input vectors, the modified FARRELL's input efficiency measure yields an indirect comparison to technically efficient activities for each input vector. For this reason the input distance function $t_I(\cdot, \mathbf{x})$, which corresponds to the reciprocal value of FARRELL's input efficiency measure, is directly defined on the basis of the convex hull $\text{conv}L(\mathbf{x})$ of the input requirement set (Section III.2.2).

Having pointed out the importance of boundary points to technical efficiency, we now seek for activities which are even efficient with respect to economic terms. An input vector $\mathbf{v}$ must not only be input efficient but also cost minimizing with respect to the production of a given commodity bundle $\mathbf{x}$. Again the relevant input vectors lie in the frontier of $\text{conv}L(\mathbf{x})$.

In principle the boundary of the convex hull of an input requirement set can be described in two ways. The first way refers to a partial factor variation and the second way to the total factor variation (Figure III.1). Starting with the partial factor variation, a function $g(\cdot|\text{conv}L(\mathbf{x}))$ is derived which implicitly represents the boundary of the set $\text{conv}L(\mathbf{x})$ (Section III.1.2). This function faces a normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ indicating the minimal factor costs in the production of the commodity bundle $\mathbf{x}$ in units of factor r. At the same time it is shown that the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ does not depend on the fact whether it is derived on the basis of the input requirement set $L(\mathbf{x})$ or its convex hull $\text{conv}L(\mathbf{x})$ (Proposition III.2). This result is not valid for the optimal solutions to the cost minimization. Not all of the optimal solutions to the cost minimization on the basis of the convex hull $\text{conv}L(\mathbf{x})$ are at the same time admissible input vectors. Nevertheless, the reverse conclusion can be proved: all of the cost minimizing input vectors belong to the set of optimal solutions with respect to $\text{conv}L(\mathbf{x})$ (Section III.1.4.2). Furthermore, there is at least one admissible cost minimizing input vector, provided the given $\mathbf{x}$ is an admissible commodity bundle.

Among other properties of the pair of functions $\big(g(\cdot|\text{conv}L(\mathbf{x})), \tilde{c}(\cdot, \mathbf{x})\big)$, it has especially been shown what requirements suffice for the normalized cost function to be differentiable at a point of normalized factor prices $\hat{\mathbf{q}}^{-r}$. In this case according to SHEPHARD's Lemma the gradient $\hat{\mathbf{v}}^{-r} = \nabla\tilde{c}(\hat{\mathbf{q}}^{-r}, \mathbf{x})$ denotes a unique cost minimizing and admissible input vector (Section III.1.4.3).

To demonstrate further properties of such pairs of dual points $(\hat{\mathbf{q}}^{-r}, \hat{\mathbf{v}}^{-r})$, we present a series of examples such that the interrelation of the pair of functions $\big(g(\cdot|\text{conv}L(\mathbf{x})), \tilde{c}(\cdot, \mathbf{x})\big)$ can be explained by graphs (Figures III.8, III.12, III.14, and III.16). It can be observed there that the generalization of SHEPHARD's Lemma only leads to unique factor demand functions if the input requirement sets are strictly convex. In particular, the normalized cost function is not differentiable at each price vector for indivisible factors. Several possible cost minimizing input vectors exist for these vectors, which can as before be determined unequivocally under the assumption of integer convex input requirement sets.

In what follows, the analysis switches over to the total factor variation, which leads to an alternative cost function $c(\cdot, \mathbf{x})$ (Section III.2). This function corresponds to the support function φ of the input requirement set $L(\mathbf{x})$ and, therefore, generates a system of hyperplanes tangent to the convex hull $\text{conv}L(\mathbf{x})$ (Proposition III.11). For each hyperplane there is a nonnegative perpendicular price vector $\mathbf{q}$ unique up to a scalar. Uniting all price vectors with the property $c(\mathbf{q}, \mathbf{x}) \geqq 1$ in the polar input requirement set $L_\circ(\mathbf{x})$ of $L(\mathbf{x})$ (Proposition III.17), a family $\big(L_\circ(\mathbf{x})|\ \mathbf{x} \in X\big)$ is generated, which can be interpreted as a cost structure in comparison to the production structure $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$. Each polar input requirement set $L_\circ(\mathbf{x})$ is in turn associated with a support function φ. Thus, according to the explanations a bipolar set $L_{\circ\circ}(\mathbf{x})$ can be generated which satisfies $L_{\circ\circ}(\mathbf{x}) = \text{conv}L(\mathbf{x})$. Analogous to the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$, it is shown that the determination of the cost function $c(\cdot, \mathbf{x})$ does not depend on the fact whether it is based on the input requirement set $L(\mathbf{x})$ or its convex hull $\text{conv}L(\mathbf{x})$. Moreover, it has been shown that the two cost functions $\tilde{c}(\cdot, \mathbf{x})$ and $c(\cdot, \mathbf{x})$ can be transformed into each other (Proposition III.12).

The introduction of the input distance function $t_I(\cdot, \mathbf{x})$ compares the cost function to a radial measure of the input efficiency of an activity. To get over the difficulties in measuring the efficiency regarding nonconvex input requirement sets, the input distance function is immediately defined with respect to the convex hull $\text{conv}L(\mathbf{x})$ (Section III.2.2). The resulting problems have already been discussed for the modified FARRELL's input efficiency measure.

It turns out that the input distance function regarding the sets $\text{conv}L(\mathbf{x})$ and $L_\circ(\mathbf{x})$ plays the reverse role of the cost function (Corollary III.18.1). The input distance function $t_I(\cdot, \mathbf{x})$ is defined as the gauge ψ of the convex hull of the input requirement set $\text{conv}L(\mathbf{x})$ and denotes the support function φ of the polar set $L_\circ(\mathbf{x})$. In contrast, the cost function $c(\cdot, \mathbf{x})$ is defined as the support function $\varphi(\cdot|L(\mathbf{x}))$ and equals the gauge $\psi(\cdot|L_\circ(\mathbf{x}))$. This dual relation is reflected by SHEPHARD's Theorem (Proposition III.19): an input vector $\hat{\mathbf{v}}$ solves the problem of cost minimization such that $c(\hat{\mathbf{q}}, \mathbf{x}) = \hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{v}}$ if and only if the input vector $\hat{\mathbf{v}}$ is efficient with respect to the convex hull $\text{conv}L(\mathbf{x})$, i.e. $t_I(\hat{\mathbf{q}}, \mathbf{x}) = 1$. In this case $(\hat{\mathbf{q}}, \hat{\mathbf{v}})$ is called a pair of polar points. Remember again that not all vectors, yielding minimal costs, correspond to admissible input vectors. However, for each admissible commodity bundle there is at least one admissible cost minimizing input vector.

The concluding examples (Section III.2.4.2) offer not only a graphical comparisons of the input requirement set $L(\mathbf{x})$ (or its convex hull $\text{conv}L(\mathbf{x})$) and its polar set $L_\circ(\mathbf{x})$, but also of the corresponding boundary functions $t_I(\mathbf{v}, \mathbf{x}) = 1$ and $c(\mathbf{q}, \mathbf{x}) = 1$ (Proposition III.17). Finally, we demonstrate the relation to the factor demand correspondences which have been derived from the normalized cost function $\tilde{c}(\cdot, \mathbf{x})$ (Figure III.36).

In the theory of the household we refrain from analyzing the expenditure structure in analogy to the cost structure of a firm. Instead, the expositions concentrate on the properties of the commodity demand which are required for the proof of the existence of a competitive equilibrium in exchange economies. In this the exchange economy consists of a set A of persons and each person possesses a continuous monotone preference ordering apart from an initial endowment $\mathbf{w}_a$. The aim is to show that a price vector exists such that the aggregate excess demand vanishes on all commodity markets. The proof succeeds when the aggregate excess demand, which is made up of the individual demand correspondences, has certain properties.

First of all, it is shown that the aggregate excess demand correspondence is homogeneous of degree zero in commodity prices $\mathbf{p}$ if every person is free of money illusion (Proposition III.21). Thus, the factor price space can be restricted to the price simplex Δ, where we have to distinguish positive price vectors $\mathbf{p} > \mathbf{0}$ from price vectors with a zero component. The aggregate excess demand correspondence is well defined for positive price vectors – i.e. $Z(\mathbf{p}) \neq \emptyset$ – but not for price vectors having a zero component (Propositions III.23 and III.28). As an alternative it is shown that the aggregate excess demand becomes infinitely large as soon as a commodity price falls to zero (Corollary III.29.1).

Finally, we present assumptions preventing the individual demand correspondences and, therefore, the aggregate excess demand correspondence from making arbitrary jumps. These jumps, caused by a marginal change of a positive price vector (Figure III.44), can be ruled out by presuming at least one divisible good in the exchange economy (Proposition III.27). For a similar reason it is assumed that each person holds a positive initial endowment $\mathbf{w}_a > \mathbf{0}$. In doing so it is assured that each person has a positive budget even when the price vector has a zero component. Otherwise the demand would suddenly disappear at that moment when the value of the initial endowment has fallen to zero. The third assumption on households' preferences also serves to avoid jumps, now ruling out that the demand suddenly becomes infinitely large as soon as a price vector has a zero component (Figure III.48).

Finally, WALRAS' law (Proposition III.30) notes conditions so that every person makes use of his whole budget. Hence, the value of aggregate excess demand is identically zero.

Chapter IV goes into the existence of competitive equilibria, where the hierarchy conforms to strictly convex, convex, and nonconvex preference sets. While in the first two cases the existence proof is successful (Propositions IV.6 and IV.7), there need not be any equilibrium with regard to indivisible goods. Nevertheless, ignoring the requirement of being integer, we can establish a feasible allocation $\mathfrak{x}^\circ$ and a price vector $\mathbf{p}^\circ > \mathbf{0}$ being suitable for an approximation of a real competitive equilibrium (Proposition IV.10). In particular, it can be shown that at the most $n-1$ persons receive no commodity bundle demanded in the so called ROTHENBERG equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ with n goods. However, they are assigned a vector $\mathbf{x}_a^\circ$ which can be expressed as a convex combination of commodity bundles demanded. In this sense the persons at hand are not "far" away from their real demand.

If it is possible to give an upper bound for the distance between two chosen commodity bundles, then the maximal degree of the accumulated deviation between the ROTHENBERG equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ and a pair $(\mathfrak{x}^d, \mathbf{p}^\circ)$ can be offered, where $\mathfrak{x}^d$ denotes an allocation which is chosen (i.e. integer) at prices $\mathbf{p}^\circ$ and usually feasible (Proposition IV.11 and Corollary IV.11.1). Note that the deviation between $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ and $(\mathfrak{x}^d, \mathbf{p}^\circ)$ is still allocated to at the most $n-1$ persons.

In the following ε-equilibria it is shown how to distribute the given deviation equally such that the burden of a nonexisting equilibrium is not borne by at the most $n-1$ persons. The two ε-equilibria stipulate that the exchange economy has a sufficiently large number of persons. In each case the ROTHENBERG equilibrium $(\mathfrak{x}^\circ, \mathbf{p}^\circ)$ serves as an initial point. The first ε-equilibrium $(\mathfrak{x}^{\circ\circ}, \mathbf{p})$ in the sense of STARR modifies the allocation $\mathfrak{x}^\circ$ to a feasible allocation $\mathfrak{x}^{\circ\circ}$ such that the distance between the vector $\mathbf{x}_a^{\circ\circ}$ and a commodity bundle $\mathbf{x}_a^d$ demanded at prices $\mathbf{p}^\circ$ is on average smaller than ε (Proposition IV.12). Moreover it is shown that this ε can be chosen even smaller, the more persons are included in the exchange economy. Although on average $\|\mathbf{x}_a^{\circ\circ} - \mathbf{x}_a^d\| < \varepsilon$ holds good for each person, this inequality can be violated for individual persons with considerable consequences. For this reason the second ε-equilibrium in the sense of HILDENBRAND and KIRMAN (Proposition IV.13) requires the preceding inequality to be satisfied for all persons. Thus, it is not surprising that now more restrictive requirements are needed with respect to the exchange economy. If the examined exchange economy is "large" under various aspects, then it can be shown that no person is assigned a vector

$\mathbf{x}_a^{\circ\circ}$ deviating by more than ε from a commodity bundle $\mathbf{x}_a^d$ demanded. Moreover, all persons satisfy their budget constraint. The allocation $\mathfrak{x}^{\circ\circ}$ derived from the ROTHENBERG equilibrium is again feasible and the number of persons receiving no commodity bundle demanded is bounded above.

The concluding extension from the exchange economy to a production economy suggests that similar results will appear; see p. 260 ff. However, SCARF stresses that when the production is characterized by important indivisibilities, then the analogous approximation for an equilibrium will become almost useless.

The concluding remarks concentrate on the criterion of PARETO efficient allocations in an exchange economy. Again, the importance of a perfectly divisible good is pointed out for the analysis of indivisible goods (Section IV.3.2.3). While the statement of the first theorem of welfare economics remains the same if there is at least one divisible good, the statement is no longer valid for exclusively indivisible goods (Figure IV.16).

The representation of indivisible goods and factors requires some comments with respect to two aspects. The first point is of a technical nature: it shows alternatives for the approximation of nonconvex sets by their convex hulls. The second point deals with the ignored problem of large, indivisible, and irreversible investments corresponding to fixed or sunk cost. At the same time the goal of optimal investment strategies is expanded by game theoretical aspects.

The theory of the firm in Sections III.1 and III.2 concentrates on implications resulting from certain properties of the production technology. If indivisible goods or factors appear, then the tool of the convex hull is used. For instance, the production technology $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$ has been approximated by $\big(\mathrm{conv}L(\mathbf{x})|\ \mathbf{x} \in X\big)$. Moreover, factor constraints are not taken into account.

This permits a further class of problems to be shown whose solution procedures are offered by discrete optimization. Suppose that the input requirement sets $L(\mathbf{x})$ are expressed as systems of inequalities

$$L(\mathbf{x}) = \left\{\mathbf{v}|\ \mathbf{v} \geqq \mathbf{A}\mathbf{x},\ \mathbf{v} \in \mathbb{R}_+^{m_d} \times \mathbb{Z}_+^{m-m_d}\right\},$$

where $\mathbf{A}$ is an $m \times n$-matrix of positive and constant input coefficients. In this case knowing the vertices of the polehydral convex set described by the inequalities $\mathbf{v} \geqq \mathbf{A}\mathbf{x}$ in order to represent $\mathrm{conv}L(\mathbf{x})$ is not sufficient.[1] Conversely, each arbitrary point $\mathbf{v}$ of the polehydral convex set satisfies the relation $\mathbf{v} \in L(\mathbf{x})$ provided this point satisfies the integer constraints, $\mathbf{v} \in \mathbb{R}_+^{m_d} \times \mathbb{Z}_+^{m-m_d}$. In all cases $\mathbf{A}\mathbf{x} = \mathbf{v}$ yields an input vector $\lceil\mathbf{v}\rceil \in L(\mathbf{x})$, where $\lceil\mathbf{v}\rceil$ denotes the "next larger input vector of $\mathbf{v}$" by Definition II.4.

If the input vector $\mathbf{v}$ is subject to further factor constraints, i.e. $\mathbf{v}$ may not exceed the given factor stocks $\mathbf{b} \in \mathbb{R}_+^{m_d} \times \mathbb{Z}_+^{m-m_d}$, then the problem of revenue maximization is given by

$$\begin{aligned} &\sup\left\{\mathbf{p}^{\mathsf{T}}\mathbf{x}|\ \mathbf{v} \in L(\mathbf{x}),\ \mathbf{v} \leqq \mathbf{b},\ \mathbf{x} \in \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}\right\} \\ = &\sup\left\{\mathbf{p}^{\mathsf{T}}\mathbf{x}|\ \mathbf{A}\mathbf{x} \leqq \mathbf{v},\ \lceil\mathbf{v}\rceil \leqq \mathbf{b},\ \mathbf{x} \in \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}\right\}. \end{aligned}$$

[1] This statement can be illustrated by Figure V.1 even though no input requirement set is shown.

Because $\lceil \mathbf{v} \rceil \leqq \lceil \mathbf{b} \rceil \equiv \mathbf{b}$ also holds for all $\mathbf{v} \leqq \mathbf{b}$, we obtain

$$\text{(V.1)} \qquad \sup \left\{ \mathbf{p}^\mathsf{T}\mathbf{x} \mid \mathbf{A}\mathbf{x} \leqq \mathbf{b},\ \mathbf{x} \in \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d} \right\}.$$

If no good is divisible, $n_d = 0$, then (V.1) is called a pure integer linear optimization problem.[2] Otherwise it is called a mixed integer linear optimization problem.

The concept of integer optimization is founded on the following observation.[3] If $\mathcal{L}$ is a polehydral convex set, i.e.

$$\mathcal{L} = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leqq \mathbf{b},\ \mathbf{x} \geqq \mathbf{0} \right\},$$

and if $[\mathcal{L}]$ denotes the set of its integer points

$$[\mathcal{L}] = \mathcal{L} \cap \left(\mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d} \right),$$

then:

1. The set $R := \operatorname{conv}[\mathcal{L}]$ is an integer polehydral set, i.e. all basic solutions or extreme points satisfy the integer constraints.
2. The idea of integer convexity in Section II.2.2.2 is reflected by $[R] = [\mathcal{L}]$.
3. The set of basic solutions R^* of the polehydral convex set R belongs to the set $[R]$, $R^* \subset [R]$. The reverse conclusion is used by the theorem of KREIN-MILMAN, p. 25: $\operatorname{conv} R^* = R$.

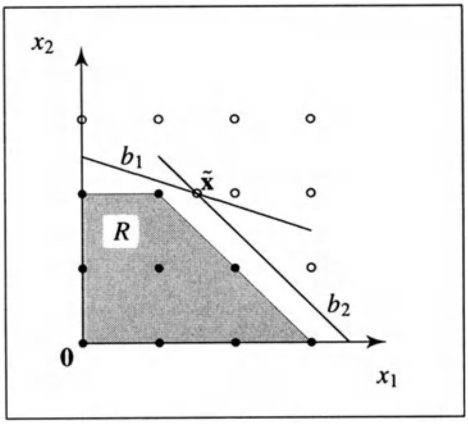

Figure V.1: The idea of discrete optimization

The Figure V.1 opposite illustrates the polehydral convex set $\mathcal{L}$ with the constraints b_1 and b_2. Thus, $[\mathcal{L}]$ corresponds to the bold points, and $R = \operatorname{conv}[\mathcal{L}]$ is described by the gray region. Note that both restrictions do not a priori permit the conclusion to $\operatorname{conv}[\mathcal{L}]$.

KORBUT and FINKELSTEIN offer a result which is the most important conclusion and very similar to the statement of Theorem II.2, p. 63. If the feasible region of problem (V.1) is denoted by $[\mathcal{L}]$ and if $\hat{\mathbf{x}}$ is an optimal basic solution to the problem

$$\text{(V.2)} \qquad \sup \left\{ \mathbf{p}^\mathsf{T}\mathbf{x} \mid \mathbf{x} \in \operatorname{conv}[\mathcal{L}] \right\},$$

then $\hat{\mathbf{x}}$ solves the original problem (V.1) subject to the feasible region $[\mathcal{L}]$, too. If we ignore the integer constraints in (V.1), then each optimal solution $\tilde{\mathbf{x}}$ to

$$\text{(V.3)} \qquad \sup \left\{ \mathbf{p}^\mathsf{T}\mathbf{x} \mid \mathbf{A}\mathbf{x} \leqq \mathbf{b},\ \mathbf{x} \geqq \mathbf{0} \right\} = \sup \left\{ \mathbf{p}^\mathsf{T}\mathbf{x} \mid \mathbf{x} \in \mathcal{L} \right\}$$

[2] A pure integer linear optimization problem is presented in KOOPMANS, BECKMANN (1957) and is based on this in REITER, SHERMAN (1962): what does an efficient allocation of n indivisible firms to n locations look like when each location can be taken by exactly one firm and each location has different advantages for the firms? What price system can be assigned to a solution of the problem?

[3] Cf. KORBUT, FINKELSTEIN (1971, p. 84, Theorem 5.1.1).

gives a value of the objective function $\mathbf{p}^T\tilde{\mathbf{x}}$, which is at the same time an upper bound for the optimal value of (V.1); see Figure V.1.[4] Moreover, provided $\tilde{\mathbf{x}}$ satisfies the ignored integer constraints, then it solves (V.1).

Now we can get an idea of how the method of cutting planes works. One must take into account according to Theorem B.11, p. 297, that the set $\mathrm{cl}(\mathrm{conv}[\mathcal{L}])$ is the intersection of all closed half-spaces containing $\mathrm{conv}[\mathcal{L}]$.

1. If $\tilde{\mathbf{x}}$ solves (V.3) and if $\tilde{\mathbf{x}}$ fulfills the integer constraints, then $\tilde{\mathbf{x}}$ solves (V.1), too.

2. Otherwise (V.3) is expanded by an additional linear inequality which eliminates $\tilde{\mathbf{x}}$ from being a feasible solution, but does not cut off any feasible integer solution. Such a cut leaves $\mathrm{conv}[\mathcal{L}]$ unchanged.

3. A finite sequence of steps is required leading eventually to an optimal solution to (V.1).

Such an algorithm has been presented by GOMORY. Note that the method of cutting planes breaks off as soon as an optimal solution is found. The introduced cuts are usually not enough to determine $\mathrm{conv}[\mathcal{L}]$ but to approximate $\mathrm{conv}[\mathcal{L}]$ "at the relevant point".

Regarding the method of cutting planes, we are now confronted by two methods:

(1) The first approach replaces the feasible region $[\mathcal{L}]$ of (V.1) with a unique integer polehydral set $\mathrm{conv}[\mathcal{L}]$ such that the optimal solutions can be computed by methods of linear programming. With a unique linear program there is also a unique dual program and the values of the dual variables do not depend on the solution procedure. The advantages of this procedure have been widely used in the theory of the firm on the supposition that the corresponding integer polehydral set $\mathrm{conv}[\mathcal{L}]$ is known. Because the ascertainment of the polehydral set $\mathrm{conv}[\mathcal{L}]$ requires considerable effort, no essential advantage for the pure computation of optimal solutions remains.

(2) The converse is true for the results derived from the introduction of additional cuts. The calculation of optimal values is simplified but it depends on the ordering of cuts.[5] Hence, a unique dual program can no longer be determined so that the dual variables have a certain degree of arbitrariness.

Apart from GOMORY's cutting plane algorithm, GOMORY, BAUMOL (1960) discuss how to interpret the dual variables associated with the extended problem. ALCALY, KLEVORICK (1966) and UEBE (1969) present a refined algorithm, which in particular assigns a zero price to each production factor with excess capacities. As shown by KIM, CHO (1988), we can also define average shadow prices. The authors derive a version of the complementary slackness theorem (Theorem 1) where a resource is assigned to an average shadow price of zero if and only if this resource has an excess capacity (Theorem 3).

[4] Cf. SALKIN, MATHUR (1989, Theorem 3.1).

[5] Alternative algorithms which do not need additional cuts are discussed in WOLSEY (1981), where the form of the price functions depends on the chosen algorithm. WILLIAMS (1989) presents an extension of WOLSEY to mixed integer problems.

A further remark refers to the algorithm for solving linear programs – the simplex algorithm. Since we know that the optimal solution is achieved at a vertex of the convex feasible region, it suffices to compare the values of the objective function at these vertices. The simplex algorithm is based on this conception[6] by systematically switching neighboring vertices such that the value of the objective function is improved by each pivot step.[7] The procedure stops if none of the neighboring vertices improves the value of the objective function.

A similar idea is pursued by SCARF (1981a), where a completely different analytical framework (basing on SPERNER's Lemma) is presented for the solution of pure integer programming problems. Here a pure integer production set Y serves as a starting point of profit maximization, where this production technology is expanded by the restrictions of given factor endowments $\mathbf{b}$.[8] Each activity (or each netput vector) $\mathbf{y} \in Y$ is connected with a *neighborhood* consisting of a finite number of activities near $\mathbf{y}$. If $\mathbf{y}$ is an activity fulfilling the constraints of the underlying problem, then the algorithm checks whether there is a *neighboring* point fulfilling the constraints as well as improving the value of the objective function. If such a point exists, then the algorithm continues with this point, otherwise an optimal solution to the problem has been found.[9]

Finally, we have to point out an extensive bibliography dealing with the implications of large discrete capacity expansions. Here attention is drawn to important indivisible inputs of a firm (for instance the number of blocks of a power station), while the output (i.e. the amount of electricity) is perfectly divisible. The problem of fixing welfare optimal prices can be described in the following way.

A continuously growing demand faces discrete production capacities. Hence, for each investment a certain period with excess capacities results until the demand permits full employment of the expanded capacity.[10] In particular BOITEUX (1964) and TURVEY (1969) recommend for these firms with respect to welfare maximization that the (held fixed) commodity price should in general be based on the long-term marginal costs. However, VICKREY (1971) argues that the price

[6]See for example COLLATZ, WETTERLING (1971, §§2–4).

[7]For each pivot step (switch of vertices) it is supposed that the slope of the objective function and therefore the relative prices do not change. At this point SCARF (1994) starts his criticism. If a pivot step goes along with large and relevant changes such that the price ratios change, then the criterion of an profitable pivot step will no longer be valid. In particular, on the basis of actual prices we cannot decide whether it is profitable to take up large and important production activities, i.e. in other words of SCARF (1994, p. 115): "... if the activity makes a negative profit at old equilibrium prices, then there is no way to use it at a discrete or continuous level so as to improve the utility of every agent in the economy. The problem arises with the converse; it is perfectly possible that the activity make[s] a positive profit at old prices and [will] still not be capable of being used at any discrete level to yield a Pareto improvement."

[8]SCARF (1981a) stresses the necessity to consider explicitly factor constraints for increasing returns to scale because of indivisible goods and factors in profit maximization. Otherwise the price mechanism could exclude profit maxima such that the existence of a WALRASian equilibrium in a production economy is impossible as well.

[9]The complexity of the algorithm for two activities is discussed in SCARF (1981b).

[10]The firm is capable of choosing between alternative indivisible investments of different size. The combination of different investments for the determination of the minimal investment costs in the electricity supply industry is used by ANDERSON (1972). For conditions on optimal growth with constant (discrete) capacity expansions in a macroeconomic model see e.g. WEITZMANN (1970).

ought to be continually adjusted to the short-term marginal costs. STARRETT (1978) presents a generalization of BOITEUX's pricing rule and furthermore deduces rules for the optimal extent of capacity expansion as well as the optimal point of time for investment.

At the same time aspects of game theory overlap the problem of large and indivisible investments. Within these approaches the forming of excess capacities by an established firm has a threatening potential to deter possible competitors from entering monopolistic or oligopolistic markets.[11] DIXIT (1980) describes the results of this game in the form of a NASH equilibrium, where the established firm has the privilege of making the first choice on capacity. Hence, up to a certain degree this VON STACKELBERG leader has the possibility to prevent his potential competitors from entering the market.

GILBERT, HARRIS (1984) follow up a similar game theoretical aspect, which is further extended by MILLS (1990). They examine the competitive behavior on markets with indivisible and irreversible investments. The market participants seek for optimal strategies with respect to the size and the points of time for investment to expand their capacity. In particular, the competition for each new investment reflects the threatening behavior against the other (established and potential) suppliers.[12] The resulting market outcome corresponds to the empirical observation of the pattern of U.S. industry in BAIN (1954). Although substantially increasing returns to scale can be detected for many sectors, they are, however, not enough to explain the observed degree of business concentration.[13]

In this regard SCARF's well founded critique must be seen in relative terms: "Both linear programming and the Walrasian model of equilibrium make the fundamental assumption that the production possibility set displays constant or decreasing returns to scale; that there are no economies associated with production at a high scale. I find this an absurd assumption, contradicted by the most casual observations."[14] In view of BAIN's empirical observations not all phenomena can be put down to increasing returns to scale such that in particular the harshness of the following statement is taken away: "If production really does obey constant returns to scale, there is nothing to be gained by organizing economic activity in large, durable, and complex units."[15] If an established firm whose production technology obeys (integer) constant returns to scale is able to deter potential competitors from market entry by its investment behavior, then it is not only the firm's size that will grow. Rather the firm will build up market power depending on the success of its investment strategy such that it is capable of earning positive profits. This case will be all the more likely, the more important the indivisibility and the irreversibility of investments are.

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[11] EATON, LIPSEY (1979) observe this behavior in the U.S. aluminum industry, which extends its capacities before they are needed for meeting demand.

[12] In GILBERT, HARRIS (1984, Section 3), it is optimal for the established firms to do investments as soon as the continuously growing demand promises a profit with respect to the investment. This outcome is even valid when the profit for the whole capacity is reduced.

[13] Cf. BAIN (1954), in particular pp. 33–35.

[14] SCARF (1994, p. 114).

[15] SCARF (1994, p. 115).

Mathematical Appendix

A Basic Concepts of Analysis

A.1 Important Properties of the Euclidean Space

The following section is primarily concerned with the analytical difficulties in treating indivisible goods and factors. For example concerning n goods of which only n_d goods are perfectly divisible, the commodity space X is defined by

$$X := \mathbb{R}_+^{n_d} \times \mathbb{Z}_+^{n-n_d}.$$

Now the question arises as to what mathematical operations may be carried out with respect to elements in X without contradicting mathematical or economic laws. Adding two elements in X results again in an element in X, but the multiplication of an element in X by a scalar cannot be interpreted as commodity bundle in every case. Probably the "space" X will be left.

With regard to this aspect it turns out to be useful to refer to the n-dimensional Euclidean space $\mathbb{R}^n$ as the basis for the technical analysis. Nevertheless, the results of mathematical operations must always be reviewed with respect to their economic meaning.

The space $\mathbb{R}^n$ satisfies the conditions of three special spaces:

1. a (real) vector space,
2. a metric space, and
3. a normed space.

Without giving an exact definition of these spaces[1] we have to explain some terms. The elements $\mathbf{x} \in \mathbb{R}^n$ are called points or vectors and they are taken as column vectors.

$$\mathbf{x} = (x_1, \dots, x_n)^\mathsf{T} \quad \text{with the coordinates} \quad x_j \in \mathbb{R} \;\; (j = 1, \dots, n)$$

The $\mathbb{R}^n$ is a (real) **vector space** or linear space if the operations "addition" and "scalar multiplication" are defined as follows: for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we set

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)^\mathsf{T} \quad \text{and} \quad \lambda\mathbf{x} = (\lambda x_1, \dots, \lambda x_n)^\mathsf{T}.$$

[1] Cf. DALLMANN, ELSTER (1991a, pp. 69–97).

