REAL ANALYSIS

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PREFACE

In teaching first courses in real analysis over the years, we have found increasingly that the classes form rather heterogeneous groups. It is no longer true that most of the students are first-year graduate students in mathematics, presenting more or less common backgrounds for the course. Indeed, nowadays we find diverse backgrounds and diverse objectives among students in such classes. Some students are undergraduates, others are more advanced. Many students are in other departments, such as statistics or engineering. Some students are seeking terminal master's degrees; others wish to become research mathematicians, not necessarily in analysis.

We have tried to write a book that is suitable for students with minimal backgrounds, one that does not presuppose that most students will eventually specialize in analysis.

We have pursued two goals. First, we would like all students to have an opportunity to obtain an appreciation of the tools, methods, and history of the subject and a sense of how the various topics we cover develop naturally. Our second objective is to provide those who will study analysis further with the necessary background in measure, integration, differentiation, metric space theory, and functional analysis.

To meet our first goal, we do several things. We provide a certain amount of historical perspective that may enable a reader to see why a theory was needed and sometimes, why the researchers of the time had difficulty obtaining the "right" theory. We try to motivate topics before we develop them and try to motivate the proofs of some of the important theorems that students often find difficult. We usually avoid proofs that may appear "magical" to students in favor of more revealing proofs that may be a bit longer. We describe the interplay of various subjects—measure, variation, integration, and differentiation. Finally, we indicate applications of abstract theorems such as the contraction mapping principle, the Baire category theorem, Ascoli's theorem, Hahn-Banach theorem, and the open mapping theorem, to concrete settings of various sorts.

We consider the exercise sections an important part of the book. Some of the exercises do no more than ask the reader to complete a proof given in the text, or to prove an easy result that we merely state. Others involve simple applications of the theorems. A number are more ambitious. Some of these exercises extend the theory that we developed or present some related material. Others provide examples that we believe are interesting and revealing, but may not be well known. In general, the problems at the ends of the chapters are more substantial. A few of these problems can form the basis of projects for further study. We have marked exercises that are referenced in later parts of the book with a $\diamondsuit$ to indicate this fact.

When we poll our students at the beginning of the course, we find there are a number of topics that some students have seen before, but many others have not. Examples are the rudiments of metric space theory, Lebesgue measure in $\mathbb{R}^1$, Riemann–Stieltjes integration, bounded variation and the elements of set theory (Zorn's lemma, well-ordering, and others). In Chapter 1, we sketch some of this material. These sections can be picked up as needed, rather than covered at the beginning of the course. We do suggest that the reader browse through Chapter 1 at the beginning, however, as it provides some historical perspective.

Text Organization

Many graduate textbooks are finely crafted works as intricate as a fabric. If some thread is pulled too severely, the whole structure begins to unravel. We have hoped to avoid this. It is reasonably safe to skip over many sections (within obvious limitations) and construct a course that covers your own choice of topics, with little fear that the student will be forced to cross reference back through a maze of earlier skipped sections.

A word about the order of the chapters. The first chapter is intended as background reading. Some topics are included to help motivate ideas that reappear later in a more abstract setting. Zorn's lemma and the axiom of choice will be needed soon enough, and a classroom reference to Sections 1.3, 1.5 and 1.11 can be used.

The course can easily start with the measure theory of Chapter 2 and proceed from there. We chose to cover measure and integration before metric space theory because so many important metric spaces involve measurable or integrable functions. The rudiments of metric space theory are needed in Chapter 3, however, so we begin that chapter with a short section containing the necessary terminology.

Instructors who wish to emphasize functional analysis and reach Chapter 9 quickly can do so by omitting much of the material in the earlier chapters. One possibility is to cover Sections 2.1 to 2.6, 4.1, 4.2, and Chapter 5 and then proceed directly to Chapter 9. This will provide enough background in measure and integration to prepare the student for the later chapters.

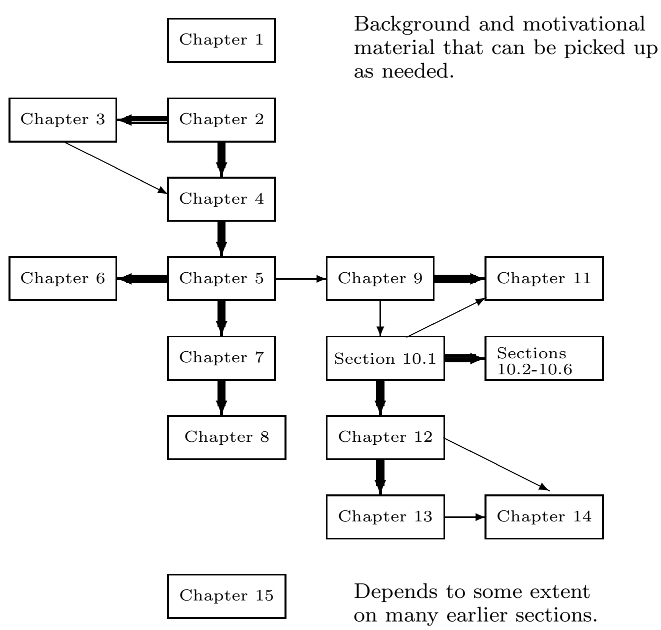
Chapter 6 on the Fubini and Tonelli theorems is used only occasionally in the sequel (Sections 8.4 and 13.9). This is presented from the outer measure point of view because it fits better with the philosophy developed in Chapters 2 and 3. One can substitute any treatment in its place. Chapter 11 on analytic sets is not needed for the later chapters, and is presented as a subject of interest on its own merits. Chapter 13 on the L_p–spaces can be bypassed in favor of Chapter 14 or 15 except for a few points. Chapter 14 on Hilbert space could be undertaken without covering Chapters 12 and 13 since all material on the spaces ℓ_2 and L_2 is repeated as needed. Chapter 15 on Fourier series does not need the Hilbert space material in order to work, but, since it is intended as a showplace for many of the methods, it does draw on many other chapters for ideas and techniques.

The dependency chart on page xiv gives a rough indication of how chapters depend on their predecessors. A strong dependency is indicated by a bold arrow, a weaker one by a fine arrow. The absence of an arrow indicates that no more than peripheral references to the earlier chapters are involved. Even when a strong dependency is indicated, the omission of certain sections near the en d of a chapter should not cause difficulties in later chapters. In addition, we have provided a number of concrete applications of abstract theorems. Many of these applications are not needed in later chapters. Thus an instructor who wishes to include material from all chapters in a year course for reasonably prepared students can do so by

1. Omitting some of the less central material such as 3.8 to 3.10, 5.10, 7.6 to 7.8, 8.4 to 8.7, 9.14 to 9.15, 10.2 to 10.6, and various material from the remaining chapters.

2. Sampling from the applications in Sections 9.8, 9.12, 9.14, 10.2 to 10.6, and 12.6.

3. Pruning sections from chapters from which no arrow emanates.



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A.M.B.
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Chapter 1

BACKGROUND AND PREVIEW

In this chapter we provide a review and historical sampling of much of the background needed to embark on a study of the theory of measure, integration, and functional analysis. The setting here is the real line. In later chapters we place most of the theory in an abstract measure space or in a metric space, but the ideas all originate in the situation on the real line. The reader will have a background in elementary analysis, including such ideas as continuity, uniform continuity, convergence, uniform convergence, and sequence limits. The emphasis at this more advanced level shifts to a study of sets of real numbers and collections of sets, and this is what we shall address first in Sections 1.1 and 1.2.

Some of the basic ideas from set theory needed throughout the text are introduced in this chapter. The rudiments of cardinal and ordinal numbers appear in Sections 1.3 to 1.5. At certain points in the text we make extensive use of cardinality arguments and transfinite induction. The axiom of choice and its equivalent versions, Zermelo's theorem and Zorn's lemma, are discussed in Sections 1.3, 1.5, and 1.11. This material should be sufficient to justify these ideas, although a proper course of instruction in these concepts is recommended. We have tried to keep these considerations both minimal and intuitive. Our business is to develop the analysis without long lingering on the set-theoretic methods that are needed.

In Sections 1.7 to 1.10 we present two contrasting and competing theories of measure on the real line: the theory of Peano–Jordan content and the theory of Lebesgue measure. They serve as an introduction to the general theory that will be developed in Chapters 2 and 3. All the material here receives its full expression in the later chapters with complete proofs in the most general setting. The reader who works through the concepts and exercises in this introductory chapter should have an easier time of it when the abstract material is presented.

The notion of category plays a fundamental role in almost all aspects of analysis nowadays. In Section 1.6 the basics of this theory on the real line are presented. We shall explore this in much more detail in Chapter 10.

Borel sets and analytic sets play a key role in measure theory. These are covered briefly in Sections 1.12 and 1.13. The latter contains only a report on the origins of the theory of analytic sets. A full treatment appears in Chapter 11.

Sections 1.15 to 1.21 present the basics of integration theory on the real line. A quick review of the integral as viewed by Newton, Cauchy, Riemann, Stieltjes, and Lebesgue is a useful prelude to an approach to the modern theory of integration. We conclude with a generalized version of the Riemann integral that helps to complete the picture on the real line. We will return to these ideas in Section 5.10.

A brief study of functions of bounded variation appears in Section 1.14. This material, often omitted from an undergraduate education, is essential background for the student of general measure theory and, in any case, cannot be avoided by anyone wishing to understand the differentiation theory of real functions.

The exercises are designed to allow the student to explore the technical details of the subject and grasp new methods. The chapter can be read superficially without doing many exercises as a fast review of the background that is needed in order to appreciate the abstract theory that follows. It may also be used more intensively as a short course in the basics of analysis on the real line.

1.1 The Real Numbers

The reader is presumed to have a working knowledge of the real number system and its elementary properties. We use $\mathbb{R}$ to denote the set of real numbers. The natural numbers (positive integers) are denoted as $\mathbb{N}$, the integers (positive, negative, and zero) as $\mathbb{Z}$, and the rational numbers as $\mathbb{Q}$. The complex numbers are written as $\mathbb{C}$ and will play a role at a number of points in our investigation, even though the topic is called real analysis.

The extended real number system $\overline{\mathbb{R}}$, that is, $\mathbb{R}$ with the two infinities $+\infty$ and $-\infty$ appended, is used extensively in measure theory and analysis. One does not try to extend too many of the real operations to $\mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$: we shall write, though,

$$c+\infty=+\infty \quad \text{and} \quad c-\infty=-\infty$$

for any $c\in\mathbb{R}$.

Limits of sequences in $\mathbb{R}$ are defined using the metric

$$\rho(x,y)=|x-y| \quad (x,y\in\mathbb{R}).$$

This metric has the properties that one expects of a distance, properties that shall be used later in Chapter 9 to develop the concept of an abstract metric space.

1. $0 \leq \rho(x,y) < +\infty$, $(x,y \in \mathbb{R})$.
2. $\rho(x,y) = 0$ if and only if $x = y$.
3. $\rho(x,y) = \rho(y,x)$.
4. $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$, $(x,y,z \in \mathbb{R})$.

We recall that sequence convergence in $\mathbb{R}$ means convergence relative to this distance. Thus $x_n \to x$ means that $\rho(x_n, x) = |x_n - x| \to 0$. A sequence $\{x_n\}$ is convergent if and only if that sequence is *Cauchy,* that is, if $\lim_{m,n\to\infty} \rho(x_m, x_n) = 0$. On the real line, sequences that are monotone and bounded are necessarily convergent. Virtually all the analysis on the real line develops from these fundamental notions.

In the theory to be studied here, we require an extensive language for classifying sets of real numbers. The reader is familiar, no doubt, with most of the following concepts, which we present here to provide an easy reference and review. All these concepts will be generalized to an abstract metric space in Chapter 9.

Set notation throughout is standard. Thus union and intersection are written $A \cup B$ and $A \cap B$. Set difference is written $A \setminus B$, and so the complement of a set $A \subset \mathbb{R}$ will be written $\mathbb{R} \setminus A$. It is convenient to have a shorthand for this sometimes and we use $\widetilde{A}$ as well for this. The union and intersection of families will appear as $\bigcup_{A\in\mathcal{A}} A$ and $\bigcap_{A\in\mathcal{A}} A$.

- A *limit point* of a set E or *point of accumulation* of a set E is any number that can be expressed as the limit of a convergent sequence of distinct points in E.
- The *closure* of a set E is the union of E together with its limit points. One writes $\overline{E}$ for the closure of E.
- An *interior point* of a set E is a point contained in an interval (a,b) that is itself entirely contained in E.
- The *interior* of a set E is the set of interior points of E. One writes E^o or perhaps $\mathrm{int}(E)$ for the interior of E.
- An *isolated point* of a set is a member of the set that is not a limit point of the set.
- A *boundary point* of a set is a point of accumulation of the set that is not also an interior point of the set.
- A set G of real numbers is *open* if every point of G is an interior point of G.
- A set F of real numbers is *closed* if F contains all its limit points.
- A set of real numbers is *perfect* if it is nonempty, closed, and has no isolated points.
- A set of real numbers is *scattered* if it is nonempty and every nonempty subset has at least one isolated point.

- A set E of real numbers is *dense* in a set E_0 if every point in E_0 is a limit point of the set E.
- A set E of real numbers is *nowhere dense* if for every interval (a,b) there is a subinterval $(c,d) \subset (a,b)$ containing no points of E. (This is the same as asserting that E is dense in no interval.)
- A set E of real numbers is a *Cantor set* if it is nonempty, bounded, perfect, and nowhere dense.

In elementary courses one learns a variety of facts about these kinds of sets. We review some of the more important of these here, and the exercises explore further facts. All will play a role in our investigations of measure theory and integration theory on the real line. To begin, one observes that the interval

$$(a,b) = \{x : a < x < b\}$$

is open and that the interval

$$[a,b] = \{x : a \leq x \leq b\}$$

is closed. It is nearly universal now for mathematicians to lean toward the letter "G" to express open sets and the letter "F" to represent closed sets. The folklore is that the custom came from the French (*fermé* for closed) and the Germans (*Gebiet* for region). The following theorem describes the fundamental properties of the families of open and closed sets.

Theorem 1.1 *Let $\mathcal{G}$ denote the family of all open subsets of the real numbers and $\mathcal{F}$ the family of all closed subsets of the real numbers. Then*

1. *Each element in $\mathcal{G}$ is the complement of a unique element in $\mathcal{F}$, and vice versa.*
2. *$\mathcal{G}$ is closed under arbitrary unions and finite intersections.*
3. *$\mathcal{F}$ is closed under finite unions and arbitrary intersections.*
4. *Every set G in $\mathcal{G}$ is the union of a sequence of disjoint open intervals (called the components of G).*
5. *Given a collection $\mathcal{C} \subset \mathcal{G}$, there is a sequence $\{G_1, G_2, G_3, \dots\}$ of sets from $\mathcal{C}$ so that*
$$\bigcup_{G \in \mathcal{C}} G = \bigcup_{i=1}^{\infty} G_i.$$

Much more complicated sets than merely open sets or closed sets arise in many questions in analysis. If $\mathcal{C}$ is a class of sets, then frequently one is led to consider sets of the form

$$E = \bigcup_{i=1}^{\infty} C_i$$

for a sequence of sets $C_i \in \mathcal{C}$. We shall write $\mathcal{C}_\sigma$ for the resulting class. Similarly, we shall write $\mathcal{C}_\delta$ for the class of sets of the form

$$E = \bigcap_{i=1}^{\infty} C_i$$

for some sequence of sets $C_i \in \mathcal{C}$. The subscript σ denotes a summation (i.e., union) and δ denotes an intersection (from the German word *Durchschnitt*).

Continuing in this fashion, we can construct classes of sets of greater and greater complexity

$$\mathcal{C},\ \mathcal{C}_\delta,\ \mathcal{C}_\sigma,\ \mathcal{C}_{\delta\sigma},\ \mathcal{C}_{\sigma\delta},\ \mathcal{C}_{\delta\sigma\delta},\ \mathcal{C}_{\sigma\delta\sigma},\dots,$$

which may play a role in the analysis of the sets $\mathcal{C}$.

These operations applied to the class $\mathcal{G}$ of open sets or the class $\mathcal{F}$ of closed sets result in sets of great importance in analysis. The class $\mathcal{G}_\delta$ and the class $\mathcal{F}_\sigma$ are just the beginning of a hierarchy of sets that form what is known as the Borel sets:

$$\mathcal{G} \subset \mathcal{G}_\delta \subset \mathcal{G}_{\delta\sigma} \subset \mathcal{G}_{\delta\sigma\delta} \subset \mathcal{G}_{\delta\sigma\delta\sigma} \dots$$

and

$$\mathcal{F} \subset \mathcal{F}_\sigma \subset \mathcal{F}_{\sigma\delta} \subset \mathcal{F}_{\sigma\delta\sigma} \subset \mathcal{F}_{\sigma\delta\sigma\delta} \dots.$$

A complete description of the class of Borel sets requires more apparatus than this might suggest, and we discuss these ideas in Section 1.12 along with some historical notes. Some elementary exercises now follow that will get the novice reader started in thinking along these lines.

Exercises

1:1.1 The classical *Cantor ternary set* is the subset of $[0,1]$ defined as

$$C = \left\{ x \in [0,1] : x = \sum_{n=1}^{\infty} \frac{i_n}{3^n} \text{ for } i_n = 0 \text{ or } 2 \right\}.$$

Show that C is perfect and nowhere dense (i.e., C is a Cantor set in the terminology of this section).

1:1.2 List the intervals complementary to the Cantor ternary set in $[0,1]$ and sum their lengths.

1:1.3 Let

$$D = \left\{ x \in [0,1] : x = \sum_{n=1}^{\infty} \frac{j_n}{3^n} \text{ for } j_n = 0 \text{ or } 1 \right\}.$$

Show $D + D = \{x + y : x, y \in D\} = [0,1]$. From this deduce, for the Cantor ternary set C, that $C + C = [0,2]$.

1:1.4 Criticize the following "argument" which is far too often seen:

> "If $G = (a, b)$ then $\overline{G} = [a, b]$. Similarly, if $G = \bigcup_{i=1}^{\infty}(a_i, b_i)$ is an open set, then $\overline{G} = \bigcup_{i=1}^{\infty}[a_i, b_i]$. It follows that an open set G and its closure $\overline{G}$ differ by at most a countable set."(?)

[Hint: Consider $G = (0, 1) \setminus C$ where C is the Cantor ternary set.]

1:1.5 Show that a scattered set is nowhere dense.

1:1.6 If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then show that the set

$$f^{-1}(C) = \{x : f(x) = y \in C\}$$

is closed for every closed set C.

1:1.7 If f is continuous, then show that the set

$$f^{-1}(G) = \{x : f(x) = y \in G\}$$

is open for every open set G.

1:1.8$\diamondsuit$ We define the *oscillation* of a real function f at a point x as

$$\omega_f(x) = \inf_{\delta>0} \sup \{|f(y) - f(z)| : y, z \in (x - \delta, x + \delta)\}.$$

Show that f is continuous at x if and only if $\omega_f(x) = 0$.

1:1.9 Show that the set $\{x : \omega_f(x) \geq \varepsilon\}$ is closed for each $\varepsilon \geq 0$.

1:1.10 For an arbitrary function f, show that the set of points where f is discontinuous is of type $\mathcal{F}_\sigma$.

1:1.11 For an arbitrary function f, show that the set of points where f is continuous is of type $\mathcal{G}_\delta$.

1:1.12 Prove the elementary parts (1, 2, and 3) of Theorem 1.1.

1:1.13 Prove part 4 of Theorem 1.1. Every open set G is the union of a unique sequence of disjoint open intervals, called the *components* of G.

1:1.14 Prove part 5 of Theorem 1.1 (Lindelöf's theorem). Given any collection $\mathcal{C}$ of open sets, there is a sequence $\{G_1, G_2, G_3, \dots\}$ of sets from $\mathcal{C}$ so that

$$\bigcup_{G\in\mathcal{C}} G = \bigcup_{i=1}^{\infty} G_i.$$

1:1.15 Show that every open interval may be expressed as the union of a sequence of closed intervals with rational endpoints. Thus every open interval is a $\mathcal{F}_\sigma$. (What about arbitrary open sets?)

1:1.16 What is $\mathcal{G} \cap \mathcal{F}$?

1:1.17 Show that $\mathcal{F} \subset \mathcal{G}_\delta$.

1:1.18 Show that $\mathcal{G} \subset \mathcal{F}_\sigma$.

1:1.19 Show that the complements of sets in $\mathcal{G}_\delta$ are in $\mathcal{F}_\sigma$, and conversely.

1:1.20 Find a set in $\mathcal{G}_\delta \cap \mathcal{F}_\sigma$ that is neither open nor closed.

1:1.21 Show that the set of zeros of a continuous function is a closed set. Given any closed set, show how to construct a continuous function that has precisely this set as its set of zeros.

1:1.22 A function f is upper semicontinuous at a point x if for every $\varepsilon > 0$ there is a $\delta > 0$ so that if $|x - y| < \delta$ then $f(y) > f(x) - \varepsilon$. Show that f is upper semicontinuous everywhere if and only if for every real α the set $\{x : f(x) \geq \alpha\}$ is closed.

1:1.23 Formulate a version of Exercise 1:1.22 for the notion of lower semicontinuity. [Hint: It should work in such a way that f is lower semicontinuous at a point if and only if $-f$ is upper semicontinuous there.]

1:1.24◇ If $f_n \to f$ at every point, then prove that

$$\{x : f(x) > \alpha\} = \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} \{x : f_n(x) \geq \alpha + 1/m\}.$$

1:1.25 Let $\{f_n\}$ be a sequence of real functions. Show that the set E of points of convergence of the sequence can be written in the form

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \left\{x : |f_n(x) - f_m(x)| \leq \tfrac{1}{k}\right\}.$$

1:1.26 Let $\{f_n\}$ be a sequence of continuous real functions. Show that the set of points of convergence of the sequence is of type $\mathcal{F}_{\sigma\delta}$.

1:1.27 Show that every scattered set is of type $\mathcal{G}_\delta$.

1:1.28 Give an example of a scattered set that is not closed nor is its closure scattered.

1:1.29 Show that every set of real numbers can be written as the union of a set that is dense in itself (i.e., has no isolated points) and a scattered set.

1:1.30 Show that the union of a finite number of Cantor sets is also a Cantor set.

1.2 Compact Sets of Real Numbers

A closed, bounded set of real numbers is said to be *compact.* The concept of compactness plays a fundamental role in nearly all aspects of analysis. On the real line the notions are particularly easy to grasp and to apply. A basic theorem, often ascribed to Cantor (1845–1918), leads easily to many applications.

Theorem 1.2 (Cantor) *If $\{[a_i, b_i]\}$ is a nested sequence of closed, bounded intervals whose lengths shrink to zero, then the intersection*

$$\bigcap_{i=1}^{\infty} [a_i, b_i]$$

contains a unique point.

Here the sequence of intervals is said to be *nested* if, for each n,

$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n].$$

The easy proof of this theorem can be obtained either by using the fact that monotone, bounded sequences converge (and hence a_n and b_n must converge) or by using the fact that Cauchy sequences converge (a sequence of points x_n chosen so that each $x_n \in [a_n, b_n]$ must be Cauchy). See Exercises 1:2.1 and 1:2.2.

Our next theorem is less well known. It was apparently first formulated by Pierre Cousin, who was a student of Henri Poincaré at the end of the nineteenth century. It asserts that a collection of intervals that contains all sufficiently small ones can be used to form a partition of any interval.

Theorem 1.3 (Cousin) *Let $\mathcal{C}$ be a collection of closed subintervals of $[a, b]$ with the property that for every $x \in [a, b]$ there is a $\delta > 0$ so that $\mathcal{C}$ contains all intervals $[c, d] \subset [a, b]$ that contain x and have length smaller than δ. Then there is a partition*

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a, b]$ so that each interval $[x_{i-1}, x_i] \in \mathcal{C}$ for all $1 \leq i \leq n$.

A proof is sketched in Exercises 1:2.3. Note that it can be made to follow from the Cantor theorem. We introduce some language that is useful in applying this theorem. Let us say that a collection of closed intervals $\mathcal{C}$ is *full* if it has the property of the theorem that it contains all sufficiently small intervals at any point x. Let us say that $\mathcal{C}$ is *additive* if whenever $[c, d]$ and $[d, e]$ are in $\mathcal{C}$ it follows that $[c, e] \in \mathcal{C}$. Then Cousin's theorem implies that any collection $\mathcal{C}$ of closed intervals that is both additive and full must contain *all* intervals.

Our remaining theorems are all consequences of the Cantor theorem or the Cousin theorem. The most economical approach to proving each is apparently provided by the Cousin theorem. In each case, define a collection $\mathcal{C}$ of closed intervals, check that it is full and additive, and conclude that $\mathcal{C}$ contains all intervals. The exercises give the necessary hints on how to start as well as explain the terminology.

Theorem 1.4 (Heine–Borel) *Every open covering of a closed and bounded set of real numbers has a finite subcover.*

Theorem 1.5 *Every collection of closed, bounded sets of real numbers that has the finite intersection property, has a nonempty intersection.*

Theorem 1.6 (Bolzano–Weierstrass) *A bounded, infinite set of real numbers has a limit point.*

By a *compactness argument* in the study of sets and functions on $\mathbb{R}$, we understand any application of one of the theorems of this section. Often one can recognize a compactness argument most clearly in the process of reducing open covers to finite subcovers (Heine–Borel) or passing from a sequence to a convergent subsequence (Bolzano–Weierstrass). The reader is encouraged to try for a variety of proofs of the exercises that ask for a compactness argument. Hints are given that allow an application of Cousin's theorem. But one should develop the other techniques too, especially since in more general settings (metric spaces, topological spaces) a version of Cousin's theorem may not be available, and a version of the Heine–Borel theorem or the Bolzano–Weierstrass theorem may be.

Exercises

1:2.1 If $\{[a_i, b_i]\}$ is a nested sequence of closed, bounded intervals whose lengths shrink to zero, then the intersection $\bigcap_{i=1}^{\infty}[a_i, b_i]$ contains a unique point. Prove this by showing that both $\lim a_i$ and $\lim b_i$ exist and are equal.

1:2.2 If $\{[a_i, b_i]\}$ is a nested sequence of closed, bounded intervals whose lengths shrink to zero, then the intersection $\bigcap_{i=1}^{\infty}[a_i, b_i]$ contains a unique point. Prove this by selecting a point x_i in each $[a_i, b_i]$ and showing that $\{x_i\}$ is Cauchy.

1:2.3 Prove Theorem 1.3. [Hint: If there is no partition of $[a, b]$, then either there is no partition of $[a, \frac{1}{2}(a+b)]$ or else there is no partition of $[\frac{1}{2}(a+b), b]$. Construct a nested sequence of intervals and obtain a contradiction.]

1:2.4 Prove Theorem 1.3. [Hint: Consider the set S of all points $z \in (a, b]$ for which there is a partition of $[a, t]$ whenever $t < z$. Write $z_0 = \sup S$. Then $z_0 \in S$ (why?), $z_0 > a$ (why?), and $z_0 < b$ is impossible (why?). Hence $z_0 = b$ and the theorem is proved.]

1:2.5 Prove the Heine–Borel theorem: Let $\mathcal{S}$ be a collection of open sets covering a closed set E. Then, for every interval $[a, b]$, there is a finite subset of $\mathcal{S}$ that covers $E \cap [a, b]$. [Hint: Let $\mathcal{C}$ be the collection of closed subintervals I of $[a, b]$ for which there is a finite subset of $\mathcal{S}$ that covers $E \cap I$.]

1:2.6 Prove Theorem 1.5 directly from the Heine–Borel theorem. Here a family of sets has the *finite intersection property* if every finite subfamily has a nonempty intersection. [Hint: Take complements of the closed sets.]

1:2.7 Prove the Bolzano–Weierstrass theorem: If a set S has no limit points, then $S \cap [a, b]$ is finite for every interval $[a, b]$. [Hint: If x

is not a limit point of S, then $S \cap [c, d]$ is finite for small intervals containing x.]

1:2.8 Show that if a function $f : \mathbb{R} \to \mathbb{R}$ is continuous, then it is uniformly continuous on every closed bounded interval. [Hint: Let $\varepsilon > 0$ and let $\mathcal{C}$ denote the set of intervals I such that, for some $\delta > 0$, $x, y \in I$ and $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Try also for other compactness arguments than Cousin's theorem.]

1:2.9 If f is continuous it is bounded on every closed bounded interval. [Hint: Let $\mathcal{C}$ denote the set of intervals I such that, for some $M > 0$ and all $x \in I$, $|f(x)| \leq M$.]

1:2.10 Prove the intermediate-value property: If f is continuous and never vanishes, then it is either always positive or always negative. [Hint: Let $\mathcal{C}$ denote the set of intervals $[a, b]$ such that $f(b)f(a) > 0$.]

1:2.11 If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $K \subset \mathbb{R}$ is compact, show that $f(K)$ is compact. Is $f^{-1}(K)$ also necessarily compact?

1:2.12 [Dini] Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is continuous for each $n = 1, 2, 3, \ldots$, and $f_1(x) \geq f_2(x) \geq f_3(x) \geq \ldots$ and $\lim_{n\to\infty} f_n(x) = 0$ at each point. Prove that the convergence is uniform on every compact interval. [Hint: Consider all intervals $[a, b]$ such that there is a p so that, for all $n \geq p$ and all $x \in [a, b]$, $f_n(x) < \varepsilon$.]

1.3 Countable Sets

The *cardinality* of a finite set is merely the number of elements that the set possesses. For infinite sets a similar notion was made available by the fundamental work of Cantor in the 1870s. We can say that a finite set S has cardinality n if the elements of S can be placed in a one-one correspondence with the elements of the set $\{1, 2, 3, 4, \ldots, n\}$.

Similarly, we say an infinite set S has cardinality $\aleph_0$ if the elements of S can be placed in a one-one correspondence with the elements of the set $\mathbb{N}$ of natural numbers. More simply put, this says that the elements of S can be listed:

$$S = \{s_1, s_2, s_3, \ldots\}.$$

A set is *countable* (some authors say it is "at most countable") if it has finite cardinality or cardinality $\aleph_0$. A set is *uncountable* if it is infinite but does not have cardinality $\aleph_0$. The choice of the first letter in the Hebrew alphabet (aleph, $\aleph$) to represent the transfinite cardinal numbers was made quite carefully by Cantor himself, and the notation is standard today.

To illustrate that these notions are not trivial, Cantor showed that any interval of real numbers is uncountable. Thus the points of an interval cannot be written in a list. The easiest and clearest proof is based on the fact that a nested sequence of intervals shrinks to a point. Cantor based his proof on a diagonal argument.

Theorem 1.7 (Cantor) *No interval* $[a, b]$ *is countable.*

Proof. Suppose not. Then the elements of $[a, b]$ can be arranged into a sequence $c_1, c_2, c_3, \dots$. Select an interval $[a_1, b_1] \subset [a, b]$ so that $c_1 \notin [a_1, b_1]$ and so that $b_1 - a_1 < 1/2$. Continuing inductively, we find a nested sequence of intervals $\{[a_i, b_i]\}$ with lengths $b_i - a_i < 2^{-i} \to 0$ and with $c_i \notin [a_i, b_i]$ for each i.

By Theorem 1.2, there is a unique point $c \in [a, b]$ common to each of the intervals. This point cannot be equal to any c_i and this is a contradiction, since the sequence $c_1, c_2, c_3, \dots$ was to contain every point of the interval $[a, b]$. ■

A comment must be made here about the method of proof. It is undoubtedly true that there is an interval $[a_1, b_1]$ with the properties that we require. It is also true that there is an interval $[a_2, b_2]$ with the properties that we require. But is it legitimate to make an infinite number of selections? One way to justify this is to make explicit in the rules of mathematics that we can make such infinite selections. This is provided by the axiom of choice that can be invoked when needed.

1.8 (Axiom of Choice) *Let* $\mathcal{C}$ *be any collection of nonempty sets. Then there is a function* f *defined on* $\mathcal{C}$ *so that* $f(E) \in E$ *for each* $E \in \mathcal{C}$.

The function f is called a choice function. That such a function exists is the same for us as the claim that an element can be chosen from each of the (perhaps) infinitely many sets. The original wording (translated from the German) of E. Zermelo from 1904 is instructive:

> For every subset M', imagine a corresponding element m'_1, which is itself a member of M' and may be called the "distinguished" *[ausgezeichnete]* element of M'.

We can invoke this axiom in order to justify the proof we have just given. Alternatively, we can puzzle over whether, in this specific instance, we can obtain our proof without using this principle. Here is how to avoid using the axiom of choice in this particular instance, replacing it with an ordinary inductive argument. Suppose that I_1, I_2, I_3, ... is a list of all the closed intervals with rational endpoints. (See Exercise 1:3.7.) Then in our proof we announce a recipe for the choice of $[a_i, b_i]$ at each stage. At the kth step in the proof we simply find the first interval I_p in the sequence I_1, I_2, I_3, ... that has the three properties that

1. $I_p \subset [a_{k-1}, b_{k-1}]$,
2. $c_k \notin I_p$, and
3. the length of I_p is less than 2^{-k}.

Then we set $[a_k, b_k] = I_p$. Since, at each stage, only a finite number of intervals need be considered in order to arrive at our interval I_p, we need much less than the full force of the axiom of choice to make the determination for us.

In most aspects of real analysis the use of the axiom of choice is unavoidable and is undertaken without apology (or perhaps even without explicit mention). Later, in Section 1.10, when we construct a nonmeasurable set we shall have to invoke the axiom of choice; there we shall mention the fact quite clearly and comment on what is known about the situation if the axiom of choice were not to be allowed. In many other parts of this work we shall follow the usual custom of real analysts and apply the axiom when needed without much concern as to whether it can be avoided or not. This attitude has taken some time to develop. The early French analysts Baire, Borel, and Lebesgue relied on the axiom implicitly in their early works and then, after Zermelo gave a formal enunciation, reacted negatively. For most of his life Lebesgue remained deeply opposed, on philosophical grounds, to its use.[1]

Further material on the axiom of choice appears in Section 1.11. This axiom is known to be independent of the rest of the axioms of set theory known as ZF (Zermelo–Fraenkel set theory, without the axiom of choice). Kurt Gödel (1906–1978) showed that the axiom of choice is consistent with the remaining axioms provided one assumes that the remaining axioms are consistent themselves. (This is something that cannot be proved, only assumed.)

Exercises

1:3.1 Show Theorem 1.7 using a diagonal argument (or find a proof in a standard text).

1:3.2 Prove that every subset of a countable set is countable.

1:3.3 Let S be countable and let S^k ($k \in \mathbb{N}$) denote the set of all sequences of length k formed of elements of S. Show that S^k is countable.

1:3.4 Prove that a union of a sequence of countable sets is countable.

1:3.5 Let S be countable. Show that the set of all sequences of finite length formed of elements of S is countable.

1:3.6 Show that the set of rational numbers is countable.

1:3.7$\diamondsuit$ Show that the set of intervals with rational numbers as endpoints is countable.

1:3.8 Show that the set of algebraic numbers is countable.

1:3.9 Show that every subset of a countable $\mathcal{G}_\delta$ set is again a countable $\mathcal{G}_\delta$ set.

1:3.10 Show that scattered sets are countable. [Hint: Consider all intervals (a, b) with rational endpoints such that $S \cap (a, b)$ is countable.]

[1] For an interesting historical essay on the subject, see G. H. Moore, "Lebesgue's measure problem and Zermelo's axiom of choice: the mathematical effect of a philosophical dispute," *Ann. N. Y. Acad. Sci.*, 412 (1983), pp. 129–154.

1:3.11 Show that every Cantor set is uncountable.

1:3.12 Prove that every infinite set contains an infinite and countable subset. [Hint: Use the axiom of choice.]

1:3.13 (Cantor–Bendixson) Show that every closed set C of real numbers can be written as the union of a perfect set and a countable set. Moreover, there is only one decomposition of C into two disjoint sets, one perfect and the other countable.

1:3.14 Show that the set of discontinuities of a monotone nondecreasing function f is (at most) countable. [Hint: Use the fact that the right-hand and left-hand limits $f(x+0)$ and $f(x-0)$ must both exist. Consider the sets

$$\{x : f(x+0) - f(x-0) < 1/n\}.$$

1:3.15 Let C be any countable set. Show that there is a monotone function f such that C is precisely the set of discontinuities of f. [Hint: Write $C = c_1, c_2, c_3, \dots$ and construct $f(x) = \sum_{c_i<x} 2^{-i}$.]

1:3.16 Show that the family of all finite subsets of a countable set is countable.

1:3.17 Let $E \subset \mathbb{R}$ and let A consist of the right-isolated points of E (that is, $x \in A$ if $x \in E$ and there exists some $y > x$ so that $(x, y) \cap E = \emptyset$). Show that A is countable.

1:3.18$\diamondsuit$ Let $\mathcal{S}$ be a collection of nondegenerate closed intervals covering a set $E \subset \mathbb{R}$. Prove that there is a countable subset of $\mathcal{S}$ that also covers E. Show by example that there need not be a finite subset of $\mathcal{S}$ that covers E. [Hint: You may wish to use Exercise 1:3.17.]

1.4 Uncountable Cardinals

Every set can be assigned a *cardinal number* that denotes its size. So far we have listed just the cardinal numbers

$$0, 1, 2, 3, 4, \dots, \aleph_0, \tag{1}$$

and we recall that the set of real numbers must have a cardinality different from these since it is infinite and is uncountable.

To handle cardinality questions for arbitrary sets, we require the following definitions and facts that can be developed from the axioms of set theory. If the elements of two sets A and B can be placed into a one-one correspondence, then we say that A and B are *equivalent* and we write $A \sim B$. For any two sets A and B, only three possibilities can arise:

1. A is equivalent to some subset of B and, in turn, B is equivalent to some subset of A.

2. A is equivalent to some subset of B, but B is equivalent to no subset of A.
3. B is equivalent to some subset of A, but A is equivalent to no subset of B.

The other possibility that might be imagined (that A is equivalent to no subset of B and B is equivalent to no subset of A) can be proved not to occur. In the first of these three cases, it can be proved that $A \sim B$ (Bernstein's theorem). These facts allow us to assign to every set A a symbol called *the cardinal number of* A. Then, if a is the cardinal number of A and if b is the cardinal number of B, cases 1, 2, and 3 can be described by the relations

1. $a = b$.
2. $a < b$.
3. $a > b$.

This orders the cardinal numbers and allows us to extend the list (1) above. We write $\aleph_1$ for the next cardinal in the list,

$$0 < 1 < 2 < 3 < 4 < \cdots < \aleph_0 < \aleph_1,$$

and we write c for the cardinality of the set $\mathbb{R}$. That the cardinals can be, in fact, written in such a list and that there is a "next" cardinal is one of the most important features of this subject. (This is called a well-order and is discussed in the next section.)

Cantor presumed that $c = \aleph_1$ but, despite great effort, was unable to prove it. It has since been established that this cannot be determined within the axioms of set theory and that those axioms are consistent if it is assumed and also consistent if it is negated. (More precisely, *if* the axioms of set theory are consistent, then they remain consistent if $c = \aleph_1$ is added or if $c > \aleph_1$ is added.) The assumption that $c = \aleph_1$ is called the *continuum hypothesis* (abbreviated CH) and is often assumed in order to construct exotic examples. But in all such cases one needs to announce clearly that the construction has invoked the continuum hypothesis.

Here are some of the rudiments of cardinal arithmetic, adequate for all the analysis that we shall pursue.

1. Let a and b be cardinal numbers for disjoint sets A and B. Then $a + b$ denotes the cardinality of the set $A \cup B$.
2. Let a and b be cardinal numbers for sets A and B. Then $a \cdot b$ denotes the cardinality of the Cartesian product set $A \times B$.
3. Let a_i $(i \in I)$ be cardinal numbers for mutually disjoint sets A_i $(i \in I)$. Then $\sum_{i \in I} a_i$ denotes the cardinality of the set $\bigcup_{i \in I} A_i$.
4. Let b be the cardinal number for a set B; then 2^b denotes the cardinality of the set of all subsets of B.

5. Finally, let a and b be cardinal numbers for sets A and B. Then a^b denotes the cardinality of the set of all functions mapping B into A.

For finite sets A and B, it is easy to count explicitly the sets in (4) and (5). There are 2^b distinct subsets of B and there are a^b distinct functions mapping B into A. Note that with $A = \{0, 1\}$, so that $a = 2$, these two meanings in (4) and (5) give the same cardinal in general. (That is, the set of all subsets of B is equivalent to the set of all mappings from $B \to \{0, 1\}$. See Exercise 1:4.5.)

This suggests a notation that we shall use throughout. By A^B we mean the set of functions mapping B into A. Hence by 2^B we mean the set of all subsets of B (sometimes called the power set of B).

One might wish to know the following theorems:

Theorem 1.9 *For every cardinal number a, $2^a > a$.*

Theorem 1.10 $\aleph_0 \cdot \aleph_0 = \aleph_0$.

Theorem 1.11 $c + \aleph_0 = c$ *and* $c + c = c$.

Theorem 1.12 $c \cdot c = c$.

Theorem 1.13 $2^{\aleph_0} = c$.

In particular, the continuum hypothesis can then be written as

$$\text{CH: } 2^{\aleph_0} = \aleph_1$$

which is its most familiar form.

Exercises

1:4.1 Prove that $(0, 1) \sim \mathbb{R}$.

1:4.2 (Bernstein's theorem) If $A \sim B_1 \subset B$ and $B \sim A_1 \subset A$, then $A \sim B$. (Not at all an easy theorem.)

1:4.3 Prove that any open interval is equivalent to any closed interval without invoking Bernstein's theorem.

1:4.4 Show that every Cantor set has cardinality c.

1:4.5 Show that the set of all subsets of B is equivalent to the set of all mappings from $B \to \{0, 1\}$. [Hint: Consider χ_A for any $A \subset B$.]

1:4.6 Show that the class of functions continuous on the interval $[0, 1]$ has cardinality c. [Hint: If two continuous functions agree on each rational in $[0, 1]$, then they are identical.]

1:4.7$\diamondsuit$ Show that the family of all closed subsets of $\mathbb{R}$ has cardinality c.

1.5 Transfinite Ordinals

The set $\mathbb{N}$ of natural numbers is the simplest, nontrivial example of what we shall call a well-ordered set. The usual order (that is, $m < n$) on the natural numbers has the following properties.

1. For any $n \in \mathbb{N}$, it is not true that $n < n$.
2. For any distinct $n, m \in \mathbb{N}$, either $m < n$ or $n < m$.
3. For any n, m, $p \in \mathbb{N}$, if $n < m$ and $m < p$, then $n < p$.
4. Every nonempty subset $S \subset \mathbb{N}$ has a first element (i.e., there is an element $n_0 \in S$ so that $n_0 < s$ for every other element s of S).

It is precisely this set of properties that allows mathematical induction. Let P be a set of integers with the following properties:

(*i*) $1 \in P$.

(*ii*) For all $n \in \mathbb{N}$, $m \in P$ for each $m < n$ implies that $n \in P$.

Then $P = \mathbb{N}$. Indeed, if P is not $\mathbb{N}$, then $P' = \mathbb{N} \setminus P$ is nonempty and so has a first element n_0. That element cannot be 1. All predecessors of n_0 are in P, which, by property (ii), implies that $n_0 \in P$, which is not possible.

Mathematical induction can be carried out on any set that has these four properties, and so we are not confined to induction on integers. We say that a set X is *linearly ordered* and that "$<$" is a *strict linear order* on X if properties (1), (2), and (3) hold for this set and this relation. We say that X is *well-ordered* if all four properties (1) to (4) hold. If X is well-ordered and x_0 is in X, then the set of all elements that precede x_0 is called an *initial segment* of X.

The following two facts are fundamental. The first can be proved from the axiom of choice and is, in fact, equivalent to the axiom of choice. The second essentially defines the countable ordinals.

1.14 (Well-ordering principle) *Every set can be well-ordered. That is, for any nonempty set X there is a relation $<$ that is a strict linear order on X making it a well-ordered set.*

1.15 (Countable ordinals) *There exists an uncountable, well-ordered set X with an order relation $<$ so that*

1. *X has a last element denoted Ω.*
2. *For every $x_0 \in X$ with $x_0 \neq \Omega$ the initial segment*

$$\{x \in X : x < x_0\}$$

is countable.

3. *There is an element $\omega \in X$ such that*

$$\{x \in X : x < \omega\} = \{0, 1, 2, 3, \dots\}$$

and $<$ has its usual meaning in the set of nonnegative integers.

Thus the set $\{0, 1, 2, 3, \dots\}$ of nonnegative integers is an initial segment of X. We can think of X as looking like a long list starting with 0 and continuing just until uncountably many elements have been listed:

$$0 < 1 < 2 < \cdots < \omega < \omega + 1 < \omega + 2 < \cdots < \omega^2 < \omega^2 + 1 < \cdots < \Omega.$$

We call all the elements of X *ordinals.* Each element prior to ω is called a *finite ordinal*. Each element from then, but prior to the last one Ω, is called a *countable ordinal*. The element Ω is called *the first uncountable ordinal.*

We can identify an element x with the initial segment consisting of the elements that precede it. Thus each element of X can be thought of as a subset of X, and we see that each element (other than the last element Ω) is finite or countable considered as a set. The first infinite ordinal is ω and the first uncountable ordinal is Ω. The cardinality of Ω (i.e., the cardinality of $X \setminus \{\Omega\}$ or, the same thing, the cardinality of X) is $\aleph_1$. Unless we assume the continuum hypothesis, we do not know if this is c.

One can develop a bit of intuition about this situation by making the following observation. Any finite collection of finite ordinals ξ_1, ξ_2, $\dots \xi_n$ will stay away from ω in the sense that there is a finite ordinal ξ so that, for each i,

$$\xi_i < \xi < \omega.$$

The reason for this is that a finite union of finite sets is again finite. Similarly any countable collection of countable ordinals ξ_1, ξ_2, $\dots$ will stay away from Ω in the sense that there is a countable ordinal ξ so that, for each i,

$$\xi_i < \xi < \Omega.$$

The reason for this is that a countable union of countable sets is again countable. This observation is most useful.

If we do assume the continuum hypothesis (CH), then the real numbers (or any set of cardinality $2^{\aleph_0}$) can be well-ordered as described above. If we do not wish to assume CH, we can still perform a transfinite induction. In this case the version of Theorem 1.15 that we shall use is the following:

Lemma 1.16 *Any set X of cardinality $2^{\aleph_0}$ can be well-ordered in such a way that for each $x \in X$ the set of all predecessors of x has cardinality strictly less than $2^{\aleph_0}$.*

Every element, except the last, of a well-ordered set has an immediate successor defined as the first element of the set of all later elements; for any $x \in X$, if x is not the last element then the immediate successor of x can be written as $x+1$. Note, however, that elements need not have immediate predecessors. Any element (ω and Ω in Theorem 1.15 are examples) that does not have an immediate predecessor is called a *limit ordinal.* We shall later define ordinals as *even* and *odd* in a way that extends the usual meaning. The first element 0 and every limit ordinal is thought of as even, a successor of an even is odd, and a successor of an odd is even. In this way every ordinal is designated as either odd or even.

As an illustration of the method of transfinite induction, let us prove a simple covering property of intervals using the ideas. We show that from a certain family of subintervals $[x, y) \subset [a, b)$ a disjoint subcover can be selected. The argument is, perhaps, the most transparent and intuitive use of a transfinite sequence.

Lemma 1.17 *Let $\mathcal{C}$ be a family of subintervals of $[a, b)$ such that for every $a \leq x < b$ there exists y, $x < y < b$ so that $[x, y) \in \mathcal{C}$. Then there is a countable disjoint subfamily $\mathcal{E} \subset \mathcal{C}$ so that*

$$\bigcup_{[x,y)\in\mathcal{E}} [x, y) = [a, b).$$

Proof. Set $x_0 = a$. By the hypotheses, we can choose an interval $[x_0, x_1) \in \mathcal{C}$ and then an interval $[x_1, x_2) \in \mathcal{C}$ and, once again, $[x_2, x_3) \in \mathcal{C}$, and so on. If $x_n \to b$, then take $\mathcal{E} = \{[x_{i-1}, x_i)\}$ and we are done. Otherwise, $x_n \to c$ with $c < b$. Then we can carry on with $[c, y_1)$, $[y_1, y_2)$, and so on, until we eventually reach b.

Well not quite! The idea seems sound, but a proper expression of this requires a transfinite sequence and transfinite induction. Set $x_0 = a$ and choose $x_1 < b$ so that $[x_0, x_1) \in \mathcal{C}$. Suppose that for each ordinal α we have chosen $x_\beta < b$ in such a way that $[x_\beta, x_{\beta+1}) \in \mathcal{C}$ for every β for which $\beta + 1 < \alpha$. Then we can choose x_α as follows: (i) If α is a limit ordinal, take $x_\alpha = \sup_{\beta<\alpha} x_\beta$. (ii) If α is not a limit ordinal, let α_0 be the immediate predecessor of α and suppose that $x_{\alpha_0} < b$. Take $x_\alpha < b$ so that $[x_{\alpha_0}, x_\alpha) \in \mathcal{C}$. The process stops if $x_{\alpha_0} = b$.

Inside each interval $[x_{\alpha-1}, x_\alpha)$ we can choose distinct rationals. Hence this process must stop in a countable number of steps. The family $\mathcal{E} = \{[x_{\alpha-1}, x_\alpha)\}$ is a countable disjoint subfamily of $\mathcal{C}$ so that $\bigcup_{[x,y)\in\mathcal{E}}[x, y) = [a, b)$. ■

This is admittedly a very sketchy introduction to the ordinals, but adequate for our purposes. The serious reader will take a course in transfinite arithmetic or consult textbooks that take the time to develop this subject from first principles.

Exercises

1:5.1 Prove the assertion 1.17 without using transfinite induction.

[Hint: Say that a point $z > a$ can be reached if there is a countable disjoint subfamily $\mathcal{E} \subset \mathcal{C}$ so that $\bigcup_{[x,y)\in\mathcal{E}}[x, y) \supset [a, z)$. Take the sup of all points that can be reached.]

1:5.2 Define a "natural" order on $\mathbb{N} \times \mathbb{N}$ and determine if it is a well-ordering.

1:5.3 Let A and B be linearly ordered sets. A natural order (the lexicographic order) on $A \times B$ is defined as $(a, b) \preceq (c, d)$ if $a \preceq c$ or if $a = c$ and $b \preceq d$. Show that this is a linear order. If A, B are well-ordered, then is this a well-ordering of $A \times B$? Describe the initial segments of $A \times B$.

1:5.4 A limit ordinal is an ordinal with no immediate predecessor. Show that ω and Ω are limit ordinals.

1.6 Category

Recall that a set E of real numbers is *nowhere dense* if for every open interval (a, b) there is a subinterval $(c, d) \subset (a, b)$ that contains no points of E. That is, it is nowhere dense if it is dense in no interval. Loosely, a nowhere dense set is shot full of holes.

A set is *first category* if it can be expressed as a union of a sequence of nowhere dense sets. Any set not of the first category is said to be of the *second category*. Nowhere dense sets are, in a certain sense, *very* small. Thus first category sets are, in the same sense, merely small. Second category sets are then not small. The complement of a first category set must apparently be quite large; such sets are said to be *residual*. Here, this notion of smallness should be taken as merely providing an intuitive guide to how these concepts can be interpreted.

A fundamental theorem of René Baire (1874–1932) proved in 1899 asserts that every interval is second category. (It was proved too by W. F. Osgood two years earlier, but credit is almost always assigned to Baire.) Note that the proof here is nearly identical with the proof of the fact that intervals are uncountable; indeed, this theorem contains Theorem 1.7.

Theorem 1.18 (Baire) *No interval $[a, b]$ is first category.*

Proof. Suppose not. Then $[a, b]$ can be written as the union of a sequence of sets $C_1, C_2, C_3, \ldots$ each of which is nowhere dense. Select an interval $[a_1, b_1] \subset [a, b]$ so that $C_1 \cap [a_1, b_1] = \emptyset$ and so that $b_1 - a_1 < 1/2$. Continuing inductively, we find a nested sequence of intervals $\{[a_i, b_i]\}$ with lengths $b_i - a_i < 2^{-i} \to 0$ and with $C_i \cap [a_i, b_i] = \emptyset$ for each i.

By Theorem 1.2, there is a unique point $c \in [a, b]$ common to each of the intervals. This point cannot belong to any C_i and this is a contradiction, since every point of the interval $[a, b]$ was to belong to some member of the sequence $C_1, C_2, C_3, \ldots$. ■

A *category argument* is one that appeals to Baire's theorem. One can prove the existence of sets or points (or even functions) by these means. It has become one of the standard tools of the analyst and plays a fundamental role in many investigations. We illustrate with an application showing that an important class of functions has certain continuity properties. A function f is said to be *in the first class of Baire* or *Baire 1* if it can be written as the pointwise limit of a sequence of continuous functions. A Baire 1 function need not be continuous. Does a Baire 1 function have any points of continuity? The existence of such points is obtained by a category argument.

Theorem 1.19 (Baire) *Every Baire 1 function is continuous except at the points of a set of the first category.*

Proof. Recall that we use $\omega_f(x)$ to denote the oscillation of the function f at a point x (see Exercise 1:1.8). The proof follows from the fact that for each $\varepsilon > 0$ the set of points

$$F(\varepsilon) = \{x : \omega_f(x) \geq \varepsilon\}$$

is nowhere dense. [This is because the set of points of discontinuity of f can be written as $\bigcup_{n=1}^{\infty} F(\frac{1}{n})$.] Let I be any interval; let us search for a subinterval $J \subset I$ that misses $F(\varepsilon)$. The proof is complete once we find J.

Let f be the pointwise limit of a sequence of continuous functions $\{f_i\}$ and write

$$E_n = \bigcap_{i=n}^{\infty} \bigcap_{j=n}^{\infty} \{x \in I : |f_i(x) - f_j(x)| \leq \varepsilon/2\}.$$

Each set E_n is closed (since the f_i are continuous), and the sequence of sets E_n expands to cover all of I (since $\{f_i\}$ converges everywhere). By Baire's theorem (Theorem 1.18), there must be an interval $J \subset I$ and a set E_n dense in J. (Otherwise, we have just expressed I as the union of a sequence of nowhere dense sets, which is impossible.) But the sets here are closed, so this means merely that E_n *contains* the interval J. For this n (which is now fixed) we have

$$|f_i(x) - f_j(x)| \leq \varepsilon/2$$

for all i, $j \geq n$ and for all $x \in J$. In this inequality set $j = n$, and let $i \to \infty$ to obtain

$$|f(x) - f_n(x)| \leq \varepsilon/2.$$

Now we see that J misses the set $F(\varepsilon)$. Our last inequality shows that f is close to the continuous function f_n on J, too close to allow the oscillation of f at any point in J to be greater than ε. Thus there is no point in J that is also in $F(\varepsilon)$. ■

Theorem 1.19 very nearly characterizes Baire 1 functions. One needs to state it in a more general form, but one that can be proved by the same method. A function f is Baire 1 if and only if f has a point of continuity relative to any perfect set.

Exercises

1:6.1 Prove Theorem 1.18 using induction in place of the axiom of choice. (We used this axiom here without comment.) [Hint: See the discussion in Section 1.3.]

1:6.2 Show that every subset of a set of first category is first category.

1:6.3 Show that every finite set is nowhere dense, and show that every countable set is first category.

1:6.4 Show that every union of a sequence of sets of first category is first category.

1:6.5 Show that every intersection of a sequence of residual sets is residual.

1:6.6 Show that the complement of a set of second category may be either first or second category.

1:6.7 If $\overline{E}$ is first category, prove that E is nowhere dense.

1:6.8 Show that a set of type $\mathcal{G}_\delta$ that is dense (briefly, "a dense $\mathcal{G}_\delta$") is residual.

1:6.9 Let $S \subset \mathbb{R}$. Call a point $x \in \mathbb{R}$ first category relative to S if there is some interval (a, b) containing x so that $(a, b) \cap S$ is first category. Show that the set

$$\{x \in S : \ x \text{ is first category relative to } S\}$$

is first category.

1:6.10 The rationals $\mathbb{Q}$ form a set of type $\mathcal{F}_\sigma$. Are they of type $\mathcal{G}_\delta$?

1:6.11 Does there exist a function continuous at every rational and discontinuous at every irrational? Does there exist a function continuous at every irrational and discontinuous at every rational? [Hint: Use Exercises 1:1.10 and 1:1.11.]

1:6.12 Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions converging pointwise to a function f. If the convergence is uniform, prove that there is a finite number M so that $|f_n(x)| < M$ for all n and all $x \in [0, 1]$. Even if the convergence is not uniform, show that there must be a subinterval $[a, b] \subset [0, 1]$ and a finite number M so that $|f_n(x)| < M$ for all n and all $x \in [a, b]$.