The opposite vector of $\mathbf{x}$ is denoted by $-\mathbf{x}$ and $\mathbf{0} = (0, \ldots, 0)^\mathsf{T}$ is called zero vector or origin. For $\mathbf{x}^1, \ldots, \mathbf{x}^m \in \mathbb{R}^n$

$$\sum_{i=1}^{m} \lambda_i \mathbf{x}^i \quad \text{with} \quad \lambda_1, \ldots, \lambda_m \in \mathbb{R}$$

is called a **linear combination** of $\mathbf{x}^1, \ldots, \mathbf{x}^m$. The vectors $\mathbf{x}^1, \ldots, \mathbf{x}^m$ are said to be **linearly independent** if

$$\sum_{i=1}^{m} \lambda_i \mathbf{x}^i = \mathbf{0} \quad \Longrightarrow \quad \lambda_i = 0 \ \ (i = 1, \ldots, m).$$

Otherwise we speak of linearly dependent vectors. In $\mathbb{R}^n$ at the most n vectors can be linearly independent. Each n linearly independent vectors are called a **base** of $\mathbb{R}^n$ and each vector in $\mathbb{R}^n$ can be expressed unequivocally as a linear combination of the base vectors. The n coordinate unit vectors of $\mathbb{R}^n$, which are denoted by

$$\mathbf{e}^1 := (1, 0, \ldots, 0)^\mathsf{T}, \ \ \mathbf{e}^2 := (0, 1, \ldots, 0)^\mathsf{T}, \ldots, \ \ \mathbf{e}^n := (0, \ldots, 0, 1)^\mathsf{T},$$

form a base of $\mathbb{R}^n$ and $\mathbf{x}$ can be expressed as

$$\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{e}^j.$$

The $\mathbb{R}^n$ is a **metric space** if we introduce the **Euclidean metric** $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$d(\mathbf{x}, \mathbf{y}) := \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}.$$

For all $\mathbf{x}, \ \mathbf{y}, \ \mathbf{z} \in \mathbb{R}^n$ this **distance** d between the points $\mathbf{x}$ and $\mathbf{y}$ satisfies the relations

[M1] $\quad d(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$;

[M2] $\quad d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$;

[M3] $\quad d(\mathbf{x}, \mathbf{y}) \leqq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$ (triangular inequality).

Analogously, the distance between two nonempty sets $C, D \subset \mathbb{R}^n$ is defined by

$$d(C, D) := \inf\{d(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in C, \ \mathbf{y} \in D\}.$$

In the special case $C = \{\mathbf{x}\}$

$$d(\mathbf{x}, D) := \inf\{d(\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in D\}$$

indicates the distance between the point $\mathbf{x}$ and the set D.

The $\mathbb{R}^n$ is a **normed space** if we establish the Euclidean norm $p\colon \mathbb{R}^n \to \mathbb{R}_+$ with

$$p(\mathbf{x}) \equiv \|\mathbf{x}\| := \sqrt{\sum_{j=1}^{n} x_j^2}\,.$$

In this case it can be shown that the requirements of a norm

[No1]	$\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$	
[No2]	$\|\mathbf{x}+\mathbf{y}\| \leqq \|\mathbf{x}\| + \|\mathbf{y}\|$	$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
[No3]	$\|\lambda\mathbf{x}\| = \lvert\lambda\rvert \cdot \|\mathbf{x}\|$	$\forall \lambda \in \mathbb{R},\ \forall \mathbf{x} \in \mathbb{R}^n$

are equivalent to the conditions [M1]–[M3], $\|\mathbf{x}\| = d(\mathbf{0}, \mathbf{x})$. The symbol $\|\mathbf{x}\|$ is frequently called the length of the vector $\mathbf{x}$, where the direction of the vector is irrelevant. Thus, [No3] yields at once $\|\mathbf{x}\| = \|-\mathbf{x}\|$ for $\lambda = -1$.

Finally, the scalar product of two vectors $\mathbf{y}, \mathbf{x} \in \mathbb{R}^n$ is defined as

$$\mathbf{y}^{\mathsf{T}}\mathbf{x} := \sum_{j=1}^{n} y_j x_j\,.$$

Bear in mind that SCHWARZ's inequality

$$\lvert\mathbf{y}^{\mathsf{T}}\mathbf{x}\rvert \leqq \|\mathbf{y}\| \cdot \|\mathbf{x}\| \tag{A.1}$$

is satisfied for all vectors $\mathbf{y}, \mathbf{x} \in \mathbb{R}^n$.

A.2 Elementary Concepts of Topology

Topologically, the presented metric is suitable for describing closed, open, and compact sets. Moreover, the convergence of a sequence of points can be characterized.

The initial point of the subsequent expositions is an arbitrary set M. If a family $\mathfrak{D} = (O_i \mid i \in I)$ satisfies the conditions[2]

[O1] $\emptyset \in \mathfrak{D}$ and $M \in \mathfrak{D}$,

[O2] for a finite number of $O_i \in \mathfrak{D}$ it follows $\bigcap_{i=1}^{r} O_i \in \mathfrak{D}$,

[O3] for an arbitrary number of $O_i \in \mathfrak{D}$ it follows $\bigcup_{i} O_i \in \mathfrak{D}$,

then the pair $(M, \mathfrak{D})$ is called a **topological space**. Each set $O_i \in \mathfrak{D}$ is said to be an **open set** (in $(M, \mathfrak{D})$). A set $C \subset M$ is **closed** (in $(M, \mathfrak{D})$) if its complement $\complement_M C := M \setminus C$ is an open set, i.e. if $\complement_M C \in \mathfrak{D}$.

The next theorem is important because the analysis frequently uses closed sets.

[2]Cf. DALLMANN, ELSTER (1991a, p. 51 ff.). For a valuable introduction to topology cf. PATTY (1993).

Theorem A.1 *The intersection of an arbitrary number of closed sets is closed. The union of a finite number of closed sets is closed.*

Various elementary terms are established for a topological space $(M, \mathfrak{D})$.

- A set $U \subset M$ is called an (open) **neighborhood of** $x \in M$ if there is an open set $O \in \mathfrak{D}$ with $x \in O$ and $O \subset U$. The set U is called a **neighborhood of the subset** $C \subset M$ if an open set $O \in \mathfrak{D}$ exists with $C \subset O \subset U$.

- A point $x \in C$ is called an **inner point** of C if there is a neighborhood U of x with $U \subset C$. The collection of all inner points is called the **interior** or **(open) core** of C and is denoted by $\operatorname{int} C$.

  The set C is open if and only if each element $x \in C$ belongs to the interior of C, i.e.

  $$C = \operatorname{int} C .$$

- A point $x^0 \in M$ is called a **cluster point** of C if every neighborhood of x^0 contains a point $x \in C$ with $x \neq x^0$. The set of all cluster point is frequently denoted by C'.

- A point $x \in C$ which is not a cluster point of C is said to be an **isolated point** of C.

- A point $x \in M$ is said to be **adherent** to C if it is either a cluster point or an isolated point of C. The **closed hull** or **closure** of the set C contains all of its adherent points and is denoted by $\operatorname{cl} C$.[3] Hence, $\operatorname{cl} C = C \cup C'$.

  A set is closed if and only if it contains all of its cluster points, that is

  $$C' \subset C \quad \text{or} \quad C = \operatorname{cl} C .$$

- A point $x \in C$ is called a **boundary point** if every neighborhood contains at least one point in C and at the same time a point in $\complement C$. The set of all boundary points is called the **boundary** or frontier ∂C of C.

  $$\partial C = \operatorname{cl} C \cap \operatorname{cl}(\complement C)$$

  Each boundary point belongs to the closure and does not belong to the interior.

  $$\partial C = \operatorname{cl} C \setminus \operatorname{int} C$$

Summarizing we have the inclusion

$$\operatorname{int} C \subset C \subset \operatorname{cl} C$$

[3]The closure of the set $C \subset \mathbb{R}^n$ is the smallest closed subset in $\mathbb{R}^n$ containing C. Cf. HILDENBRAND, KIRMAN (1988, p. 244).

for each set C in the topological space $(M, \mathfrak{D})$. All of the concepts can be transferred to the metric space $\mathbb{R}^n$ by introducing the term **ball**. A ball K of radius $r \in \mathbb{R}_{++}$ centered at the point $\mathbf{x} \in \mathbb{R}^n$ is given by

$$K(\mathbf{x}, r) := \left\{\mathbf{y} \in \mathbb{R}^n \mid d(\mathbf{x}, \mathbf{y}) < r\right\} \qquad \text{(open ball)},$$
$$K[\mathbf{x}, r] := \left\{\mathbf{y} \in \mathbb{R}^n \mid d(\mathbf{x}, \mathbf{y}) \leqq r\right\} \qquad \text{(closed ball)}.$$

If we denote the intersection of a finite number of open balls or the union of an arbitrary number of open balls as an open sets, then these open sets fulfill the conditions [O1]–[O3]. We speak of the natural or ordinary topology or the topology induced by the metric d. In this way each set $X \subset \mathbb{R}^n$ can be converted into a topological space by declaring the set $\hat{C} := C \cap X$ to be open in X for each open set C (in $\mathbb{R}^n$).

An open ball centered at point $\mathbf{x}$ immediately provides a neighborhood of the vector $\mathbf{x}$. The ball $K(\mathbf{x}, \varepsilon) \equiv U_\varepsilon(\mathbf{x})$ is then said to be an (open) ε-neighborhood of $\mathbf{x}$.

A subset $C \subset \mathbb{R}^n$ is now said to be **open** (with respect to the metric d) if either

(A.2) $$\mathbf{x} \in C \implies \exists r > 0 \quad \text{with} \quad K(\mathbf{x}, r) \subset C$$

or $C = \emptyset$. A subset $C \subset \mathbb{R}^n$ is said to be **closed** (with respect to the metric d) if either the complement $\complement C$ is open or $C = \emptyset$. By (A.2) the interior of the set C can be expressed as

(A.3) $$\operatorname{int} C = \{\mathbf{x} \in C \mid \exists \varepsilon > 0: \; K(\mathbf{x}, \varepsilon) \subset C\}.$$

The characterization of specific points follows from

Theorem A.2 *Given a point $\mathbf{x} \in \mathbb{R}^n$ and a nonempty set $C \subset \mathbb{R}^n$, then we have for the space $\mathbb{R}^n$ endowed with a metric d*

1. *The point $\mathbf{x}$ is an adherent point of C iff $d(\mathbf{x}, C) = 0$.*
2. *The point $\mathbf{x}$ is an inner point of C iff $d(\mathbf{x}, \complement C) > 0$.*
3. *The point $\mathbf{x}$ is an inner point of $\complement C$ (outer point of C) iff $d(\mathbf{x}, C) > 0$.*
4. *The point $\mathbf{x}$ is a boundary point of C iff $d(\mathbf{x}, C) = d(\mathbf{x}, \complement C) = 0$.*

A.3 Convergence in Metric Spaces

The description of sequences is fundamental to the concept of convergence in the metric space $\mathbb{R}^n$.

A unique mapping ϕ of the set $\mathbb{N}$ of natural numbers into the set $\mathbb{R}$ of real numbers is called a **sequence of numbers** or, in short, sequence. If we put $\phi(\nu) = x^\nu$ for all $\nu \in \mathbb{N}$, then x^ν is called an **element of the sequence** and we write $\phi = \{x^\nu\}$. Similarly, $\{\mathbf{x}^\nu\}$ denotes a **sequence of points** in $\mathbb{R}^n$.

Monotonic sequences are subdivided into monotonically increasing and decreasing sequences. A sequence $\{x^\nu\}$ is said to be monotonically decreasing or nonincreasing if $x^\nu \geqq x^{\nu+1}$ holds for all $\nu \in \mathbb{N}$. Monotonically increasing sequences are defined analogously. If the strict inequality holds, then we speak of strict monotonicity. Letting $\{\nu_k\}$ be a strictly monotonically increasing sequence with $\nu_k \in \mathbb{N}$ for all $k \in \mathbb{N}$, then $\{x^{\nu_k}\}$ is called a **subsequence** of $\{x^\nu\}$.[4]
A sequence of numbers $\{x^\nu\}$ is said to be **bounded** (bounded below, bounded above) if there is a number $r \in \mathbb{R}$ such that for all $\nu \in \mathbb{N}$ we have $|x^\nu| \leqq r$ ($x^\nu \geqq r$; $x^\nu \leqq r$). Accordingly, a sequence of points $\{\mathbf{x}^\nu\}$ is said to be bounded if there is a number $r \in \mathbb{R}$ such that $\|\mathbf{x}^\nu\| \leqq r$ for all $\nu \in \mathbb{N}$.

A number α is called a **cluster point** or a **partial limit of the sequence of numbers** $\{x^\nu\}$ if the sequence is frequently in every ε-neighborhood of α, i.e. if there is an infinite number of indices ν for each ε such that $|x^\nu - \alpha| < \varepsilon$. If the sequence is eventually in every ε-neighborhood of α, then the number α is called the **limit** of the sequence.
A bounded sequence of numbers $\{x^\nu\}$ is said to be **convergent**, if it has exactly one cluster point α, which is moreover equivalent to the limit of the sequence of numbers;[5] we now write $x^\nu \to x^0$ or $\lim\limits_{\nu \to +\infty} x^\nu = x^0$. If there is no number x^0 such that $x^\nu \to x^0$, then we say that the sequence is **divergent**. The sequence of numbers $\{x^\nu\}$ converges if and only if for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|x^n - x^m| < \varepsilon$ for all n, m with $n \geqq n_0$ and $m \geqq n_0$ (**CAUCHY's criterion of convergence**). The sequence of numbers $\{x^\nu\}$ converges to x^0 if for each positive number ε an index n_0 exists such that always $|x^\nu - x^0| < \varepsilon$ for all $\nu > n_0$. That is, a sequence of numbers converges to x^0 if the sequence is eventually in every ε-neighborhood of x^0 and only a finite number[6] of elements of the sequence does not lie in the ε-neighborhood. Frequently, sequences of numbers converging to zero are of particular interest.

A sequence of points $\{\mathbf{x}^\nu\}$ in $\mathbb{R}^n$ approaches the limit $\mathbf{x}^0 \in \mathbb{R}^n$ if and only if the sequence of numbers $\{d(\mathbf{x}^\nu, \mathbf{x}^0)\}$ converges to zero.

$$d(\mathbf{x}^\nu, \mathbf{x}^0) \to 0 \iff \mathbf{x}^\nu \to \mathbf{x}^0 .$$

Equivalently,[7] the sequence of points must converge coordinatewise to $\mathbf{x}^0$.

$$\mathbf{x}^\nu \to \mathbf{x}^0 \iff x_j^\nu \to x_j^0 \quad (j = 1, \ldots, n)$$

A point $\mathbf{x}^0 \in \mathbb{R}^n$ is called a **cluster point**[8] **of the sequence of points** $\{\mathbf{x}^\nu\}$ if there is a subsequence $\{\mathbf{x}^{\nu_k}\}$ with $\mathbf{x}^{\nu_k} \to \mathbf{x}^0$. The only cluster point of a convergent sequence is its limit.

Theorem A.3[9] *A point* $\mathbf{x}^0 \in \mathbb{R}^n$ *is a cluster point of the set C if and only if there*

[4] Cf. BRONSTEIN, SEMENDJAJEW (1987, p. 245 f.) and DALLMANN, ELSTER (1991b, p. 110 ff.).

[5] Accordingly, a sequence can have at the most one limit.

[6] If all but the most elements of the sequence lie in the ε-neighborhood, then all except a finite number of elements of the sequence have this property.

[7] Cf. HILDENBRAND, KIRMAN (1988, p. 242).

[8] LEWIN, LEWIN (1993, p. 113) speak of partial limits instead of cluster points.

[9] DALLMANN, ELSTER (1991a, p. 82, Theorem 4.7).

is a sequence of points $\{\mathbf{x}^\nu\} \subset C$ *with* $\mathbf{x}^\nu \neq \mathbf{x}^0$ *for all* $\nu \in \mathbb{N}$ *converging to the limit* $\mathbf{x}^0$, *i.e.* $\mathbf{x}^\nu \to \mathbf{x}^0$.

Theorem A.4 (BOLZANO, WEIERSTRASS)[10] *Each bounded sequence* $\{\mathbf{x}^\nu\}$ *in* $\mathbb{R}^n$ *(equipped with an arbitrary norm) contains a convergent subsequence.*

A sequence having $-\infty$ or $+\infty$ as a sole cluster point is said to be **divergent**[11], we then write $x^\nu \to -\infty$ or $x^\nu \to +\infty$. If a sequence is convergent or divergent with limit $-\infty$ or $+\infty$, then we say that it achieves a limit in $[-\infty, +\infty]$.
The subsequent statements are also valid for sequences of points. Note that the elements of a sequence need not be distinct.

- Each bounded sequence of numbers $\{x^\nu\}$ has at least one cluster point.[12]
- By Theorem A.4 (BOLZANO, WEIERSTRASS) each bounded sequence of numbers $\{x^\nu\}$ contains a convergent subsequence $\{x^{\nu_k}\}$.[13]
- The number α is a cluster point of the sequence $\{x^\nu\}$ if and only if $\{x^\nu\}$ contains a convergent subsequence $\{x^{\nu_k}\}$ such that α is the limit of this subsequence, $x^{\nu_k} \to \alpha$. In doing so the number α itself does not need to belong to the examined sequence.
- Every subsequence $\{x^{\nu_k}\}$ of a convergent sequence of numbers $\{x^\nu\}$ as well converges to the limit of $\{x^\nu\}$.
- Each monotonic and bounded sequence of numbers $\{x^\nu\}$ is convergent.
- Each monotonic and nonbounded sequence of numbers $\{x^\nu\}$ diverges to infinity; if the sequence monotonically increases (decreases), then it diverges to $+\infty$ $(-\infty)$.

The set of all cluster points of a given sequence of numbers is denoted by $L\{x^\nu\}$ and each bounded sequence of numbers has a greatest and a smallest cluster point. The greatest and the smallest cluster point are called **limes superior** $(\limsup_{\nu \to +\infty} x^\nu = \sup L\{x^\nu\})$ and **limes inferior** $(\liminf_{\nu \to +\infty} x^\nu = \inf L\{x^\nu\})$ respectively.[14] From these definitions follows at once

$$\liminf_{\nu \to +\infty} x^\nu \leqq x^0 \leqq \limsup_{\nu \to +\infty} x^\nu$$

for each element of the set of cluster points $x^0 \in L\{x^\nu\}$.

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[10] Cf. HEUSER (1992, p. 22, Theorem 109.15).

[11] Sequences with more than one cluster point are also called divergent. Cf. DALLMANN, ELSTER (1991b, p. 122).

[12] Cf. DALLMANN, ELSTER (1991b, p. 116 f.).

[13] Cf. in what follows DALLMANN, ELSTER (1991b, pp. 125–131).

[14] For the empty set we declare $\inf \emptyset := +\infty$ and $\sup \emptyset := -\infty$.

If the sequence $\{x^\nu\}$ is not bounded above (or below), then[15] $\limsup_{\nu\to+\infty} x^\nu = +\infty$ (or $\liminf_{\nu\to+\infty} x^\nu = -\infty$). Obviously, $\limsup_{\nu\to+\infty} x^\nu = +\infty$ holds if and only if $\{x^\nu\}$ contains a subsequence $\{x^{\nu_k}\}$ diverging to infinity, $x^{\nu_k} \to +\infty$. Hence, for a monotonically increasing and bounded above sequence of numbers $\liminf_{\nu\to+\infty} x^\nu = \limsup_{\nu\to+\infty} x^\nu = +\infty$.

A bounded sequence of number $\{x^\nu\}$ converges to x^0 if and only if

$$\limsup_{\nu\to+\infty} x^\nu = x^0 = \liminf_{\nu\to+\infty} x^\nu .$$

Examples: Each real number is a cluster point for the sequence of the rational numbers $\mathbb{Q}$. The sequence of the rational numbers in $[0, 1[$ has each $x^0 \in [0, 1]$ as a cluster point. Thus, we have $\liminf_{\nu\to+\infty} x^\nu = 0$ and $\limsup_{\nu\to+\infty} x^\nu = 1$. For the sequence of the positive rational numbers $\mathbb{Q}_{++}$ we have $\liminf_{\nu\to+\infty} x^\nu = 0$ and $\limsup_{\nu\to+\infty} x^\nu = +\infty$. The sequence of the natural numbers $\mathbb{N}$ yields $\liminf_{\nu\to+\infty} x^\nu = \limsup_{\nu\to+\infty} x^\nu = +\infty$ since there is no finite cluster point.

Because each closed set in an arbitrary topological space contains all of its cluster points, the closedness in the metric space $\mathbb{R}^n$ can be described with respect to the characterization of cluster points by Theorem A.3.

Theorem A.5[16] *A subset* $C \subset \mathbb{R}^n$ *is closed if and only if the limit* $\mathbf{x}^0$ *of a convergent sequence* $\{\mathbf{x}^\nu\}$ *with* $\mathbf{x}^\nu \in C$ *always satisfies* $\mathbf{x}^0 \in C$.

$$[\mathbf{x}^\nu \to \mathbf{x}^0,\ \mathbf{x}^\nu \in C] \implies \mathbf{x}^0 \in C$$

A set $C \subset \mathbb{R}^n$ is **bounded** if a positive real number r exists and a point $\mathbf{y} \in \mathbb{R}^n$ with $d(\mathbf{x}, \mathbf{y}) < r$ for each $\mathbf{x} \in C$. Accordingly, for each bounded set C there is an open ball $K(\mathbf{y}, r)$ of finite radius r containing C, i.e. $C \subset K(\mathbf{y}, r)$. Together with Theorem A.4 each sequence in a bounded set C contains a convergent subsequence.

A.4 Compact Sets

A set $C \subset \mathbb{R}^n$ is said to be **compact** if one of the following equivalent statements[17] is satisfied.

(1) The set C is closed and bounded.[18]

[15] Cf. HEUSER (1982, p. 185).

[16] DALLMANN, ELSTER (1991a, p. 82, Theorem 4.8).

[17] The statements are equivalent as long as C is a subset in $\mathbb{R}^n$. The equivalence is not valid for more general spaces.

[18] Cf. HEUSER (1992, p. 33, Theorem 111.6).

(2) Each open covering of C contains a finite covering (*covering compactness* of HEINE, BOREL).[19]

(3) Each sequence of points $\{\mathbf{x}^{\nu}\}$ in C contains a convergent subsequence $\{\mathbf{x}^{\nu_k}\}$ whose limit belongs to C (*sequentially compactness*).

(4) If $(C_i|\, i \in I)$ is an arbitrary family of closed sets in C such that each finite intersection of sets of the family is not empty, then $\bigcap_{i\in I} C_i \neq \emptyset$ (*finite intersection property*).

Since the finite intersection property is an unusual criterion for the compactness of a set C according to (4), we now prove the equivalence of (2) and (4). All of the remaining proof is omitted.
The necessary part proves by contradiction that (4) is implied by (2): let $(C_i|\, i \in I)$ be a family of closed sets in C such that each finite intersection of sets of the family is not empty. For $\bigcap_{i\in I} C_i = \emptyset$ DE MORGAN's complementation rules yield[20]

$$\bigcap_{i\in I} C_i = \emptyset \implies C = \bigcup_{i\in I} (C \setminus C_i) \overset{(2)}{\implies} C = \bigcup_{i=1}^{m} (C \setminus C_i) \implies \bigcap_{i=1}^{m} C_i = \emptyset\,,$$

so that a contradiction to the premise results.
In the sufficient part (4) is given and (2) is concluded by contradiction: Suppose $(C_i|\, i \in I)$ is an open covering of C containing no finite covering, i.e. for each finite family $(C_i|\, i = 1,\dots, m)$ we have

$$\exists\, \mathbf{x} \in C\colon \mathbf{x} \notin \bigcup_{i=1}^{m} C_i \iff C \setminus \left(\bigcup_{i=1}^{m} C_i\right) = \bigcap_{i=1}^{m} (C \setminus C_i) \neq \emptyset\,.$$

Then the family of closed sets $(C \setminus C_i|\, i \in I)$ satisfies $\bigcap_{i\in I} (C \setminus C_i) \neq \emptyset$ by (4). In view of DE MORGAN's complementation rules follows

$$\bigcap_{i\in I} (C \setminus C_i) = C \setminus \left(\bigcup_{i\in I} C_i\right) \neq \emptyset\,.$$

As $C \not\subset \bigcup_{i\in I} C_i$ contradicts the assumption, $(C_i|\, i \in I)$ cannot be an open covering of C.

[19] A family $(C_i|\, i \in I)$ of subsets in $\mathbb{R}^n$ with $C \subset \bigcup_{i\in I} C_i$ is called a covering of C. If the family consists of open sets, then we speak of an open covering. Accordingly, the family is called a finite covering if it consists of a finite number of sets.

[20] DE MORGAN's complementation rules state

$$C \setminus \left(\bigcup_{i\in I} C_i\right) = \bigcap_{i\in I} (C \setminus C_i) \quad \text{and} \quad C \setminus \left(\bigcap_{i\in I} C_i\right) = \bigcup_{i\in I} (C \setminus C_i).$$

see HEUSER (1982, p. 20).

B Convex Analysis

B.1 Affine Sets

Let C and D be two nonempty subsets in $\mathbb{R}^n$. The subsequent operations are defined for a vector $\mathbf{a} \in \mathbb{R}^n$ and a real number λ.[21]

$$C + \mathbf{a} := \{\mathbf{x} + \mathbf{a} \mid \mathbf{x} \in C\}$$
$$C + D := \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in C,\ \mathbf{y} \in D\}$$
$$\lambda C := \{\lambda \mathbf{x} \mid \mathbf{x} \in C\}$$

Moreover, the following rules must be taken into account.

$$\begin{aligned} C + \emptyset &= \emptyset \\ \lambda\emptyset &= \emptyset \\ -C &= (-1)C \\ C - D &= C + (-D) \end{aligned}$$

A subset C in $\mathbb{R}^n$ is called an **affine set**[22], if each two points $\mathbf{x}^1$ and $\mathbf{x}^2$ in C imply

$$(1-\lambda)\mathbf{x}^1 + \lambda\mathbf{x}^2 \in C \qquad \forall \lambda \in \mathbb{R}.$$

Subspaces in $\mathbb{R}^n$ are affine sets containing the origin of ordinates. If we define the dimension of a nonempty affine set as dimension of the half-space parallel to the examined set, then affine sets of dimension 0, 1, or 2 are called a **point**, a **line** or a **plane**. For the empty set we declare $\dim\emptyset = -1$. An $(n-1)$-dimensional affine subset in $\mathbb{R}^n$ is called a **hyperplane**.

Theorem B.1[23] *For given* $\alpha \in \mathbb{R}$ *and* $\mathbf{y} \in \mathbb{R}^n$ *with* $\mathbf{y} \neq \mathbf{0}$ *the set*

$$H(\mathbf{y}, \alpha) := \left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{y}^\mathsf{T}\mathbf{x} = \alpha\right\}$$

is a hyperplane in $\mathbb{R}^n$. *Every hyperplane may be represented in this way, with* α *and* $\mathbf{y}$ *unique up to a common nonzero multiple* $\lambda \neq 0$.

The vector $\mathbf{y}$ is **normal** to the hyperplane H. All vectors which are normal to H may be expressed as $\lambda\mathbf{y}$, where the scalar $\lambda \in \mathbb{R}$ may not be zero. A hyperplane $H(\mathbf{y}, \alpha)$ is called a **supporting hyperplane**of a set $C \subset \mathbb{R}^n$ if both of the following conditions are satisfied.

$$\begin{aligned} &\mathbf{y}^\mathsf{T}\mathbf{x} \geqq \alpha && \text{for all } \mathbf{x} \in C \\ \text{and } &\mathbf{y}^\mathsf{T}\mathbf{x} = \alpha && \text{for at least one } \mathbf{x} \in C \end{aligned}$$

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[21] Cf. BERGE (1963, p. 131).

[22] Cf. ROCKAFELLAR (1972, p. 3).

[23] ROCKAFELLAR (1972, p. 5, Theorem 1.3).

By a **nontrivial supporting hyperplane** of the set C we understand a supporting hyperplane H of C with $C \not\subset H$.

Apart from the **coordinate representation** of a hyperplane H in $\mathbb{R}^n$ according to Theorem B.1 there is a **parametric representation** of H. Thus, each hyperplane in $\mathbb{R}^n$ can be expressed by a vector $\mathbf{b}^0 \in \mathbb{R}^n$ and $n-1$ linearly independent directions $\mathbf{b}^i \neq \mathbf{0}$ $(i = 1,\dots,n-1)$.

$$H = \left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{b}^0 + \sum_{i=1}^{n-1} \lambda_i \mathbf{b}^i \text{ with } \lambda_i \in \mathbb{R}\ (i = 1,\dots,n-1)\right\}$$

An equivalent representation of H is given by n affinely independent vectors. Let $\mathbf{x}^1,\dots,\mathbf{x}^m$ be distinct vectors in $\mathbb{R}^n$, then the vectors are said to be **affinely independent** if

$$\sum_{i=1}^{m} \lambda_i \mathbf{x}^i = \mathbf{0},\ \sum_{i=1}^{m} \lambda_i = 0 \quad \Longrightarrow \quad \lambda_i = 0\ (i = 1,\dots,m).$$

One can show, that the definition of affine independence is equivalent to linear independence[24] of the vectors $\mathbf{x}^2 - \mathbf{x}^1,\dots,\mathbf{x}^m - \mathbf{x}^1$, i.e.

$$\sum_{i=2}^{m} \lambda_i^* (\mathbf{x}^i - \mathbf{x}^1) = \mathbf{0} \quad \Longrightarrow \quad \lambda_i^* = 0\ (i = 1,\dots,m).$$

Moreover, for each set $C \subset \mathbb{R}^n$ one can show that there is an unique smallest affine set containing C, which is denoted the **affine hull** aff$C$. Over and above the points $\mathbf{x}^0, \mathbf{x}^1,\dots,\mathbf{x}^m \in \mathbb{R}^n$ are affinely independent if its affine hull $\text{aff}\{\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^m\}$ is m-dimensional.

If the set C consists of three different points in $\mathbb{R}^n$ and if these points do not lie in a line, then the plane through these points is the affine hull of the set C. If these points lie on a line, then the line itself determines the affine hull of the set C. For three identical points this point would also be the affine hull of the points.

Theorem B.2[25] *The affine hull of a set $C \subset \mathbb{R}^n$ is given by the set of all affinely linear combinations of points in C, i.e.*

$$\text{aff}C = \left\{\sum_{i=1}^{m} \lambda_i \mathbf{x}^i \mid \mathbf{x}^i \in C,\ \sum_{i=1}^{m} \lambda_i = 1,\ \ m = 1, 2, \dots\right\}.$$

[24]In $\mathbb{R}^3$ the vector $\mathbf{b}^0$ and the two linearly independent directions $\mathbf{b}^1 \neq \mathbf{0}$ and $\mathbf{b}^2 \neq \mathbf{0}$ give a parametric representation of the plane

$$H = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{b}^0 + \lambda_1 \mathbf{b}^1 + \lambda_2 \mathbf{b}^2 \quad \text{with} \quad \lambda_1, \lambda_2 \in \mathbb{R}\right\}.$$

Alternatively, three affinely independent vectors $\mathbf{a}^0 = \mathbf{b}^0$, $\mathbf{a}^1$, $\mathbf{a}^2$ suffice for the representation of the hyperplane. The directions are then given by $\mathbf{b}^1 = \mathbf{a}^1 - \mathbf{a}^0$ and $\mathbf{b}^2 = \mathbf{a}^2 - \mathbf{a}^0$.
According to Theorem B.1 we obtain the coordinate representation of the hyperplane by choosing a vector $\mathbf{y}$ perpendicular to the directions $\mathbf{b}^1$ and $\mathbf{b}^2$, i.e. $\mathbf{y}^\mathrm{T}\mathbf{b}^1 = 0$ and $\mathbf{y}^\mathrm{T}\mathbf{b}^2 = 0$. Finally, we put $\alpha = \mathbf{y}^\mathrm{T}\mathbf{b}^0$.

[25]Cf. Elster, Reinhardt, Schäuble, Donath (1977, p. 34 f., Theorem 2.7).

B.2 Convex Sets

For the class of convex sets two definitions are of particular interest.[26] A point $\mathbf{x} \in \mathbb{R}^n$ with

$$\text{(B.1)} \qquad \mathbf{x} = \sum_{i=1}^{m} \lambda_i \mathbf{x}^i \qquad \text{with} \quad \sum_{i=1}^{m} \lambda_i = 1, \ \lambda_i \geqq 0 \ \ (i = 1, \ldots, m)$$

is called a **convex (linear) combination** or a **centroid** of the points $\mathbf{x}^1, \ldots, \mathbf{x}^m \in \mathbb{R}^n$. A set $C \subset \mathbb{R}^n$ is said to be **convex** if

$$\mathbf{x}^1, \mathbf{x}^2 \in C, \ \ 0 < \lambda < 1 \implies \lambda \mathbf{x}^1 + (1 - \lambda)\mathbf{x}^2 \in C .$$

The empty set is also declared to be a convex set in $\mathbb{R}^n$. For convex sets the following theorems hold good.

- The intersection of an arbitrary number of convex sets is convex.
- A subset in $\mathbb{R}^n$ is said to be convex if and only if it contains all of the convex combinations of its elements.
- Let C be a convex subset in $\mathbb{R}^n$. If $\mathbf{a} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, then the sets $C + \mathbf{a}$ and λC are convex, too.
- Let C_i $(i = 1, \ldots, r)$ be convex subsets in $\mathbb{R}^n$. Then each linear combination $\sum_{i=1}^{r} \lambda_i C_i$ is convex, too.

The **relative interior**[27] of a convex set $C \in \mathbb{R}^n$ is determined by the interior of the set C regarding its affine hull aff$C$. If we denote the relative interior of the set C by rint C, then analogous to (A.3) it ensues[28]

$$\text{(B.2)} \qquad \operatorname{rint} C := \{\mathbf{x} \in \operatorname{aff} C \mid \exists \varepsilon > 0 : \ K(\mathbf{x}, \varepsilon) \cap \operatorname{aff} C \subset C\} .$$

In the special case of a solitary point $C = \{\mathbf{x}\}$ the relative interior of this set consists of the element $\mathbf{x}$ itself, $\operatorname{rint} C = \{\mathbf{x}\}$.
The **relative boundary** of C results from the set difference of the closed hull of C and the relative interior of C, i.e. $\operatorname{cl} C \setminus \operatorname{rint} C$. A set is called **relatively open** (with respect to aff$C$) if $\operatorname{rint} C = C$. [29] Regarding the separation theorems, the following property is important: for each convex set $C \subset \mathbb{R}^n$ we have

$$\text{(B.3)} \qquad \operatorname{cl}(\operatorname{rint} C) = \operatorname{cl} C \quad \text{and} \quad \operatorname{rint}(\operatorname{cl} C) = \operatorname{rint} C .$$

[26] Unless otherwise specified the expositions are taken from ROCKAFELLAR (1972).

[27] A line between two distinct points in $\mathbb{R}^3$ has no interior points. However, all points of the line except the end points are included as relatively interior points.

[28] Cf. ROCKAFELLAR (1972, p. 44).

[29] See p. 283.

The dimension dim C of a convex set C is the dimension of its affine hull. Thus, for an n-dimensional convex subset $C \subset \mathbb{R}^n$ we need not to distinguish its relative interior by (B.2) from its interior by (A.3):

$$\text{(B.4)} \qquad \operatorname{rint} C = \operatorname{int} C.$$

In the class of convex subsets in $\mathbb{R}^n$ especially **convex cones** stand out. The nonnegative orthant $\mathbb{R}^n_+$ as well as the positive orthant $\mathbb{R}^n_{++}$ with

$$\mathbb{R}^n_+ := \{\mathbf{x} \in \mathbb{R}^n |\, \mathbf{x} \geqq \mathbf{0}\} \quad \text{and} \quad \mathbb{R}^n_{++} := \{\mathbf{x} \in \mathbb{R}^n |\, \mathbf{x} > \mathbf{0}\}$$

belong to the perhaps most significant convex cones of $\mathbb{R}^n$. Further examples of convex cones are half-spaces. For $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \neq \mathbf{0}$ and $\alpha \in \mathbb{R}$ the sets

$$\{\mathbf{x} \in \mathbb{R}^n |\, \mathbf{y}^\mathrm{T}\mathbf{x} \leqq \alpha\} \quad \text{and} \quad \{\mathbf{x} \in \mathbb{R}^n |\, \mathbf{y}^\mathrm{T}\mathbf{x} \geqq \alpha\}$$

are called closed half-spaces. Accordingly, the following sets denote open half-spaces:

$$\{\mathbf{x} \in \mathbb{R}^n |\, \mathbf{y}^\mathrm{T}\mathbf{x} < \alpha\} \quad \text{and} \quad \{\mathbf{x} \in \mathbb{R}^n |\, \mathbf{y}^\mathrm{T}\mathbf{x} > \alpha\}$$

In many cases we need a representation of nonconvex sets by convex sets. For this purpose we establish the convex hull of a set analogous to the affine hull.

The **convex hull** of a set $C \subset \mathbb{R}^n$ is defined by the intersection of all convex sets containing C and is denoted by conv$C$. Equivalently, conv$C$ is the smallest convex set containing the original set C.

The convex hull of a set is always convex. We have $\operatorname{conv}C = C$ if and only if C is a convex subset in $\mathbb{R}^n$. The convex hull of two points $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n$ is the line segment $[\mathbf{x}^1, \mathbf{x}^2]$. The convex hull of the three points (not lying on a straight line) $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathbb{R}^n$ is the triangle determined by these points. Summarizing we have in correspondence with Theorem B.2

Theorem B.3[30] *The convex hull* conv$C$ *of a set* C *consists of the set of all convex combinations of points in* C.

$$\operatorname{conv}C = \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{x}^i |\, \mathbf{x}^i \in C, \;\; \lambda_i \geqq 0 \;\; (i = 1, \dots, m), \;\; \sum_{i=1}^{m} \lambda_i = 1, \;\; m = 1, 2, \dots \right\}$$

The convex hull of a set C containing a finite number of points is called a **polehydral set**. If C consists of $m + 1$ affinely independent vectors, then the convex hull of C is called an m-dimensional simplex. The vectors $\{\mathbf{x}^0, \dots, \mathbf{x}^m\}$ are called **vertices** of the simplex. Each point of a simplex can be expressed unequivocally as a convex combination of the vertices. To simplify notation, we define the n-dimensional **unit simplex** Λ^{n+1} with the coordinate unit vectors as its vertices:[31]

$$\text{(B.5)} \qquad \Lambda^{n+1} := \left\{ \boldsymbol{\lambda} \in \mathbb{R}^{n+1}_+ |\, \boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_n)^\mathrm{T}, \; \sum_{i=0}^{n} \lambda_i = 1 \right\}$$

[30]Cf. ELSTER, REINHARDT, SCHÄUBLE, DONATH (1977, p. 34 f., Theorem 2.7).

[31]For n goods the price simplex $\Delta := \{\mathbf{p} \in \mathbb{R}^n_+ |\, \sum_{j=1}^n p_j = 1\}$ is an $(n-1)$-dimensional unit simplex with the unit vectors as vertices.

Thus, the statement of Theorem B.3 can be rewritten as

Theorem B.4 (CARATHÉODORY's Theorem)[32] *Let C be a nonempty subset in $\mathbb{R}^n$. Then each point $\mathbf{x} \in \operatorname{conv} C$ can be expressed as a convex combination of no more than $n+1$ points in C which are not necessarily distinct.*

$$\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}^i \qquad \textit{with} \quad \mathbf{x}^i \in C, \;\; \boldsymbol{\lambda} \in \Lambda^{n+1}$$

In particular, $\operatorname{conv} C$ *corresponds to the union of all d-dimensional simplices with vertices in C, where* $d = \dim C$.

Note that this theorem especially holds if C is convex, i.e. $C = \operatorname{conv} C$.

Corollary B.4.1 *If $(C_i \mid i \in I)$ is an arbitrary family of nonempty convex subsets in $\mathbb{R}^n$, then each point $\mathbf{x} \in \operatorname{conv} \bigcup_{i \in I} C_i$ can be expressed as a convex combination of no more than $n+1$ distinct points $\mathbf{x}^i$, where each $\mathbf{x}^i$ belongs to a diverse set C_i.*

Proof: According to Theorem B.4 there are no more than $n+1$ not necessarily distinct points $\mathbf{x}^i$ in $\bigcup_{i \in I} C_i$ such that $\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}^i$ with $\boldsymbol{\lambda} \in \Lambda^{n+1}$. Having eliminated all points weighted by zero, all pairs of points belonging to the same set can be removed as follows: assume without loss of generality that $\mathbf{x}^0$ and $\mathbf{x}^i$ belong to the set C_i. Substituting $\mu \tilde{\mathbf{x}}^i := \lambda_0 \mathbf{x}^0 + \lambda_i \mathbf{x}^i$ by $\lambda_0 + \lambda_i = \mu$ yields

$$\tilde{\mathbf{x}}^i = (\lambda_0/\mu)\mathbf{x}^0 + (\lambda_i/\mu)\mathbf{x}^i \in C_i.$$

Therefore, the expression of $\mathbf{x}$ can be reduced to points belonging to diverse sets. ■

In particular, if the family $(C_i \mid i \in I)$ consist of m sets, where $m \leqq n$ by assumption, then each point in $\operatorname{conv} \bigcup_{i \in I} C_i$ can be expressed as a convex combination of no more than m distinct points, where again each point belongs to a diverse set. For the special case of two sets we obtain

Corollary B.4.2[33] *For two nonempty convex subsets C and D in $\mathbb{R}^n$ we have*

$$\operatorname{conv}(C \cup D) = \{\lambda \mathbf{x} + \mu \mathbf{y} \mid \mathbf{x} \in C, \; \mathbf{y} \in D, \; \lambda \geqq 0, \; \mu \geqq 0, \; \lambda + \mu = 1\}.$$

Corollary B.4.3 *Provided $\mathbf{y}^{\mathsf{T}} \mathbf{x} \leqq \alpha$ holds for all $\mathbf{x} \in C$ with $C \subset \mathbb{R}^n$, then the inequality is also satisfied for all $\mathbf{x} \in \operatorname{conv} C$. The analogous statement is valid if the inequality is substituted by an equation.*

Proof: By Theorem B.4 each $\mathbf{x} \in \operatorname{conv} C$ can be expressed as

$$\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}^i \qquad \text{with} \quad \mathbf{x}^i \in C, \;\; \boldsymbol{\lambda} \in \Lambda^{n+1}.$$

[32] Cf. NOŽIČKA, GRYGAROVÁ, LOMMATZSCH (1988, p. 31) and ROCKAFELLAR (1972, p. 155).
[33] Cf. NOŽIČKA, GRYGAROVÁ, LOMMATZSCH (1988, p. 34).

Therefore, $\mathbf{y}^\mathsf{T}\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{y}^\mathsf{T}\mathbf{x}^i \leqq \alpha$ ensues from $\mathbf{y}^\mathsf{T}\mathbf{x}^i \leqq \alpha \;\; (i = 0, \ldots, n)$. ∎

Theorem B.5[34] *For two nonempty subsets C and D in* $\mathbb{R}^n$ *and any two numbers* λ *and* μ *we have*

$$\mathrm{conv}(\lambda C + \mu D) = \lambda \mathrm{conv} C + \mu \mathrm{conv} D\,.$$

For appropriate sets this theorem yields especially

$$\mathrm{conv}(C + D) = \mathrm{conv} C + \mathrm{conv} D \quad \text{and} \quad \lambda \mathrm{conv} C = \mathrm{conv}(\lambda C).$$

Topologically, some important properties must be noted. Letting C and D be subsets in $\mathbb{R}^n$, then the following statements hold good:

- If C is open, then $C + D$ is open, too.[35]
- The sum of two closed sets C and D need not be closed. This becomes obvious by

$$C + D = \bigcup_{\mathbf{x} \in D} (C + \mathbf{x})$$

  with respect to Theorem A.1. But if one of the two sets is bounded – i.e. compact – then $C + D$ is closed, too.
- If C and D are compact, then $C + D$ is compact, too.[36]
- If C is open (closed, compact), then λC is open (closed, compact) with the exception of $\lambda = 0$ in the case of an open set C.[37]
- If C is open, then conv$C$ is open.[38]
- If C is bounded, then conv$C$ is bounded and $\mathrm{cl}(\mathrm{conv} C) = \mathrm{conv}(\mathrm{cl}\, C)$. Thus, conv$C$ is compact for compact C.[39]

The assertion that the convex hull of a closed set $C \subset \mathbb{R}^n$ should be closed is wrong. A counterexample given by LEICHTWEISS refers to a (closed) hyperplane H and a point $\mathbf{x}$ not lying in this hyperplane $(\mathbf{x} \notin H)$.

[34] Cf. NOŽIČKA, GRYGAROVÁ, LOMMATZSCH (1988, p. 32).
[35] BERGE (1963, p. 161, Corollary 1).
[36] BERGE (1963, p. 161, Corollary 2).
[37] BERGE (1963, p. 161, Corollary 4).
[38] Cf. LEICHTWEISS (1980, p. 24).
[39] ROCKAFELLAR (1972, p. 158).

The convex hull of the closed set $H \cup \{\mathbf{x}\}$ corresponds to a set composed of the semi-open stripe between H and the parallel hyperplane through $\mathbf{x}$ in union with point $\mathbf{x}$ itself. As illustrated by the Figure B.1, this convex hull is not closed.

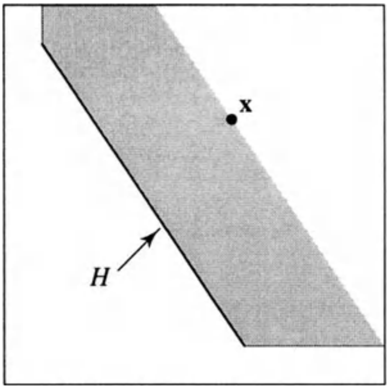

Figure B.1: Counterexample

Finally, some combinatory results can be given. First of all, it is known that vectors $\mathbf{x}^i \in C_i$ $(i = 1, \ldots, m)$ exist for each point $\mathbf{x} \in \sum_i C_i$ such that $\mathbf{x} = \sum_i \mathbf{x}^i$ holds. If we replace the set C_i with its convex hull, then problems result which are estimated by the next theorem.

Theorem B.6 (Shapley, Folkman)[40] *Let C_i $(i = 1, \ldots, m)$ be a compact subset in $\mathbb{R}^n$, where $m > n$. Then vectors $\mathbf{x}^i \in \operatorname{conv} C_i$ exist for each $\mathbf{x} \in \operatorname{conv}(\sum_i C_i)$ such that*

$$\mathbf{x} = \sum_{i=1}^{m} \mathbf{x}^i \quad \textit{and} \quad \#\{\, i \mid \mathbf{x}^i \notin C_i\} \leqq n .$$

For a further estimation of this fault we define the **radius** of a nonempty set $C \subset \mathbb{R}^n$ by

$$\operatorname{rad} C := \inf_{\mathbf{y} \in \mathbb{R}^n} \sup_{\mathbf{x} \in C} \|\mathbf{y} - \mathbf{x}\| .$$

For each compact set C the symbol $\operatorname{rad} C$ denotes the radius of the smallest closed ball containing C. Thus, the radius of an unbounded set C is declared to be $\operatorname{rad} C = +\infty$. If C consists of a sole point, then $\operatorname{rad} C = 0$. A "movement" of the set C does not change its radius, $\operatorname{rad} C = \operatorname{rad}(C + \mathbf{x})$.

Theorem B.7[41] *Let C_i $(i = 1, \ldots, m)$ be nonempty subsets in $\mathbb{R}^n$. If $\mathbf{x} \in \operatorname{conv}(\sum_i C_i)$, then points $\mathbf{x}^i \in C_i$ $(i = 1 \ldots, m)$ exist such that*

$$\|\mathbf{x} - \sum_{i=1}^{m} \mathbf{x}^i\|^2 \leqq \sum_{i=1}^{m} \operatorname{rad}^2 C_i .$$

Corollary B.7.1[42] *Let C_i $(i = 1, \ldots, m)$ be nonempty compact subsets in $\mathbb{R}^n$. If $\mathbf{x} \in \operatorname{conv}(\sum_i C_i)$, then points $\mathbf{x}^i \in C_i$ $i = 1 \ldots, m)$ exist such that*

$$\|\mathbf{x} - \sum_{i=1}^{m} \mathbf{x}^i\|^2 \leqq R ,$$

[40] Theorem and proof may be found in Starr (1969, p. 35).

[41] Theorem and proof may be found in Starr (1969, p. 36).

[42] Cf. Starr (1969, p. 36).

where R is defined as the sum of the $\min\{m, n\}$ *greatest* $\operatorname{rad}^2 C_i$.

Proof: For $m \leqq n$ Theorem B.7 must be used. In the other case with $m > n$ there are $\tilde{\mathbf{x}}^i \in \operatorname{conv} C_i$ with $\mathbf{x} = \sum_i \tilde{\mathbf{x}}^i$ (Theorem B.6). If we choose $\mathbf{x}^i = \tilde{\mathbf{x}}^i$ for $\tilde{\mathbf{x}}^i \in C_i$ in the corollary, then at the most n vectors with $\tilde{\mathbf{x}}^i \notin C_i$ remain whose indices are collected in T. Again Theorem B.7 yields the needed result,

$$\|\sum_{i=1}^{m}(\tilde{\mathbf{x}}^i - \mathbf{x}^i)\|^2 = \|\sum_{i\in T}(\tilde{\mathbf{x}}^i - \mathbf{x}^i)\|^2 = \|\tilde{\tilde{\mathbf{x}}} - \sum_{i\in T}\mathbf{x}^i\|^2 \leqq \sum_{i\in T}\operatorname{rad}^2 C_i \leqq R,$$

where $\tilde{\tilde{\mathbf{x}}} := \sum_{i\in T}\tilde{\mathbf{x}}^i$ and $\#T \leqq n < m$ must be taken into account. ■

B.3 Separation Theorems

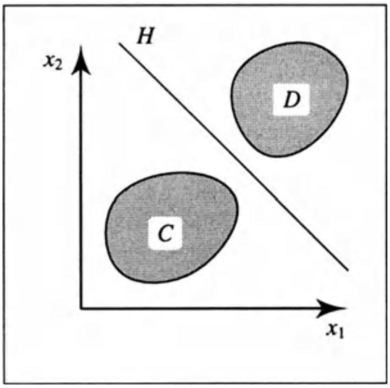

Figure B.2: Separation of two sets

Two nonempty subsets C and D in $\mathbb{R}^n$ are **separated** by the hyperplane $H(\mathbf{y}, \alpha)$ if

$$\inf\left\{\mathbf{y}^{\mathrm{T}}\mathbf{x} \mid \mathbf{x} \in C\right\} \geqq \alpha \geqq \sup\left\{\mathbf{y}^{\mathrm{T}}\mathbf{x} \mid \mathbf{x} \in D\right\}.$$

If the strict inequality holds, then the sets C and D are **strongly separated** by the hyperplane H. The hyperplane H **separates** C and D **properly** if H separates both sets and $C \cup D \not\subset H$. Regarding convex sets some separation theorems result.

Theorem B.8 (MINKOWSKI)[43] *Let C and D be nonempty disjoint convex subsets in $\mathbb{R}^n$. Then a hyperplane H exists **separating** these sets.*

Theorem B.9 (First Separation Theorem)[44] *Let C and D be nonempty convex subsets in $\mathbb{R}^n$. Then a hyperplane H exists **separating** the sets **properly** if and only if* $\operatorname{rint} C \cap \operatorname{rint} D = \emptyset$.

Theorem B.10 (Second Separation Theorem)[45] *Let C and D be nonempty disjoint closed convex subsets in $\mathbb{R}^n$. If one of the sets is bounded (and therefore compact), then a hyperplane H exists **separating** both sets **strongly**.*

The separation theorems yield numerous conclusions, of which only the most important ones are quoted at this point. If we interpret Theorems B.3 and B.4 (CARATHEODORY) as "inner" representation of a convex set C, then the following two theorems can be seen as "outer" representation.

[43] Cf. BERGE (1963, p. 163).

[44] Cf. ROCKAFELLAR (1972, p. 97, Theorem 11.3). Note particularly (B.3), $\operatorname{rint}(\operatorname{cl} C) = \operatorname{rint} C$.

[45] Cf. BERGE (1963, p. 163).
For two bounded sets see ROCKAFELLAR (1972, p. 98, Corollary 11.4.1).

Theorem B.11[46] *A closed convex set C is the intersection of all closed half-spaces containing C.*

Inspecting an arbitrary set $D \subset \mathbb{R}^n$, Theorem B.11 is valid with respect to the set $C = \text{cl}(\text{conv} D)$. A stronger version of Theorem B.11 yields

Theorem B.12[47] *An n-dimensional closed subset C in $\mathbb{R}^n$ is the intersection of all closed half-spaces tangent to C.*

Theorem B.12 may be interpreted as follows: each boundary point $\mathbf{x} \in \partial C$ of an n-dimensional closed convex set $C \subset \mathbb{R}^n$ is at the same time a supporting point of C, i.e. there is a supporting hyperplane of C at $\mathbf{x}$. In the class of supporting points the exposed points introduced on p. 26 stand out. At this point we also presented conditions such that the convex hull of the exposed points of a set C equals the set C itself; see for instance (II.9).

C Mappings

C.1 Functions as Single-Valued Mappings

This section introduces two kinds of mappings. Single-valued mappings or functions face multi-valued mappings or correspondences. Although functions may be interpreted as special cases of correspondences it is useful to treat functions and correspondences separately.

The expositions on the concept of functions are expanded by important properties of functions with respect to the theory of conjugate functions. Apart from proper functions we introduce convex, closed, and linearly homogeneous functions.

Let X and Y be any two nonempty sets. Each rule f with the property[48]

"Each $x \in X$ is assigned to precisely one $y \in Y$."

is called a **mapping from X into Y**, a **single-valued function** or simply a **function**. The abbreviation $f\colon X \to Y$ fixes the set of departure X and the set of arrival Y. The set X is called the **domain** of f. It is compared to the **range** Range f of f.

$$\text{Range}\, f \equiv f(X) = \{y \mid y = f(x),\ x \in X\} \subset Y$$

Given two sets $C \subset X$ and $D \subset Y$, we call $f(C)$ the **image** of C and $f^{-1}(D)$ the **inverse image** of D under the mapping f, where

$$f(C) := \{f(x) \mid x \in C\} \subset Y \quad \text{and} \quad f^{-1}(D) := \{x \mid f(x) \in D\} \subset X\,.$$

[46] ROCKAFELLAR (1972, p. 99, Theorem 11.5).

[47] ROCKAFELLAR (1972, p. 169, Theorem 18.8). A closed half-space is tangent to the set C if it contains C and if it has a boundary point in common with C. The boundary of such a half-space has already been introduced as supporting hyperplane.

[48] Cf. DALLMANN, ELSTER (1991b, p. 71).

Finally, the **graph** of f is defined by

$$\operatorname{graph} f := \left\{ \binom{x}{y} \mid x \in X,\ y = f(x) \right\} \subset X \times Y.$$

If a function f can only take finite real values and if it is defined on a subset X in $\mathbb{R}^n$, then $f\colon X \to \mathbb{R}$ is called a **real-valued function** with the domain X and the range Range f.

The objects $+\infty$ and $-\infty$ are no elements of the set of real numbers $\mathbb{R}$. To embrace functions which may take the functional values $\pm\infty$, we define the set of extended real numbers,

$$\mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \equiv [-\infty, +\infty]\,.$$

Besides $-(-\infty) = +\infty$ we declare the following rules of calculation.

$$\begin{array}{ll} +\infty + x = x + \infty = +\infty & \forall x \in]-\infty, +\infty]\,, \\ -\infty + x = x - \infty = -\infty & \forall x \in [-\infty, +\infty[\,, \\ x \cdot (\pm\infty) = \pm\infty \cdot x = \pm\infty, & \forall x \in]0, +\infty]\,, \\ x \cdot (\pm\infty) = \pm\infty \cdot x = \mp\infty, & \forall x \in [-\infty, 0[\,, \\ (\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0. & \end{array}$$

Terms like $-\infty + \infty$ or $+\infty - \infty$ are not defined.[49]

A positive real-valued x is indicated by $x \in \mathbb{R}_{++} =]0, +\infty[$. However, the notation $x \in \mathbb{R}_{++} \cup \{+\infty\} =]0, +\infty]$ means that x is allowed to take the value $+\infty$.

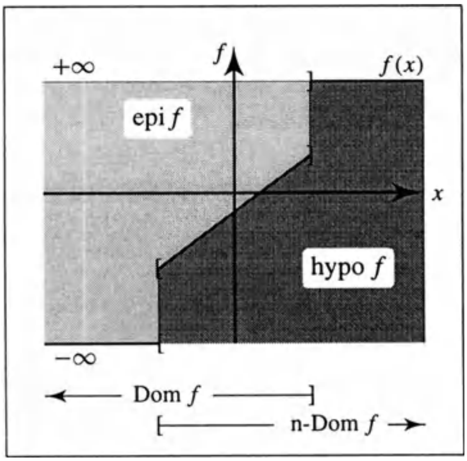

Figure C.1: Graphical representation of a hypograph and an epigraph

If a function f can take real values or the values $\pm\infty$ and if it is defined on a set $X \subset \mathbb{R}^n$, then $f\colon X \to [-\infty, +\infty]$ is called an **extended real-valued function** or generalized numerical function.

Each function f is associated with an **epigraph**

$$\operatorname{epi} f := \left\{ \binom{\mathbf{x}}{y} \in X \times \mathbb{R} \mid y \geqq f(\mathbf{x}) \right\},$$

which faces the **hypograph** of f.

$$\operatorname{hypo} f := \left\{ \binom{\mathbf{x}}{y} \in X \times \mathbb{R} \mid y \leqq f(\mathbf{x}) \right\}$$

Figure C.1 shows both the epigraph and the hypograph of a function f.[50]

[49] Cf. HEWITT, STROMBERG (1969, p. 54 f.).

[50] The vertical separating line between epi f and hypo f indicates that both sets are not closed in the present case. We refer to this at a later stage.

The **effective domain** Dom f of a function $f\colon X \to [-\infty, +\infty]$ is given by[51]

$$\begin{aligned}\operatorname{Dom} f &:= \left\{\mathbf{x} \in X \mid \exists\, y \in \mathbb{R} : \tbinom{\mathbf{x}}{y} \in \operatorname{epi} f\right\}\\ &= \{\mathbf{x} \in X \mid f(\mathbf{x}) < +\infty\}.\end{aligned}$$

Moreover, the set n-Dom f, which will be required at a later stage, is established as follows:

$$\text{n-Dom}\, f := \{\mathbf{x} \in X \mid f(\mathbf{x}) > -\infty\}$$

Figure C.1 includes also Dom f and n-Dom f.

Definition C.1[52] *An extended real-valued function $f\colon X \to [-\infty, +\infty]$ is said to be **proper** if it achieves the value $-\infty$ nowhere on X and if it is finite for at least one point,*

$$\begin{aligned} & \exists\, \mathbf{x} \in X : \ f(\mathbf{x}) < +\infty \ (\iff \operatorname{epi} f \neq \emptyset)\\ \textit{and}\quad & \forall\, \mathbf{x} \in X : \ f(\mathbf{x}) > -\infty\,.\end{aligned}$$

Functions which are not proper are said to be **improper**. This distinction is needed to rule out cases with undefined terms like $+\infty - \infty$ or $-\infty + \infty$. For this purpose other cases only inspect functions $f\colon X \to [-\infty, +\infty]$ which are ***n*-proper**, i.e. they take the value $+\infty$ nowhere on X and they are finite for at least one point. Improper functions cannot simply be excluded from the analysis since the transforms of proper functions may imply improper functions.

Comment C.1 Each extended real-valued function $f\colon X \to [-\infty, +\infty]$ with $X \subset \mathbb{R}^n$ – especially each real-valued function – can be extended to the entire Euclidean space $\mathbb{R}^n$ with respect to its domain X. With a suitable fixing of the functional values for $\mathbf{x} \notin X$ two new functions arise.

(a) From $\tilde{f}(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in X$ and $\tilde{f}(\mathbf{x}) = +\infty$ for all $\mathbf{x} \notin X$ we obtain an extended real-valued function $\tilde{f}\colon \mathbb{R}^n \to [-\infty, +\infty]$. The function $\tilde{f}$ is proper if f is also proper. Moreover, we have $\operatorname{Dom} f = \operatorname{Dom} \tilde{f}$ for the effective domain of the resulting function.

(b) Analogously, the setting $\tilde{f}(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in X$ and $\tilde{f}(\mathbf{x}) = -\infty$ for all $\mathbf{x} \notin X$ yields an extended real-valued function $\tilde{f}\colon \mathbb{R}^n \to [-\infty, +\infty]$ which is n-proper if f is an n-proper function f. Moreover, we have $\text{n-Dom}\, f = \text{n-Dom}\, \tilde{f}$.

Given two functions $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ and $g\colon \mathbb{R}^n \to [-\infty, +\infty]$. If the relation $f(\mathbf{x}) = g(\mathbf{x})$ holds for all $\mathbf{x} \in \mathbb{R}^n$, then we abbreviate $f = g$. The relations $f \leqq g$, $f \geqq g$ and $f \equiv +\infty$ are defined similarly. If a function $f\colon \mathbb{R}^n \times \mathbb{R}^m \to [-\infty, +\infty]$ depends on the parameters $\mathbf{y} \in \mathbb{R}^m$, then $f(\cdot, \mathbf{y})\colon \mathbb{R}^n \to [-\infty, +\infty]$ denotes the corresponding function with respect to the variables $\mathbf{x}$.

[51] Cf. ROCKAFELLAR (1972, p. 23). If a convex function f takes nowhere the value $+\infty$, then $X = \operatorname{Dom} f$.

[52] Cf. ROCKAFELLAR (1972, p. 24) and WETS (1976, p. 2).

Definition C.2 *The function* $f\colon X \to [-\infty, +\infty]$ *is said to be* ***convex*** *in X if its epigraph* epi f *is a convex subset in* $\mathbb{R}^{n+1}$.
Analogously, a function $g\colon X \to [-\infty, +\infty]$ *is said to be* ***concave*** *on X if* $-g$ *is convex in X. In this case the hypograph* hypo g *is a convex set.*
Functions being both convex and concave are called ***affine***.[53]

For a convex function f bear in mind that not only the epigraph epi f but also the effective domain

$$\operatorname{Dom} f = \{\mathbf{x} \in X \mid f(\mathbf{x}) < +\infty\}$$

is a convex set. Moreover, each convex function is continuous in the relative interior of its domain rint(Dom f). In the more convenient definition of convex functions we have to avoid terms of the form $+\infty - \infty$.

Theorem C.1[54] *A function* $f\colon X \to]-\infty, +\infty]$ *is* ***convex*** *on the convex set* $X \subseteq \mathbb{R}^n$ *if and only if*

$$\lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \geqq f\big(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}\big) \quad \forall \lambda \in [0, 1].$$

holds for arbitrary $\mathbf{x}, \mathbf{y} \in X$.
Regarding $g\colon X \to [-\infty, +\infty[$ *with the inverse inequality the function g is said to be* ***concave***.