1:6.13 Theorem 1.19 as stated does not characterize Baire 1 functions. Show that a function is discontinuous except at the points of a first category set if and only if it is continuous at a dense set of points.

1:6.14 (Fort's theorem) If f is discontinuous at the points of a dense set, show that the set of points x, where $f'(x)$ exists, is of the first category.

1:6.15 If f is Baire 1, show that every set of the form $\{x : f(x) > \alpha\}$ is of type $\mathcal{F}_\sigma$ and every set of the form $\{x : f(x) \geq \alpha\}$ is of type $\mathcal{G}_\delta$. (The converse is also true.) [Hint: Use Exercise 1:1.24.]

1.7 Outer Measure and Outer Content

By the 1880s it was recognized that integration theory was intimately linked to the notion of measuring the "length" of subsets of $\mathbb{R}$ or the "area" of subsets of $\mathbb{R}^2$. Peano (1858–1932), Jordan (1838–1922), Cantor (1845–1918), Borel (1871–1956) and Lebesgue (1875–1941) are the main contributors to this development, but many authors addressed these problems.

At the end of the century there were two main competing notions that allowed the concept of length to be applied to all sets of real numbers. The Peano–Cantor–Jordan treatment defines a notion of *outer content* in terms of approximations that employ finite sequences of intervals. The Borel–Lebesgue method defines a notion of *outer measure* in terms of approximations that employ infinite sequences of intervals. The two methods are closely related, and it is, perhaps, best to study them together. The outer measure concept now dominates analysis and has left the outer content idea as a historical curiosity. Nonetheless, by seeing the two together and appreciating the difficulties that the early mathematicians had in coming to the correct ideas about measure, we can more easily learn this theory.

For any interval I we shall write $|I|$ for its length. Thus $|[a,b]| = |(a,b)| = b-a$ and $|(-\infty,a)| = |(b,+\infty)| = +\infty$. We include the empty set as an open interval and consider it to have zero length.

Definition 1.20 Let E be an arbitrary set of real numbers. We write

$$c^*(E) = \inf\left\{\sum_{i=1}^{n} |I_i| : E \subset \bigcup_{i=1}^{n} I_i\right\}$$

and

$$\lambda^*(E) = \inf\left\{\sum_{i=1}^{\infty} |I_i| : E \subset \bigcup_{i=1}^{\infty} I_i\right\},$$

where in the two cases $\{I_i\}$ is a finite (infinite) sequence of open intervals covering E.

We refer to the set function c^* as the outer content (or Peano-Jordan content) and λ^* as (Lebesgue) outer measure. Note that c^* is not of much interest for unbounded sets since it must assign the value $+\infty$ to each. Each of these set functions assigns a value (thought of as a "length") to each subset $E \subset \mathbb{R}$.

The following properties are essential and can readily be proved directly from the definitions. All the properties claimed for the Lebesgue outer measure in this chapter will be fully justified in Chapters 2 and 3.

Theorem 1.21 *The outer content and the outer measure have the following properties:*

1. $c^*(\emptyset) = \lambda^*(\emptyset) = 0$.

2. *For every interval I, $c^*(I) = \lambda^*(I) = |I|$.*
3. *For every set E, $c^*(E) \geq \lambda^*(E)$.*
4. *For every compact set K, $c^*(K) = \lambda^*(K)$.*
5. *For a finite sequence of sets $\{E_i\}$, $c^*(\bigcup_{i=1}^n E_i) \leq \sum_{i=1}^n c^*(E_i)$.*
6. *For any sequence of sets $\{E_i\}$, $\lambda^*(\bigcup_{i=1}^\infty E_i) \leq \sum_{i=1}^\infty \lambda^*(E_i)$.*
7. *Both c^* and λ^* are translation invariant.*
8. *For any set E, $c^*(E) = c^*(\overline{E})$.*

This last property, $c^*(E) = c^*(\overline{E})$, would nowadays be considered a flaw in the definition of a generalized length function. For a long time, though, it was felt that this property was essential: if a set $A \subset B$ is dense in B, then "surely" the two sets should be assigned the same length.

Exercises

1:7.1 Show that, for every interval I, $c^*(I) = \lambda^*(I) = |I|$.

1:7.2 Show that, for every set E, $c^*(E) \geq \lambda^*(E)$, and give an example to show that the inequality can occur.

1:7.3 Show that, for every compact set K, $c^*(K) = \lambda^*(K)$.

1:7.4 Show that, for any set E, $c^*(E) = c^*(\overline{E})$.

1:7.5◇ Show that, for every finite sequence of sets $\{E_i\}$,
$$c^*\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n c^*(E_i).$$

1:7.6◇ Show that, for every infinite sequence of sets $\{E_i\}$,
$$\lambda^*\left(\bigcup_{i=1}^\infty E_i\right) \leq \sum_{i=1}^\infty \lambda^*(E_i).$$

1:7.7 Show that both c^* and λ^* are translation invariant.

1:7.8◇ Let G be an open set with components $\{(a_i, b_i)\}$. Show that
$$\lambda^*(G) = \sum_{i=1}^\infty (b_i - a_i),$$
but that $c^*(G)$ may be strictly larger.

1:7.9◇ Let G be an open subset of an interval $[a, b]$ and write $K = [a, b] \setminus G$. Show that
$$c^*(K) = \lambda^*(K) = b - a - \lambda^*(G)$$
but that $c^*(K) = b - a - c^*(G)$ may be false.

1.8 Small Sets

In many studies of analysis there is a natural class of sets whose members are "small" or "negligible" for some purposes. We have already encountered the classes of countable sets, nowhere dense sets, and first category sets that can, with some justice, be considered small. In addition, the class of sets of zero outer content and the class of sets of zero outer measure also play the role of small sets in many investigations. Each of these classes enters into certain problems in that if a set is small in one of these senses it may be neglected in the analysis.

After some thought, one expects that in order to apply the term "small" to the members of some class of sets $\mathcal{S}$ one would require that finite (or perhaps countable) unions of small sets be small, that subsets of small sets be small, and that no interval be allowed to be small. More formally, the properties of $\mathcal{S}$ that seem to be desirable are as follows:

1. The union of a finite [countable] collection of sets in $\mathcal{S}$ is itself in $\mathcal{S}$.
2. Any subset of a set in $\mathcal{S}$ is itself in $\mathcal{S}$.
3. No interval (a, b) belongs to $\mathcal{S}$.

We say that $\mathcal{S}$ is an *ideal* of sets if properties (1) and (2) hold. If the stronger version of (1) holds (with countable unions), then we say that $\mathcal{S}$ is a *σ-ideal* of sets. We have, by now, a number of different ideals of sets that can be viewed as composed of small sets. Let us summarize.

Theorem 1.22

1. *The nowhere dense sets form an ideal.*
2. *The first category sets form a σ-ideal.*
3. *The finite sets form an ideal.*
4. *The countable sets form a σ-ideal.*
5. *The sets of outer content zero form an ideal.*
6. *The sets of outer measure zero form a σ-ideal.*

There are some obvious connections and some surprising contrasts. Certainly, finite sets are nowhere dense and of outer content zero. Countable sets are first category and of outer measure zero. The other relations are not so easy or so immediate. Let us first compare perfect, nowhere dense sets and sets of outer content zero.

In the early days of the study of the Riemann integral (before the 1870s) it was recognized that sets of zero outer content played an important role as the sets that could be neglected in arguments. Nowhere dense sets at first appeared to be equally negligible, and there was some confusion as to the distinction. It is easy to check that a set of zero outer content must be nowhere dense; lacking any easy examples to the contrary, one might assume, as did a number of mathematicians, that the converse is also true. The following construction then comes as a bit of a surprise and shook the intuition of many nineteenth-century mathematicians. This shows that Cantor sets (nonempty, perfect, nowhere dense sets) can have relatively large measure (or content, since the two notions agree for compact sets) even though they appear to be small in some other sense. Constructions of this sort were given by H. J. Smith (1826–1883), du Bois-Reymond (1831–1889) and others.

Theorem 1.23 *Let $0 \leq \alpha < 1$. Then there is a Cantor set $C \subset [0,1]$ whose outer content (measure) is exactly α.*

Proof. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of positive numbers with

$$\sum_{k=1}^{\infty} \alpha_k = 1 - \alpha.$$

Let I_1 be an open subinterval of $I_0 = [0,1]$, with $|I_1| = \alpha_1$ chosen in such a way that the set $A_1 = I_0 \setminus I_1$ consists of two closed intervals, each of length less than $1/2$. At the second stage we shall remove from A_1 two further intervals, one from inside each of the two closed intervals, leaving $A_2 = I_0 \setminus (I_1 \cup I_2 \cup I_3)$ consisting of four intervals. We define the procedure inductively. After the nth stage, we have selected

$$1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$$

nonoverlapping open intervals $I_1, \ldots, I_{2^n-1}$ with

$$\sum_{k=1}^{2^n-1} |I_k| = \sum_{i=1}^{n} \alpha_i,$$

and the set

$$A_n = I_0 \setminus \bigcup_{k=1}^{2^n-1} I_k$$

consists of 2^n closed intervals, each of length less than $1/n$, and $\lambda^*(A_n) = 1 - \sum_{i=1}^{n} \alpha_i$. (Note that the lengths of the closed intervals go to zero as n goes to infinity.)

Now let $C = \bigcap_{n=1}^{\infty} A_n$ and $B = I_0 \setminus C$. Then C is closed, B is open, and $B = \bigcup_{k=1}^{\infty} I_k$, with the intervals I_k pairwise disjoint. We see, by Exercise 1:7.8, that

$$\lambda^*(B) = \sum_{k=1}^{\infty} |I_k| = \sum_{k=1}^{\infty} \alpha_k = 1 - \alpha$$

and hence, by Exercise 1:7.9, that

$$\lambda^*(C) = 1 - \lambda^*(B) = \alpha.$$

Thus C is a nowhere dense closed subset of I_0 of measure α, and B is a dense open subset of I_0 of measure $1-\alpha$. ■

Theorem 1.23 shows the contrast between sets of zero content and nowhere dense sets. As a result, we should not be surprised that there is a similar contrast between sets of outer measure zero and sets of the first category. The next theorem expresses this in a remarkable way. Every set of reals can be expressed as the union of two "small" sets (small in different ways). Be sure to notice that we are using outer *measure*, not outer content, in the theorem.

Theorem 1.24 *Every set of real numbers can be written as the disjoint union of a set of outer measure zero and a set of the first category.*

Proof. Let $\{q_i\}$ be a listing of all the rational numbers. Denote by I_{ij} that open interval centered at q_i and with length 2^{-i-j}. Write $G_j = \bigcup_{i=1}^{\infty} I_{ij}$ and $B = \bigcap_{j=1}^{\infty} G_j$. Each G_j is a dense open set, and so B is residual and hence its complement $\mathbb{R} \setminus B$ is first category. But it is easy to check that B has measure zero. Thus every set $A \subset \mathbb{R}$ can be written as

$$A = (A \cap B) \cup (A \setminus B)$$

which is, evidently, the union of a set of outer measure zero and a set of the first category. ■

Exercises

1:8.1 Show that every set of outer content zero is nowhere dense, but there exist dense sets of outer measure zero.

1:8.2 Show that every set of outer measure zero that is also of type $\mathcal{F}_\sigma$ is first category.

1:8.3 Show that no interval can be written as the union of a set of outer content zero and a set of the first category.

1:8.4 Show that a set E of real numbers has outer measure zero if and only if there is a sequence of intervals $\{I_k\}$ such that each point of E belongs to infinitely many of the intervals and $\sum_{k=1}^{\infty} |I_k| < +\infty$.

1:8.5 Let B and C be the sets referenced in the proof of Theorem 1.23.

(a) Prove that B is dense and open in $[0,1]$, so C is nowhere dense and closed.

(b) Prove that C is perfect.

(c) Let $\{q_i\}$ be a listing of all the rational numbers. Denote by I_{ij} that open interval centered at q_i and with length 2^{-i-j}. Write $G_j = \bigcup_{i=1}^{\infty} I_{ij}$ and $B = \bigcap_{j=1}^{\infty} G_j$. Show that $\lambda^*(B) \le \lambda^*(G_j) \le 2^{-j}$ for each j, and deduce that $\lambda^*(B) = 0$.

(d) Prove Theorem 1.24 by using the fact that, in every interval $[a, b]$ and for every $\varepsilon > 0$, there is a Cantor set $C \subset [a, b]$ with measure exceeding $b - a - \varepsilon$.

1:8.6 Let $\mathcal{Z}$ be the class of all sets of real numbers that are expressible as countable unions of sets of outer content zero.

(a) Show that $\mathcal{Z}$ is a σ-ideal.

(b) Show that $\mathcal{Z}$ is precisely the σ-ideal of subsets of sets that are outer measure zero and $\mathcal{F}_\sigma$.

(c) Show that $\mathcal{Z}$ is not the σ-ideal of sets that are outer measure zero.

[Hint: Let C be a Cantor set whose intersection with each open interval is either empty or of positive outer measure. Choose a countable subset $D \subset C$, dense in C, and a $\mathcal{G}_\delta$ set $E \supset D$ of outer measure zero. Then $E \cap C$ is also outer measure zero but cannot be in $\mathcal{Z}$. (Use a Baire category argument.)]

1.9 Measurable Sets of Real Numbers

The outer measure and outer content have many desirable properties, but lack one that would seem to be an essential ingredient of a theory of lengths. They are not additive. If E_1 and E_2 are disjoint sets, then one expects the length of the union $E_1 \cup E_2$ to be the sum of the two lengths. In general, we have only that

$$c^*(E_1 \cup E_2) \le c^*(E_1) + c^*(E_2)$$

and

$$\lambda^*(E_1 \cup E_2) \le \lambda^*(E_1) + \lambda^*(E_2).$$

It is, however, not difficult to see that if E_1 and E_2 are not too "intertangled," then equality would hold. One seeks a class of sets on which the outer content or the outer measure is additive.

The key to creating these classes rests on a notion used by the Greeks in their investigations into area of plane figures. They considered that the area had been successfully found only if it had been computed by successive approximations from outside and by successive approximations from inside and that the two methods gave the same answer. Here our outer measure and outer content are obtained from outside approximations. Evidently, we should introduce an *inside* approximation, hence an inner measure and an inner content, and look for the class of sets on which the outer and inner estimates agree. In the case of content, this theory is due to Peano and Jordan. In the case of measure, the corresponding definition was used by Lebesgue.

Definition 1.25 Let E be a bounded set contained in an interval $[a, b]$. We write

$$c_*(E) = b - a - c^*([a, b] \setminus E)$$

and refer to $c_*(E)$ as the *inner content* of E and the set function c_* as the *inner content*.

Definition 1.26 Let E be a bounded set contained in an interval $[a, b]$. We write

$$\lambda_*(E) = b - a - \lambda^*([a, b] \setminus E)$$

and refer to $\lambda_*(E)$ as the *inner measure* of E and the set function λ_* as the *inner measure*.

It is left as an exercise to show that, in these two definitions, the particular interval $[a, b]$ that is chosen to contain the set E need not be specified. Measurability for bounded sets is defined as agreement of the inner and outer estimates.

Definition 1.27 A bounded set E is said to be *Peano–Jordan measurable* if $c_*(E) = c^*(E)$. A bounded set E is said to be *Lebesgue measurable* if $\lambda_*(E) = \lambda^*(E)$. An unbounded set E is measurable (in either sense) if $E \cap [a, b]$ is measurable in the same sense for each interval $[a, b]$. The class of Peano–Jordan measurable sets shall be denoted as $\mathcal{PJ}$. The class of Lebesgue measurable sets shall be denoted as $\mathcal{L}$.

When the inner and outer estimates agree, it makes sense to drop the subscripts and superscripts. Thus on the sets where $c_* = c^*$ we write $c = c_* = c^*$ and refer to c as the *content* or perhaps Peano–Jordan content. Similarly, on the Lebesgue measurable sets we write $\lambda = \lambda_* = \lambda^*$ and refer to λ as *Lebesgue measure.*

The families of sets so formed have strong properties, and the set functions c and λ defined on those families will have our desired additive properties. To have some language to express these facts, we shall use the following:

Definition 1.28 Let X be any set, and let $\mathcal{A}$ be a nonempty class of subsets of X. We say $\mathcal{A}$ is an *algebra of sets* if it satisfies the following conditions:

1. $\emptyset \in \mathcal{A}$.
2. If $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.
3. If $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A}$.

It is easy to verify that an algebra of sets is closed also under differences, finite unions, and finite intersections. For any set X, the class 2^X of all subsets of X is obviously an algebra. So is the class $\mathcal{A} = \{\emptyset, X\}$. An algebra that is also closed under *countable* unions is said to be a σ*–algebra.* Many of the classes of sets that arise in measure theory are algebras or σ–algebras.

Definition 1.29 Let $\mathcal{A}$ be an algebra of sets and let ν be an extended real-valued function defined on $\mathcal{A}$. If ν satisfies the following conditions, we say that ν is an *additive set function*.

1. $\nu(\emptyset) = 0$.
2. If $A \in \mathcal{A}$, $B \in \mathcal{A}$, and $A \cap B = \emptyset$, then $\nu(A \cup B) = \nu(A) + \nu(B)$.

A nonnegative additive set function is often called a *finitely additive measure*. Note that, for an additive set function ν and every finite disjoint sequence $\{E_1, E_2, \dots E_n\}$ of sets from $\mathcal{M}$,

$$\nu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \nu(E_i).$$

In general, we shall prefer a countable version of this definition. We say that ν is a *countably additive set function* if, for every infinite disjoint sequence $\{E_1, E_2, \dots\}$ of sets from $\mathcal{M}$ whose union $\bigcup_{i=1}^{\infty} E_i$ is also in $\mathcal{M}$,

$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i).$$

Using this language, we can now describe the classical measure theory developed in the nineteenth century by Peano, Jordan, and others and by Lebesgue at the beginning of the twentieth century. Peano–Jordan content is a finitely additive set function on an algebra of sets; Lebesgue measure is a countably additive set function on a σ–algebra of sets. The theorems that now follow describe this formally. The first is not difficult. The second will be proved in full as part of our more general development in Chapter 2. It is worth attempting a proof of these two theorems now in order to appreciate the technical problems that arise in the subject.

Theorem 1.30 *Let $\mathcal{PJ}[a,b]$ denote the family of all Peano–Jordan measurable subsets of an interval $[a,b]$. Then the class $\mathcal{PJ}[a,b]$ forms an algebra of subsets of $[a,b]$, and $c = c_* = c^*$ is a finitely additive set function on that algebra.*

Theorem 1.31 *The class $\mathcal{L}$ forms a σ–algebra of subsets of $\mathbb{R}$, and $\lambda = \lambda_* = \lambda^*$ is a countably additive set function on that σ–algebra.*

Theorem 1.30 is largely a historical curiosity. Theorem 1.31 is one of the fundamental results of elementary measure theory. Chapter 2 contains a complete proof of this in a more general setting.

Exercises

1:9.1 Let E be a bounded set contained in an interval $[a,b] \subset [a_1, b_1]$. Show that

$$c_*(E) = b - a - c^*([a,b] \setminus E) = b_1 - a_1 - c^*([a_1,b_1] \setminus E).$$

This shows that the definition of the inner content does not depend on the containing interval.

1:9.2 Let E be a bounded set contained in an interval $[a,b] \subset [a_1,b_1]$. Show that

$$\lambda_*(E) = b - a - \lambda^*([a,b] \setminus E) = b_1 - a_1 - \lambda^*([a_1,b_1] \setminus E).$$

This shows that the definition of the inner measure does not depend on the containing interval.

1:9.3 Verify that an algebra of sets is closed also under differences, finite unions, and finite intersections.

1:9.4 Show that each of the following classes of subsets of a set X is an algebra:

(a) The class $\{\emptyset, X\}$.

(b) The class of all subsets of X.

(c) The class of subsets E of X such that either E or $X \setminus E$ is finite.

(d) The class of subsets of X that have outer content zero or whose complement has outer content zero (here $X \subset \mathbb{R}$).

1:9.5 Show that each of the following classes of subsets of a set X is a σ–algebra:

(a) The class of all subsets of X.

(b) The class of all subsets of X that are countable or have a countable complement.

(c) The class of subsets of X that have outer measure zero or whose complement has outer measure zero (here $X \subset \mathbb{R}$).

1:9.6 Let $\mathcal{A}_i$ be an algebra of subsets of a set X for each $i \in I$. Show that $\bigcap_{i\in I} \mathcal{A}_i$ is also an algebra.

1:9.7 Let $\mathcal{A}_i$ be a σ–algebra of subsets of a set X for each $i \in I$. Show that $\bigcap_{i\in I} \mathcal{A}_i$ is also a σ–algebra.

1:9.8◇ Let $\mathcal{S}$ be a collection of subsets of a set X. Show that there is a smallest σ–algebra containing $\mathcal{S}$. (We call this the σ–algebra *generated by* $\mathcal{S}$.) [Hint: Consider the family of all σ–algebras that contain $\mathcal{S}$ (are there any?) and use Exercise 1:9.7.]

1:9.9 Show that every interval (closed, open, or half-closed) is both Peano–Jordan measurable and Lebesgue measurable.

1:9.10 Show that every set of outer content zero is Peano–Jordan measurable.

1:9.11 Show that every set of outer measure zero is Lebesgue measurable.

1:9.12◇ Suppose that a set E is Peano–Jordan measurable or Lebesgue measurable. Show that every translate $E + r = \{x + r : x \in E\}$ is also measurable in the same sense and has the same measure.

1:9.13◇ Show that the class of Peano–Jordan measurable sets and the class of Lebesgue measurable sets must both have cardinality 2^c. [Hint: Consider the subsets of a Cantor set of measure zero.]

1:9.14 Show that every Peano–Jordan measurable set is also Lebesgue measurable, but not conversely.

1:9.15 Theorems 1.30 and 1.31 might be misrepresented by saying that "c is merely finitely additive while λ is countably additive." Explain why it is that c is also countably additive.

1:9.16◇ Let E be a bounded subset of $\mathbb{R}$. Show that

$$\lambda_*(E) = \sup\{\lambda^*(F) : F \subset E, F \text{ closed}\}.$$

1:9.17 Prove that if $E_1 \subset E_2$ then $\lambda^*(E_1) \leq \lambda^*(E_2)$ and $\lambda_*(E_1) \leq \lambda_*(E_2)$.

1:9.18 Prove that both outer measure λ^* and inner measure λ_* are translation invariant functions defined on the class of all subsets of $\mathbb{R}$.

1:9.19 Show that $\lambda_*(E) \leq \lambda^*(E)$ for all $E \subset \mathbb{R}$.

1:9.20 Show that every σ–algebra of sets has either finitely many elements or uncountably many elements.

1.10 Nonmeasurable Sets

The measurability concept allows us to restrict the set functions c^* and λ^* to certain algebras of sets on which they are well behaved, in particular on which they are additive. Have we excluded any sets from consideration by this device? Are there sets that are so badly misbehaved with respect to the measurability definition that we cannot use them?

It is easy enough to characterize the class of Peano–Jordan measurable sets. Then we easily see which sets are not measurable and we see how to construct nonmeasurable sets. We address this first. The situation for Lebesgue measure is considerably more subtle and requires entirely different arguments.

Theorem 1.32 *A bounded set E of real numbers is Peano–Jordan measurable if and only if its set of boundary points has outer content zero.*

Proof. We may suppose that $\overline{E} \subset (a, b)$. Let $E_1 = \text{int}(E)$, $E_2 = \overline{E} \setminus E_1$, and $E_3 = (a, b) \setminus \overline{E}$. Suppose that $c^*(E_2) = 0$; we show that E is Peano–Jordan measurable. Let $\varepsilon > 0$. Choose a finite collection of disjoint open subintervals $\{I_i\}$ of (a, b) covering E_2 so that $\sum |I_i| < \varepsilon$. Let us consider the intervals complementary to $\{\overline{I_i}\}$ in (a, b). These are of two types, the ones interior to E_1 and the ones interior to E_3. We call the former $\{J_i\}$ and the latter $\{K_i\}$. Note that $\{I_i\}$, $\{J_i\}$ together cover E and $\{I_i\}$, $\{K_i\}$ together cover $(a,b) \setminus E$.

We have

$$b - a = \sum |I_i| + \sum |J_i| + \sum |K_i|.$$

Hence

$$\begin{aligned} b - a &= \left(\sum |I_i| + \sum |J_i|\right) + \left(\sum |I_i| + \sum |K_i|\right) - \sum |I_i| \\ &\geq c^*(E) + c^*((a,b) \setminus E) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, we can deduce that

$$c^*(E) + c^*((a,b) \setminus E) \leq b - a.$$

But the inequality

$$c^*(E) + c^*([a,b] \setminus E) \geq b - a$$

is true and

$$c^*([a,b] \setminus E) = c^*((a,b) \setminus E).$$

Thus $c^*(E) + c^*((a,b) \setminus E) = b - a$, and this establishes the measurability of the set E.

Conversely, suppose that we have this equality. Take a partition $\{I_i\}$ of $[a,b]$ using open intervals in such a way that

$$\sum \{|I_i| : I_i \cap E \neq \emptyset\} \leq c^*(E) + \varepsilon$$

and

$$\sum \{|I_i| : I_i \cap ([a,b] \setminus E) \neq \emptyset\} \leq c^*([a,b] \setminus E) + \varepsilon.$$

(We can do this by refining two partitions that handle each inequality separately.) Note that intervals that are used in both of these sums must contain a boundary point of E. Thus, because $b - a = \sum |I_i|$ and $c^*(E) + c^*([a,b] \setminus E) = b - a$, we can argue that

$$c^*(\overline{E} \setminus \mathrm{int}(E)) \leq \sum \{|I_i| : I_i \text{ contains a boundary point of } E\} \leq 2\varepsilon.$$

Since ε is arbitrary, $c^*(\overline{E} \setminus \mathrm{int}(E)) = 0$ as required. ■

In particular, note that it is an easy matter now to exhibit sets that are not Peano–Jordan measurable. The set of rational numbers in any interval must be nonmeasurable since every point is a boundary point. For a more interesting example, any Cantor set C will be Peano–Jordan measurable if and only if $c^*(C) = 0$ (see Exercise 1:10.1). We have seen in Theorem 1.23 how to construct Cantor sets in $[0,1]$ of positive outer content.

We turn now to a search for Lebesgue nonmeasurable sets. We can characterize Lebesgue measurable sets in a variety of ways. None of these, however, does anything to help to see whether there might exist sets that are nonmeasurable. The first proof that nonmeasurable sets must exist is due to G. Vitali (1875–1932). He showed that there cannot possibly exist a set function defined for all subsets of real numbers that is translation invariant, is countably additive, and extends the usual notion of length.

Theorem 1.33 *There exist subsets of $\mathbb{R}$ that are not Lebesgue measurable.*

Proof. Let $I = [-\frac{1}{2}, \frac{1}{2}]$. For $x, y \in I$, write $x \sim y$ if $x - y \in \mathbb{Q}$. For all $x \in I$, let

$$K(x) = \{y \in I : x - y \in \mathbb{Q}\} = \{x + r \in I : r \in \mathbb{Q}\}.$$

We show that $\sim$ is an equivalence relation. It is clear that $x \sim x$ for all $x \in I$ and that if $x \sim y$ then $y \sim x$. To show transitivity of $\sim$, suppose that $x, y, z \in I$ and $x - y = r_1$ and $y - z = r_2$ for $r_1, r_2 \in \mathbb{Q}$. Then $x - z = (x - y) + (y - z) = r_1 + r_2$, so $x \sim z$. Thus the set of all equivalence classes $K(x)$ forms a partition of I: $\bigcup_{x \in I} K(x) = I$, and if $K(x) \neq K(y)$, then $K(x) \cap K(y) = \emptyset$.

Let A be a set containing exactly one member of each equivalence class. (The existence of such a set A follows from the axiom of choice.) We show that A is nonmeasurable. Let $0 = r_0, r_1, r_2, \ldots$ be an enumeration of $\mathbb{Q} \cap [-1, 1]$, and define

$$A_k = \{x + r_k : x \in A\}$$

so that A_k is obtained from A by the translation $x \to x + r_k$.

Then

$$[-\tfrac{1}{2}, \tfrac{1}{2}] \subset \bigcup_{k=0}^{\infty} A_k \subset [-\tfrac{3}{2}, \tfrac{3}{2}]. \tag{2}$$

To verify the first inclusion, let $x \in [-\frac{1}{2}, \frac{1}{2}]$ and let x_0 be the representative of $K(x)$ in A. We have $\{x_0\} = A \cap K(x)$. Then $x - x_0 \in \mathbb{Q} \cap [-1, 1]$, so there exists k such that $x - x_0 = r_k$. Thus $x \in A_k$. The second inclusion is immediate: the set A_k is the translation of $A \subset [-\frac{1}{2}, \frac{1}{2}]$ by the rational number $r_k \in [-1, 1]$.

Suppose now that A is measurable. It follows (Exercise 1:9.12) that each of the translated sets A_k is also measurable and that $\lambda(A_k) = \lambda(A)$ for every k. But the sets $\{A_i\}$ are pairwise disjoint. If $z \in A_i \cap A_j$ for $i \neq j$, then $x_i = z - r_i$ and $x_j = z - r_j$ are in different equivalence classes. This is impossible, since $x_i - x_j \in \mathbb{Q}$. It now follows from (2) and the countable additivity of λ on $\mathcal{L}$ that

$$1 = \lambda([-\tfrac{1}{2}, \tfrac{1}{2}]) \leq \lambda(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \lambda(A_k) \leq \lambda([-\tfrac{3}{2}, \tfrac{3}{2}]) = 3. \tag{3}$$

Let $\alpha = \lambda(A) = \lambda(A_k)$. From (3), we infer that

$$1 \leq \alpha + \alpha + \cdots \leq 3. \tag{4}$$

But it is clear that no number α can satisfy both inequalities in (4). The first inequality implies that $\alpha > 0$, but the second implies that $\alpha = 0$. Thus A is nonmeasurable.

A variant of our argument (using Exercise 1:22.11) shows that $\lambda_*(A) = 0$ while $\lambda^*(A) > 0$. This, again, reveals why it is that A is nonmeasurable. ■

Many of the ideas that appear in this section, including the exercises, will reappear, in abstract settings as well as in concrete settings, in later chapters.

The proof has invoked the axiom of choice in order to construct the nonmeasurable set. One might ask whether it is possible to give a more constructive proof, one that does not use this principle. This question belongs to the subject of logic rather than analysis, and the logicians have answered it. In 1964, R. M. Solovay showed that, in Zermelo–Fraenkel set theory with a weaker assumption than the axiom of choice, it is consistent that all sets are Lebesgue measurable. On the other hand, the existence of nonmeasurable sets does not imply the axiom of choice. Thus it is no accident that our proof had to rely on the axiom of choice: it would have to appeal to some further logical principle in any case.

Exercises

1:10.1 Show that a Cantor set is Peano–Jordan measurable if and only if it has outer content zero.

1:10.2 Show that every set of positive outer measure contains a nonmeasurable set.

1:10.3 Show that there exist disjoint sets $\{E_k\}$ so that

$$\lambda^*\left(\bigcup_{k=1}^{\infty} E_k\right) < \sum_{k=1}^{\infty} \lambda^*(E_k).$$

1:10.4 Show that there exists a decreasing sequence of sets $E_1 \supset E_2 \supset E_3 \dots$ so that each $\lambda^*(E_k) < +\infty$ and

$$\lambda^*\left(\bigcap_{k=1}^{\infty} E_k\right) < \lim_{k\to\infty} \lambda^*(E_k).$$

1.11 Zorn's Lemma

In our brief survey we have already seen several points where an appeal to the axiom of choice was needed. This fundamental logical principle can be formulated in a variety of equivalent ways, each of use in certain situations.

The form we shall discuss now is called Zorn's lemma after Max Zorn (1906–1994). To express this, we need some terms from the language of partially ordered sets. A partially ordered set is a relaxation of a linearly ordered set as defined in Section 1.5. A relation $a \preceq b$, defined for certain pairs in a set S, is said to be a *partial order* on S, and $(S, \preceq)$ is said to be a *partially ordered set* if

1. For all $a \in S$, $a \preceq a$.
2. If $a \preceq b$ and $b \preceq a$, then $a = b$.
3. If $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

The word "partial" indicates that not all pairs of elements need be comparable, only that the three properties here hold. A *maximal element* in a partially ordered set is an element $m \in S$ with nothing further in the order; that is, if $m \preceq a$ is true, then $a = m$.

The existence of maximal elements in partially ordered sets is of great importance. Zorn's lemma provides a criterion that can be checked in order to claim the existence of maximal elements. A *chain* in a partially ordered set is any subset that is itself linearly ordered. An upper bound of a chain is simply an element beyond every element in the chain. The language is suggestive, and pictures should help keep the concepts in mind.

Lemma 1.34 (Zorn) *If every chain in a partially ordered set has an upper bound, then the set has a maximal element.*

This assertion is, in fact, equivalent to the axiom of choice. We shall prove one direction just as an indication of how Zorn's lemma can be used in practice.

Let $\{A_i : i \in I\}$ be a collection of sets, each nonempty. We wish to show the existence of a choice function, that is, a function f with domain I such that $f(i) \in A_i$ for each $i \in I$. For any single given element $i_1 \in I$, we are assured that A_{i_1} is nonempty and hence we can choose some element $f(i_1) \in A_{i_1}$. We could do the same for any finite collection $\{i_1, i_2, \ldots, i_n\}$, but without appealing to some logical principle we cannot do this for *all* elements of I.

Zorn's lemma offers a technique. Define $\mathcal{F}$ as the family of all functions f such that

1. The domain of f is contained in I.
2. $f(i) \in A_i$ for each i in the domain of f.

We already know that there are some functions in $\mathcal{F}$. The choice function we want is presumably there too: it is any element of $\mathcal{F}$ with domain I.

Use $\operatorname{dom} f$ to denote the domain of a function f. Define a partial order on $\mathcal{F}$ by writing $f \preceq g$ to mean that $\operatorname{dom} f \subset \operatorname{dom} g$ and g is an extension of f. A maximal element of $\mathcal{F}$ must be our choice function. For, if f is maximal and yet the domain of f is not all of I, we can choose $i_0 \in I \setminus \operatorname{dom} f$ and some $x_{i_0} \in A_{i_0}$. Define g on $\operatorname{dom} f \cup \{i_0\}$ so that $g(i_0) = x_{i_0}$. Then g is an extension of f, and this contradicts the fact that f is to be maximal.

How do we prove the existence of a maximal element? Zorn's lemma allows us merely to verify that every chain has an upper bound. If $\mathcal{C} \subset \mathcal{F}$

is a chain, then there is a function h defined on $\bigcup_{g\in\mathcal{C}} \text{dom } g$ so that h is an extension of each $g \in \mathcal{C}$. Simply take $h(i) = g(i)$ for any $g \in \mathcal{C}$ for which $i \in \text{dom } g$. The fact that $\mathcal{C}$ is linearly ordered shows that this definition is unambiguous.

This completes the proof that Zorn's lemma implies the axiom of choice. All applications of Zorn's lemma will look something like this. The cleverness that may be needed is to interpret the problem at hand as a maximal problem in an appropriate partially ordered set.

Exercises

1:11.1 Let 2^X denote the set of all subsets of a nonempty set X. Show that the relation $A \subset B$ is a partial order on 2^X. Is it ever a linear order?

1:11.2 Let $\mathcal{F}$ denote the family of all functions $f : X \to Y$. Write $f \preceq g$ if the domain of g includes the domain of f and g is an extension of f. Show in detail that $(\mathcal{F}, \preceq)$ is a partially ordered set in which every chain has an upper bound.

1:11.3◇ Prove that there is a Hamel basis for the real numbers; that is, there exists a set $H \subset \mathbb{R}$ that is linearly independent over the rationals and that spans $\mathbb{R}$. (A set H is linearly independent over the rationals if given distinct elements $h_1,\ h_2,\ \dots h_n \in H$ and any $r_1,\ r_2,\ \dots r_n \in \mathbb{Q}$ with $\sum_{i=1}^n r_i h_i = 0$ then necessarily

$$r_1 = r_2 = \cdots = r_n = 0.$$

A set H spans $\mathbb{R}$ if for any $x \in \mathbb{R}$ there exist

$$h_1, h_2, \dots h_n \in H \text{ and } r_1, r_2, \dots r_n \in \mathbb{Q}$$

so that $\sum_{i=1}^n r_i h_i = x$.) [Hint: Find a maximal linearly independent set.]

1:11.4 Prove the axiom of choice assuming the well-ordering principle (that every set can be well-ordered). [Hint: Given $\{A_i : i \in I\}$ a collection of sets, each nonempty, well order the set $\bigcup_{i\in I} A_i$. Consider $c(A_i)$ as the first element in the set A_i in the order.]

1:11.5 Show that the following statement is equivalent to the axiom of choice: If $\mathcal{C}$ is a family of disjoint, nonempty subsets of a set X, then there is a set C that has exactly one element in common with each set in $\mathcal{C}$.

1.12 Borel Sets of Real Numbers

We have already defined several classes of sets that form the start of what is known as the Borel sets:

$$\mathcal{G} \subset \mathcal{G}_\delta \subset \mathcal{G}_{\delta\sigma} \subset \mathcal{G}_{\delta\sigma\delta} \subset \mathcal{G}_{\delta\sigma\delta\sigma} \dots$$

and

$$\mathcal{F} \subset \mathcal{F}_\sigma \subset \mathcal{F}_{\sigma\delta} \subset \mathcal{F}_{\sigma\delta\sigma} \subset \mathcal{F}_{\sigma\delta\sigma\delta} \dots .$$

Now, with transfinite ordinals available to us, we can continue this construction. The reason the transfinite ordinals are needed is that this process, which evidently can continue following a sequence of operations, does not terminate using an ordinary sequence.

The notation used above, while useful at the start of the process, will not serve us for long. Recall that the first ordinal 0 and every limit ordinal is thought of as even, the successor of an even ordinal is odd, and a successor of an odd ordinal is even.

We define the classes $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ for every ordinal $\alpha < \Omega$. We start by writing $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{G}_0 = \mathcal{G}$, $\mathcal{F}_1 = \mathcal{F}_\sigma$ and $\mathcal{G}_1 = \mathcal{G}_\delta$, $\mathcal{F}_2 = \mathcal{F}_{\sigma\delta}$ and $\mathcal{G}_2 = \mathcal{G}_{\delta\sigma}$. The classes $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ for every ordinal α are defined by taking countable intersections or countable unions of sets from the corresponding classes $\mathcal{F}_\beta$ and $\mathcal{G}_\beta$ for ordinals $\beta < \alpha$. If α is odd, then take $\mathcal{F}_\alpha$ as the class formed from countable unions of members from any classes $\mathcal{F}_\beta$ for $\beta < \alpha$. If α is even, then take $\mathcal{F}_\alpha$ as the class formed from countable intersections of members from any classes $\mathcal{F}_\beta$ for $\beta < \alpha$.

Similarly, if α is odd, then take $\mathcal{G}_\alpha$ as the class formed from countable intersections of members from any classes $\mathcal{G}_\beta$ for $\beta < \alpha$. If α is even, then take $\mathcal{G}_\alpha$ as the class formed from countable unions of members from any classes $\mathcal{G}_\beta$ for $\beta < \alpha$.

This process continues through all the countable ordinals by transfinite induction. For $\alpha = \Omega$, we find that the formation of countable intersections (to form $\mathcal{F}_\Omega$) or countable unions (to form $\mathcal{G}_\Omega$) does not create new sets (see Exercise 1:12.5). The collection of all sets formed by this process is called the Borel sets.

We list without proof some properties of the Borel sets on the line to give the flavor of the theory.

1.35 *The complement of a set of type $\mathcal{F}_\alpha$ is a set of type $\mathcal{G}_\alpha$, and the complement of a set of type $\mathcal{G}_\alpha$ is a set of type $\mathcal{F}_\alpha$.*

1.36 *The union and intersection of a* finite *number of sets of type $\mathcal{F}_\alpha$ ($\mathcal{G}_\alpha$) is of the same type.*

1.37 *Let $\alpha < \Omega$ be odd. Then the union of a* countable *number of sets of type $\mathcal{F}_\alpha$ is of the same type, and the intersection of a* countable *number of sets of type $\mathcal{G}_\alpha$ is of the same type.*

1.38 *Every set of type $\mathcal{F}_\alpha$ is of type $\mathcal{G}_{\alpha+1}$. Every set of type $\mathcal{G}_\alpha$ is of type $\mathcal{F}_{\alpha+1}$.*

1.39 *The Borel sets form the smallest σ–algebra of sets that contains the closed sets (the open sets).*

Thus one says that the Borel sets are generated by the closed sets (or by the open sets). (Exercise 1:9.8 shows that there must exist, independent of this theorem, a "smallest" σ–algebra containing any given collection of sets.) It is this form that we take as a definition in Chapter 3 for the Borel sets in a metric space.

Exercises

1:12.1 Show that the Borel sets form the smallest family of subsets of $\mathbb{R}$ that (i) contains the closed sets, (ii) is closed under countable unions, and (iii) is closed under countable intersections.

1:12.2 Show that the Borel sets form the smallest family of subsets of $\mathbb{R}$ that (i) contains the closed sets, (ii) is closed under countable *disjoint* unions, and (iii) is closed under countable intersections.

1:12.3 Show that the collection of all Borel sets has cardinality c.

1:12.4 Show that there must exist Lebesgue measurable sets that are not Borel sets. [Hint: Use Exercise 1:9.13.]

1:12.5 Show that the formation of countable intersections (to form $\mathcal{F}_\Omega$) or countable unions (to form $\mathcal{G}_\Omega$) does not create new sets. [Hint: All members of any sequence of sets from these classes must belong to one of the classes.]

1.13 Analytic Sets of Real Numbers

The Borel sets *clearly* form the largest class of respectable sets. This class is closed under all the reasonable operations that one might perform in analysis. Or so it seems.

In an important paper in 1905, Lebesgue made the observation that the projections of Borel sets in $\mathbb{R}^2$ onto the line are again Borel sets. The statement seems so reasonable and expected that he gave no detailed proof, assuming it to follow by methods he just sketched. The reader may know that the projection of a compact set in $\mathbb{R}^2$ is a compact set in $\mathbb{R}$ (any continuous image of a compact set is compact), and so any set that is a countable union of compact sets must project to a Borel set. It seems likely that one could prove that projections of other Borel sets must also be Borel by some obvious argument.

Lebesgue's assertion went unchallenged for ten years until the error was spotted by a young student in Moscow. Suslin, a student of Lusin, not only found the error, but reported to his professor that he was able to characterize the sets that could be expressed as projections of Borel sets and that he could produce an example that was not a Borel set.

Suslin calls a set $E \subset \mathbb{R}$ *analytic* if it can be expressed in the form

$$E = \bigcup_{(n_1,n_2,n_3,\dots)} \bigcap_{k=1}^{\infty} I_{n_1,n_2,n_3,\dots,n_k}$$

where each $I_{n_1,n_2,n_3,\dots,n_k}$ is a nonempty, closed interval for each

$$(n_1, n_2, n_3, \dots, n_k) \in \mathbb{N}^k$$

and each $k \in \mathbb{N}$, and where the union is taken over all possible sequences $(n_1, n_2, n_3, \dots)$ of natural numbers. Note that while the family of sets under consideration, $\{I_{n_1,n_2,n_3,\dots,n_k}\}$, is countable the union involves uncountably many sets. Accordingly, this operation is substantially more complicated than the operations that preserve Borel sets. We shall call this the *Suslin operation*, although some authors, following Suslin himself, call it operation A.

In a short space of time Suslin, with the evident assistance of Lusin, established the basic properties of analytic sets and laid the groundwork for a vast amount of mathematics that has proved to be of importance for analysts, topologists, and logicians. We shall study this in some detail in Chapter 11. Here let us merely announce some of his discoveries. He obtained each of the following facts about analytic sets:

- All Borel sets are analytic.
- There is an analytic set that is not Borel.
- A set is Borel if and only if it and its complement are both analytic.
- Every analytic set in $\mathbb{R}$ is the projection of some $\mathcal{G}_\delta$ set in $\mathbb{R}^2$.
- Every uncountable analytic set has cardinality c.
- The projections of analytic sets are again analytic.

Thus in his short career (he died in 1919) Suslin established the fundamental properties of analytic sets, properties that exhibit the role that they must play. Lusin and his Polish colleague Sierpiński carried on the study in subsequent years, and by the end of the 1930s the study was quite complete and extensive. Let us mention two of their results that are important from the perspective of measure theory.

- All analytic sets are Lebesgue measurable.
- The Suslin operation applied to a family of Lebesgue measurable sets produces again a Lebesgue measurable set.

The study of analytic sets was well developed and well known in certain circles (mostly in Poland), but it did not receive a great deal of general attention until two main developments. In the 1950s a number of important problems in analysis were solved by employing the techniques associated with the study of analytic sets. In another direction it was discovered that most of the theory played an essential role in the study of descriptive set theory; since then all the methods and results of Suslin, Lusin, Sierpiński, and others have been absorbed by the logicians in their development of this subject.

We shall return to these ideas in Chapter 11 where we will explore the methods used to prove the statements listed here.

1.14 Bounded Variation

The following two problems attracted some attention in the latter years of the nineteenth century.

1.40 *What is the smallest linear space containing the monotonic functions?*

1.41 *For what class of functions f does the graph*

$$\{(x, y) : y = f(x)\}$$

have finite length?

Du Bois-Reymond, for one, attempted to solve Problem 1.40. He noted that, for a function f that is the integral of its derivative, one could write

$$f(x) = f(a) + \int_a^x [f'(t)]^+ \, dt - \int_a^x [f'(t)]^- \, dt,$$

where we are using the useful notation

$$[a]^+ = \max\{a, 0\} \quad \text{and} \quad [a]^- = \max\{-a, 0\}.$$

Clearly, this expresses f as a difference of monotone functions. This led him to a more difficult problem, which he was unable to resolve: Which functions are indefinite integrals of their derivatives? Unfortunately, this leads to a problem that will not resolve the original problem in any case.

Camille Jordan (1838–1922) solved both problems by introducing the class of functions of bounded variation. The functions of bounded variation play a central role in many investigations, notably in studies of rectifiability (as Problem 1.41 would suggest) and fundamental questions involving integrals and derivatives. They also lead to natural generalizations in the abstract study of measure and integration. For that reason, the student should be aware of the basic facts and methods that are developed in the exercises.

Let f be a real-valued function defined on $[a, b]$, and let

$$P = \{x_0, x_1, \ldots, x_n\}$$

be a partition of $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Let

$$V(f, P) = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|.$$

The *variation* of f on $[a, b]$ is defined as

$$V(f; [a, b]) = \sup\{V(f, P) : P \text{ is a partition of } [a, b]\}.$$

When $V(f;[a,b])$ is finite, we say that f is of *bounded variation* on $[a,b]$. We then write f is BV on $[a,b]$, or f is BV when the interval is understood. (The variant VB is also in common usage because of the French *variation bornée*.)

The function $T(x) = V(f;[a,x])$ measures the variation on the interval $[a,x]$ and evidently is an increasing function. This is called the *total variation* of f. It is this that allows the solution of Problem 1.40, for one shows that

$$f(x) = T(x) - (T(x) - f(x))$$

expresses f as a difference of monotone functions (Exercise 1:14.10).

For the problems on arc length, we need the following definitions. Let f and g be real functions on an interval $[a,b]$. A curve C in the plane is considered to be the pair of parametric equations

$$x = f(t),\ y = g(t) \qquad (a \le t \le b).$$

The graph of the curve C is the set of points

$$\{(x,y) : x = f(t),\, y = g(t)\ (a \le t \le b)\}.$$

The length $\ell(C)$ of the curve C is defined as

$$\sup \sum_{j=1}^{n} \sqrt{\left(f(x_j) - f(x_{j-1})\right)^2 + \left(g(x_j) - g(x_{j-1})\right)^2},$$

where, as above, the supremum is taken over all partitions of $[a,b]$. The curve is said to be *rectifiable* if this is finite. Such a curve is rectifiable precisely when both functions f and g have bounded variation (Exercise 1:14.14). The graph of a function f is rectifiable precisely when f has bounded variation (Exercise 1:14.16).

Exercises

1:14.1 Show that a monotonic function on $[a,b]$ is BV.

1:14.2 Show that a continuous function with a finite number of local maxima and minima on $[a,b]$ is BV.

1:14.3 Show that a continuously differentiable function on $[a,b]$ is BV.

1:14.4 Show that a function that satisfies a Lipschitz condition on $[a,b]$ is BV. [A function f is said to satisfy a Lipschitz condition if, for some constant M, $|f(x) - f(y)| \le M|y - x|$. These conditions were introduced by R. Lipschitz in an 1876 study of differential equations.]

1:14.5 Estimate the variation of the function $f(x) = x \sin x^{-1}$, $f(0) = 0$, on the interval $[0,1]$.

1:14.6 Estimate the variation of the function $f(x) = x^2 \sin x^{-1}$, $f(0) = 0$, on the interval $[0,1]$.

1:14.7 If f is BV on $[a,b]$, then prove that f is bounded on $[a,b]$.

1:14.8 Show that the class of functions of bounded variation on $[a,b]$ is closed under addition, subtraction, and multiplication. If f and g are BV, and g is bounded away from zero, then f/g is BV.

1:14.9$\diamondsuit$ Show that if f is BV on $[a,b]$ and $a \le c \le b$, then

$$V(f;[a,b]) = V(f;[a,c]) + V(f;[c,b]).$$

1:14.10$\diamondsuit$ Show that a function f is BV on $[a,b]$ if and only if there exist functions f_1 and f_2 that are nondecreasing on $[a,b]$, and $f(x) = f_1(x) - f_2(x)$ for all $x \in [a,b]$. [Hint: Let $V(x) = V(f;[a,x])$. Verify that $V - f$ is nondecreasing on $[a,b]$ and use $f = V - (V - f)$.]

1:14.11 Show that the set of discontinuities of a function of bounded variation is (at most) countable. [Hint: See Exercise 1:3.14.]

1:14.12 Show that if f is BV on $[a,b]$, with variation $V(x) = V(f;[a,x])$, then

$$\{x : f \text{ is right continuous at } x\} = \{x : V \text{ is right continuous at } x\}.$$

1:14.13 Let $\{f_n\}$ be a sequence of functions, each BV on $[a,b]$ with variation less than or equal to some number M. If $f_n \to f$ pointwise on $[a,b]$, show that f is BV on $[a,b]$ with variation no greater than M.

1:14.14 Show that the graph of a curve C in the plane, given by the pair of parametric equations

$$x = f(t),\ y = g(t) \qquad (a \le t \le b)$$

is rectifiable if and only if both f and g have bounded variation on $[a,b]$. [Hint: $|x|$, $|y| \le \sqrt{x^2+y^2} \le |x| + |y|$.]

1:14.15 Show that the length of a curve C in the plane, given by the pair of parametric equations $x = f(t)$, $y = g(t)$ $(a \le t \le b)$, is the integral

$$\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

if f and g are continuously differentiable.

1:14.16 Show that the graph of a function f is rectifiable if and only if f has bounded variation on $[a,b]$.

1:14.17$\diamondsuit$ Let $f : [a,b] \to \mathbb{R}$. We say that f is *absolutely continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\{[a_n,b_n]\}$ is any finite or countable collection of nonoverlapping closed intervals in $[a,b]$ with $\sum_{k=1}^{\infty}(b_k - a_k) < \delta$, then

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \varepsilon.$$

This concept plays a significant role in the integration theory of real functions. Show that an absolutely continuous function is both continuous and of bounded variation.

1:14.18 Give a natural definition for a complex-valued function on a real interval $[a, b]$ to have bounded variation. Prove that a complex-valued function has bounded variation if and only if its real and imaginary parts have bounded variation.

1.15 Newton's Integral

We embark now on a tour of classical integration theory leading up to the Lebesgue integral. The reader will be familiar to various degrees with much of this material, since it appears in a variety of undergraduate courses. Here we need to clarify many different themes that come together in an advanced course in measure and integration.

The simplest starting point is the integral as conceived by Newton. For him the integral is just an inversion of the derivative. In the same spirit (but not in the same technical way that he would have done it) we shall make the following definition.

Definition 1.42 A real-valued function f defined on an interval $[a, b]$ is said to be Newton integrable on $[a, b]$ if there exists an antiderivative of f, that is, a function F on $[a, b]$ with $F'(x) = f(x)$ everywhere there. Then we write

$$(N)\int_a^b f(x)\,dx = F(b) - F(a).$$

The mean-value theorem shows that the value is well defined and does not depend on the particular primitive function F chosen to evaluate the integral. This integral must be considered *descriptive* in the sense that the property of integrability and the value of the integral are determined by the existence of some object for which no construction or recipe is available. If, perchance, such a function F can be found, then the value of the integral is determined, but otherwise there is no hope, a priori, of finding the integral or even of knowing whether it exists.

One might wish to call this the *calculus integral* since, in spite of the many texts that teach constructive definitions for integrals, most freshman calculus students hardly ever view an integral as anything more than a determination of an antiderivative.

At this point let us remark that this integral is handling functions that are not handled by other methods. The integrals of Cauchy and of Riemann, discussed next, require a fair bit of continuity in the function and do not tolerate much unboundedness. But derivatives can be unbounded and derivatives can be badly discontinuous. We know that a derivative is Baire 1 and that Baire 1 functions are continuous except at the points of a first category set; this first category set can, however, have positive measure, and this will interfere with integrability in the senses of Cauchy or Riemann. Thus, while this integral may seem quite simple and unassuming, it is involved in a process that is more mysterious than might appear at first glance. Attempts to understand this integral will take us on a long journey.

Exercises

1:15.1 Show that the mean-value theorem can be used to justify the definition of the Newton integral.

1:15.2 Show that a derivative f' of a continuous function f is Baire 1 and has the intermediate-value property. [Hint: Consider $f_n(x) = n^{-1}(f(x + n^{-1}) - f(x))$. The intermediate-value property can be deduced from the mean-value theorem.]

1:15.3 Show that a derivative on a finite interval can be unbounded.

1:15.4 Which of the elementary properties of the Riemann integral hold for the Newton integral? For example, can we write

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx?$$

1.16 Cauchy's Integral

A first course in calculus will include a proper definition of the integral that dates back to the middle of the nineteenth century and is generally attributed to Bernhard Riemann (1826–1866). Actually, Augustin Cauchy (1789–1857) had conceived of such an integral a bit earlier, but Cauchy limited his study to continuous functions. Here is Cauchy's definition, stated in modern language but essentially as he would have given it in 1823 in his lessons at the École Polytechnique.

Let f be continuous on $[a, b]$ and consider a partition P of this interval:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Form the sum

$$S(f, P) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Let $\|P\| = \max_{1\le i\le n}(x_i - x_{i-1})$ and define

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} S(f, P).$$

Cauchy showed that this limit exists.

Prior to Cauchy, such a definition of integral might not have been possible. The modern notion of "continuity" was not available (it was advanced by Cauchy in 1821), and even the proper definition of "function" was in dispute. Cauchy also established a form of the fundamental theorem of calculus.

Theorem 1.43 *Let f be continuous on $[a,b]$, and let*

$$F(x) = \int_a^x f(t)\ dt \quad (a \leq x \leq b).$$

Then F is differentiable on $[a,b]$, and $F'(x) = f(x)$ for all $x \in [a,b]$.

Theorem 1.44 *Let F be continuously differentiable on $[a,b]$. Then*

$$F(b) - F(a) = \int_a^b F'(x)\ dx.$$

Thus, for continuous functions, Cauchy offers an integral that is constructive and agrees with the Newton integral. There are, however, unbounded derivatives, and so the Newton integral remains more general than Cauchy's version. To handle unbounded functions, Cauchy introduces the following idea, one that survives to this day in elementary calculus courses, usually under the unfortunate term "improper integral." Let us introduce it in a more formal manner, one that leads to a better understanding of the structure.

Let f be a real function on an interval $[a,b]$. A point $x_0 \in [a,b]$ is a point of unboundedness of f if f is unbounded in every open interval containing x_0. Let S_f denote the set of points of unboundedness. If S_f is a finite set and f is continuous at every point of $[a,b] \setminus S_f$, there is some hope of obtaining an integral of f. Certainly, we know the value of $\int_c^d f(t)\,dt$ for every interval $[c,d]$ disjoint from S_f. It is a matter of extending these values. Cauchy's idea is to obtain, for any $c, d \in S_f$ with $(c,d) \cap S_f = \emptyset$,

$$\int_c^d f(t)\,dt = \lim_{\varepsilon_1 \searrow 0,\, \varepsilon_2 \searrow 0} \int_{c+\varepsilon_1}^{d-\varepsilon_2} f(t)\,dt$$

Then, in a finite number of steps, one can extend the integral to $[a,b]$, providing only that each limit as above exists. A function is *Cauchy integrable* on an interval $[a,b]$ provided that S_f is finite, f is continuous at each point of $[a,b]$ excepting the points in S_f and all the limits above exist.