The **convex hull of a function** $f\colon X \to [-\infty, +\infty]$ is denoted by conv f and indicates the greatest convex function with $\operatorname{conv} f \leqq f$. Accordingly, the convex hull of the function f, as illustrated in Figure C.2, is described by the (not marked) curve $\overline{ABCD}$. Geometrically, the epigraph of conv f is given by the convex hull of the epigraph of f, i.e.

$$\operatorname{epi}(\operatorname{conv} f) = \operatorname{conv}(\operatorname{epi} f).$$

Apart from functions with convex epigraphs we can establish a further class of functions whose epigraphs are closed. Regarding Theorem C.2 we need the criterion of a semi-continuous function.[55]

Definition C.3[56] *Let* $\mathbf{x}^0$ *be a point in the domain* $X \subset \mathbb{R}^n$ *of a function* $f\colon X \to [-\infty, +\infty]$. *The function f is called*

[53] Each vector-valued function $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ with $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ is called affine, where $\mathbf{A}$ is an $m \times n$-matrix and $\mathbf{b} \in \mathbb{R}^m$. In the case of $\mathbf{b} = \mathbf{0}$ the function is said to be linear.

[54] ROCKAFELLAR (1972, p. 25, Theorem 4.1).

[55] Each proper lower semi-continuous function $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ achieves its minimum on a compact subset C in its effective domain Dom f, $\inf\{f(\mathbf{x}) \mid \mathbf{x} \in C\} = \min\{f(\mathbf{x}) \mid \mathbf{x} \in C\}$. See BERGE (1963, p. 76).

[56] Cf. ROCKAFELLAR (1972, p. 51 f.). Equivalent definitions of real-valued functions may be found in BERGE (1963, p. 74) and SHEPHARD (1953, p. 295 f.).

- ***lower semi-continuous at*** $\mathbf{x}^0$ *if*

$$f(\mathbf{x}^0) = \liminf_{\mathbf{x}^\nu \to \mathbf{x}^0} f(\mathbf{x}^\nu) = \lim_{\varepsilon \downarrow 0} \left(\inf\left\{ f(\mathbf{x}^\nu) |\ \|\mathbf{x}^\nu - \mathbf{x}^0\| \leqq \varepsilon \right\}\right),$$

- ***upper semi-continuous at*** $\mathbf{x}^0$ *if*

$$f(\mathbf{x}^0) = \limsup_{\mathbf{x}^\nu \to \mathbf{x}^0} f(\mathbf{x}^\nu) = \lim_{\varepsilon \downarrow 0} \left(\sup\left\{ f(\mathbf{x}^\nu) |\ \|\mathbf{x}^\nu - \mathbf{x}^0\| \leqq \varepsilon \right\}\right).$$

The next theorem is helpful to identify semi-continuous functions as illustrated by Figure C.2 for $\mu = \mu^*$ with the closed level set $\{x \in \mathbb{R} |\ f(x) \leqq \mu^*\} = [a', b']$.

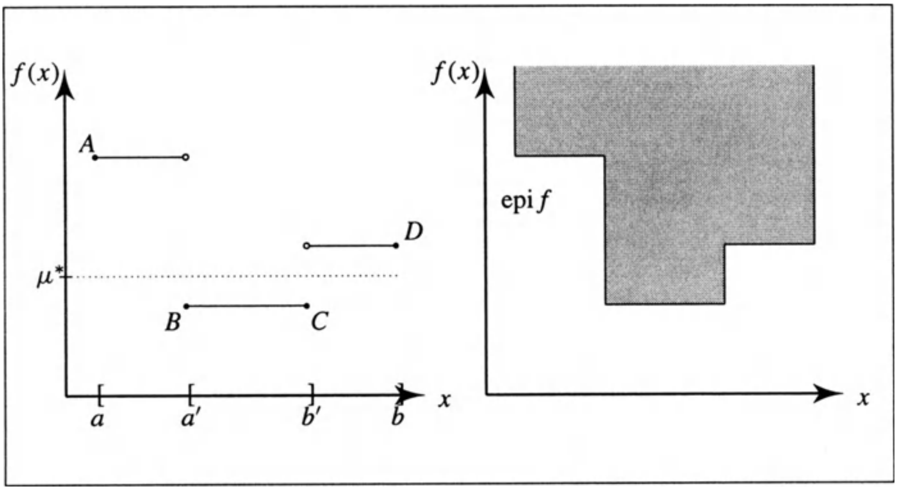

Figure C.2: Semi-continuity of a function

Theorem C.2[57] *For each extended real-valued function* $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ *the following conditions are equivalent:*

1. *The function* f *is lower semi-continuous throughout* $\mathbb{R}^n$.
2. *The level set* $\{\mathbf{x} \in \mathbb{R}^n |\ f(\mathbf{x}) \leqq \mu\}$ *is closed for every* $\mu \in \mathbb{R}$.
3. *The epigraph* $\operatorname{epi} f = \left\{ \binom{\mathbf{x}}{\mu} \in \mathbb{R}^{n+1} |\ f(\mathbf{x}) \leqq \mu \right\}$ *is a closed set in* $\mathbb{R}^{n+1}$.

Remember the properties of lower semi-continuous functions passing on to upper semi-continuous functions by substituting f by $-f$. In doing so the three equivalent conditions can be rewritten for a function $g\colon \mathbb{R}^n \to [-\infty, +\infty]$:

1. The function g is upper semi-continuous throughout $\mathbb{R}^n$.

[57] Cf. ROCKAFELLAR (1972, p. 51) and BERGE (1963, p. 76).

2. The level set $\{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) \geqq \mu\}$ is closed for every $\mu \in \mathbb{R}$.

3. The hypograph $\operatorname{hypo} g = \left\{\binom{\mathbf{x}}{\mu} \in \mathbb{R}^{n+1} \mid g(\mathbf{x}) \geqq \mu\right\}$ is a closed set in $\mathbb{R}^{n+1}$.

Bearing that in mind, we can establish the lower semi-continuous hull $\operatorname{lsc} f$ of a function f. Letting $\operatorname{lsc} f$ denote the greatest lower semi-continuous function with $\operatorname{lsc} f \leqq f$, then by definition

$$\operatorname{epi}(\operatorname{lsc} f) = \operatorname{cl}(\operatorname{epi} f)$$

or, equivalently,

$$\operatorname{lsc} f(\mathbf{x}) = \liminf_{\mathbf{x}^\nu \to \mathbf{x}} f(\mathbf{x}^\nu).$$

Thus, the function f is lower semi-continuous if and only if $f = \operatorname{lsc} f$.

Referring to proper functions, which attain nowhere the value $-\infty$, the following operation, which is slightly modified, turns out to be more useful for convex analysis.

Definition C.4[58] *The **closure*** $\operatorname{cl} f$ *of a function* $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ *is defined by*

$$\operatorname{cl} f(\mathbf{x}) := \begin{cases} \operatorname{lsc} f(\mathbf{x}) & \textit{for all } \mathbf{x}, \quad \textit{if} \quad \operatorname{lsc} f(\mathbf{x}) > -\infty \quad \textit{for all } \mathbf{x} \\ -\infty & \textit{for all } \mathbf{x}, \quad \textit{if} \quad \operatorname{lsc} f(\mathbf{x}) = -\infty \quad \textit{for one } \mathbf{x}. \end{cases}$$

*A function is **closed** if* $\operatorname{cl} f = f$.

Accordingly, a function f is closed if $\operatorname{lsc} f$ takes nowhere the value $-\infty$ or if $f \equiv -\infty$.[59] To protect n-proper functions against the exclusion from the analysis, the upper closure is compared to the lower closure. Regarding the upper semi-continuous hull of a function g

$$\operatorname{usc} g(\mathbf{x}) = \limsup_{\mathbf{x}^\nu \to \mathbf{x}} g(\mathbf{x}^\nu)$$

we now put[60]

$$\operatorname{cl} g(\mathbf{x}) := \begin{cases} \operatorname{usc} g(\mathbf{x}) & \text{for all } \mathbf{x}, \quad \text{if} \quad \operatorname{usc} g(\mathbf{x}) < +\infty \quad \text{for all } \mathbf{x} \\ +\infty & \text{for all } \mathbf{x}, \quad \text{if} \quad \operatorname{usc} g(\mathbf{x}) = +\infty \quad \text{for one } \mathbf{x}. \end{cases}$$

Thus, the closure operation is no longer unique, but for the analysis of proper and n-proper functions it is enough to identify the closure uniquely by the following criteria.[61]

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[58] Cf. ROCKAFELLAR (1972, p. 52).

[59] The identity $f \equiv \pm\infty$ implies $\operatorname{cl} f \equiv \pm\infty$.

[60] The identity $g \equiv \pm\infty$ implies $\operatorname{cl} g \equiv \pm\infty$.

[61] IOFFE, TIHOMIROV (1979, p. 169) define the closedness of a function by the closedness of its epigraph.

1. *A proper function* $f\colon X \to\ ]-\infty, +\infty]$ *with* $X \subset \mathbb{R}^n$ *is closed if and only if its epigraph* epi f *is closed.*

$$f = \operatorname{lsc} f = \operatorname{cl} f \iff \operatorname{epi} f = \operatorname{epi}(\operatorname{lsc} f) = \operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f)$$

Proof: If the function f is proper, $f > -\infty$, and closed, $\operatorname{cl} f = f$, then the equation $\operatorname{lsc} f = \operatorname{cl} f$ ensues from $\operatorname{cl} f > -\infty$. Thus, f is lower semi-continuous, $\operatorname{lsc} f = f$, such that the epigraph epi f is closed (Theorem C.2).
If the proper function $f > -\infty$ has a closed epigraph epi f, then the function is lower semi-continuous by Theorem C.2, $\operatorname{lsc} f = f$. Thus, lsc f takes nowhere the value $-\infty$ such that $f = \operatorname{lsc} f = \operatorname{cl} f$. ■

2. *An n-proper function* $g\colon X \to [-\infty, +\infty[$ *with* $X \subset \mathbb{R}^n$ *is closed if and only if its hypograph* hypo g *is closed.*

$$g = \operatorname{usc} g = \operatorname{cl} g \iff \operatorname{hypo} g = \operatorname{hypo}(\operatorname{usc} g) = \operatorname{hypo}(\operatorname{cl} g) = \operatorname{cl}(\operatorname{hypo} g)$$

The proof parallels the conclusions on proper functions.

For proper, convex functions we do not need to distinguish the criterion of lower semi-continuity from the criterion of closedness.[62] If X is a nonempty convex subset in $\mathbb{R}^n$, then the closure cl f of a proper convex function $f\colon X \to [-\infty, +\infty]$ is given by[63]

$$\operatorname{cl} f(\mathbf{x}) = \liminf_{\mathbf{x}^\nu \to \mathbf{x}} f(\mathbf{x}^\nu) \qquad \forall \mathbf{x} \in \operatorname{cl} X .$$

The closure cl g of an n-proper concave function $g\colon X \to [-\infty, +\infty]$ results from

$$\operatorname{cl} g(\mathbf{x}) = \limsup_{\mathbf{x}^\nu \to \mathbf{x}} g(\mathbf{x}^\nu) \qquad \forall \mathbf{x} \in \operatorname{cl} X .$$

The closure of a convex function f may be seen as regularization of this function since the functional values of f and cl f differ at the most at the relative boundary of the effective domain Dom f.

Before this statement is noted by Theorem C.3, a graphical representation is given by Figure C.3. Here the graph of a function f is illustrated with $\tilde{x}$ not belonging to the (effective) domain of f. Note that epi f is neither open nor closed. Furthermore, the closure of the epigraph contains the left (dotted) vertical line such that cl f is defined by $\operatorname{cl} f(\tilde{x}) = a$ at point $\tilde{x}$.

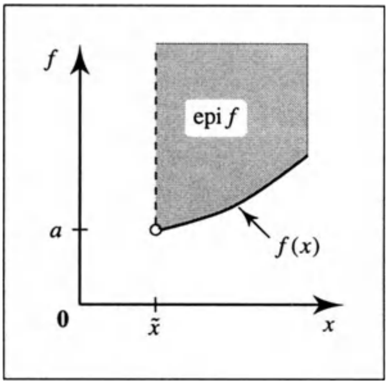

Figure C.3: Closure of a function

[62]The advantage to permit only real-valued convex functions becomes noticeable particularly in BLUM, ÖTTLI (1975, p. 154 ff.).

[63]Cf. ROCKAFELLAR (1974, pp. 14 and 17) and NOŽIČKA, GRYGAROVÁ, LOMMATZSCH (1988, p. 243, Theorem 18.2).

Theorem C.3[64] *Let* $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ *be a proper convex function. Then* $\operatorname{cl} f$ *is a proper closed convex function with*

$$\operatorname{cl} f(\mathbf{x}) = f(\mathbf{x}) \qquad \forall\, \mathbf{x} \in \operatorname{rint}(\operatorname{Dom} f).$$

Let $g\colon \mathbb{R}^n \to [-\infty, +\infty]$ *be an n-proper concave function. Then* $\operatorname{cl} g$ *is an n-proper closed concave function with*

$$\operatorname{cl} g(\mathbf{x}) = g(\mathbf{x}) \qquad \forall\, \mathbf{x} \in \operatorname{rint}(\text{n-Dom}\, g).$$

Apart from the discussed classes of functions linearly homogeneous functions are important for convex analysis. A function $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ is said to be positively homogeneous of degree 1 or **linearly homogeneous**[65], if

$$f(\lambda\mathbf{x}) = \lambda f(\mathbf{x}) \qquad 0 < \lambda < +\infty$$

for every $\mathbf{x}$. Form the point of view of the epigraph of the function f

$$\operatorname{epi} f = \left\{\binom{\mathbf{x}}{y} \in \mathbb{R}^n \times \mathbb{R} \mid y \geqq f(\mathbf{x})\right\}$$

the class of linearly homogeneous functions corresponds to the class of functions whose epigraphs are cones[66]; see the left hand part of Figure C.4. That is, the function f is linearly homogeneous if and only if its epigraph $\operatorname{epi} f$ is a cone.

$$\begin{pmatrix}\mathbf{x}\\ f(\mathbf{x})\end{pmatrix} \in \operatorname{epi} f,\ \lambda > 0 \implies \begin{pmatrix}\lambda\mathbf{x}\\ f(\lambda\mathbf{x})\end{pmatrix} = \begin{pmatrix}\lambda\mathbf{x}\\ \lambda f(\mathbf{x})\end{pmatrix} = \lambda\begin{pmatrix}\mathbf{x}\\ f(\mathbf{x})\end{pmatrix} \in \operatorname{epi} f$$

Considering Theorem C.1, we obtain a modified criterion for linearly homogeneous functions to be convex.

Theorem C.4[67] *A linearly homogeneous function* $f\colon \mathbb{R}^n \to\ ]-\infty, +\infty]$ *is convex if and only if*

$$f(\mathbf{x}+\mathbf{y}) \leqq f(\mathbf{x}) + f(\mathbf{y}) \qquad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{C.1}$$

Proof: If formula (C.1) holds for $\mathbf{x} = \lambda\tilde{\mathbf{x}}$ and $\mathbf{y} = (1-\lambda)\tilde{\mathbf{y}}$, then the function f is concave since

$$f\big(\lambda\tilde{\mathbf{x}} + (1-\lambda)\tilde{\mathbf{y}}\big) \leqq f(\lambda\tilde{\mathbf{x}}) + f\big((1-\lambda)\tilde{\mathbf{y}}\big) \qquad \forall\, \lambda\tilde{\mathbf{x}}, (1-\lambda)\tilde{\mathbf{y}} \in \mathbb{R}^n$$

implies with respect to linear homogeneity the criterion of convexity

$$f\big(\lambda\tilde{\mathbf{x}} + (1-\lambda)\tilde{\mathbf{y}}\big) \leqq \lambda f(\tilde{\mathbf{x}}) + (1-\lambda) f(\tilde{\mathbf{y}}) \qquad \forall\, \tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbb{R}^n,\ \lambda \in [0,1]. \tag{C.2}$$

[64] Cf. NOŽIČKA, GRYGAROVÁ, LOMMATZSCH (1988, p. 243) or ROCKAFELLAR (1972, p. 56).

[65] A function is called homogeneous of degree 1 if the above criterion is satisfied for all $\lambda \in \mathbb{R}$. If only $\lambda > 0$ is required, then the function is called *positively* homogeneous of degree 1. The addition *positively* is omitted in future as long as the context is unambiguous.

[66] At this point the definition of a cone, not necessarily containing its vertex $\mathbf{0}$, turns out to be useful as linearly homogeneous functions may take an arbitrary functional value at the point $\mathbf{x} = \mathbf{0}$.

[67] Cf. ROCKAFELLAR (1972, p. 30, Theorem 4.7).

Suppose the convexity of the function f, then (C.1) induces (C.2) by choosing $\lambda = 1/2$ and using again the linear homogeneity. ■

Figure C.4 shows two forms of scalar multiplication. Interpreting λf as a new function, then the convexity of f passes on to the function λf by the operation $f \to \lambda f$. [68] This form of left scalar multiplication is shown in the left hand part of Figure C.4 for the scalar $\lambda = 1/2$.

Given a convex function f, we can moreover define a function $f\lambda$ with

$$(f\lambda)(\mathbf{x}) := \lambda f(\mathbf{x}/\lambda) \qquad \forall \lambda > 0, \tag{C.3a}$$

where the operation $f \to f\lambda$ is defined as right scalar multiplication.[69] As depicted in the right hand part of Figure C.4 (for $\lambda = 1/2$), the operation $f \to f\lambda$ is equivalent to the multiplication of the epigraph $\operatorname{epi} f$ by a scalar $\lambda > 0$, i.e. $\lambda \operatorname{epi} f = \operatorname{epi}(f\lambda)$. According to the figure with $\tilde{\mathbf{x}} = \lambda \mathbf{x}$, the equivalence relation can technically be expressed as

$$\begin{aligned}\begin{pmatrix}\mathbf{x}\\ f(\mathbf{x})\end{pmatrix} \in \operatorname{epi} f \iff & \begin{pmatrix}\lambda\mathbf{x}\\ \lambda f(\mathbf{x})\end{pmatrix} \in \lambda \operatorname{epi} f = \operatorname{epi}(f\lambda)\\ \iff & \begin{pmatrix}\tilde{\mathbf{x}}\\ (f\lambda)(\tilde{\mathbf{x}})\end{pmatrix} \in \operatorname{epi}(f\lambda).\end{aligned}$$

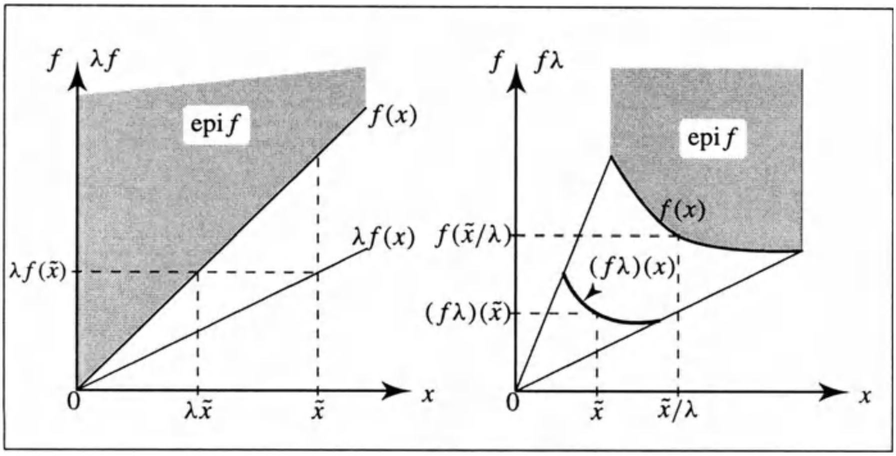

Figure C.4: Scalar multiplications

Consequently, we may now examine the parameter value $\lambda = 0$ presuming

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[68]The operation $f \to \lambda f$ is defined by $(\lambda f)(\mathbf{x}) = \lambda(f(\mathbf{x}))$ and is called the **left** scalar multiplication in ROCKAFELLAR (1972).

[69]Cf. ROCKAFELLAR (1972, p. 35) for the definition of the **right** scalar multiplication.
The operation $f \to f\lambda$ is picked up and extended in Section D.2.1; see for instance (D.8).

$f \not\equiv +\infty$ or $\operatorname{epi} f \neq \emptyset$:

(C.3b) $$(f0)(\mathbf{x}) = \delta(\mathbf{x}|\mathbf{0}) := \begin{cases} 0 & \text{for } \mathbf{x} = \mathbf{0} \\ +\infty & \text{for } \mathbf{x} \neq \mathbf{0}. \end{cases}$$

For $f \equiv +\infty$ we set $f0 \equiv +\infty$.

Such as the left scalar multiplication the right scalar multiplication implies a convex function $f\lambda$ provided the function f is convex. Note that a function f is linearly homogeneous if and only if $f\lambda = f$ for all $\lambda > 0$. Accordingly, the right hand part of Figure C.4 depicts no linearly homogeneous function f. Neither the epigraph $\operatorname{epi} f$ is a cone nor $f\lambda = f$.

C.2 Correspondences as Multi-Valued Mappings

The economic theory often looks at mappings, where no unique functional value exists. For instance, the point at which the objective function of a convex program achieves its optimal value does not need to be unique provided the scrutinized program has a solution at all. In order to note all optimal solutions or the *set* of solutions we introduce correspondences.

Again X and Y denote any nonempty sets. If the rule Γ maps each element in X into precisely one subset in Y, then we speak of a **multi-valued mapping** or **correspondence**. Using the power set $\mathfrak{P}(Y)$, the correspondence $\Gamma: X \to \mathfrak{P}(Y)$ denotes the mapping rule of X into Y. The set $\Gamma(x)$ is the so called image of x under the mapping Γ.[70]

A comparison to the concept of (single-valued) functions shows that each function $f: X \to Y$ can be interpreted as special case of a correspondence $\Gamma: X \to \mathfrak{P}(Y)$. If every set $\Gamma(x) \subset Y$ consists of one and only one element $y \in Y$, then the function f with $f(x) = y$ corresponds to the "vector-valued" correspondence Γ with the only difference of representation.

$$y = f(x) \in Y \qquad \text{or} \qquad \{y\} = \Gamma(x) \subset Y$$

Besides the domain

$$\operatorname{Dom}\Gamma := \{x \in X \mid \exists\, y \in Y:\ y \in \Gamma(x)\} = \{x \in X \mid \Gamma(x) \neq \emptyset\}$$

the range of the correspondence Γ is defined by[71]

$$\operatorname{Range}\Gamma := \{y \in Y \mid \exists\, x \in X:\ y \in \Gamma(x)\} = \bigcup_{x \in X} \Gamma(x).$$

The **graph** of the correspondence Γ is given by

$$\operatorname{graph}\Gamma := \{(x, y) \mid x \in X,\ y \in \Gamma(x)\}.$$

[70] Some authors refrain from a symbolic distinction between functions and correspondences. Cf. TAKAYAMA (1990, p. 7) or SHEPHARD (1953, p. 298).

[71] Some authors call the mapping $\Gamma: X \to \mathfrak{P}(Y)$ a correspondence only if $\Gamma(\mathbf{x}) \neq \emptyset$ for all $\mathbf{x} \in X$. The mapping Γ of X onto Y then has the range $\operatorname{Range}\Gamma = Y$. Cf. e.g. HEUSER (1992, p. 609).

The multi-valued mapping $\Gamma^{-1}: Y \rightarrow \mathfrak{P}(X)$ is called the **inverse correspondence** of Γ if

$$\Gamma^{-1}(y) = \{x \in X | \, y \in \Gamma(x)\}$$

holds for every $y \in Y$. Accordingly, inverse correspondences satisfy the equivalence relation

$$y \in \Gamma(x) \quad \Longleftrightarrow \quad x \in \Gamma^{-1}(y).$$

Finally, the **composition product** $\Gamma_1 \circ \Gamma_2$ of two correspondences $\Gamma_1: Y \rightarrow \mathfrak{P}(Z)$ and $\Gamma_2: X \rightarrow \mathfrak{P}(Y)$ is defined by

$$\Gamma_1 \circ \Gamma_2(x) := \Gamma_1\big(\Gamma_2(x)\big) = \bigcup_{y \in \Gamma_2(x)} \Gamma_1(y).$$

If the correspondence is defined on a set $X \subset \mathbb{R}^n$ such that $\Gamma(\mathbf{x})$ is a subset in $Y \subset \mathbb{R}^m$ for every $\mathbf{x} \in X$, then two kinds of correspondences can immediately be emphasized. A correspondence $\Gamma: X \rightarrow \mathfrak{P}(Y)$ is **compact-valued** if the sets $\Gamma(\mathbf{x})$ are compact for all $\mathbf{x} \in X$. Analogously, a correspondence $\Gamma: X \rightarrow \mathfrak{P}(Y)$ is **convex-valued** if the sets $\Gamma(\mathbf{x})$ are convex for all $\mathbf{x} \in X$.

Moreover, semi-continuous and closed correspondences must be stressed. Bear in mind that we have to distinguish *strictly* between the concepts of a semi-continuous (extended) function and the semi-continuity of a correspondence! For this reason HILDENBRAND, KIRMAN[72] speak of *semi*-continuous functions and *hemi*-continuous correspondences.

Definition C.5[73] *Let $\Gamma: X \rightarrow \mathfrak{P}(Y)$ be a correspondence which maps each vector in $X \subset \mathbb{R}^n$ into a nonempty subset in $Y \subset \mathbb{R}^m$. Suppose $\mathbf{x}^0$ to be a point in the domain X. The correspondence Γ is called*

- ***lower semi-continuous at** $\mathbf{x}^0$ if for each open set V with $\Gamma(\mathbf{x}^0) \subset V \subset Y$ there is a neighborhood $U(\mathbf{x}^0)$ such that $\Gamma(\mathbf{x}) \subset V$ for all $\mathbf{x} \in U(\mathbf{x}^0)$.*

- ***lower semi-continuous at** $\mathbf{x}^0$ if for each open set $V \subset Y$ with $\Gamma(\mathbf{x}^0) \cap V \neq \emptyset$ there is a neighborhood $U(\mathbf{x}^0)$ such that $\Gamma(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in U(\mathbf{x}^0)$.*[74]

- ***continuous at** $\mathbf{x}^0$ if it is upper semi-continuous and lower semi-continuous at $\mathbf{x}^0$.*[75]

- ***quasi upper semi-continuous** (in X) if it is upper semi-continuous at $\mathbf{x}$ for all $\mathbf{x} \in X$.*

[72]Cf. HILDENBRAND, KIRMAN (1988, p. 189 f.).

[73]Cf. BERGE (1963, p. 109) or MOORE (1968, p. 130).

[74]BERGE writes in his definition "open set V meeting $\Gamma(\mathbf{x}^0)$", what has been taken falsely by TAKAYAMA as "open set V containing $\Gamma(\mathbf{x}^0)$".

[75]When Γ is a single-valued mapping, then Γ is continuous in X if and only if for each open (closed) set $V \in Y$ the inverse image $\Gamma^{-1}(V)$ is open (closed) in X. Cf. e.g. TAKAYAMA (1990, p. 255).

- ***upper semi-continuous*** *(in X) if it is upper semi-continuous at every* $\mathbf{x} \in X$ *and if* $\Gamma(\mathbf{x})$ *is compact*[76] *for all* $\mathbf{x} \in X$.

- ***lower semi-continuous (continuous)*** *(in X) if it is lower semi-continuous (continuous) at* $\mathbf{x}$ *for all* $\mathbf{x} \in X$.

A graphical representation of the concept is given in Section 3.2.1(c) by explaining the upper semi-continuity of demand correspondences.

If a correspondence $\Gamma\colon X \to \mathfrak{P}(Y)$ with $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}$ and a function $f\colon X \to Y$ fulfill the relation $\Gamma(\mathbf{x}) = \{f(\mathbf{x})\}$, then the subsequent relation can be proved. The definitions of upper and lower semi-continuity of Γ at point $\mathbf{x}^0$ are equivalent and they equal the definition of the continuity of f at point $\mathbf{x}^0$.[77]

The following result will be important with respect to Theorem C.17 (DEBREU, GALE, NIKAIDO):

Theorem C.5[78] *Letting* $\Gamma\colon X \to \mathfrak{P}(Y)$ *be a compact-valued and upper semi-continuous correspondence, then the set*

$$\widehat{\Gamma}(S) := \bigcup_{\mathbf{x}\in S} \Gamma(\mathbf{x})$$

is compact for each compact $S \subset X$. *Thus, the correspondence* $\widehat{\Gamma}$ *is bounded, provided* $\widehat{\Gamma}$ *is the correspondence* Γ *which is restricted to the region S.*

The semi-continuity of a correspondence must be distinguished from the property of closedness.

Definition C.6[79] *The correspondence* $\Gamma\colon X \to \mathfrak{P}(Y)$ *with the topological spaces X and Y is said to be* ***closed*** *if it satisfies the following condition: whenever* $x^0 \in X$, $y^0 \in Y$ *and* $y^0 \notin \Gamma(x^0)$, *then neighborhoods* $U(x^0)$ *and* $V(y^0)$ *exist such that* $x \in U(x^0)$ *implies* $\Gamma(x) \cap V(y^0) = \emptyset$.

Specifying the topological spaces X and Y by subsets in the Euclidean spaces, we obtain

Theorem C.6[80] *A correspondence* $\Gamma\colon X \to \mathfrak{P}(Y)$ *with* $X \subset \mathbb{R}^n$ *and* $Y \subset \mathbb{R}^m$ *is* ***closed at point*** $\mathbf{x}^0 \in X$ *if the sequences* $\{\mathbf{x}^\nu\}$ *and* $\{\mathbf{y}^\nu\}$ *satisfy*

$$\left[\mathbf{x}^\nu \to \mathbf{x}^0,\ \mathbf{y}^\nu \to \mathbf{y}^0,\ \mathbf{y}^\nu \in \Gamma(\mathbf{x}^\nu)\right] \implies \mathbf{y}^0 \in \Gamma(\mathbf{x}^0).$$

The correspondence Γ is closed (in X) if it is closed for every point in X.

[76] Some authors dispense with requiring a compact-valued correspondence. Cf. HILDENBRAND, KIRMAN (1988, p. 262 ff.) in contrast with TAKAYAMA (1990, p. 251).

[77] Cf. TAKAYAMA (1990, p. 252). For the definition of continuous functions see DALLMANN, ELSTER (1991a, p. 59).

[78] DALLMANN, ELSTER (1991a, p. 609).

[79] Cf. BERGE (1963, p. 111).

[80] Cf. BERGE (1963, p. 111, Theorem 4) or HEUSER (1992, p. 612).

Thus, a closed correspondence implies

$$\left[(\mathbf{x}^\nu, \mathbf{y}^\nu) \to (\mathbf{x}^0, \mathbf{y}^0),\ \ (\mathbf{x}^\nu, \mathbf{y}^\nu) \in \operatorname{graph}\Gamma \right] \implies (\mathbf{x}^0, \mathbf{y}^0) \in \operatorname{graph}\Gamma,$$

i.e. the graph $\operatorname{graph}\Gamma := \{(\mathbf{x}, \mathbf{y}) | \mathbf{x} \in X,\ \mathbf{y} \in \Gamma(\mathbf{x})\}$ is a closed set in $X \times Y$. Hence, the level sets $\Gamma(\mathbf{x})$ must also be closed in Y.

The next results emphasize the relationship of semi-continuous and closed correspondences.

Theorem C.7[81] *Every upper semi-continuous correspondence is closed.*

The reverse conclusion regarding this theorem is in general not admissible, i.e.:

Theorem C.8[82] *Let Y be a compact space. Then the correspondence $\Gamma\colon X \to \mathfrak{P}(Y)$ is upper semi-continuous if and only if it is closed.*

The next theorem serves for the identification of semi-continuous correspondences.

Theorem C.9[83] *A compact-valued correspondence $\Gamma\colon X \to \mathfrak{P}(Y)$ is upper semi-continuous at point $\mathbf{x}^0$ if and only if for every sequence $\{\mathbf{x}^\nu\}$ in X with limit $\mathbf{x}^0$ and for each sequence $\{\mathbf{y}^\nu\}$ with $\mathbf{y}^\nu \in \Gamma(\mathbf{x}^\nu)$ there is a convergent subsequence $\{\mathbf{y}^{\nu_k}\}$ whose limit $\mathbf{y}^0$ belongs to $\Gamma(\mathbf{x}^0)$, i.e. $\mathbf{y}^{\nu_k} \to \mathbf{y}^0 \in \Gamma(\mathbf{x}^0)$.*

Theorem C.10[84] *Let Y be a compact subset in $\mathbb{R}^m$. Then two statement are true for the correspondence $\Gamma\colon X \to \mathfrak{P}(Y)$ with $X \subset \mathbb{R}^n$.*

1. *The correspondence Γ is upper semi-continuous at $\mathbf{x}^0$ if and only if the sequences $\{\mathbf{x}^\nu\}$ and $\{\mathbf{y}^\nu\}$ yield*
$$\left[\mathbf{x}^\nu \to \mathbf{x}^0,\ \mathbf{y}^\nu \to \mathbf{y}^0,\ \mathbf{y}^\nu \in \Gamma(\mathbf{x}^\nu)\right] \implies \mathbf{y}^0 \in \Gamma(\mathbf{x}^0).$$
2. *The correspondence Γ is lower semi-continuous at $\mathbf{x}^0$ if and only if the sequence $\{\mathbf{x}^\nu\}$ yields*
$$\left[\mathbf{x}^\nu \to \mathbf{x}^0,\ \mathbf{y}^0 \in \Gamma(\mathbf{x}^0)\right] \implies \left[\exists\{\mathbf{y}^\nu\}\colon\ \mathbf{y}^\nu \to \mathbf{y}^0,\ \mathbf{y}^\nu \in \Gamma(\mathbf{x}^\nu)\right].$$

The following properties are presented without proof:[85]

- Letting $\Gamma_1\colon Y \to \mathfrak{P}(Z)$ and $\Gamma_2\colon X \to \mathfrak{P}(Y)$ be lower (upper) semi-continuous, then $\Gamma_1 \circ \Gamma_2\colon X \to \mathfrak{P}(Z)$ is as well lower (upper) semi-continuous.
- The union $\Gamma = \bigcup_{i \in I} \Gamma_i$ of a family of lower semi-continuous correspondences $\Gamma_i\colon X \to \mathfrak{P}(Y)$ is lower semi-continuous with $\Gamma\colon X \to \mathfrak{P}(Y)$.

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[81] BERGE (1963, p. 112).
[82] Cf. BERGE (1963, p. 112) or MOORE (1968, p. 130).
[83] Cf. HILDENBRAND, KIRMAN (1988, p. 262).
[84] Cf. TAKAYAMA (1990, pp. 239 and 252).
[85] See BERGE (1963, p. 113 ff.).

- The intersection $\Gamma = \bigcap_{i \in I} \Gamma_i$ of a family of upper semi-continuous correspondences $\Gamma_i: X \to \mathfrak{P}(Y)$ is also upper semi-continuous with $\Gamma: X \to \mathfrak{P}(Y)$.

- The union $\Gamma = \bigcup_{i=1}^{n} \Gamma_i$ of a finite family of upper semi-continuous correspondences $\Gamma_i: X \to \mathfrak{P}(Y)$ is also upper semi-continuous with $\Gamma: X \to \mathfrak{P}(Y)$.

- The Cartesian product $\Gamma = \underset{i=1}{\overset{n}{\times}} \Gamma_i$ of a finite family of lower (upper) semi-continuous correspondences $\Gamma_i: X \to \mathfrak{P}(Y_i)$ is also a lower (upper) semi-continuous correspondence $\Gamma: X \to \mathfrak{P}(Y)$ with $Y = \underset{i=1}{\overset{n}{\times}} Y_i$.

- The sum $\Gamma = \sum_{i=1}^{n} \Gamma_i$ of a finite family of upper semi-continuous correspondences $\Gamma_i: X \to \mathfrak{P}(\mathbb{R}^m)$ is as well upper semi-continuous with $\Gamma: X \to \mathfrak{P}(\mathbb{R}^m)$.

Theorem C.11[86] *Provided a correspondence* $\Gamma: X \to \mathfrak{P}(\mathbb{R}^n)$ *is compact-valued and upper semi-continuous at point* $\mathbf{x}^0$*, then the correspondence* Γ_{co} *with*

$$\Gamma_{co}(\mathbf{x}) := \operatorname{conv}\Gamma(\mathbf{x}) \qquad \forall \mathbf{x} \in X$$

has the same properties.

Proof: Since the convex hull of a compact set is compact (see p. 294), the new correspondence Γ_{co} must also be compact-valued at point $\mathbf{x}$. Thus, it remains to be shown that Γ_{co} is upper semi-continuous at point $\mathbf{x}$.
By Theorem C.9 for every sequence $\{\mathbf{x}^\nu\}$ in X with limit $\mathbf{x}^0$ and for every sequence $\{\mathbf{y}^\nu\}$ with $\mathbf{y}^\nu \in \Gamma_{co}(\mathbf{x}^\nu)$ there must be a convergent subsequence $\{\mathbf{y}^{\nu_k}\}$ whose limit $\mathbf{y}^0$ belongs to $\Gamma_{co}(\mathbf{x}^0)$, $\mathbf{y}^{\nu_k} \to \mathbf{y}^0 \in \Gamma_{co}(\mathbf{x}^0)$. According to Theorem B.4 (CARATHÉODORY) each element of the sequence of points $\{\mathbf{y}^\nu\}$ with $\mathbf{y}^\nu \in \Gamma_{co}(\mathbf{x}^\nu)$ can be expressed as a convex combination of no more than $n+1$ points in $\Gamma(\mathbf{x}^\nu)$.

$$\mathbf{y}^\nu = \sum_{i=0}^{n} \lambda_i^\nu \mathbf{y}_i^\nu \quad \text{with} \quad \mathbf{y}_i^\nu \in \Gamma(\mathbf{x}^\nu),\ \boldsymbol{\lambda} \in \Lambda^{n+1}$$

Theorem C.9 assures that each sequence of points $\{\mathbf{y}_i^\nu\}$ with $\mathbf{y}_i^\nu \in \Gamma(\mathbf{x}^\nu)$ contains a convergent subsequence $\{\mathbf{y}_i^{\nu_k}\}$ whose limit $\mathbf{y}_i^0$ belongs to $\Gamma(\mathbf{x}^0)$. The bounded sequence $\{\boldsymbol{\lambda}^\nu\}$ also contains a convergent subsequence $\{\boldsymbol{\lambda}^{\nu_k}\}$ whose limit $\boldsymbol{\lambda}^0$ satisfies $\lambda_0^0 + \lambda_1^0 + \cdots + \lambda_n^0 = 1$. Thus, for

$$\mathbf{y}^{\nu_k} = \sum_{i=0}^{n} \lambda_i^{\nu_k} \mathbf{y}_i^{\nu_k} \quad \text{with} \quad \mathbf{y}_i^{\nu_k} \in \Gamma(\mathbf{x}^{\nu_k}),\ \boldsymbol{\lambda}^{\nu_k} \in \Lambda^{n+1}$$

[86] Cf. HILDENBRAND, KIRMAN (1988, p. 266).

we have a convergent subsequence of $\{\mathbf{y}^\nu\}$ whose limit

$$\mathbf{y}^0 = \sum_{i=0}^{n} \lambda_i^0 \mathbf{y}_i^0 \quad \text{with} \quad \mathbf{y}_i^0 \in \Gamma(\mathbf{x}^0),\ \boldsymbol{\lambda}^0 \in \Lambda^{n+1}$$

belongs to $\Gamma_{co}(\mathbf{x}^0) = \mathrm{conv}\Gamma(\mathbf{x}^0)$. ■

C.3 Fixed-Point Theorems

The semi-continuity of correspondences yields frequently used implications, which can be noted by the Maximum Theorem as well as the fixed-point theorems of BROUWER and KAKUTANI. Moreover, the maximum theorem can be applied to the proof of KAKUTANI's fixed-point theorem.

Theorem C.12[87] *Let* $\phi\colon X \times Y \to \mathbb{R}$ *be a lower semi-continuous function and* $\Gamma\colon X \to \mathfrak{P}(Y)$ *be a lower semi-continuous correspondence with* $\Gamma(\mathbf{x}) \neq \emptyset$ *for all* $\mathbf{x} \in X$. *Then a lower semi-continuous function* $\psi\colon X \to \mathbb{R}$ *results with*

$$\psi(\mathbf{x}) := \sup\{\phi(\mathbf{x}, \mathbf{y}) |\, \mathbf{y} \in \Gamma(\mathbf{x})\}.$$

Theorem C.13[88] *Let* $\phi\colon X \times Y \to \mathbb{R}$ *be an upper semi-continuous function and* $\Gamma\colon X \to \mathfrak{P}(Y)$ *be an upper semi-continuous correspondence with* $\Gamma(\mathbf{x}) \neq \emptyset$ *for all* $\mathbf{x} \in X$. *Then an upper semi-continuous function* $\psi\colon X \to \mathbb{R}$ *results with*

$$\psi(\mathbf{x}) := \max\{\phi(\mathbf{x}, \mathbf{y}) |\, \mathbf{y} \in \Gamma(\mathbf{x})\}.$$

Theorem C.14 (Maximum Theorem)[89] *Let* $\phi\colon X \times Y \to \mathbb{R}$ *be a continuous function and* $\Gamma\colon X \to \mathfrak{P}(Y)$ *be a continuous correspondence with* $\Gamma(\mathbf{x}) \neq \emptyset$ *for all* $\mathbf{x} \in X$. *Then the function* $\psi\colon X \to \mathbb{R}$ *with*

$$\psi(\mathbf{x}) := \max\{\phi(\mathbf{x}, \mathbf{y}) |\, \mathbf{y} \in \Gamma(\mathbf{x})\}$$

is continuous in X and the correspondence $\Psi\colon X \to \mathfrak{P}(Y)$ *with*

$$\Psi(\mathbf{x}) := \{\mathbf{y} |\, \mathbf{y} \in \Gamma(\mathbf{x}),\ \phi(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x})\}$$

is upper semi-continuous.

Proof: The continuity of ψ ensues from applying the two preceding theorems. To prove the upper semi-continuity of Ψ, we look at two sequences $\{\mathbf{x}^\nu\} \subset X$, $\{\mathbf{y}^\nu\} \subset Y$ with $\mathbf{x}^\nu \to \mathbf{x}^0$, $\mathbf{y}^\nu \to \mathbf{y}^0$ and $\mathbf{y}^\nu \in \Psi(\mathbf{x}^\nu)$. This yields:

[87] Cf. BERGE (1963, p. 115). Recall the difference between an upper semi-continuous function and a lower semi-continuous correspondence.

[88] Cf. BERGE (1963, p. 116). Remind that upper semi-continuous correspondences have compact level sets by definition!

[89] BERGE states the Maximum Theorem for the function $\phi\colon Y \to \mathbb{R}$. Cf. BERGE (1963, p. 116). The proof of the modified theorem is taken from DEBREU (1982, p. 701, Lemma 1).
As in Theorem C.13, a continuous correspondence requires compact level sets!

1. Because of $\Psi(\mathbf{x}^\nu) \subset \Gamma(\mathbf{x}^\nu)$ the upper semi-continuity of Γ (Theorem C.10) results in
$$\left[\mathbf{x}^\nu \to \mathbf{x}^0,\ \mathbf{y}^\nu \to \mathbf{y}^0,\ \mathbf{y}^\nu \in \Gamma(\mathbf{x}^\nu)\right] \implies \mathbf{y}^0 \in \Gamma(\mathbf{x}^0).$$
2. For each $\mathbf{z}^0 \in \Gamma(\mathbf{x}^0)$ the lower semi-continuity of Γ (Theorem C.10) yields
$$\left[\mathbf{x}^\nu \to \mathbf{x}^0,\ \mathbf{z}^0 \in \Gamma(\mathbf{x}^0)\right] \implies \left[\exists\{\mathbf{z}^\nu\} :\ \mathbf{z}^\nu \to \mathbf{z}^0,\ \mathbf{z}^\nu \in \Gamma(\mathbf{x}^\nu)\right].$$
3. According to 2., for each $\mathbf{z}^0 \in \Gamma(\mathbf{x}^0)$ the sequence $\{\mathbf{z}^\nu\}$ gives with continuity of ϕ
$$\psi(\mathbf{x}^\nu) = \phi(\mathbf{x}^\nu, \mathbf{y}^\nu) \geqq \phi(\mathbf{x}^\nu, \mathbf{z}^\nu) \xrightarrow{\nu\to\infty} \psi(\mathbf{x}^0) = \phi(\mathbf{x}^0, \mathbf{y}^0) \geqq \phi(\mathbf{x}^0, \mathbf{z}^0),$$
where by point 1. we are especially allowed to put $\mathbf{z}^0 = \mathbf{y}^0$.

The relation $\mathbf{y}^0 \in \Psi(\mathbf{x}^0)$ ensues from point 3. Thus, Ψ is upper semi-continuous.
■

Corollary C.14.1 *Let S be a nonempty compact subset in Y with $X = \mathbb{R}^n = Y$. Then the function $\psi: X \to \mathbb{R}$ with $\psi(\mathbf{x}) := \max\left\{\mathbf{x}^\mathsf{T}\mathbf{y} \mid \mathbf{y} \in S\right\}$ is continuous in X. Furthermore, the correspondence $\Psi: X \to \mathfrak{P}(Y)$ with $\Psi(\mathbf{x}) := \left\{\mathbf{y} \mid \mathbf{y} \in S,\ \mathbf{x}^\mathsf{T}\mathbf{y} = \psi(\mathbf{x})\right\}$ is upper semi-continuous.*

Proof: The statement of Corollary C.14.1 is an immediate consequence of the Maximum Theorem by putting $\phi(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\mathsf{T}\mathbf{y}$ and $\Gamma(\mathbf{x}) = S$ for all $\mathbf{x} \in X$.
■

Theorem C.15 (BROUWER's Fixed-Point Theorem) *Let C be a nonempty compact convex subset in $\mathbb{R}^n$. If the function $\mathbf{f}: C \to C$ is continuous, then $\mathbf{f}$ has a fixed-point $\mathbf{x}^\circ$, i.e. $\mathbf{x}^\circ = \mathbf{f}(\mathbf{x}^\circ)$.*

Accordingly, each continuous function $\mathbf{f}: \Delta \to \Delta$ has a fixed-point $\mathbf{x}^\circ$ if Δ denotes the unit simplex in $\mathbb{R}^n$, i.e. $\Delta := \left\{\mathbf{p} \in \mathbb{R}^n_+ \mid \sum_{j=1}^n p_j = 1\right\}$.

Corollary C.15.1 *Let $\boldsymbol{\psi}: \Delta \to \mathbb{R}^n$ be a continuous function such that the inequality $\mathbf{p}^\mathsf{T}\boldsymbol{\psi}(\mathbf{p}) \leqq 0$ holds for all $\mathbf{p} \in \Delta$. Then there is a $\mathbf{p}^\circ \in \Delta$ with $\boldsymbol{\psi}(\mathbf{p}^\circ) \leqq \mathbf{0}$.*

Proof: Define the function $\mathbf{g}$ with the components
$$g_j(\mathbf{p}) = \frac{p_j + \max\left\{0, \psi_j(\mathbf{p})\right\}}{1 + \sum_{k=1}^n \max\{0, \psi_k(\mathbf{p})\}} \qquad (j = 1, \ldots, n).$$

Since $\boldsymbol{\psi}$ and, therefore, the ψ_j are continuous[90], the maxima of all ψ_j exist by Theorem II.1 (WEIERSTRASS), p. 63. With that and because of the continuity of the maximum function the g_j are also continuous functions. Moreover, $\mathbf{g}(\mathbf{p})$ is a

[90]Cf. DALLMANN, ELSTER (1991b, p. 199, Theorem 7.7): If the function $f: [a, b] \to \mathbb{R}$ is continuous in $[a, b]$, then f is bounded on $[a, b]$.

point in Δ since $\sum_{j=1}^{n} g_j(\mathbf{p}) = 1$ holds good and all of the g_j are nonnegative. Therefore, the function $\mathbf{g}: \Delta \to \Delta$ satisfies the conditions of BROUWER's fixed-point theorem and a fixed-point $\mathbf{p}^\circ$ exists with $\mathbf{p}^\circ = \mathbf{g}(\mathbf{p}^\circ)$. This fixed-point means for each component of $\mathbf{g}$

$$p_j^\circ = \frac{p_j^\circ + \max\left\{0, \psi_j(\mathbf{p}^\circ)\right\}}{1 + \sum_{k=1}^{n} \max\{0, \psi_k(\mathbf{p}^\circ)\}} \qquad (j = 1, \dots, n).$$

Multiplication by the denominator yields

$$p_j^\circ \sum_{k=1}^{n} \max\{0, \psi_k(\mathbf{p}^\circ)\} = \max\left\{0, \psi_j(\mathbf{p}^\circ)\right\}$$

$$\iff \quad \psi_j(\mathbf{p}^\circ) p_j^\circ \sum_{k=1}^{n} \max\{0, \psi_k(\mathbf{p}^\circ)\} = \psi_j(\mathbf{p}^\circ) \max\left\{0, \psi_j(\mathbf{p}^\circ)\right\}$$

$$\implies \quad \sum_{j=1}^{n} \psi_j(\mathbf{p}^\circ) p_j^\circ \sum_{k=1}^{n} \max\{0, \psi_k(\mathbf{p}^\circ)\} = \sum_{j=1}^{n} \psi_j(\mathbf{p}^\circ) \max\left\{0, \psi_j(\mathbf{p}^\circ)\right\}$$

for each $j = 1, \dots, n$. Under the assumptions of the corollary we have $\mathbf{p}^{\circ\mathrm{T}} \boldsymbol{\psi}(\mathbf{p}^\circ) \leqq 0$ and, therefore,

$$0 \geqq \sum_{j=1}^{n} \psi_j(\mathbf{p}^\circ) \max\left\{0, \psi_j(\mathbf{p}^\circ)\right\} .$$

Since each summand equals either zero or $(\psi_j(\mathbf{p}^\circ))^2$, we get $\boldsymbol{\psi}(\mathbf{p}^\circ) \leqq \mathbf{0}$ for the fixed-point. ■

Comment: Corollary C.15.1 requires a continuous function $\boldsymbol{\psi}$. This is the reason why the function $\boldsymbol{\psi}$ cannot immediately be identified with the aggregate excess demand function $\mathbf{z}$. Since $\mathbf{z}(\mathbf{p})$ is not defined for all price vectors $\mathbf{p}$ having a zero component, the problem $\max\left\{0, z_j(\mathbf{p})\right\}$ cannot be solved for these price vectors.

The transference of BROUWER's fixed-point theorem to convex valued correspondences is given by KAKUTANI's fixed-point theorem. As known from HEUSER (1992, p. 616), the two fixed-point theorems are completely equivalent.

Theorem C.16 (KAKUTANI's Fixed-Point Theorem)[91] *Let C be a nonempty compact convex subset in $\mathbb{R}^n$. If the correspondence $\Gamma: C \to \mathfrak{P}(C)$ is upper semi-continuous and if the sets $\Gamma(\mathbf{x})$ are not empty and convex for all $\mathbf{x} \in C$, then Γ has a fixed-point $\mathbf{x}^\circ$.*

$$\exists \mathbf{x}^\circ \in C : \ \mathbf{x}^\circ \in \Gamma(\mathbf{x}^\circ)$$

[91] Cf. BERGE (1963, p. 174). HEUSER (1992, p. 614) speaks of a closed correspondence, where by Theorem C.8 it must be noted, that for compact C the correspondence Γ is closed if and only if it is upper semi-continuous.

Although the next theorem has been derived independently of Theorem C.16 (KAKUTANI), UZAWA (1962) has proved that both theorems are equivalent.

Theorem C.17 (DEBREU, GALE, NIKAIDO)[92] *Let S be a nonempty closed convex subset in the unit simplex $\Delta \subset \mathbb{R}^n$. Suppose the correspondence Ψ: $S \to \mathfrak{P}(\mathbb{R}^n)$ has the following properties:*

1. *The sets $\Psi(\mathbf{p})$ are not empty and convex for all $\mathbf{p} \in S$.*
2. *The correspondence Ψ is closed.*
3. *The correspondence Ψ is bounded, i.e. a (closed) ball $K \subset \mathbb{R}^n$ exists such that $\Psi(\mathbf{p}) \subset K$ for every $\mathbf{p} \in S$.*
4. *We have $\mathbf{p}^{\mathsf{T}}\mathbf{z} \leqq 0$ for each $\mathbf{p} \in S$ and each $\mathbf{z} \in \Psi(\mathbf{p})$.*

Then $\mathbf{p}^\circ \in S$ and $\mathbf{z}^\circ \in \Psi(\mathbf{p}^\circ)$ exist such that $\mathbf{p}^{\mathsf{T}}\mathbf{z}^\circ \leqq 0$ for all $\mathbf{p} \in S$.

Proof: First of all, we define the correspondence $\Gamma\colon S \times K \to \mathfrak{P}(S \times K)$ with $\Gamma(\mathbf{p}, \mathbf{z}) := M(\mathbf{z}) \times \Psi(\mathbf{p})$ and $M(\mathbf{z}) := \{\mathbf{p} \mid \mathbf{p} \in S,\ \mathbf{p}^{\mathsf{T}}\mathbf{z} = \max\{\mathbf{q}^{\mathsf{T}}\mathbf{z} \mid \mathbf{q} \in S\}\}$. In this $S \times K$ is a nonempty compact convex set.

1. The function $\mathbf{q}^{\mathsf{T}}\mathbf{z}$ is continuous at every $\mathbf{z}$ such that $M(\mathbf{z}) \neq \emptyset$ holds for all $\mathbf{z} \in K$ according to Theorem II.1 (WEIERSTRASS), p. 63. Since $\Psi(\mathbf{p}) \neq \emptyset$ for all $\mathbf{p} \in S$, we get $\Gamma(\mathbf{p}, \mathbf{z}) \neq \emptyset$ for all $(\mathbf{p}, \mathbf{z}) \in S \times K$.
2. For a convex set S the sets $M(\mathbf{z})$ are convex for every $\mathbf{z}$.[93] Since $\Psi(\mathbf{p})$ is convex for each $\mathbf{p} \in S$ by assumption, $\Gamma(\mathbf{p}, \mathbf{z}) = M(\mathbf{z}) \times \Psi(\mathbf{p})$ must be convex.
3. By Theorem C.14 (Maximum Theorem) the correspondence M is upper semi-continuous. In accordance with Theorem C.8 for a (compact) ball $K \subset \mathbb{R}^n$ the correspondence $\Psi\colon S \to \mathfrak{P}(K)$ is upper semi-continuous if and only if it is closed. Thus, the cross product Γ must be upper semi-continuous.[94]

Now we can apply KAKUTANI's fixed-point theorem to Γ

$$\begin{aligned}
& \exists(\mathbf{p}^\circ, \mathbf{z}^\circ) \in S \times K : (\mathbf{p}^\circ, \mathbf{z}^\circ) \in \Gamma(\mathbf{p}^\circ, \mathbf{z}^\circ) \\
\Longleftrightarrow\quad & [\exists\, \mathbf{p}^\circ \in S \text{ and } \exists\, \mathbf{z}^\circ \in K] : [\mathbf{p}^\circ \in M(\mathbf{z}^\circ) \text{ and } \mathbf{z}^\circ \in \Psi(\mathbf{p}^\circ)] \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{with } \Psi(\mathbf{p}^\circ) \subset K \\
\Longleftrightarrow\quad & [\exists\, \mathbf{p}^\circ \in S \text{ and } \exists\, \mathbf{z}^\circ \in \Psi(\mathbf{p}^\circ)] : \mathbf{p}^\circ \in M(\mathbf{z}^\circ) \\
\Longleftrightarrow\quad & [\exists\, \mathbf{p}^\circ \in S \text{ and } \exists\, \mathbf{z}^\circ \in \Psi(\mathbf{p}^\circ)] : [\forall\, \mathbf{p} \in S : \mathbf{p}^{\mathsf{T}}\mathbf{z}^\circ \leqq \mathbf{p}^{\circ\mathsf{T}}\mathbf{z}^\circ]
\end{aligned}$$

[92] Cf. HILDENBRAND, KIRMAN (1988, p. 278) and HEUSER (1992, p. 631).

[93] If C is a convex subset in $\mathbb{R}^n$ and if the function f is convex in C, then $\min\{f(\mathbf{x}) \mid \mathbf{x} \in C\}$ is called a convex program. The set of all optimal solutions to this program is convex (but possibly empty). Cf. BLUM, ÖTTLI (1975, p. 5, Theorem 1).

[94] See p. 310.

The proof concludes with a consideration of the fourth assumption. ■

A detailed discussion of further fixed-point theorems may be found in BORDER (1985). Besides the discussion of the relationships between fixed-point theorems the author presents a series of economic results. For example, the book contains conditions for the existence of WALRASian equilibria, NASH equilibria, and equilibria in cooperative games.

D Duality Theory

D.1 Duality of Conjugate Functions

The expositions on duality theory are based on ROCKAFELLAR (1972). Continuing with the results of convex analysis in Appendix B, the following sections deal with implications which result from the properties of functions introduced in Appendix C.1. In order to emphasize the dual aspects, we set

$$X = \mathbb{R}^n = Y,$$

where X and Y are the spaces of primal and dual variables respectively. Theorem B.11 serves as the initial point. Its meaning for closed convex sets can be transferred to convex functions in the following way.

Theorem D.1[95] *A closed convex function f is the pointwise supremum over the class of all affine functions h such that $h \leqq f$.*

The analogy to Theorem B.11 becomes lucid if we imagine the epigraph of the function f as intersection, where the intersection is taken over the epigraphs of all affine functions $h(\mathbf{x}) = \mathbf{y}^\mathsf{T}\mathbf{x} + \alpha \quad (\mathbf{y} \neq \mathbf{0},\ \alpha \in \mathbb{R})$ satisfying $h \leqq f$. Choosing the supremum over the set of all functional values $h(\mathbf{x})$ for each point $\mathbf{x}$, where h denotes the class of all affine functions with $h \leqq f$, the function f results. Cf. the left hand part of Figure D.1.

If we only allow for linear functions with $\alpha = 0$ instead of affine functions $h(\mathbf{x}) = \mathbf{y}^\mathsf{T}\mathbf{x} + \alpha \quad (\mathbf{y} \neq \mathbf{0},\ \alpha \in \mathbb{R})$, then the next definition of conjugate functions can be justified corresponding to the right hand part of Figure D.1.

Definition D.1[96] *Let $f\colon X \to [-\infty, +\infty]$ be an arbitrary function. Then the* ***convex conjugate function*** *$f^*\colon Y \to [-\infty, +\infty]$ of f is given by*

$$f^*(\mathbf{y}) := \sup\left\{\mathbf{y}^\mathsf{T}\mathbf{x} - f(\mathbf{x}) \mid \mathbf{x} \in X\right\}.$$

[95] Cf. ROCKAFELLAR (1972, p. 102).

[96] Cf. ROCKAFELLAR (1974, pp. 15 and 18). Comment: for a convex differentiable function the FENCHEL transform is closely related to the LEGENDRE transform; see ROCKAFELLAR (1972, p. 251 ff.).

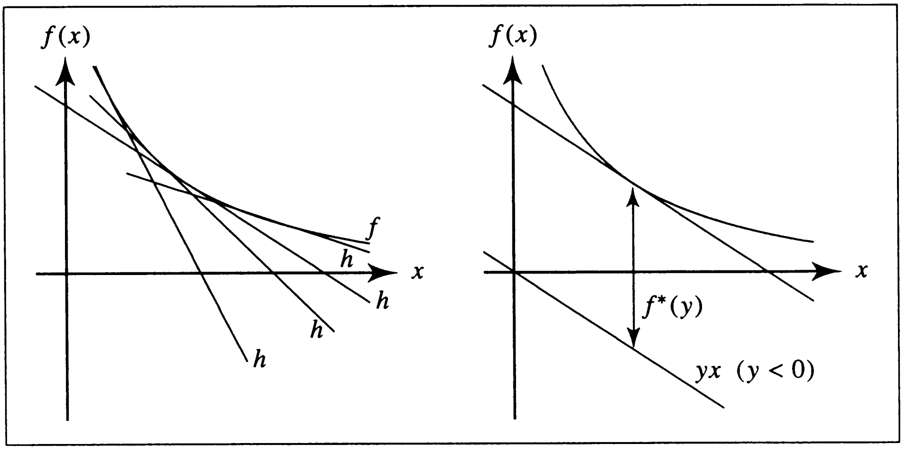

Figure D.1: Illustration of a convex function

The operation $f \to f^*$ *is called the* ***(convex)*** **FENCHEL** ***transform****. The* ***convex biconjugate function*** *of f is the convex conjugate function f^{**} of f^* (second convex* FENCHEL *transform).*

$$f^{**}(\mathbf{x}) := \sup \{\mathbf{y}^\mathsf{T}\mathbf{x} - f^*(\mathbf{y}) \mid \mathbf{y} \in Y\}$$

Let $g\colon X \to [-\infty, +\infty]$ *be an arbitrary function. Then* $g_*\colon Y \to [-\infty, +\infty]$ *with*

$$g_*(\mathbf{y}) := \inf \{\mathbf{y}^\mathsf{T}\mathbf{x} - g(\mathbf{x}) \mid \mathbf{x} \in X\}$$

is called the ***concave conjugate function*** *of g. The operation* $g \to g_*$ *is called the* ***concave*** **FENCHEL** ***transform****. Moreover, we define the* ***concave biconjugate function*** g_{**} *by*

$$g_{**}(\mathbf{x}) := \inf \{\mathbf{y}^\mathsf{T}\mathbf{x} - g_*(\mathbf{y}) \mid \mathbf{y} \in Y\}.$$

The definition assumes an arbitrary function f or g. Remember according to Comment C.1, p. 299, that each real-valued function $f\colon C \to \mathbb{R}$ with $C \subset \mathbb{R}^n$ can be expanded to a function $\tilde{f}\colon X \to [-\infty, +\infty]$ with $X = \mathbb{R}^n$. If we define $\tilde{f}(\mathbf{x}) = +\infty$ for all $\mathbf{x} \notin C$, then each vector $\mathbf{x} \notin C$ is irrelevant for the calculation of the convex conjugate function $\tilde{f}^*$.[97]

In order to apply the duality of conjugate functions the next theorem is decisive.

Theorem D.2[98] *The convex conjugate function* $f^*\colon Y \to [-\infty, +\infty]$ *of an* arbitrary *function* $f\colon X \to [-\infty, +\infty]$ *is a closed convex function. Moreover,*

[97] Cf. BLUM, ÖTTLI (1975, p. 155 f.). There it is found: if f is a convex function defined on the convex set $C \subset \mathbb{R}^n$, then $f^*(\mathbf{y}) := \sup \{\mathbf{y}^\mathsf{T}\mathbf{x} - f(\mathbf{x}) \mid \mathbf{x} \in C\}$ is called the convex conjugate function of f. Remember, moreover, that the effective domain of an extended real-valued convex function is a convex set.