One important feature of this integral is its *nonabsolute* character. A function f may be integrable in Cauchy's sense on an interval $[a,b]$ and yet the absolute value $|f|$ may not be. An easy example is the function $f(x) = F'(x)$ on $[0,1]$, where $F(x) = x^2 \sin x^{-2}$. Here $S_f = \{0\}$ and f is continuous away from 0. Obviously, f is Cauchy integrable on $[0,1]$, and yet $|f|$ is not. Somehow the "cancellations" that take place for integrating f do not occur for $|f|$, since

$$\lim_{\varepsilon \searrow 0} \int_\varepsilon^1 |f(t)|\,dt = +\infty.$$

This can be considered as the analog in integration theory of the fact that $\sum_{i=1}^\infty (-1)^i/i$ exists and yet $\sum_{i=1}^\infty 1/i = +\infty$.

Finally, we mention Cauchy's method for handling unbounded intervals. The procedure above for determining the integral of a continuous function on a bounded interval $[a, b]$ does not immediately extend to the unbounded intervals $(-\infty, a]$, $[a, +\infty)$, or $(-\infty, +\infty)$. Cauchy handled these in a now familiar way. He defines

$$\int_{-\infty}^{+\infty} f(x)\, dx = \lim_{s,t\to+\infty} \int_{-s}^{t} f(x)\, dx.$$

Note that this integral, too, is a nonabsolute integral.

Exercises

1:16.1 Let S_f denote the set of points of unboundedness of a function f. Show that S_f is closed.

1:16.2 Cauchy also considered symmetric limits of the form

$$\lim_{t\to0+} \left(\int_{a}^{b-t} f(x)\, dx + \int_{b+t}^{c} f(x)\, dx \right)$$

as "principal-value" limits. Give an example to show that these can exist when the ordinary Cauchy integral does not.

1:16.3 Cauchy also considered symmetric limits for unbounded intervals

$$\lim_{t\to+\infty} \int_{-t}^{t} f(x)\, dx.$$

as "principal value" limits. Give an example to show that this can exist when the ordinary Cauchy integral does not.

1:16.4 Let $f(x) = x^2 \sin x^{-2}$, $f(0) = 0$ and show that f' is an unbounded derivative on $[0, 1]$ integrable by both Cauchy and Newton's methods to the same value. Show that $|f|$ is not integrable by either method.

1.17 Riemann's Integral

Riemann extended Cauchy's concept of integral to include some bounded functions that are discontinuous. All the definitions one finds in standard calculus texts are equivalent to his. Using exactly the language we have given for one of the results of Cauchy from the preceding section, we can give a definition of Riemann's integral. Note that it merely turns a theorem (for continuous functions) into a definition of the meaning of the integral for discontinuous functions. This shift represents a quite modern point of view, one that Cauchy and his contemporaries would never have made.

Definition 1.45 Let f be a real-valued function defined on $[a, b]$, and consider a partition P of this interval

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

supplied with associated points $\xi_i \in [x_{i-1}, x_i]$. Form the sum

$$S(f,P) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

and let

$$\|P\| = \max_{1 \le i \le n} (x_i - x_{i-1}).$$

Then we define

$$\int_a^b f(x)\ dx = \lim_{\|P\| \to 0} S(f,P)$$

and call f *Riemann integrable* if this limit exists.

The structure of Riemann integrable functions is quite easy to grasp. They are bounded (this is evident from the definition) and they are "mostly" continuous. This was established by Riemann himself. His analysis of the continuity properties of integrable functions lacked only an appropriate language in which to express it. With Lebesgue measure at our disposal, the characterization is immediate and compelling. It reveals too just why the Riemann integral must be considered so limited in application.

Theorem 1.46 *A necessary and sufficient condition for a function f to be Riemann integrable on an interval $[a, b]$ is that f is bounded and that its set of points of discontinuity in $[a, b]$ forms a set of Lebesgue measure zero.*

Perhaps we should give a version of this theorem that would be more accessible to the mathematicians of the nineteenth century, who would have known Peano–Jordan content but not Lebesgue measure. The set of points of discontinuity has an easy structure: it is the countable union $\bigcup_{n=1}^{\infty} F_n$ of the sequence of closed sets

$$F_n = \{x : \omega_f(x) \ge 1/n\},$$

where the oscillation of the function is greater than the positive value $1/n$. [Exercise 1:1.8 defines $\omega_f(x)$.] That the set of points of continuity of f has measure zero is seen to be equivalent to each of the sets F_n having content zero. Thus the theorem could have been expressed in this, rather more clumsy, way. Note that, so expressed, one may miss the obvious fact that it is only the nature of the set of discontinuity points itself that plays a role, not some other geometric property of the function. In particular, this serves as a good illustration of the merits of the Lebesgue measure over the Peano–Jordan content.

Exercises

1:17.1 Show that a Riemann integrable function must be bounded.

1:17.2◇ (Riemann) Let f be a real-valued function defined on $[a, b]$, and consider a partition P of this interval:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Form the sum

$$O(f, P) = \sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}),$$

where

$$\omega(f, I) = \sup\{|f(x) - f(y)| : x, y \in I\}$$

is called the oscillation of f on the interval I. Show that in order for f to be Riemann integrable on $[a, b]$ it is necessary and sufficient that

$$\lim_{\|P\| \to 0} O(f, P) = 0.$$

1:17.3 Relate Exercise 1:17.2 to the problem of finding the Peano–Jordan content (Lebesgue measure) of the closed set of points where the oscillation $\omega_f(x)$ of f is greater or equal to some positive number c.

1:17.4 Relate Exercise 1:17.2 to the problem of finding the Lebesgue measure of the set of points where f is continuous (i.e., where the oscillation ω_f of f is zero).

1:17.5 Riemann's integral does not handle unbounded functions. Define a Cauchy–Riemann integral using Cauchy's extension method to handle unbounded functions.

1:17.6 Let S_f denote the set of points of unboundedness of a function f in an interval $[a, b]$. Suppose that S_f has content zero (i.e., measure zero since it is closed) and that f is Riemann integrable in every interval $[c, d] \subset [a, b]$ disjoint from S_f. Define $f_{st}(x) = f(x)$ if $-s \le f(x) \le t$, $f_{st}(x) = t$ if $f(x) > t$ and $f_{st}(x) = -s$ if $-s > f(x)$. Define

$$\int_a^b f(x)\,dx = \lim_{s,t \to +\infty} \int_a^b f_{st}(x)\,dx$$

if this exists. Show that $\int_a^b f(x)\,dx$ does exist under these assumptions. This is the way de la Vallée Poussin proposed to handle unbounded functions. Show that this method is different from the Cauchy–Riemann integral by showing that this integral is an absolutely convergent integral.

1:17.7 Prove that a function f on an interval $[a, b]$ is Riemann integrable if f has a finite limit at every point.

1:17.8 Prove that a bounded function on an interval $[a, b]$ is Riemann integrable if and only if f has a finite right-hand limit at every point except only a set of measure zero. [Hint: The set of points at which f is discontinuous and yet has a finite right-hand limit is countable.]

1.18 Volterra's Example

By the end of the nineteenth century, many limitations to Riemann's approach were apparent. All these flaws related to the fact that the class of Riemann integrable functions is too small for many purposes.

The most obvious problem is that a Riemann integrable function must be bounded. Much attention was given to the problem of integrating unbounded functions by the analysts of the last century and less to the fact that, even for bounded functions, the integrability criteria were too strict. This fact was put into startling clarity by an example of Volterra. He produces an everywhere differentiable function F such that F' is bounded but not Riemann integrable. Thus the fundamental theorem of calculus fails for this function, and the formula

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

is invalid.

Here are some of the details of a construction due to C. Goffman. For a version closer to Volterra's actual construction, see Exercise 5:5.5. Note that we have only to construct a derivative F' that is discontinuous on a set of positive measure (or a closed set of positive content). For this we take a Cantor set of positive measure (Theorem 1.23). It was the existence of such sets that provided the key to Volterra's construction.

Let $C \subset [0,1]$ be a Cantor set of measure $1/2$ and let $\{I_n\}$ denote the sequence of open intervals complementary to C in $(0,1)$. Then $\sum_{i=1}^{\infty} |I_i| = 1/2$. Choose a closed subinterval $J_n \subset I_n$ centered in I_n such that $|J_n| = |I_n|^2$. Define a function f on $[0,1]$ with values $0 \leq f(x) \leq 1$ such that f is continuous on each interval J_n and is 1 at the centers of each interval J_n and vanishes outside of every J_n. It is straightforward to check that f cannot be Riemann integrable on $[0,1]$. Indeed, since the intervals $\{I_n\}$ are dense and have total length $1/2$, and the oscillation of f is 1 on each I_n, this function violates Riemann's criterion (Exercise 1:17.2).

That f is a derivative follows immediately from advanced considerations (it is bounded and everywhere approximately continuous and hence the derivative of its Lebesgue integral). This can also be seen without any technical apparatus. We can construct a continuous primitive function F for f on each interval J_n. To define a primitive F on all of $[0,1]$, we write

$$F(x) = \sum_{n=1}^{\infty} \int_{J_n \cap [0,x]} f(t)\,dt.$$

Let $I \subset [0,1]$ be an interval that meets the Cantor set C, and let n be any integer so that $I \cap J_n \neq \emptyset$. Let $\ell_n = |I_n|$. Since $\ell_n \leq \frac{1}{2}$, it follows that

$$|I \cap I_n| \geq \tfrac{1}{2}(\ell_n - \ell_n^2) \geq \tfrac{1}{4}\ell_n.$$

Then

$$|I \cap J_n| \le |J_n| = \ell_n^2 \le 16|I \cap I_n|^2.$$

If N is the set of integers n for which $I \cap J_n \neq \emptyset$, then

$$\sum_{n \in N} |I \cap J_n| \le \sum_{n \in N} 16|I \cap I_n|^2 \le 16|I|^2.$$

From this we can check that $F'(x) = f(x) = 0$ for each $x \in C$. For $x \in [0,1] \setminus C$, it is obvious that $F'(x) = f(x)$. Thus f is a derivative and bounded (between 0 and 1).

Other flaws that reveal the narrowness of the Riemann integral emerge by comparison with later theories. One would like useful theorems that assert a series of functions can be integrated term by term. More precisely, if $\{f_n\}$ is a sequence of integrable functions on $[a,b]$, and $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then f is integrable, and

$$\int_a^b f(x)\ dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\ dx.$$

Riemann's integral does not do very well in this connection since the limit function f can be badly discontinuous even if the functions f_n are themselves each continuous. Many authors in the first half of the nineteenth century routinely assumed the permissibility of term-by-term integration. It was not until 1841 that the notion of uniform convergence appeared, and its role in theorems about term-by-term integration, continuity of the sum, and the like, followed soon thereafter. By the end of the century there was felt a strong need to go beyond uniform convergence in theorems of this kind.

Yet another type of limitation is that Riemann's integral is defined only over intervals. For many purposes, one needs to be able to deal with the integral over a set E that need not be an interval. The Riemann integral can, in fact, be defined over Peano–Jordan measurable sets, but we have seen that this class of sets is rather limited and does not embrace many sets (Cantor sets of positive measure for example) that arise in applications. One often needs a larger class of sets over which an integral makes sense.

We shall deal in this text with a notion of integral, essentially due to Henri Lebesgue, that does much better. The class of integrable functions is sufficiently large to remove, or at least reduce, the limitations we discussed, and it allows natural generalizations to functions defined on spaces much more general than the real line.

Exercises

1:18.1 Check the details of the construction of the function F whose derivative is bounded and not Riemann integrable.

1:18.2 Construct a sequence of continuous functions converging pointwise to a function that is not Riemann integrable.

1:18.3 Define
$$\int_E f(x)\,dx = \int_a^b \chi_E(x)f(x)\,dx$$
when $E \subset [a,b]$ and f is continuous on $[a,b]$. For what sets E is this generally possible?

1.19 Riemann–Stieltjes Integral

T. J. Stieltjes (1856–1894) introduced a generalization of the Riemann integral that would seem entirely natural. He introduced a weight function g into the definition and considered limits of sums of the form

$$\sum_{i=1}^{n} f(\xi_i)\,(g(x_i) - g(x_{i-1}))$$

where, as usual, x_0, x_1, $\dots$, x_n is a partition of an interval and each $\xi_i \in [x_{i-1}, x_i]$. Although it was introduced for the specific purpose of representing functions in a problem in continued fractions, it should have been clear that this object (the Riemann–Stieltjes integral) had some independent merit. Stieltjes himself died before the appearance of his paper, and the idea attracted almost no attention for the next 15 years. Then F. Riesz showed that this integral gave a precise characterization of the general continuous linear functions on the space of continuous function on an interval. (See Section 12.8.) Since then it has become a mainstream tool of analysis. It also played a fundamental role in the development [notably by J. Radon (1887-1956) and M. Fréchet (1878-1973)] of the abstract theory of measure and integration. For these reasons the student should know at least the rudiments of the theory as presented here.

Definition 1.47 Let f, g be real-valued functions defined on $[a,b]$, and consider a partition P of this interval

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

supplied with associated points $\xi_i \in [x_{i-1}, x_i]$. Form the sum

$$S(f, dg, P) = \sum_{i=1}^{n} f(\xi_i)\,(g(x_i) - g(x_{i-1}))$$

and let
$$\|P\| = \max_{1\le i\le n}(x_i - x_{i-1}).$$

Then we define
$$\int_a^b f(x)\,dg(x) = \lim_{\|P\|\to 0} S(f, dg, P)$$
and call f *Riemann–Stieltjes integrable with respect to* g if this limit exists.

Clearly, the case $g(x) = x$ is just the Riemann integral. For g continuously differentiable, the integral reduces to a Riemann integral of the form

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)g'(x)\,dx.$$

If g is of a very simple form, then the integral can be computed by hand. Suppose that g is a step function; that is, for some partition P of this interval,

$$a = c_0 < c_1 < c_2 < \cdots < c_{k-1} < c_k = b,$$

the function g is constant on each interval (c_{i-1}, c_i). Let j_i be the jumps of g at c_i; that is $j_0 = g(c_0+) - g(c_0)$, $j_k = g(c_k) - g(c_k-)$, and $j_i = g(c_i+) - g(c_i-)$ for $1 \leq i \leq k-1$. Then one easily checks for a continuous function f that

$$\int_a^b f(x)\,dg(x) = \sum_{i=1}^{k} f(c_i)j_i.$$

The most natural applications of this integral occur for f continuous and g of bounded variation. In this case the integral exists and there is a useful estimate for its magnitude. We state this as a theorem; it is assigned as an exercise in Section 12.8 where it is needed. We leave the rest of the theoretical development of the integral to the exercises.

Theorem 1.48 *If f is continuous and g has bounded variation on an interval $[a, b]$, then f is Riemann–Stieltjes integrable with respect to g and*

$$\left|\int_a^b f(x)\,dg(x)\right| \leq \left(\max_{x\in[a,b]} |f(x)|\right) V(g; [a, b]).$$

The exercises can be used to sense the structure of the theory that emerges without working through the details. We do not require this theory in the sequel; but, as there are many applications of the Riemann–Stieltjes integral in analysis, the reader should emerge with some familiarity with the ideas, if not a full technical appreciation of how the proofs go. The study of $\int_a^b f(x)\,dg(x)$ is easiest if f is continuous and g monotonic (or of bounded variation). The details are harder if one wants more generality.

Exercises

1:19.1 What is $\int_a^b f(x)\,dg(x)$ if f is constant? If g is constant?

1:19.2 Writing

$$I(f, g) = \int_a^b f(x)\,dg(x)$$

establish the linearity of $f \to I(f, g)$ and $g \to I(f, g)$; that is, show that $I(f_1 + f_2, g) = I(f_1, g) + I(f_2, g)$, $I(cf, g) = I(f, cg) = cI(f, g)$, and $I(f, g_1 + g_2) = I(f, g_1) + I(f, g_2)$.

1:19.3 Give an example to show that both $\int_a^b f(x)\,dg(x)$ and $\int_b^c f(x)\,dg(x)$ may exist and yet $\int_a^c f(x)\,dg(x)$ may not.

1:19.4 Show that

$$\int_a^c f(x)\,dg(x) = \int_a^b f(x)\,dg(x) + \int_b^c f(x)\,dg(x)$$

under appropriate assumptions.

1:19.5 Suppose that g is continuously differentiable and f is continuous. Prove that

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)g'(x)\,dx.$$

[Hint: Write $f(\xi_i)(g(x_i) - g(x_{i-1}))$ as $f(\xi_i)g'(\eta_i)(x_i - x_{i-1})$, where ξ_i, $\eta_i \in [x_{i-1}, x_i]$ using the mean-value theorem.]

1:19.6 Let g be a step function, constant on each interval (c_{i-1}, c_i) of the partition

$$a = c_0 < c_1 < c_2 < \cdots < c_{k-1} < c_k = b.$$

Then, for a continuous function f,

$$\int_a^b f(x)\,dg(x) = \sum_{i=1}^{k} f(c_i)j_i,$$

where j_i are the jumps of g at c_i; that is, $j_0 = g(c_0+) - g(c_0)$, $j_k = g(c_k) - g(c_k-)$, and $j_i = g(c_i+) - g(c_i-)$ for $1 \le i \le k-1$.

1:19.7 Show that if $\int_a^b f(x)\,dg(x)$ exists then f and g have no common point of discontinuity.

1:19.8 (Integration by parts) Establish the formula

$$\int_a^b f(x)\,dg(x) + \int_a^b g(x)\,df(x) = f(b)g(b) - f(a)g(a)$$

under appropriate assumptions on f and g.

1:19.9 (Mean-value theorem) Show that

$$\int_a^b f(x)\,dg(x) = f(\xi)(g(b) - g(a))$$

for some $\xi \in [a, b]$ under appropriate assumptions on f and g.

1:19.10 Suppose that f_1, f_2 are continuous and g is of bounded variation on $[a, b]$, and define

$$h(x) = \int_a^x f_1(t)\,dg(t)$$

for $a \le x \le b$. Show that

$$\int_a^b f_2(t)\,dh(t) = \int_a^b f_1(t)f_2(t)\,dg(t).$$

1:19.11 Let $g, g_1, g_2, \ldots$ be BV functions on $[a,b]$ such that $g(a) = g_1(a) = \cdots = 0$. Suppose that the variation of $g - g_n$ on $[a,b]$ tends to zero as $n \to \infty$. Show that

$$\lim_{n\to\infty} \int_a^b f(x)\, dg_n(x) = \int_a^b f(x)\, dg(x)$$

for every continuous f. [Hint: Use Theorem 1.48.]

1.20 Lebesgue's Integral

The mainstream of modern integration theory is based on the notion of integral due to Lebesgue. A formal development of the integral must wait until Chapter 5, where it is done in full generality. Here we give some insight into what is involved.

Suppose that you have several coins in your pocket to count: 4 dimes, 2 nickels, and 3 pennies. There are two natural ways to count the total value of the coins.

Computation 1. Count the coins in the order in which they appear as you pull them from your pocket, for example,

$$10 + 10 + 5 + 10 + 1 + 5 + 10 + 1 + 1 = 53.$$

Computation 2. Group the coins by value, and compute

$$(10)(4) + (5)(2) + (1)(3) = 53.$$

Computation 1 corresponds to Riemann integration, while computation 2 corresponds to Lebesgue integration. Let's look at this a bit more closely. Figure 1.1 is the graph of a function that models our counting problem using the order from computation 1.

One can check easily that $\int_0^9 f(x)\, dx = 53$, the integral being Riemann's. Because of the simple nature of this function, one sees that one needs no finer partition than the partition obtained by dividing $[0,9]$ into 9 congruent intervals. This partition gives the sum corresponding to the first method.

To consider the second method of counting, we use the notation of measure theory. If I is an interval, we write, as usual, $\lambda(I)$ for the length of I. If E is a finite union of pairwise-disjoint intervals, $E = I_1 \cup \cdots \cup I_n$, then the measure of E is given by the sum

$$\lambda(E) = \lambda(I_1) + \cdots + \lambda(I_n).$$

Now let

$$E_1 = \{x : f(x) = 1\},$$
$$E_5 = \{x : f(x) = 5\},$$

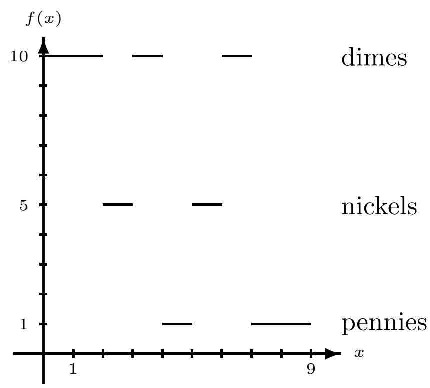

Figure 1.1: A function that models our counting problem.

and

$$E_{10} = \{x : f(x) = 10\}.$$

Then $\lambda(E_1) = 3$, $\lambda(E_5) = 2$, and $\lambda(E_{10}) = 4$. In computation 2 we formed the sum

$$(1)\lambda(E_1) + (5)\lambda(E_5) + (10)\lambda(E_{10}).$$

Note that the numbers 1, 5, and 10 represent the values of the function f, and $\lambda(E_i)$ indicates "how often" the value i is taken on.

We have belabored this simple example because it contains the seed of the Lebesgue integral. Let us try to imitate this example for an arbitrary bounded function f defined on $[a, b]$. Suppose that $m \leq f(x) < M$ for all $x \in [a, b]$. Instead of partitioning the interval $[a, b]$, we partition the interval $[m, M]$:

$$m = y_0 < y_1 < \cdots < y_n = M.$$

For $k = 1, \ldots, n$, let

$$E_k = \{x : y_{k-1} \leq f(x) < y_k\}.$$

Thus the partition of the range induces a partition of the interval $[a, b]$:

$$[a, b] = E_1 \cup E_2 \cup \cdots \cup E_n$$

where the sets $\{E_k\}$ are clearly pairwise disjoint. We can form the sums

$$\sum y_k \, \lambda(E_k) \quad \text{and} \quad \sum y_{k-1} \, \lambda(E_k)$$

in the expectation that these can be used to approximate our integral, the first from above and the second from below. We hope two things: that such approximating sums approach a limit as the norm of the partition approaches zero and that the two limits are the same. If each of the sets E_k happens to be always a finite union of intervals (e.g., if f is a polynomial), then the upper and lower sums do have the same limit. This is just another way of describing a well-known development of the Riemann integral via upper and lower sums.

But the sets E_k may be much more complicated than this. For example, each E_k might contain no interval. Thus one needs to know in advance the measure of quite arbitrary sets. This attempt at an integral will break down unless we restrict things in such a way that the sets that arise are Lebesgue measurable. This means we must restrict our attention to classes of functions for which all such sets are measurable, the measurable functions (Chapter 4).

After we understand the basic ideas of measures (Chapter 2) and measurable functions (Chapter 4), we will be ready to develop the integral. The idea of considering sums of the form

$$\sum y_k \, \lambda(E_k) \quad \text{and} \quad \sum y_{k-1} \, \lambda(E_k)$$

taken over a partition of the interval

$$[a, b] = E_1 \cup E_2 \cup \cdots \cup E_n$$

did not originate with Lebesgue; Peano had used it earlier. But the idea of partitioning the range in order to induce this partition seems to be Lebesgue's contribution, and it points out very clearly the class of functions that should be considered; that is, functions f for which the associated sets

$$E = \{x : \alpha \leq f(x) < \beta\}$$

are Lebesgue measurable.

The preceding paragraphs represent an outline of how one could arrive at the Lebesgue integral. Our development will be more general; it will include a theory of integration that applies to functions defined on general "measure spaces." The fascinating evolution of the theory of integration is delineated in Hawkins book on this subject.[2] A reading of this book allows one to admire the genius of some leading mathematicians of the time. It also allows one to sympathize with their misconceptions and the frustration these misconceptions must have caused.

1.21 The Generalized Riemann Integral

The main motivation that Lebesgue gave for generalizing the Riemann integral was Volterra's example of a bounded derivative that is not Riemann

[2] T. Hawkins, *Lebesgue's Theory of Integration*, Chelsea Publishing Co., (1979).

integrable. Lebesgue was able to prove that his integral would handle all bounded derivatives. His integral is, however, by its very nature an absolute integral. That is, in order for $\int_a^b f(x)\,dx$ to exist, it must be true that

$$\int_a^b |f(x)|\,dx$$

also exists. The problem of inverting derivatives cannot be solved by an absolute integral, as we know from the elementary example F' with $F(x) = x^2 \sin x^{-2}$.

Thus we are still left with a curious situation. Despite a century of the best work on the subject, the integration theories of Cauchy, Riemann, and Lebesgue do not include the original Newton integral. There are derivatives (necessarily unbounded) that are not integrable in any of these three senses. In general, how can one invert a derivative then?

To answer this, we can take a completely naive approach and start with the definition of the derivative itself. If $F' = f$ everywhere, then, at each point ξ and for every $\varepsilon > 0$, there is a $\delta > 0$ so that

$$|F(x'') - F(x') - f(\xi)(x'' - x')| < \varepsilon(x'' - x') \tag{5}$$

for $x' \leq \xi \leq x''$ and $0 < x'' - x' < \delta$.

We shall attempt to recover $F(b) - F(a)$ as a limit of Riemann sums for f, even though this is a misguided attempt, since we know that the Riemann integral must fail in general to accomplish this. Even so, let us see where the attempt takes us.

Let

$$a = x_0 < x_1 < x_2 \ldots x_n = b$$

be a partition of $[a, b]$, and let $\xi_i \in [x_{i-1}, x_i]$. Then

$$F(b) - F(a) = \sum_{i=1}^{n} (F(x_{i-1}) - F(x_i)) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) + R$$

where

$$R = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}) - f(\xi_i)(x_i - x_{i-1}))\,.$$

Thus $F(b) - F(a)$ has been given as a Riemann sum for f plus some error term R. But it appears now that, if the partition is finer than the number δ so that (5) may be used, we have

$$\begin{aligned} |R| &\leq \sum_{i=1}^{n} \Big| F(x_i) - F(x_{i-1}) - f(\xi_i)(x_i - x_{i-1}) \Big| \\ &< \sum_{i=1}^{n} \varepsilon(x_i - x_{i-1}) = \varepsilon(b - a). \end{aligned}$$

Evidently, then, *if there are no mistakes here* we have just proved that f is Riemann integrable and that $\int_a^b f(t)\,dt = F(b) - F(a)$.

This is false of course. Even the Lebesgue integral does not invert all derivatives, and the Riemann integral cannot invert even all bounded derivatives. The error is that the choice of δ depends on the point ξ considered and so is not a constant. But, instead of abandoning the argument, one can change the definition of the Riemann integral to allow a variable δ. The definition then changes to look like this.

Definition 1.49 A function f is *generalized Riemann integrable* on $[a, b]$ with value I if for every ε there is a positive function δ on $[a, b]$ so that

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - I \right| < \varepsilon$$

whenever $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ is a partition of $[a, b]$ with $\xi_i \in [x_{i-1}, x_i]$ and $0 < x_i - x_{i-1} < \delta(\xi_i)$.

To justify the definition requires knowing that such partitions actually exist for any such gauge δ; this is supplied by the Cousin theorem (Theorem 1.3).

This defines a Riemann-type integral that includes the Lebesgue integral and the Newton integral. It is equivalent to the integrals invented by A. Denjoy and O. Perron in 1912. The generalized Riemann integral was discovered in the 1950s, independently, by R. Henstock and J. Kurzweil, and these ideas have led to a number of other integration theories that exploit the geometry of the underlying space in the same way that this integral exploits the geometry of derivatives on the real line.

In Section 5.10 we shall present a property of the Lebesgue integral that shows how it is included in a generalized Riemann integral. We do not develop this theme any further as these ideas should be considered, for the moment anyway, as rather specialized. A development of these ideas can be found in the recent monographs of Pfeffer[3] or Gordon.[4] The main tool of modern analysis is the standard theory of measure and integration developed in subsequent chapters, and we confine our interests in integration theory to its exposition.

Exercises

1:21.1 Develop the elementary properties of the generalized Riemann integral directly from its definition (e.g., the integral of a sum $f + g$, the integral formula $\int_a^b + \int_b^c = \int_a^c$, etc.).

[3] W. F. Pfeffer, *The Riemann Approach to Integration: Local Geometric Theory.* Cambridge (1993).

[4] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron and Henstock.* Grad. Studies in Math, Vol. 4, Amer. Math. Soc. (1994).

1:21.2 Show directly from the definition that the function f defined as $f(x) = 0$ for x rational and $f(x) = 1$ for x irrational is not Riemann integrable, but is generalized Riemann integrable on any interval, and that $\int_0^1 f(x)\,dx = 1$.

1:21.3 Show that the generalized Riemann integral is closed under the extension procedure of Cauchy from Section 1.16.

1.22 Additional Problems for Chapter 1

1:22.1 For an arbitrary function $F : \mathbb{R} \to \mathbb{R}$, prove that the set

$$\{x : F \text{ assumes a strict local maximum or minimum at } x\}$$

is countable. [Hint: Consider

$$A_n = \left\{ x : F(t) < F(x)\ \forall\, t \neq x \text{ in } \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \right\}.$$

1:22.2 For an arbitrary function $F : \mathbb{R} \to \mathbb{R}$, prove that the set

$$\left\{ x : \limsup_{t \to x} F(t) > \limsup_{t \to x+} F(t) \right\}$$

is countable.

1:22.3 For an arbitrary function $F : \mathbb{R} \to \mathbb{R}$, prove that the set

$$\left\{ x : F(x) \notin \left[\liminf_{t \to x} F(t), \limsup_{t \to x} F(t) \right] \right\}$$

is countable.

1:22.4 For an arbitrary function $F : \mathbb{R} \to \mathbb{R}$, prove that the set

$$\left\{ x : F \text{ is discontinuous at } x \text{ and } \lim_{t \to x} F(t) \text{ exists} \right\}$$

is countable.

1:22.5 Show that the set of irrationals in $[0, 1]$ has inner measure 1 and the set of rationals in $[0, 1]$ has outer measure 0.

1:22.6 Prove (or find somewhere a proof) that the three logical principles (i) the axiom of choice, (ii) the well-ordering principle [Zermelo's theorem], and (iii) Zorn's lemma are equivalent.

1:22.7$\diamondsuit$ An uncountable set S of real numbers is said to be *totally imperfect* if it contains no perfect set. A set S of real numbers is said to be a *Bernstein set* if neither S nor $\mathbb{R} \setminus S$ contains a perfect set. Prove the existence of such sets assuming the continuum hypothesis and using Statement 1.15. (Incidentally, no Borel set can be totally imperfect.) [Hint: Let $\mathcal{C}$ be the collection of all perfect sets. This has cardinality c (see Exercise 1:4.7). Under CH we can well order $\mathcal{C}$ as in Statement 1.15, say indexing as $\{P_\alpha\}$, so that each element has only countably many predecessors. Construct S by picking two distinct points x_α, y_α from each P_α in such a way that at each stage we pick new points. (You will have to justify this by a cardinality argument.) Put the x_α in S.]

1:22.8$\diamondsuit$ Show the existence of Bernstein sets (without assuming CH by using Lemma 1.16). [Hint: Use basically the same proof as Exercise 1:22.7, but with a little more attention to the cardinality arguments.]

1:22.9$\diamondsuit$ Assuming CH, show that there is an uncountable set U of real numbers (called a *Lusin set*) such that every dense open set contains all but countably many points from U. [Hint: Let $\{G_\alpha\}$ be a well ordering of the open dense sets so that every element has only countably many predecessors. Choose distinct points x_α from $\bigcap_{\beta\le\alpha} G_\beta$. Then U consists of all the points x_α. (The steps have to be justified. Remember that a countable intersection of dense open sets is residual and therefore uncountable.)]

1:22.10 Recall (Exercise 1:7.5) that the outer content c^* is *finitely subadditive*; that is, if $\{E_k\}$ is a sequence of subsets of $\mathbb{R}$, then

$$c^*\left(\bigcup_{k=1}^{n} E_k\right) \le \sum_{k=1}^{n} c^*(E_k).$$

Show that c_* is *finitely superadditive*; that is, if $\{E_k\}$ is a *disjoint* sequence of subsets of $\mathbb{R}$,

$$c_*\left(\bigcup_{k=1}^{n} E_k\right) \ge \sum_{k=1}^{n} c_*(E_k).$$

1:22.11 Recall (Exercise 1:7.6) that the outer measure λ^* is *countably subadditive*; that is, if $\{E_k\}$ is a sequence of subsets of $\mathbb{R}$, then

$$\lambda^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} \lambda^*\left(E_k\right).$$

Similarly, show that λ_* is *countably superadditive*; that is, if $\{E_k\}$ is a *disjoint* sequence of subsets of an interval $[a,b]$, then

$$\lambda_*\left(\bigcup_{k=1}^{\infty} E_k\right) \ge \sum_{k=1}^{\infty} \lambda_*(E_k).$$

[Hint: Use Exercise 1:9.16.]

1:22.12 Let $\{c_k\}$ be complex numbers with $\sum_{k=1}^\infty |c_k| < +\infty$ and write $f(z) = \sum_{k=1}^\infty c_k z^k$ for $|z| \le 1$. Show that f is BV on each radius of the circle $|z| = 1$.

1:22.13$\diamondsuit$ Let C and B be the sets referenced in the proof of Theorem 1.23. Define a function f in the following way. On I_1, let $f = 1/2$; on I_2,

$f = 1/4$; on I_3, $f = 3/4$. Proceed inductively. On the $2^{n-1} - 1$ open intervals appearing at the nth stage, define f to satisfy the following conditions:

(*i*) f is constant on each of these intervals.

(*ii*) f takes the values
$$\frac{1}{2^n}, \frac{3}{2^n}, \dots, \frac{2^n - 1}{2^n}$$
on these intervals.

(*iii*) If x and y are members of different nth-stage intervals with $x < y$, then $f(x) < f(y)$.

This description defines f on B. Extend f to all of $[0, 1]$ by defining $f(0) = 0$ and, for $x \neq 0$, $f(x) = \sup\{f(t) : t \in B, t < x\}$.

(a) Show that $f(B)$ is dense in I_0.

(b) Show that f is nondecreasing on I_0.

(c) Infer from (a) and (b) that f is continuous on I_0.

(d) Show that $f(C) = I_0$, and thus C has the same cardinality as I_0.

As an example, Figure 1.2 corresponds to the case in which, every time an interval I_k is selected, it is the middle third of the closed component of A_n from which it is chosen. In this case, the set C is called the *Cantor set* (or Cantor ternary set) and f is called the *Cantor function*. The set and function are named for the German mathematician Georg Cantor (1845–1918). Observe that f "does all its rising" on the set C, which here has measure zero. More precisely, $\lambda(f(B)) = 0$, $\lambda(f(C)) = 1$. This example will be important in several places in Chapters 4 and 5.

1:22.14 Using some of the ideas in the construction of the Cantor function (Exercise 1:22.13), obtain a continuous function that is not of bounded variation on any subinterval of $[0, 1]$.

1:22.15 Using some of the ideas in the construction of the Cantor function (Exercise 1:22.13), obtain a continuous function that is of bounded variation on $[0, 1]$, but is not monotone on any subinterval of $[0, 1]$.

1:22.16 Show that the Cantor function is not absolutely continuous (Exercise 1:14.17).

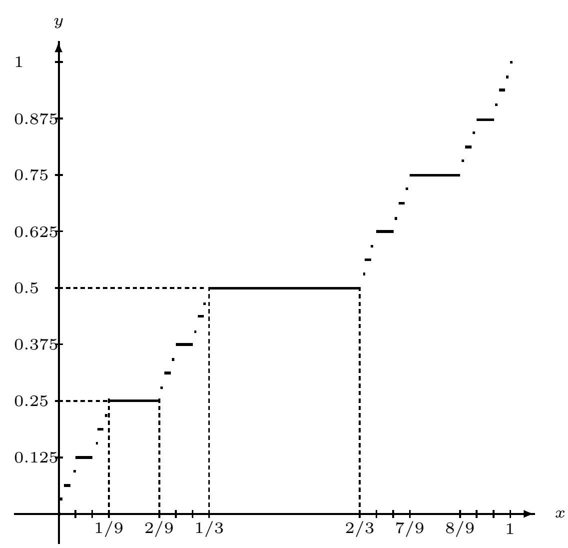

Figure 1.2: The Cantor function.

Chapter 2

MEASURE SPACES

With the help of the Riemann version of the integral, calculus students can study such notions as the length of a curve, the area of a region in the plane, the volume of a region in space, and mass distributions on the line, in the plane, or in space. These notions, as well as many others, can be studied within the framework of measure theory.

In this framework, one has a set X, a class $\mathcal{M}$ of subsets of X, and a measure μ defined on $\mathcal{M}$. The class $\mathcal{M}$ satisfies certain natural conditions (See Sections 2.2 and 2.3), and μ satisfies conditions one would expect of such notions as length, area, volume, or mass.

Our objective in this chapter is to provide the reader with a working knowledge of basic measure theory. In Section 2.1, we provide an outline of Lebesgue measure on the line via the notions of *inner measure* and *outer measure*. Then, in Sections 2.2 and 2.3, we begin our development of abstract measure theory by extracting features of Lebesgue measure that one would want for any notion of measure.

This abstract approach has the advantage of being quite general and therefore of being applicable to a variety of phenomena. But it does not tell us *how* to obtain a measure with which to model a given phenomenon. Here we take our cue from the development in Section 2.1. We find that a measure can always be obtained from an *outer measure* (Section 2.7).

We also find that when we have a primitive notion of our phenomenon, for example, length of an interval, area of a square, volume of a cube, or mass in a square or cube, this primitive notion determines an outer measure in a natural way. The outer measure, in turn, defines a measure that extends this primitive notion to a large class of sets $\mathcal{M}$ that is suitable for a coherent theory.

Many measures possess special properties that make them particularly useful. Lebesgue measure has most of these. For example, the Lebesgue outer measure of any set E can be obtained as the Lebesgue measure of a larger set $H \supset E$ that is measurable. Every subset of a set of Lebesgue measure zero is measurable and has, again, Lebesgue measure zero. In

Sections 2.9 to 2.12 we develop such properties abstractly. Finally, Section 2.10 addresses the problem of nonmeasurable sets in a very general setting.

2.1 One-Dimensional Lebesgue Measure

We begin our study of measures with a heuristic development of Lebesgue measure in $\mathbb{R}$ that will provide a concrete example that we can recall when we develop the abstract theory. This is independent of the sketch given in the first chapter. Our development will be heuristic for two reasons. First, a development including all details would obscure the major steps we wish to highlight. Some of these details are covered by the exercises. Second, our development of the abstract theory in the remainder of the chapter, which does not depend on Lebesgue measure in any way, will verify the correctness of our claims. Thus Lebesgue measure serves as our motivating example to guide the development of the theory and our illustrative example to show the theory in application.

We begin with the primitive notion of the length of an interval. We then extend this notion in a natural way first to open sets, then to closed sets. Finally, by the method of inner and outer measures, this is extended to a large class of "measurable" sets.

1. We define
$$\lambda(I) = b - a,$$
where I denotes the open interval (a, b). This is the beginning of a process that can, with some adjustments, be applied to a variety of situations.

2. Define
$$\lambda(G) = \sum \lambda(I_k),$$
where G is an open set and $\{I_k\}$ is the sequence of component intervals of G. If one of the components is unbounded, we let $\lambda(G) = \infty$. [If $G \neq \emptyset$, then G can be expressed as a finite or countably infinite disjoint union of open intervals: $G = \bigcup I_k$. If $G = \emptyset$, the empty set, define $\lambda(G) = 0$.] This definition is a natural one; it conforms to our intuitive requirement that "the whole is equal to the sum of the parts."

3. Define
$$\lambda(E) = b - a - \lambda((a, b) \setminus E),$$
where E is a bounded closed set and $[a, b]$ is the smallest closed interval containing E. Since $[a, b] = E \cup ([a, b] \setminus E)$, our intuition would demand that
$$\lambda(E) + \lambda((a, b) \setminus E) = b - a$$
and this becomes our definition.

So far, we have a notion of measure for arbitrary open sets and for bounded closed sets. We shall presently use these notions to extend the measure to a larger class of sets—the measurable sets. Let us pause first to look at an intuitive example.

Example 2.1 Let $0 \le \alpha < 1$. There is a nowhere dense closed set $C \subset [0,1]$ that is of measure α. (For the full details of the construction see Section 1.8.) Its complement $B = [0,1] \setminus C$ is a dense open subset of $[0,1]$ of measure $1-\alpha$. In particular, if $\alpha > 0$, C has positive measure. In any case, C is a nonempty nowhere dense perfect subset of $[0,1]$ and therefore has cardinality of the continuum. (See Exercise 1:22.13.)

While the construction of the set C is relatively simple, the existence of such sets was not known until late in the nineteenth century. Prior to that, mathematicians recognized that a nowhere dense set could have limit points, even limit points of limit points, but could not conceive of a nowhere dense set as possibly having positive measure. Since dense sets were perceived as large and nowhere dense sets as small, this example, with $\alpha > 0$, would have begun the process of clarifying the ideas that would lead to a coherent development of measure theory.

We shall now use our definitions of measure for bounded open sets and bounded closed sets to obtain a large class $\mathcal{L}$ of Lebesgue measurable sets to which the measure λ can be extended. To each set $E \in \mathcal{L}$, we shall assign a nonnegative number $\lambda(E)$ called the Lebesgue measure of E. Our intuition demands that a certain "monotonicity" condition be satisfied for measurable sets: if E_1 and E_2 are measurable and $E_1 \subset E_2$, then

$$\lambda(E_1) \le \lambda(E_2).$$

In particular, if G is any open set containing a set E, we would want $\lambda(E) \le \lambda(G)$, so $\lambda(G)$ provides an upper bound for $\lambda(E)$, if E is to be measurable. We can now define the *outer measure* of an arbitrary set E by choosing G "economically."

Definition 2.2 Let E be an arbitrary subset of $\mathbb{R}$. Let

$$\lambda^*(E) = \inf\{\lambda(G) : \ E \subset G,\, G \text{ open}\}.$$

Then $\lambda^*(E)$ is called the Lebesgue *outer measure* of E.

We point out, for later reference, that the outer measure can also be obtained by approximating from outside with sequences of open intervals (Exercise 2:1.10):

$$\lambda^*(E) = \inf\left\{\sum_{k=1}^{\infty}\lambda(I_k) : E \subset \textstyle\bigcup_{k=1}^{\infty} I_k, \text{ each } I_k \text{ an open interval}\right\}.$$

Now $\lambda^*(E)$ may seem like a good candidate for $\lambda(E)$. It meets the monotonicity requirement and it is well defined for all bounded subsets of $\mathbb{R}$.

It is also true, but by no means obvious, that $\lambda^*(E) = \lambda(E)$ when E is open or closed. (See Exercise 2:1.4.) But λ^* lacks an essential property: we cannot conclude for a pair of disjoint sets E_1, E_2 that

$$\lambda^*(E_1 \cup E_2) = \lambda^*(E_1) + \lambda^*(E_2).$$

The whole need not equal the sum of its parts.

Here is how Lebesgue remedied this flaw. So far we have used only part of what is available to us—*outside* approximation of E by *open* sets. Now we use *inside* approximation by *closed* sets.

Definition 2.3 Let E be an arbitrary subset of $\mathbb{R}$. Let

$$\lambda_*(E) = \sup\{\lambda(F) : F \subset E,\ F \text{ compact}\}.$$

Then $\lambda_*(E)$ is called the Lebesgue *inner measure* of E.

Since E need not contain any intervals, there is no inner approximation by intervals, analogous to the approximation of the outer measure by intervals. We have, however, the following formula for a bounded set E.

2.4 Let $[a, b]$ be the smallest interval containing a bounded set E. Then

$$\lambda_*(E) = b - a - \lambda^*([a, b] \setminus E).$$

This shows the important fact that the inner measure is definable directly in terms of the outer measure. In particular, it suggests already that a theory based on the outer measure alone may be feasible. We illustrate these definitions with an example.

Example 2.5 Let $I_0 = [0, 1]$, and let $\mathbb{Q}$ denote the rational numbers in I_0. Let $\varepsilon > 0$ and let $\{q_k\}$ be an enumeration of $\mathbb{Q}$. For each positive integer n, let I_n be an open interval such that $q_n \in I_n$ and $\lambda(I_n) < \varepsilon/2^n$. Then $\mathbb{Q} \subseteq \bigcup I_n$ and $\sum \lambda(I_n) < \varepsilon$. Thus $\lambda^*(\mathbb{Q}) = 0$. The set $P = I_0 \setminus \bigcup I_k$ is closed, and $P \subset I_0 \setminus \mathbb{Q}$. We see, using the assertion 2.4 and Exercise 2:1.12, that $\lambda(P) > 1 - \varepsilon$. It follows that

$$1 - \varepsilon < \lambda_*(P) \le \lambda_*(I_0 \setminus \mathbb{Q}),$$

so that $\lambda_*(I_0 \setminus \mathbb{Q}) = 1$. Thus the set of irrationals in I_0 has inner measure 1, and the set of rationals has outer measure 0.

Inner measure λ_* has the same flaw as outer measure λ^*. The key to obtaining a large class of measurable sets lies in the observation that we would like outside approximation to give the same result as inside approximation.

Definition 2.6 Let E be a bounded subset of $\mathbb{R}$, and let $\lambda^*(E)$ and $\lambda_*(E)$ denote the outer and inner measures of E. If

$$\lambda^*(E) = \lambda_*(E),$$

we say that E is *Lebesgue measurable* with *Lebesgue measure* $\lambda(E) = \lambda^*(E)$. If E is unbounded, we say that E is measurable if $E \cap I$ is measurable for every interval I and again write $\lambda(E) = \lambda^*(E)$.

One can verify that the class $\mathcal{L}$ of Lebesgue measurable sets is closed under countable unions and under set difference. If $\{E_k\}$ is a sequence of measurable sets, so is $\bigcup E_k$, and the difference of two measurable sets is measurable. In addition, Lebesgue measure λ is *countably additive on the class* $\mathcal{L}$: if $\{E_k\}$ is a sequence of pairwise disjoint sets from $\mathcal{L}$, then

$$\lambda(\bigcup E_k) = \sum \lambda(E_k).$$

We shall not prove these statements at this time. They will emerge as consequences of the theory developed in Section 2.9. Observe for later reference that λ^* *is* countably additive on $\mathcal{L}$, since $\lambda^* = \lambda$ on $\mathcal{L}$. Thus we can view λ as the restriction of λ^*, which is defined for *all* subsets of $\mathbb{R}$, to $\mathcal{L}$, the class of Lebesgue measurable sets.

Not all subsets of $\mathbb{R}$ can be measurable. In Section 1.10 we have given the details of the proof of this fact. But we shall discover that all sets that arise in practice are measurable.

Many of the ideas that appear in this section, including the exercises, will reappear, in abstract settings as well as in concrete settings, throughout the remainder of this chapter.

Exercises

2:1.1 In the definition of $\lambda(G)$ for G a bounded open set, how do we know that the sum $\sum \lambda(I_k)$ is finite?

2:1.2 Prove that both the outer measure and inner measure are monotone: If $E_1 \subset E_2$, then $\lambda^*(E_1) \leq \lambda^*(E_2)$ and $\lambda_*(E_1) \leq \lambda_*(E_2)$.

2:1.3 Prove that the outer measure λ^* and inner measure λ_* are translation-invariant functions defined on the class of all subsets of $\mathbb{R}$.

2:1.4 Prove that $\lambda^*(E) = \lambda_*(E) = \lambda(E)$ when E is open or closed and bounded. (Thus the definition of measure for open sets and for compact sets in terms of λ^* and λ_* is consistent with the definition given at the beginning of the section.) [Hint: If E is an open set with component intervals $\{(a_i, b_i)\}$, then show how $\lambda_*(E)$ can be approximated by the measure of a compact set of the form

$$\bigcup_{i=1}^{N} \left[a_i + \varepsilon 2^{-i}, b_i - \varepsilon 2^{-i}\right]$$

for large N and small $\varepsilon > 0$.]

2:1.5 Let $[a, b]$ be the smallest interval containing a bounded set E. Prove that

$$\lambda_*(E) = b - a - \lambda^*([a, b] \setminus E).$$

[Hint: Split the equality into two inequalities and prove each directly from the definition.]

2:1.6 For all $E \subset \mathbb{R}$, show that $\lambda_*(E) \le \lambda^*(E)$. [Hint: If $F \subset E \subset G$ with F compact and G open, we know already that $\lambda(F) \le \lambda(G)$. Take first the infimum over G and then the supremum over F.]

2:1.7 Show that if $\lambda^*(E) = 0$ then E and all its subsets are measurable.

2:1.8 Show that there exist 2^c Lebesgue measurable sets (where c is, as usual, the cardinality of the real numbers).

2:1.9 Show that if $\{G_k\}$ is a sequence of open subsets of $\mathbb{R}$ then

$$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \le \sum_{k=1}^{\infty} \lambda(G_k).$$

[Hint: If $(a,b) \subset \bigcup_{k=1}^{\infty} G_k$, show that $b - a \le \sum_{k=1}^{\infty} \lambda(G_k)$ by considering that

$$[a+\varepsilon, b-\varepsilon] \subset \bigcup_{k=1}^{N} G_k$$

for small ε and sufficiently large N.]

2:1.10 Using Exercise 2:1.9, show that

$$\lambda^*(E) = \inf\left\{\sum_{k=1}^{\infty} \lambda(I_k) : E \subset \bigcup_{k=1}^{\infty} I_k, \text{ each } I_k \text{ an open interval}\right\}.$$

2:1.11 Show that if $\{F_k\}$ is a sequence of compact disjoint subsets of $\mathbb{R}$ then

$$\lambda\left(\bigcup_{k=1}^{n} F_k\right) \ge \sum_{k=1}^{n} \lambda(F_k).$$

[Hint: If F_1 and F_2 are disjoint compact sets, then there are disjoint open sets $G_1 \supset F_1$ and $G_2 \supset F_2$.]

2:1.12 Show that λ^* is *countably subadditive*: if $\{E_k\}$ is a sequence of subsets of $\mathbb{R}$, then

$$\lambda^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} \lambda^*(E_k).$$

[Hint: Choose open sets $G_k \supset E_k$ so that $\lambda^*(E_k) + \varepsilon 2^{-k} \ge \lambda(G_k)$ and use Exercise 2:1.9.]

2:1.13 Similarly to Exercise 2:1.12, show that λ_* is *countably superadditive*: if $\{E_k\}$ is a *disjoint* sequence of subsets of $\mathbb{R}$,

$$\lambda_*\left(\bigcup_{k=1}^{\infty} E_k\right) \ge \sum_{k=1}^{\infty} \lambda_*(E_k).$$

[Hint: Choose compact sets $F_k \subset E_k$ so that $\lambda_*(E_k) - \varepsilon 2^{-k} \ge \lambda(F_k)$ and use Exercise 2:1.11.]

2:1.14◇ We recall that a set is of type $\mathcal{F}_\sigma$ if it can be expressed as a countable union of closed sets, and it is of type $\mathcal{G}_\delta$ if it can be expressed as a countable intersection of open sets. (See the discussion of these ideas in Sections 1.1 and 1.12.)

(a) Prove that every closed set $F \subset \mathbb{R}$ is of type $\mathcal{G}_\delta$ and every open set $G \subset \mathbb{R}$ is of type $\mathcal{F}_\sigma$.

(b) Prove that for every set $E \subset \mathbb{R}$ there exists a set K of type $\mathcal{F}_\sigma$ and a set H of type $\mathcal{G}_\delta$ such that $K \subset E \subset H$ and

$$\lambda(K) = \lambda_*(E) \leq \lambda^*(E) = \lambda(H).$$

The set K is called a *measurable kernel* of E, while the set H is called a *measurable cover* for E.

(c) Prove that if $E \in \mathcal{L}$ there exist K, H as above such that

$$\lambda(K) = \lambda(E) = \lambda(H).$$

[The point of this exercise is to show that one can approximate measurable sets by relatively simple sets on the inside and on the outside. By use of the Baire category theorem (see Section 1.6), one can show that the roles played by sets of type $\mathcal{F}_\sigma$ and of type $\mathcal{G}_\delta$ cannot be exchanged in parts (b) and (c).]

(d) Show that "$\mathcal{F}_\sigma$" cannot be improved to "closed" and "$\mathcal{G}_\delta$" cannot be improved to "open" in parts (b) and (c).

2:1.15 Give an example of a nonmeasurable set E for which $\lambda_*(E) = \lambda^*(E) = \infty$. [Hint: Use Theorem 1.33.]

2.2 Additive Set Functions

We begin now our study of structures suggested by Lebesgue measure. The class of sets that are Lebesgue measurable has certain natural properties: it is closed under the formation of unions, intersections, and set differences. This leads to our first abstract definition.

Definition 2.7 Let X be any set, and let $\mathcal{A}$ be a nonempty family of subsets of X. We say $\mathcal{A}$ is an *algebra of sets* if it satisfies the following conditions:

1. $\emptyset \in \mathcal{A}$.
2. If $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.
3. If $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A}$.

It is easy to verify that an algebra of sets is closed also under differences, finite unions, and finite intersections. (See Exercise 2:2.1.) For any set X, the family 2^X of all subsets of X is obviously an algebra. So is the family $\mathcal{A} = \{\emptyset, X\}$. We have noted that the family $\mathcal{L}$ of Lebesgue measurable sets is an algebra. Here is another example, to which we shall return later.

Example 2.8 Let $X = (0, 1]$. Let $\mathcal{A}$ consist of $\emptyset$ and all finite unions of half-open intervals $(a, b]$ contained in X. Then $\mathcal{A}$ is an algebra of sets.

Our next notion, that of *additive set function*, might be viewed as the forerunner of the notion of measure. If we wish to model phenomena such as area, volume, or mass, we would like our model to conform to physical laws, reflect our intuition, and make precise concepts, such as "the whole is the sum of its parts." We can do this as follows.

Definition 2.9 Let $\mathcal{A}$ be an algebra of sets and let ν be an extended real-valued function defined on $\mathcal{A}$. If ν satisfies the following conditions, we say ν is an *additive set function*.

1. $\nu(\emptyset) = 0$.
2. If A, $B \in \mathcal{A}$ and $A \cap B = \emptyset$, then $\nu(A \cup B) = \nu(A) + \nu(B)$.

Note that such a function is allowed to take on infinite values, but cannot take on both $-\infty$ and ∞ as values. (See Exercise 2:2.8.) A nonnegative additive set function is often called a *finitely additive measure*.

Example 2.10 Let $X = (0, 1]$ and $\mathcal{A}$ be as in Example 2.8. Let f be an arbitrary function on $[0, 1]$. Define $\nu_f((a, b]) = f(b) - f(a)$, and extend ν_f to be additive on $\mathcal{A}$. Then ν_f is an additive set function. (See Exercise 2:2.14.)

Example 2.10 plays an important role in the general theory, both for applications and to illustrate many ideas. Note that if f is nondecreasing, then the set function ν_f is nonnegative and can model many concepts. If $f(x) = x$ for all $x \in X$, then $\nu_f(A) = \lambda(A)$ for all $A \in \mathcal{A}$. Here, ν_f models a uniform distribution of mass—the amount of mass in an interval is proportional to the length of the interval. Another nondecreasing function would give rise to a different mass distribution. For example, if $f(x) = x^2$, $\nu_f((0, \frac{1}{2}]) = \frac{1}{4}$, while $\nu_f((\frac{1}{2}, 1]) = \frac{3}{4}$; in this case the mass is not uniformly distributed. As yet another example, let

$$f(x) = \begin{cases} 0, & 0 \le x < x_0 < 1; \\ 1, & x_0 \le x \le 1. \end{cases}$$

Then f has a jump discontinuity at x_0, and

$$\nu_f(A) = \begin{cases} 0, & \text{if } x_0 \notin A; \\ 1, & \text{if } x_0 \in A. \end{cases}$$

We would like to say that x_0 is a "point mass" and that the set function assigns the value 1 to the singleton set $\{x_0\}$, but $\{x_0\} \notin \mathcal{A}$. Since point masses arise naturally as models in nature, this algebra $\mathcal{A}$ is not fully adequate to discuss finite mass distributions on $(0, 1]$. This flaw will disappear when we consider *measures* on *σ-algebras* in Section 2.3. In that setting, $\{x_0\}$ will be a member of the σ-algebra and will have unit mass. These ideas are the forerunner of Lebesgue–Stieltjes measures, which we study in Section 3.5.