[98] For the proof, see ROCKAFELLAR (1974, pp. 16 and 18) or ROCKAFELLAR (1972, p. 104 f.).

f^* is proper if and only if f is proper. If f is convex, then, furthermore,

$$f^{**} = \operatorname{cl} f \quad \textit{and} \quad (\operatorname{cl} f)^* = f^* .$$

Analogously, the concave conjugate function $g_*: Y \to [-\infty, +\infty]$ to an arbitrary *function $g: X \to [-\infty, +\infty]$ is closed and concave. Moreover, g_* is n-proper if and only if g has this property. For a concave function g we also have*

$$g_{**} = \operatorname{cl} g \quad \textit{and} \quad (\operatorname{cl} g)_* = g_* .$$

This theorem includes two major cases with respect to an arbitrary function f.

1. An improper function f implies an improper function f^*. Again we have to distinguish between two cases.[99]
   (a) If an $\hat{\mathbf{x}} \in X$ with $f(\hat{\mathbf{x}}) = -\infty$ exists, then $f^*(\mathbf{y}) = +\infty$ holds good for all $\mathbf{y} \in Y$. The function f^* is improper. This implication is no contradiction to Theorem D.2 since $f^* \equiv -\infty$ is a closed convex function with the (closed convex) epigraph $\operatorname{epi} f^* = \mathbb{R}^{n+1}$.
   (b) Supposing $f \equiv +\infty$, at once an improper function $f^* \equiv -\infty$ results. Even this implication is no contradiction to Theorem D.2 as $f^* \equiv +\infty$ is a closed convex function with the (closed convex) epigraph $\operatorname{epi} f^* = \emptyset$.

2. A proper function f implies f^* to be proper.
   (a) Since $f > -\infty$, $f^*(\mathbf{0}) = \sup\{-f(\mathbf{x}) \mid \mathbf{x} \in X\} < +\infty$ must hold.
   (b) If an $\hat{\mathbf{x}}$ exists with finite $f(\hat{\mathbf{x}})$, then we get

$$-\infty < \mathbf{y}^{\mathsf{T}}\hat{\mathbf{x}} - f(\hat{\mathbf{x}}) \leqq \sup\left\{\mathbf{y}^{\mathsf{T}}\mathbf{x} - f(\mathbf{x}) \mid \mathbf{x} \in X\right\} = f^*(\mathbf{y})$$

   for all $\mathbf{y} \in Y$. From (b) $f^* > -\infty$ and (a) $f^* \not\equiv +\infty$ we obtain a proper function f^*.

The definition of conjugate functions immediately yields the **YOUNG-FENCHEL inequalities** satisfied for each pair of convex conjugate functions (f, f^*) and for each pair of concave conjugate functions (g, g_*) respectively.

(D.1a) $$f(\mathbf{x}) + f^*(\mathbf{y}) \geqq \mathbf{y}^{\mathsf{T}}\mathbf{x} \qquad \forall \mathbf{x} \in X,\ \forall \mathbf{y} \in Y$$

(D.1b) $$g(\mathbf{x}) + g_*(\mathbf{y}) \leqq \mathbf{y}^{\mathsf{T}}\mathbf{x} \qquad \forall \mathbf{x} \in X,\ \forall \mathbf{y} \in Y$$

Thus,

(D.2a) $$\left[f^{**}(\mathbf{x}) = \sup\left\{\mathbf{y}^{\mathsf{T}}\mathbf{x} - f^*(\mathbf{y})\right\} \leqq f(\mathbf{x}) \quad \forall \mathbf{x} \in X\right] \iff f^{**} \leqq f,$$

(D.2b) $$\left[g_{**}(\mathbf{x}) = \inf\left\{\mathbf{y}^{\mathsf{T}}\mathbf{x} - g_*(\mathbf{y})\right\} \geqq g(\mathbf{x}) \quad \forall \mathbf{x} \in X\right] \iff g_{**} \geqq g.$$

How to interpret the function f^{**}? The answer can be illustrated by understanding the second convex FENCHEL transform as an operation which assigns the greatest closed convex function f^{**} with $f^{**} \leqq f$ to an arbitrary function f

[99] Illustrate the given epigraphs by Figure C.1, p. 298.

(convexification of f). If $\mathrm{cl}(\mathrm{conv} f)$ is a proper function, then this statement can be made more precise,[100]

$$f^{**} = \mathrm{cl}(\mathrm{conv} f).$$

Transferred to the epigraphs we equivalently gain

$$\mathrm{epi} f^{**} = \mathrm{cl}\left(\mathrm{conv}(\mathrm{epi} f)\right).$$

Analogously, g_{**} corresponds to the smallest closed concave function satisfying $g_{**} \geqq g$ (concavification of g). As before, g_{**} can be determined by the relation $\mathrm{hypo}\, g_{**} = \mathrm{cl}\left(\mathrm{conv}(\mathrm{hypo}\, g)\right)$ for an n-proper function g. With that we get an idea of the next two theorems.

Theorem D.3 (FENCHEL-MOREAU Theorem)[101] *Let* $f\colon X \to [-\infty, +\infty]$ *be a proper function. The* FENCHEL *transform is a symmetric operation* $f \to h$ *with* $h = f^*(= f^{***})$ *and* $h^* = f^{**} = f$ *if and only if* f *is a closed convex function.*

Corollary D.3.1 (FENCHEL-MOREAU Theorem, Concave Version)[102]
Let $g\colon X \to [-\infty, +\infty]$ *be an n-proper function. The concave* FENCHEL *transform is a symmetric operation* $g \to h$ *with* $h = g_*(= g_{***})$ *and* $h_* = g_{**} = g$ *if and only if* g *is a closed concave function.*

In order to prove this corollary ROCKAFELLAR refers to the following relation: for an n-proper concave function $g\colon X \to [-\infty, +\infty]$ we obtain a proper convex function f by $g = -f$. The convex conjugate function f^* then induces $g_*(\mathbf{y}) = -f^*(-\mathbf{y})$.

Further duality propositions result from the examination of subgradients and supergradients of f at a point in the effective domain.

Definition D.2[103] *A vector* $\mathbf{y}$ *is called a* ***subgradient*** *of the convex function* f *at point* $\hat{\mathbf{x}} \in X$ *if the following inequality is satisfied:*

$$f(\mathbf{x}) \geqq f(\hat{\mathbf{x}}) + \mathbf{y}^{\mathsf{T}}(\mathbf{x} - \hat{\mathbf{x}}) \qquad \forall \mathbf{x} \in X$$

The set of all subgradients of f *at point* $\hat{\mathbf{x}}$ *is called the subdifferential of* f *at point* $\hat{\mathbf{x}}$ *and is denoted by* $\partial f(\hat{\mathbf{x}})$. *The correspondence* $\partial f\colon X \to \mathfrak{P}(Y)$ *is called the* ***subdifferential*** *of* f.
A vector $\mathbf{y}$ *is called a* ***supergradient*** *of the concave function* g *at point* $\hat{\mathbf{x}} \in X$ *if*

$$g(\mathbf{x}) \leqq g(\hat{\mathbf{x}}) + \mathbf{y}^{\mathsf{T}}(\mathbf{x} - \hat{\mathbf{x}}) \qquad \forall \mathbf{x} \in X.$$

[100] Cf. IOFFE, TIHOMIROV (1979, p. 177, Corollary 2).

[101] Cf. IOFFE, TIHOMIROV (1979, p. 174 f.).

[102] Cf. NEWMAN (1987b, p. 927). Note for NEWMAN's Theorem 2 that every convex (concave) closed (n-)proper function is also lower (upper) semi-continuous.

[103] The definitions are taken from ROCKAFELLAR (1972, p. 214 f.) and NEWMAN (1987b, p. 928).

The set of all supergradients of g at point $\hat{\mathbf{x}}$ is called the superdifferential of g at point $\hat{\mathbf{x}}$ and is denoted by $\Delta g(\hat{\mathbf{x}})$. The correspondence $\Delta g\colon X \to \mathfrak{P}(Y)$ is called ***superdifferential*** *of g.*

Note that a convex function f achieves its minimum at point $\hat{\mathbf{x}}$ if and only if $\mathbf{0} \in \partial f(\hat{\mathbf{x}})$ since the subgradient $\mathbf{y} = \mathbf{0}$ satisfies

$$f(\mathbf{x}) \geqq f(\hat{\mathbf{x}}) \qquad \forall \mathbf{x} \in X$$

by definition. Conversely, a concave function g attains its maximum at point $\hat{\mathbf{x}}$ if and only if $\mathbf{0} \in \Delta g(\hat{\mathbf{x}})$.

If $\mathbf{y}$ is a subgradient of f at point $\hat{\mathbf{x}}$, $\mathbf{y} \in \partial f(\hat{\mathbf{x}})$, then the affine function

$$h(\mathbf{x}; \hat{\mathbf{x}}) = f(\hat{\mathbf{x}}) + \mathbf{y}^{\mathsf{T}}(\mathbf{x} - \hat{\mathbf{x}}) \tag{D.3}$$

is called a support function of f at point $\hat{\mathbf{x}}$.[104] Geometrically, the graph of this function $h(\mathbf{x}; \hat{\mathbf{x}})$ can be illustrated by a supporting hyperplane of the convex set epi f at point $\binom{\hat{\mathbf{x}}}{f(\hat{\mathbf{x}})}$. The subgradient $\mathbf{y}$ determines a normal vector to this supporting hyperplanes by $\binom{\mathbf{y}}{-1}$;[105] see the comments on Theorem B.1.

For a convex function f the subdifferential $\partial f(\mathbf{x})$ is a closed[106] convex set. If $\partial f(\hat{\mathbf{x}}) \neq \emptyset$, then f is said to be **subdifferentiable** at point $\hat{\mathbf{x}}$. Moreover, for a proper convex function f we have[107]

$$\begin{aligned}
\mathbf{x} \notin \operatorname{Dom} f \quad &\Longrightarrow \quad \partial f(\mathbf{x}) = \emptyset, \\
\mathbf{x} \in \operatorname{rint}(\operatorname{Dom} f) \quad &\Longrightarrow \quad \partial f(\mathbf{x}) \neq \emptyset, \\
\mathbf{x} \in \operatorname{int}(\operatorname{Dom} f) \quad &\Longleftrightarrow \quad \partial f(\mathbf{x}) \neq \emptyset \quad \text{and bounded.}
\end{aligned}$$

The next theorem is useful to classify subgradients and supergradients.

Theorem D.4[108] *Let $\mathbf{x}$ be a point at which the convex function f is finite. If f is differentiable at point $\mathbf{x}$, then the* ***gradient*** *$\nabla f(\mathbf{x})$ is the unique subgradient of f at $\mathbf{x}$, $\partial f(\mathbf{x}) = \{\nabla f(\mathbf{x})\}$. Conversely, if a convex function f has a unique subgradient $\mathbf{y}(\mathbf{x})$ at point $\mathbf{x}$, then f is differentiable at $\mathbf{x}$ and $\mathbf{y}(\mathbf{x}) = \nabla f(\mathbf{x})$.*

If the function f is differentiable at $\hat{\mathbf{x}}$, then according to (D.3) $h(\mathbf{x}; \hat{\mathbf{x}}) = f(\hat{\mathbf{x}}) + \nabla f(\hat{\mathbf{x}})^{\mathsf{T}}(\mathbf{x} - \hat{\mathbf{x}})$ is called a linearization of f at $\hat{\mathbf{x}}$. The main results on the duality of conjugate functions may now be summarized by

[104] Cf. BLUM, ÖTTLI (1975, pp. 49–50). A graphical representation of the corresponding supergradient may be found in Figure III.9, p. 104.

[105] Cf. ELSTER, REINHARDT, SCHÄUBLE, DONATH (1977, p. 84 f.).
Converting all subgradients $\mathbf{y} \in \partial f(\hat{\mathbf{x}})$ of the convex function f into $\binom{\mathbf{y}}{-1}$, then a convex cone results containing all vectors which are normal to the set epi f at point $\binom{\hat{\mathbf{x}}}{f(\hat{\mathbf{x}})}$. This cone is frequently called *Clark's normal cone* with respect to CLARKE (1975).

[106] The empty set is open and closed by definition.

[107] Cf. ROCKAFELLAR (1972, p. 217, Theorem 23.4).

[108] Cf. BLUM, ÖTTLI (1975, p. 50) or ROCKAFELLAR (1972, p. 242). The transference to the concave case is omitted.

Theorem D.5[109] *Given a proper convex function f and a vector $\hat{\mathbf{x}}$. Then the following relations on a vector $\hat{\mathbf{y}}$ are equivalent to each other.*[110]

- $\hat{\mathbf{y}} \in \partial f(\hat{\mathbf{x}})$,
- $\hat{\mathbf{y}}^{\mathsf{T}}\mathbf{x} - f(\mathbf{x})$ *achieves the supremum at* $\mathbf{x} = \hat{\mathbf{x}}$,
- $f(\hat{\mathbf{x}}) + f^*(\hat{\mathbf{y}}) = \hat{\mathbf{y}}^{\mathsf{T}}\hat{\mathbf{x}}$.

Dually, three more relations are equivalent to each other.

- $\hat{\mathbf{x}} \in \partial f^*(\hat{\mathbf{y}})$,
- $\mathbf{y}^{\mathsf{T}}\hat{\mathbf{x}} - f^*(\mathbf{y})$ *achieves the supremum at* $\mathbf{y} = \hat{\mathbf{y}}$,
- $f^{**}(\hat{\mathbf{y}}) + f^*(\hat{\mathbf{x}}) = \hat{\mathbf{y}}^{\mathsf{T}}\hat{\mathbf{x}}$.

Moreover, if $\operatorname{cl} f(\hat{\mathbf{x}}) = f(\hat{\mathbf{x}})$ *at point* $\hat{\mathbf{x}}$, *then all of the relations are equivalent and can be expanded by* $\hat{\mathbf{y}} \in \partial(\operatorname{cl} f)(\hat{\mathbf{x}})$.

In particular, if the inspected function f is closed, i.e. $\operatorname{cl} f = f$ by Definition C.4, p. 302, then

Corollary D.5.1 *Let f be a proper, convex,* and *closed function. Then* $\partial f : X \to \mathfrak{P}(Y)$ *and* $\partial f^* : Y \to \mathfrak{P}(X)$ *are inverse correspondences.*

$$\mathbf{y} \in \partial f(\mathbf{x}) \quad \Longleftrightarrow \quad \mathbf{x} \in \partial f^*(\mathbf{y})$$

A pair of points $(\mathbf{x}, \mathbf{y})$ *satisfying this condition is called a* ***pair of dual points***.

Corollary D.5.2 (Concave Version)[111] *Given an n-proper concave function g and a vector $\hat{\mathbf{x}}$. Then the following relations are equivalent to each other with respect to a vector $\hat{\mathbf{y}}$.*

- $\hat{\mathbf{y}} \in \Delta g(\hat{\mathbf{x}})$,
- $\hat{\mathbf{y}}^{\mathsf{T}}\mathbf{x} - g(\mathbf{x})$ *achieves the infimum at* $\mathbf{x} = \hat{\mathbf{x}}$,
- $g(\hat{\mathbf{x}}) + g_*(\hat{\mathbf{y}}) = \hat{\mathbf{y}}^{\mathsf{T}}\hat{\mathbf{x}}$.

Dually, three more relations are equivalent to each other.

- $\hat{\mathbf{x}} \in \Delta g_*(\hat{\mathbf{y}})$,
- $\mathbf{y}^{\mathsf{T}}\hat{\mathbf{x}} - g_*(\mathbf{y})$ *achieves the infimum at* $\mathbf{y} = \hat{\mathbf{y}}$,
- $g_{**}(\hat{\mathbf{x}}) + g_*(\hat{\mathbf{y}}) = \hat{\mathbf{y}}^{\mathsf{T}}\hat{\mathbf{x}}$.

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[109] Cf. ROCKAFELLAR (1972, p. 218, Theorem 23.5).

[110] If $\sup\{f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n\}$ is finite and if the supremum is attained at $\hat{\mathbf{x}} \in \mathbb{R}^n$, then $\hat{\mathbf{x}}$ is called an optimal solution to the problem and we write $f(\hat{\mathbf{x}}) = \max\{f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n\}$. See the discussion in Section 2.4.2.

[111] Cf. NEWMAN (1987b, p. 929).

If, moreover, $\operatorname{cl} g(\hat{\mathbf{x}}) = g(\hat{\mathbf{x}})$ *holds at point* $\hat{\mathbf{x}}$*, then all of the six relations are equivalent and can be expanded by* $\hat{\mathbf{y}} \in \Delta(\operatorname{cl} g)(\hat{\mathbf{x}})$*. In particular, if the function* g *is closed, i.e.* $\operatorname{cl} g = g$*, then the correspondences* $\Delta g\colon X \to \mathfrak{P}(Y)$ *and* $\Delta g_*\colon Y \to \mathfrak{P}(X)$ *are inverse to each other.*

$$\mathbf{y} \in \Delta g(\mathbf{x}) \quad \Longleftrightarrow \quad \mathbf{x} \in \Delta g_*(\mathbf{y})$$

The properties of conjugate functions have numerous implications especially reflected in treating dual programs. To present some aspects of this duality theory, the following pair of dual programs underlies the concluding explanations:

$$\text{(P1)} \qquad \inf\left\{ f(\mathbf{x}) - g(\mathbf{x}) \mid \mathbf{x} \in X_f \cap X_g \right\}$$

$$\text{(P1*)} \qquad \sup\left\{ g_*(\mathbf{y}) - f^*(\mathbf{y}) \mid \mathbf{y} \in Y_f \cap Y_g \right\},$$

where X_f and X_g are convex subsets in $\mathbb{R}^n$. If the function $f\colon X_f \to \mathbb{R}$ is convex on X_f and if the function $g\colon X_g \to \mathbb{R}$ concave in X_g such that

$$f^*(\mathbf{y}) = \sup\left\{ \mathbf{y}^{\mathsf{T}}\mathbf{x} - f(\mathbf{x}) \mid \mathbf{x} \in X_f \right\} \quad \text{with} \quad Y_f := \left\{ \mathbf{y} \mid f^*(\mathbf{y}) < +\infty \right\}$$

$$g_*(\mathbf{y}) = \inf\left\{ \mathbf{y}^{\mathsf{T}}\mathbf{x} - g(\mathbf{x}) \mid \mathbf{x} \in X_g \right\} \quad \text{with} \quad Y_g := \left\{ \mathbf{y} \mid g_*(\mathbf{y}) > -\infty \right\},$$

then FENCHEL's duality theorems hold for optimal values to both programs inf(P1) and sup(P1*).[112]

1. If $X_f \cap X_g \neq \emptyset$ and $Y_f \cap Y_g \neq \emptyset$, then inf(P1) is bounded below and sup(P1*) is bounded above. Moreover, inf(P1) = sup(P1*).

2. If X_f and X_g as well as Y_f and Y_g have relatively interior points in common and if the functions f and g are closed, then (P1) and (P1*) have optimal solutions. Moreover, min(P1) = max(P1*).

A more general approach is presented by WALK (1989) including the duality theory of linear programs as a limit case. He examines a LAGRANGEan function

$$\Phi(\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{x}, \mathbf{y}) - \alpha(\mathbf{x}) - \beta(\mathbf{y}),$$

where it is supposed that $\alpha\colon X_\alpha \to \mathbb{R}$, $\beta\colon Y_\beta \to \mathbb{R}$, $\varphi\colon X_\alpha \times Y_\beta \to \mathbb{R}$ and $X_\alpha \times Y_\beta \subset \mathbb{R}^n \times \mathbb{R}^m$. If we define the *generalized* conjugate functions

$$\alpha_*(\mathbf{y}) := \inf\left\{ \varphi(\mathbf{x}, \mathbf{y}) - \alpha(\mathbf{x}) \mid \mathbf{x} \in X_\alpha \right\} \quad \text{with} \quad \widehat{Y}_\beta := \left\{ \mathbf{y} \in Y_\beta \mid \alpha_*(\mathbf{y}) > -\infty \right\}$$

$$\beta^*(\mathbf{x}) := \sup\left\{ \varphi(\mathbf{x}, \mathbf{y}) - \beta(\mathbf{y}) \mid \mathbf{y} \in Y_\beta \right\} \quad \text{with} \quad \widehat{X}_\alpha := \left\{ \mathbf{x} \in X_\alpha \mid \beta^*(\mathbf{x}) < +\infty \right\},$$

then besides $\widehat{X}_\alpha \times \widehat{Y}_\beta \subset X_\alpha \times Y_\beta$ it follows

$$\inf_{\mathbf{x} \in X_\alpha} \Phi(\mathbf{x}, \mathbf{y}) = \alpha_*(\mathbf{y}) - \beta(\mathbf{y}) \quad \text{with} \quad \mathbf{y} \in \widehat{Y}_\beta,$$

$$\sup_{\mathbf{y} \in Y_\beta} \Phi(\mathbf{x}, \mathbf{y}) = \beta^*(\mathbf{x}) - \alpha(\mathbf{x}) \quad \text{with} \quad \mathbf{x} \in \widehat{X}_\alpha.$$

[112] Cf. FENCHEL (1953, p. 105 ff.). For further expositions see BLUM, ÖTTLI (1975, pp. 156–160).

For the pair of dual programs[113]

(P2) $\inf\left\{\beta^*(\mathbf{x}) - \alpha(\mathbf{x}) \mid \mathbf{x} \in \widehat{X}_\alpha\right\}$

(P2*) $\sup\left\{\alpha_*(\mathbf{y}) - \beta(\mathbf{y}) \mid \mathbf{y} \in \widehat{Y}_\beta\right\}$

the subsequent duality theorem can be offered.

Theorem D.6[114] *The following statements are equivalent:*

1. *There is a pair* $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in X_\alpha \times Y_\beta$ *such that*

$$\Phi(\mathbf{x}, \hat{\mathbf{y}}) \leqq \Phi(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \leqq \Phi(\hat{\mathbf{x}}, \mathbf{y})$$

for all $(\mathbf{x}, \mathbf{y}) \in X_\alpha \times Y_\beta$.

2. *We have* $\widehat{X}_\alpha \neq \emptyset$ *and* $\widehat{Y}_\beta \neq \emptyset$. *Both problems* (P2) *and* (P2*) *are feasible and*

$$\min\left\{\beta^*(\mathbf{x}) - \alpha(\mathbf{x}) \mid \mathbf{x} \in \widehat{X}_\alpha\right\} = \max\left\{\alpha_*(\mathbf{y}) - \beta(\mathbf{y}) \mid \mathbf{y} \in \widehat{Y}_\beta\right\}.$$

3. *There is a pair* $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \widehat{X}_\alpha \times \widehat{Y}_\beta$ *such that*

$$\varphi(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \alpha(\hat{\mathbf{x}}) + \alpha^*(\hat{\mathbf{y}}) \quad \textit{and} \quad \varphi(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \beta(\hat{\mathbf{y}}) + \beta_*(\hat{\mathbf{x}}).$$

D.2 Duality of Polar Gauges

D.2.1 Properties of the Support Function

After the duality of conjugate functions has been introduced, the next appendix deals with the duality of polar gauges where the term "polar gauge" is given by Definition D.6, p. 334. Although the two theories have points in common there is a big difference between their initial points. Whereas the theory of conjugate functions examines "best" inequalities of the form $\mathbf{y}^\mathsf{T}\mathbf{x} \leqq f(\mathbf{x}) + g(\mathbf{y})$ the theory of polar gauges deals with "best" inequalities of the form $\mathbf{y}^\mathsf{T}\mathbf{x} \leqq f(\mathbf{x})\, g(\mathbf{y})$. The YOUNG-FENCHEL inequality (D.1a) is compared to MAHLER's inequality (D.16). All of the main issues are summarized in Figure D.3, p. 336. A simple example stresses the most important properties; see Figure D.4.

As before, the dual spaces are $X = \mathbb{R}^n = Y$ so that the dual aspects of the presented functions can explicitly be stressed. In fact, the easiest representation of a set $C \subset X$ is given by the **indicator function** $\delta(\cdot|C)\colon X \to [0, +\infty]$

$$\delta(\mathbf{x}|C) := \begin{cases} 0 & \text{for } \mathbf{x} \in C \\ +\infty & \text{for } \mathbf{x} \notin C\,. \end{cases}$$

An alternative way to represent C follows from

[113] In the case of $m = n$ with $\varphi(\mathbf{x}, \mathbf{y}) = \mathbf{y}^\mathsf{T}\mathbf{x}$ the relation to (P1) and (P1*) becomes apparent by putting $\alpha = g$ and $\beta = f^*$.

[114] Cf. VOGEL (1968, p. 4, Main Theorem 1).

Definition D.3 *The* ***support function*** $\sigma(\cdot|C)\colon Y \to [-\infty, +\infty]$ *of a set* $C \subset X$ *is defined by*

$$\sigma(\mathbf{y}|C) := \sup\left\{\mathbf{y}^{\mathsf{T}}\mathbf{x} \mid \mathbf{x} \in C\right\}.$$

Analogously, $\varphi(\cdot|C)\colon Y \to [-\infty, +\infty]$ *with*

$$\varphi(\mathbf{y}|C) := \inf\left\{\mathbf{y}^{\mathsf{T}}\mathbf{x} \mid \mathbf{x} \in C\right\}$$

or, equivalently, $\varphi(\mathbf{y}|C) = -\sigma(-\mathbf{y}|C)$ *is called the* ***reciprocal support function*** *of the set C.*

The relationship between the (linearly homogeneous) indicator function $\delta(\cdot|C)$ and the linearly homogeneous support function[115] $\sigma(\cdot|C)$ results from

$$\begin{aligned}\sigma(\mathbf{y}|C) &= \sup\left\{\mathbf{y}^{\mathsf{T}}\mathbf{x} \mid \mathbf{x} \in C\right\}\\ &= \sup\left\{\mathbf{y}^{\mathsf{T}}\mathbf{x} - \delta(\mathbf{x}|C) \mid \mathbf{x} \in X\right\}\\ &= \delta^*(\mathbf{y}|C)\end{aligned} \tag{D.4}$$

where Theorem D.8 explicitly refers to this result. If the set C is convex, then the support function yields

$$\sigma(\cdot|C) = \sigma(\cdot|\operatorname{cl} C) = \sigma(\cdot|\operatorname{rint} C)$$

despite the inclusion $\operatorname{rint} C \subset C \subset \operatorname{cl} C$. Moreover, by the separation theorems in Appendix B.3 we can show:

Theorem D.7[116] *Provided a convex set* $C \subset X$, *the following equivalence relations are valid.*

$$\begin{aligned}\mathbf{x} \in \operatorname{cl} C &\iff \left[\mathbf{y}^{\mathsf{T}}\mathbf{x} \leqq \sigma(\mathbf{y}|C) \quad \forall \mathbf{y} \in Y\right],\\ \mathbf{x} \in \operatorname{rint} C &\iff \left[\mathbf{y}^{\mathsf{T}}\mathbf{x} < \sigma(\mathbf{y}|C) \quad \forall \mathbf{y} \in Y \text{ with } \sigma(\mathbf{y}|C) \neq -\sigma(-\mathbf{y}|C)\right],\\ \mathbf{x} \in \operatorname{int} C &\iff \left[\mathbf{y}^{\mathsf{T}}\mathbf{x} < \sigma(\mathbf{y}|C) \quad \forall \mathbf{y} \neq \mathbf{0}\right].\end{aligned}$$

The following corollary is not really an application of Theorem D.7. It merely stresses the one-to-one relation between closed convex sets and their support functions.

Corollary D.7.1[117] *Let* $C \subset X$ *be a closed convex set. Then this set can be expressed as system of inequalities given by its support function* $\sigma(\cdot|C)$.

$$\begin{aligned}C &= \left\{\mathbf{x} \mid \mathbf{y}^{\mathsf{T}}\mathbf{x} \leqq \sigma(\mathbf{y}|C) \ \forall \mathbf{y} \in Y\right\}\\ &= \bigcap_{\mathbf{y}\in Y} \left\{\mathbf{x} \mid \mathbf{y}^{\mathsf{T}}\mathbf{x} \leqq \sigma(\mathbf{y}|C)\right\}\end{aligned} \tag{D.5}$$

[115] The linear homogeneity of the support function at once yields $\lambda\sigma(\mathbf{y}|C) = \sigma(\lambda\mathbf{y}|C) = \sigma(\mathbf{y}|\lambda C)$ for all $\lambda > 0$.

[116] Cf. ROCKAFELLAR (1972, p. 112 f., Theorem 13.1)

[117] Cf. ROCKAFELLAR (1972, p. 113) and ROBINSON (1987, p. 655).

Thus, each closed convex set $C \subset X$ is completely determined by its support function.

Proof: First of all, the system of inequalities given by the support function $\sigma(\mathbf{y}|C)$ is rewritten as follows:[118]

$$\begin{aligned} &\left\{\mathbf{x}|\ \mathbf{y}^{\mathrm{T}}\mathbf{x} \leqq \sigma(\mathbf{y}|C)\ \ \forall \mathbf{y} \in Y\right\} && \text{Theorem D.7} \\ = &\left\{\mathbf{x}|\ \sup\left\{\mathbf{y}^{\mathrm{T}}\mathbf{x} - \sigma(\mathbf{y}|C)|\ \mathbf{y} \in Y\right\} \leqq 0\right\} && \\ = &\left\{\mathbf{x}|\ \sigma^*(\mathbf{x}|C) \leqq 0\right\} && \text{Definition D.1} \end{aligned}$$

Considering (D.4), we obtain $\sigma^*(\cdot|C) = \delta^{**}(\cdot|C)$. If C is a convex set, then Theorem D.2 gives $\delta^{**}(\cdot|C) = \mathrm{cl}\,\delta(\cdot|C)$. Moreover, $\delta^{**}(\cdot|C) = \delta(\cdot|C)$ holds by Theorem D.3 for a closed convex set C. Thus, the system of inequalities represents the set C,

$$\left\{\mathbf{x}|\ \sigma^*(\mathbf{x}|C) \leqq 0\right\} = \{\mathbf{x}|\ \delta(\mathbf{x}|C) \leqq 0\} = C. \quad \blacksquare$$

If the set C is convex but not necessarily closed, the support function $\sigma(\cdot|C)$ furthermore implies

$$\sigma^*(\cdot|C) = \delta^{**}(\cdot|C) = \mathrm{cl}\,\delta(\cdot|C) = \delta(\cdot|\,\mathrm{cl}\,C).$$

Moreover, concerning two convex sets C and D the equation (D.5) at once yields[119]

$$\text{(D.6)} \qquad \mathrm{cl}\,C \subset \mathrm{cl}\,D \iff \sigma(\cdot|C) \leqq \sigma(\cdot|D),$$

since $\left\{\mathbf{x}|\ \mathbf{y}^{\mathrm{T}}\mathbf{x} \leqq \sigma(\mathbf{y}|C)\ \ \forall \mathbf{y} \in Y\right\} \subset \left\{\mathbf{x}|\ \mathbf{y}^{\mathrm{T}}\mathbf{x} \leqq \sigma(\mathbf{y}|D)\ \ \forall \mathbf{y} \in Y\right\}$.

Theorem D.8[120] *The indicator function and the support function of a closed convex set $C \subset X$ are convex conjugate to each other,*

$$\sigma(\cdot|C) = \delta^*(\cdot|C) \quad \textit{and} \quad \delta(\cdot|C) = \sigma^*(\cdot|C).$$

The support function of a nonempty set is proper, closed, and convex. Conversely, each proper closed convex function which is linearly homogeneous is the support function of a nonempty convex set.

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[118] Instead of a proof ROCKAFELLAR refers to Theorem B.11.

[119] Cf. ROCKAFELLAR (1972, p. 113, Corollary 13.1.1).
For two nonempty sets C and D we may summarize further properties of the support function as follows:

$$\sigma(\cdot|C+D) = \sigma(\cdot|C) + \sigma(\cdot|D) \quad \text{and} \quad \sigma(\cdot|C-D) = \sigma(\cdot|C) - \varphi(\cdot|D).$$

[120] Cf. ROCKAFELLAR (1972, p. 114, Theorem 13.2). The example in Figure D.4 illustrates this correlation. The switch-over to a reciprocal indicator function $\varrho(\cdot|C)$ and a reciprocal support function $\varphi(\cdot|C)$ is given by Proposition III.6, p. 132.

Proof: Apart from (D.4) only some comments are given. By (D.4) each support function $\sigma(\cdot|C)$ has to be closed and convex (Theorem D.2). Furthermore, the functions $\delta(\cdot|C)$ and $\sigma(\cdot|C)$ are proper if and only if $C \neq \emptyset$ (Theorem D.2).

The indicator function $\delta(\cdot|C)$ is closed and convex if and only if the set C is closed and convex. For $C = \emptyset$ we obtain $\delta(\cdot|\emptyset) = \sigma^*(\cdot|\emptyset) = +\infty$ and $\delta^*(\cdot|\emptyset) = \sigma(\cdot|\emptyset) = -\infty$. For $C \neq \emptyset$ Theorem D.3 yields $\sigma^*(\cdot|C) = \delta^{**}(\cdot|C) = \delta(\cdot|C)$. ■

Comment: A proper closed convex function f is an indicator function (of a nonempty set C) if and only if the conjugate function (i.e. the support function of the set C) is linearly homogeneous.

Given the linearly homogeneous and convex function $\sigma(\cdot|C)$, Theorem C.4 yields

$$\sigma(\mathbf{y}^1 + \mathbf{y}^2|C) \leqq \sigma(\mathbf{y}^1|C) + \sigma(\mathbf{y}^2|C) \qquad \forall\, \mathbf{y}^1, \mathbf{y}^2.$$

Corollary D.8.1[121] *Let f be a linearly homogeneous convex function with $f \not\equiv +\infty$. Then* cl f *is the support function of a certain closed convex set $C \subset Y$, namely*

$$C = \left\{\mathbf{y} |\, \mathbf{y}^{\mathsf{T}}\mathbf{x} \leqq f(\mathbf{x}) \;\; \forall\, \mathbf{x} \in X\right\}.$$

Proof: If $\operatorname{cl} f \equiv -\infty$,[122] then $\operatorname{cl} f$ is the support function of the empty set. Otherwise, $\operatorname{cl} f$ is a proper closed convex function which is linearly homogeneous. Theorem D.8 says that the convex conjugate function $f^* = (\operatorname{cl} f)^*$ (Theorem D.2) is an indicator function, namely of

$$C = \left\{\mathbf{y} |\, f^*(\mathbf{y}) \leqq 0\right\} = \left\{\mathbf{y} |\, \mathbf{y}^{\mathsf{T}}\mathbf{x} - f(\mathbf{x}) \leqq 0 \;\; \forall\, \mathbf{x} \in X\right\}.$$

This set is closed and convex because its indicator function $\delta(\cdot|C) = f^*$ is closed and convex. The following rearrangement completes the proof.

$$\begin{aligned} \sigma(\cdot|C) &= \delta^*(\cdot|C) && \text{Theorem D.8} \\ &= f^{**} && \text{because of } \delta(\cdot|C) = f^* \\ &= \operatorname{cl} f && \text{Theorem D.2} \qquad ■ \end{aligned}$$

If the function f, mentioned in Corollary D.8.1, is not linearly homogeneous, then, however, the statement of the corollary is not affected when f is replaced with a certain function, which is important for the following sections. Given a convex function f with the epigraph

$$\operatorname{epi} f = \left\{\tbinom{\mathbf{x}}{\mu} \in \mathbb{R}^n \times \mathbb{R} |\, \mu \geqq f(\mathbf{x})\right\},$$

[121]Cf. ROCKAFELLAR (1972, p. 114, Corollary 13.2.1).
The transference to a linearly homogeneous concave function g with $g \not\equiv -\infty$ causes no problems. Thus, cl g may be called the reciprocal support function of the closed convex set

$$D = \{\mathbf{y} |\, g_*(\mathbf{y}) \geqq 0\} = \{\mathbf{y} |\, \mathbf{y}^{\mathsf{T}}\mathbf{x} \geqq g(\mathbf{x}) \;\; \forall\, \mathbf{x} \in X\}.$$

In this case we obtain $\operatorname{cl} g = \varphi(\cdot|D) = \varrho(\cdot|D)$.

[122]This case must explicitly be considered according to Definition C.4.

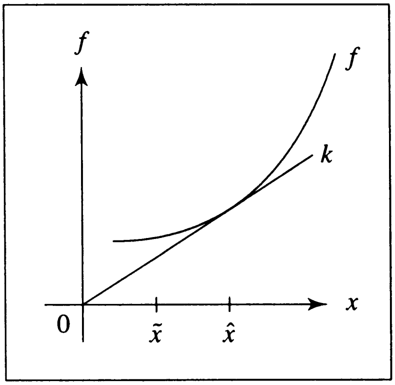

Figure D.2: Greatest linearly homogeneous convex function k generated by f.

the smallest convex cone K generated by this epigraph and containing the origin $\mathbf{0}$ is

$$K := \left\{\lambda\boldsymbol{\xi} \mid \lambda \geqq 0,\ \boldsymbol{\xi} \in \operatorname{epi} f\right\}.$$

If we set

$$k(\mathbf{x}) = \inf\left\{\mu \mid \binom{\mathbf{x}}{\mu} \in K\right\}$$

analogous to (III.2), p. 85, then k is the greatest of all linearly homogeneous convex functions h with $h(\mathbf{0}) \leqq 0$ and $h \leqq f$. Thus, the function k is called the **greatest linearly homogeneous convex function generated by** f;[123] see the figure opposite. By recursive substitution we get

$$k(\mathbf{x}) = \inf\left\{\mu \mid \binom{\mathbf{x}}{\mu} \in \{\lambda\boldsymbol{\xi} \mid \lambda \geqq 0,\ \boldsymbol{\xi} \in \operatorname{epi} f\}\right\}.$$

For $\lambda = 0$ we have to distinguish between two cases where $\operatorname{epi} f \neq \emptyset$ is assumed. Using $k_1(\mathbf{x}) := \inf\left\{\mu \mid \binom{\mathbf{x}}{\mu} \in \mathbf{0}\right\}$ it ensues

(D.7a)
$$k_1(\mathbf{x}) = \delta(\mathbf{x}|\mathbf{0}) := \begin{cases} 0 & \text{for } \mathbf{x} = \mathbf{0} \\ +\infty & \text{for } \mathbf{x} \neq \mathbf{0}. \end{cases}$$

If $\lambda > 0$, then

(D.7b)
$$\begin{aligned} k_2(\mathbf{x}) &:= \inf\left\{\mu \mid \binom{\mathbf{x}}{\mu} \in \left\{\lambda\binom{\tilde{\mathbf{x}}}{\tilde{\mu}} \mid \lambda > 0,\ \tilde{\mu} \geqq f(\tilde{\mathbf{x}})\right\}\right\} \\ &= \inf\{\mu \mid \lambda > 0,\ \mu/\lambda \geqq f(\mathbf{x}/\lambda)\} \\ &= \inf\{\lambda f(\mathbf{x}/\lambda) \mid \lambda > 0\}. \end{aligned}$$

where $\mathbf{x} = \lambda\tilde{\mathbf{x}}$ and $\mu = \lambda\tilde{\mu}$. Finally, using $k(\mathbf{x}) = \inf\{k_1(\mathbf{x}), k_2(\mathbf{x})\}$ (D.7a) and (D.7b) yield the greatest linearly homogeneous convex function generated by $f \not\equiv +\infty$, i.e.

$$k(\mathbf{x}) = \inf_{\lambda \geqq 0} \left\{\begin{matrix} \lambda f(\mathbf{x}/\lambda) & \text{for } \lambda > 0 \\ \delta(\mathbf{x}|\mathbf{0}) & \text{for } \lambda = 0 \end{matrix}\right\}.$$

An alternative representation of the function k results from the convention introduced in (C.3a) and (C.3b).

(D.8)
$$k(\mathbf{x}) = \inf\{(f\lambda)(\mathbf{x}) \mid \lambda \geqq 0\}$$

In this $\lambda = 0$ can be ignored in the infimum if $\mathbf{x} \neq \mathbf{0}$ or $f(\mathbf{0}) < +\infty$ is fulfilled. The resulting function k is illustrated in the right hand part of Figure C.4,

[123] Cf. ROCKAFELLAR (1972, p. 35).
Note that a linearly homogeneous function h with $h > -\infty$ can be convex at the most for $h(\mathbf{0}) \geqq 0$; see Theorem C.4. Conversely, the function k with $k(\mathbf{0}) = 0$ can only be called the greatest of all linearly homogeneous convex functions h with $h \leqq f$, when we require $h(\mathbf{0}) \leqq 0$.
If $f(\mathbf{0}) < 0$, then we put $k(\mathbf{0}) = -\infty$. For instance, we obtain $k \equiv -\infty$ for $f(x) = x^2 - 1$.

p. 305. There k corresponds to the flatter ray through the origin analogous to Figure D.2.

In particular, a linearly homogeneous function f yields $k(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{0}$. Provided $f \not\equiv +\infty$, we have $k(\mathbf{0}) = 0$ at point $\mathbf{x} = \mathbf{0}$ by definition. Figure D.2 results in $k(\hat{x}) = f(\hat{x})$. For $2\tilde{x} = \hat{x}$ we have $k(\tilde{x}) = 1/2 f(2\tilde{x}) = 1/2 f(\hat{x})$ and $x = 0$ yields $k(0) = 0$.

The presented procedure will be of major importance for two applications. While Theorem D.12 goes into the relationship between the gauge and the indicator function of a set, the following remarks serve as a preparation for Corollary D.9.1. This corollary describes the relationship between a function f and the support function of the set hypo$(-f^*)$.

Given a proper convex function $f\colon \mathbb{R}^n \to]-\infty, +\infty]$,

$$\text{(D.9)} \qquad \mathfrak{k}(\mathbf{x}, \lambda) = \begin{cases} (f\lambda)(\mathbf{x}) & \text{for } \lambda \geqq 0 \\ +\infty & \text{for } \lambda < 0 \end{cases}$$

denotes the greatest linearly homogeneous convex function $\mathfrak{k}$ generated by[124]

$$\mathfrak{f}(\mathbf{x}, \lambda) = f(\mathbf{x}) + \delta(\lambda|1) = \begin{cases} f(\mathbf{x}) & \text{for } \lambda = 1 \\ +\infty & \text{for } \lambda \neq 1\,. \end{cases}$$

Because firstly Theorem D.9 and implicitly Corollary D.9.1 refer to the closure of the function $\mathfrak{k}$ and secondly the determination of the closure $\mathrm{cl}\,\mathfrak{k}$ requires a more extensive analytical framework, the function $\mathrm{cl}\,\mathfrak{k}$ is only established for a special case, which is enough for the presented purposes.[125] If f is a proper closed convex function with $\mathbf{0} \in \mathrm{Dom}\, f$, then

$$\mathrm{cl}\,\mathfrak{k}(\mathbf{x}, \lambda) := \begin{cases} (f\lambda)(\mathbf{x}) & \text{for } \lambda > 0 \\ \lim\limits_{\lambda \downarrow 0} (f\lambda)(\mathbf{x}) & \text{for } \lambda = 0 \\ +\infty & \text{for } \lambda < 0\,. \end{cases}$$

Theorem D.9[126] *Let f be a proper closed convex function. If k_1 is the greatest linearly homogeneous convex function generated by f^*, then $\mathrm{cl}\,k_1$ is the support function of the set $\{\mathbf{x}|\ f(\mathbf{x}) \leqq 0\}$. If, dually, k_2 is the greatest linearly homogeneous convex function generated by f, then $\mathrm{cl}\,k_2$ is the support function of $\{\mathbf{y}|\ f^*(\mathbf{y}) \leqq 0\}$.*

[124]Applying (D.8) to $\mathfrak{f}$, remember that $(\mathfrak{f}\lambda)(\mathbf{x}, \lambda) = \lambda \mathfrak{f}(\mathbf{x}/\lambda, 1) = \lambda f(\mathbf{x}/\lambda) = (f\lambda)(\mathbf{x})$.

[125]Cf. ROCKAFELLAR (1972, p. 66 f.), in particular Corollary 8.5.2.

[126]Cf. ROCKAFELLAR (1972, p. 118, Theorem 13.5 and p. 67) or JORGENSON, LAU (1974, p. 193). For the transference of this theorem to a concave function g it must be taken into account, that we now have to apply the smallest linearly homogeneous concave function generated by g; see Section III.2.1.3. The proof of the next theorem ensues analogous to Theorem D.9 with opposite signs.

Concave Version of Theorem D.9 *Let g be an n-proper closed concave function. If k_1 is the smallest linearly homogeneous concave function generated by g_*, then $\mathrm{cl}\,k_1$ is the reciprocal support function of the set $\{\mathbf{x}|\ g(\mathbf{x}) \geqq 0\}$. If, dually, k_2 is the smallest linearly homogeneous concave function generated by g, then $\mathrm{cl}\,k_2$ is the support function of the set $\{\mathbf{y}|\ g_*(\mathbf{y}) \geqq 0\}$.*

Proof: If we can show that the subsequent sets C and D are equal, then the proof is complete because of the symmetry between f and f^*.
By Corollary D.8.1 follows that $\operatorname{cl} k_2$ is the support function of the set

$$C := \{\mathbf{y} \mid \mathbf{y}^\mathsf{T}\mathbf{x} \leqq k_2(\mathbf{x}) \;\; \forall \mathbf{x} \in X\} = \bigcap_{\mathbf{x}\in X} \{\mathbf{y} \mid \mathbf{y}^\mathsf{T}\mathbf{x} \leqq k_2(\mathbf{x})\}.$$

In accordance with the theorem this set faces the set D.

$$\begin{aligned} D &:= \{\mathbf{y} \mid f^*(\mathbf{y}) \leqq 0\} \\ &= \{\mathbf{y} \mid \mathbf{y}^\mathsf{T}\mathbf{x} \leqq f(\mathbf{x}) \;\; \forall \mathbf{x} \in X\} = \bigcap_{\mathbf{x}\in X} \{\mathbf{y} \mid \mathbf{y}^\mathsf{T}\mathbf{x} \leqq f(\mathbf{x})\} \end{aligned}$$

Because of $k_2 \leqq f$, we have $\{\mathbf{y} \mid \mathbf{y}^\mathsf{T}\mathbf{x} \leqq k_2(\mathbf{x})\} \subset \{\mathbf{y} \mid \mathbf{y}^\mathsf{T}\mathbf{x} \leqq f(\mathbf{x})\}$ and, therefore, $C \subset D$.
It remains to be shown, that $\tilde{\mathbf{y}} \in D$ and $\tilde{\mathbf{y}} \notin C$ imply a contradiction. Defining the linearly homogeneous function $h(\mathbf{x}) := \tilde{\mathbf{y}}^\mathsf{T}\mathbf{x}$, the sets D and C induce

$$\left.\begin{aligned} \forall \mathbf{x} \in X &: \; h(\mathbf{x}) \leqq f(\mathbf{x}) \\ \exists \tilde{\mathbf{x}} \in X &: \; h(\tilde{\mathbf{x}}) > k_2(\tilde{\mathbf{x}}) \end{aligned}\right\} \Longrightarrow k_2(\tilde{\mathbf{x}}) < h(\tilde{\mathbf{x}}) \leqq f(\tilde{\mathbf{x}}).$$

Contrary to its definition, k_2 cannot be the greatest of all linearly homogeneous functions h with $h(\mathbf{0}) \leqq 0$ and $h \leqq f$. ∎

The meaning of Theorem D.9 as link between the theory of conjugate functions and the theory of support functions is expressed by the next corollary. Moreover, Theorem D.9 will later serves as the link to the analysis of gauges.

Corollary D.9.1[127] *Given a closed proper convex function* $f\colon X \to [-\infty, +\infty]$ *with* $\mathbf{0} \in \operatorname{Dom} f$. *Then the function* $k\colon \mathbb{R}^{n+1} \to [-\infty, +\infty]$ *with*

$$k(\mathbf{x}, \lambda) := \begin{cases} (f\lambda)(\mathbf{x}) & \textit{for } \lambda > 0 \\ \lim\limits_{\lambda \downarrow 0} (f\lambda)(\mathbf{x}) & \textit{for } \lambda = 0 \\ +\infty & \textit{for } \lambda < 0\,. \end{cases}$$

is the support function of the set $\left\{\binom{\mathbf{y}}{\mu} \mid \mu \leqq -f^*(\mathbf{y})\right\} = \operatorname{hypo}(-f^*) \subset \mathbb{R}^{n+1}$.

Proof: By (D.9) the function $k(\mathbf{x}, \lambda)$ is the closure of the greatest linearly homogeneous function generated by $\mathsf{f}(\mathbf{x}, \lambda) = f(\mathbf{x}) + \delta(\lambda \mid 1)$. Thus, according to Theorem D.9 $k(\mathbf{x}, \lambda)$ is the support function of the set $\left\{\binom{\mathbf{y}}{\mu} \mid \mathsf{f}^*(\mathbf{y}, \mu) \leqq 0\right\}$. Consider

[127] Cf. ROCKAFELLAR (1972, p. 119, Corollary 13.5.1), though he uses the recession function $f0^+$. Without defining the function $f0^+$, $\mathbf{x} \in \operatorname{Dom} f$ guarantees by ROCKAFELLAR (1972, p. 67, Corollary 8.5.2) that $(f0^+)(\mathbf{x}) = \lim\limits_{\lambda \downarrow 0} \lambda f(\mathbf{x}/\lambda)$.

now

$$\begin{aligned} \mathfrak{f}^*(\mathbf{y}, \mu) &= \sup\left\{\mathbf{y}^{\mathsf{T}}\mathbf{x} + \mu\lambda - \mathfrak{f}(\mathbf{x}, \lambda) \mid \binom{\mathbf{x}}{\lambda} \in \mathbb{R}^{n+1}\right\} \\ &= \sup\left\{\mathbf{y}^{\mathsf{T}}\mathbf{x} + \mu\lambda - f(\mathbf{x}) - \delta(\lambda|1) \mid \binom{\mathbf{x}}{\lambda} \in \mathbb{R}^{n+1}\right\} \\ &= \sup\left\{\mathbf{y}^{\mathsf{T}}\mathbf{x} + \mu - f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^{n}\right\} \\ &= \mu + f^*(\mathbf{y}). \quad \blacksquare \end{aligned}$$

D.2.2 Properties of the Gauge

Definition D.4[128] *The* ***gauge*** $\gamma(\cdot|C)\colon X \to [0, +\infty]$ *of a nonempty set* $C \subset X$ *is defined by*

$$\gamma(\mathbf{x}|C) := \inf\{\lambda \geqq 0 \mid \mathbf{x} \in \lambda C\}.$$

However, $\psi(\cdot|C)\colon X \to [0, +\infty] \cup \{-\infty\}$ *with*

$$\psi(\mathbf{x}|C) := \sup\{\lambda \geqq 0 \mid \mathbf{x} \in \lambda C\}$$

is called the ***reciprocal gauge***.[129]

Theorem D.10[130] *Let C be a nonempty star-shaped subsets in X, then the gauge* $\gamma(\cdot|C)$ *has the following properties.*

1. *The function* $\gamma(\cdot|C)$ *is proper,* $0 \leqq \gamma(\cdot|C) \leqq +\infty \quad \forall \mathbf{x} \in X$.
2. *The function* $\gamma(\cdot|C)$ *is linearly homogeneous,* $\lambda\gamma(\mathbf{x}|C) = \gamma(\lambda\mathbf{x}|C) \quad \forall \lambda > 0$.
3. *We have* $\gamma(\mathbf{x}|C) = 0$ *if and only if* $\lambda\mathbf{x} \in C$ *for all* $\lambda > 0$.
4. *If C is closed, then* $\gamma(\cdot|C)$ *is lower semi-continuous and*
$$\gamma(\mathbf{x}|C) \leqq 1 \iff \mathbf{x} \in C.$$
5. *Provided C is convex, then* $\gamma(\cdot|C)$ *is convex, too.*

The statements of the theorem remain the same for each set $C \subset X$ if we supersede the inspected set C by its star hull star C:

$$\gamma(\cdot|\operatorname{star} C) = \gamma(\cdot|C)$$

Three more remarks are useful. First, if $\gamma(\cdot|C)$ is proper and lower semi-continuous, then $\gamma(\cdot|C)$ is closed by Definition C.4, p. 302.[131] Second, the

[128] The genuine German term *Distanzfunktion* was introduced by MINKOWSKI. Hence, gauges are frequently called MINKOWSKI functions.

[129] Note $\sup \emptyset := -\infty$ for the unusual range.
In honor of SHEPHARD, who introduced this function 1953 into economics, some authors speak of s-gauges; cf. e.g. NEWMAN (1987c, p. 484).

[130] Cf. PHELPS (1963, p. 394). For the definition of star-shaped sets see (II.15), p. 28.

[131] A similar result is noted by Theorem D.13.

linearly homogeneous gauge is convex by Theorem C.4, p. 304, if and only if it is subadditive.

$$\gamma(\mathbf{x}^1+\mathbf{x}^2|C) \leqq \gamma(\mathbf{x}^1|C)+\gamma(\mathbf{x}^2|C) \qquad \forall\, \mathbf{x}^1, \mathbf{x}^2 \in X$$

Third, the relation $\mathbf{0} \in \operatorname{int} C$ induces a unique expression for the boundary ∂C of the closed set C.

$$\partial C = \{\mathbf{x} |\ \gamma(\mathbf{x}|C) = 1\}$$

Theorem D.11[132] *Let C be a nonempty subset in X with $\mathbf{0} \notin \operatorname{cl} C$. Then the reciprocal gauge $\psi(\cdot|C)$ has the following properties.*

1. *The function $\psi(\cdot|C)$ is n-proper,* $-\infty \leqq \psi(\cdot|C) < +\infty \quad \forall\, \mathbf{x} \in X$.
2. *The function $\psi(\cdot|C)$ is linearly homogeneous,* $\lambda\psi(\mathbf{x}|C) = \psi(\lambda\mathbf{x}|C) \quad \forall\, \lambda > 0$.
   Furthermore, we have $\psi(\cdot|C) = \lambda\psi(\cdot|\lambda C) \quad \forall\, \lambda > 0$.
3. *We have* $\psi(\mathbf{x}|C) = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$ *and, similarly,* $\psi(\mathbf{x}|C) > 0$ *if and only if* $\mathbf{x} \in \operatorname{cone} C \setminus \{\mathbf{0}\}$.
   Thus, the effective domain is $\text{n-Dom}\, \psi(\cdot|C) = \operatorname{cone} C$.
4. *If C is closed and* $\psi(\mathbf{x}|C) \geqq 0$, *then the supremum is achieved, i.e.* $\psi(\mathbf{x}|C) = \max\{\lambda \geqq 0 |\ \mathbf{x} \in \lambda C\}$.
5. *If C is closed, then $\psi(\cdot|C)$ is lower semi-continuous at every point $\mathbf{x}$ with* $\psi(\mathbf{x}|C) \geqq 0$, *i.e. in* $\operatorname{cone} C$.
6. *For convex C the function $\psi(\cdot|C)$ is concave and, therefore, superadditive.*

   $$\psi(\mathbf{x}^1+\mathbf{x}^2|C) \geqq \psi(\mathbf{x}^1|C)+\psi(\mathbf{x}^2|C) \qquad \forall\, \mathbf{x}^1, \mathbf{x}^2 \in X$$

7. *Given* $\operatorname{int} C \neq \emptyset$, *then $\psi(\cdot|C)$ is continuous on* $\operatorname{cone}(\operatorname{int} C)$.
8. *The star-shaped set* $C' = \mathbb{R}^n \setminus \operatorname{aur} C$ *yields* $\gamma(\mathbf{x}|C') = \psi(\mathbf{x}|C)$ *for all* $\mathbf{x} \in X$ *with* $\psi(\mathbf{x}|C) > -\infty$. *Provided* $\mathbf{x} \neq \mathbf{0}$, *then* $\gamma(\mathbf{x}|C') = 0$ *if and only if* $\psi(\mathbf{x}|C) = -\infty$.

Again the statements of the theorem remain unchanged for each set $C \subset X$ with $\mathbf{0} \notin \operatorname{cl} C$ if the inspected set is superseded by its aureoled hull $\operatorname{aur} C$:[133]

$$\psi(\cdot|\operatorname{aur} C) = \psi(\cdot|C)$$

The next theorem notes the correlation between a gauge of a set and its indicator function.

Theorem D.12 *The gauge $\gamma(\cdot|C)$ of a nonempty convex set $C \subset X$ is the greatest linearly homogeneous convex function generated by the modified indicator function* $\delta(\cdot|C)+1$.

[132] Cf. PHELPS (1963, p. 398).

[133] See (II.16) on p. 28 for aureoled sets. Note that a closed set C implies $\operatorname{aur} C$ to be closed.

Proof: According to (D.8) the greatest linearly homogeneous convex function generated by $\delta(\cdot|C) + 1$ is

$$k(\mathbf{x}) = \inf\{(\delta\lambda)(\mathbf{x}|C) + \lambda |\ \lambda \geqq 0\}.$$

Considering

$$\begin{aligned} (\delta\lambda)(\mathbf{x}|C) &= \lambda\delta(\mathbf{x}/\lambda|C) = \delta(\mathbf{x}/\lambda|C) = \delta(\mathbf{x}|\lambda C) && \text{for } \lambda > 0 \\ (\delta 0)(\mathbf{x}|C) &= \delta(\mathbf{x}|\mathbf{0}) && \text{for } \lambda = 0, \end{aligned}$$

it ensues (because of $\delta(\mathbf{x}|\lambda C) = \delta(\mathbf{x}|\mathbf{0})$ for $\lambda = 0$)

$$\begin{aligned} k(\mathbf{x}) &= \inf\{\delta(\mathbf{x}|\lambda C) + \lambda |\ \lambda \geqq 0\} \\ &= \inf\{\lambda |\ \lambda \geqq 0,\ \delta(\mathbf{x}|\lambda C) = 0\} \qquad \text{by } \delta(\cdot|C) \not\equiv +\infty \\ &= \inf\{\lambda \geqq 0 |\ \mathbf{x} \in \lambda C\} \end{aligned}$$

and, therefore, $k = \gamma(\cdot|C)$. In particular, $k(\mathbf{0}) = 0$ is valid. Note that the convexity of the set C implies a convex indicator function $\delta(\cdot|C)$. Moreover, the right scalar multiplication transfers the convexity to $(\delta\lambda)(\cdot|C)$ and, therefore, to k.
■

Theorem D.13[134] *Let* $C \subset X$ *be a closed convex set containing the origin* $\mathbf{0}$. *Then the gauge is closed,* $\gamma(\cdot|C) = \operatorname{cl}\gamma(\cdot|C)$, *moreover,*

(D.10a) $\qquad \{\mathbf{x}|\ \gamma(\mathbf{x}|C) \leqq \lambda\} = \lambda C \qquad \forall\lambda > 0$

(D.10b) $\qquad \{\mathbf{x}|\ \gamma(\mathbf{x}|C) = 0\} = 0^+C.$

Based on the results of Theorems D.12 and D.13, we can justify to call each function $k\colon X \to [-\infty, +\infty]$ a gauge, which is nonnegative, linearly homogeneous, and convex and which takes the functional value $k(\mathbf{0}) = 0$. [135] The epigraph epi$k$ of such a function is then a convex cone in $\mathbb{R}^{n+1}$ with vertex $\mathbf{0}$, containing no point $\binom{\mathbf{x}}{\lambda}$ with $\lambda < 0$. Therefore, the class of gauges is given by the set of all functions k with

$$k(\mathbf{x}) = \gamma(\mathbf{x}|C) = \inf\{\lambda \geqq 0 |\ \mathbf{x} \in \lambda C\},$$

where C is an arbitrary nonempty convex subset in X. This statement is illustrated in the left hand part of Figure D.4, p. 337. Note that the set C contains the origin, $0 \in C$. Accordingly, the illustration has to be modified if $0 \notin C$. Although the set C is in general not uniquely determined by k, we gain a set $C = \{\mathbf{x}|\ k(\mathbf{x}) \leqq 1\}$ by assuming a gauge k according to (D.10a) such that $\gamma(\cdot|C) = k$ results from the reverse conclusion.

[134] Cf. ROCKAFELLAR (1972, p. 79, Corollary 9.7.1). The set 0^+C is called recession cone of C and defined by (II.11), p. 27. Comparing to Theorem D.10 it has to be noted that each convex set containing the origin is star-shaped.

[135] CASSELS (1971) calls each nonnegative continuous function k with $\lambda k(\mathbf{x}) = k(\lambda\mathbf{x})$ for all $\lambda \geqq 0$ a gauge.

D.2.3 Polar Sets and Functions

Definition D.5[136] *Given a nonempty set* $C \subset X$. *The* ***polar set*** $C^\circ \subset Y$ *of* C *is defined by*

$$C^\circ := \left\{\mathbf{y} \mid \mathbf{y}^\mathsf{T}\mathbf{x} \leqq 1 \;\; \forall \mathbf{x} \in C\right\}.$$

The ***reciprocally polar set*** $C_\circ \subset Y$ *of* C *is given by*

$$C_\circ := \left\{\mathbf{y} \mid \mathbf{y}^\mathsf{T}\mathbf{x} \geqq 1 \;\; \forall \mathbf{x} \in C\right\}.$$

The respective ***bipolar sets*** *follow from* $C^{\circ\circ} = (C^\circ)^\circ$ *and* $C_{\circ\circ} = (C_\circ)_\circ$.

Both polar sets and reciprocally polar sets are closed and convex since they are defined as intersection of closed convex sets. The polar set of a nonempty set C always contains the origin $\mathbf{0} \in C^\circ$. In particular, if C consists of the origin as singleton, then

$$C = \{\mathbf{0}\} \implies C^\circ = Y \implies C^{\circ\circ} = \{\mathbf{0}\}.$$

However, the reciprocally polar set of a nonempty set C never contains the origin, $\mathbf{0} \notin C_\circ$. Moreover, reciprocally polar sets satisfy the following equivalence relation[137]

$$\text{(D.11)} \qquad \mathbf{0} \in \mathrm{cl}(\mathrm{conv}C) \iff C_\circ = \emptyset.$$

In view of the support functions defined by Definition D.3 we get an alternative representation of polar sets

$$\text{(D.12a)} \qquad C^\circ = \{\mathbf{y} \mid \sigma(\mathbf{y} \mid C) \leqq 1\},$$

$$\text{(D.12b)} \qquad C_\circ = \{\mathbf{y} \mid \varphi(\mathbf{y} \mid C) \geqq 1\}.$$

Because of $\sigma(\cdot \mid \emptyset) \equiv -\infty$ and $\varphi(\cdot \mid \emptyset) \equiv +\infty$, now the empty set may be taken into account.

$$\text{(D.13a)} \qquad C = \emptyset \implies C^\circ = Y \implies C^{\circ\circ} = \{\mathbf{0}\}$$

$$\text{(D.13b)} \qquad C = \emptyset \implies C_\circ = Y \implies C_{\circ\circ} = \emptyset$$

We obtain the following relations for each nonempty convex set $C \subset X$. The closure of the support functions, given by Definition D.4, satisfies

$$\text{(D.14a)} \qquad \mathrm{cl}\,\gamma(\cdot \mid C) = \sigma(\cdot \mid C^\circ)$$

$$\text{(D.14b)} \qquad \mathrm{cl}\,\psi(\cdot \mid C) = \varphi(\cdot \mid C_\circ) \qquad \text{if } C_\circ \neq \emptyset.$$

---

[136] Cf. ROCKAFELLAR (1972, p. 125) and RUYS, WEDDEPOHL (1979, p. 50).
We have to distinguish polar sets from polar cones, $K^\circ := \left\{\mathbf{y} \mid \mathbf{y}^\mathsf{T}\mathbf{x} \leqq 0 \;\; \forall \mathbf{x} \in K\right\}$.