In Example 2.10 we can take f *nonincreasing* and we can model "negative mass." This is analogous to the situation in elementary calculus where one often interprets an integral $\int_a^b g(x)\,dx$ in terms of negative area when the integrand is negative on the interval.

One can combine positive and negative mass. If f has a decomposition into a difference of monotonic functions

$$f = f_1 - f_2 \text{ with } f_1 \text{ and } f_2 \text{ nondecreasing on } X, \tag{1}$$

then it is easy to check that ν_f has a similar decomposition:

$$\nu_f = \nu_{f_1} - \nu_{f_2}.$$

Unless f is monotonic on X, there will be intervals of positive mass and intervals of negative mass. Functions f that admit the representation (1) are those that are of *bounded variation*. (We have reviewed some properties of such functions in Section 1.14. Note particularly Exercise 1:14.10.) It appears then that we can model a mass distribution ν_f on $[a, b]$ that involves both positive and negative mass as a difference of two nonnegative mass distributions. This is so if, in Example 2.10, f has bounded variation; is it true for an arbitrary function f?

This leads us to variational questions for additive set functions that parallel the ideas and methods employed in the study of functions of bounded variation.

Definition 2.11 Let X be any set, let $\mathcal{A}$ be an algebra of subsets of X and let ν be additive on $\mathcal{A}$. For $E \in \mathcal{A}$, we define the *upper variation of ν on E* by

$$\overline{V}(\nu, E) = \sup\{\nu(A) : A \in \mathcal{A}, A \subset E\}.$$

Similarly, we define the *lower variation of ν on E* by

$$\underline{V}(\nu, E) = \inf\{\nu(A) : A \in \mathcal{A}, A \subset E\}.$$

Finally, we define the (total) *variation of ν on E* by

$$V(\nu, E) = \overline{V}(\nu, E) - \underline{V}(\nu, E).$$

Since $\nu(\emptyset) = 0$, $\underline{V}(\nu, E) \leq 0 \leq \overline{V}(\nu, E)$; thus the total variation is the sum of two nonnegative terms. Exercise 2:2.16 displays $V(\nu, E)$ in an equivalent form reminiscent of the variation of a real-valued function.

Theorem 2.12 *If ν is additive on an algebra $\mathcal{A}$ of subsets of X, then all the variations are additive set functions on $\mathcal{A}$.*

Proof. We show that the upper variation is additive on $\mathcal{A}$, the other proofs being similar. That $\overline{V}(\nu, \emptyset) = 0$ is clear. To verify condition 2 of Definition 2.9, let A and B be disjoint members of $\mathcal{A}$. Assume first that

$$\overline{V}(\nu, A \cup B) < \infty.$$

Let $\varepsilon > 0$. There exist A' and B' in $\mathcal{A}$ such that $A' \subset A$, $B' \subset B$, $\nu(A') > \overline{V}(\nu, A) - \varepsilon/2$, and $\nu(B') > \overline{V}(\nu, B) - \varepsilon/2$. Thus

$$\begin{aligned} \overline{V}(\nu, A \cup B) \geq \nu(A' \cup B') &= \nu(A') + \nu(B') \\ &> \overline{V}(\nu, A) + \overline{V}(\nu, B) - \varepsilon. \end{aligned} \tag{2}$$

In the other direction, there exists a set $C \in \mathcal{A}$ such that $C \subset A \cup B$ and $\nu(C) > \overline{V}(\nu, A \cup B) - \varepsilon$. Thus

$$\begin{aligned} \overline{V}(\nu, A \cup B) - \varepsilon < \nu(C) &= \nu(A \cap C) + \nu(B \cap C) \\ &\leq \overline{V}(\nu, A) + \overline{V}(\nu, B). \end{aligned} \tag{3}$$

Since ε is arbitrary, it follows from (2) and (3) that

$$\overline{V}(\nu, A \cup B) = \overline{V}(\nu, A) + \overline{V}(\nu, B).$$

It remains to consider the case $\overline{V}(\nu, A \cup B) = \infty$. Here one can easily verify that either $\overline{V}(\nu, A) = \infty$ or $\overline{V}(\nu, B) = \infty$, and the conclusion follows. ■

Theorem 2.13 provides an abstract version in the setting of additive set functions of the Jordan decomposition theorem for functions of bounded variation (Exercise 1:14.10). It indicates how, in many cases, a mass distribution can be decomposed into the difference of two nonnegative mass distributions. Observe that $\underline{V}(\nu, A)$ is *nonpositive*, so one can view this decomposition as a difference of two nonnegative additive set functions.

Theorem 2.13 (Jordan decomposition) *Let ν be an additive set function on an algebra $\mathcal{A}$ of subsets of X, and suppose that ν has finite total variation. Then, for all $A \in \mathcal{A}$,*

$$\nu(A) = \overline{V}(\nu, A) + \underline{V}(\nu, A). \tag{4}$$

Proof. Let A, $E \in \mathcal{A}$ and $E \subset A$. Since

$$\nu(E) = \nu(A) - \nu(A \setminus E),$$

we have

$$\nu(A) - \overline{V}(\nu, A) \leq \nu(E) \leq \nu(A) - \underline{V}(\nu, A). \tag{5}$$

Expression (5) is valid for all $E \in \mathcal{A}$, $E \subset A$. Noting the definition of $\overline{V}(\nu, A)$, we see from the second inequality that

$$\overline{V}(\nu, A) \leq \nu(A) - \underline{V}(\nu, A). \tag{6}$$

Similarly, from the first inequality, we infer that

$$\underline{V}(\nu, A) \geq \nu(A) - \overline{V}(\nu, A). \tag{7}$$

Comparing (6) and (7), we obtain our desired conclusion, (4). ■

Exercises

2:2.1 Show that an algebra of sets is closed under differences, finite unions, and finite intersections.

2:2.2 Let X be a nonempty set. Show that 2^X (the family of all subsets of X) and $\{\emptyset, X\}$ are both algebras of sets, in fact the largest and the smallest of the algebras of subsets of X.

2:2.3$\diamondsuit$ Let $\mathcal{S}$ be any family of subsets of a nonempty set X. The smallest algebra containing $\mathcal{S}$ is called *the algebra generated by* $\mathcal{S}$. Show that this exists. [Hint: This can be described as the intersection of all algebras containing $\mathcal{S}$. Make sure to check that there are such algebras and that the intersection of a collection of algebras is again an algebra.]

2:2.4$\diamondsuit$ Let $\mathcal{S}$ be a family of subsets of a nonempty set X such that (i) $\emptyset$, $X \in \mathcal{S}$ and (ii) if A, $B \in \mathcal{S}$ then both $A \cap B$ and $A \cup B$ are in $\mathcal{S}$. Show that the algebra generated by $\mathcal{S}$ is the family of all sets of the form $\bigcup_{i=1}^{n} A_i \setminus B_i$ for A_i, $B_i \in \mathcal{S}$ with $B_i \subset A_i$.

2:2.5$\diamondsuit$ Let X be an arbitrary nonempty set, and let $\mathcal{A}$ be the family of all subsets $A \subset X$ such that either A or $X \setminus A$ is finite. Show that $\mathcal{A}$ is the algebra generated by the singleton sets $\mathcal{S} = \{\{x\} : x \in X\}$.

2:2.6$\diamondsuit$ Let X be an arbitrary nonempty set, and let $\mathcal{A}$ be the algebra generated by a collection $\mathcal{S}$ of subsets of X. Let A be an arbitrary element of $\mathcal{A}$. Show that there is a finite family $\mathcal{S}_0 \subset \mathcal{S}$ so that A belongs to the algebra generated by $\mathcal{S}_0$. [Hint: Consider the union of all the algebras generated by finite subfamilies of $\mathcal{S}$.]

2:2.7 Show that Example 2.8 provides an algebra of sets.

2:2.8$\diamondsuit$ Show how it follows from Definition 2.9 that an additive set function ν cannot take on both $-\infty$ and ∞ as values. [Hint: If $\nu(A) = -\nu(B) = +\infty$, then find disjoint subsets A', B' with $\nu(A') = +\infty$ and $\nu(B') = -\infty$. Consider what this means for $\nu(A' \cup B')$.]

2:2.9 Suppose that ν is an additive set function on an algebra $\mathcal{A}$. Let E_1 and E_2 be members of $\mathcal{A}$ with $E_1 \subset E_2$ and $\nu(E_2)$ finite. Show that

$$\nu(E_2 \setminus E_1) = \nu(E_2) - \nu(E_1).$$

2:2.10 Let μ be a finitely additive measure and suppose that A, B and C are sets in the domain of μ with $\mu(A)$ finite. Show that

$$|\mu(A \cap B) - \mu(A \cap C)| \leq \mu(B \triangle C)$$

where $B \triangle C = (B \setminus C) \cup (C \setminus B)$ is called the *symmetric difference* of B and C.

2:2.11$\diamondsuit$ Suppose that ν is additive on an algebra $\mathcal{A}$. If $B \subset A$ with A, $B \in \mathcal{A}$ and $\nu(B) = +\infty$, then $\nu(A) = +\infty$.

2:2.12 Use Exercise 2:2.9 to show that the condition $\nu(\emptyset) = 0$ in Definition 2.9 is superfluous unless ν is identically infinite.

2:2.13 Let X be any infinite set, and let $\mathcal{A} = 2^X$. For $A \subset X$, let

$$\nu(A) = \begin{cases} 0, & \text{if } A \text{ is finite;} \\ \infty, & \text{if } A \text{ is infinite.} \end{cases}$$

Show that ν is additive. Let

$$\mathcal{B} = \{A \subset X : A \text{ is finite or } X \setminus A \text{ is finite}\},$$

let $B \in \mathcal{B}$, and let

$$\tau(B) = \begin{cases} 0, & \text{if } B \text{ is finite;} \\ \infty, & \text{if } X \setminus B \text{ is finite.} \end{cases}$$

Show that $\mathcal{B}$ is an algebra and τ is additive.

2:2.14$\diamondsuit$ Show that, in Example 2.10, ν_f is additive on $\mathcal{A}$ and ν_f is nonnegative if and only if f is nondecreasing. [Hint: This involves verifying that, for $A \in \mathcal{A}$, $\nu_f(A)$ does not depend on the choice of intervals whose union is A.]

2:2.15 Complete the proof of Theorem 2.12 by showing that the lower and total variations are additive on $\mathcal{A}$.

2:2.16 Establish the formula

$$V(\nu, E) = \sup \sum_{k=1}^{n} |\nu(A_k)|,$$

where the supremum is taken over all finite collections of pairwise disjoint subsets A_k of E, with each A_k in $\mathcal{A}$.

2:2.17 Suppose that ν is additive on $\mathcal{A}$ and is bounded above. Prove that $\overline{V}(\nu, A)$ is finite for all $A \in \mathcal{A}$. Similarly, if ν is bounded from below, $\underline{V}(\nu, A)$ is finite for all $A \in \mathcal{A}$.

2:2.18 Use Exercise 2:2.17 to obtain the Jordan decomposition for additive set functions that are bounded either above or below.

2:2.19 Show that to every finitely additive set function of finite total variation on the algebra of Example 2.8 corresponds a function f of bounded variation, such that $\nu(\,(a,b]\,) = f(b) - f(a)$ for every $(a,b] \in \mathcal{A}$.

2:2.20 We have already seen that if f is BV on $[0,1]$ then Example 2.10 models a finite mass distribution that may have negative, as well as positive, mass. What happens if f is not of bounded variation? Is there necessarily a decomposition into a difference of nonnegative additive set functions then?

2.3 Measures and Signed Measures

Additive set functions defined on algebras have limitations as models for mass distributions or areas. These limitations are in some way similar to limitations of the Riemann integral. The Riemann integral fails to integrate enough functions. Similarly, an algebra of sets may not include all the sets that one expects to be able to handle. In Example 2.10, for example, one can discuss the mass of an interval or a finite union of intervals, but one cannot define mass for more general sets.

We have mentioned several times that to obtain a coherent theory of measure the class of measurable sets should be "large." What do we mean by that statement? Roughly, we should require that the class of sets to be considered measurable encompass all the sets that one reasonably expects to encounter while applying the normal operations of analysis. The situation on the real line with Lebesgue measure will illustrate.

In a study of a continuous function $f : \mathbb{R} \to \mathbb{R}$ we could expect to investigate sets of the form $\{x : f(x) \geq c\}$ or $\{x : f(x) > c\}$. The first of these is closed and the second open if f is continuous. We would hope that these sets are measurable, as indeed they are for Lebesgue measure. In Chapter 3 we shall make the measurability of closed and open sets a key requirement in our study of general measures on metric spaces.

Again, if f is the limit of a convergent sequence of continuous functions (a common enough operation in analysis), what can we expect for the set $\{x : f(x) > c\}$? We can rewrite this as

$$\{x : f(x) > c\} = \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} \{x : f_n(x) \geq c + 1/m\}$$

(using Exercise 1:1.24). It follows that the set that we are interested in is measurable provided that the class of measurable sets is closed under the operations of taking countable unions and countable intersections. An algebra of sets need only be closed under the operations of taking finite unions and finite intersections. This, and other considerations, leads us to Definition 2.14. We shall see that with this definition we can develop a coherent theory of measure and integration.

Definition 2.14 Let X be a set, and let $\mathcal{M}$ be a family of subsets of X. We say that $\mathcal{M}$ is a *σ–algebra of sets* if $\mathcal{M}$ is an algebra of sets and $\mathcal{M}$ is closed under countable unions; that is, if $\{A_k\} \subset \mathcal{M}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$.

It is now natural to replace the notion of additive set function with *countably* additive set function or *signed measure.*

Definition 2.15 Let $\mathcal{M}$ be a σ-algebra of subsets of a set X, and let μ be an extended real-valued function on $\mathcal{M}$. We say that μ is a *signed measure* if $\mu(\emptyset) = 0$, and whenever $\{A_k\}$ is a sequence of pairwise disjoint elements of $\mathcal{M}$, then $\sum_{n=1}^{\infty} \mu(A_n)$ is defined as an extended real number with

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n). \tag{8}$$

If $\mu(A) \geq 0$ for all $A \in \mathcal{M}$, we say that μ is a *measure*. In this case we call the triple $(X, \mathcal{M}, \mu)$ a *measure space*. The members of $\mathcal{M}$ are called *measurable sets*.

We mention that the term *countably additive set function* μ indicates that μ satisfies (8). We shall also use the term *σ-additive set function*.

Example 2.16 Let $X = \mathbb{N}$ (the set of natural numbers) and $\mathcal{M} = 2^{\mathbb{N}}$, the family of all subsets of $\mathbb{N}$. It is clear that $\mathcal{M}$ is a σ-algebra of sets. For $A \in \mathcal{M}$, let

$\mu_1(A) = \sum_{n \in A} 1/2^n$, $\qquad \mu_2(A) = \sum_{n \in A} 1/n$,

$\mu_3(A) = \sum_{n \in A} (-1)^n/2^n$, $\qquad \mu_4(A) = \sum_{n \in A} (-1)^n/n$.

One verifies easily that μ_1 and μ_2 are measures, with $\mu_1(X) = 1$ and $\mu_2(X) = \infty$. The set function μ_3 is a signed measure. Since the series $\sum_{n=1}^{\infty} (-1)^n/n$ is conditionally convergent, $\mu_4(A)$ is not defined for all subsets of X, and μ_4 is not a signed measure.

An inspection of the example μ_3 reveals that it is the difference of two measures,

$$\mu_3(A) = \sum_{n \,\in\, A,\; n \text{ even}} 1/2^n - \sum_{n \,\in\, A,\; n \text{ odd}} 1/2^n,$$

just as we have seen that every additive set function is the difference of two nonnegative additive set functions. In Section 2.5 we will show that this is always the case for signed measures; thus we will be able to reduce the study of signed measures to the study of measures. Signed measures will again return to a position of importance in Chapter 5. At the moment, our focus will be on measures.

We shall require immediately some skill in handling measures. Often we are faced with a set expressed as a countable union of measurable sets. If the sets are disjoint, then the measure of the union can be obtained as a sum. What do we do if the sets are not pairwise disjoint? Our first theorem shows how to unscramble these sets in a useful way. (We leave the straightforward proof of Theorem 2.17 as Exercise 2:3.11. Recall that we use $\mathbb{N}$ to denote the set of natural numbers.)

Theorem 2.17 *Let $\{A_n\}$ be a sequence of subsets of a set X, and let $A = \bigcup_{n=1}^{\infty} A_n$. Let $B_1 = A_1$ and, for all $n \in \mathbb{N}$, $n \geq 2$, let*

$$B_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1}).$$

Then $A = \bigcup_{n=1}^{\infty} B_n$, the sets B_n are pairwise disjoint and $B_n \subset A_n$ for all $n \in \mathbb{N}$. If the sets A_n are members of an algebra $\mathcal{M}$, then $B_n \in \mathcal{M}$ for all $n \in \mathbb{N}$.

We next show that measures are *monotonic* and *countably subadditive.*

Theorem 2.18 *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

1. *If $A, B \in \mathcal{M}$ with $B \subset A$, then $\mu(B) \leq \mu(A)$. If, in addition, $\mu(B) < \infty$, then $\mu(A \setminus B) = \mu(A) - \mu(B)$.*
2. *If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{M}$, then $\mu(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$.*

Proof. Part (1) follows from the representation

$$A = B \cup (A \setminus B).$$

To verify part (2), let $\{A_k\} \in \mathcal{M}$, and let $A = \bigcup_{k=1}^{\infty} A_k$. Let $\{B_k\}$ be the sequence of sets appearing in Theorem 2.17. Since $\mathcal{M}$ is an algebra of sets, $B_k \in \mathcal{M}$ for all $k \in \mathbb{N}$. It follows that $A = \bigcup_{k=1}^{\infty} B_k$ and that the sets B_k are pairwise disjoint. Since μ is a measure, $\mu(A) = \sum_{k=1}^{\infty} \mu(B_k)$. But for each $k \in \mathbb{N}$, $\mu(B_k) \leq \mu(A_k)$, by part (1). Thus $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$. ■

We end this section with the observation that any family $\mathcal{S}$ of subsets of a nonempty set X is contained in the σ-algebra 2^X of all subsets of X. The smallest σ-algebra containing $\mathcal{S}$ is called *the σ-algebra generated by* $\mathcal{S}$. It can be described as the intersection of all σ-algebras containing $\mathcal{S}$. The σ-algebra generated by the open (or closed) subsets of $\mathbb{R}$ is called the class of *Borel sets*. It contains all sets of type $\mathcal{F}_\sigma$ or of type $\mathcal{G}_\delta$, but it also contains many other sets. The σ-algebra generated by the algebra $\mathcal{A}$ of Example 2.10 also consists of the Borel sets.

Exercises

2:3.1 Let X be a nonempty set. Show that 2^X (the family of all subsets of X) and $\{\emptyset, X\}$ are both σ–algebras of sets, in fact the largest and the smallest of the σ–algebras of subsets of X.

2:3.2 Let $\mathcal{S}$ be any family of subsets of a nonempty set X. The smallest σ–algebra containing $\mathcal{S}$ is called *the σ–algebra generated by* $\mathcal{S}$. Show that this exists. [Hint: This is described in the last paragraph of this section. Compare with Exercise 2:2.3.]

2:3.3 Let $\mathcal{S}$ be a family of subsets of a nonempty set X such that (i) $\emptyset$, $X \in \mathcal{S}$ and (ii) if A, $B \in \mathcal{S}$, then both $A \cap B$ and $A \cup B$ are in $\mathcal{S}$. Show that the σ–algebra generated by $\mathcal{S}$ is, in general, *not* the family of all sets of the form $\bigcup_{i=1}^{\infty} A_i \setminus B_i$ for A_i, $B_i \in \mathcal{S}$ with $B_i \subset A_i$. This contrasts with what one might have expected in view of Exercise 2:2.4. [Hint: Take $\mathcal{S}$ as the collection of intervals $[0, n^{-1}]$ along with $\emptyset$.]

2:3.4 Let X be an arbitrary nonempty set, and let $\mathcal{A}$ be the family of all subsets $A \subset X$ such that either A or $X \setminus A$ is countable. Show that $\mathcal{A}$ is the σ–algebra generated by the singleton sets $\mathcal{S} = \{\{x\} : x \in X\}$.

2:3.5 Let X be an arbitrary nonempty set, and let $\mathcal{A}$ be the σ–algebra generated by a collection $\mathcal{S}$ of subsets of X. Let A be an arbitrary element of $\mathcal{A}$. Show that there is a countable family $\mathcal{S}_0 \subset \mathcal{S}$ so that A belongs to the σ–algebra generated by $\mathcal{S}_0$. [Hint: Compare with Exercise 2:2.6.]

2:3.6 Let $\mathcal{A}$ be an algebra of subsets of a set X. If $\mathcal{A}$ is finite, prove that $\mathcal{A}$ is in fact a σ–algebra. How many elements can $\mathcal{A}$ have?

2:3.7 Describe the domain of the set function μ_4 defined in Example 2.16.

2:3.8 Show that a σ-algebra of sets is closed under countable intersections.

2:3.9$\diamondsuit$ Let X be any set, and let $\mu(A)$ be the number of elements in A if A is finite and ∞ if A is infinite. Show that μ is a measure. (Commonly, μ is called the *counting measure* on X.)

2:3.10$\diamondsuit$ Let μ be a signed measure on a σ-algebra. Show that the associated variations are countably additive. Thus, by Theorem 2.13, each signed measure of finite total variation is a difference of two measures. (See Theorem 2.22 for an improvement of this statement.)

2:3.11 Prove Theorem 2.17.

2:3.12 Let ν be a signed measure on a σ-algebra. If $E_0 \subset E_1 \subset E_2 \dots$ are members of the σ–algebra, then the limit $\lim_{n\to\infty} E_n$ of the sequence is defined to be $\bigcup_{n=0}^{\infty} E_n$. Prove that

$$\nu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \nu(E_n).$$

[The method of Theorem 2.20 can be used, but try to prove without looking ahead. The same remark applies to the next exercise.]

2:3.13$\diamondsuit$ Let ν be a signed measure on a σ-algebra. If $E_0 \supset E_1 \supset E_2 \dots$ are members of the σ–algebra, then the limit $\lim_{n\to\infty} E_n$ of the sequence is defined to be $\bigcap_{n=0}^{\infty} E_n$. Prove that if $\nu(E_0)$ is finite then

$$\nu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \nu(E_n).$$

2.4 Limit Theorems

The countable additivity of a signed measure allows a number of limit theorems not possible for the general additive set function. To formulate some of these theorems, we need a bit of set-theoretic terminology. First, recall that if A is a subset of a set X then the *characteristic function* of A is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \in X \setminus A. \end{cases}$$

Suppose, now, that we are given a sequence $\{A_n\}$ of subsets of X. Then there exist sets B_1 and B_2 with

$$\chi_{B_1} = \limsup \chi_{A_n}$$

and

$$\chi_{B_2} = \liminf \chi_{A_n}.$$

The set B_1 consists of those $x \in X$ that belong to infinitely many of the sets A_n, while the set B_2 consists of those $x \in X$ that belong to all but a finite number of the sets A_n. We call these sets the $\limsup A_n$ and $\liminf A_n$, respectively. Our formal definition has the advantage of involving only set-theoretic notions.

Definition 2.19 Let $\{A_n\}$ be a sequence of subsets of a set X. We define

$$\limsup A_n = \bigcap_{m=1}^{\infty}\left(\bigcup_{n=m}^{\infty} A_n\right)$$

and

$$\liminf A_n = \bigcup_{m=1}^{\infty}\left(\bigcap_{n=m}^{\infty} A_n\right).$$

If

$$\limsup A_n = \liminf A_n = A,$$

we say that *the sequence* $\{A_n\}$ *converges to* A and we write

$$A = \lim A_n.$$

Observe that monotone sequences, either expanding or contracting, converge to their union and intersection, respectively. Furthermore, if all the sets A_n belong to a σ-algebra $\mathcal{M}$, then $\limsup A_n \in \mathcal{M}$ and $\liminf A_n \in \mathcal{M}$.

For monotone sequences of measurable sets, limit theorems are intuitively clear.

Theorem 2.20 *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{A_n\}$ be a sequence of measurable sets.*

1. *If $A_1 \subset A_2 \subset \ldots$, then $\lim \mu(A_n) = \mu(\lim A_n)$.*
2. *If $A_1 \supset A_2 \supset \ldots$ and $\mu(A_m) < \infty$ for some $m \in \mathbb{N}$, then $\lim \mu(A_n) = \mu(\lim A_n)$.*

Proof. Let $A_0 = \emptyset$. Then

$$\lim_n A_n = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A_n \setminus A_{n-1}).$$

Since the last union is a disjoint union, we can infer that

$$\begin{aligned}\mu(\lim_n A_n) &= \sum_{n=1}^{\infty} \mu(A_n \setminus A_{n-1}) = \lim_k \sum_{n=1}^{k} \mu(A_n \setminus A_{n-1}) \\ &= \lim_k \mu\left(\bigcup_{n=1}^{k} (A_n \setminus A_{n-1})\right) = \lim_k \mu(A_k).\end{aligned}$$

This proves part (1). For part (2), choose m so that $\mu(A_m) < \infty$. A similar argument shows that

$$\mu(A_m \setminus \lim_n A_n) = \lim_n (\mu(A_m) - \mu(A_n)).$$

Because these are finite, assertion (2) follows. ■

Theorem 2.21 *Let μ be a measure on $\mathcal{M}$, and let $\{A_n\}$ be a sequence of sets from $\mathcal{M}$. Then*

1. $\mu(\liminf A_n) \leq \liminf \mu(A_n)$*;*
2. *if* $\mu(\bigcup_{n=1}^{\infty} A_n) < \infty$*, then* $\mu(\limsup A_n) \geq \limsup \mu(A_n)$*;*
3. *if* $\{A_n\}$ *converges and* $\mu(\bigcup_{n=1}^{\infty} A_n) < \infty$*, then*

$$\mu(\lim A_n) = \lim \mu(A_n).$$

Proof. We prove (1), the remaining parts following readily. For $m \in \mathbb{N}$, let $B_m = \bigcap_{n=m}^{\infty} A_n$. Since $B_m \subset A_m$, $\mu(B_m) \leq \mu(A_m)$. It follows that

$$\liminf \mu(B_m) \leq \liminf \mu(A_m). \tag{9}$$

The sequence$\{B_m\}$ is expanding, so $\lim_m B_m = \bigcup_{m=1}^{\infty} B_m$. Using Theorem 2.20, we then obtain

$$\mu(\lim_m B_m) = \lim_m \mu(B_m).$$

Thus

$$\begin{aligned}\mu(\liminf A_n) &= \mu\left(\bigcup_{m=1}^{\infty} B_m\right) = \mu(\lim_m B_m) = \lim_m \mu(B_m) \\ &= \liminf \mu(B_m) \leq \liminf \mu(A_m),\end{aligned}$$

the last inequality being (9). ■

Exercises

2:4.1 Verify that in Definition 2.19
$$\limsup_{n\to\infty} A_n = \{x : x \in A_n \text{ for infinitely many } n\}$$
and
$$\liminf_{n\to\infty} A_n = \{x : x \in A_n \text{ for all but finitely many } n\}\,.$$

2:4.2 Supply all the details needed to prove part (2) of Theorem 2.20.

2:4.3 For any $A \subset \mathbb{N}$, let
$$\nu(A) = \begin{cases} \sum_{n\in A} 2^{-n}, & \text{if } A \text{ is finite;} \\ \infty, & \text{if } A \text{ is infinite.} \end{cases}$$

(a) Show that ν is an additive set function, but not a measure on $2^{\mathbb{N}}$.

(b) Show that ν does not have the limit property expressed in part (1) of Theorem 2.20 for measures.

2:4.4 Verify parts (2) and (3) of Theorem 2.21.

2:4.5 Show that the finiteness assumptions in parts 2 and 3 of Theorem 2.21 cannot be dropped.

2:4.6 State and prove an analog for Theorem 2.20 for signed measures.

2:4.7◇ Verify the following criterion for an additive set function to be a signed measure: If ν is additive on a σ-algebra $\mathcal{M}$, and $\lim_n \nu(A_n) = \nu(\lim_n A_n)$ for every expanding sequence $\{A_n\}$ of sets from $\mathcal{M}$, then ν is a signed measure on $\mathcal{M}$.

2:4.8◇ (Borel–Cantelli lemma) Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{A_n\}$ be a sequence of sets with $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Then
$$\mu(\limsup A_n) = 0.$$

2:4.9 Let C be a Cantor set in $[0,1]$ of measure α $(0 \le \alpha < 1)$ (see Example 2.1). Does there exist a sequence $\{J_k\}$ of intervals with $\sum_{k=1}^{\infty} \lambda(J_k) < \infty$ such that every point of the set C lies in infinitely many of the intervals J_k?

2:4.10◇ Let $\mathcal{A}$ be the algebra of Example 2.10, let
$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < x_0 < 1; \\ 1, & \text{if } x_0 \le x \le 1. \end{cases}\,,$$
and let ν_f be as in that example. We shall see later that ν_f can be extended to a measure μ_f defined on the σ-algebra $\mathcal{B}$ of Borel sets in $(0,1]$. Assume this, for the moment. Show that $\mu_f(\{x_0\}) = 1$; thus $\{x_0\}$ represents a point mass.

2.5 Jordan and Hahn Decomposition

Let us return to the Jordan decomposition theorem, but applied now to signed measures. Certainly, since a signed measure is also an additive set function, we see that any signed measure with finite variation can be expressed as the difference of two nonnegative additive set functions. We expect the latter to be measures, but this does not yet follow. In the setting of signed measures there is also a technical simplification that comes about. An additive set function may be itself finite and yet have both of its variations infinite. For this reason, in the proof of Theorem 2.13, we needed to assume that both variations were finite; otherwise, the proof collapsed. For signed measures this does not occur. Thus we have the correct version of the decomposition for signed measures, with better hypotheses and a stronger conclusion.

Theorem 2.22 (Jordan decomposition) *Let ν be a signed measure on a σ–algebra $\mathcal{A}$ of subsets of X. Then, for all $A \in \mathcal{A}$,*

$$\nu(A) = \overline{V}\,(\nu, A) + \underline{V}\,(\nu, A)$$

and the set functions $\overline{V}\,(\nu, \cdot)$ and $-\underline{V}\,(\nu, \cdot)$ are measures, at least one of which must be finite.

Proof. This follows by the same methods used in the proof of Theorem 2.13, provided we establish some facts. We can prove (see Exercise 2:3.10) that if ν is σ–additive on $\mathcal{A}$ then so too are both variations. We prove also that if ν is finite then both variations are finite. Thus, with these two facts, the theorem (for finite-valued signed measures) follows directly from Theorem 2.13.

If ν is not finite, then we shall show that precisely one of the two variations is infinite. In fact, if $\nu(E)$ takes the value $+\infty$, then $\overline{V}\,(\nu, E) = +\infty$ and $-\underline{V}\,(\nu, \cdot)$ is everywhere finite. With this information the proof of Theorem 2.13 can be repeated to obtain the decomposition.

Evidently then, the theorem can be obtained from the following assertion which we will now prove.

2.23 *Let ν be a signed measure on a σ–algebra $\mathcal{A}$ of subsets of X. If $E \in \mathcal{A}$ and $\overline{V}\,(\nu, E) = +\infty$, then $\nu(E) = +\infty$. If $E \in \mathcal{A}$ and $\underline{V}\,(\nu, E) = -\infty$, then $\nu(E) = -\infty$.*

It is sufficient to prove the first statement. Suppose that $\overline{V}\,(\nu, E) = +\infty$. Because of Exercise 2:2.11, we may obtain that $\nu(E) = +\infty$ by finding a subset $A \subset E$ with $\nu(A) = +\infty$. There must exist a set $E_1 \subset E$ such that

$$\nu(E_1) > 1.$$

As $\overline{V}\,(\nu, \cdot)$ is additive and $\overline{V}\,(\nu, E) = +\infty$, it follows that either $\overline{V}\,(\nu, E_1) = \infty$ or else $\overline{V}\,(\nu, E \setminus E_1) = \infty$. Choose A_1 to be either E_1 or $E \setminus E_1$, according to which of these two is infinite, so that $\overline{V}\,(\nu, A_1) = +\infty$.

Inductively choose $E_n \subset A_{n-1}$ so that

$$\nu(E_n) > n$$

and choose A_n to be either E_n or $A_{n-1} \setminus E_n$ according to which of these two is infinite so that $\overline{V}(\nu, A_n) = +\infty$.

There are two case to consider:

1. For an infinite number of n, $A_n = A_{n-1} \setminus E_n$.
2. For all sufficiently large n (say for $n \geq n_0$), $A_n = E_n$.

In the first of these cases we obtain a sequence of disjoint sets $\{E_{n_k}\}$ so that we can use the σ–additivity of ν to obtain

$$\nu\left(\bigcup_{k=1}^{\infty} E_{n_k}\right) = \sum_{k=1}^{\infty} \nu\left(E_{n_k}\right) \geq \sum_{k=1}^{\infty} n_k = +\infty.$$

This would give us a subset of E with infinite ν measure so that $\nu(E) = +\infty$ as required.

In the second case we have obtained a sequence

$$E \supset E_{n_0} \supset E_{n_0+1} \supset E_{n_0+2} \ldots.$$

If $\nu(E_{n_0}) = +\infty$, we once again have a subset of E with infinite ν measure so that $\nu(E) = +\infty$ as required. If $\nu(E_{n_0}) < +\infty$, then we can use Exercise 2:3.13 to obtain

$$\nu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \nu(E_n) \geq \lim_{n\to\infty} n = +\infty$$

and yet again have a subset of E with infinite ν measure so that $\nu(E) = +\infty$. This exhausts all possibilities and so the proof of assertion 2.23 is complete. The main theorem now follows. ■

The Jordan decomposition theorem is one of the primary tools of general measure theory. It can be clarified considerably by a further analysis due originally to H. Hahn (1879–1934). In fact, our proof invokes the Jordan decomposition, but Hahn's theorem could be proved first and then one can derive the Jordan decomposition from it. This decomposition is, again, one of the main tools of general measure theory; we shall have occasion to use it later in our discussion of the Radon–Nikodym theorem in Section 5.8.

Theorem 2.24 (Hahn decomposition) *Let ν be a signed measure on a σ-algebra $\mathcal{M}$. Then there exists a set $P \in \mathcal{M}$ such that $\nu(A) \geq 0$ whenever $A \subset P$, $A \in \mathcal{M}$, and $\nu(A) \leq 0$ whenever $A \subset X \setminus P$, $A \in \mathcal{M}$.*

We call the set P a *positive* set for ν, the set $N = X \setminus P$ a *negative* set for ν, and the pair (P, N) a *Hahn decomposition* for ν.

Proof. Using Exercise 2:2.8, we see that ν cannot take both the values $+\infty$ and $-\infty$. Assume for definiteness that $\nu(E) < \infty$ for all $E \in \mathcal{M}$. It follows that $\overline{V}(\nu, X)$ is finite. We construct a set P for which

$$\overline{V}\left(\nu, \widetilde{P}\right) = \underline{V}(\nu, P) = 0,$$

where $\overline{V}$ and $\underline{V}$ denote the upper and lower variations of ν as defined in Section 2.2. We know that $\overline{V}$ and $-\underline{V}$ are measures. (Recall the notation $\widetilde{P}$ for the complement of a set.)

For each $n \in \mathbb{N}$, there exists $P_n \in \mathcal{M}$ such that

$$\nu(P_n) > \overline{V}(\nu, X) - \frac{1}{2^n}. \tag{10}$$

Define $P = \limsup_{n\to\infty} P_n$, so that $\widetilde{P} = \liminf_{n\to\infty} \widetilde{P_n}$. Then, from the inequality (10), we have

$$\overline{V}\left(\nu, \widetilde{P_n}\right) = \overline{V}(\nu, X) - \overline{V}(\nu, P_n) \le \overline{V}(\nu, X) - \nu(P_n) \le \frac{1}{2^n}.$$

Using Theorem 2.21 (1), we infer that

$$0 \le \overline{V}\left(\nu, \widetilde{P}\right) \le \liminf_{n\to\infty} \overline{V}\left(\nu, \widetilde{P_n}\right) \le \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

Thus $\overline{V}\left(\nu, \widetilde{P}\right) = 0$.

It remains to show that $\underline{V}(\nu, P) = 0$. First, note that

$$-\underline{V}(\nu, P_n) = \overline{V}(\nu, P_n) - \nu(P_n) \le \overline{V}(\nu, X) - \nu(P_n) \le \frac{1}{2^n}.$$

Hence, for every $k \in \mathbb{N}$,

$$\begin{aligned} 0 &\le -\underline{V}(\nu, P) \le -\underline{V}\left(\nu, \bigcup_{n=k}^{\infty} P_n\right) \\ &\le -\sum_{n=k}^{\infty} \underline{V}(\nu, P_n) \le \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}}. \end{aligned}$$

It follows that $\underline{V}(\nu, P) = 0$ as required. ■

Note the connection with variation both in the proof of this theorem and in the decomposition itself. For any signed measure ν we shall use its Hahn decomposition (P, N) to define three further measures ν^+, ν^- and $|\nu|$ by writing for each $E \in \mathcal{M}$,

$$\nu^+(E) = \nu(E \cap P) = \overline{V}(\nu, E) \quad \text{[positive variation]}$$

$$\nu^-(E) = -\nu(E \cap N) = -\underline{V}(\nu, E) \quad \text{[negative variation]}$$

and

$$|\nu|(E) = \nu^+(E) + \nu^-(E) \qquad \text{[total variation]}.$$

Observe that the positive, negative, and total variations of ν are measures (not merely signed measures) and that the following obvious relations hold among them:

$$\nu = \nu^+ - \nu^-$$

$$|\nu| = \nu^+ + \nu^-.$$

Two measures α and β on $\mathcal{M}$ are called *mutually singular*, written as $\alpha \perp \beta$, if there are disjoint measurable sets A and B such that $X = A \cup B$ and $\alpha(B) = \beta(A) = 0$; that is, the measures are concentrated on two different disjoint sets. The measures here ν^+ and ν^- are mutually singular, since $\nu^+(N) = \nu^-(P) = 0$.

Exercises

2:5.1 A set E is a *null set* for a signed measure ν if $|\nu|(E) = 0$. Show that if (P, N) and (P_1, N_1) are Hahn decompositions for ν then P and P_1 (and similarly N and N_1) differ by a null set [i.e., $|\nu|(P \setminus P_1) = |\nu|(P_1 \setminus P) = 0$].

2:5.2 Exhibit a Hahn decomposition for each of the signed measures μ_3 and $3\mu_1 - \mu_2$, where μ_1, μ_2, and μ_3 have been given in Example 2.16.

2:5.3 Let F be the Cantor function on [0,1] (defined in Exercise 1:22.13). Suppose that μ_F is a measure on the Borel subsets of $(0, 1]$ for which $\mu_F((a, b]) = F(b) - F(a)$ for any $(a, b] \subset (0, 1]$. Let λ be Lebesgue measure restricted to the Borel sets.

(a) Show that $\mu_F \perp \lambda$.

(b) Exhibit a Hahn decomposition for $\lambda - \mu_F$.

2.6 Complete Measures

Consider for a moment Lebesgue measure λ on $[0, 1]$. Since λ is the restriction of λ^* to the family $\mathcal{L}$ of Lebesgue measurable sets, every subset of a zero measure set has measure zero. But, for a general measure space $(X, \mathcal{M}, \mu)$, it need not be the case that subsets of zero measure sets are necessarily measurable.

This is illustrated by the space $(X, \mathcal{B}, \lambda)$, where X is $[0, 1]$ and $\mathcal{B}$ is the class of Borel sets in $[0, 1]$: that is, $\mathcal{B}$ is the σ-algebra generated by the open sets. A cardinality argument (Exercise 2:6.1) shows that, while the Cantor ternary set K has 2^c subsets, only c of them are Borel sets, yet $\lambda(K) = 0$. It follows that there are Lebesgue measurable sets of measure zero that are not Borel sets. Thus $(X, \mathcal{B}, \lambda)$ is not *complete* according to the following definition.

Definition 2.25 Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is called *complete* if the conditions $Z \subset A$ and $\mu(A) = 0$ imply that $Z \in \mathcal{M}$. In that case, $(X, \mathcal{M}, \mu)$ is called a *complete measure space*.

Completeness of a measure refers to the domain $\mathcal{M}$ and so, properly speaking, it is $\mathcal{M}$ that might be called complete; but it is common usage to refer directly to a complete measure.

It is clear from the monotonicity of μ that, when a subset of a zero measurable set is measurable, its measure must be zero. When a measure space is not complete, it possesses subsets E that intuition demands be small, but that do not happen to be in the domain of the measure μ. It may seem that such sets should have measure zero, but the measure is not defined for such sets. It would be convenient if one could always deal with a complete space. Instead of saying that a property is valid except on a "subset of a set of measure zero," we could correctly say "except on a set of measure zero." Fortunately, every measure space can be *completed* naturally by extending μ to a measure $\overline{\mu}$ defined on the σ-algebra generated by $\mathcal{M}$ and the family of subsets of sets of measure zero.

Theorem 2.26 *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let*

$$\mathcal{Z} = \{Z : \exists N \in \mathcal{M} \text{ for which } Z \subset N \text{ and } \mu(N) = 0\}.$$

Let $\overline{\mathcal{M}} = \{M \cup Z : M \in \mathcal{M}, Z \in \mathcal{Z}\}$. Define $\overline{\mu}$ on $\overline{\mathcal{M}}$ by

$$\overline{\mu}(M \cup Z) = \mu(M).$$

Then

1. *$\overline{\mathcal{M}}$ is a σ-algebra containing $\mathcal{M}$ and $\mathcal{Z}$.*
2. *$\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$ and agrees with μ on $\mathcal{M}$.*
3. *$\overline{\mu}$ is complete.*

Proof. (1) It is clear that $\overline{\mathcal{M}}$ contains $\mathcal{M}$ and $\mathcal{Z}$. To show that $\overline{\mathcal{M}}$ is closed under complementation, let $A = M \cup Z$ with $M \in \mathcal{M}$, $Z \subset N$ and $\mu(N) = 0$. Now

$$\widetilde{A} = \widetilde{M} \cap \widetilde{Z} = (\widetilde{M} \cap \widetilde{N}) \cup (N \cap \widetilde{M} \cap \widetilde{Z}).$$

Since $\widetilde{M} \cap \widetilde{N} \in \mathcal{M}$ and $N \cap \widetilde{M} \cap \widetilde{Z} \subset N \in \mathcal{Z}$, we see from the definition of $\overline{\mathcal{M}}$ that $\widetilde{A} \in \overline{\mathcal{M}}$.

Finally, we show that $\overline{\mathcal{M}}$ is closed under countable unions. Let $\{A_n\}$ be a sequence of sets in $\overline{\mathcal{M}}$. For each $n \in \mathbb{N}$, write

$$A_n = M_n \cup Z_n$$

with $M_n \in \mathcal{M}$, $Z_n \in \mathcal{Z}$. Then

$$\bigcup A_n = \bigcup (M_n \cup Z_n) = \left(\bigcup M_n\right) \cup \left(\bigcup Z_n\right).$$

We have $M_n \in \mathcal{M}$ and $Z_n \subset N_n \in \mathcal{M} \cap \mathcal{Z}$, so $\bigcup M_n \in \mathcal{M}$ and

$$\bigcup Z_n \subset \bigcup N_n \in \mathcal{M} \cap \mathcal{Z}.$$

Thus $\bigcup A_n$ has the required representation. This completes the verification of (1).

(2) We first check that $\overline{\mu}$ is well defined. Suppose that A has two different representations:

$$A = M_1 \cup Z_1 = M_2 \cup Z_2$$

for M_1, $M_2 \in \mathcal{M}$, Z_1, $Z_2 \in \mathcal{Z}$. We show $\mu(M_1) = \mu(M_2)$. Now

$$M_1 \subset A = M_2 \cup Z_2 \subset M_2 \cup N_2 \text{ with } \mu(N_2) = 0.$$

Thus

$$\mu(M_1) \leq \mu(M_2) + \mu(N_2) = \mu(M_2).$$

Similarly, $\mu(M_2) \leq \mu(M_1)$, so $\overline{\mu}$ is well defined.

To show that $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$, we verify countable additivity, the remaining requirements being trivial to verify. Let $\{A_n\}$ be a sequence of pairwise disjoint sets in $\overline{\mathcal{M}}$. For every $n \in \mathbb{N}$, we can write $A_n = M_n \cup Z_n$ for sets $M_n \in \mathcal{M}$, $Z_n \in \mathcal{Z}$. Note that the union $\bigcup_{n=1}^{\infty} M_n$ belongs to $\mathcal{M}$ and that $\bigcup_{n=1}^{\infty} Z_n$ belongs to $\mathcal{Z}$. Then

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty} (M_n \cup Z_n)\right) = \overline{\mu}\left(\left(\bigcup_{n=1}^{\infty} M_n\right) \cup \left(\bigcup_{n=1}^{\infty} Z_n\right)\right)$$

$$= \mu\left(\bigcup_{n=1}^{\infty} M_n\right) = \sum_{n=1}^{\infty} \mu\left(M_n\right) = \sum_{n=1}^{\infty} \overline{\mu}\left(A_n\right).$$

Thus $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$. It is clear from the representation $A = M \cup Z$ and the definition of $\overline{\mu}$ that $\overline{\mu} = \mu$ on $\mathcal{M}$.

(3) Let $\overline{\mu}(A) = 0$ and let $B \subset A$. We show that $\overline{\mu}(B) = 0$. Write $A = M \cup Z$, $M \in \mathcal{M}$, $Z \in \mathcal{Z}$. Since $\overline{\mu}(A) = 0$, $\mu(M) = 0$, so $A = M \cup Z \in \mathcal{Z}$. It follows that $B \in \mathcal{Z} \subset \overline{\mathcal{M}}$, and so $\overline{\mu}$ is complete as required. ■

Exercises

2:6.1 Prove each of the following assertions:

(a) The cardinality of the class $\mathcal{G}$ of open subsets of $[0, 1]$ is c.
(b) The cardinality of the class $\mathcal{B}$ of Borel sets in $[0, 1]$, is also c.
(c) The zero measure Cantor set has subsets that are not Borel sets.
(d) The measure space $(X, \mathcal{B}, \lambda)$ is not complete.

2:6.2 Let $\mathcal{B}$ denote the Borel sets in $[0, 1]$, and let λ be Lebesgue measure on $\mathcal{B}$. Prove that

$$([0, 1], \overline{\mathcal{B}}, \overline{\lambda}) = ([0, 1], \mathcal{L}, \lambda).$$

2.7 Outer Measures

We turn now to the following general problem. Suppose that we have a primitive notion for some phenomenon that we wish to model in the setting of a suitable measure space. How can we construct such a space? We can abstract some ideas from Lebesgue's approach (given in Section 2.1). That procedure involved three steps. The primitive notion of the length of an open interval was the starting point. This was used to provide an outer measure defined on all subsets of $\mathbb{R}$. That, in turn, led to an inner measure and then, finally, the class of measurable sets was defined as the collection of sets on which the inner and outer measures agreed. In this section and the next we shall see that this same procedure can be used quite generally. Only one important variant is necessary—we must circumvent the use of inner measure. The reason for this will become apparent.

We begin by abstracting the essential properties of the Lebesgue outer measure. A method for constructing outer measures similar to that used to construct the Lebesgue outer measure will be developed in the next section.

Definition 2.27 Let X be a set, and let μ^* be an extended real-valued function defined on 2^X such that

1. $\mu^*(\emptyset) = 0$.
2. If $A \subset B \subset X$, then $\mu^*(A) \le \mu^*(B)$.
3. If $\{A_n\}$ is a sequence of subsets of X, then

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n).$$

Then μ^* is called an *outer measure* on X.

It follows from the first two conditions that an outer measure is nonnegative. Condition 3 is called *countable subadditivity*.

Let us first address the question of how we obtain a measure from an outer measure. The simple example that follows may be instructive.

Example 2.28 Let $X = \{1, 2, 3\}$. Let $\mu^*(\emptyset) = 0$, $\mu^*(X) = 2$, and $\mu^*(A) = 1$ for every other set $A \subset X$. It is a routine matter to verify that μ^* is an outer measure. Suppose now that we wish to mimic the procedure that worked so well for the Peano–Jordan content and the Lebesgue measure. We could take our cue from the formula in assertion 2.4 and define a version of the inner measure for this example as

$$\mu_*(A) = \mu^*(X) - \mu^*(X \setminus A) = 2 - \mu^*(X \setminus A).$$

If we then call A measurable provided that $\mu_*(A) = \mu^*(A)$, and let

$$\mu(A) = \mu^*(A)$$

for such sets, our process is complete. We find that *all* eight subsets of X are measurable by this definition, but μ is clearly not additive on 2^X. The classical inner–outer measure procedure completely fails to work in this simple example!

A bit of reflection pinpoints the problem. The inner–outer measure approach puts a set A to the following test stated solely in terms of μ^*: is it true that

$$\mu^*(A) + \mu^*(X \setminus A) = \mu^*(X)?$$

In Example 2.28, every $A \subset X$ passed this test. But, for $A = \{1\}$ and $E = \{1, 2\}$, we see that

$$\mu^*(A) + \mu^*(E \setminus A) = 2 > 1 = \mu^*(E).$$

Thus, while μ^* is additive with respect to A and its complement in X, it is not with respect to A and its complement in E.

These considerations lead naturally to the following criterion of measurability. It is due to Constantin Carathéodory (1873–1950).

Definition 2.29 Let μ^* be an outer measure on X. A set $A \subset X$ is μ^**-measurable* if, for all sets $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A). \tag{11}$$

This definition of the measurability of a set A requires testing the set A against every subset E of the space. In contrast the inner–outer measure approach requires only that equation (11) of Definition 2.29 be valid for the single "test set" $E = X$.

Example 2.30 Let X and μ^* be as in Example 2.28. Consider $a, b \in X$, with $a \neq b$. If $E = \{a, b\}$ is examined as the test set in (11) of Definition 2.29, we see that $\{a\}$ is not μ^*-measurable. Similarly, we find that no two-point set is μ^*-measurable. Thus only $\emptyset$ and X are μ^*-measurable. This is the best one could hope for if some kind of additivity of μ^* over the measurable sets is to occur. Note, also, that unlike Lebesgue measure, nonmeasurable sets in X have no measurable covers or measurable kernels. (See Exercise 2:1.14.)

Definition 2.29 defining measurability involves an additivity requirement of μ^*, but not any kind of σ-additivity. It may therefore be surprising that this simple modification of the inner–outer measure approach suffices to provide a σ-algebra $\mathcal{M}$ of measurable sets on which μ^* is σ-additive.

Theorem 2.31 *Let X be a set, μ^* an outer measure on X, and $\mathcal{M}$ the class of μ^*-measurable sets. Then $\mathcal{M}$ is a σ-algebra and μ^* is countably additive on $\mathcal{M}$. Thus the set function μ defined on $\mathcal{M}$ by $\mu(A) = \mu^*(A)$ for all $A \in \mathcal{M}$ is a measure.*

Proof. It follows immediately from the condition (11) in Definition 2.29 that $\emptyset \in \mathcal{M}$ and that $\mathcal{M}$ is closed under complementation. Now let $\{A_j\}$

be a sequence of measurable sets. To verify that $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$, we let $E \subset X$ and show that Definition 2.29 is satisfied. For convenience, define $\bigcup_{i=1}^{0} A_i = \emptyset$. Observe that

$$E \cap A = E \cap \bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} \left(\left(E \setminus \bigcup_{i=1}^{j-1} A_i \right) \cap A_j \right). \tag{12}$$

It follows from the subadditivity of μ^* that

$$\mu^*(E) \leq \mu^* \left(E \cap \bigcup_{j=1}^{\infty} A_j \right) + \mu^* \left(E \setminus \bigcup_{j=1}^{\infty} A_j \right).$$

Using the subadditivity of μ^* once more and noting (12), we see that

$$\mu^*(E) \leq \sum_{j=1}^{\infty} \mu^* \left(\left(E \setminus \bigcup_{i=1}^{j-1} A_i \right) \cap A_j \right) + \mu^* \left(E \setminus \bigcup_{j=1}^{\infty} A_j \right). \tag{13}$$

Since A_1 and A_2 are members of $\mathcal{M}$, we have

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A_1) + \mu^*(E \setminus A_1) \\ &= \mu^*(E \cap A_1) + \mu^*((E \setminus A_1) \cap A_2) + \mu^*(E \setminus (A_1 \cup A_2)). \end{aligned}$$

Proceeding inductively, it follows from the measurability of the sets A_i that, for all $k \in \mathbb{N}$,

$$\mu^*(E) = \sum_{j=1}^{k} \mu^* \left(\left(E \setminus \bigcup_{i=1}^{j-1} A_i \right) \cap A_j \right) + \mu^* \left(E \setminus \bigcup_{j=1}^{k} A_j \right). \tag{14}$$

Because of condition 2 in Definition 2.27, we can infer that

$$\mu^*(E) \geq \sum_{j=1}^{k} \mu^* \left(\left(E \setminus \bigcup_{i=1}^{j-1} A_i \right) \cap A_j \right) + \mu^* \left(E \setminus \bigcup_{j=1}^{\infty} A_j \right).$$

This last inequality is valid for all $k \in \mathbb{N}$. Thus

$$\mu^*(E) \geq \sum_{j=1}^{\infty} \mu^* \left(\left(E \setminus \bigcup_{i=1}^{j-1} A_i \right) \cap A_j \right) + \mu^* \left(E \setminus \bigcup_{j=1}^{\infty} A_j \right). \tag{15}$$

This inequality is the reverse of (13). Noting (12), we see that (13) and (15) imply that A satisfies the test of measurability, condition (11) of Definition 2.29. The proof that $\mathcal{M}$ is a σ-algebra is now complete.

It remains to show that $\mu = \mu^*$ is a measure on $\mathcal{M}$. That $\mu(\emptyset) = 0$ is clear from condition 1 of Definition 2.27. To show that μ is countably additive on $\mathcal{M}$, let $\{A_j\}$ be a sequence of pairwise disjoint members of $\mathcal{M}$. Let $E = \bigcup_{j=1}^{\infty} A_j$. Then, for all $j \in \mathbb{N}$,

$$E \setminus \bigcup_{i=1}^{j-1} A_i = \bigcup_{i=j}^{\infty} A_i$$

since the sets $\{A_i\}$ are pairwise disjoint. It follows that

$$\left(E \setminus \bigcup_{i=1}^{j-1} A_i\right) \cap A_j = A_j \quad \text{and} \quad E \setminus \bigcup_{j=1}^{\infty} A_j = \emptyset. \tag{16}$$

Substituting (16) into the inequalities (13) and (15), which are valid for every subset of X, we find that

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu^*(A_j).$$

■

Exercises

2:7.1 Verify formula (14).

2:7.2 Let X be an uncountable set. Let $\mu^*(A) = 0$ if A is countable and $\mu^*(A) = 1$ if A is uncountable. Show that μ^* is an outer measure, and determine the class of measurable sets.

2:7.3 Let μ^* be an outer measure on X, and let Y be a μ^*-measurable subset of X. Let $\nu^*(A) = \mu^*(A)$ for all $A \subset Y$. Show that ν^* is an outer measure on Y, and a set $A \subset Y$ is ν^*-measurable if and only if A is μ^*-measurable. Thus, for example, a subset A of $[0, 1]$ is Lebesgue measurable (as a subset of $[0, 1]$) if and only if it is Lebesgue measurable as a subset of $\mathbb{R}$.

2:7.4$\diamondsuit$ Prove that if $A \subset X$ and $\mu^*(A) = 0$ then A is μ^*-measurable. Consequently, the measure space generated by any outer measure is complete.

2.8 Method I

In Section 2.7 we have seen how one can obtain a measure μ from an outer measure μ^*. We still have the problem of determining how to obtain an outer measure μ^* so that the resulting measure μ is compatible with whatever primitive notion we wish to extend.

Once again, we can abstract this from Lebesgue's procedure. Suppose that we have a set X, a family $\mathcal{T}$ of subsets of X, and a nonnegative function $\tau : \mathcal{T} \to [0, \infty]$. We view $\mathcal{T}$ as the family of sets for which we have a primitive notion of "size" and $\tau(T)$ as a measure of that size. We shall call τ a premeasure to indicate the role that it takes in defining a measure. In order for our methods to work, we need assume no more of a premeasure τ than that it is nonnegative and vanishes on the empty set. [In the Lebesgue framework of Section 2.1, for example, we can take $X = [0, 1]$, $\mathcal{T}$ as the family of open intervals, and the premeasure $\tau(T)$ as the length of the open interval T.]

Here is a more formal development of these ideas.

Definition 2.32 Let X be a set, and let $\mathcal{T}$ be a family of subsets of X such that $\emptyset \in \mathcal{T}$. A nonnegative function τ defined on $\mathcal{T}$ so that $\tau(\emptyset) = 0$ is called a *premeasure*, and we refer to the family $\mathcal{T}$ as a *covering family* for X.