[137] Cf. NEWMAN (1987b, p. 486, Theorem 2).
This statement can be illustrated by expressing the origin as a convex combination of points in C.

The proof of the first equation is omitted at this point. But we get an idea of it[138] by applying Theorem D.9 to the modified indicator function $f = \delta(\cdot|C) + 1$. The equivalent proof of (D.14b) yields Proposition III.15 by using the concave version of Theorem D.9.

The subsequent properties are valid for polar and reciprocally polar sets.[139]

$C^\circ = (\mathrm{rint}\, C)^\circ = (\mathrm{cl}\, C)^\circ = (\mathrm{conv} C)^\circ,$	$C_\circ = (\mathrm{rint}\, C)_\circ = (\mathrm{cl}\, C)_\circ = (\mathrm{conv} C)_\circ,$
$C^\circ = (\mathrm{star}\, C)^\circ = \mathrm{star}\, C^\circ,$	$C_\circ = (\mathrm{aur}\, C)_\circ = \mathrm{aur}\, C_\circ,$
$\mathbf{y} \in \partial C^\circ \iff H(\mathbf{y}, 1)$ supports $C,$	$\mathbf{y} \in \partial C_\circ \iff H(\mathbf{y}, 1)$ supports $C,$
$C \subset D \implies C^\circ \supseteq D^\circ,\ C^{\circ\circ} \subset D^{\circ\circ},$	$C \subset D \implies C_\circ \supseteq D_\circ,\ C_{\circ\circ} \subset D_{\circ\circ},$
$(\lambda C)^\circ = \lambda^{-1} C^\circ \ \ \forall \lambda \in \mathbb{R}_{++},$	$(\lambda C)_\circ = \lambda^{-1} C_\circ \ \ \forall \lambda \in \mathbb{R}_{++},$
$(C \cup D)^\circ = C^\circ \cap D^\circ,$	$(C \cup D)_\circ = C_\circ \cap D_\circ,$
$(C \cap D)^\circ = \mathrm{conv}(C^\circ \cap D^\circ),$	$(C \cap D)_\circ = \mathrm{conv}(C_\circ \cap D_\circ),$
$\mathbf{0} \in \mathrm{int}(\mathrm{conv} C) \iff C^\circ$ is bounded,	$\mathbf{0} \in \mathrm{cl}(\mathrm{conv} C) \iff C_\circ = \emptyset,$

Theorem D.14 (Bipolar Theorem)[140] *For each set $C \subset X$ the following statements are true:*

1. The polar set of $C^\circ \subset Y$ is

$$C^{\circ\circ} = \mathrm{cl}\left(\mathrm{conv}(C \cup \{\mathbf{0}\})\right) = \mathrm{cl}\left(\mathrm{conv}(\mathrm{star}\, C)\right) \subset X .$$

2. If $\mathbf{0} \notin \mathrm{cl}(\mathrm{conv} C)$, then the reciprocally polar set of $C_\circ \subset Y$ is given by

$$C_{\circ\circ} = \mathrm{cl}\left(\mathrm{conv}(\mathrm{aur}\, C)\right) \subset X .$$

In the presented theory those cases are of special interest in which the bipolar sets equal the original sets,[141] i.e. the polar sets C° and $C_\circ$ represent C without loss of information. The main outcomes are summarized in the following

Corollary D.14.1 *Theorem D.14 at once implies two statements.*[142]

1. *For the bipolar set $C^{\circ\circ}$ of a closed convex set $C \subset X$ the equation $C^{\circ\circ} = C$ holds if and only if C contains the origin $\mathbf{0} \in C$. In particular, $C^\circ = C^{\circ\circ\circ}$.*

2. *The reciprocally bipolar set $C_{\circ\circ}$ of a convex closed set $C \subset X$ fulfills $C_{\circ\circ} = C$ if and only if C does not contain the origin, $\mathbf{0} \notin C$, and if $C = \mathrm{aur}\, C$.*

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[138]Cf. ROCKAFELLAR (1972), p. 125.

[139]Cf. RUYS, WEDDEPOHL (1979, p. 52 f.).
The hyperplane H is defined by $H(\mathbf{y}, 1) = \{\mathbf{x} |\ \mathbf{x}^T\mathbf{y} = 1\}$; see Appendix B.

[140]Cf. NEWMAN (1987c, p. 485).

[141]In all cases $C \subset C^{\circ\circ}$ and $C \subset C_{\circ\circ}$ hold good (Theorem D.14). Remember $C_{\circ\circ} = X$ if $\mathbf{0} \in \mathrm{cl}(\mathrm{conv} C)$ or $C_\circ = \emptyset$, see (D.13b).

[142]An alternative proof may be found in WEDDEPOHL (1972, p. 170).

Proof: Under the assumptions of the first statement we have $C = \operatorname{cl}\big(\operatorname{conv}(C \cup \{\mathbf{0}\})\big)$. Thus, Theorem D.14 yields $C^{\circ\circ} = C$. The proof of the second statement ensues analogously. ■

According to the first statement of the corollary, the relations $\mathbf{0} \in C$ and $C = \operatorname{star} C$ are equivalent for a convex set C, see the first part in Theorem D.14. On the basis of the second statement of the corollary the determination of $C_{\circ\circ}$ becomes superfluous if the set C contains the origin since then $C_\circ = \emptyset$.

Theorem D.15 *For an arbitrary set* $C \subset X$ *we obtain*

$$\gamma(\cdot\,|C^\circ) = \sigma(\cdot\,|C^{\circ\circ}) \quad \textit{and} \quad \gamma(\cdot\,|C^{\circ\circ}) = \sigma(\cdot\,|C^\circ).$$

Proof: Theorem D.13 implies $\operatorname{cl}\gamma(\cdot\,|C^\circ) = \gamma(\cdot\,|C^\circ)$. Hence, (D.14a) yields $\gamma(\cdot\,|C^\circ) = \sigma(\cdot\,|C^{\circ\circ})$. Analogously, Theorem D.13 gives $\operatorname{cl}\gamma(\cdot\,|C^{\circ\circ}) = \gamma(\cdot\,|C^{\circ\circ})$. Again, $\gamma(\cdot\,|C^{\circ\circ}) = \sigma(\cdot\,|C^{\circ\circ\circ}) = \sigma(\cdot\,|C^\circ)$ is implied by (D.14a). ■

Theorem D.15 is even valid for $C = \emptyset$ as (D.13a) implies $\gamma(\cdot\,|Y) = 0 = \sigma(\cdot\,|\{\mathbf{0}\})$.

The issue in Theorem D.15 does not surprise because of

$$\begin{aligned} C^{\circ\circ} &= \{\mathbf{x}|\ \sigma(\mathbf{x}|C^\circ) \leqq 1\} && \text{by (D.12a)} \\ &= \{\mathbf{x}|\ \operatorname{cl}\gamma(\mathbf{x}|C) \leqq 1\} && \text{by (D.14a)} \\ C^{\circ\circ} &= \{\mathbf{x}|\ \gamma(\mathbf{x}|C^{\circ\circ}) \leqq 1\} && \text{by (D.10a)} \end{aligned}$$

Corollary D.15.1[143] *If C is a closed convex subset in X containing the origin* $\mathbf{0}$, *then the support function of C is at the same time the gauge of C°,*

$$\sigma(\cdot\,|C) = \gamma(\cdot\,|C^\circ).$$

In dual view the support function of C° is at the same time the gauge of C,

$$\sigma(\cdot\,|C^\circ) = \gamma(\cdot\,|C).$$

Proof: From Corollary D.14.1 we obtain $C = C^{\circ\circ}$. ■

With the results of this corollary it is now inquired to what extent the functions $\gamma(\cdot\,|C)$ and $\sigma(\cdot\,|C)$ are related to each other. For this reason each gauge k is compared to a nonnegative function k° satisfying an optimality criterion according to the next definition.

Definition D.6 *Provided* $k\colon X \to [-\infty, +\infty]$ *is a gauge, then the* ***polar gauge*** $k^\circ\colon Y \to [-\infty, +\infty]$ *is defined by*

$$k^\circ(\mathbf{y}) := \inf\left\{\mu \geqq 0 |\ \mathbf{y}^{\mathsf{T}}\mathbf{x} \leqq \mu k(\mathbf{x})\ \ \forall\, \mathbf{x} \in X\right\}.$$

[143]Cf. ROCKAFELLAR (1972, p. 125, Theorem 14.5).
The example in Figure D.4 illustrates this correlation.

Therefore, each pair of polar functions (k, k°) has the property

(D.15) $\quad \mathbf{y}^\mathsf{T}\mathbf{x} \leqq k(\mathbf{x})k^\circ(\mathbf{y}) \qquad \forall\, \mathbf{x} \in \operatorname{Dom} k, \;\; \forall\, \mathbf{y} \in \operatorname{Dom} k^\circ,$

and one can show for each $\hat{\mathbf{x}} \neq \mathbf{0}$ that there is a $\hat{\mathbf{y}} \neq \mathbf{0}$ such that $\hat{\mathbf{y}}^\mathsf{T}\hat{\mathbf{x}} = k(\hat{\mathbf{x}})\,k^\circ(\hat{\mathbf{y}})$ holds, et vice versa.[144] As confirmed by Theorem D.16, it is well-founded to call the polar function k° a gauge.

Theorem D.16[145] *Let k be a gauge. Then the polar function k° of k is a closed gauge and we have $k^{\circ\circ} = \operatorname{cl} k$. If C is a nonempty convex set with $k = \gamma(\cdot|C)$, then the polar set C° of C yields the polar gauge $k^\circ = \gamma(\cdot|C^\circ)$.*

The inequality (D.15) is now

(D.16) $\quad \mathbf{y}^\mathsf{T}\mathbf{x} \leqq \gamma(\mathbf{x}|C)\gamma(\mathbf{y}|C^\circ) \qquad \forall\, \mathbf{x} \in \operatorname{Dom} \gamma(\cdot|C), \;\; \forall\, \mathbf{y} \in \operatorname{Dom} \gamma(\cdot|C^\circ).$

For a gauge $\gamma(\cdot|C)$ being finite and positive everywhere except at the origin we can rewrite the inequality of Definition D.6.

$$\gamma(\mathbf{y}|C^\circ) = \gamma^\circ(\mathbf{y}|C) = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{y}^\mathsf{T}\mathbf{x}}{\gamma(\mathbf{x}|C)}$$

Corollary D.16.1 (Gauge Duality)[146] *In the class of closed gauges the following symmetry between k and $h = k^\circ$ holds good.*

$$h^\circ = k^{\circ\circ} = k \quad \textit{and} \quad k^\circ = h^{\circ\circ} = h$$

Two closed convex sets containing the origin are polar to each other,

$$C = C^{\circ\circ} = D^\circ \quad \textit{and} \quad D = D^{\circ\circ} = C^\circ,$$

if and only if their gauges $\gamma(\cdot|C)$ and $\gamma(\cdot|D) = \gamma^\circ(\cdot|C)$ are polar to each other.

Corollary D.16.2[147] *Let C be a closed convex set in X containing the origin* $\mathbf{0}$. *Then the gauge $\gamma(\cdot|C)$ and the support function $\sigma(\cdot|C)$ are polar gauges.*

(D.17) $\quad k = \gamma(\cdot|C) \iff k^\circ = \sigma(\cdot|C)$

Proof: Because of $\sigma(\cdot|C) = \gamma(\cdot|C^\circ)$ by Corollary D.15.1, the equivalence relation (D.17) directly ensues from Theorem D.16. ■

Under the assumptions of Corollary D.16.2 the equivalence relation (D.17) can be described by **MAHLER's inequality**, which is equivalent to (D.16).[148]

(D.18) $\quad \mathbf{y}^\mathsf{T}\mathbf{x} \leqq \sigma(\mathbf{y}|C)\,\gamma(\mathbf{x}|C^\circ) \qquad \forall\, \mathbf{y} \in Y, \;\; \forall\, \mathbf{x} \in X$

---

[144] Cf. CASSELS (1971, p. 114, Corollary 1).

[145] Cf. ROCKAFELLAR (1972, p. 128, Theorem 15.1).

[146] Cf. ROCKAFELLAR (1972, p. 129, Corollary 15.1.1).

[147] Cf. ROCKAFELLAR (1972, p. 129, Corollary 15.1.2). If C is the unit ball $\{\mathbf{x}|\ \|\mathbf{x}\| \leqq 1\}$, then we can show that k corresponds to the Euclidean norm and $k = k^\circ$. Moreover, (D.17) and (D.18) reduce to SCHWARZ's inequality $|\mathbf{y}^\mathsf{T}\mathbf{x}| \leqq \|\mathbf{y}\| \cdot \|\mathbf{x}\|$.

[148] Cf. AUBIN (1979, p. 41, Proposition 12).

Before going into the concluding example, the main outcomes of Appendix D.2 are summarized in the following figure.

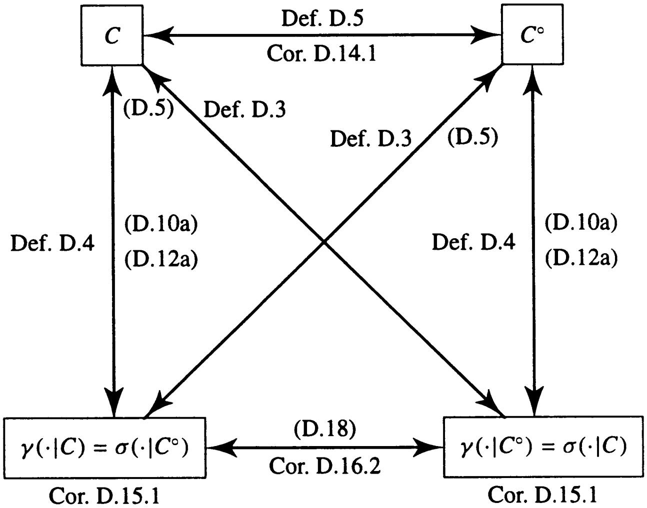

Figure D.3: Dual relationships in the sense of ROCKAFELLAR

Finally, an easy example is discussed with respect to the set $C = \{x|\ a \leqq x < b\}$ *in order to emphasize the main aspects of the presented theory. The results are summarized in Figure D.4. The linearly homogeneous convex support function of the set C satisfies*

$$\begin{aligned}
\sigma(y|C) &= \sup\{yx|\ x \in C\} \\
&= \sup\{yx - \delta(x|C)|\ x \in \mathbb{R}\} \\
&= \sup_{x\in\mathbb{R}} \left\{ yx - \begin{cases} 0 & \text{for } a \leqq x < b \\ +\infty & \text{otherwise} \end{cases} \right\} = \begin{cases} yb & \text{for } y > 0 \\ 0 & \text{for } y = 0 \\ ya & \text{for } y < 0. \end{cases}
\end{aligned}$$

Writing out in full, two chains of equations are compared, which are equivalent to each other under the above conditions.

$$k(\mathbf{x}) = \gamma(\mathbf{x}|C) = \gamma^\circ(\mathbf{x}|C^\circ) = \sigma(\mathbf{x}|C^\circ) = \sigma^\circ(\mathbf{x}|C) = \sup_{\mathbf{y}\neq\mathbf{0}} \frac{\mathbf{y}^{\mathrm{T}}\mathbf{x}}{k^\circ(\mathbf{y})}$$

$$k^\circ(\mathbf{y}) = \sigma(\mathbf{y}|C) = \sigma^\circ(\mathbf{y}|C^\circ) = \gamma(\mathbf{y}|C^\circ) = \gamma^\circ(\mathbf{y}|C) = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{y}^{\mathrm{T}}\mathbf{x}}{k(\mathbf{x})}$$

The convex conjugate function $\sigma^*(\cdot|C)$ is

$$\begin{aligned}\sigma^*(x|C) &= \sup\{yx - \sigma(y|C) \mid y \in \mathbb{R}\}\\ &= \sup_{y\in\mathbb{R}} \begin{Bmatrix} y(x-b) & \text{for } y>0\\ yx & \text{for } y=0\\ y(x-a) & \text{for } y<0\end{Bmatrix} = \begin{cases} +\infty & \text{for } x>b\\ 0 & \text{for } a \leqq x \leqq b\\ +\infty & \text{for } x<a.\end{cases}\end{aligned}$$

In fact $\sigma^*(\cdot|C) = \operatorname{cl}\delta(\cdot|C)$ holds, see Theorem D.8. A graphical representation is given by Figure D.4. If in the above mentioned figure it is supposed that C contains the origin, then $a < 0 < b$ must hold. The polar set is given by $C^\circ = \{y \mid 1/a \leqq x \leqq 1/b\}$ and we have $C^{\circ\circ} = \operatorname{cl} C$, see Theorem D.14. Apart from these polar sets the following figure shows the adjoined gauge and support functions, see Theorem D.15.

$$\gamma(x|C) = \sigma(x|C^\circ) = \begin{cases} x/a & \text{for } x<0\\ 0 & \text{for } x=0\\ x/b & \text{for } x>0\end{cases}$$

$$\gamma(y|C^\circ) = \sigma(y|C) = \begin{cases} ay & \text{for } y<0\\ 0 & \text{for } y=0\\ by & \text{for } y>0.\end{cases}$$

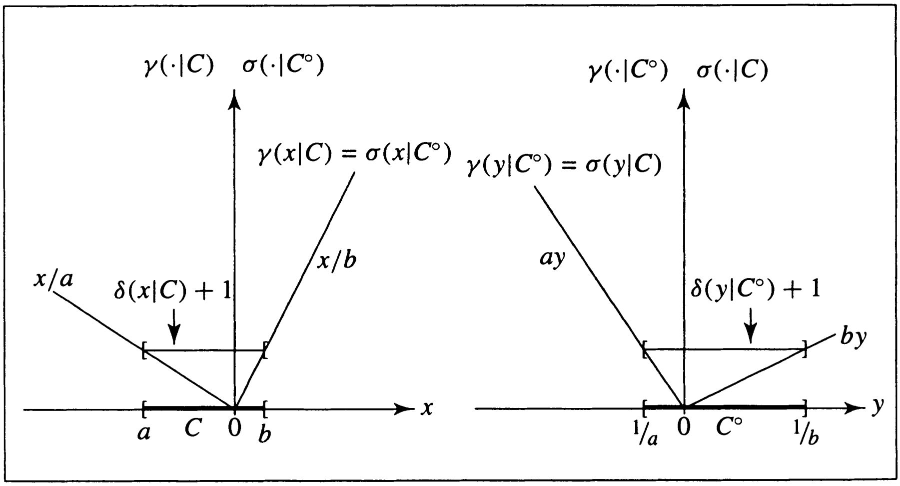

Figure D.4: Graphical representation of polar sets

Tables

List of Symbols

1. Symbols with economic meaning

a	index of persons ($a \in A$)
A	set of all persons
b	index of firms ($b \in B$)
B	set of all firms
$B\colon \Delta \times \mathbb{R}^n_+ \to \mathfrak{P}(X)$	budget correspondence
$B(\mathbf{p}, \mathbf{w}_a)$	person a's budget set
$B^s\colon \Delta \times \mathbb{R}^n_+ \to \mathfrak{P}(X^s_1)$	synthetic budget correspondence
$c\colon \overline{Q} \times \overline{X} \to [-\infty, +\infty]$	cost function (see Section III.2)
$\tilde{c}\colon \mathcal{Q} \times X \to [-\infty, +\infty]$	normed cost function (see Section III.1)
$D\colon Q \times X \to \mathfrak{P}(V)$	factor demand correspondence (see Section III.2)
$\tilde{D}\colon \mathcal{Q} \times X \to \mathfrak{P}(\mathcal{V})$	factor demand correspondence (see Section III.1)
$D_a\colon \Delta \times \mathbb{R}^n_+ \to \mathfrak{P}(X)$	individual demand correspondence of person a basing on the preference ordering $\succcurlyeq_a$
$D^{co}_a\colon \Delta \times \mathbb{R}^n_+ \to \mathfrak{P}(\mathbb{R}^n_+)$	convex-valued individual demand correspondence
$D^s_a\colon \Delta \times \mathbb{R}^n_+ \to \mathfrak{P}(X^s_1)$	synthetic demand correspondence
$\widehat{D}\colon \Delta \to \mathfrak{P}(X)$	aggregate demand correspondence
D^{sup}	measure for the degree of nonconvexity
$D_B(C, R)$	set of best elements in the set C with respect to relation R
$D_M(C, R)$	set of maximal elements in the set C with respect to relation R
$\mathcal{E}\colon A \to \Pi \times \mathbb{R}^n_+$	exchange economy
$\mathcal{F}_I\colon V \times X \setminus \{\mathbf{0}\} \to [0, +\infty]$	FARRELL's input efficiency measure
$\mathcal{F}_O\colon X \setminus \{\mathbf{0}\} \times V \to [0, +\infty]$	FARRELL's output efficiency measure
$G\colon \Delta \times \mathbb{R}^n_+ \to \mathfrak{P}(\mathbb{R}^n_+)$	correspondence containing the correspondence B
$G(\mathbf{p}, \mathbf{w}_a)$	set of all $\mathbf{x} \in \mathbb{R}^n_+$ with $\mathbf{p}^\mathsf{T}\mathbf{x} \leqq \mathbf{p}^\mathsf{T}\mathbf{w}_a$
GR	graph of the production technology
$L\colon X \to \mathfrak{P}(V)$	input correspondence
$L(\mathbf{x})$	input requirement set
$\big(L(\mathbf{x}) \mid \mathbf{x} \in X\big)$	production structure

$L_\circ(\mathbf{x})$	(reciprocally) polar input requirement set
$(L_\circ(\mathbf{x}) \mid \mathbf{x} \in X)$	cost structure
m	number of production factors
m_d	number of divisible production factors
n	number of goods
n_d	number of divisible goods
p_j	price of commodity j ($j = 1,\ldots, n$)
$\mathbf{p} = (p_1,\ldots, p_n)^\mathsf{T}$	commodity price vector
$P_\mathbf{p} = \mathbb{R}^n_+$	space of nonnegative commodity prices
$\mathcal{P}_\mathbf{p} = \mathbb{R}^{n-1}$	space of normed commodity prices
$\overline{P}_\mathbf{p} = \mathbb{R}^n$	space of commodity prices
$\mathcal{P}_a\colon X \to \mathfrak{P}(X)$	$\mathcal{P}_a(\mathbf{x})$ set of all commodity bundles, person a thinks to be not worse than $\mathbf{x}$ (preference set)
$P\colon V \to \mathfrak{P}(X)$	output correspondence
$P(\mathbf{v})$	production possibility set
q_i	price of factor i ($i = 1,\ldots, m$)
$\mathbf{q} = (q_1,\ldots, q_m)^\mathsf{T}$	factor price vector
$Q = \mathbb{R}^m_+$	space of nonnegative factor prices
$\mathcal{Q} = \mathbb{R}^{m-1}$	space of normed factor prices
$\overline{Q} = \mathbb{R}^m$	space of factor prices
$r\colon \overline{P}_\mathbf{p} \times \overline{V} \to \mathbb{R}$	revenue function
$\mathcal{R}\colon V \times X \setminus \{\mathbf{0}\} \to [0, +\infty]$	RUSSELL's input efficiency measure
$t_I\colon \overline{V} \times X \to \mathbb{R}_+$	input distance function
$t_O\colon \overline{X} \times V \to \mathbb{R}_+$	output distance function
$u_a\colon X \to \mathbb{R}$	utility function representing person a's preferences
v_i	quantity of factor i ($i = 1,\ldots, m$)
$\mathbf{v} = (v_1,\ldots, v_m)^\mathsf{T}$	vector of input quantities
$(\mathbf{v}, \mathbf{x}) \in V \times X$	possible activity
$V = \mathbb{R}^{m_d}_+ \times \mathbb{Z}^{m-m_d}_+$	factor space with respect to integer constraints
$\mathcal{V} = \mathbb{R}^{m-1}$	factor space (see Section III.1)
$\overline{V} = \mathbb{R}^m$	factor space (see Section III.2)
$V_\mathbf{b} = \{\mathbf{v} \in V \mid \mathbf{v} \leqq \mathbf{b}\}$	restricted factor space
$\mathbf{w}_a = (w_{1a},\ldots, w_{na})^\mathsf{T}$	person a's initial endowment
$\mathbf{w}_A = (w_{1A},\ldots, w_{nA})^\mathsf{T}$	total endowment of the economy
x_j	quantity of good j ($j = 1,\ldots, n$)
$\mathbf{x} = (x_1,\ldots, x_n)^\mathsf{T}$	commodity vector
$\mathbf{x}^d_a\colon \Delta \times \mathbb{R}^n_+ \to X$	individual (vector-valued) demand function of person a basing on the preference ordering $\succcurlyeq_a$
$\mathfrak{x} = (\mathbf{x}_a)_{a\in A}$	commodity allocation
$X = \mathbb{R}^{n_d}_+ \times \mathbb{Z}^{n-n_d}_+$	commodity space with respect to integer constraints
$\mathcal{X} = \mathbb{R}^{m-1}$	commodity space (see Section III.1)
$\overline{X} = \mathbb{R}^m$	commodity space (see Section III.2)
X^s_j ($j = 1, 2, 3$)	subsets in the commodity space X
X_a	person a's consumption set
$\mathfrak{X}$	set of all feasible allocations

Y_b	firm b's production set
$\mathbf{y}_b$	firm b's netput vector
$\mathfrak{y} = (\mathbf{y}_b)_{b \in B}$	production allocation
z_j	aggregate quantity of excess demand for good j ($j = 1, \ldots, n$)
$\mathbf{z} = (z_1, \ldots, z_n)^{\mathsf{T}}$	vector of aggregate excess demand
$\mathbf{z}_a = (z_{1a}, \ldots, z_{na})^{\mathsf{T}}$	person a's vector of excess demand
$Z\colon \Delta \to \mathfrak{P}(\mathbb{R}^n)$	aggregate excess demand correspondence
$Z_{co}\colon \Delta \to \mathfrak{P}(\mathbb{R}^n)$	convex-valued aggregate excess demand correspondence
$Z^s\colon \Delta \to \mathfrak{P}(X_3^s)$	synthetic aggregate excess demand correspondence
$\succcurlyeq_a$	person a's preference ordering $\mathbf{x} \succcurlyeq_a \mathbf{x}'$ person a thinks $\mathbf{x}$ to be at least as good as $\mathbf{x}'$ $\mathbf{x} \succ_a \mathbf{x}'$ person a prefers $\mathbf{x}$ to $\mathbf{x}'$ $\mathbf{x} \sim_a \mathbf{x}'$ person a is indifferent to $\mathbf{x}$ and $\mathbf{x}'$
$\succ_A$	binary relation between two allocations
δ_a	deviation between the chosen commodity bundle $\mathbf{x}_a^d$ and a vector $\mathbf{x}_a^\circ$
δ_A	deviation between an allocation and the total endowment $\mathbf{w}_A$
Δ	price simplex
Δ^+	(relative) interior of the price simplex
$\partial\Delta$	(relative) boundary of the price simplex
$\pi_b\colon \Delta \to \mathbb{R}$	firm b's profit
Π	set of all continuous preference orderings
$\Pi_{(s)mo}^{(s)co}$	set of all continuous (strongly) monotone and (strictly) convex preference orderings

2. Symbols in analysis

2.1 Symbols with respect to the set C

$\#C$	number of elements of C
$\complement C$	complement of C
C°	polar set of C
$C_\circ$	reciprocally polar set of C
$\operatorname{aff} C$	affine hull of C
$\operatorname{aur} C$	aureoled hull of C
$\operatorname{cl} C$	closure of t C
$\operatorname{cone} C$	cone generated by C
$\operatorname{conv} C$	convex hull of C
$d(\mathbf{x}, C)$	distance between the point $\mathbf{x}$ and the set C
$\operatorname{int} C$	interior of C
$K(C)$	cone generated by C without the vertex $\mathbf{0}$
$\mathfrak{P}(C)$	power set of C; set of all subsets in C

rint C	relative interior of C
star C	star-shaped hull of C
$\gamma(\cdot\,\|C)\colon \mathbb{R}^n \to [0,+\infty]$	gauge of $C \subset \mathbb{R}^n$
$\delta(\cdot\,\|C)\colon \mathbb{R}^n \to [0,+\infty]$	indicator function of $C \subset \mathbb{R}^n$
∂C	boundary of C
$\mu(C)$	measure of C
$\varrho(\cdot\,\|C)\colon \mathbb{R}^n \to [-\infty,0]$	reciprocal indicator function of $C \subset \mathbb{R}^n$
$\sigma(\cdot\,\|C)\colon \mathbb{R}^n \to [-\infty,+\infty]$	support function of $C \subset \mathbb{R}^n$
$\varphi(\cdot\,\|C)\colon \mathbb{R}^n \to [-\infty,+\infty]$	reciprocal support function of $C \subset \mathbb{R}^n$
$\psi(\cdot\,\|C)\colon \mathbb{R}^n \to [-\infty,+\infty]$	reciprocal gauge of $C \subset \mathbb{R}^n$

2.2 Symbols with respect to the function f

$f\colon X \to Y$	function or single-valued mapping
f°	polar function of f
$f_\circ$	reciprocally polar function of f
f^*	convex conjugate function of f
f_*	concave conjugate function of f
cl f	closure of f
conv f	convex hull of f
Dom f	effective domain of f
epi f	epigraph of f
hypo f	hypograph of f
n-Dom f	effective domain of f
Range f	range of f
$\Delta f\colon X \to \mathfrak{P}(Y)$	superdifferential of f
$\partial f\colon X \to \mathfrak{P}(Y)$	subdifferential of f
λf	left scalar multiplication
$f\lambda$	right scalar multiplication
$\nabla f\colon X \to Y$	gradient of f

2.3 Other Symbols

$\mathbf{e}^j$	j-th unit vector
H	hyperplane
$K(\mathbf{x}, r)$	open ball centered at point $\mathbf{x}$ of radius r
$K[\mathbf{x}, r]$	closed ball centered at point $\mathbf{x}$ of radius r
$\mathbb{N}$	set of positive integers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of nonnegative real numbers
$\mathbb{R}_{++}$	set of positive real numbers
$\mathbb{Z}$	set of integer numbers
$\Gamma\colon X \to \mathfrak{P}(Y)$	correspondence or multi-valued mapping
Λ^{n+1}	n-dimensional unit simplex
$\|x\|$	absolute value of the number x
$\\|\mathbf{x}\\|$	Euclidean metric, length of the vector $\mathbf{x}$

$\lceil x \rceil$	smallest integer not smaller than x
$\lfloor x \rfloor$	greatest integer not greater than x
$\{x^\nu\}$	sequence of numbers $x^1, x^2, \ldots$
$\{x^{\nu_k}\}$	subsequence of the sequence of numbers $\{x^\nu\}$
$x^\nu \to x$	sequence of numbers $\{x^\nu\}$ with limit x
$\{\mathbf{x}^\nu\}$	sequence of points $\mathbf{x}^1, \mathbf{x}^2, \ldots$
$\mathbf{x}^\nu \to \mathbf{x}$	sequence of points $\{\mathbf{x}^\nu\}$ converging to $\mathbf{x}$ with respect to each component

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