Note that hardly anything is assumed about the properties of a premeasure and a covering family. The terminology is employed just to indicate the intended use: we use the members of the family to cover sets, and we use the premeasure to generate an outer measure. The process, defined in the following theorem, of constructing outer measures is often called Method I in the literature. Note that a set A not contained in any countable union of sets from the covering family $\mathcal{T}$ is assigned an infinite outer measure. Note too that, while the definition of the outer measure uses countable covers, finite covers are included as well since $\emptyset \in \mathcal{T}$ and $\tau(\emptyset) = 0$.

Theorem 2.33 (Method I construction of outer measure) *Let $\mathcal{T}$ be a covering family for a set X, and let $\tau : \mathcal{T} \to [0, \infty]$ with $\tau(\emptyset) = 0$. For $A \subset X$, let*

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \tau(T_n) : \ T_n \in \mathcal{T} \ \text{and} \ A \subset \bigcup_{n=1}^{\infty} T_n \right\}, \tag{17}$$

where an empty infimum is taken as ∞. Then μ^* is an outer measure on X.

Proof. It is clear that $\mu^*(\emptyset) = 0$ and that μ^* is monotone. To verify that μ^* is countably subadditive, let $\{A_n\}$ be a sequence of subsets of X. We show that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

If any $\mu^*(A_n) = \infty$, there is nothing to prove, so we suppose that each is finite. Let $\varepsilon > 0$. For every $n \in \mathbb{N}$, there exists a sequence $\{T_{nk}\}_{k=1}^{\infty}$ of sets from $\mathcal{T}$ such that $A_n \subset \bigcup_{k=1}^{\infty} T_{nk}$, and

$$\sum_{k=1}^{\infty} \tau(T_{nk}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}. \tag{18}$$

Now

$$\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} T_{nk},$$

so by (17) and (18)

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \tau(T_{nk}) \leq \sum_{n=1}^{\infty} \left[\mu^*(A_n) + \frac{\varepsilon}{2^n}\right] = \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

We conclude that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

since ε is an arbitrary positive number. ∎

Method I is very useful, but it can have an important flaw when X is a metric space. In Section 3.2 we shall discuss this flaw and see how a variant, called Method II, overcomes this problem.

It is now easy to see how we can use Method I and Theorem 2.31 to obtain models that extend various sorts of primitive notions. For example, if we wish a measure-theoretic model for area in the Euclidean plane $\mathbb{R}^2$, we could start with $\mathcal{T}$ as the family of squares (along with $\emptyset$) and with $\tau(T)$ as the area of the square T. We apply Method I to obtain an outer measure λ_2^* in $\mathbb{R}^2$. We then restrict λ_2^* to the class $\mathcal{L}_2$ of measurable sets, and we have Lebesgue's two-dimensional measure λ_2.

We would be assured at this point of having a σ-algebra of measurable sets $\mathcal{L}_2$, but we would need to do more work to show that $\mathcal{L}_2$ possesses certain desirable properties. Nothing in our general work so far guarantees, for example, that members of the original family $\mathcal{T}$ are in $\mathcal{L}_2$ (i.e., the members of $\mathcal{T}$ are measurable) or, indeed, that the measure of a square T is the original value $\tau(T)$ with which we started. In the case of $\mathcal{L}_2$, it would be unfortunate if open squares were not measurable by the criterion of Definition 2.29 and worse still if the measure of a square were not its area. We shall see later that no such problem exists for Lebesgue measure in $\mathbb{R}^n$ or for a variety of other important measures.

Exercises 2:8.3 to 2:8.5 illustrate that the members of $\mathcal{T}$ need not, in general, be measurable and that $\tau(T)$ need not equal $\mu(T)$, even when $T \in \mathcal{T}$ is measurable.

Exercises

2:8.1 Verify that the set function μ^* as defined in (17) satisfies conditions 1 and 2 of Definition 2.27.

2:8.2◇ Refer to Example 2.10. Let $\mathcal{T}$ consist of $\emptyset$ and the half-open intervals $(a,b] \subset (0,1]$, and let $\tau = \nu_f$. Apply Method I to obtain μ^* and $\mathcal{M}$. Assuming that $\mathcal{T} \subset \mathcal{M}$ and $\mu = \tau$ on $\mathcal{T}$, this now provides a model for mass distributions on $(0,1]$. Let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q} \cap (0,1]$. Construct a function f, so that for all $A \subset (0,1]$,

$$\mu(A) = \sum_{q_n \in A} \frac{1}{2^n},$$

where μ is obtained from τ by our process, and $\tau((a,b]) = f(b) - f(a)$.

2:8.3◇ Let $X = \{1,2,3\}$, $\mathcal{T}$ consist of $\emptyset$, X and all doubleton sets, with $\tau(\emptyset) = 0$, $\tau(\{x,y\}) = 1$, for all $x \neq y \in X$, and $\tau(X) = 2$. Show that Method I results in the outer measure μ^* of Example 2.28. How do things change if $\tau(X) = 3$?

2:8.4 Let $X = \mathbb{N}$, $\mathcal{T}$ consist of $\emptyset$, X, and all singleton sets. Let $\tau(\emptyset) = 0$, $\tau(\{x\}) = 1$, for all $x \in X$, and

(a) $\tau(X) = 2$.

(b) $\tau(X) = \infty$.

In each case, apply Method I and determine the family of measurable sets.

2:8.5 Repeat Exercise 2:8.4 with the modification that

$$\tau(\{x\}) = \frac{1}{2^{x-1}}.$$

[Note in part (b), that $X \in \mathcal{M}$, but $\tau(X) \neq \mu(X)$.] How do things change if $\tau(X) = 1$?

2:8.6 Show that if $\mathcal{T} \subset \mathcal{M}$ then $\mu(T) \leq \tau(T)$ for all $T \in \mathcal{T}$.

2.9 Regular Outer Measures

We saw in Section 2.7 that the inner–outer measure approach does not, in general, give rise to a measure on a σ-algebra. There are, however, many situations in which the class of sets whose inner and outer measures are the same is identical to the class of sets measurable according to Definition 2.29.

Definition 2.34 An outer measure μ^* is called *regular* if for every $E \subset X$ there exists a measurable set $H \supset E$ such that $\mu(H) = \mu^*(E)$. The set H is called a *measurable cover* for E.

Theorem 2.35 *Let μ^* be a regular outer measure on X and suppose that $\mu^*(X) < \infty$. A necessary and sufficient condition that a set $A \subset X$ be measurable is that*

$$\mu^*(X) = \mu^*(A) + \mu^*(X \setminus A). \tag{19}$$

Proof. The necessity is clear from Definition 2.29. To prove that the condition is sufficient, let A be a subset of X satisfying (19), let E be any subset of X, and let H be a measurable cover for E. It suffices to verify that

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A), \tag{20}$$

the reverse inequality being automatically satisfied because of the subadditivity of μ^*.

Observe first that

$$\mu^*(A \setminus H) + \mu^*((X \setminus A) \setminus H) \geq \mu^*(X \setminus H). \tag{21}$$

Since H is measurable, we have

$$\mu^*(A) = \mu^*(A \cap H) + \mu^*(A \setminus H) \tag{22}$$

and

$$\mu^*(X \setminus A) = \mu^*(H \setminus A) + \mu^*((X \setminus H) \setminus A). \tag{23}$$

Now $\mu^*(X) = \mu^*(A) + \mu^*(X \setminus A)$ by (19). Thus, from equations (22) and (23) and the subadditivity of μ^*, we infer that

$$\begin{aligned} \mu(X) &= \mu^*(A \cap H) + \mu^*(A \setminus H) + \mu^*(H \setminus A) + \mu^*((X \setminus H) \setminus A) \\ &\geq \mu(H) + \mu(X \setminus H) = \mu(X). \end{aligned}$$

It follows that the one inequality above is actually an equality. Subtracting the inequality (21) from this equality, we obtain

$$\mu^*(H \cap A) + \mu^*(H \setminus A) \leq \mu(H). \tag{24}$$

This subtraction is justified since all the quantities involved are finite. Because $E \subset H$, we see from (24) that

$$\mu^*(E \cap A) + \mu^*(E \setminus A) \leq \mu^*(H \cap A) + \mu^*(H \setminus A) \leq \mu(H) = \mu^*(E).$$

This verifies (20). ■

In Section 2.1, we gave a sketch of one-dimensional Lebesgue measure and promised there to justify those aspects of the development that we did not verify at the time. The material in Section 2.7 provides a framework for developing Lebesgue measure using the Carathéodory criterion of Definition 2.29 and Method I. But it does not justify the inner–outer measure approach of Section 2.1. For that, we need to verify that λ^* is regular and then invoke Theorem 2.35.

It is not the case that every outer measure obtained by Method I is regular. Example 2.28 and Exercise 2:8.3 show this. Theorem 2.36 is useful in showing that, when Method I is invoked for the purpose of extending the primitive notions that we have already mentioned (length, area, volume, and mass) the resulting outer measures will be regular.

Theorem 2.36 *Let μ^* be constructed by Method I from $\mathcal{T}$ and τ. If all members of $\mathcal{T}$ are μ^*-measurable, then μ^* is regular.*

Proof. Let $A \subset X$. We find a measurable cover for A. If $\mu^*(A) = \infty$, then X is a measurable cover. Suppose then that $\mu^*(A) < \infty$. For each $m \in \mathbb{N}$, let $\{T_{mn}\}_{n=1}^{\infty}$ be a sequence of sets from the covering class $\mathcal{T}$ such that

$$A \subset \bigcup_{n=1}^{\infty} T_{mn} \quad \text{and} \quad \sum_{n=1}^{\infty} \tau(T_{mn}) < \mu^*(A) + \frac{1}{m}.$$

Let

$$T_m = \bigcup_{n=1}^{\infty} T_{mn} \quad \text{and} \quad H = \bigcap_{m=1}^{\infty} T_m.$$

Since each of the sets T_{mn} is measurable, so too is H. We show that H is a measurable cover for A.

Clearly, $A \subset H$ and so $\mu^*(A) \leq \mu(H)$. For the opposite inequality, we have, for each $m \in \mathbb{N}$,

$$\mu^*(T_m) \leq \sum_{n=1}^{\infty} \mu^*(T_{mn}) \leq \sum_{n=1}^{\infty} \tau(T_{mn}) \leq \mu^*(A) + \frac{1}{m}.$$

For each $m \in \mathbb{N}$, $H \subset T_m$, and so

$$\mu(H) \leq \mu^*(T_m) \leq \mu^*(A) + \frac{1}{m}.$$

This last inequality is true for all $m \in \mathbb{N}$, so $\mu(H) \leq \mu^*(A)$. Thus $\mu(H) = \mu^*(A)$, and H is a measurable cover for A. ■

Corollary 2.37 *Lebesgue outer measure λ^* on $\mathbb{R}$ is regular.*

Proof. Here $\mathcal{T}$ consists of $\emptyset$ and the open intervals, and $\tau(T)$ is the length of the interval T. Because of Theorem 2.36, it suffices to show that each interval (a, b) is measurable by Carathéodory's criterion (Definition 2.29).

Let $E \subset \mathbb{R}$ and let $\varepsilon > 0$. There is a sequence $\{T_n\} \subset \mathcal{T}$ that covers E for which

$$\sum_{n=1}^{\infty} \tau(T_n) \leq \lambda^*(E) + \frac{\varepsilon}{2}.$$

Take

$$\begin{aligned}
\mathcal{U}_1 &= \{T_n \cap (a, b) : n \in \mathbb{N}\}, \\
\mathcal{U}_2 &= \{T_n \cap (-\infty, a) : n \in \mathbb{N}\}, \\
\mathcal{U}_3 &= \{T_n \cap (b, \infty) : n \in \mathbb{N}\},
\end{aligned}$$

and

$$\mathcal{U}_4 = \left\{ \left(a - \tfrac{1}{8}\varepsilon, a + \tfrac{1}{8}\varepsilon\right), \left(b - \tfrac{1}{8}\varepsilon, b + \tfrac{1}{8}\varepsilon\right) \right\}.$$

Then $\mathcal{U}_1$ covers $E \cap (a,b)$ and $\mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4$ covers $E \setminus (a,b)$. The total length of the intervals in $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ is the same as for the original sequence, and the additional lengths from $\mathcal{U}_4$ have total length equal to $\varepsilon/2$. Hence

$$\lambda^*(E \cap (a,b)) + \lambda^*(E \setminus (a,b)) \le \sum_{n=1}^{\infty} \tau(T_n) + \varepsilon/2 \le \lambda^*(E) + \varepsilon.$$

Since ε is arbitrary, we have

$$\lambda^*(E \cap (a,b)) + \lambda^*(E \setminus (a,b)) \le \lambda^*(E)$$

for any $E \subset \mathbb{R}$, and it follows that (a,b) must be measurable. ■

Let us summarize some of the ideas in Sections 2.7 and 2.9, insofar as they relate to the important case of Lebesgue measure on an interval. We start with the covering family $\mathcal{T}$ of open intervals and with the primitive notion $\tau(T)$ as the length of the interval T. Upon applying Method I, this gives rise to an outer measure μ^*. We then apply the Carathéodory process to obtain a class $\mathcal{M}$ of measurable sets and a measure μ that equals μ^* on $\mathcal{M}$. To verify that our primitive notion of length is not destroyed by the process, we show, as in the proof of Corollary 2.37, that open intervals are measurable. It is then almost trivial to verify that the measure of an interval is its length. Theorem 2.36 now tells us that μ^* is regular; thus we could have used the inner–outer measure approach of Section 2.1. This would result in the same class of measurable sets and the same measure as provided by the Carathéodory process.

Exercises

2:9.1 Prove that, if μ^* is a regular outer measure and $\{A_n\}$ is a sequence of sets in X, then $\mu^*(\liminf A_n) \le \liminf \mu^*(A_n)$. Compare with Theorem 2.21 (1).

2:9.2◇ Prove that, if μ^* is a regular outer measure and $\{A_n\}$ is an expanding sequence of sets, then $\mu^*(\lim_n A_n) = \lim_n \mu^*(A_n)$. Compare with Theorem 2.20 (1).

2:9.3 Show that the conclusions of Exercises 2:9.1 and 2:9.2 are not valid for arbitrary outer measures.

2:9.4 Let $X = \mathbb{N}$, $\mu^*(\emptyset) = 0$, and $\mu^*(E) = 1$ for all $E \neq \emptyset$.

(a) Show that μ^* is a regular outer measure.

(b) Let $\{A_n\}$ be a sequence of subsets of X (not assumed measurable). Show that, while the analog of part (1) of Theorem 2.21 does hold (Exercise 2:9.1), the analogs of parts (2) and (3) do not hold.

2:9.5 Let $X = \mathbb{N}$, and let $0 = a_0, a_1 = \frac{1}{2} < a_2 < a_3 < \cdots$ with $\lim_n a_n = 1$. If E has n members, let $\mu^*(E) = a_n$. If E is infinite, let $\mu^*(E) = 1$.

(a) Show that μ^* is an outer measure, but that μ^* is not regular.

(b) Show that the conclusions of Exercise 2:9.2 and Theorem 2.35 hold.

2:9.6 Prove the following variant of Theorem 2.35: Let μ^* be a regular outer measure, let H be measurable with $\mu(H) < \infty$, and let $A \subset H$. If $\mu(H) = \mu^*(H \cap A) + \mu^*(H \setminus A)$, then A is measurable.

2:9.7◇ Let $X = (0,1]$, $\mathcal{T}$ consist of the half-open intervals $(a,b]$ contained in $(0,1]$, and f be increasing and right continuous on $(0,1]$ with $\lim_{x\to 0} f(x) = 0$. Let $\tau((a,b]) = f(b) - f(a)$. Apply Method I to obtain an outer measure μ_f^*. Prove that $\mathcal{T} \subset \mathcal{M}$ and μ_f^* is regular and thus the inner–outer measure approach works here. Observe that all open sets as well as all closed sets are μ_f^* measurable. In particular, such measures can be used to model mass distributions on $\mathbb{R}$. (See Exercise 2:4.10, and Example 2.10 and the discussion following it.)

2:9.8◇ Let $\mathcal{T}$ be a covering family for X. Prove that, if Method I is applied to $\mathcal{T}$ and τ to obtain the outer measure μ^*, then for each $E \subset X$ with $\mu^*(E) < \infty$ there exists $S \in \mathcal{T}_{\sigma\delta}$ such that $E \subset S$ and $\mu^*(S) = \mu^*(E)$. (In particular, if X is a metric space and $\mathcal{T}$ consists of open sets, S can be taken to be of type G_δ.) [Hint: See the proof of Theorem 2.36.]

2.10 Nonmeasurable Sets

In any particular setting, can we determine the existence of nonmeasurable sets? Certainly, it is easy to give artificial examples where all sets are measurable or where nonmeasurable sets exist. But in important applications we would like some generally applicable methods.

The special case of Lebesgue nonmeasurable sets should be instructive. Vitali was the first to demonstrate the existence of such sets using the axiom of choice. Let $0 = r_0, r_1, r_2, \ldots$ be an enumeration of $\mathbb{Q} \cap [-1,1]$. Using this sequence, he finds a set $A \subset [-\frac{1}{2}, \frac{1}{2}]$ so that the collection of sets

$$A_k = \{x + r_k : x \in A\}$$

forms a disjoint sequence covering the interval $[-\frac{1}{2}, \frac{1}{2}]$. As Lebesgue measure is translation invariant and countably additive, the set A cannot be measurable. (See Section 1.10 for the details.) In Section 12.6 we will encounter an example of a *finitely* additive measure that extends Lebesgue measure to *all* subsets of $[0,1]$ and is translation invariant. This set function cannot be a measure, however, because of the Vitali construction. Unfortunately, this discussion does little to help us in general as it focuses attention on the additive group structure of $\mathbb{R}$ and the invariance of λ.

Another example may help more. We have seen a proof of the existence of Bernstein sets, that is, a set of real numbers such that neither it nor its complement contains any perfect set. (See Exercises 1:22.7 and 1:22.8.) Such a set cannot be Lebesgue measurable. To see this, remember that the outer measure of any set can be approximated from above by open sets; consequently, the measure of a measurable set can be approximated from inside by closed (or perfect) sets. But a Bernstein set and its complement contain no perfect set, and so both would have to have measure zero if they were measurable.

This example does contain a clue, albeit somewhat obliquely. The example suggests that some topological property (relating to closed and open sets) of Lebesgue measure is intimately related to the existence of nonmeasurable sets. But the proof of the existence of Bernstein sets simply employed a cardinality argument and did not invoke any deep topological properties of the real line. In fact, the nonmeasurability question reduces in many cases, surprisingly, to one of cardinality.

The following result of S. M. Ulam illustrates the first step in this direction. Ultimately, we wish to ask, for a set X, when is it possible to have a finite measure defined on all subsets of X, but that assigns zero measure to each singleton set?

Theorem 2.38 (Ulam) *Let Ω be the first uncountable ordinal, and let $X = [0, \Omega)$. If μ is a finite measure defined on all subsets of X and such that $\mu(\{x\}) = 0$ for each $x \in X$, then μ is the zero measure.*

Proof. For any $y \in X$, write $A_y = \{x \in X : x < y\}$, the set of all predecessors of y. Then each set A_y is countable, and so there is a injection

$$f(\cdot, y) : A_y \to \mathbb{N}.$$

Define for each $x \in X$ and $n \in \mathbb{N}$

$$B_{x,n} = \{z \in X : x < z,\ f(x, z) = n\}.$$

If x_1, x_2 are distinct points in X, then evidently the sets $B_{x_1,n}$ and $B_{x_2,n}$ are disjoint. Since μ is finite, this means that, for each integer n, $\mu(B_{x,n}) > 0$ for only countably many $x \in X$. This means, since X is uncountable, that there must be some $x_0 \in X$ for which $\mu(B_{x_0,n}) = 0$ for each integer n.

Consider the union

$$B_0 = \bigcup_{n=1}^{\infty} B_{x_0,n}$$

and observe that $\mu(B_0) = 0$. If $y > x_0$, then $f(x_0, y) = n$ for some $n \in \mathbb{N}$. Hence $\{y \in X : x_0 < y\} \subset B_0$. Thus

$$X = B_0 \cup \{y \in X : y \leq x_0\},$$

and this expresses X as the union of a set of μ measure zero and a countable set. Hence $\mu(X) = 0$ as required. ■

If we assume CH (the continuum hypothesis), it follows from Ulam's theorem that there is no finite measure defined on *all* subsets of the real line and vanishing at points except for the zero measure itself. This applies not just to the real line, then, but to any set of cardinality **c**. This is true even without invoking the continuum hypothesis, but requires other axioms of set theory. Note that this means that it is not the invariance of Lebesgue measure or its properties relative to open and closed sets that does not allow it to be defined on all subsets of the reals. There is no nontrivial finite measure defined on all subsets of an interval of the real line that vanishes on singleton sets.

These ideas can be generalized to spaces of higher cardinality. We define an *Ulam number* to be a cardinal number with the property of the theorem.

Definition 2.39 A cardinal number $\aleph$ is an *Ulam number* if whenever X is a set of cardinality $\aleph$ and μ is a finite measure defined on all subsets of X and such that $\mu(\{x\}) = 0$ for each $x \in X$ then μ is the zero measure.

Certainly, $\aleph_0$ is an Ulam number. We have seen in Theorem 2.38 that $\aleph_1$ is also an Ulam number. The class of all Ulam numbers forms a very large initial segment in the class of all cardinal numbers. It will take more set theory than we choose to develop to investigate this further,[1] but some have argued that one could consider safely that all cardinal numbers that one expects to encounter in analysis are Ulam numbers.

Exercises

2:10.1 Show that every set of real numbers that has positive Lebesgue outer measure contains a nonmeasurable set.

2:10.2 Show that there exist disjoint sets $\{E_k\}$ so that

$$\lambda^*\left(\bigcup_{k=1}^{\infty} E_k\right) < \sum_{k=1}^{\infty} \lambda^*\left(E_k\right).$$

2:10.3 Show that there exist sets $E_1 \supset E_2 \supset E_3 \ldots$ so that $\lambda^*(E_k) < +\infty$, for each k, and

$$\lambda^*\left(\bigcap_{k=1}^{\infty} E_k\right) < \lim_{k\to\infty} \lambda^*\left(E_k\right).$$

2:10.4 Let E be a measurable set of positive Lebesgue measure. Show that E can be written as the disjoint union of two sets $E = E_1 \cup E_2$ so that $\lambda(E) = \lambda^*(E_1) = \lambda^*(E_2)$.

2:10.5 Let H be a Hamel basis (see Exercise 1:11.3) and H_0 a nonempty finite or countable subset of H. Show that the set of rational linear combinations of elements of $H \setminus H_0$ is nonmeasurable.

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[1] See K. Ciesielski, "How good is Lebesgue measure?", *Math. Intelligencer* 11(2), 1989, pp. 54–58, for a discussion of material related to this section and for references to the literature. Also, in Section 12.6 we return to some related measure problems.

2:10.6 Every totally imperfect set of real numbers contains no Cantor set but does contain an uncountable measurable set.

2:10.7 Exercise 2:10.6 suggests asking whether there can exist an uncountable set of real numbers that contains no uncountable measurable subset. Such a set (if it exists) is called a *Sierpiński set* and must clearly be nonmeasurable.

(a) Let X be a set of power $2^{\aleph_0}$ and let $\mathcal{E}$ be a family of subsets of X, also of power $2^{\aleph_0}$, with the property that X is the union of the family $\mathcal{E}$, but is not the union of any countable subfamily. Assuming CH, show that there is an uncountable subset of X that has at most countably many points in common with each member of $\mathcal{E}$.

(b) By applying (a) to the family of measure zero G_δ subsets of $\mathbb{R}$, show that, assuming CH, there exists a Sierpiński set.

2:10.8 Let μ^* be an outer measure on a set X, and suppose that $E \subset X$ is not μ^*–measurable. Show that

$$\inf\{\mu^*(A \cap B) : \ A,\, B\ \mu^*\text{–measurable},\, A \supset E,\, B \supset X \setminus E\} > 0.$$

2:10.9 A cardinal number $\aleph$ is an Ulam number if and only if the following: if μ^* is an outer measure on a set X and $\mathcal{C}$ is a disjointed family of subsets of X with (i) $\operatorname{card}(\mathcal{C}) \leq \aleph$, (ii) the union of every subfamily of $\mathcal{C}$ is μ^*–measurable, (iii) $\mu^*(C) = 0$ for each $C \in \mathcal{C}$, and (iv) $\mu\left(\bigcup_{C\in\mathcal{C}} C\right) < \infty$, then

$$\mu\left(\bigcup_{C\in\mathcal{C}} C\right) = 0.$$

2:10.10 If $\mathcal{S}$ is a set of Ulam numbers and $\operatorname{card}(\mathcal{S})$ is an Ulam number then the least upper bound of $\mathcal{S}$ is an Ulam number.

2:10.11 The successor of any Ulam number is an Ulam number. [Hint: See Federer, *Geometric Measure Theory*, Springer (1969), pp. 58–59, for a proof of these last three exercises.]

2.11 More About Method I

Let us review briefly our work to this point from the perspective of building a measure-theoretic framework for modeling some geometric or physical phenomena. In an attempt to satisfy our sense that "the whole should be the sum of its parts," we created the structure of an algebra of sets $\mathcal{A}$ with an additive set function defined on $\mathcal{A}$. This structure had limitations—the algebra might be too small for our purposes. For example, the algebra generated by the half-open intervals on $(0, 1]$ consisted only of finite unions of such intervals (and $\emptyset$ of course). Even singletons are not in the algebra.

The notion of countable additivity in place of additivity helped here—it gave rise to a σ-algebra of sets and a measure.

We then turned to the problem of how to obtain a measure space that could serve as a model for a given phenomenon for which we had a "primitive notion." We saw that we can always obtain a measure from an outer measure via the Carathéodory process and that Method I might be useful in obtaining an outer measure suitable for modeling our phenomenon. We say "might be useful" instead of "is useful" because there still are two unpleasant possibilities: our "primitive" sets T need not be measurable and, even if they are, it need not be true that

$$\tau(T) = \mu(T)$$

for all $T \in \mathcal{T}$. Such flaws might not be surprising insofar as we have placed only minimal requirements on τ and $\mathcal{T}$. What sorts of further restrictions will eliminate these two flaws?

Let us return to the family of half-open intervals on $(0, 1]$. Here we have an increasing function f defined on $[0, 1]$, and we obtain τ from f by

$$\tau((a, b]) = f(b) - f(a),$$

with τ extended to be additive on the algebra $\mathcal{T}$ generated by the half-open intervals. In this natural setting, we have some additional structure. The family $\mathcal{T}$ is an algebra of sets, and τ is additive on $\mathcal{T}$. This structure suffices to eliminate one of the unpleasant possibilities. Note that the proof is nearly identical to that for Corollary 2.37, but there, since the open intervals that were used for the covering family did not form an algebra, it was not so easy to carve up the sets.

Theorem 2.40 *Let μ^* be constructed from a covering family $\mathcal{T}$ and a premeasure τ by Method I, and let $(X, \mathcal{M}, \mu)$ be the resulting measure space. If $\mathcal{T}$ is an algebra and τ is additive on $\mathcal{T}$, then $\mathcal{T} \subset \mathcal{M}$ and μ^* is regular.*

Proof. By Theorem 2.36, it is enough to check that each member of $\mathcal{T}$ is μ^*–measurable. Let $T \in \mathcal{T}$. To obtain that $T \in \mathcal{M}$, it suffices to show that, for each $E \subset X$ for which $\mu^*(E) < \infty$,

$$\mu^*(E) \geq \mu^*(E \cap T) + \mu^*(E \cap \widetilde{T}). \tag{25}$$

Let $\varepsilon > 0$. Choose a sequence $\{T_n\}$ from $\mathcal{T}$ such that

$$E \subset \bigcup_{n=1}^{\infty} T_n$$

and

$$\sum_{n=1}^{\infty} \tau(T_n) < \mu^*(E) + \varepsilon.$$

Since τ is additive on $\mathcal{T}$, we have, for all $n \in \mathbb{N}$

$$\tau(T_n) = \tau(T_n \cap T) + \tau(T_n \cap \widetilde{T}).$$

But

$$E \cap T \subset \bigcup_{n=1}^{\infty} (T_n \cap T) \quad \text{and} \quad E \cap \widetilde{T} \subset \bigcup_{n=1}^{\infty} (T_n \cap \widetilde{T}). \tag{26}$$

Thus

$$\begin{aligned} \mu^*(E) + \varepsilon > \sum_{n=1}^{\infty} \tau(T_n) &= \sum_{n=1}^{\infty} \tau(T_n \cap T) + \sum_{n=1}^{\infty} \tau(T_n \cap \widetilde{T}) \\ &\geq \sum_{n=1}^{\infty} \mu^*(T_n \cap T) + \sum_{n=1}^{\infty} \mu^*(T_n \cap \widetilde{T}) \\ &\geq \mu^*(E \cap T) + \mu^*(E \cap \widetilde{T}), \end{aligned}$$

the last inequality following from (26). Since ε is arbitrary, (25) follows. ■

Primitive notions like area, volume, and mass that are fundamentally additive might well lead to a τ, $\mathcal{T}$ combination that satisfies the hypotheses of Theorem 2.40.

We next ask whether the hypotheses of Theorem 2.40 remove the other flaw that we mentioned: $\tau(T)$ need not equal $\mu(T)$. To address this question, we look ahead.

A result of Section 12.6 enters our discussion. There is a finitely additive measure τ defined on *all* subsets of $[0, 1]$ such that $\tau = \lambda$ on the class $\mathcal{L}$ of Lebesgue measurable sets. We mentioned this example in Section 2.10, where we proved too that, if μ is a finite measure on $2^{[0,1]}$ with $\mu(\{x\}) = 0$ for all $x \in [0, 1]$, then $\mu(E) = 0$ for all $E \subset [0, 1]$.

Suppose now that we take $\mathcal{T} = 2^{[0,1]}$ and τ the finitely additive extension of λ mentioned above and apply Method I to obtain μ^* and μ. Theorem 2.40 guarantees that all members of $\mathcal{T}$ are measurable. But this means that *every* subset of $[0, 1]$ is measurable. From the material in Section 2.10 just mentioned, this implies that $\mu \equiv 0$. Since $\tau = \lambda$ on $\mathcal{L}$, τ and μ cannot agree on any set of positive Lebesgue measure. Thus, even though $\mathcal{T}$ and τ had enough structure to guarantee *all* subsets of $[0, 1]$ measurable, the measure μ did not retain anything of the primitive notion of length provided by τ!

Our development of Lebesgue measure on $[0, 1]$ actually provides a clue for removing the remaining flaw. Recall that in Section 2.1 we first extended the primitive notion of $\lambda(I)$, the length of an interval, to $\lambda(G)$, G open. This anticipated a form of σ-additivity. We then defined $\lambda(F)$, F closed. We can extend λ by additivity to the algebra $\mathcal{T}$ generated by the family of open sets (or, equivalently, by the family of closed sets). Taking $\tau = \lambda$ on $\mathcal{T}$, one can show that τ is σ-additive according to the following definition.

Definition 2.41 Let $\mathcal{A}$ be an algebra of sets, and let α be additive on $\mathcal{A}$. If

$$\alpha(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \alpha(A_n)$$

whenever $\{A_n\}$ is a sequence of pairwise disjoint sets from $\mathcal{A}$ for which

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A},$$

we say that α is *σ-additive* on $\mathcal{A}$.

Thus if $\alpha \geq 0$, it can fail to be a measure only when $\mathcal{A}$ is not a σ-algebra. It may well happen that when a concept is "fundamentally" additive, a τ, $\mathcal{T}$ combination can be found such that τ is σ-additive on $\mathcal{T}$. See Exercise 2:12.4.

Theorem 2.42 *Under the hypotheses of Theorem 2.40, if τ is σ-additive on $\mathcal{T}$, then $\mu(T) = \tau(T)$ for all $T \in \mathcal{T}$.*

Proof. We first show that, if $\{T_n\}$ is any sequence of sets in $\mathcal{T}$, $T \in \mathcal{T}$ and $T \subset \bigcup_{n=1}^{\infty} T_n$, then

$$\tau(T) \leq \sum_{n=1}^{\infty} \tau(T_n). \tag{27}$$

Let $B_1 = T \cap T_1$ and, for $n \geq 2$, let

$$B_n = T \cap T_n \setminus (T_1 \cup \cdots \cup T_{n-1}).$$

Then, for all $n \in \mathbb{N}$, $B_n \subset T \cap T_n$, $B_n \in \mathcal{T}$, the sets B_n are pairwise disjoint, and $T = \bigcup_{n=1}^{\infty} B_n$. Since τ is σ-additive on $\mathcal{T}$,

$$\tau(T) = \sum_{n=1}^{\infty} \tau(B_n) \leq \sum_{n=1}^{\infty} \tau(T_n).$$

This verifies (27). It now follows that

$$\tau(T) \leq \inf \left\{ \sum_{n=1}^{\infty} \tau(T_n) : \bigcup_{n=1}^{\infty} T_n \supset T, T_n \in \mathcal{T} \right\} = \mu^*(T).$$

But since $\{T\}$ covers the set T, $\mu^*(T) \leq \tau(T)$. Thus $\tau(T) = \mu^*(T)$. Since T is measurable by Theorem 2.40, $\mu^*(T) = \mu(T)$. ■

Exercises

2:11.1 Following the proof of Theorem 2.40, we gave an example of a τ, $\mathcal{T}$ combination, $\mathcal{T} = 2^{[0,1]}$ and $\tau = \lambda$ on $\mathcal{L}$, such that the μ resulting from Method I had little connection to length on $\mathcal{L}$. What would happen if we took the same τ but restricted τ to $\mathcal{T} = \mathcal{L}$?

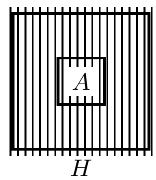

Figure 2.1: The set N is a measurable cover for $H \setminus A$.

2.12 Completions

Our presentation of Method I in Section 2.7 seemed simple and natural. It required little of τ and $\mathcal{T}$. But it had flaws that we removed in Section 2.11 by imposing additional additivity conditions on τ and $\mathcal{T}$. These conditions seemed natural because τ often represents a primitive notion of size that is intuitively additive. Exercise 2:12.4 provides a possible example of how we might naturally be led to use Theorems 2.40 and 2.42. On the other hand, these conditions seem to impose serious restrictions on the use of Method I. One might ask, what measure spaces $(X, \mathcal{M}, \mu)$ are the Method I result of a τ, $\mathcal{T}$ combination that satisfies such additivity conditions?

Such a space must be complete because any Method I measure is complete. We next show that the only other restriction on $(X, \mathcal{M}, \mu)$ is that X not be "too large."

Definition 2.43 Let $(X, \mathcal{M}, \mu)$ be a measure space. If $\mu(X) < \infty$, then we say that the measure space is *finite*. If $X = \bigcup_{n=1}^{\infty} X_n$ with $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$, then we say that the space is *σ-finite*.

Theorem 2.44 *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $\mathcal{T} = \mathcal{M}$ and $\tau = \mu$, and apply Method I to obtain an outer measure $\hat{\mu}^*$ and a measure space $(X, \widehat{\mathcal{M}}, \hat{\mu})$. Then*

1. *If $A \in \widehat{\mathcal{M}}$, then $A = M \cup Z$, where $M \in \mathcal{M}$ and $Z \subset N \in \mathcal{M}$ with $\mu(N) = 0$. Thus $(X, \widehat{\mathcal{M}}, \hat{\mu})$ is the completion of $(X, \mathcal{M}, \mu)$.*
2. *If μ is the restriction of a regular outer measure μ^* to its class of measurable sets, then $\hat{\mu}^* = \mu^*$.*

Proof. To prove (1), assume first that $\mu(X) < \infty$. Let $A \in \widehat{\mathcal{M}}$. Now $\mathcal{M} \subset \widehat{\mathcal{M}}$ by Theorem 2.40. Thus $\hat{\mu}^*$ is regular by Theorem 2.36, so A has a $\hat{\mu}^*$-measurable cover H. Since $\mathcal{M}$ is a σ-algebra, Theorem 2.36 and Exercise 2:9.8 show that H can be taken in $\mathcal{M}$. Because $X \in \mathcal{M}$, our assumption that $\mu(X) < \infty$ implies that $\hat{\mu}^*(A) < \infty$. Since $\hat{\mu}^*$ is additive on $\widehat{\mathcal{M}}$,

$$\hat{\mu}^*(H \setminus A) = \hat{\mu}^*(H) - \hat{\mu}^*(A) = 0.$$

Now let N be a measurable cover in $\mathcal{M}$ for $H \setminus A$. See Figure 2.1.

By Theorem 2.42, $\hat{\mu}^*(N) = \mu(N)$, so $\mu(N) = \hat{\mu}^*(H \setminus A) = 0$. But

$$A = (H \setminus N) \cup (A \cap N).$$

To verify this, observe first that if $x \in A$, but $x \notin N$, then

$$x \in A \setminus N \subset H \setminus N.$$

In the other direction, since $N \supset H \setminus A$, any $x \in H \setminus N$ must be in A, and obviously $A \cap N \subset A$.

Now let $M = H \setminus N$, and let $Z = A \cap N$. Then $M \in \mathcal{M}$ and $Z \subset N$ with $\mu(N) = 0$. The equality $A = M \cup Z$ is the required one, and the proof of part (1) of the theorem is complete when $\mu(X) < \infty$. The proof when $\mu(X) = \infty$ is left as Exercise 2:12.1.

To prove (2), let $A \subset X$. By hypothesis, μ comes from a regular outer measure μ^*. Thus there exists a measurable cover $M \in \mathcal{M}$ for A. By the definition of $\hat{\mu}^*$,

$$\hat{\mu}^*(A) \le \mu(M) = \mu^*(A).$$

In the other direction, observe first that, since $\mathcal{M}$ is a σ-algebra,

$$\hat{\mu}^*(A) = \inf\{\mu(B) : A \subset B \in \mathcal{M}\}.$$

But if $A \subset B \in \mathcal{M}$, then $\mu^*(A) \le \mu^*(B) = \mu(B)$, so

$$\mu^*(A) \le \inf\{\mu(B) : A \subset B \in \mathcal{M}\}.$$

Therefore, $\hat{\mu}^*(A) = \mu^*(A)$. ■

Corollary 2.45 *Every complete σ-finite measure space $(X, \mathcal{M}, \mu)$ is its own Method I Carathéodory extension. That is, an application of Method I to $\mathcal{T} = \mathcal{M}$ and $\tau = \mu$ results in the space $(X, \mathcal{M}, \mu)$.*

Proof. Observe that the completion of a complete measure space is the space itself and apply part (1) of Theorem 2.44. ■

The hypotheses of Theorem 2.44 and Corollary 2.45 cannot be dropped. See Exercises 2:12.2 and 2:12.3.

Exercises

2:12.1 Prove part (1) of Theorem 2.44 when $\mu(X) = \infty$.

2:12.2 Let $X = \mathbb{R}$, $\mathcal{M} = \left\{A : A \text{ is countable or } \widetilde{A} \text{ is countable}\right\}$, and define

$$\mu(A) = \begin{cases} \text{cardinality } A, & A \text{ is finite;} \\ \infty, & A \text{ is infinite.} \end{cases}$$

(a) Show that μ is a complete measure on $\mathcal{M}$.

(b) Show that $\hat{\mu}$ (See Theorem 2.44) is not the completion of μ.

(c) Show that μ is not the restriction to its measurable sets of any outer measure.

(d) Reconcile these with Theorem 2.44 and Corollary 2.45.

2:12.3 Let $(X, \mathcal{M}, \mu)$ be as in Example 2.28. Apply the process of Theorem 2.44 and determine whether $\hat{\mu}^* = \mu^*$.

2:12.4$\diamondsuit$ Suppose that we have a mass distribution on the half-open square $S = (0,1] \times (0,1]$ in $\mathbb{R}^2$, and we know how to compute the mass in any half-open "interval" $(a,b] \times (c,d]$. Suppose that singleton sets have zero mass. We wish to obtain a measure space $(X, \mathcal{M}, \mu)$ to model this distribution *based only on the ideas we have developed so far.*

First try: Take $\mathcal{T}$ as the half-open intervals in S, together with $\emptyset$, and let $\tau(T)$ be the mass of T for $T \in \mathcal{T}$. Apply Method I to get μ^* and then $(X, \mathcal{M}, \mu)$.

(a) Can we be sure that $\mathcal{M}$ is a σ-algebra and μ is a measure on $\mathcal{M}$? Can we be sure that $\mathcal{T} \subset \mathcal{M}$? If $T \in \mathcal{M}$, must $\mu(T) = \tau(T)$?

Second try: We note that τ is intuitively additive. So let $\mathcal{T}_1$ be the algebra generated by $\mathcal{T}$, and extend τ to τ_1 so that τ_1 is additive on $\mathcal{T}_1$.

(b) Can we do this? That is, can we be sure that $\tau_1(T_1)$, $T_1 \in \mathcal{T}_1$, does not depend on the decomposition of T_1 into a union of members of $\mathcal{T}$? If so, what are the answers to the questions posed in part (a) when we apply Method I to $\mathcal{T}_1$ and τ_1?

Third try: We believe mass is fundamentally σ-additive. But $\mathcal{T}_1$ is only an algebra. So we verify that τ_1 is σ-additive on $\mathcal{T}_1$. Can we now answer the three questions in part (a) affirmatively?

2.13 Additional Problems for Chapter 2

2:13.1 Criticize the following "argument" which is far too often seen:

> "If $G = (a,b)$ then $\overline{G} = [a,b]$. Similarly, if $G = \bigcup_{i=1}^{\infty}(a_i, b_i)$ is an open set, then $\overline{G} = \bigcup_{i=1}^{\infty}[a_i, b_i]$ so that G and $\overline{G}$ differ by a countable set. Since every countable set has Lebesgue measure zero, it follows that an open set G and its closure $\overline{G}$ have the same Lebesgue measure."(?)

2:13.2 Let A be a set of real numbers of Lebesgue measure zero. Show that the set $\{x^2 : x \in A\}$ also has measure zero.

2:13.3 Let A be the set of real numbers in the interval $(0,1)$ that have a decimal expansion that contains the number 3. Show that A is a Borel set and find its Lebesgue measure.

2:13.4 Let E be a Lebesgue measurable subset of $[0,1]$, and define

$$B = \{x \in [0,1] : \lambda(E \cap (x-\varepsilon, x+\varepsilon)) > 0 \text{ for all } \varepsilon > 0\}.$$

Show that B is perfect.

2:13.5 Let E be a Lebesgue measurable subset of $[0,1]$ and let $c > 0$. If $\lambda(E \cap I) \geq c\lambda(I)$ for all open intervals $I \subset [0,1]$, show that $\lambda(E) = 1$.

2:13.6 Let A_n be a sequence of Lebesgue measurable subsets of $[0,1]$ and suppose that $\limsup_{n\to\infty} \lambda(A_n) = 1$. Show that there is some subsequence with

$$\lambda\left(\bigcap_{k=1}^{\infty} A_{n_k}\right) > 0.$$

[Hint: Arrange for $\sum_{k=1}^{\infty} (1 - \lambda(A_{n_k})) < 1$.]

2:13.7$\diamondsuit$ Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $A \in \mathcal{M}$ is called an *atom*, if $\mu(A) > 0$ and, for all measurable sets $B \subset A$, $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. The measure space is *nonatomic* if there are no atoms.

(a) For any $x \in X$, if $\{x\} \in \mathcal{M}$ and $\mu(\{x\}) > 0$, then $\{x\}$ is an atom.

(b) Determine all atoms for the counting measure. (The counting measure is defined in Exercise 2:3.9.)

(c) Show that if $A \in \mathcal{M}$ is an atom then every subset $B \subset A$ with $B \in \mathcal{M}$ and $\mu(B) > 0$ is also an atom.

(d) Show that if A_1, $A_2 \in \mathcal{M}$ are atoms then, up to a set of μ–measure zero, either A_1 and A_2 are equal or disjoint.

(e) Suppose that μ is σ-finite. Show that there is a set $X_0 \subset X$ such that X_0 is a disjoint union of countably many atoms of $(X, \mathcal{M}, \mu)$ and $X \setminus X_0$ contains no atoms.

(f) Show that the Lebesgue measure space is nonatomic.

(g) Give an example of a nontrivial measure space $(X, \mathcal{M}, \mu)$ with $\mu(\{x\}) = 0$ for all $x \in X$ and so that every set of positive measure is an atom. [Hint: Construct a measure using Exercise 2:2.5.]

2:13.8$\diamondsuit$ (Liaponoff's theorem) Let $\mu_1, \dots, \mu_n$ be nonatomic measures on $(X, \mathcal{M})$, with $\mu_i(X) = 1$ for all $i = 1, \dots, n$. These measures can be viewed as giving rise to a *vector measure*

$$\mu : \mathcal{M} \to [0,1]^n = [0,1] \times [0,1] \times \cdots [0,1]$$

on $(X, \mathcal{M})$ defined by

$$\mu(A) = (\mu_1(A), \dots, \mu_n(A))$$

for each $A \in \mathcal{M}$. A theorem of Liaponoff (1940) states that

The set S of n-tuples $(x_1, \ldots, x_n)$ for which there exists $A \in \mathcal{M}$ such that $\mu(A) = (x_1, \ldots, x_n)$ is a convex subset of $[0,1]^n$.

(a) Let $(X, \mathcal{M}, \mu)$ be a nonatomic measure space with $\mu(X) = 1$. Show that for each $\gamma \in [0,1]$ there is a set $E_\gamma \subset X$ such that $\mu(E_\gamma) = \gamma$. [Hint: Use some form of Zorn's lemma (Section 1.11) or transfinite induction.]

(b) Show that part (a) follows from Liaponoff's theorem.

(c) Show that $(1/n, 1/n, \ldots, 1/n) \in S$. You may assume the validity of Liaponoff's Theorem.

(d) Interpret part (c) to obtain the following result, indicating the technical meanings of the terms in quotation marks.

Given a cake with n ingredients (e.g., butter, sugar, chocolate, garlic, etc.), each nonatomic and of unit mass and mixed together in any "reasonable" way, it is possible to "cut the cake into n pieces" such that each of the pieces contains its "share" of each of the ingredients.

2:13.9$\diamondsuit$ Show that there exists a set $E \subset [0,1]$ such that, for every open interval $I \subset [0,1]$, $\lambda(I \cap E) > 0$ and $\lambda(I \setminus E) > 0$.

2:13.10 Let $\{E_n\}$ be a sequence of measurable sets in a measure space $(X, \mathcal{M}, \mu)$ with each $0 < \mu(E_n) < \infty$. When is it generally possible to select a set $A \in \mathcal{M}$ with each $\mu(A \cap E_n) > 0$ and each $\mu(E_n \setminus A) > 0$?

2:13.11 Let K be the Cantor set. Each point $x \in K$ has a unique ternary expansion of the form

$$x = .a_1 a_2 a_3 \ldots \qquad (a_i = 0 \text{ or } a_i = 2, \quad i \in \mathbb{N}).$$

Let $b_i = a_i/2$ and let $f(x) = .b_1 b_2 b_3 \ldots$, interpreted in base 2. For example, if $x = \frac{2}{9} = 0.0200\ldots$ (base 3), then we would have $f(x) = \frac{1}{4} = 0.0100\ldots$ (base 2). Show that if f is extended to be linear and continuous on the closure of each interval complementary to K, then the the extended function $\overline{f}$ is continuous on $[0,1]$. Determine the relationship of this function $\overline{f}$ to the Cantor function (Exercise 1:22.13).

2:13.12 Let $X = [0,1]$ and let $\tau = \lambda^*$. In each case apply Method I to the family $\mathcal{T}$ and determine μ^* and $\mathcal{M}$. How do things change if $\tau = \lambda_*$ in part (f)?

(a) $\mathcal{T}$ consists of $\emptyset$ and $[0,1]$.

(b) $\mathcal{T}$ consists of $\emptyset$ and the family of all open subintervals.

(c) $\mathcal{T}$ consists of $\emptyset$ and all nondegenerate subintervals.

(d) $\mathcal{T}$ is $\mathcal{B}$.

(e) $\mathcal{T}$ is $\mathcal{L}$.

(f) $\mathcal{T}$ is 2^X.

[Hint for (f): The nonmeasurable set A discussed in Section 1.10 has $\lambda_*(A) = 0$.]

2:13.13$\diamondsuit$ Show that every set $E \subset \mathbb{R}$ with $\lambda^*(E) > 0$ contains a set that is nonmeasurable. [Hint: Let $E \subset [-\frac{1}{2}, \frac{1}{2}]$, and let $E_k = E \cap A_k$, where $\{A_k\}$ is the family of sets appearing in our proof in Section 1.10 of the existence of sets in $\mathbb{R}$ that are not Lebesgue measurable.]

2:13.14 Suppose that μ^* is the outer measure on X obtained by Method I from $\mathcal{T}$ and τ, and suppose that μ_1^* is any other outer measure on X satisfying $\mu_1^*(T) \leq \tau(T)$ for all $T \in \mathcal{T}$. Prove that $\mu_1^* \leq \mu^*$. Give an example for which $\mu_1^*(T) = \tau(T)$ for all $T \in \mathcal{T}$ and $\mu_1^* \neq \mu^*$. [Hint: Let $\mathcal{T} = \{\emptyset, [0,1]\}$ and $\mu_1^* = \lambda^*$.]

2:13.15$\diamondsuit$ Let $\mathcal{T}$ be a covering family, and let τ_1 and τ_2 be nonnegative functions on $\mathcal{T}$. Let μ_1^* and μ_2^* be the associated Method I outer measures. Prove that if $\mu_1^*(T) = \mu_2^*(T)$ for all $T \in \mathcal{T}$ then $\mu_1^* = \mu_2^*$.

2:13.16 Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$, and suppose that $\mu(M) > 0$ for each nonempty $M \in \mathcal{M}$. For each $x \in X$, let

$$\alpha(x) = \inf\{\mu(E) : E \in \mathcal{M}, x \in E\}.$$

(a) Show that there is a set $A_x \in \mathcal{M}$ such that $x \in A_x$ and $\mu(A_x) = \alpha(x)$.

(b) Prove that the sets $\{A_x\}$ are either disjoint or identical.

Chapter 3

METRIC OUTER MEASURES

In Chapter 2 we studied the basic abstract structure of a measure space. The only ingredients are a set X, a σ–algebra of subsets of X, and a measure defined on the σ–algebra. In almost all cases the set X will have some other structure that is of interest. Our example of Lebesgue measure on the real line illustrates this well. While $(\mathbb{R}, \mathcal{L}, \lambda)$ is a measure space, we should remember that $\mathbb{R}$ also has a great deal of other structure and that this measure space is influenced by that other structure. For instance $\mathbb{R}$ is linearly ordered, is a metric space, and also has a number of algebraic structures. Lebesgue measure, naturally, interacts with each of these.

In this chapter we study measures in a general metric space. As it happens, the only measures that are of any genuine interest are those that interact with the metric structure in a consistent way. In Section 3.2 we introduce the concepts of metric outer measure and Borel measure, which capture this interaction in the most convenient and useful way. In Section 3.3 we give an extension of the Method I construction that allows us to obtain metric outer measures. Section 3.4 explores how the measure of sets in a metric space can be approximated by the measure of less complicated sets, notably open sets or closed sets or simple Borel sets. The remaining sections develop some applications of the theory to important special measures, the Lebesgue–Stieltjes measures on the real line and Lebesgue–Stieltjes measures and Hausdorff measures in $\mathbb{R}^n$.

We begin with a brief review of metric space theory. In this chapter, only the most rudimentary properties of a metric space need be used. Even so the reader will feel more comfortable in the ensuing discussion after obtaining some familiarity with the concepts. A full treatment of metric spaces begins in Chapter 9. Some readers may prefer to gain some expertise in that general theory before studying measures on metric spaces. Abstract theories, such as metric spaces, allow for deep and subtle generalizations. But one can also view them as simplifications in that they permit one to focus on essentials of the structure.

3.1 Metric Space

Sequence limits in $\mathbb{R}$ are defined using the metric

$$\rho(x,y) = |x-y| \quad (x,y \in \mathbb{R})$$

which describes distances between pairs of points in $\mathbb{R}$. In higher dimensions one develops a similar theory, but using for distance the familiar expression

$$\rho(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \quad (x,y \in \mathbb{R}^n).$$

The only properties of these distance functions that are needed to develop an adequate theory in an abstract setting are those we have listed in Section 1.1. We can take these as forming our definition.

Definition 3.1 Let X be a set and let $\rho : X \times X \to \mathbb{R}$. If ρ satisfies the following conditions, then we say ρ is a *metric* on X and call the pair (X,ρ) a *metric space*.

1. $\rho(x,y) \geq 0$ for all $x,y \in X$.
2. $\rho(x,y) = 0$ if and only if $x = y$.
3. $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$.
4. $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x,y,z \in X$ (triangle inequality).

A metric space is a pair (X,ρ), where X is a set equipped with a metric ρ; in many cases one simply says that X is a metric space when the context makes it clear what metric is to be used. Sequence convergence in a metric space (X,ρ) means convergence relative to this distance. Thus $x_n \to x$ means that $\rho(x_n,x) \to 0$. The role that intervals on the real line play is assumed in an abstract metric space by the analogous notion of *an open ball*; that is, a set of the form

$$B(x_0,\varepsilon) = \{x : \rho(x,x_0) < \varepsilon\},$$

which can be thought of as the interior of a sphere centered at x_0 and with radius ε; avoid, however, too much geometric intuition, since "spheres" are not "round" and do not have the kind of closure properties that one may expect.

The language of metric space theory is just an extension of that for real numbers. Throughout (X,ρ) is a fixed metric space. For this chapter we need to understand the notions of diameter, open sets, and closed sets.

- For $x_0 \in X$ and $r > 0$, the set
$$B(x_0, r) = \{x \in X : \rho(x_0, x) < r\}$$
is called the *open ball* with center x_0 and radius r.
- For $x_0 \in X$ and $r > 0$, the set
$$B[x_0, r] = \{x \in X : \rho(x_0, x) \le r\}$$
is called the *closed ball* with center x_0 and radius r.
- A set $G \subset X$ is called *open* if for each $x_0 \in G$ there exists $r > 0$ such that $B(x_0, r) \subset G$.
- A set F is called *closed* if its complement $\widetilde{F}$ is open.
- A set is *bounded* if it is contained in some open ball.
- A *neighborhood* of x_0 is any open set G containing x_0.
- If $G = B(x_0, \varepsilon)$, we call G the ε-*neighborhood* of x_0.
- The point x_0 is called an *interior point* of a set A if x_0 has a neighborhood contained in A.
- The *interior* of A consists of all interior points of A and is denoted by A^o or, occasionally, $\mathrm{int}(A)$. It is the largest open set contained in A; it might be empty.
- A point $x_0 \in X$ is a *limit point* or *point of accumulation* of a set A if every neighborhood of x_0 contains points of A distinct from x_0.
- The *closure*, $\overline{A}$, of a set A consists of all points that are either in A or limit points of A. (It is the smallest closed set containing A.) One verifies easily that $x_0 \in \overline{A}$ if and only if there exists a sequence $\{x_n\}$ of points in A such that $x_n \to x_0$.
- A *boundary point* of A is a point x_0 such that every neighborhood of x_0 contains points of A as well as points of $\widetilde{A}$.
- The *diameter* of a set $E \subset X$ is defined as
$$\text{diameter}\,(E) = \sup\{\rho(x, y) : x, y \in E\}.$$
[We shall take diameter $(\emptyset) = 0$].
- An *isolated point* of a set is a member of the set that is not a limit point of the set.
- A set is *perfect* if it is nonempty, closed, and has no isolated points.
- A set $E \subset X$ is *dense* in a set $E_0 \subset X$ if every point in E_0 is a limit point of the set E.

- The *distance* between a point $x \in X$ and a nonempty set $A \subset X$ is defined as
$$\operatorname{dist}(x, A) = \inf\{\rho(x, y) : y \in A\}.$$
- The *distance* between two nonempty sets A, $B \subset X$ is defined as
$$\operatorname{dist}(A, B) = \inf\{\rho(x, y) : x \in A,\ y \in B\}.$$
- Two nonempty sets A, $B \subset X$ are said to be *separated* if they are a positive distance apart [i.e., if $\operatorname{dist}(A, B) > 0$].

The last three of these notions play an important role in the discussion in Section 3.2, where they are discussed in more detail. Here we should note that "dist" is not itself a metric on the subsets of X since the second condition of Definition 3.1 is violated if $A \cap B \neq \emptyset$ but $A \neq B$.

The *Borel sets* in a metric space are defined in the same manner as on the real line and have much the same properties. We shall use the following formal definition.

Definition 3.2 Let (X, ρ) be a metric space. The family of Borel subsets of (X, ρ) is the smallest σ–algebra that contains all the open sets in X.

It is convenient to have other expressions for the Borel sets. The family of Borel sets can be seen to be the smallest σ–algebra that contains all the closed sets in X. But for some applications we shall need the following characterization.

Theorem 3.3 *The family of Borel subsets of a metric space (X, ρ) is the smallest class $\mathcal{B}$ of subsets of X with the properties*

1. *If $E_1, E_2, E_3, \dots$ belong to $\mathcal{B}$, then so too does $\bigcup_{i=1}^{\infty} E_i$.*
2. *If $E_1, E_2, E_3, \dots$ belong to $\mathcal{B}$, then so too does $\bigcap_{i=1}^{\infty} E_i$.*
3. *$\mathcal{B}$ contains all the closed sets in X.*

We can also introduce the transfinite sequence of the Borel hierarchy
$$\mathcal{G} \subset \mathcal{G}_\delta \subset \mathcal{G}_{\delta\sigma} \subset \mathcal{G}_{\delta\sigma\delta} \subset \mathcal{G}_{\delta\sigma\delta\sigma} \dots$$
and
$$\mathcal{F} \subset \mathcal{F}_\sigma \subset \mathcal{F}_{\sigma\delta} \subset \mathcal{F}_{\sigma\delta\sigma} \subset \mathcal{F}_{\sigma\delta\sigma\delta} \dots,$$
just as we did in Section 1.12. Of these, we would normally not go beyond the second stage or perhaps the third stage in any of our applications.

Exercises

3:1.1 In a metric space every closed set is a $\mathcal{G}_\delta$.

3:1.2 In a metric space every open set is an $\mathcal{F}_\sigma$.

3:1.3 Prove Theorem 3.3.

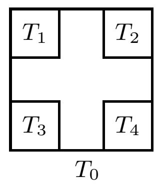

Figure 3.1: The square T_0.

3:1.4◊ Prove that the family of Borel subsets of X is the smallest class $\mathcal{C}$ of subsets of X with the following properties:

(a) If $E_1, E_2, E_3, \dots$ are disjoint and belong to $\mathcal{C}$, then so too does $\bigcup_{i=1}^{\infty} E_i$.

(b) If $E_1, E_2, E_3, \dots$ belong to $\mathcal{C}$, then so too does $\bigcap_{i=1}^{\infty} E_i$.

(c) $\mathcal{C}$ contains all the open sets in X. (This is true if $\mathcal{C}$ contains all the closed sets, but is harder to prove.)

3:1.5 A metric space (X, d) is said to be *separable* if there exists a countable subset of X that is dense in X. In a separable metric space, show that there are no more than $2^{\aleph_0}$ open sets and $2^{\aleph_0}$ closed sets.

3:1.6 In a separable metric space, show that there are no more than $2^{\aleph_0}$ Borel sets. [Hint: Use transfinite induction, the ideas of Section 1.12, and Exercise 3:1.5.]

3.2 Metric Outer Measures

We begin our discussion with an example of a Method I construction that produces a measure badly incompatible with the metric structure of $\mathbb{R}^2$. We use this to draw a number of conclusions. It will give us an insight into the conditions that we might wish to impose on measures defined on a metric space. It also gives us an important clue as to how Method I should be improved to recognize the metric structure.

Example 3.4 Take $X = \mathbb{R}^2$, let $\mathcal{T}$ be the family of open squares in X, and choose as a premeasure $\tau(T)$ to be the diameter of T. We apply Method I to obtain an outer measure μ^* and then a measure space $(\mathbb{R}^2, \mathcal{M}, \mu)$. What would we expect about the measurability of sets in $\mathcal{T}$? Since diameter is essentially a one-dimensional concept, while $\mathcal{T}$ consists of two-dimensional sets, perhaps we expect that every nonempty T has infinite measure.

Let $T_0 \in \mathcal{T}$ have side length 3, and let T_1, T_2, T_3 and T_4 be in $\mathcal{T}$, each with side length 1, and as shown in Figure 3.1. Then $\tau(T_0) = 3\sqrt{2}$, while $\tau(T_i) = \sqrt{2}$ for $i = 1, 2, 3, 4$. It is easy to verify that, for all $T \in \mathcal{T}$,

$\mu^*(T) = \tau(T)$ and that

$$\mu^*\left(\bigcup_{i=1}^{4} T_i\right) \le \mu^*(T_0) = 3\sqrt{2} < 4\sqrt{2} = \sum_{i=1}^{4} \mu^*(T_i).$$

It follows that none of the sets T_i, $i = 1, 2, 3, 4$, is measurable. A moment's reflection shows that no nonempty member of $\mathcal{T}$ can be measurable.

We note two significant features of this example.

1. The squares T_i are not only pairwise disjoint, but they are also separated from each other by positive distances: if $x \in T_i$, $y \in T_j$, and $i \ne j$, then the distance between x and y exceeds 1. As we saw, μ^* is not additive on these sets. Now we know outer measures are not additive in general, but for Lebesgue outer measure, if $\mu^*(A \cup B) \ne \mu^*(A) + \mu^*(B)$ and $A \cap B = \emptyset$, then the sets A and B are badly intertwined, not separated.

2. The class $\mathcal{M}$ of measurable sets is incompatible with the topology on $\mathbb{R}^2$: open sets need not be measurable.

Indeed, these two features, we shall soon discover, are intimately linked. If we wish open sets to be measurable, we must have an outer measure which is additive on separated sets, and conversely. We take the latter requirement as our definition of a metric outer measure. Recall that in a metric space we use

$$\operatorname{dist}(A, B) = \inf\{\rho(x, y) : x \in A \text{ and } y \in B\}$$

as a measure of the distance between two sets A and B. When $A = \{x\}$, we write $\operatorname{dist}(x, B)$ in place of $\operatorname{dist}(\{x\}, B)$. Although we call $\operatorname{dist}(A, B)$ the distance between A and B, dist is not a metric on the subsets of X. Recall, too, that if $\operatorname{dist}(A, B) > 0$, then we say that A and B are *separated* sets. For example, the sets T_i appearing in Example 3.4 are pairwise separated; indeed, $\operatorname{dist}(T_i, T_j) \ge 1$ if $i \ne j$.

Definition 3.5 Let μ^* be an outer measure on a metric space X. If

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

whenever A and B are separated subsets of X, then μ^* is called a *metric outer measure.*

Thus metric outer measures are designed to avoid the unpleasant possibility (1) that we observed for the Method I outer measure μ^* in our example. In Theorem 3.7 we show that the second unpleasant possibility of our example cannot occur: Borel sets will always be measurable for metric outer measures. We begin with a lemma due to Carathéodory.

Lemma 3.6 *Let μ^* be a metric outer measure on X. Let G be a proper open subset of X, and let $A \subset G$. Let*

$$A_n = \left\{ x \in A : \operatorname{dist}(x, \widetilde{G}) \geq 1/n \right\}.$$

Then

$$\mu^*(A) = \lim_{n\to\infty} \mu^*(A_n).$$

Proof. Recall that $\widetilde{G}$ denotes the set complementary to G, which in this case must be closed since G is open. The existence of the limit follows from the monotonicity of μ^* and the fact that $\{A_n\}$ is an expanding sequence of sets. Since $A_n \subset A$ for all $n \in \mathbb{N}$, $\mu^*(A) \geq \lim_{n\to\infty} \mu^*(A_n)$. It remains to verify that

$$\mu^*(A) \leq \lim_{n\to\infty} \mu^*(A_n).$$

Since G is open, $\operatorname{dist}(x, \widetilde{G}) > 0$ for all $x \in A$, so there exists $n \in \mathbb{N}$ such that $x \in A_n$. It follows that $A = \bigcup_{n=1}^{\infty} A_n$.

For each n, let

$$B_n = A_{n+1} \setminus A_n = \left\{ x : \frac{1}{n+1} \leq \operatorname{dist}(x, \widetilde{G}) < \frac{1}{n} \right\}.$$

Then

$$A = A_{2n} \cup \bigcup_{k=2n}^{\infty} B_k = A_{2n} \cup \bigcup_{k=n}^{\infty} B_{2k} \cup \bigcup_{k=n}^{\infty} B_{2k+1}.$$

Thus

$$\mu^*(A) \leq \mu^*(A_{2n}) + \sum_{k=n}^{\infty} \mu^*(B_{2k}) + \sum_{k=n}^{\infty} \mu^*(B_{2k+1}).$$

If the series are convergent, then

$$\mu^*(A) \leq \lim_{n\to\infty} \mu^*(A_{2n}) = \lim_{n\to\infty} \mu^*(A_n),$$

as was to be proved.

The argument to this point is valid for any outer measure. We now invoke our hypothesis that μ^* is a metric outer measure. Suppose that one of the series diverges, say

$$\sum_{k=1}^{\infty} \mu^*(B_{2k}) = \infty. \tag{1}$$

It follows from the definition of the sets B_k that, for each $k \in \mathbb{N}$,

$$\operatorname{dist}(B_{2k}, B_{2k+2}) \geq \frac{1}{2k+1} - \frac{1}{2k+2} > 0,$$

so these sets are separated. Thus

$$\mu^*\left(\bigcup_{k=1}^{n-1} B_{2k}\right) = \sum_{k=1}^{n-1} \mu^*(B_{2k}). \tag{2}$$

But $A_{2n} \supset \bigcup_{k=1}^{n-1} B_{2k}$, so

$$\mu^*(A_{2n}) \geq \mu^*\left(\bigcup_{k=1}^{n-1} B_{2k}\right). \tag{3}$$

Combining (2) and (3), we see that

$$\mu^*(A_{2n}) \geq \sum_{k=1}^{n-1} \mu^*(B_{2k}).$$

It follows from our assumption (1) that $\lim_{n\to\infty} \mu^*(A_{2n}) = \infty$, so

$$\lim_{n\to\infty} \mu^*(A_n) \geq \mu^*(A).$$

Finally, if it is the series $\sum_{k=1}^{\infty} \mu^*(B_{2k+1})$ that diverges, the argument is similar. We omit the details. ■

Theorem 3.7 *Let μ^* be an outer measure on a metric space X. Then every Borel set in X is measurable if and only if μ^* is a metric outer measure.*

Proof. Assume first that μ^* is a metric outer measure. Since the class of Borel sets is the σ-algebra generated by the closed sets, it suffices to verify that every closed set is measurable. Let F be a nonempty closed set and let $G = \widetilde{F}$. Then G is open. We show that F satisfies the measurability condition of Definition 2.29. Let $E \subset X$, let $A = E \setminus F$, and let $\{A_n\}$ be the sequence of sets appearing in Lemma 3.6. Then $\operatorname{dist}(A_n, F) \geq 1/n$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \mu^*(A_n) = \mu^*(E \setminus F). \tag{4}$$

Since μ^* is a metric outer measure and the sets A_n are separated from F, we have, for each $n \in \mathbb{N}$,

$$\mu^*(E) \geq \mu^*((E \cap F) \cup A_n) = \mu^*(E \cap F) + \mu^*(A_n).$$

From (4) we see that

$$\mu^*(E) \geq \mu^*(E \cap F) + \mu^*(E \setminus F).$$

The reverse inequality is obvious. Thus F is measurable.

To prove the converse, assume that all Borel sets are measurable. Let A_1 and A_2 be separated sets, say $\text{dist}(A_1, A_2) = \gamma > 0$. For each $x \in A_1$, let

$$G(x) = \{z : \rho(x, z) < \gamma/2\},$$

and let

$$G = \bigcup_{x \in A_1} G(x).$$

Then G is open, $A_1 \subset G$, and $G \cap A_2 = \emptyset$. Since G is measurable, it satisfies the measurability condition of Definition 2.29 for the set $E = A_1 \cup A_2$; that is,

$$\mu^*(A_1 \cup A_2) = \mu^*((A_1 \cup A_2) \cap G) + \mu^*((A_1 \cup A_2) \cap \widetilde{G}). \tag{5}$$

But $A_1 \subset G$ and $G \cap A_2 = \emptyset$, so $(A_1 \cup A_2) \cap G = A_1$ and

$$(A_1 \cup A_2) \cap \widetilde{G} = A_2,$$

and (5) becomes

$$\mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*(A_2),$$

as was to be shown. ■

Theorem 3.7 shows that metric outer measures give rise to *Borel measures*, that is, measures for which every Borel set is measurable. This does not rule out the possibility that there exist measurable sets that are not Borel sets. Some authors reserve the term Borel measure for a measure satisfying rather more. For example, one might wish compact sets to have finite measure or one might demand further approximation properties. The term Radon measure is also used in this context to denote Borel measures with special properties relative to the compact sets.

Exercises

3:2.1 Let us try to fix the problems that arose in connection with Example 3.4 that began this section. Let $\mathcal{T}$ be the family of half-open squares in $(0,1] \times (0,1]$ of the form $(a,b] \times (c,d]$, $b - a = d - c$, together with $\emptyset$, and let $\tau(T)$ be the diameter of T. Do the finite unions of elements of $\mathcal{T}$ form an algebra of sets? Can τ be extended to the algebra generated by $\mathcal{T}$ so as to be additive on this algebra? Can we use Theorem 2.40 effectively?

3:2.2 Let $X = \mathbb{R}^2$, let $\mathcal{T}$ consist of the half-open *intervals*

$$T = (a,b] \times (c,d]$$

in X, and let $\tau(T)$ be the area of T. Let μ^* be obtained from $\mathcal{T}$ and τ by Method I. Prove that μ^* is a metric outer measure. The resulting measure is called *two-dimensional Lebesgue measure.*

3.3 Method II

As we have seen, the Method I construction applied in a metric space can fail to produce a metric outer measure. We now seek to modify Method I in such a manner so as to guarantee that the resulting outer measure is metric. The modified construction will be called Method II.

Let us return to Example 3.4 involving squares in $\mathbb{R}^2$, with $\tau(T)$ the diameter of the square T. To obtain $\mu^*(T)$, we observe we can do no better than to cover T with itself. If, for example, we cover a square T of side length 1 with smaller squares, say ones of diameter no greater than $1/n$, we find that we need more than n^2 squares to do the job, and the estimate for $\mu^*(T)$ obtained from these squares exceeds $n\sqrt{2}$. The smaller the squares we use in the cover of T, the larger the estimate for $\mu^*(T)$. We do best by simply taking one square, T, for the cover. Thus the small squares are irrelevant and play no role in the construction, and yet it is precisely these that should have an influence on the size of the measure. This is the source of our problem. We now present a new method for obtaining measures from outer measures that explicitly addresses this by forcing the sets of small diameter to be taken into account.

Let $\mathcal{T}$ be a covering family on a metric space X. For each $n \in \mathbb{N}$, let

$$\mathcal{T}_n = \{T \in \mathcal{T} : \text{diameter}\,(T) \leq 1/n\}\,.$$

Then $\mathcal{T}_n$ is also a covering family for X for each $n \in \mathbb{N}$. Let τ be a premeasure defined on the family $\mathcal{T}$. For every $n \in \mathbb{N}$, we construct μ_n^* by Method I from $\mathcal{T}_n$ and τ. Since $\mathcal{T}_{n+1} \subset \mathcal{T}_n$,

$$\mu_{n+1}^*(E) \geq \mu_n^*(E)$$

for all $n \in \mathbb{N}$ and for each $E \subset X$. Thus the sequence $\{\mu_n^*(E)\}$ approaches a finite or infinite limit. We define μ_0^* as $\lim_{n\to\infty} \mu_n^*$ and refer to this as the outer measure determined by Method II from τ and $\mathcal{T}$.

Theorem 3.8 shows that this process always gives rise to a metric outer measure.

Theorem 3.8 *Let μ_0^* be the measure determined by Method II from a premeasure τ and a family $\mathcal{T}$. Then μ_0^* is a metric outer measure.*

Proof. We first show that μ_0^* is an outer measure. That $\mu_0^*(\emptyset) = 0$, and that $\mu_0^*(A) \leq \mu_0^*(B)$ if $A \subset B$ are immediate. To verify that μ_0^* is countably subadditive, let $\{A_k\}$ be a sequence of subsets of X. Since $\mu_0^*(E) \geq \mu_n^*(E)$ for all $E \subset X$ and $n \in \mathbb{N}$, we have

$$\mu_n^*\left(\bigcup_{k=1}^\infty A_k\right) \leq \sum_{k=1}^\infty \mu_n^*(A_k) \leq \sum_{k=1}^\infty \mu_0^*(A_k).$$

Thus

$$\mu_0^*\left(\bigcup_{k=1}^\infty A_k\right) = \lim_{n\to\infty} \mu_n^*\left(\bigcup_{k=1}^\infty A_k\right) \leq \sum_{k=1}^\infty \mu_0^*(A_k).$$

This verifies that μ_0^* is an outer measure.

It remains to show that if A and B are separated then

$$\mu_0^*(A \cup B) = \mu_0^*(A) + \mu_0^*(B).$$

Certainly,

$$\mu_0^*(A \cup B) \leq \mu_0^*(A) + \mu_0^*(B),$$

and so it is enough to establish the opposite inequality. We may assume that $\mu_0^*(A \cup B)$ is finite. Suppose then that $\operatorname{dist}(A, B) > 0$. Choose $N \in \mathbb{N}$ such that $\operatorname{dist}(A, B) > 1/N$. Let $\varepsilon > 0$. For every $n \in \mathbb{N}$ there exists a sequence $\{T_{nk}\}$ from $\mathcal{T}_n$ such that $\bigcup_{k=1}^{\infty} T_{nk} \supset A \cup B$ and

$$\sum_{k=1}^{\infty} \tau(T_{nk}) \leq \mu_n^*(A \cup B) + \varepsilon.$$

Then, for $n \geq N$ and $k \in \mathbb{N}$, no set T_{nk} can meet both A and B and hence $T_{nk} \cap A = \emptyset$ or else $T_{nk} \cap B = \emptyset$. Let

$$\mathbb{N}_1 = \{k \in \mathbb{N} : T_{nk} \cap A \neq \emptyset\}$$

and

$$\mathbb{N}_2 = \{k \in \mathbb{N} : T_{nk} \cap B \neq \emptyset\}.$$

Then

$$\mu_n^*(A) \leq \sum_{k \in \mathbb{N}_1} \tau(T_{nk})$$

and

$$\mu_n^*(B) \leq \sum_{k \in \mathbb{N}_2} \tau(T_{nk}).$$

Therefore,

$$\mu_n^*(A) + \mu_n^*(B) \leq \sum_{k=1}^{\infty} \tau(T_{nk}) \leq \mu_n^*(A \cup B) + \varepsilon.$$

Since this is true for every $\varepsilon > 0$, we have, for $n \geq N$,

$$\mu_n^*(A) + \mu_n^*(B) \leq \mu_n^*(A \cup B).$$

Because this holds for all $n \geq N$, $\mu_0^*(A) + \mu_0^*(B) \leq \mu_0^*(A \cup B)$. Thus μ_0^* is a metric outer measure. ■

Let us return to Example 3.4. Our previous discussion involving covers of a square T with smaller squares suggests that $\mu_0^*(T) = \infty$ for every square T. This is, in fact, the case. If T is an open square with unit side length, $\mu_n^*(T) = n\sqrt{2}$. Thus

$$\mu_0^*(T) = \lim_{n\to\infty} \mu_n^*(T) = \infty.$$

A similar argument shows that $\mu_0^*(T) = \infty$ for all $T \in \mathcal{T}$. This may be no surprise since we have used a "one-dimensional" concept (diameter) as a premeasure for a two-dimensional set T. Recall that the Method I outer measure μ^* had $\mu^*(T) = \tau(T)$, since we could efficiently cover T by itself. In this example, small squares cannot cover large squares efficiently, and the Method I outcome differs from that of Method II. Our next result, Theorem 3.9, shows that if "small squares *can* cover large squares efficiently" then the Method I and Method II measures do agree.

Theorem 3.9 *Let μ_0^* be the measure determined by Method II from a premeasure τ and a family $\mathcal{T}$ and let μ^* be the Method I measure constructed from τ and $\mathcal{T}$. A necessary and sufficient condition that $\mu_0^* = \mu^*$ is that for each choice of $\varepsilon > 0$, $T \in \mathcal{T}$, and $n \in \mathbb{N}$, there is a sequence $\{T_k\}$ from $\mathcal{T}_n$ such that $T \subset \bigcup_{k=1}^\infty T_k$ and*

$$\sum_{k=1}^\infty \tau(T_k) \le \tau(T) + \varepsilon.$$

Proof. Necessity is clear. If the condition fails for some ε, T, and n, then $\mu_0^*(T) > \mu^*(T)$. To prove sufficiency, observe first that, since $\mathcal{T}_n \subset \mathcal{T}$ for all $n \in \mathbb{N}$,

$$\mu^* \le \mu_n^* \le \mu_0^*. \tag{6}$$

To verify the reverse inequality, let $A \subset X$ and let $\varepsilon > 0$. We may assume that $\mu^*(A) < \infty$. Let $\{T_i\}$ be a sequence of sets from $\mathcal{T}$ such that $A \subset \bigcup_{i=1} T_i$ and

$$\sum_{i=1}^\infty \tau(T_i) \le \mu^*(A) + \frac{\varepsilon}{2}. \tag{7}$$

Let $n \in \mathbb{N}$. Using our hypotheses, we have, for each $i \in \mathbb{N}$, a sequence $\{S_{ik}\}$ of sets from $\mathcal{T}_n$ covering T_i such that

$$\sum_{k=1}^\infty \tau(S_{ik}) \le \tau(T_i) + \frac{\varepsilon}{2^{i+1}}. \tag{8}$$

Now $A \subset \bigcup_{i=1}^\infty \bigcup_{k=1}^\infty S_{ik}$, so by (7) and (8) we have

$$\mu_n^*(A) \le \sum_{i=1}^\infty \sum_{k=1}^\infty \tau(S_{ik}) \le \sum_{i=1}^\infty \left[\tau(T_i) + \frac{\varepsilon}{2^{i+1}}\right] \le \mu^*(A) + \varepsilon.$$

Since ε is arbitrary, $\mu_n^*(A) \le \mu^*(A)$. This is true for every $n \in \mathbb{N}$, so

$$\mu_0^*(A) = \lim_{n\to\infty} \mu_n^*(A) \le \mu^*(A). \tag{9}$$

From (6) and (9), we see that $\mu^* = \mu_0^*$. ■

Corollary 3.10 *Under the hypotheses of Theorem 3.9, Method I results in a metric outer measure.*

Method II also has a regularity result identical to Theorem 2.36. We leave the details as Exercise 3:3.4.

Theorem 3.11 *Let μ_0^* be constructed from $\mathcal{T}$ and τ by Method II. If all members of $\mathcal{T}$ are measurable, then μ_0^* is regular. In particular, if each $T \in \mathcal{T}$ is an open set, the measurable covers can be chosen to be Borel sets of type $\mathcal{G}_\delta$.*

Exercises

3:3.1 In the proof of Theorem 3.8, verify that $\mu_0^*(\emptyset) = 0$ and $\mu_0^*(A) \le \mu_0^*(B)$ if $A \subset B$.

3:3.2 Let $\mathcal{T}$ consist of $\emptyset$ and the open intervals in $X = (-1, 1)$, and let $\tau((a,b)) = |b^2 - a^2|$. Apply Method I to obtain μ^* and Method II to obtain μ_0^*.

(a) Determine the class of μ^*-measurable sets.

(b) Calculate $\mu^*((0,1))$ and $\mu_0^*((0,1))$.

3:3.3 Let $X = \mathbb{R}$, $\mathcal{T}$ consist of $\emptyset$ and the open intervals in $\mathbb{R}$. Let $\tau(\emptyset) = 0$ and let $\tau((a,b)) = (b-a)^{-1}$ for all other $(a,b) \in \mathcal{T}$. Let μ_1 and μ_2 be the measures obtained from $\mathcal{T}$ and τ by Methods I and II, respectively.

(a) Show that $\mu_1(E) = 0$ for all $E \subset X$.

(b) Show that $\mu_2(E) = \infty$ for every nonempty set $E \subset X$.

Note $\tau(T)$, $\mu_1(T)$, and $\mu_2(T)$ are all different in this example. While Method I always results in $\mu^*(T) \le \tau(T)$, this inequality is not valid in general when Method II is used. We had already seen this in our example with squares.

3:3.4 Prove Theorem 3.11.

3:3.5 Verify that in Theorem 3.11, if we do not assume that the sets in $\mathcal{T}$ are measurable, we can still conclude that each set $A \subset X$ with finite measure has a cover in $\mathcal{T}_{\sigma\delta}$. (Compare with Exercise 2:9.8.)

3.4 Approximations

In most settings the measure of a measurable set can be approximated from inside or outside by simpler sets, perhaps open sets or $\mathcal{G}_\delta$ sets, as we were able to do on $\mathbb{R}$ with Lebesgue measure. By the use of Theorems 2.35 and 3.11, one can obtain such approximations from sets that were used in the first place to construct the measure. The approximation theorem that follows is of a different sort, however, in that it does not involve Methods I or II, or outer measures. We show how to approximate the measure of any Borel set first from inside by closed sets and then from outside by open sets for any Borel measure. Recall that for μ to be a Borel measure requires merely that μ be a measure whose σ–algebra of measurable sets includes all Borel sets.

Theorem 3.12 *Let X be a metric space, μ a Borel measure on X, $\varepsilon > 0$ and B a Borel set with $\mu(B) < \infty$. Then B contains a closed set F with $\mu(B \setminus F) < \varepsilon$.*

Proof. We may assume that $\mu(X) < \infty$. Let $\mathcal{E}$ consist of those sets $E \subset X$ that have the property that for any $\gamma > 0$ there is a closed subset K of E for which $\mu(E \setminus K) < \gamma$. We claim that every Borel set $B \subset X$ is a member of $\mathcal{E}$ and the theorem follows. We show that $\mathcal{E}$ contains the closed sets and that it is closed under countable unions and closed under countable intersections. By Theorem 3.3, it follows that $\mathcal{E}$ must contain all the Borel sets.

It is clear that $\mathcal{E}$ contains the closed sets. Suppose now that E_1, E_2, …belong to $\mathcal{E}$. There must exist closed sets $K_i \subset E_i$ with $\mu(E_i \setminus K_i) < \varepsilon 2^{-i}$. We get immediately that

$$\mu\left(\bigcap_{i=1}^{\infty} E_i \setminus \bigcap_{i=1}^{\infty} K_i\right) \leq \mu\left(\bigcup_{i=1}^{\infty} (E_i \setminus K_i)\right) < \sum_{i=1}^{\infty} \varepsilon 2^{-i} = \varepsilon.$$

Since $\bigcap_{i=1}^{\infty} K_i$ is a closed subset of $\bigcap_{i=1}^{\infty} E_i$, we see that the intersection of the sequence $\{E_i\}$ belongs to $\mathcal{E}$.

The union can be handled similarly but requires an extra step, since countable unions of closed sets are not necessarily closed. Note that

$$\lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{\infty} E_i \setminus \bigcup_{i=1}^{n} K_i\right) = \mu\left(\bigcup_{i=1}^{\infty} E_i \setminus \bigcup_{i=1}^{\infty} K_i\right)$$
$$\leq \mu\left(\bigcup_{i=1}^{\infty} (E_i \setminus K_i)\right) < \sum_{i=1}^{\infty} \varepsilon 2^{-i} = \varepsilon.$$

(It is here that we are using the finiteness assumption, since to invoke the limit requires Theorem 2.20.) Thus, for sufficiently large n, we must have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i \setminus \bigcup_{i=1}^{n} K_i\right) < \varepsilon,$$

and this set, $\bigcup_{i=1}^{n} K_i$, is a closed subset of $\bigcup_{i=1}^{\infty} E_i$. ■

We mention that the discussion following Theorem 3.19 will show that the σ-algebra $\mathcal{E}$ in the proof just given need not consist of all measurable sets. See also Exercise 3:6.3. We now turn to the approximation from the outside by open sets.

Theorem 3.13 *Let X be a metric space, μ a Borel measure on X, $\varepsilon > 0$, and B a Borel set. If $\mu(X) < \infty$ or, more generally, if B is contained in the union of countably many open sets O_i each of finite μ-measure, then B is contained in an open set G with $\mu(G \setminus B) < \varepsilon$.*

Proof. This theorem follows from the preceding. Choose each closed set $C_i \subset O_i \setminus B$ in such a way that

$$\mu\left((O_i \setminus C_i) \setminus B\right) = \mu\left((O_i \setminus B) \setminus C_i\right) < \varepsilon 2^{-i}.$$

Here $B \cap O_i$ is a subset of the open set $O_i \setminus C_i$. Define

$$G = \bigcup_{i=1}^{\infty} (O_i \setminus C_i).$$

Then G is open, G contains B, and $\mu(G \setminus B) < \varepsilon$. ■

For reference let us put the two theorems together to derive a corollary, valid in spaces of finite measure.

Corollary 3.14 *Let X be a metric space and μ a Borel measure with $\mu(X) < \infty$. For every $\varepsilon > 0$ and every Borel set B, there is a closed set F and an open set G such that*

$$F \subset B \subset G,$$

with

$$\mu(B) - \varepsilon < \mu(F) \le \mu(B) \le \mu(G) < \mu(B) + \varepsilon.$$

From these two theorems we easily derive an approximation theorem using slightly larger classes of sets than the open and closed sets.

Theorem 3.15 *Let X be a metric space, and μ a Borel measure on X such that $\mu(X) < \infty$. Then every Borel set $B \subset X$ has a subset K of type $\mathcal{F}_\sigma$ and a superset H of type $\mathcal{G}_\delta$, such that*

$$\mu(K) = \mu(B) = \mu(H).$$

In terms of the language of Exercise 2:1.14, every Borel set in X has a measurable cover of type $\mathcal{G}_\delta$ and a measurable kernel of type $\mathcal{F}_\sigma$. The requirement that $\mu(X) < \infty$ in the statement of Theorem 3.15 cannot be dropped. See Exercise 3:4.3.

Corollary 3.14 and Theorem 3.15 involve approximations of Borel sets by simpler sets. If we know that measurable sets can be approximated by Borel sets, then the conclusions of 3.14 and 3.15 can be sharpened. For example, under the hypotheses of Theorem 3.11, if $\mathcal{T}$ consists of Borel sets, every measurable set M has a cover $H \in \mathcal{B}$. If $\mu(X) < \infty$, H has a cover H' of type $\mathcal{G}_\delta$. Thus H' is a cover for M as well. If one wished, one could combine the hypotheses of 3.11, 3.14, and 3.15 suitably to obtain various results concerning approximations of measurable sets by Borel sets, sets of type $\mathcal{G}_\delta$, open sets, and so on.

Exercises

3:4.1 Prove Theorem 3.13 in the simplest case where $\mu(X) < \infty$.

3:4.2 Prove Theorem 3.15.

3:4.3 Let $\mathcal{B}$ denote the Borel sets in $\mathbb{R}$. Recall that part of the Baire category theorem for $\mathbb{R}$ that asserts that a set of type $\mathcal{G}_\delta$ that is dense in some interval cannot be expressed as a countable union of nowhere dense sets. For $E \in \mathcal{B}$, let $\mu(E) = \lambda(E)$ if E is a countable union of nowhere dense sets, $\mu(E) = \infty$ otherwise. Show that $(\mathbb{R}, \mathcal{B}, \mu)$ is a measure space for which the conclusion of Theorem 3.15 fails.

3:4.4 Let μ be a finite Borel measure on a metric space X. Prove that, for every Borel set $B \subset X$,

$$\mu(B) = \inf \{\mu(G) : B \subset G,\, G \text{ open}\}$$

and

$$\mu(B) = \sup\{\mu(F) : F \subset B,\, F \text{ closed}\}.$$

3.5 Construction of Lebesgue–Stieltjes Measures

The most important class of Borel measures on $\mathbb{R}^n$ are those that are finite on bounded sets. Often these are called Lebesgue–Stieltjes measures after the Dutch mathematician, T. J. Stieltjes (1856–1894), whose integral (see Section 1.19) played a key role in the development of measure theory by J. Radon (1887–1956) in the second decade of this century. For the same reason, they have also been called Radon measures. Certain of the Hausdorff measures that we discuss in Section 3.8 are, in contrast, examples of important Borel measures that are infinite on *every* open set.

Lebesgue–Stieltjes measures are Borel measures in $\mathbb{R}^n$ that can serve to model mass distributions. Some previews can be found in Example 2.10 and Exercises 2:2.14, 2:8.2, and 2:9.7. We can now use the machinery we have developed to obtain such models rigorously and compatibly with our intuition. We consider the one-dimensional situation in detail here and then outline the construction for $\mathbb{R}^n$ in Section 3.7.

Suppose, for each $x \in \mathbb{R}$, that we know the mass of intervals of the form $(0,x]$ or of the form $(x,0]$ and that all such masses are finite. Let

$$f(x) = \begin{cases} \text{mass } (0,x], & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -\text{mass } (x,0], & \text{if } x < 0. \end{cases} \tag{10}$$

Then f is a nondecreasing function on $\mathbb{R}$. While f need not be continuous, we require f to be right continuous. Since monotonic functions have left and right limits at every point, this just fixes the value of f at its countably many points of discontinuity in a particular way.

We now carry out a program similar to the one we outlined in Exercise 2:12.4. Here we are dealing with intervals in $\mathbb{R}$, rather than in $\mathbb{R}^2$. Let $\mathcal{T}$ consist of the half-open intervals of the form $(a,b]$, the empty set, and the unbounded intervals of the form $(-\infty,b]$ and (a,∞). For a premeasure $\tau : \mathcal{T} \to [0,\infty]$, we shall use

$$\tau(T) = \begin{cases} 0, & \text{if } T = \emptyset; \\ f(b) - f(a), & \text{if } T = (a,b]; \\ f(b) - \lim_{a\to-\infty} f(a), & \text{if } T = (-\infty,b]; \\ \lim_{b\to\infty} f(b) - f(a), & \text{if } T = (a,\infty). \end{cases} \tag{11}$$

The limits involved exist, finite or infinite, because f is nondecreasing.

Continuing the program, we let $\mathcal{T}_1$ be the algebra generated by $\mathcal{T}$. One sees immediately that $\mathcal{T}_1$ consists of all finite unions of elements of $\mathcal{T}$. We wish to extend the premeasure τ to an additive function $\tau_1 : \mathcal{T}_1 \to [0,\infty]$. For $T \in \mathcal{T}_1$, write

$$T = T_1 \cup T_2 \cup \cdots \cup T_n,$$

with $T_i \in \mathcal{T}$ for each $i = 1,\ldots,n$, and $T_i \cap T_j = \emptyset$ if $i \neq j$. We "define"

$$\tau_1(T) = \tau(T_1) + \tau(T_2) + \cdots + \tau(T_n). \tag{12}$$

The quotes indicate that we must verify that (12) is unambiguous. (Recall our example of squares in Section 3.2 when τ was the diameter of the square.)

3.16 *The set function τ_1 is well defined on $\mathcal{T}_1$.*

Proof. Consider first the case that $T \in \mathcal{T}$. Let

$$T = (a,b] = \bigcup_{i=1}^{n} (a_i, b_i]$$

with $a_1 = a$, $b_n = b$, and $a_{i+1} = b_i$ for all $i = 1,\ldots,n-1$. Thus

$$\tau((a,b]) = f(b) - f(a) = \sum_{i=1}^{n} (f(b_i) - f(a_i)) = \sum_{i=1}^{n} \tau((a_i,b_i]).$$

A similar argument shows that if an unbounded interval $T \in \mathcal{T}$ is decomposed into finitely many members of $\mathcal{T}$ then (12) holds. Finally, any $T \in \mathcal{T}_1$ is a finite union of members of $\mathcal{T}$. These members can be appropriately combined, if necessary, to become a disjoint collection

$$\{(a_i, b_i]\}_{i=1}^{n} \text{ with } b_i < a_{i+1}. \tag{13}$$

Here it is possible that $a_1 = -\infty$ or $b_n = \infty$. Suppose that T is decomposed into a finite disjoint union of sets in $\mathcal{T}$, say $T = \bigcup_{j=1}^{m} T_j$. Let

$$A_i = \{j : T_j \subset (a_i, b_i]\}.$$

Then, $(a_i, b_i] = \bigcup_{j \in A_i} T_j$. We have already seen that, for all $i = 1, \ldots, n$,

$$\tau((a_i, b_i]) = \sum_{j \in A_i} \tau(T_j).$$

Since any representation of T as a finite disjoint union of members of $\mathcal{T}$ heads to the same collection (13), the sum in (12) does not depend on the representation for T. ■

Because of Theorem 2.40, we now know that an application of Method I would lead to a measure space in which every member of $\mathcal{T}$ is measurable. This implies that every Borel set is measurable. To see this, note that an open interval is a countable union of half-open intervals,

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b_n],$$

where $a < b_1 < b_2 < \cdots < b$ and $\lim_{n \to \infty} b_n = b$. It follows from Theorem 3.7 that μ^* is a metric outer measure. From Theorem 2.36 we see that μ^* is also regular and from Exercise 2:9.8 that each set $A \subset \mathbb{R}$ has a Borel set B as a measurable cover. It now follows readily from Theorem 3.15 that B can be taken to be of type $\mathcal{G}_\delta$ (left as Exercise 3:5.1). What we do not yet know is that the members of $\mathcal{T}_1$, or even of $\mathcal{T}$, have the right measure; that is, that $\mu^*(T) = \tau(T)$. To obtain this result, it suffices to show that τ_1 is σ-additive on $\mathcal{T}_1$. We can then invoke Theorem 2.42.

3.17 *The set function τ_1 is σ-additive on $\mathcal{T}_1$.*

Proof. To show that τ_1 is σ-additive on $\mathcal{T}_1$, we must show that, if $\{T_n\}$ is a sequence of pairwise disjoint sets in $\mathcal{T}_1$ whose union T is also in $\mathcal{T}_1$, then

$$\tau_1(T) = \sum_{n=1}^{\infty} \tau_1(T_n).$$

Observe that it is sufficient to consider only the case that T is a single interval $(a, b]$. For *finite* additivity, our work was simplified by the fact that if $(a,b] = \bigcup_{i=1}^{n}(a_i, b_i]$, with the sets $\{(a_i, b_i]\}$ pairwise disjoint,

$$f(b) - f(a) = \sum_{i=1}^{n}(f(b_i) - f(a_i)),$$

because the intervals must form a partition of $(a, b]$.

This telescoping of the sum is not always possible when dealing with an infinite decomposition of the form $(a,b] = \bigcup_{i=1}^{\infty}(a_i, b_i]$ with the sets $\{(a_i, b_i]\}$ pairwise disjoint. For example, consider

$$(-1, 1] = (-1, 0] \cup \bigcup_{n=1}^{\infty} \left((n+1)^{-1}, n^{-1}\right].$$

Here 0 is a right endpoint of an interval in the collection, but not a left endpoint of any other interval. It is still true that

$$f(1) - f(-1) = f(0) - f(-1) + \sum_{n=1}^{\infty} \left[f(n^{-1}) - f((n+1)^{-1})\right],$$

but this requires handling right-hand limits at 0. In general, if for some $i \in \mathbb{N}$, b_i is a limit point of the set $\{a_j\}_{j=1}^{\infty}$, then $b_i \neq a_j$ for any $j \in \mathbb{N}$. Thus we do not get the cancellations from which we benefited when we had telescoping sums. Moreover, there can be infinitely many points of this type to handle. Note that it is only the right endpoints that have this feature.

Let us look at the situation in some detail. Let $A = \{a_i\}$ and $B = \{b_i\}$. Then $A \subset B \cup \{a\}$, but B is not necessarily contained in A. A simple diagram can illustrate that $B \setminus A$ can be infinite. Now

$$[a, b] = \bigcup (a_k, b_k) \cup B \cup \{a\}.$$

It follows that $B \cup \{a\}$ is a countable closed set. Let $J_0 = [f(a), f(b)]$ and, for $k \in \mathbb{N}$, let $J_k = [f(a_k), f(b_k)]$. Since f is nondecreasing, $\bigcup_{k=1}^{\infty} J_k \subset J_0$, and the intervals J_k have no interior points in common. Because f is right continuous at $x = a$,

$$J_0 \subset \bigcup_{k=1}^{\infty} J_k \cup f(B) \cup \{f(a)\}.$$

B is countable, so $f(B)$ is also countable, and hence

$$\lambda(f(B) \cup \{f(a)\}) = 0,$$

where, as usual, λ denotes the Lebesgue measure. It follows that

$$\begin{aligned}\sum_{k=1}^{\infty}(f(b_k)-f(a_k)) &= \lambda\left(\bigcup_{k=1}^{\infty} J_k\right) \le \lambda(J_0)\\ &\le \lambda\left(\bigcup_{k=1}^{\infty} J_k \cup f(B)\cup\{f(a)\}\right)\\ &= \sum_{k=1}^{\infty}\lambda(J_k) = \sum_{k=1}^{\infty}(f(b_k)-f(a_k)).\end{aligned}$$

Thus $f(b)-f(a)=\lambda(J_0)=\sum_{k=1}^{\infty}(f(b_k)-f(a_k))$, so that

$$\tau_1((a,b]) = \sum_{k=1}^{\infty}\tau_1((a_k,b_k])$$

as required. ■

We have now completed the program. We can finally conclude that an application of Method I will give rise to an outer measure μ_f^* and then to a measure space $(X,\mathcal{M}_f,\mu_f)$ with

$$\mu_f((a,b]) = f(b)-f(a).$$

We call μ_f the *Lebesgue–Stieltjes measure with distribution function* f. We shall also use such phrases as μ_f is the measure "induced by" f or "associated with" f. Observe that for $c\in\mathbb{R}$ the function $f+c$ can also serve as a distribution function for μ_f. When dealing with finite Lebesgue–Stieltjes measures, it is often convenient to choose f so that $\lim_{x\to-\infty} f(x)=0$. Moreover, when all the measure is located in some interval I, it may be convenient merely to specify f only on I itself (as, for example, we do in Exercise 3:5.5). Technically, this amounts to extending f to all of $\mathbb{R}$ in such a way that $\mu_f(\mathbb{R}\setminus I)=0$. (Such an extension would be required for Exercise 3:11.5.)

Example 3.18 A *probability space* is a measure space of total measure 1. If $X=\mathbb{R}$, the distribution function can be chosen so that $\lim_{x\to-\infty} f(x)=0$ and will then satisfy $\lim_{x\to\infty} f(x)=1$. For a measurable set A, $\mu_f(A)$ represents the probability that a random variable lies in A. As a concrete example, if ϕ is the standard normal density (bell-shaped curve),

$$\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\qquad(-\infty<x<\infty),$$

then $\int_{-\infty}^{\infty}\phi(x)\,dx=1$, and one can take $f(x)=\int_{-\infty}^{x}\phi(t)\,dt$ as an associated distribution function.

In the setting of probability, the "mass" of a Borel set A is interpreted as the probability of the "event" A occurring. Thus the probability that a standard normal random variable Z satisfies $a < Z \le b$ is

$$\Pr(a < Z \le b) = f(b) - f(a) = \int_a^b \phi(x)\ dx.$$

More generally, for any Borel set A we would have

$$\Pr(Z \in A) = \mu_f(A) = \int_A \phi(x)\, dx,$$

where the integral must be interpreted in the Lebesgue sense. (We will have to wait until Chapter 5 for this.)

Exercises

3:5.1 Prove that, for any Lebesgue–Stieltjes measure μ, every $A \subset X$ has a measurable cover of type $\mathcal{G}_\delta$ and a measurable kernel of type $\mathcal{F}_\sigma$.

3:5.2 Use Theorems 3.8 and 3.9 to give another proof that a Lebesgue–Stieltjes outer measure μ_f^* is a metric outer measure.

3:5.3 Let

$$f(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } 0 \le x < 1; \\ 2, & \text{if } x \ge 1. \end{cases}$$

Show that $\mu_f((0,1)) < \mu_f((0,1]) < \mu_f([0,1])$.

3:5.4 Let $X = \mathbb{R}$ and

$$\mu(A) = \begin{cases} n, & \text{if card } A \cap \mathbb{N} = n; \\ \infty, & \text{if } A \cap \mathbb{N} \text{ is infinite.} \end{cases}$$

Construct a distribution function f such that $\mu_f = \mu$.

3:5.5 Let f be the Cantor function, and let μ_f be the associated Lebesgue–Stieltjes measure. Calculate $\mu_f((\frac{1}{3}, \frac{2}{3}))$ and $\mu_f((K \cap (\frac{2}{9}, \frac{1}{3}))$, where K is the Cantor ternary set.

3:5.6 Let μ_f be a Lebesgue–Stieltjes measure. Show that

$$\mu_f((a,b)) = \lim_{x \to b-} (f(x) - f(a))$$

and calculate $\mu_f(\{b\})$.

3:5.7 The term Lebesgue–Stieltjes measure is often used to apply to what would more properly be called "Lebesgue–Stieltjes signed measure." What should we mean by that term? Let

$$f(x) = \begin{cases} 1, & \text{if } x < -1; \\ x^2, & \text{if } -1 \le x \le 1; \\ 1, & \text{if } 1 < x. \end{cases}$$

Let μ_f be the associated Lebesgue–Stieltjes measure. Calculate the Jordan decomposition for the signed measure μ_f, and compute $\mu_f((-1,1))$ and $V(\mu_f,(-1,1))$. Note that functions of bounded variation give rise to Lebesgue–Stieltjes signed measures via their decomposition into a difference of two nondecreasing functions.

3:5.8◇ Let $(X,\mathcal{M},\mu)$ be a measure space. A set $A \in \mathcal{M}$ is called an *atom* if $\mu(A) > 0$ and for all measurable sets $B \subset A$, $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. (See Exercise 2:13.7.)

(a) Give an example of a space $(\mathbb{R},\mathcal{M},\mu)$ for which $[0,1]$ is an atom.

(b) Let $(\mathbb{R},\mathcal{M}_f,\mu_f)$ be a Lebesgue–Stieltjes measure space. Prove that, if A is an atom in this space, A contains a singleton atom with the same measure. That is, there exists $a \in A$ for which $\mu_f(\{a\}) = \mu_f(A)$. One also uses the term "point mass" to describe a singleton atom of μ_f.

(c) A measure μ is *nonatomic* if there are no atoms. Prove that a Lebesgue–Stieltjes measure is nonatomic if and only if its distribution function is continuous.

3.6 Properties of Lebesgue–Stieltjes Measures

We investigate now some of the important properties of Lebesgue–Stieltjes measures in one dimension. The first theorem provides a sense of the generality of such measures.

Theorem 3.19 *Let f be nondecreasing and right continuous on $\mathbb{R}$. Let μ_f^* be the associated Method I outer measure, and let $(\mathbb{R},\mathcal{M}_f,\mu_f)$ be the resulting measure space. Then*

1. *μ_f^* is a metric outer measure and thus all Borel sets are μ_f^*-measurable.*
2. *If A is a bounded Borel set, then $\mu_f(A) < \infty$.*
3. *Each set $A \subset \mathbb{R}$ has a measurable cover of type $\mathcal{G}_\delta$.*
4. *For every half-open interval $(a,b]$, $\mu_f((a,b]) = f(b) - f(a)$.*

Conversely, let μ^* be an outer measure on $\mathbb{R}$ with $(X,\mathcal{M},\mu)$ the resulting measure space. If conditions (1), (2), and (3) are satisfied by μ^* and μ, then there exists a nondecreasing, right-continuous function f defined on $\mathbb{R}$ such that $\mu_f^*(A) = \mu^*(A)$ for all $A \subset \mathbb{R}$. In particular, $\mu_f(A) = \mu(A)$ for all $A \in \mathcal{M}$.

Proof. Most of the proof of the first half of the theorem is contained in our development. The converse direction needs some justification, since our concept of "mass" was not made precise. Define f on $\mathbb{R}$ by

$$f(x) = \begin{cases} \mu((0,x]), & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -\mu((x,0]), & \text{if } x < 0. \end{cases}$$

It is clear that f is nondecreasing. To verify that f is right continuous, let $x \in \mathbb{R}$ and let $\{\delta_n\}$ be a sequence of positive numbers decreasing to zero. Suppose, without loss of generality, that $x > 0$. Then

$$(0,x] = \bigcap_{n=1}^{\infty} (0, x+\delta_n].$$

Since $\mu((0,x+\delta_1]) < \infty$ by (2), we see from Theorem 2.20, part (2), that

$$\mu((0,x]) = \lim_{n\to\infty} \mu((0,x+\delta_n]),$$

that is, $f(x) = \lim_{n\to\infty} f(x+\delta_n)$, and f is right continuous.

To show that $\mu_f^* = \mu^*$, we proceed in stages. We start by showing agreement on half-open intervals, then open intervals, open sets, bounded $\mathcal{G}_\delta$ sets, bounded sets, and finally arbitrary sets.

First, it follows from the definition of f that

$$\mu_f((a,b]) = \mu((a,b])$$

for every finite half-open interval $(a,b]$. Next, observe that, since both μ and μ_f are σ-additive, and every open interval is a countable disjoint union of half-open intervals, $\mu(G) = \mu_f(G)$ for every open interval G. This extends immediately to all open sets G. Now let H be any bounded set of type $\mathcal{G}_\delta$. Write $H = \bigcap_{n=1}^{\infty} G_n$, where $\{G_n\}$ is a decreasing sequence of bounded open sets. That the sequence $\{G_n\}$ can be chosen decreasing follows from the fact that the intersection of a finite number of open sets containing H is also an open set containing H. Since $\mu_f(G_n) = \mu(G_n)$ for every $n \in \mathbb{N}$, it follows from (2) and Theorem 2.20, part (2), that $\mu_f(H) = \mu(H)$. Thus μ_f and μ agree on all bounded sets of type $\mathcal{G}_\delta$. (We needed these sets to be bounded so that we could apply the limit theorem.)

Now let A be any bounded subset of $\mathbb{R}$. By (3), there exist sets H_1 and H_2 of type $\mathcal{G}_\delta$ such that $H_1 \supset A$, $H_2 \supset A$, $\mu_f(H_1) = \mu_f^*(A)$, and $\mu(H_2) = \mu^*(A)$. Let $H = H_1 \cap H_2$. Then $A \subset H$. It follows that

$$\mu_f^*(A) = \mu(H) = \mu^*(A).$$

Finally, let A be any subset of $\mathbb{R}$. For $n \in \mathbb{N}$, let

$$A_n = A \cap [-n,n].$$

Then $\mu_f^*(A_n) = \mu^*(A_n)$. Since both μ_f^* and μ^* are regular outer measures, we obtain

$$\mu_f^*(A) = \lim_{n\to\infty} \mu_f^*(A_n) = \lim_{n\to\infty} \mu^*(A_n) = \mu^*(A)$$

from Exercise 2:9.2. ■

We should add here a word about regularity of Borel measures. One might expect, given the nice approximation properties of Borel measures, that in any setting in which the Borel sets are measurable one would find a Borel regular measure. This is not the case; a Borel measure may behave quite weirdly on the non-Borel sets. Our next example gives such a construction that shows in particular that condition (3) in Theorem 3.19 cannot be dropped.

Example 3.20 Let $(\mathbb{R}, \mathcal{M}, \mu)$ be an extension of Lebesgue measure λ to a σ-algebra larger than $\mathcal{L}$. (See Exercise 3:11.13.) Thus $\mathcal{L}$ is a proper subset of $\mathcal{M}$, and $\mu = \lambda$ on $\mathcal{L}$. Let $A \in \mathcal{M}$, and suppose that A is bounded, say $A \subset I = [a, b]$. Suppose further that A and $I \setminus A$ have Borel covers with respect to μ. Let H_1 and H_2 be such covers. Thus $A \subset H_1$, $I \setminus A \subset H_2$, $\mu(H_1) = \mu(A)$, and $\mu(H_2) = \mu(I \setminus A)$. We may assume that H_1 and H_2 are also λ^*-covers of A and $I \setminus A$, respectively, since we could intersect H_1 and H_2 with such Borel covers. Since $\mu = \lambda$ on $\mathcal{L}$,

$$\mu(A) = \mu(H_1) = \lambda(H_1) = \lambda^*(A)$$

and

$$\mu(I \setminus A) = \mu(H_2) = \lambda(H_2) = \lambda^*(I \setminus A).$$

Then

$$\mu(I) = \mu(A) + \mu(I \setminus A) = \lambda^*(A) + \lambda^*(I \setminus A).$$

We see from the regularity of λ^* that $A \in \mathcal{L}$. It follows that there are μ-measurable sets A without Borel covers: if $A \subset B \in \mathcal{B}$, then $\mu(B) > \mu(A)$.

We can apply this discussion to the converse part of Theorem 3.19 to show that the regularity condition (3) cannot be dropped. Let us first apply the machinery of Theorem 2.44. We arrive at the complete measure space $(\mathbb{R}, \widehat{\mathcal{M}}, \hat{\mu})$. It is clear that $\hat{\mu}$ is a Borel measure that is finite on bounded Borel sets, but not every $A \in \widehat{\mathcal{M}}$ has a Borel cover with respect to $\hat{\mu}$. We show that there is no nondecreasing, right-continuous function f such that

$$\mu_f = \hat{\mu} \text{ on } \widehat{\mathcal{M}}. \tag{14}$$

Thus, for all such functions, $\mu_f^* \neq \mu^*$.

Suppose, by way of contradiction, that there is a function f so that $\mu_f = \hat{\mu}$ on $\widehat{\mathcal{M}}$. Since $\hat{\mu} = \lambda$ on $\mathcal{L}$, the function f must be of the form $f(x) = x + c$, $c \in \mathbb{R}$. Otherwise, there would be an interval $(a, b]$ such that

$$\mu_f((a, b]) = f(b) - f(a) \neq b - a = \lambda((a, b]).$$

It follows that μ_f is Lebesgue measure. But $\widehat{\mathcal{M}}$ contains sets that are not Lebesgue measurable, so μ_f is not defined on all of $\widehat{\mathcal{M}}$, contradicting (14).

We do, however, have the following theorem that illustrates the generality of Lebesgue–Stieltjes measures. In particular, every finite Borel measure on $\mathbb{R}$ agrees with some Lebesgue–Stieltjes measure on the class of Borel sets. This is of interest in certain disciplines, such as probability, in which measure space models have finite measure. See Exercise 3:11.4 for an improvement of Theorem 3.21.

Theorem 3.21 *Let μ be a Borel measure on $\mathbb{R}$ with $\mu(B) < \infty$ for every bounded Borel set B. Then there exists a nondecreasing, right-continuous function f such that $\mu_f(B) = \mu(B)$ for every Borel set $B \subset \mathbb{R}$.*

Proof. We leave the proof as Exercise 3:6.1. ■

Let us return to Theorem 3.19. From condition (4) we see that

$$\mu_f((a,b]) = f(b) - f(a)$$

for every half-open interval $(a,b]$. If f is continuous, $\mu_f(\{x\}) = 0$ for all $x \in \mathbb{R}$ (see Exercise 3:5.8), and the four intervals with endpoints a and b have the same μ_f-measure. We can interpret that measure as the "growth" of f on the interval: $\mu_f(I) = \lambda(f(I))$. If one replaces the intervals by arbitrary sets E, one might expect $\mu_f^*(E) = \lambda^*(f(E))$; the outer measure of E is the amount of "growth" of f on E. This is, in fact, the case.

Theorem 3.22 *Let f be continuous and nondecreasing on $\mathbb{R}$, and let μ_f^* be the associated Lebesgue–Stieltjes outer measure. For every set $E \subset \mathbb{R}$,*

$$\mu_f^*(E) = \lambda^*(f(E)).$$

Proof. Let $E \subset \mathbb{R}$ and let $\varepsilon > 0$. Cover E with a sequence of intervals $\{(a_n, b_n]\}$ so that

$$\sum_{n=1}^{\infty}(f(b_n) - f(a_n)) \le \mu_f^*(E) + \varepsilon.$$

Let $J_n = f((a_n, b_n])$. Since f is continuous and nondecreasing, each interval J_n has endpoints $f(a_n)$ and $f(b_n)$. Now

$$f(E) \subset \bigcup_{n=1}^{\infty} J_n$$

so,

$$\lambda^*(f(E)) \le \lambda^*\left(\bigcup_{n=1}^{\infty} J_n\right) \le \sum_{n=1}^{\infty}(f(b_n) - f(a_n)) \le \mu_f^*(E) + \varepsilon.$$

Since ε is arbitrary,

$$\lambda^*(f(E)) \le \mu_f^*(E). \tag{15}$$

To prove the reverse inequality, let G be an open set containing $f(E)$ so that $\lambda(G) \le \lambda^*(f(E)) + \varepsilon$. Let $\{J_n\}$ be the sequence of open component intervals of G. For each $n \in \mathbb{N}$, let $I_n = f^{-1}(J_n)$. Since f is continuous, each I_n is open and, since f is nondecreasing, I_n is an interval. It is clear that $E \subset \bigcup_{n=1}^{\infty} I_n$. Thus, for $I_n = (a_n, b_n)$, we have

$$\begin{aligned}\mu_f^*(E) &\le \mu_f\left(\bigcup_{n=1}^{\infty} I_n\right) \le \sum_{n=1}^{\infty} \mu_f(I_n) \\ &= \sum_{n=1}^{\infty}(f(b_n) - f(a_n)) = \sum_{n=1}^{\infty} \lambda(J_n) = \lambda(G) \le \lambda^*(f(E)) + \varepsilon.\end{aligned}$$

Since ε is arbitrary,

$$\mu_f^*(E) \le \lambda^*(f(E)). \tag{16}$$

The desired conclusion follows from (15) and (16). ■

The hypothesis that f be continuous is essential in the statement of Theorem 3.22. Exercise 3:6.4 provides a version that handles discontinuities.

Exercises

3:6.1 Prove Theorem 3.21. [Hint: Follow the proof of Theorem 3.19 to the point that a measurable cover of type $\mathcal{G}_\delta$ is not available.]

3:6.2 Give an example of a σ-finite measure μ on the Borel sets in $\mathbb{R}$ for which no Lebesgue–Stieltjes measure agrees with μ on the Borel sets. [Hint: Let $\mu(\{x\}) = 1$ for all $x \in \mathbb{Q}$.]

3:6.3 Show that there exists a measure space $(X, \mathcal{M}, \mu)$ with $\mu(X) < \infty$ and all Borel sets measurable, which also meets the following condition. There exists a measurable set M and an $\varepsilon > 0$ such that if G is open and $G \supset M$ then $\mu(G) > \mu(M) + \varepsilon$. Compare with Corollary 3.14. [Hint: See the discussion following Theorem 3.19.]

3:6.4 Let f be nondecreasing, and let μ_f denote its associated Lebesgue–Stieltjes measure.

(a) Prove that the set of atoms of μ_f is at most countable.

(b) Let A be the set of atoms of μ_f. Prove that, for every $E \subset X$,

$$\mu_f^*(E) = \lambda^*(f(E)) + \sum_{a \in A \cap E} \mu_f(\{a\}).$$

[Hint: See Exercise 3:5.8 and Theorem 3.22.]

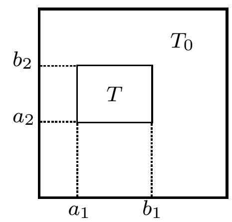

Figure 3.2: Define $\tau(T) = f(b_1, b_2) - f(a_1, b_2) - f(b_1, a_2) + f(a_1, a_2)$.

3.7 Lebesgue–Stieltjes Measures in $\mathbb{R}^n$

We turn now to a brief, simplified discussion of Lebesgue–Stieltjes measures in n-dimensional Euclidean space $\mathbb{R}^n$. As before, we are interested in Borel measures that assume finite values on bounded sets.

For ease of exposition, we limit our discussion to the case $n = 2$. We wish to model a mass distribution or probability distribution on $\mathbb{R}^2$. As a further concession to simplification, let us assume finite total mass, all contained in the half-open square

$$T_0 = (0, 1] \times (0, 1] = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 \leq 1,\ 0 < x_2 \leq 1\}.$$

Let $\mathcal{T}$ denote the family of half-open intervals $(a_1, b_1] \times (a_2, b_2]$ contained in T_0; that is, sets of the form

$$(a, b] = \{(x_1, x_2) : 0 < a_1 < x_1 \leq b_1 \leq 1,\ 0 < a_2 < x_2 \leq b_2 \leq 1\},$$

where $a = (a_1, a_2)$, $b = (b_1, b_2)$. Since $\emptyset = (a, a]$ for any $a \in T_0$, $\emptyset \in \mathcal{T}$.

Suppose now that for all $b \in T_0$ we know the mass "up to b"; more precisely, we have a function $f : T_0 \to \mathbb{R}$ such that $f(b)$ represents the mass of $(0, b]$. We wish to obtain τ from f as we did in the one-dimensional setting. This will provide a means of measuring our primitive notion of mass. Since two or more intervals can be pieced together to form a single interval, τ must be additive on such intervals. We achieve this in the following way. Let $T = (a, b] \in \mathcal{T}$. Two of the corners of T are $a = (a_1, a_2)$ and $b = (b_1, b_2)$. The other two corners are (a_1, b_2) and (b_1, a_2). Define a premeasure τ on the covering family $\mathcal{T}$ by

$$\tau(T) = f(b_1, b_2) - f(a_1, b_2) - f(b_1, a_2) + f(a_1, a_2). \tag{17}$$

Figure 3.2 illustrates.

We can now proceed as we did before. We extend τ to the algebra $\mathcal{T}_1$ generated by $\mathcal{T}$. This algebra consists of all finite unions of half-open intervals contained in T_0. We then extend τ to τ_1 by additivity and verify that τ_1 is actually σ-additive on $\mathcal{T}_1$. The ideas are the same as those in the one-dimensional case, but the details are messy. Method I leads to a metric outer measure μ_f^*, and each $A \subset T_0$ has a measurable cover of type $\mathcal{G}_\delta$. Furthermore, every interval $(a, b]$ is measurable, and

$$\mu_f((a,b]) = \tau((a,b]),$$

with $\tau((a, b])$ as given in (17).

In our preceding discussion, we chose the function f to satisfy our intuitive notion of "the mass up to b." Suppose that we turn the problem around. We ask which functions f can serve as such distributions. In the one-dimensional case, it sufficed to require that f be nondecreasing and right continuous. The monotonicity of f guaranteed that μ_f be nonnegative, and right continuity followed from Theorem 2.20 and the equality

$$(0, x] = \bigcap_{\delta>0} (0, x + \delta].$$

In the present setting, f must lead to $\tau(T) \geq 0$ in expression (17). This replaces the monotonicity requirement in the one-dimensional case. Right continuity is needed for the same reason that it is needed in one dimension. Here this means right continuity of f in each variable separately.

Exercises 3:7.1 to 3:7.3 provide illustrations of Lebesgue–Stieltjes measures on T_0.

Exercises

3:7.1 Let f be defined on T_0 by

$$f(x, y) = \begin{cases} x\sqrt{2}, & \text{for } y > x; \\ y\sqrt{2}, & \text{for } y \leq x. \end{cases}$$

Let μ_f be the associated Lebesgue–Stieltjes measure. Prove that for every Borel set $B \subset T_0$,

$$\mu_f(B) = \lambda(B \cap L),$$

where L is the line with equation $y = x$ and λ is one-dimensional Lebesgue measure on L. Observe that f is continuous, yet certain closed rectangles with one side on L have larger measures than their interiors.

3:7.2 Let f be defined on T_0 by

$$f(x, y) = \begin{cases} x, & \text{if } y \geq \frac{1}{2}; \\ 0, & \text{if } y < \frac{1}{2}. \end{cases}$$

Let μ_f be the associated Lebesgue–Stieltjes measure. Show that μ_f represents a mass all of which is located on the line $y = \frac{1}{2}$.

3:7.3 Let f be defined on T_0 by

$$f(x, y) = \begin{cases} x + y, & \text{if } x + y < 1; \\ 1, & \text{if } x + y \geq 1. \end{cases}$$

Show f is increasing in each variable separately, but that the resulting τ takes on some negative values. In particular, $\tau(T_0) = -1$.

3.8 Hausdorff Measures and Hausdorff Dimension

The measures and dimensional concepts that we shall describe here go back to the work of Felix Hausdorff in 1919, based on earlier work of Carathéodory, who had developed a notion of "length" for sets in $\mathbb{R}^N$. In our language, the length of a set $E \subset \mathbb{R}^N$ will be its Hausdorff one-dimensional outer measure, $\mu^{*(1)}$. Considerable advances were made in the years following, particularly by A. S. Besicovitch and his students. In recent years, the subject has attracted a large number of researchers because of its fundamental importance in the study of fractal geometry. A development of this subject would take us too far afield. For such developments, we refer the reader to the many excellent recent books on the subject.[1] Here we give only an indication of how to construct the Hausdorff measures, how the dimensional ideas arise, and an indication of how the dimension of a set can provide a more delicate sense of the size of a set in $\mathbb{R}^N$ than Lebesgue measure provides.

Let us return once again to our illustration with squares in Section 3.2. This time, however, in anticipation of our needs, we change the covering family $\mathcal{T}$. We take $\mathcal{T}$ to consist of *all* open sets in $\mathbb{R}^2$, with $\tau(T) = \text{diameter}\,(T)$, the diameter of the set $T \in \mathcal{T}$. Method II gives rise to a metric outer measure μ_0^* such that $\mu_0^*(T) = \infty$ for all open squares $T \in \mathcal{T}$. This might have been expected, since diameter is a one-dimensional notion and open squares are two-dimensional.

Suppose that we take, instead, a different premeasure

$$\tau(T) = (\text{diameter}\,(T))^3$$

which is smaller for sets of diameter smaller than 1. Perhaps, now, Method II will give rise to an outer measure for which squares will have *zero* measure, a two-dimensional object being measured by a "three-dimensional" measure. Let T_0 be a square of unit diameter, and let $m, n \in \mathbb{N}$. We cover T_0 with $(n+1)^2$ open squares T_i $(i = 1, \ldots, (n+1)^2)$, each of diameter $1/n$, and find for all $m \leq n$ that

$$\mu_m^*(T_0) \leq \sum_{i=1}^{(n+1)^2} \tau(T_i) = \frac{(n+1)^2}{n^3}. \tag{18}$$

Consequently, each measure has $\mu_m^*(T_0) = 0$ and

$$\mu_0^*(T_0) = \lim_{m\to\infty} \mu_m^*(T_0) = 0.$$

The same is true of any open square. In fact, $\mu_0^*(\mathbb{R}^2) = 0$.

[1] For example, C. A. Rogers, *Hausdorff Measures,* Cambridge (1970) and K. J. Falconer, *The Geometry of Fractal Sets,* Cambridge (1985).

Consider now a further choice of premeasure

$$\tau(T) = (\text{diameter}\,(T))^2,$$

which is intermediate between the two preceding examples. A similar analysis shows that

$$\mu_m^*(T_0) \leq \frac{(n+1)^2}{n^2} \qquad (m \leq n),$$

so

$$\mu_0^*(T_0) \leq 1 = \tau(T_0) = 2\lambda_2(T_0), \tag{19}$$

where λ_2 denotes two-dimensional Lebesgue measure. On the other hand, if $T_0 \subset \bigcup_{k=1}^{\infty} T_k$ and $T_k \in \mathcal{T}_n$, then

$$\begin{aligned} \sum_{k=1}^{\infty} \tau(T_k) &= \sum_{k=1}^{\infty} (\text{diameter}\,(T_k))^2 \geq \sum_{k=1}^{\infty} \lambda_2(T_k) \\ &\geq \lambda_2(\bigcup_{k=1}^{\infty} T_k) \geq \lambda_2(T_0), \end{aligned}$$

the first inequality following from the fact that any set of finite diameter δ is contained in a square of the side length δ. It follows that $\lambda_2(T_0) \leq \mu_0^*(T_0)$. Combine this inequality with (19) and recognize that T_0 is μ_0^*-measurable to obtain

$$\lambda_2(T_0) \leq \mu_0(T_0) \leq 2\lambda_2(T_0).$$

Let us take a more general viewpoint. Let $\mathcal{T}$ consist of the open sets in $\mathbb{R}^N$. For each real $s > 0$, let

$$\tau(T) = (\text{diameter}\,(T))^s,$$

and let $\mu^{(s)}$ be the measure obtained from τ and $\mathcal{T}$ by Method II. A bit of reflection suggests several facts. In the space $\mathbb{R}^2$ ($N = 2$), we have

$$\mu^{*(s)}(T) = \begin{cases} 0, & \text{if } s > 2; \\ \infty, & \text{if } s < 2. \end{cases} \quad \text{for every } T \in \mathcal{T},$$

and

$$2 = \sup\{s : \mu^{(s)}(\mathbb{R}^2) = \infty\} = \inf\{s : \mu^{(s)}(\mathbb{R}^2) = 0\}.$$

Similarly, for arbitrary N, we have

$$N = \sup\{s : \mu^{(s)}(\mathbb{R}^N) = \infty\} = \inf\{s : \mu^{(s)}(\mathbb{R}^N) = 0\}.$$

The proofs of the last three assertions are not difficult. One can actually show that if λ_N is Lebesgue N-dimensional measure in $\mathbb{R}^N$ and if we use

$$\tau(T) = (\text{diameter}\,(T))^N$$

then $\mu^{(N)}$ is a multiple of λ_N, a multiple that depends on the dimension N. For example, in $\mathbb{R}^2$ ($N = 2$), this multiple can be proved to be $4/\pi$. In the special case on the real line $\mathbb{R}$ ($N = 1$), we are using as premeasure

$$\tau(T) = \text{diameter}\,(T),$$

which is just the length if T is an open interval. Method II reduces to Method I in this case and we have $\mu^{(1)} = \lambda$. Thus the multiple connecting Lebesgue one-dimensional measure and $\mu^{(1)}$ is 1.

These concepts can be extended to a more general setting and will allow us to define a notion of dimension for subsets of a metric space.

Definition 3.23 Let X be a metric space, let $\mathcal{T}$ denote the family of all open subsets of X, and let $s > 0$. Define a premeasure τ on $\mathcal{T}$ by

$$\tau(T) = (\text{diameter}\,(T))^s.$$

Then the outer measure $\mu^{*(s)}$ obtained from τ and $\mathcal{T}$ by Method II is called the *Hausdorff s-dimensional outer measure*, and the resulting measure $\mu^{(s)}$, the *Hausdorff s-dimensional measure*.

We know that $\mu^{*(s)}$ is a metric outer measure by Theorem 3.8 and that it is regular, with covers in $\mathcal{G}_\delta$ by Theorem 3.11. These measures are all translation invariant, since the premeasures are easily seen to be so. We could have taken $\mathcal{T} = 2^X$ in the definition, but our work in Section 3.2 indicates advantages to having $\mathcal{T}$ consist of open sets. Furthermore, for $E \subset X$, $s > 0$, and $\varepsilon > 0$, there exists an open set $G \supset E$ such that

$$(\text{diameter}\,(G))^s < (\text{diameter}\,(E))^s + \varepsilon.$$

It follows (Exercise 3:8.1) that the outer measures $\mu^{*(s)}$ that we obtain do not depend on whether we take for our covering family $\mathcal{T} = 2^X$ or $\mathcal{T} = \mathcal{G}$, the family of open sets in X.

Our first theorem shows that, in general, the behavior we have seen in $\mathbb{R}^N$ using $s = 1, 2, 3$ must occur. For any set $E \subset X$, there is a number s_0 so that for $s > s_0$ the assigned s-dimensional measure is zero, while for $s < s_0$ the s-dimensional measure is infinite.

Theorem 3.24 *If $\mu^{*(s)}(E) < \infty$ and $t > s$, then $\mu^{*(t)}(E) = 0$.*

Proof. Write $\delta(T)$ for diameter (T), where T is any subset of our metric space X. Let $n \in \mathbb{N}$ and let $\{T_i\}$ be a sequence from $\mathcal{T}$ such that $E \subset \bigcup_{i=1}^{\infty} T_i$ and $\delta(T_i) \leq 1/n$, for all $i \in \mathbb{N}$. Then, for all $i \in \mathbb{N}$,

$$\frac{(\delta(T_i))^t}{(\delta(T_i))^s} = (\delta(T_i))^{t-s} \leq \left(\frac{1}{n}\right)^{t-s},$$

and

$$\mu_n^{*(t)}(E) \leq \sum_{i=1}^{\infty} (\delta(T_i))^t \leq \left(\frac{1}{n}\right)^{t-s} \sum_{i=1}^{\infty} (\delta(T_i))^s. \tag{20}$$

Since (20) is valid for every covering of E by sets in $\mathcal{T}_n$,

$$\mu_n^{*(t)}(E) \leq \left(\frac{1}{n}\right)^{t-s} \mu_n^{*(s)}(E).$$

Now let $n \to \infty$ to obtain $\lim_{n\to\infty} \mu_n^{*(t)}(E) = \mu^{*(t)}(E) = 0$. ■

Note that this theorem shows that, for $s < 1$, $\mu^{(s)}$ is a Borel measure on $\mathbb{R}$ that assigns infinite measure to every open set. In fact, $\mu^{(s)}$ is not even σ-finite on $\mathbb{R}$ (Exercise 3:8.8). Thus we have an important example of regular Borel measures on $\mathbb{R}$ that are not Lebesgue–Stieltjes measures.

Theorem 3.24 justifies Definition 3.25.

Definition 3.25 Let E be a subset of a metric space X, and let $\mu^{*(s)}(E)$ denote the Hausdorff s-dimensional outer measure of E. If there is no value $s > 0$ for which $\mu^{*(s)}(E) = \infty$, then we let $\dim(E) = 0$. Otherwise, let

$$\dim(E) = \sup\{s : \mu^{*(s)}(E) = \infty\}.$$

Then $\dim(E)$ is called the *Hausdorff dimension* of E.

Suppose that K is a Cantor set, that is a nonempty, bounded nowhere dense perfect set in $\mathbb{R}$. It is possible that $\lambda(K) > 0$, in which case $\mu^{(1)}(K) = \lambda(K)$, but if $\lambda(K) = 0$, Lebesgue measure can contribute no additional information as to its size. Hausdorff dimension, however, provides a more delicate sense of size. Exercises 3:8.2 and 3:8.3 show that there exists Cantor sets in $[0, 1]$ of dimension 1 and 0 respectively. Exercise 3:8.11 shows that the Cantor ternary set has dimension $\log 2/\log 3$. Moreover, one can show that for every $s \in [0, 1]$ there exists a Cantor set of dimension s. If

$$\dim(K_1) = s_1 < s_2 = \dim(K_2),$$

then for $t \in (s_1, s_2)$, $\mu^{(t)}(K_1) = 0$, while $\mu^{(t)}(K_2) = \infty$. Thus the measure $\mu^{(t)}$ distinguishes between the sizes of K_1 and K_2.

Hausdorff dimension has an intuitive appeal when familiar objects are under consideration. We have noted, for example, that $\dim(\mathbb{R}^N) = n$. What about $\dim(\mathcal{C})$, where $\mathcal{C}$ is a curve, say in $\mathbb{R}^3$? Before we jump to the conclusion that $\dim(\mathcal{C}) = 1$, we should recall that there are curves in $\mathbb{R}^3$ that fill a cube.[2] Such curves must have dimension 3. And there are curves in $\mathbb{R}^2$, even graphs of continuous functions $f : \mathbb{R} \to \mathbb{R}$, that are of dimension strictly between 1 and 2. But for rectifiable curves, that is, curves of finite arc length, we have the expected result, which we present in Theorem 3.26.

[2] See, for example, G. Edgar, *Measure, Topology and Fractal Geometry*, Springer (1990), for the construction of such curves.

First, we review a definition of the length of a curve. By a curve in a metric space (X, ρ), we mean a continuous function $f : [0,1] \to X$. The *length* of the curve is

$$\sup \sum_{i=1}^{m} \rho(f(x_{i-1}), f(x_i)),$$

where the supremum is taken over all partitions

$$0 = x_0 < x_1 < \cdots < x_m = 1$$

of $[0,1]$. The set of points $\mathcal{C} = f[0,1]$ is a subset of X, and it is the dimension of the set $\mathcal{C}$ that is our concern. The proof uses elementary knowledge of compact sets in metric spaces. The continuous image of a compact set is again compact. Also, the diameter of a compact set K is attained; that is, there are points x, $y \in K$ so that $\rho(x,y)$ is the diameter of K.

Theorem 3.26 *Let $f : [0,1] \to X$ be a continuous, nonconstant curve in a metric space X, and let ℓ be its arc length. Then, for $\mathcal{C} = f([0,1])$,*

1. $0 < \mu^{(1)}(\mathcal{C}) \leq \ell$.
2. *If f is one to one, then $\mu^{(1)}(\mathcal{C}) = \ell$.*

Thus, if $\ell < \infty$, $\dim(\mathcal{C}) = 1$.

Proof. Write $\delta(T)$ for diameter(T) for any set $T \subset X$. We prove first that $\mu^{(1)}(\mathcal{C}) \leq \ell$. If $\ell = \infty$, there is nothing to prove, so that assume $\ell < \infty$. It is convenient here to use the result of Exercise 3:8.1 and to use coverings of $\mathcal{C}$ by arcs of $\mathcal{C}$. If $A_1, \ldots, A_m$ is a collection of subarcs of $\mathcal{C}$ such that $\mathcal{C} = \bigcup_{i=1}^{m} A_i$, and $\delta(A_i) \leq 1/n$, for all $i = 1, \ldots, m$, then

$$\mu_n^{*(1)}(\mathcal{C}) \leq \sum_{i=1}^{m} \delta(A_i). \tag{21}$$

We wish to relate the right side of (21) to the definition of ℓ.

First, let us obtain the arcs A_i formally. Let $n \in \mathbb{N}$. Since f is uniformly continuous, there exists $\gamma > 0$ such that

$$\rho(f(x), f(y)) < \frac{1}{n}$$

whenever $x, y \in [0,1]$ and $|x - y| < \gamma$. Let

$$0 = x_0 < x_1 \cdots < x_m = 1$$

be a partition of $[0,1]$ with $|x_i - x_{i-1}| < \gamma$ for all $i = 1, \ldots, m$. Then the arcs $A_i = f([x_{i-1}, x_i])$ cover $\mathcal{C}$, and

$$\frac{1}{n} > \delta(A_i) \geq \rho(f(x_{i-1}), f(x_i))$$

for all $i = 1, \ldots, m$. It follows from the compactness of $[x_{i-1}, x_i]$ that A_i is compact for each i. Thus the diameter of A_i is actually achieved by points $f(y_i)$ and $f(z_i)$, with

$$x_{i-1} \le y_i \le z_i \le x_i.$$

This means that

$$\delta(A_i) = \rho(f(y_i), f(z_i)).$$

We now use the partition

$$0 \le y_1 \le z_1 \le y_2 \le z_2 \le \cdots \le y_m \le z_m \le 1$$

to obtain a lower estimate for ℓ. Continuing (21), we have

$$\mu_n^{*(1)}(\mathcal{C}) \le \sum_{i=1}^{m} \delta(A_i) = \sum_{i=1}^{m} \rho(f(y_i), f(z_i)) \le \ell. \tag{22}$$

Letting $n \to \infty$, we infer that

$$\mu^{*(1)}(\mathcal{C}) = \lim_{n\to\infty} \mu_n^{*(1)}(\mathcal{C}) \le \ell.$$

That

$$\mu^{(1)}(\mathcal{C}) \le \ell$$

now follows from the fact that $\mathcal{C}$ is $\mu^{*(1)}$-measurable. That $0 < \mu^{(1)}(\mathcal{C})$ follows from the fact that if $0 \le a < b \le 1$ then

$$\mu^{(1)}(f[a,b]) \ge \rho(f(a), f(b)). \tag{23}$$

(See Exercise 3:8.7.) This completes the proof of part (1).

Suppose now that f is one-one. Let

$$0 = x_0 < x_1 < \cdots < x_m = 1$$

be a partition of $[0,1]$, and note that the sets $f([x_{i-1}, x_i))$ are pairwise disjoint Borel sets. Thus, using (23) on each arc,

$$\begin{aligned} \sum_{i=1}^{m} \rho(f(x_{i-1}), f(x_i)) &\le \sum_{i=1}^{m} \mu^{(1)}(f([x_{i-1}, x_i))) \\ &= \mu^{(1)}\left(\bigcup_{i=1}^{m} f([x_{i-1}, x_i))\right) \\ &= \mu^{(1)}(f([0,1))) \\ &= \mu^{(1)}(f([0,1])) = \mu^{(1)}(\mathcal{C}). \end{aligned}$$

This is valid for all partitions, and so $\ell \le \mu^{(1)}(\mathcal{C})$. In view of part (1), $\ell = \mu^{(1)}(\mathcal{C})$. ■

We end this section with a comment about "exceptional sets". Consider the following statements about a nondecreasing function f defined on an interval I. Let

$$\begin{aligned} D &= \{x : f \text{ is discontinuous at } x\}, \\ N &= \{x : f \text{ is nondifferentiable at } x\}, \\ N' &= \{x : f \text{ has no derivative, finite or infinite, at } x\}. \end{aligned}$$

Then

1. D is countable.
2. $\lambda(N) = 0$.
3. $\mu^{(1)}(G) = 0$ where $G \subset \mathbb{R}^2$ consists of the points on the graph of f corresponding to points of continuity in N'.

Each of these statements indicates that a nondecreasing function has some desirable property outside some small exceptional set. The notion of smallness differs in these three statements. Observe that statement (3) involves a subset of $\mathbb{R}^2$. The weaker statement, that $\lambda_2(G) = 0$, provides much less information than statement (3). We shall prove a theorem corresponding to assertion (2) later in Chapter 7.

We shall encounter a number of theorems involving exceptional sets. Cardinality and measure are only two of the many frameworks for expressing a sense in which a set may be small. The notion of first category set is another such framework; we study this intensively in Chapter 10. We mention another sense of smallness involving "porosity" in Exercise 7:9.12

Exceptional sets of measure zero are encountered so frequently that we employ special terminology for dealing with them. Suppose that a function-theoretic property is valid except, perhaps, on a set of μ–measure zero. We then say that this property holds *almost everywhere*, or perhaps μ–*almost everywhere* or even *for almost all* members of X. This is frequently abbreviated as *a.e.* For example, statement (2) above could be expressed as "f is differentiable a.e."

Exercises

3:8.1 Verify that, for all $s > 0$ and $E \subset X$, $\mu^{*(s)}(E)$ has the same value when $\mathcal{T} = \mathcal{G}$ as when $\mathcal{T} = 2^X$.

3:8.2 Let P be a Cantor set in $\mathbb{R}$ with $\lambda(P) > 0$. What is $\dim(P)$?

3:8.3 Construct a Cantor set in $\mathbb{R}$ of dimension zero. [Hint: Control the sizes of the intervals comprising the sets A_n in Example 2.1.]

3:8.4 Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called a *Lipschitz function* if there exists $M > 0$ such that $|f(y) - f(x)| \leq M|y - x|$ for all $x, y \in \mathbb{R}$. Show that if f is a Lipschitz function then, for all $E \subset \mathbb{R}$, $\dim(f(E)) \leq \dim(E)$.

3:8.5 Show how to construct a set A in $\mathbb{R}$ such that $\lambda(A) = 0$, but $\dim(A) = 1$.

3:8.6 Give an example of a continuous curve $\mathcal{C}$ of finite length such that $\mu^{(1)}(\mathcal{C}) < \ell$.

3:8.7 Prove that if $f : [0,1] \to X$ is a continuous curve in X and $0 \le a < b \le 1$ then

$$\mu^{(1)}(f[a,b]) \ge \rho(f(a), f(b)).$$

[Hint: Define $g : f([a,b]) \to \mathbb{R}$ by $g(w) = \rho(f(a), w)$. Use g to obtain a comparison between $\mu^{(1)}(f[a,b])$ and the length of the interval $[0, \rho(f(a), f(b))]$.]

3:8.8 Show that, for $s < 1$, $(\mathbb{R}, \mathcal{B}, \mu^{*(s)})$ is not a σ-finite measure space.

3:8.9 Let $X = \mathbb{R}$ but supplied with the metric $\rho(x,y) = 1$ if $x \ne y$. What is the result of applying Method II to $\mathcal{T} = \mathcal{G}$, $\tau = \text{diameter}\,(T)$. (What are the families $\mathcal{T}_n$?)

3:8.10 Suppose that we were trying to measure the length of a hike. We count our steps, each of which is exactly 1 meter long, and arrive at a distance that we publish in our trail guide. A mouse does the same thing, but its steps are only 1 centimeter long. Since it must walk around rocks and other objects that we ignore, it will report a longer length. An insect's measurement would be still longer, and a germ, noticing every tiny undulation, would measure the distance as enormous. Probably, the actual distance along an ideal curve covering the trail is infinite. A better sense of "size of the trail" can be given by its Hausdorff dimension. Benoit Mandelbrot[3] discusses the differences in reported lengths of borders between countries. He also provides estimates of the dimensions (between 1 and 2) of these borders. Express our fanciful discussion of trail length in the more precise language of coverings, Hausdorff measures, and Hausdorff dimension.

3:8.11 Let K be the Cantor set, and let $s = \log 2/\log 3$. Cover K with 2^n intervals, each of length 3^{-n}. Show that

$$\mu^{*(s)}_{3^n}(K) \le 1.$$

Show that these intervals are the most economical ones with which to cover K. Deduce that $\dim(K) = \log 2/\log 3$ and $\mu^{*(s)}(K) = 1$.

3.9 Methods III and IV

In applications of measure theory to analysis, one may need to construct an appropriate measure to serve as a tool in the investigation. We have already

[3]B. Mandelbrot, *The Fractal Geometry of Nature,* W. H. Freeman and Co. (1982).

seen the usefulness of Methods I and II, both of which were developed by Carathéodory. In this section we extend this collection of methods by adopting a new approach, but one built on the same theme of refining some crude premeasure into a useful outer measure. These methods can also be used to develop Lebesgue–Stieltjes and Hausdorff measures. We shall use them in Section 7.6 to construct total variation measures for arbitrary continuous functions.

Let $\mathcal{T}$ be a collection of subsets of a metric space X, and suppose that a premeasure τ is defined on $\mathcal{T}$. We assume, just as for Methods I and II, that there is no structure on τ, only that it assigns a number $0 \leq \tau(C) \leq \infty$ to each member $C \in \mathcal{T}$ and $\tau(\emptyset) = 0$, and that this crude measure will be refined into a genuine outer measure by some kind of approximation process.

Here, however, we shall use packings rather than coverings. The idea of a covering estimate is to approximate the measure of a set E by some minimal covering of E using sets $\{C_i\}$ from $\mathcal{T}$. Naturally, overlapping of sets would occur even in a good covering. For a packing, we allow no overlap.

Define, for any subclass $\mathcal{T}_0 \subset \mathcal{T}$,

$$V(\tau, \mathcal{T}_0) = \sup \sum_{i=1}^{\infty} \tau(C_i),$$

where the supremum is with regard to all sequences $\{C_i\} \subset \mathcal{T}_0$ and where the elements in the sequence are pairwise disjoint. We shall find ways of using the estimates $V(\tau, \mathcal{T}_0)$ to obtain our measures.

In this setting we shall assume that there is a relationship "x is attached to C" defined for points $x \in X$ and sets $C \in \mathcal{T}$. For example, a simple and useful such relation would be to take x is attached to C if $x \in C$; a slight variant would have $x \in \overline{C}$. If the sets in $\mathcal{T}$ are balls, then a useful version is to have x is attached to C to mean that C is centered at x. In general, the geometry and the application dictate how this can be interpreted. No special assumptions are needed on the relationship in general. Recall the notation $B(x, \delta)$ for the open ball centered at x and with radius δ.

Definition 3.27 Let $\mathcal{T}$ be a collection of subsets of a metric space X, and suppose that there is given a relationship "x is attached to C" defined for $x \in X$ and $C \in \mathcal{T}$. Let $E \subset X$.

1. A family $\mathcal{C} \subset \mathcal{T}$ is said to be a *full cover* of E (relative to $\mathcal{T}$) if for every $x \in E$ there is a $\delta > 0$ so that

$$C \in \mathcal{T},\ x \text{ is attached to } C \text{ and } \ C \subset B(x, \delta) \Rightarrow C \in \mathcal{C}.$$

2. A family $\mathcal{C} \subset \mathcal{T}$ is said to be a *fine cover* of E (relative to $\mathcal{T}$) if, for every $x \in E$ and every $\varepsilon > 0$, either

$$\exists C \in \mathcal{C},\ x \text{ is attached to } C \text{ and } \ C \subset B(x, \varepsilon)$$

or else no such set C exists in all of $\mathcal{T}$.

The fine covers are often called Vitali covers in the literature. They will play a key role in the differentiation chapters. We now define our two methods of constructing outer measures.

Definition 3.28 Let $\mathcal{T}$ be a collection of subsets of a metric space X and τ a premeasure on $\mathcal{T}$. For every $E \subset X$, we define

1. $\tau^\bullet(E) = \inf\{V(\tau, \mathcal{C}) : \mathcal{C}$ a full cover of $E\}$.
2. $\tau^\circ(E) = \inf\{V(\tau, \mathcal{C}) : \mathcal{C}$ a fine cover of $E\}$.

The set functions $\tau^\bullet$ and τ° will be called the Method III and Method IV outer measures (respectively) generated by τ, $\mathcal{T}$ and the relation of attachment.

Theorem 3.29 *Let $\mathcal{T}$ be a collection of subsets of a metric space X and τ a premeasure on $\mathcal{T}$. Then $\tau^\bullet$ and τ° are metric outer measures on X and $\tau^\circ \leq \tau^\bullet$.*

Proof. Most of the details of the proof are either elementary or routine. Here are two details that may not be seen immediately.

First, let us check the countable subadditivity of $\tau^\bullet$. Suppose that E is contained in a union $\bigcup_{n=1}^\infty E_n$ and that each $\tau^\bullet(E_n)$ is finite. Then for any $\varepsilon > 0$ there are full covers $\mathcal{C}_n$ of E_n so that

$$V(\tau, \mathcal{C}_n) \leq \tau^\bullet(E_n) + \varepsilon 2^{-n}.$$

Since $\mathcal{C} = \bigcup_{n=1}^\infty \mathcal{C}_n$ is a full cover of E, we must have

$$\tau^\bullet(E) \leq V(\tau, \mathcal{C}) \leq \sum_{n=1}^\infty V(\tau, \mathcal{C}_n) \leq \sum_{n=1}^\infty \left(\tau^\bullet(E_n) + \varepsilon 2^{-n}\right).$$

From this one sees that

$$\tau^\bullet(E) \leq \sum_{n=1}^\infty \tau^\bullet(E_n).$$

Second, let us consider how to prove that $\tau^\bullet$ is a *metric* outer measure. Suppose that A, B are subsets of X a positive distance apart. Let $\mathcal{C}$ be a full cover of $A \cup B$ with

$$V(\tau, \mathcal{C}) \leq \tau^\bullet(A \cup B) + \varepsilon.$$

Because of this separation, we may choose two disjoint open sets G_1 and G_2 covering A and B, respectively. Consider the families

$$\mathcal{C}_1 = \{C \in \mathcal{C} : C \subset G_1\}$$

and

$$\mathcal{C}_2 = \{C \in \mathcal{C} : C \subset G_2\}.$$

Then $\mathcal{C}_1$ is a full cover of A and $\mathcal{C}_2$ is a full cover of B. No set in $\mathcal{C}_1$ meets any set in $\mathcal{C}_2$. This means that

$$\tau^\bullet(A) + \tau^\bullet(B) \le V(\tau, \mathcal{C}_1) + V(\tau, \mathcal{C}_2) \le V(\tau, \mathcal{C}) \le \tau^\bullet(A \cup B) + \varepsilon.$$

From this inequality and the subadditivity of $\tau^\bullet$ the identity,

$$\tau^\bullet(A \cup B) = \tau^\bullet(A) + \tau^\bullet(B)$$

can be readily obtained. The remaining details of the proof are left as exercises. ■

Here is a simple regularity theorem that illustrates some methods that can be used in the study of these measures. In any application, one would need to adjust the ideas to the geometry of the situation.

Theorem 3.30 *Let $\mathcal{T}$ be a collection of subsets of a metric space X and τ a premeasure on $\mathcal{T}$. Suppose that the given relation "x is attached to C" means that x is an interior point of C. Let $E \subset X$ with $\tau^\bullet(E) < \infty$ and let $\varepsilon > 0$. Then there are an $\mathcal{F}_\sigma$ set $C_1 \supset E$ and an $\mathcal{F}_{\sigma\delta}$ set $C_2 \supset E$ such that*

$$\tau^\bullet(C_1) < \tau^\bullet(E) + \varepsilon \textit{ and } \tau^\bullet(C_2) = \tau^\bullet(E).$$

Proof. There is a full cover $\mathcal{C} \subset \mathcal{T}$ of E so that

$$V(\tau, \mathcal{C}) < \tau^\bullet(E) + \varepsilon.$$

Choose $\delta(x) > 0$ for each $x \in E$ so that

$$C \in \mathcal{T},\ x \in \operatorname{int}(C), \text{ and } C \subset B(x, \delta) \Rightarrow C \in \mathcal{C}.$$

Define

$$E_n = \{x \in E : \delta(x) > 1/n\}$$

and consider the closed sets $\{\overline{E_n}\}$. One checks, directly from the definition, that $\mathcal{C}$ is a full cover of each set $\overline{E_n}$. Thus

$$\tau^\bullet\left(\overline{E_n}\right) \le V(\tau, \mathcal{C}) < \tau^\bullet(E) + \varepsilon$$

and so also

$$\tau^\bullet\left(\bigcup_{n=1}^\infty \overline{E_n}\right) \le \tau^\bullet(E) + \varepsilon.$$

The set $C_1 = \bigcup_{n=1}^\infty \overline{E_n}$ is an $\mathcal{F}_\sigma$ set that contains E and affords our desired approximation to $\tau^\bullet(E)$. The set C_2 of the theorem can now be obtained by taking an intersection of such sets. ■

We conclude with some examples. In each case the relation defining the attachment can be taken as ordinary set membership.

Example 3.31 Let $\mathcal{T}$ denote the set of all intervals $(a, b]$ of real numbers, and take for τ the length of the interval so that $\tau((a, b]) = b - a$. Then

$$\tau^\circ = \tau^\bullet = \lambda^*.$$

That is, both measures recover the Lebesgue outer measure. This will be discussed in greater detail in Section 7.6.

Example 3.32 Let $\mathcal{T}$ denote the set of all intervals $(a, b]$ of real numbers, and for τ take, $\tau((a, b]) = g(b) - g(a)$, where g is a continuous nondecreasing function. Then

$$\tau^\circ = \tau^\bullet = \mu_g^*.$$

That is, both measures recover the Lebesgue–Stieltjes outer measure μ_g^* generated by the monotonic function g. This too will be discussed in greater detail in Section 7.6.

Example 3.33 Let $\mathcal{T}$ denote the set of all intervals $(a, b]$ of real numbers, and for τ take, $\tau((a, b]) = (b - a)^s$, where $0 < s < 1$. Then τ° can be shown to be exactly the s-dimensional Hausdorff measure, and the larger measure $\tau^\bullet$ is indeed larger and plays a role in many investigations under the name "packing measure."

Exercises

3:9.1 Show that every full cover of a set is also a fine cover of that set.

3:9.2 Let $\mathcal{C}$ be a full (fine) cover of E and suppose that G is an open set containing E. Then

$$\mathcal{C}_1 = \{C \in \mathcal{C} : C \subset G\}$$

is also a full (fine) cover of E.

3:9.3◊ Let $\mathcal{T}$ be the collection of all intervals $(a, b]$ (a, $b \in \mathbb{R}$). Let "x is attached to $(a, b]$" mean that $x = a$, the left endpoint. Suppose that f is a real function. Show that the collection

$$\mathcal{C} = \{(x, y] : f(y) - f(x) > c(y - x)\}$$

is a full cover of the set

$$\left\{x : \liminf_{y \to x+} \frac{f(y) - f(x)}{y - x} > c\right\}$$

and a fine cover of the (larger) set

$$\left\{x : \limsup_{y \to x+} \frac{f(y) - f(x)}{y - x} > c\right\}.$$

3:9.4 If $\mathcal{C}_n$ is a full (fine) cover of E_n for each $n = 1, 2, 3, \dots$ then $\bigcup_{n=1}^\infty \mathcal{C}_n$ is a full (fine) cover of $\bigcup_{n=1}^\infty E_n$.

3:9.5 If $\mathcal{C}_1$, $\mathcal{C}_2$, ... are families of sets, then

$$V\left(\tau, \bigcup_{n=1}^{\infty} \mathcal{C}_n\right) \leq \sum_{n=1}^{\infty} V(\tau, \mathcal{C}_n).$$

3:9.6 If $\mathcal{C}_1$ is a full cover of E, and $\mathcal{C}_2$ is a full cover of E then $\mathcal{C}_1 \cap \mathcal{C}_2$ is a full cover of E.

3:9.7$\diamondsuit$ If $\mathcal{C}_1$ is a fine cover of E, and $\mathcal{C}_2$ is a full cover of E then $\mathcal{C}_1 \cap \mathcal{C}_2$ is a fine cover of E.

3:9.8 If $\mathcal{C}_1$ is a fine cover of E, and $\mathcal{C}_2$ is a fine cover of E then $\mathcal{C}_1 \cap \mathcal{C}_2$ need not be a fine cover of E.

3:9.9 Complete all the needed details for a proof of Theorem 3.29.

3:9.10 In the proof of Theorem 3.30, show in detail that $\mathcal{C}$ is a full cover of each set $\overline{E_n}$.

3.10 Additional Remarks

We end this chapter with some additional remarks concerning monotonic functions, Cantor sets, and nonatomic measures. For simplicity, we work on the interval $[0, 1]$.

We have already discussed, in Exercise 1:22.13, Cantor-like functions. These are continuous nondecreasing functions that map a Cantor set onto an interval. Speaking loosely, we can say that Cantor functions do all their rising on a Cantor set, that is, a nonempty, bounded, perfect, nowhere dense set.

Our first theorem gives an indication of the role of Cantor sets in the rising of a nondecreasing function.

Theorem 3.34 *Let $A \subset [0, 1]$, and let $f : A \to \mathbb{R}$ be a nondecreasing function. If $\lambda_*(f(A)) > 0$, then A contains a Cantor set.*

Proof. We may assume that f is bounded on A. Otherwise, we do our work on an appropriate smaller interval I. We begin by extending f to a nondecreasing function $\overline{f}$ defined on all of $[0, 1]$. Let

$$\overline{f}(x) = \begin{cases} \inf\ f, & \text{for } 0 \leq x \leq \inf\ A; \\ \sup\{f(t) : t \in A, t \leq x\}, & \text{for } \inf\ A < x \leq 1. \end{cases}$$

Then $\overline{f}$ is nondecreasing on $[0, 1]$.

Our objective is to find a Cantor set P of positive measure such that $P \subset f(A)$ and f^{-1} maps P homeomorphically into A. To do this, we first remove from consideration any points of discontinuity of $\overline{f}$, as well as any intervals on which $\overline{f}$ is constant. Since $\overline{f}$ is nondecreasing, its set D of points of discontinuity is countable. Thus

$$\lambda(\overline{f}(D)) = 0. \tag{24}$$

Now, for each $y \in f(A)$, the set $\overline{f}^{-1}(y)$ is an interval, since $\overline{f}$ is nondecreasing. Let $\mathcal{I}$ be the family of such intervals that are not degenerate. The intervals in $\mathcal{I}$ are pairwise disjoint and each has positive length. Thus $\mathcal{I}$ is countable, say $\mathcal{I} = \{I_k\}$. Let $G = \bigcup_{k=1}^{\infty} I_k$. Since $\overline{f}$ is constant on each member of $\mathcal{I}$, $\overline{f}(G)$ is countable and

$$\lambda(\overline{f}(G)) = 0. \tag{25}$$

Let $M = f(A) \setminus \overline{f}(D \cup G)$. It follows from (24) and (25) that $\lambda_*(M) > 0$. Let $y \in M$. There exists $x \in A$ such that $f(x) = y$. We see from the definition of the set M that

$$\overline{f}(t) < y \text{ for } t < x \quad \text{and} \quad \overline{f}(t) > y \text{ for } t > x.$$

Thus $\overline{f}^{-1}(y) = \{x\}$. It follows that $\overline{f}^{-1}$ is strictly increasing on the set M, and $\overline{f}^{-1}(M) \subset A$. Note that, since $M \subset f(A)$ and $\overline{f}^{-1}(M) \subset A$, $\overline{f}^{-1} = f^{-1}$ on M.

The set E of points of discontinuity of $f^{-1} : M \to A$ is countable. Thus there is a Cantor set P of positive measure contained in $M \setminus E$. Since f^{-1} is continuous and strictly increasing on P, the set $F = f^{-1}(P)$ is also a Cantor set, and $F \subset A$. It is clear that f maps the Cantor set F onto the set P of positive measure. ■

Exercise 3:11.14 at the end of this chapter shows that we cannot replace the monotonicity hypothesis with one of continuity in Theorem 3.34.

We observed in Section 2.1 how nineteenth century misconceptions about nowhere dense subsets of $\mathbb{R}$ may have delayed the development of measure theory. Cantor sets were not part of the mathematical repertoire until late in the nineteenth century. Nowadays, Cantor sets appear in diverse areas of mathematics. Our familiarity with them makes it difficult to visualize an uncountable set that does not contain a Cantor set, though this is, in fact, possible. We have earlier (e.g., Exercises 1:22.7 and 1:22.8) discussed totally imperfect sets; that is, an uncountable set of real numbers that contains no Cantor set. We have shown the existence of Bernstein sets (a set such that neither it nor its complement contains a Cantor set). The existence can be obtained by a cardinality argument (which is especially simple under the continuum hypothesis).

Bernstein sets have interesting properties relative to Lebesgue measure and Lebesgue–Stieltjes measures. Let f be continuous and nondecreasing on $[0, 1]$, with $f([0, 1]) = [0, 1]$. Suppose that neither A nor $\widetilde{A}$ contains a Cantor set. Then

$$\lambda_*(A) = \lambda_*(\widetilde{A}) = 0.$$

It follows that

$$\lambda^*(A) = \lambda^*(\widetilde{A}) = 1.$$

Now $f(A) \cup f(\widetilde{A}) = [0, 1]$. By Theorem 3.34,

$$\lambda_*(f(A)) = \lambda_*(f(\widetilde{A})) = 0.$$

Thus
$$\lambda^*(f(A)) = \lambda^*(f(\widetilde{A})) = 1$$
and the set A cannot be measurable with respect to any nonatomic Lebesgue–Stieltjes measure except the zero measure. We know, by Exercise 3:11.13, that there are extensions $\overline{\overline{\lambda}}$ of λ for which the set A is $\overline{\overline{\lambda}}$-measurable. Similarly, there are extensions $\overline{\overline{\mu}}_f$ of any given Lebesgue–Stieltjes measure for which A is $\overline{\overline{\mu}}_f$-measurable. But such extensions are not Lebesgue–Stieltjes measures. See the discussion following the proof of Theorem 3.19.

Arguments similar to the ones we have given show that if A is totally imperfect then, for every nonatomic Lebesgue–Stieltjes measure μ_f, either $\mu_f(A) = 0$ or A is not μ_f-measurable. Which alternative applies depends on whether $\lambda(f(A)) = 0$ or $\lambda^*(f(A)) > 0$.

We turn now to the opposite phenomenon. Are there sets that are measurable with respect to *every* nonatomic Lebesgue–Stieltjes measure? Since Lebesgue–Stieltjes measures are Borel measures, the question should be asked about non-Borel sets.

To address this question, we construct another example of an unusual set of real numbers (cf. Exercise 1:22.9), called occasionally a Lusin set.

Lemma 3.35 *Assuming the continuum hypothesis, there exists a set X of real numbers such that X has cardinality c, yet every nowhere dense subset of X is countable.*

Proof. We shall construct a set $X \subset [0,1]$ so that, if A is a nowhere dense subset of the space X using the Euclidean metric, then A is countable. To construct the set X, arrange the nowhere dense closed subsets of [0,1] into a transfinite sequence $\{F_\alpha\}$, $0 \le \alpha < \Omega$, where Ω is the first uncountable ordinal. For each $\alpha < \Omega$, consider the difference
$$F_\alpha \setminus \bigcup_{\beta<\alpha} F_\beta.$$
Since the interval [0,1] is complete, uncountably many of these differences must be nonempty. Let X be a set that contains exactly one point from each such difference. Then X has cardinality c.

We now show that if N is a nowhere dense subset of [0,1] then $N \cap X$ is countable. Since $\overline{N}$ is also nowhere dense in [0,1], there exists $\alpha < \Omega$ such that $\overline{N} = F_\alpha$. The construction of X implies that, for $\gamma > \alpha$, $X \cap F_\gamma \cap F_\alpha = \emptyset$. Thus
$$\overline{N} \cap X \subset \bigcup_{\beta\le\alpha} F_\beta,$$
so $\overline{N} \cap X$ is countable. The same is true of $N \cap X$. Since any set that is nowhere dense in X is also nowhere dense in [0,1], we infer that every nowhere dense subset of X is countable. ■

For this space X, we have the following.

Theorem 3.36 *The space X has the following properties.*

1. *The only finite nonatomic Borel measure μ on X is the zero measure.*
2. *Any nondecreasing function f on X maps X onto a set of measure zero.*
3. *For every nonatomic Lebesgue–Stieltjes measure μ_f on $\mathbb{R}$, X is μ_f-measurable and $\mu_f(X) = 0$.*

Proof. Let D be a countable dense subset of X, and let $\varepsilon > 0$. Since μ is nonatomic, $\mu(D) = 0$. By Corollary 3.14, there exists an open set G containing D such that $\mu(G) < \varepsilon$. The set G is a dense and open subset of X. Thus $X \setminus G$ is nowhere dense in X. But for this space X, this implies that $X \setminus G$ is countable. Since μ is nonatomic, $\mu(X \setminus G) = 0$. It follows that

$$\mu(X) = \mu(G) + \mu(X \setminus G) < \varepsilon.$$

Since ε is arbitrary, $\mu(X) = 0$. This proves (1). The proof of (2) is similar. We leave it as Exercise 3:10.2. Part (3) follows directly from part (2) and Theorem 3.22. ■

It is a fact (proved later in Theorem 11.11) that every uncountable analytic set in $\mathbb{R}$ contains a Cantor set. Since all Borel sets are analytic, it follows that every uncountable Borel set in $\mathbb{R}$ has positive measure with respect to some nonatomic Lebesgue–Stieltjes measure. The space X is not a Borel subset of $\mathbb{R}$. It has cardinality c, yet has *universal measure* zero. This means every finite, nonatomic Lebesgue–Stieltjes measure gives X measure zero. The space X can be used to show that there is no nontrivial nonatomic measure defined on all subsets of $[0, 1]$. This gives another proof of Theorem 2.38 of Ulam, here using the continuum hypothesis.

Theorem 3.37 *If μ is a nonatomic, finite measure defined on all subsets of $[0, 1]$, then $\mu([0, 1]) = 0$.*

Proof. Let h be a one-to-one function mapping X onto $[0, 1]$. Define ν on 2^X by

$$\nu(E) = \mu(h(E)).$$

Then ν is a finite, nonatomic measure on 2^X. By Theorem 3.36 (1), $\nu(X) = 0$. In particular, $\mu([0, 1]) = \mu(h(X)) = \nu(X) = 0$. ■

There is nothing special about the interval $[0, 1]$. The proof of Theorem 3.37 works equally well for any set of cardinality c. Nontrivial finite, nonatomic measures cannot be defined for all subsets of any set Y of cardinality c. It is perhaps curious that this statement is one of pure set theory: no metric or topological conditions are imposed on Y. The proof here, however, did make heavy use of a strange property of the metric space X.

Exercises

3:10.1 Show that if $A \subset [0,1]$ is totally imperfect then, for every Lebesgue–Stieltjes measure μ_f, either $\mu_f(A) = 0$ or A is not μ_f-measurable. [Hint: For the second alternative, apply Theorem 3.22 to A and $\widetilde{A}$.]

3:10.2 Verify part (2) of Theorem 3.36.

3:10.3 The only finite, nonatomic Borel measure on the space X appearing in Theorem 3.36 is the zero measure. If one tries to imitate the proof of Theorem 3.37 to show that every nonatomic, finite Borel measure on $[0,1]$ is the zero measure, one step fails. Which is it?

3:10.4$\diamondsuit$ Let h be continuous and strictly increasing on $\mathbb{R}$. Prove that $h(B)$ is a Borel set if and only if B is a Borel set. [Hint: Let $\mathcal{S}$ be the family of all sets $A \subset \mathbb{R}$ such that $h(A)$ is a Borel set. Show that $\mathcal{S}$ is a σ-algebra that contains the closed sets. For the "only if" part, consider h^{-1}.]

3.11 Additional Problems for Chapter 3

3:11.1 Let μ be a regular Borel measure on a compact metric space X such that $\mu(X) = 1$, and let $\mathcal{E}$ be the family of all closed subsets F of X such that $\mu(F) = 1$.

(a) Prove that the intersection of any finite subcollection of $\mathcal{E}$ also belongs to $\mathcal{E}$.

(b) Prove that the intersection H of the sets in $\mathcal{E}$ is a nonempty compact set.

(c) Prove that $\mu(H) = 1$.

(d) Prove that $\mu(H \cap V) > 0$ for each open set V with $H \cap V \neq \emptyset$.

(e) Prove that if K is a compact subset of X such that $\mu(K) = 1$ and $\mu(K \cap V) > 0$ for each open set V with $K \cap V \neq \emptyset$ then $H = K$.

3:11.2 Let X be a well-ordered set that has a last element Ω such that if $x \in X$ then the set of predecessors of x,

$$\{y \in X : y < x\},$$

is countable. Let $Y = \{y \in X : y < \Omega\}$, and let $\mathcal{M}$ be a σ–algebra of subsets of Y that contains at least all singleton sets. Prove that for any measure on $\mathcal{M}$ the following assertions are equivalent:

(a) For every $a \in Y$, $\mu(\{x \in Y : x \leq a\}) < \infty$.

(b) The set $P = \{x \in Y : \mu(\{x\}) > 0\}$ is countable and $\mu(P) < \infty$.

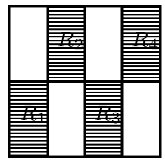

Figure 3.3: The rectangles R^0, and R_i ($i = 1 \ldots 4$) in Exercise 3:11.7.

3:11.3 Let A and B be sets. The set

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

is called the *symmetric difference* of A and B.

Prove that there exists a countable family $\mathcal{A}$ of open sets in $[0, 1]$ with the following property: For every $\varepsilon > 0$ and $E \in \mathcal{L}$, there exists $A \in \mathcal{A}$ with $\lambda(A \triangle E) < \varepsilon$. Thus the countable family $\mathcal{A}$ can be used to approximate all members of $\mathcal{L}$. We shall see later that $\lambda(A \triangle B)$ is "almost" a metric on $\mathcal{L}$.

3:11.4 Let $\mathcal{E}$ be defined as in the proof of Theorem 3.12. Let $(X, \hat{\mathcal{B}}, \hat{\mu})$ be the completion of $(X, \mathcal{B}, \mu)$.

(a) Show that $\mathcal{E} \supset \hat{\mathcal{B}}$. [Hint: Use Theorems 2.36, 2.44, and 3.15.]

(b) Use part (a) to improve Theorem 3.21 to give the conclusion $\mu_f(E) = \hat{\mu}(E)$ for all $E \in \mathcal{E}$.

3:11.5 Let I be an interval in $\mathbb{R}$. Show how one can reduce a theory of Lebesgue–Stieltjes measures on I to the theory that we developed for Lebesgue–Stieltjes measures on $\mathbb{R}$.

3:11.6◇ Let f be continuous on $[0, 1]$. Let $\mathcal{T}$ consist of $\emptyset$ and the closed intervals in $[0, 1]$. Let $\tau([a, b]) = |f(b) - f(a)|$, and let μ_1^* and μ_2^* be the associated Method I and Method II outer measures, respectively.

(a) Is μ_1^* equal to μ_2^*?

(b) What relationship exists between the measure μ_2 and the variation of f?

(c) What is the answer to (b) if f is piecewise monotonic?

3:11.7 Let R^0 be the unit square. Divide R^0 into 8 rectangles of height $\frac{1}{2}$ and width $\frac{1}{4}$, as indicated in Figure 3.3. Now divide each of the rectangles R_i into 8 or 10 rectangles, giving rise to the situation depicted in Figure 3.4 for R^2. Continue this process by cutting heights in half and widths into 4 or 5 parts in such a way that $R^{k+1} \subset R^k$, and R^k is compact and connected. Let $R = \bigcap_{k=1}^{\infty} R^k$.

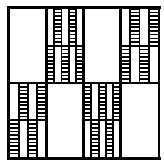

Figure 3.4: The rectangles R^2 (the shaded region).

(a) Show that this intersection R is the graph of a continuous function g. (The construction of this function is due to James Foran.)

(b) Show that for each $c \in [0,1]$ the set $\{x : g(x) = c\}$ is a Cantor set.

(c) Let $\mathcal{T}$ consist of $\emptyset$ and the closed intervals in $[0,1]$, and let $\tau([a,b]) = |g(b)-g(a)|$. Let μ_0^* be the Method II outer measure obtained from $\mathcal{T}$ and τ. Calculate $\mu_0^*(E)$ for $E \subset [0,1]$. [Hint: Calculate $\mu_0^*([0,1])$.]

(d) Compare your answer to part (c) with your answer to part (b) of Exercise 3:11.6.

3:11.8$\diamondsuit$ Prove that there exists a set $E \subset [0,1]$ with $E \in \mathcal{L}$, but $F(E) \notin \mathcal{L}$, where F is the Cantor function. [Hint: Use Exercise 2:13.13.]

3:11.9$\diamondsuit$ Let f be continuous on $[a,b]$. Prove that the following statements are equivalent.

(a) There exists $E \subset [a,b]$ such that $E \in \mathcal{L}$, but $f(E) \notin \mathcal{L}$.

(b) There exists $E \subset [a,b]$ such that $\lambda(E) = 0$, but $\lambda^*(f(E)) \neq 0$.

3:11.10$\diamondsuit$ Let μ_1 and μ_2 be measures defined on a common σ-algebra $\mathcal{M}$. We say that μ_1 is *absolutely continuous* with respect to μ_2, written $\mu_1 \ll \mu_2$, if $\mu_1(E) = 0$ whenever $\mu_2(E) = 0$, $E \in \mathcal{M}$. Let $\mathcal{M} = \mathcal{B}$, and let F be the Cantor function. Is $\mu_F \ll \lambda$? Is $\lambda \ll \mu_F$?

3:11.11$\diamondsuit$ Refer to Exercise 3:11.10. Let μ_g be a continuous Lebesgue–Stieltjes measure on $\mathcal{B}$.

(a) Prove that $\mu_g \ll \lambda$ if and only if, for $E \in \mathcal{B}$ and $\lambda(E) = 0$, $\lambda(g(E)) = 0$.

(b) Prove that if $\lambda \ll \mu_g$ then g is strictly increasing.

3:11.12 Let $\{L_n\}$ be a sequence of pairwise disjoint Lebesgue measurable sets in $\mathbb{R}$, let $L = \bigcup_{n=1}^{\infty} L_n$, and let $E \subset \mathbb{R}$.

(a) Prove that $\lambda^*(L \cap E) = \sum_{n=1}^{\infty} \lambda^*(L_n \cap E)$. [Hint: Let H be a measurable cover for $L \cap E$, H_n for $L_n \cap E$ with the sets H_n pairwise disjoint.]

(b) Prove that $\lambda_*(L \cap E) = \sum_{n=1}^{\infty} \lambda_*(L_n \cap E)$.
[Outline of proof: Let K be a measurable kernel for $L \cap E$. Justify the inequalities

$$\begin{aligned}\lambda_*(L \cap E) &= \lambda(K) = \sum_{n=1}^{\infty} \lambda(L_n \cap K) \\ &\leq \sum_{n=1}^{\infty} \lambda_*(L_n \cap E) \leq \lambda_*(L \cap E).\end{aligned}$$

3:11.13$\diamondsuit$ **(Extending $\mathcal{L}$ and λ)** Let $X = [0, 1]$.

(a) Prove that, for each $E \subset X$ and $L \in \mathcal{L}$,

$$\lambda(L) = \lambda_*(L \cap E) + \lambda^*(L \cap \widetilde{E}).$$

(b) Let $E \subset X$, $E \notin \mathcal{L}$. Let $\overline{\mathcal{L}}$ be the algebra generated by $\mathcal{L}$ and $\{E\}$. Show that $\overline{\mathcal{L}}$ consists of all sets of the form

$$\overline{L} = (L_1 \cap E) \cup (L_2 \cap \widetilde{E}) \text{ with } L_1, L_2 \in \mathcal{L}.$$

(c) Define $\overline{\lambda}$ on $\overline{\mathcal{L}}$ by

$$\overline{\lambda}(\overline{L}) = \lambda^*(\overline{L} \cap E) + \lambda_*(\overline{L} \cap \widetilde{E}).$$

Let $\mathcal{T} = \overline{\mathcal{L}}$, $\tau = \overline{\lambda}$ and let $(X, \overline{\overline{\mathcal{L}}}, \overline{\overline{\lambda}})$ be the measure space obtained by Method I. Prove that $\overline{\overline{\lambda}} = \lambda$ on $\mathcal{L}$. Thus $(X, \overline{\overline{\mathcal{L}}}, \overline{\overline{\lambda}})$ is an extension of $(X, \mathcal{L}, \lambda)$ and contains sets not in $\mathcal{L}$.

(d) Show that $\overline{\overline{\lambda}}(E) = \lambda^*(E)$. Thus E has a $\mathcal{G}_\delta$ cover with respect to $\overline{\overline{\lambda}}$. That is, there exists $H \in \mathcal{G}_\delta$ such that $H \supset E$ and $\overline{\overline{\lambda}}(H) = \overline{\overline{\lambda}}(E) = \lambda^*(E)$. Does $\widetilde{E}$ also have such a cover in $\mathcal{G}_\delta$?

3:11.14 We stated Theorem 3.34 for nondecreasing functions. That hypothesis cannot be dropped. Show that, for the continuous function g of Exercise 3:11.7, there exists a totally imperfect set A such that $g(A) = [0, 1]$. This exercise shows that, unlike monotonic functions, continuous functions *can* rise on totally imperfect sets. [Hint: A proof can be based on the continuum hypothesis and transfinite induction. Let $\{y_\alpha\}$, $\alpha < \Omega$, be a well-ordering of the Cantor sets in $[0, 1]$. Choose a_1 such that $f(a_1) = y_1$. Now choose $b_1 \in P_1 \setminus \{a_1\}$. Proceed inductively. If we have $\{a_\beta\} \subset [0, 1]$ and $\{b_\beta\} \subset [0, 1]$ for all $\beta < \alpha$, choose

$$a_\alpha \in [0, 1] \setminus \bigcup_{\beta < \alpha} (\{a_\beta\} \cup \{b_\beta\})$$

such that $f(a_\alpha) = y_\alpha$. Then choose

$$b_\alpha \in [0,1] \setminus \left(\bigcup_{\beta \le \alpha} \{a_\beta\} \cup \bigcup_{\beta < \alpha} \{b_\beta\} \right)$$

such that $b_\alpha \in P_\alpha$. Let

$$A = \bigcup_{\alpha < \Omega} \{a_\alpha\} \quad \text{and} \quad B = \bigcup_{\alpha < \Omega} \{b_\alpha\}.$$

Then $f(A) = [0,1]$. If P is a Cantor set in $[0,1]$, there exists α such that $P = P_\alpha$. By construction, $b_\alpha \in P_\alpha$ and $A \cap B = \emptyset$. Thus $b_\alpha \notin A$, so A does not contain P.]

3:11.15 Use the continuum hypothesis to prove the existence of a set A of real numbers such that A and $\widetilde{A}$ are both totally imperfect. [Hint: Modify the argument in Exercise 3:11.14 to choose points a_α and b_α from P_α.]

Chapter 4

MEASURABLE FUNCTIONS

We saw in Section 1.20 that the definition of the Lebesgue integral of a function f involves the measure of sets such as

$$\{x : \alpha \leq f(x) < \beta\}.$$

We devote this chapter to the study of functions for which these sets, and others defined by similar inequalities, are necessarily measurable. These will be called *measurable functions*. We shall see that, for a given measure space $(X, \mathcal{M}, \mu)$, the class of μ-measurable functions is well behaved with respect to the elementary algebraic operations and with respect to various operations involving limits. The proofs here will follow readily from our requirement that $\mathcal{M}$ be a σ-algebra, together with a bit of set-theoretic algebra. We provide the necessary development in Sections 4.1 and 4.2.

In Chapters 2 and 3 we saw that, while measurable sets can be quite complicated, one can under certain circumstances approximate measurable sets, and even nonmeasurable sets, by simpler sets. For example, when dealing with the Lebesgue–Stieltjes measure space $(\mathbb{R}, \mathcal{M}_f, \mu_f)$, we know that every set $M \in \mathcal{M}_f$ has a $\mathcal{G}_\delta$ cover and an $\mathcal{F}_\sigma$ kernel, and we know that for every $\varepsilon > 0$ there exists an open set G and a closed set F such that $F \subset M \subset G$ and $\mu_f(G \setminus F) < \varepsilon$. Such approximations have allowed us to deal with measurable sets that might be unwieldy to combine or manipulate by replacing them with simpler sets that we can handle. Similar simplifications are also available when dealing with measurable functions. In Sections 4.4 and 4.5 we see that, under suitable hypotheses on a measure space $(X, \mathcal{M}, \mu)$, measurable functions can be approximated by simpler functions in several ways. In particular, for many important classes of measure spaces, the approximating functions can be taken to be continuous.

We also need to discuss convergence of sequences of measurable functions. Of the several notions of convergence that we encounter in Section 4.2, the "preferred" notion may be uniform convergence. It became apparent in the middle of the nineteenth century that a number of theorems that are easy to prove when uniform convergence is assumed in appropriate places are either false or more difficult to prove when weaker forms of convergence are hypothesized. In Section 4.3 we show that, on a finite measure space, a sequence $\{f_n\}$ of measurable functions that is known to converge in some weaker sense actually converges "almost uniformly," that is, uniformly when one ignores a set of small measure.

Thus three fundamental concepts in analysis—set, convergence, and function—allow approximations by more tractable objects. Although one gives up a bit at the stages where one makes the approximation, the conclusion reached at the end of the argument is still often the best possible.

4.1 Definitions and Basic Properties

We begin with Lebesgue's original definition of a measurable function.

Definition 4.1 Let $(X, \mathcal{M}, \mu)$ be a measure space, and let

$$f : X \to [-\infty, \infty].$$

The function f is *measurable* if for every $\alpha \in \mathbb{R}$ the set

$$E_\alpha(f) = \{x : f(x) > \alpha\}$$

is a measurable set.

A special case of this definition has its own terminology.

Definition 4.2 Let X be a metric space, and let $f : X \to [-\infty, \infty]$. The function f is a *Borel function* or is *Borel measurable* if the set

$$E_\alpha(f) = \{x : f(x) > \alpha\}$$

is a Borel set for every $\alpha \in \mathbb{R}$.

Observe that measurability of f depends on the σ-algebra $\mathcal{M}$ under consideration, but not on the measure μ. Nonetheless, one often sees phrases such as "f is μ-measurable."

Example 4.3 Take $(\mathbb{R}, \mathcal{L}, \lambda)$ as the measure space. Let f be a continuous function, g be a discontinuous increasing function, and $h = \chi_A$ for some set $A \subset \mathbb{R}$. Then, for every $\alpha \in \mathbb{R}$, $E_\alpha(f)$ is open and $E_\alpha(g)$ is an interval. Thus both f and g are measurable. For h we find that

$$E_\alpha(h) = \begin{cases} \emptyset, & \text{if } \alpha \geq 1, \\ A, & \text{if } 0 \leq \alpha < 1, \\ \mathbb{R}, & \text{if } \alpha < 0. \end{cases}$$

Hence h is λ-measurable if and only if $A \in \mathcal{L}$. If we had taken $(\mathbb{R}, \mathcal{B}, \lambda)$ as our measure space, then f and g are measurable (and hence Borel functions) because open sets and arbitrary intervals are Borel sets, and h is measurable if and only if A is a Borel set.

Example 4.4 If $\mathcal{M} = \{\emptyset, X\}$, only constant functions are measurable, while if $\mathcal{M} = 2^X$, all functions are measurable. In particular, if X is countable and each singleton set is measurable, then every function on X is measurable.

Theorem 4.5 shows that there is nothing special about the specific inequality we chose in Definition 4.1.

Theorem 4.5 *Let $(X, \mathcal{M}, \mu)$ be a measure space. The following conditions on a function f are equivalent.*

1. *f is measurable.*
2. *For all $\alpha \in \mathbb{R}$, the set $\{x : f(x) \geq \alpha\} \in \mathcal{M}$.*
3. *For all $\alpha \in \mathbb{R}$, the set $\{x : f(x) < \alpha\} \in \mathcal{M}$.*
4. *For all $\alpha \in \mathbb{R}$, the set $\{x : f(x) \leq \alpha\} \in \mathcal{M}$.*

Proof. Suppose that f is measurable and let $\alpha \in \mathbb{R}$. Observe that

$$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) > \alpha - \frac{1}{n}\right\}.$$

Since f is measurable, each set in the intersection is measurable and, hence, so is the intersection itself. This proves that (1)$\Rightarrow$(2). The implication (2)$\Rightarrow$(3) follows directly from the equality

$$\{x : f(x) < \alpha\} = X - \{x : f(x) \geq \alpha\}.$$

The implication (3)$\Rightarrow$(4) follows from the equality

$$\{x : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) < \alpha + \frac{1}{n}\right\}.$$

Finally, the implication (4)$\Rightarrow$(1) follows by complementation in (3). It now follows that all four statements are equivalent. ■

Simple arguments show that various other sets associated with a measurable function f are measurable, for example, the sets

$$\{x : f(x) = \alpha\} \quad \text{and} \quad \{x : \alpha \leq f(x) \leq \beta\}.$$

Note that measurability of a function f is related to the mapping properties of f^{-1}. In fact, measurability of f is equivalent to the condition that f^{-1} take Borel sets to measurable sets. (The proof is left as Exercise 4:1.2.)

Theorem 4.6 *Let $(X, \mathcal{M}, \mu)$ be a measure space and f a real-valued function on X. Then f is measurable if and only if $f^{-1}(B) \in \mathcal{M}$ for every Borel set $B \subset \mathbb{R}$.*

Our next example shows that we cannot replace Borel sets with arbitrary measurable sets in this theorem. It also shows that the mapping properties of f (as opposed to f^{-1}) may be quite different for measurable functions. (The reader may wish to consult Exercises 2:13.13 and 3:11.8 to 3:11.10 before proceeding with this example.)

Example 4.7 We work with the Lebesgue measure space $(\mathbb{R}, \mathcal{L}, \lambda)$. Let K be the Cantor ternary set, and let P be a Cantor set of positive measure. Write $a = \min\{x : x \in P\}$ and $b = \max\{x : x \in P\}$. Exercise 4:1.10 shows that there exists a strictly increasing continuous function h that maps $[a, b]$ onto $[0, 1]$ and maps P onto K.

Let A be a nonmeasurable subset of P, and let $E = h(A)$. Since $E \subset K$, $\lambda(E) = 0$ and, in particular, E is Lebesgue measurable. It follows that

1. $h^{-1}(E) = A$. Thus, even for the strictly increasing continuous function h, the inverse image of a measurable set need not be measurable.

2. The function h^{-1} is also continuous and strictly increasing. It maps the zero measure set E onto a nonmeasurable set.

3. Let $f = h^{-1}$ and let μ_f be the associated Lebesgue–Stieltjes measure on $[0, 1]$. Then μ_f is not absolutely continuous with respect to λ, since $\lambda(K) = 0$, but

$$\mu_f(K) = \lambda(f(K)) = \lambda(P) > 0.$$

Observe that part (1) offers another proof that there are Lebesgue measurable sets that are not Borel sets. The set E is Lebesgue measurable. If it were a Borel set, then $A = h^{-1}(E)$ would also be measurable by Theorem 4.6.

We next consider various ways that measurable functions combine to give rise to other measurable functions.

Theorem 4.8 *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f and g be measurable functions on X. Let $\phi : \mathbb{R} \to \mathbb{R}$ be continuous, and let $c \in \mathbb{R}$. Then*

1. *cf is measurable.*

2. *$f + g$ is measurable.*

3. *$\phi \circ f$ is measurable if f is finite.*

4. *fg is measurable.*

Proof. The proof of (1) is trivial. To verify (2), observe first that for any $\alpha \in \mathbb{R}$ the function $\alpha - g$ is measurable. Now let $\{q_k\}$ be an enumeration of the rational numbers. Then

$$\{x : f(x) + g(x) > \alpha\} = \{x : f(x) > \alpha - g(x)\}$$

$$= \bigcup_{k=1}^{\infty} (\{x : f(x) > q_k\} \cap \{x : g(x) > \alpha - q_k\}).$$

This set is clearly measurable. Since this is true for all $\alpha \in \mathbb{R}$, $f + g$ is measurable.

To verify (3), let $\alpha \in \mathbb{R}$, and observe that

$$(\phi \circ f)^{-1}((\alpha, \infty)) = f^{-1}(\phi^{-1}((\alpha, \infty))).$$

Since ϕ is continuous, the set $G = \phi^{-1}((\alpha, \infty))$ is open, and since f is measurable, $f^{-1}(G) \in \mathcal{M}$. This verifies (3).

Part (4) follows immediately from parts (1) and (2), the continuity of the function x^2, and the identity

$$4fg = (f + g)^2 - (f - g)^2.$$

■

In part (3) of Theorem 4.8, the order of composition does matter. See Exercise 4:1.7.

Exercises

4:1.1 Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that for an arbitrary function f on X the class $\{A \subset \mathbb{R} : f^{-1}(A) \in \mathcal{M}\}$ is a σ-algebra.

4:1.2◇ Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that a function f on X is measurable if and only if $\{A \subset \mathbb{R} : f^{-1}(A) \in \mathcal{M}\}$ contains all Borel sets.

4:1.3 Suppose that, for each rational number q, the set $\{x : f(x) > q\}$ is measurable. Can we conclude that f is measurable?

4:1.4 Let $\mathcal{S}_0$ be a family of subsets of $\mathbb{R}$ such that all open sets belong to the smallest σ-algebra containing $\mathcal{S}_0$. If $f^{-1}(E)$ is measurable for all $E \in \mathcal{S}_0$ then f is measurable. Apply this to obtain another proof of the preceding exercise and another proof of Theorem 4.5.

4:1.5 Show that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that, for each $\alpha \in \mathbb{R}$, the set $\{x : f(x) = \alpha\}$ is in $\mathcal{L}$, but f is not Lebesgue measurable. [Hint: Map a nonmeasurable set onto $(0, 1)$ and its complement onto $(1, 2)$ in an appropriate manner.]

4:1.6 Provide conditions under which a quotient of measurable functions is measurable.

4:1.7 Give an example of a continuous function ϕ and a Lebesgue measurable function f, both defined on $[0, 1]$, such that $f \circ \phi$ is not measurable. Give an example of a nonmeasurable function f on $[0, 1]$ such that $|f|$ is measurable. [Hint: See Example 4.7.]

4:1.8 Let $(X, \mathcal{M}, \mu)$ be a measure space. Suggest conditions under which there can exist a nonmeasurable function f on X for which $|f|$ is measurable.

4:1.9 Show that a measurable function f defined on $[0,1]$ has the property that for every $\varepsilon > 0$ there is a $M_\varepsilon > 0$ so that

$$\lambda\left(\{x \in [0,1] : |f(x)| \leq M_\varepsilon\}\right) \geq 1 - \varepsilon$$

if and only if f is finite almost everywhere.

4:1.10$\diamondsuit$ Let E and F be any two Cantor sets in $\mathbb{R}$. Let $\mathcal{I} = \{I_k\}$ and $\mathcal{J} = \{J_k\}$ be the sequences of intervals complementary to E and F, respectively.

(a) Show that to each pair of distinct intervals I_i and I_k in $\mathcal{I}$ there exists an interval $I_j \in \mathcal{I}$ between I_i and I_k.

(b) Use part (a) to show that there exists an order-preserving correspondence between $\mathcal{I}$ and $\mathcal{J}$. That is, there exists a function γ mapping $\mathcal{I}$ onto $\mathcal{J}$ such that if $I, I' \in \mathcal{I}$ and $J = \gamma(I)$, while $J' = \gamma(I')$, then J is to the left of J' if and only if I is to the left of I'.

(c) For each $I_i \in \mathcal{I}$, let f_i be continuous and strictly increasing on I_i, and map I_i onto the interval $\gamma(I_i)$. Use the functions f_i to obtain a strictly increasing continuous function f mapping $\bigcup_{i=1}^\infty I_i$ onto $\bigcup_{i=1}^\infty J_i$.

(d) Extend f to be a continuous strictly increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$ and E onto F.

4:1.11 Let $\mathcal{T}$ consist of $\emptyset$ and the open squares in $\mathbb{R}^2$, and let $\tau(T)$ be the diameter of T. Use Method I to obtain an outer measure μ^* and a measure space $(\mathbb{R}^2, \mathcal{M}, \mu)$. Is every continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ measurable with respect to $\mathcal{M}$? What would your answer be if we had used Method II instead of Method I?

4:1.12 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

(a) Show that f maps compact sets to compact sets.

(b) Show that f maps sets of type F_σ to sets of the same type.

(c) If f is also one-one, show that f maps Borel sets to Borel sets.

(d) If f is also Lipschitz, show that f maps sets of Lebesgue measure zero to sets of the same type.

(e) If f is Lipschitz, show that f maps Lebesgue measurable sets to sets of the same type.

(We have seen in Example 4.7 that a one-to-one continuous function $f : \mathbb{R} \to \mathbb{R}$ need not map Lebesgue measurable sets to Lebesgue measurable sets. We mention that, without the assumption that f be one to one, we cannot be sure that f maps Borel sets to Borel sets. It is true that a continuous function f maps Borel sets onto Lebesgue measurable sets. Proofs appear in Chapter 11.)

4:1.13 Let $(X, \mathcal{M}, \mu)$ be a complete measure space with X a metric space.

(a) Prove that if all Borel sets are measurable each function f that is continuous a.e. is measurable.

(b) Prove that if every continuous function $f : X \to \mathbb{R}$ is measurable then $\mathcal{M} \supset \mathcal{B}$. [Hint: Let G be open in X. Let $f(x) = \operatorname{dist}(x, \widetilde{G})$. See Section 3.2. Show that f is continuous and $f^{-1}((0,\infty)) = G$.]

(c) Let $X = [0,1]$, $\mathcal{M} = \{\emptyset, X\}$, and let $f(x) = x$. Is f measurable?

4:1.14 Using the continuum hypothesis, one can prove that there exists a Lebesgue nonmeasurable subset E of $\mathbb{R}^2$ such that E intersects every horizontal or vertical line in exactly one point. Use this set to show that there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that f is Borel measurable in each variable separately, yet f is not Lebesgue measurable. Note also that the restriction of f to any horizontal or vertical line has only one point of discontinuity. Compare with Exercise 4:1.13 (a).

4:1.15 In part (3) of Theorem 4.8 we had to assume f finite. Otherwise the function $\phi \circ f$ is not defined on the set $\{x : f(x) = \pm\infty\}$. Suppose that $(X, \mathcal{M}, \mu)$ is complete. Since the measurability of a function does not depend on its values on a set of measure zero, one can discuss the measurability of functions defined only a.e. Formulate how this can be done, and then prove part (3) of Theorem 4.8 under the assumption that f is finite a.e.

4:1.16 Let $(X, \mathcal{M}, \mu)$ be a measure space and Y a metric space. Give a reasonable definition for a function $f : X \to Y$ to be measurable. How much of the theory of this section and the next can be done in this generality?

4.2 Sequences of Measurable Functions

Several forms of convergence of a sequence of functions are important in the theory of integration. Two of these forms, *pointwise convergence* and *uniform convergence*, form part of the standard material of courses in elementary analysis. We assume that the reader is familiar with these forms of convergence. We discuss two other forms in this section: *almost everywhere convergence* and *convergence in measure*. We first show that the class of measurable functions is closed under certain operations on sequences.

Theorem 4.9 *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions on X. Then each of the functions $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$ and $\liminf_n f_n$ is measurable.*

Proof. Since

$$\left\{x : \sup_n f_n(x) \le \alpha\right\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) \le \alpha\},$$

the function $\sup_n f_n$ is measurable. That $\inf_n f_n$ is measurable follows from the identity

$$\inf_n f_n = -\sup_n(-f_n).$$

The identities

$$\limsup_n \, f_n = \inf_k \sup_{n\ge k} f_n \quad \text{and} \quad \liminf_n \, f_n = \sup_k \inf_{n\ge k} f_n$$

supply the measurability of the other two functions. ∎

It follows that the set $\{x : \limsup_n \, f_n(x) = \liminf_n \, f_n(x)\}$ is a measurable set. This is the set of convergence of the sequence $\{f_n\}$. Here one must allow the possibility that $f_n(x) \to \pm\infty$. It is also true that the set on which $\{f_n\}$ converges to a finite limit is measurable. See Exercise 4:2.4. It follows readily that if $\{f_n(x)\}$ converges for all $x \in X$ then the limit function $f(x) = \lim_n f_n(x)$ is measurable.

We shall see in Chapter 5 that the integral of a function f does not depend on the values that f assumes on a set of measure zero. It is also true that one can often assert no more than that the sequence $\{f_n\}$ converges for *almost every* $x \in X$. This form of convergence suffices in many applications. We present a formal definition.

Definition 4.10 Let $\{f_n\}$ be a sequence of finite a.e., measurable functions on a measurable set $E \subset X$. If there exists a function f such that

$$\lim_{n\to\infty} |f_n(x) - f(x)| = 0$$

for almost all $x \in E$, we say that $\{f_n\}$ *converges* to f *almost everywhere* on E, and we write

$$\lim_n f_n = f \text{ [a.e.]} \quad \text{or} \quad f_n \to f \text{ [a.e.] on } E.$$

The usual slight variation in language applies when $E = X$.

It is now clear that if $f_n \to f$ [a.e.] then f is measurable. A bit of care is needed in interpreting this statement if the measure space is not complete. Removing the set of measure zero on which $\{f_n\}$ does not converge to f leaves a measurable set on which the sequence converges pointwise, and f is measurable on that set.

We mention that some authors provide slightly different definitions for convergence [a.e.]. For example, the concept makes sense without the functions being measurable or finite a.e., so more inclusive definitions are possible. We shall rarely deal with nonmeasurable functions or with functions that take on infinite values on sets of positive measure. By imposing the extra restrictions in our definition, we focus on the way convergence [a.e.] actually arises in our development. Observe that if $f_n \to f$ [a.e.] then our definition guarantees that f is finite a.e.

We turn now to another form of convergence, *convergence in measure*.

Definition 4.11 Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $E \in \mathcal{M}$. Let $\{f_n\}$ be a sequence of finite a.e., measurable functions on E. We say that $\{f_n\}$ *converges in measure* on E to the function f and write

$$\lim_n f_n = f \text{ [meas]} \quad \text{or} \quad f_n \to f \text{ [meas] on } E$$

if for any pair (ε, η) of positive numbers there corresponds $N \in \mathbb{N}$ such that, if $n \geq N$, then

$$\mu(\{x : |f_n(x) - f(x)| \geq \eta\}) < \varepsilon.$$

Equivalently, $f_n \to f$ [meas] if, for every $\eta > 0$,

$$\lim_n \mu(\{x : |f_n(x) - f(x)| \geq \eta\}) = 0.$$

These notions of convergence are used, too, in probability theory. There convergence a.e. is called "convergence almost surely" and convergence in measure is called "convergence in probability." We shall see in Section 4.3 that, when $\mu(X) < \infty$, convergence [a.e.] implies convergence [meas]. Thus in probability theory where the space has measure 1, almost sure convergence always implies convergence in probability. In general, this is not so, as the next example shows.

Example 4.12 Let

$$f_n(x) = \frac{x}{n}.$$

Each function f_n is finite and Lebesgue measurable on $\mathbb{R}$. One verifies easily that $f_n \to 0$ [a.e.], but $\{f_n\}$ does not converge in measure to any function on $\mathbb{R}$.

Our next example shows that it is possible for $f_n \to 0$ [meas] without $\{f_n(x)\}$ converging for any x. This example also illustrates a feature of this convergence that will play a role in integration theory. Even though the sequence has no pointwise limit, we can still write

$$\lim_{m\to\infty} \int_0^1 f_m \, d\lambda = 0 = \int_0^1 \lim_{m\to\infty} f_m \, d\lambda,$$

provided that $\lim_{m\to\infty} f_m$ is taken in the sense of convergence in measure.

Example 4.13 (A sliding sequence of functions) For nonnegative integers n, k, with $0 \leq k < 2^n$ and $m = 2^n + k$, let

$$E_m = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right].$$

Let $f_1 = \chi_{[0,1]}$ and, for $n > 1$, $f_m = \chi_{E_m}$. We see that

$$\begin{aligned} f_2 &= \chi_{[0,\frac{1}{2}]}, \quad f_3 = \chi_{[\frac{1}{2},1]}, \\ f_4 &= \chi_{[0,\frac{1}{4}]}, \quad f_5 = \chi_{[\frac{1}{4},\frac{1}{2}]}, \quad f_6 = \chi_{[\frac{1}{2},\frac{3}{4}]}, \quad f_7 = \chi_{[\frac{3}{4},1]}, \\ f_8 &= \chi_{[0,\frac{1}{8}]}, \ \ldots \end{aligned}$$

Every point $x \in [0,1]$ belongs to infinitely many of the sets E_m, and so $\limsup_m f_m(x) = 1$, while $\liminf_m f_m(x) = 0$. Thus $\{f_m\}$ converges at no point in $[0,1]$, yet $\lambda(E_m) = 2^{-n}$ for $m = 2^n + k$. As $m \to \infty$, $n \to \infty$ also. For every $\eta > 0$,

$$\lambda(\{x : f_m(x) \geq \eta\}) \leq \frac{1}{2^n}.$$

It follows that $f_m \to 0$ [meas] on the interval $[0,1]$.

If we study Example 4.13 further, we might note that, while the sequence $\{f_m\}$ converges at no point, suitable subsequences converge [a.e.]. For example, $f_{2^n}(x) \to 0$ for each $x \neq 0$. It is true, in general, that such convergent subsequences exist. This is the first of our attempts at finding relations among the various notions of convergence.

Theorem 4.14 *If $f_n \to f$ [meas], there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ [a.e.].*

Proof. For each $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ such that

$$\mu\left(\left\{x : |f_n(x) - f(x)| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}$$

for every $n \geq n_k$. We choose the sequence $\{n_k\}$ to be increasing. Let

$$A_k = \left\{x : |f_{n_k}(x) - f(x)| \geq \frac{1}{2^k}\right\},$$

and let $A = \limsup_k A_k$. Since $\sum_{k=1}^{\infty} \mu(A_k) < 1 < \infty$, it follows that $\mu(A) = 0$ by the Borel–Cantelli lemma (Exercise 2:4.8). Let $x \notin A$. Then x is a member of only finitely many of the sets A_k. Thus there exists K such that, if $k \geq K$,

$$|f_{n_k}(x) - f(x)| < \frac{1}{2^k}.$$

It follows that $\{f_{n_k}\} \to f$ [a.e.]. ∎

In Section 4.3 we shall introduce yet another form of convergence and obtain some more relations that exist among the various modes of convergence.

Exercises

4:2.1 Let $\{f_n\}$ be a sequence of finite functions on a space X, and let $\alpha \in \mathbb{R}$. Prove that

$$\left\{x : \liminf_n f_n(x) > \alpha\right\} = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \left\{x : f_n(x) - \alpha \geq \frac{1}{m}\right\}.$$

Use this to provide another proof of the fact that a pointwise limit of measurable functions is measurable.

4:2.2 Let $\{A_n\}$ be a sequence of measurable sets, and write $f_n(x) = \chi_{A_n}(x)$. Describe in terms of the sets $\{A_n\}$ what it means for the sequence of functions $\{f_n\}$ (a) to converge pointwise, (b) to converge uniformly, (c) to converge almost everywhere, and (d) to converge in measure.

4:2.3 Characterize convergence in measure in the case where the measure is the counting measure.

4:2.4 Show that if $\{f_n\}$ is a sequence of measurable functions then the set of points x at which $\{f_n(x)\}$ converges to a finite limit is measurable.

4:2.5 Prove that if, for each $n \in \mathbb{N}$, f_n is finite a.e. and if $f_n \to f$ [a.e.] then f is finite [a.e.]. [Hint: This is a feature of Definition 4.10 and may not be true for other definitions of a.e. convergence.]

4:2.6 Verify that the sequence $\{f_n\}$ in Example 4.12 converges to 0 [a.e.], but does not converge [meas].

4:2.7 Prove that if $f_n \to f$ and $g_n \to g$ both in measure then $f_n + g_n \to f + g$ in measure.

4:2.8 (a) Prove that if $f_n \to f$ [meas], $g_n \to g$ [meas], and $\mu(X) < \infty$ then $f_n g_n \to fg$ [meas]. [Hint. Consider first the case that $f_n \to 0$ [meas] and $g_n \to 0$ [meas].]

(b) Use $f_n(x) = x$ and $g_n(x) = 1/n$ to show that the finiteness assumption in part (a) cannot be dropped.

4:2.9 Let $X = \mathbb{N}$, $\mathcal{M} = 2^{\mathbb{N}}$, and $\mu(\{n\}) = 2^{-n}$. Determine which of the four modes of convergence coincide in this case. [Hint: Uniform and pointwise do not coincide here.]

4:2.10 Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Prove that $f_n \to f$ [meas] if and only if every subsequence $\{f_{n_k}\}$ of $\{f_n\}$ has a subsequence $\{f_{n_{k_j}}\}$ such that $f_{n_{k_j}} \to f$ [a.e.].

4:2.11 Let $\{f_n\}$ be a sequence of measurable functions on a finite measure space $(X, \mathcal{M}, \mu)$, and let α_n be a sequence of positive numbers. Suppose that

$$\sum_{n=1}^{\infty} \mu\left(\{x \in X : |f_n(x)| > \alpha_n\}\right) < \infty.$$

Prove that

$$-1 \leq \liminf_{n\to\infty} \frac{f_n(x)}{\alpha_n} \leq \limsup_{n\to\infty} \frac{f_n(x)}{\alpha_n} \leq 1$$

for μ–almost every $x \in X$.

4.3 Egoroff's Theorem

We saw in Section 4.2 that neither of the two forms of convergence, convergence a.e. and convergence in measure, implies the other. We now develop a third form of convergence that is stronger than these two, but weaker than uniform convergence. If $\{f_n\}$ converges *uniformly* to f on X, we write

$$\lim_n f_n = f \text{ [unif] } \text{ or } f_n \to f \text{ [unif]}.$$

Almost uniform convergence is just uniform convergence off a set of arbitrarily small measure.

Definition 4.15 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of finite a.e., measurable functions on X. We say that $\{f_n\}$ converges *almost uniformly* to f on X if for every $\varepsilon > 0$ there exists a measurable set E such that $\mu(X \setminus E) < \varepsilon$ and $\{f_n\}$ converges uniformly to f on E. We then write

$$\lim_n f_n = f \text{ [a.u.] } \text{ or } f_n \to f \text{ [a.u.]}.$$

It is instructive to compare convergence [a.u.] with convergence [meas]. Suppose that $f_n \to f$ [meas] on X. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon$$

for all x in a set A_n with $\mu(X \setminus A_n) < \varepsilon$. The sets A_n can vary with n. In Example 4.13, the sets $X \setminus A_n$ "slide" so much that $\{f_n(x)\}$ converge for no $x \in [0, 1]$. Convergence [a.u.] requires that a *single* set E suffice for all sufficiently large n: the set E does not depend on n.

Almost uniform convergence implies both convergence [a.e.] and convergence [meas]. (We leave verification of these facts as Exercise 4:3.1.) Neither converse is true. Example 4.13 and the functions $f_n(x) = x/n$, $x \in \mathbb{R}$, show this.

On a finite measure space convergence [a.u.] and convergence [a.e.] are equivalent. This is a form of a theorem due to D. Egoroff (1869–1931) (also transliterated sometimes as Egorov). One obtains the immediate corollary that, when $\mu(X) < \infty$, convergence [a.e.] implies convergence [meas].

Theorem 4.16 (Egoroff) *Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence of finite a.e., measurable functions such that $f_n \to f$ [a.e.]. Then $f_n \to f$ [a.u.].*

Proof. For every $n, k \in \mathbb{N}$, let

$$A_{nk} = \bigcap_{m=n}^{\infty} \left\{ x : |f_m(x) - f(x)| < \frac{1}{k} \right\}.$$

The function f is measurable, from which it follows that each of the sets A_{nk} is measurable. Let

$$E = \left\{ x : \lim_n |f_n(x) - f(x)| = 0 \right\}.$$

Since $f_n \to f$ [a.e.], E is measurable, $\mu(E) = \mu(X)$, and for each $k \in \mathbb{N}$, $E \subset \bigcup_{n=1}^{\infty} A_{nk}$. For fixed k, the sequence $\{A_{nk}\}_{n=1}^{\infty}$ is expanding, so that

$$\lim_n \mu(A_{nk}) = \mu\left(\bigcup_{n=1}^{\infty} A_{nk} \right) \geq \mu(E) = \mu(X).$$

Since $\mu(X) < \infty$,

$$\lim_n \mu(X \setminus A_{nk}) = 0. \tag{1}$$

Now let $\varepsilon > 0$. It follows from (1) that there exists $n_k \in \mathbb{N}$ such that

$$\mu(X \setminus A_{n_k k}) < \varepsilon 2^{-k}. \tag{2}$$

We have shown that for each $\varepsilon > 0$ there exists $n_k \in \mathbb{N}$ such that inequality (2) holds. Let

$$A = \bigcap_{k=1}^{\infty} A_{n_k k}.$$

We now show that $\mu(X \setminus A) < \varepsilon$ and that $f_n \to f$ [unif] on A. It is clear that A is measurable. Furthermore,

$$\mu(X \setminus A) = \mu\left(\bigcup_{k=1}^{\infty} (X \setminus A_{n_k k}) \right) \leq \sum_{k=1}^{\infty} \mu\left(X \setminus A_{n_k k}\right) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

We see from the definition of the sets A_{nk} that, for $m \geq n_k$,

$$|f_m(x) - f(x)| < \frac{1}{k}$$

for every $x \in A_{n_k k}$ and therefore for every $x \in A$. Thus $f_n \to f$ [unif] on A as we wished to show. ■

One often restricts one's attention to some measurable subset E of X. It is clear how the concepts and results of this section apply to this setting. For example, if $\mu(E) < \infty$, then $f_n \to f$ [a.u.] on E whenever $f_n \to f$ [a.e.] on E, even if $\mu(X) = \infty$.

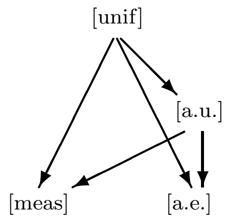

Figure 4.1: Comparison of modes of convergence in a general measure space.

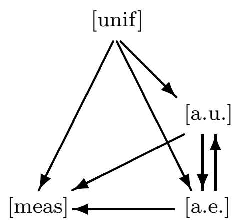

Figure 4.2: Comparison of modes of convergence in a *finite* measure space.

We summarize our comparison of the modes of convergence with two figures.[1] In each case, we assume that $\{f_n\}$ is a sequence of finite a.e., measurable functions on X. Figure 4.1 shows the situation in a general measure space. Figure 4.2 gives the implications that are valid when $\mu(X) < \infty$. Where an arrow is missing, a counterexample is needed. The sliding sequence of Example 4.13 shows that convergence in measure does not imply any of the other forms of convergence, even when $\mu(X) < \infty$. The sequence $\{x^n\}$ shows uniform convergence is not implied by any other form of convergence. Finally, the sequence $\{x/n\}$ shows that convergence [a.e.] does not in general imply convergence [a.u.] or convergence [meas].

We view the implications given in the figures as preliminary comparisons of four forms of convergence. In Chapter 5, we shall study a fifth form of convergence, called *mean convergence*, and indicate its "place" in the diagrams. We shall also provide a third diagram that applies even when $\mu(X) = \infty$ if functions in the sequence are suitably dominated by some integrable function. Exercise 4:3.4 provides an example in this spirit, but not expressed in the language of integration.

[1]These figures have been popular for many years, since appearing in M. E. Munroe, *Introduction to Measure and Integration*, Addison-Wesley (1953).

Exercises

4:3.1 Prove that if $f_n \to f$ [a.u.] on X then $f_n \to f$ [a.e.] on X and $f_n \to f$ [meas] on X.

4:3.2 By quoting results of this section or by other means, verify each implication appearing in the figures. Also verify that no additional implications can be added to the diagrams.

4:3.3 Let α_n be a sequence of positive numbers converging to zero. If f is continuous, then certainly $f(x - \alpha_n)$ converges to $f(x)$. Find a bounded measurable function on $[0, 1]$ such that the sequence of functions $f_n(x) = f(x - \alpha_n)$ is not a.e. convergent to f. [Hint: Take the characteristic function of a Cantor set of positive measure.]

4:3.4◇ Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $[0, \infty)$ such that $|f_n(x)| \leq e^{-x}$ for all $x \in [0, \infty)$. If $f_n \to 0$ [a.e.], then $f_n \to 0$ [a.u.]. [Hint: The only place where we used our assumption that $\mu(X) < \infty$ in the proof of Theorem 4.16 was to obtain the limit in equation (1).]

4:3.5 Prove another version of Egoroff's theorem:

> **Theorem** *Let $(X, \mathcal{M}, \mu)$ be a finite or σ-finite measure space. Let $\{f_n\}$ be a sequence of finite a.e., measurable functions such that $f_n \to f$ [a.e.]. Then there is a partition of X into a sequence E_0, E_1, E_2, ... of disjoint measurable sets such that $\mu(E_0) = 0$ and $f_n \to f$ uniformly on each E_i, $i \geq 1$.*

4.4 Approximations by Simple Functions

A recurring theme in our development has been to find approximations to complicated objects by simpler ones. Naturally, we wish to do the same for measurable functions. The simplest measurable functions in a general space are those that are linear combinations of characteristic functions of measurable sets. In this section we show that these *simple* functions can be used to approximate general measurable functions. The simplest measurable functions in a metric space are continuous. In the next section we show that all measurable functions in a metric space furnished with an appropriate measure can be approximated by continuous functions.

We have not seen many examples of measurable functions and may not appreciate just how they come about or just how complicated they may, at first, appear. Thus it is instructive to begin with an example that exhibits some interesting features.

Example 4.17 We work on the interval $I_0 = (0, 1)$. Each $x \in I_0$ has a unique base 2 expansion $x = .a_1a_2\ldots$ that does not end in a string of

1's. For each $i \in \mathbb{N}$, a_i is a function of x with only a finite number of discontinuities. Thus a_i is Borel measurable. For each $n \in \mathbb{N}$, let

$$f_n(x) = \frac{a_1(x) + a_2(x) + \cdots + a_n(x)}{n}.$$

Finally, let

$$f(x) = \limsup_n f_n(x).$$

One verifies easily that f is Borel measurable. Observe that, while $f_n(x)$ depends only on the first n bits in the binary expansion of x, f depends only on the "tail" of the expansion. If

$$x = .a_1 a_2 \ldots \quad \text{and} \quad y = .b_1 b_2 \ldots$$

and if there exists $j, N \in \mathbb{N}$ such that

$$b_{k+j} = a_k \quad \text{for all} \quad k \geq N,$$

then $f(x) = f(y)$. One can also verify that, for every nondegenerate interval $I \subset I_0$, f maps I onto the interval [0,1]. For example, any x whose expansion has the tail $.\overline{1000}$ will map onto $\frac{1}{4}$ (decimal), and the set of all such x is dense in I_0. (Some other features of f and related functions appear in Exercise 4:4.1.)

Notice one remarkable feature of the Borel measurable function f: it takes every one of its values on a dense set.[2] Despite this apparent complexity, we can still approximate such a function by much simpler functions, indeed by a continuous function as we will see in the next section.

Definition 4.18 Let $E_1, E_2, \ldots, E_n$ be pairwise disjoint measurable sets, and let $c_1, c_2, \ldots, c_n$ be real numbers. Let

$$f(x) = c_1 \chi_{E_1} + \cdots + c_n \chi_{E_n}.$$

Then f is called a *simple* function.

Thus a simple function is one that takes on only finitely many real values, each on a measurable set. Our restriction that the sets E_i be measurable guarantees that simple functions are measurable.

Theorem 4.19 (Approximation by simple functions)
Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be measurable on X. Then there exists a sequence $\{f_n\}$ of simple functions such that

$$\lim_n f_n(x) = f(x) \quad \textit{for all} \quad x \in X.$$

If $f(x) \geq 0$ for all $x \in X$, the sequence $\{f_n\}$ can be chosen to be a nondecreasing sequence, so that $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$. If f is bounded on X, then $f_n \to f$ [unif].

[2]Functions with this property can arise quite naturally. See Exercise 7:9.15.

Proof. Suppose first that f is nonnegative. Fix $n \in \mathbb{N}$. For each $k = 1, 2, \ldots, n2^n$, let

$$J_k = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right).$$

Let

$$f_n(x) = \begin{cases} \frac{k-1}{2^n}, & \text{if } f(x) \in J_k; \\ n, & \text{otherwise.} \end{cases}$$

The intervals J_k are pairwise disjoint, and

$$\bigcup_{k=1}^{n2^n} J_k = [0, n).$$

Since f is measurable, so is the function f_n. It is clear that f_n is a simple function and that $f_n(x) \leq f(x)$ for all $x \in X$. It is also clear for every $x \in X$, that $f_{n+1}(x) \geq f_n(x)$. Also

$$f_{n+1}(x) - f_n(x) \leq \frac{1}{2^{n+1}}$$

if $f(x) \leq n$, and

$$f_{n+1}(x) - f_n(x) \leq 1$$

if $n < f(x)$. It follows that

$$\lim f_n(x) = f(x)$$

and that the convergence is uniform if f is bounded. [Indeed, if $0 \leq f(x) \leq M$ for all $x \in X$, then

$$f_{n+1}(x) - f(x) \leq \frac{1}{2^n}$$

for all $n \geq M$, so that the convergence is uniform.]

In the general case, f need not be nonnegative. Let

$$f^+(x) = \begin{cases} f(x), & \text{if } f(x) \geq 0; \\ 0, & \text{if } f(x) < 0; \end{cases}$$

and let

$$f^-(x) = \begin{cases} -f(x), & \text{if } f(x) < 0; \\ 0, & \text{if } f(x) \geq 0. \end{cases}$$

Then $f = f^+ - f^-$. Each of the functions f^+ and f^- is measurable and nonnegative. Thus there exist sequences $\{g_n\}$ and $\{h_n\}$ of simple functions having the desired properties with respect to f^+ and f^-, respectively. For each $n \in \mathbb{N}$, let

$$f_n = g_n - h_n.$$

The sequence $\{f_n\}$ has all the required properties. ∎

Our next result provides a sense of how measurable functions that are finite a.e. can be approximated by bounded measurable functions. Since these, in turn, can be approximated uniformly by simple functions, we are close to an understanding of the structure of arbitrary measurable functions.

Theorem 4.20 *Suppose that f is finite a.e. and measurable on X with $\mu(X) < \infty$, and let $\varepsilon > 0$. Then there exists a bounded measurable function g such that*

$$\mu\left(\{x : g(x) \neq f(x)\}\right) < \varepsilon.$$

Proof. Let

$$A_\infty = \{x : |f(x)| = \infty\},$$

and for every $k \in \mathbb{N}$ let

$$A_k = \{x : |f(x)| > k\}.$$

By hypothesis, $\mu(A_\infty) = 0$. The sequence $\{A_k\}$ is a descending sequence of measurable sets, and $A_\infty = \bigcap_{k=1}^{\infty} A_k$. Since $\mu(X) < \infty$, it follows from Theorem 2.20 (2) that

$$\lim_k \mu(A_k) = \mu(A_\infty) = 0.$$

Thus there exists $K \in \mathbb{N}$ such that $\mu(A_K) < \varepsilon$. Let

$$g(x) = \begin{cases} f(x), & \text{if } x \in \widetilde{A_K}; \\ 0, & \text{if } x \in A_K. \end{cases}$$

Then g is measurable, and $|g(x)| \leq K$ for all $x \in X$. Now

$$\{x : g(x) \neq f(x)\} = A_K$$

and $\mu(A_K) < \varepsilon$, so g is the required function. ■

Exercises

4:4.1◇ Let f be the function on $(0,1)$ defined in Example 4.17.

(a) Prove that $f(I) = [0,1]$ for every open interval $I \subset I_0$. That is, for every $c \in [0,1]$, the set $f^{-1}(c)$ is dense in I_0.

(b) Prove that the graph of f is dense in $I_0 \times [0,1]$.

(c) Let

$$g(x) = \begin{cases} f(x), & \text{if } f(x) \neq x; \\ 0, & \text{if } f(x) = x. \end{cases}$$

Show that g has the properties of f given in (a) and (b).

(d) Show that the graph of g is not a connected subset of $\mathbb{R}^2$.

(e) Show that $h(x) = g(x) - x$ does not have the Darboux property.

We have mentioned that some nineteenth century mathematicians believed that the Darboux property (intermediate-value property) should be taken as a definition of continuity. They obviously were not aware of functions such as f and g above, nor of the function $h(x) = g(x) - x$. The function h is the sum of a Darboux function with a genuinely continuous function.

4:4.2 Show that the class of simple functions on a measure space is closed under linear combinations and products.

4:4.3 Characterize those functions that can be expressed as uniform limits of simple functions.

4:4.4 Let $I_1, I_2, \dots, I_n$ be pairwise disjoint intervals with $[a,b] = \bigcup_{k=1}^n I_k$, and let $c_1, \dots, c_n$ be real numbers. Let $f = \sum_{k=1}^n c_k \chi_{I_k}$. Then f is called a *step function*.

(a) Show that every step function is a simple function for Lebesgue measure.

(b) Show that the proof of Theorem 4.19 applied to the function $f(x) = x$ on $[a,b]$ shows that f can be expressed as a uniform limit of step functions.

(c) Can every bounded measurable function on $[a,b]$ be expressed as a uniform limit of step functions?

(d) Characterize those functions that can be expressed as uniform limits of step functions. (This is harder.)

4:4.5 Let $f : X \to [0, +\infty]$ be measurable, and let $\{r_k\}$ be any sequence of positive numbers for which $r_k \to 0$ and $\sum_{k=1}^\infty r_k = +\infty$. Then there are measurable sets $\{A_k\}$ so that

$$f(x) = \sum_{k=1}^\infty r_k \chi_{A_k}(x)$$

at every $x \in X$. [Hint: Inductively define the sets

$$A_k = \left\{ x \in X : f(x) \geq r_k + \sum_{j<k} r_j \chi_{A_j}(x) \right\}.$$

4.5 Approximation by Continuous Functions

We turn now to the problem of approximating a measurable function by a continuous one. We shall show that, under suitable hypotheses, we can redefine a measurable function f on a small set so that the new function g is continuous.

Throughout this section we take X to be a metric space and μ to be a Borel measure with $\mu(X) < \infty$. We also assume the following.

4.21 *If E is measurable and $\varepsilon > 0$, then there exists a closed set $F \subset E$ such that $\mu(E \setminus F) < \varepsilon$.*

We recall that when E is also a Borel set this inner approximation by a closed set is always available (see Corollary 3.14). The force of this assumption is that all *measurable* sets are assumed to have the same property. For example, if μ is a Lebesgue–Stieltjes measure on $\mathbb{R}$ with $\mu(\mathbb{R}) < \infty$, Theorem 3.19 (3) can be used to show that assertion 4.21 applies.

Before we embark on our program of approximating measurable functions, even badly behaved ones like the function f of Example 4.17, by continuous functions, we discuss briefly the notions of *relative continuity* and *extendibility*.

Suppose that X is a metric space, and $S \subset X$. Let $f : X \to \mathbb{R}$, and let $s_0 \in S$. The statement that f is continuous at s_0 means that

$$\lim_{x \to s_0} f(x) = f(s_0).$$

It may be that f is discontinuous at s_0, but continuous at s_0 relative to the set S, that is

$$\lim_{x \to s_0,\ x \in S} f(x) = f(s_0).$$

In other words, the restriction of the function f to the set S is continuous at s_0. It is possible that $f|S$ is continuous, but cannot be extended to a function continuous on all of X. For example, $f(x) = \sin x^{-1}$ is continuous on $S = (0, 1]$, but cannot be extended to a continuous function on $[0, 1]$. For that, one needs f to be uniformly continuous on S. We make use of the Tietze extension theorem that we will establish in Chapter 9 in greater generality for functions defined on metric spaces. We prove it here only for the case of functions on the real line.

Theorem 4.22 (Tietze extension theorem) *Let S be a closed subset of a metric space X and suppose that $f : S \to \mathbb{R}$ is continuous. Then f can be extended to a continuous function g defined on all of X. Furthermore, if $|f(x)| \leq M$ on S, then $|g(x)| \leq M$ on X.*

Proof. For $X = \mathbb{R}$, this is easy to prove. Let $\{(a_n, b_n)\}$ be the sequence of intervals complementary to S. Define g to be equal to f on S, and to be linear and continuous on each interval $[a_n, b_n]$ if $-\infty < a_n < b_n < \infty$. If $a_n = -\infty$ or $b_n = \infty$, we define g to be the appropriate constant on $(-\infty, b_n]$ or $[a_n, \infty)$. One verifies easily that g is continuous on $\mathbb{R}$. Note also that if $|f(x)| \leq M$ on S then $|g(x)| \leq M$ on $\mathbb{R}$. ■

We shall use the Tietze extension theorem in conjunction with "inside" approximation of measurable sets by closed sets. For this we shall use Corollary 3.14. We approximate X by closed sets. On these closed sets we shall obtain continuous functions that approximate our measurable function f. These functions can, in turn, be extended to functions continuous on all of X. We shall obtain a succession of theorems, each improving the sense of approximation of f by continuous functions. Each of these theorems is of interest in itself. The theorems culminate in an important theorem discovered independently by Guiseppe Vitali (1875–1932) and Nikolai Lusin (1883–1950). It is almost universally called Lusin's theorem. It asserts that for every $\varepsilon > 0$ there is a continuous function g defined on X such that $g = f$ except on a set of measure less than ε. (Lusin, often transliterated as Luzin, was a student of Egoroff, who is known mainly for the theorem on almost uniform convergence that we have just seen in the preceding section.)

Since we have not yet proved the Tietze extension theorem in a general metric space, the reader may wish to take X in the theorem to be an interval $[a, b]$ in $\mathbb{R}$.

Theorem 4.23 *Let $(X, \mathcal{M}, \mu)$ be a finite measure space with X a metric space and μ a Borel measure. Suppose that $\mathcal{M}$ satisfies condition 4.21. Let f be finite a.e. and measurable on X. Then to each pair (ε, η) of positive numbers corresponds a bounded, continuous function g such that*

$$\mu(\{x : |f(x) - g(x)| \geq \eta\}) < \varepsilon.$$

Furthermore, if $|f(x)| \leq M$ on X, then one can choose g so that $|g(x)| \leq M$ on X.

Proof. Suppose first that $|f(x)| \leq M$ on X. By Theorem 4.19 there exists a simple function h, also bounded by M, such that

$$|h(x) - f(x)| < \eta \qquad (x \in X).$$

Let $c_1, \ldots, c_m$ be the values that h assumes on X, and for each $i = 1, \ldots, m$ let

$$E_i = \{x : h(x) = c_i\}.$$

The sets E_i are pairwise disjoint and cover X. Choose closed sets $F_1, \ldots, F_m$ such that, for each $i = 1, \ldots, m$, $F_i \subset E_i$ and

$$\mu(E_i \setminus F_i) < \frac{\varepsilon}{m}.$$

Let

$$F = F_1 \cup \cdots \cup F_m.$$

Then F is closed, $F \subset X$ and $\mu(X \setminus F) < \varepsilon$. Furthermore, the restriction of h to F_i, $h|F_i$, is constant for $i = 1, \ldots, m$. It follows that $h|F$ is continuous.

To see this, we need only note that, if $x_0 \in F_i$ and $x_n \to x_0$ with $x_n \in F$ for all n, then for n sufficiently large $x_n \in F_i$, a set on which h is constant. By the Tietze extension theorem the function $h|F$ can be extended to a function g continuous on X with $|g(x)| \leq M$ on X. Since

$$\mu(X \setminus F) < \varepsilon,$$

g is the desired function.

The general case in which we do not assume f bounded now follows readily from Theorem 4.20. ■

Theorem 4.24 *Let $(X, \mathcal{M}, \mu)$ be a finite measure space with X a metric space and μ a Borel measure. Suppose that $\mathcal{M}$ satisfies condition 4.21. Let f be finite a.e. and measurable on X. There exists a sequence $\{g_k\}$ of bounded, continuous functions for which $g_k \to f$ [a.u.].*

Proof. It follows immediately from Theorem 4.23 that there exists a sequence $\{f_n\}$ of continuous functions for which $f_n \to f$[meas]. By Theorem 4.14, there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ [a.e.]. The desired conclusion now follows from Egoroff's theorem, by defining $g_k = f_{n_k}$. ■

We are now ready to state and prove the main theorem of this section.

Theorem 4.25 (Lusin) *Let $(X, \mathcal{M}, \mu)$ be a finite measure space with X a metric space and μ a Borel measure. Suppose that $\mathcal{M}$ satisfies condition 4.21. Let f be finite a.e. and measurable on X, and let $\varepsilon > 0$. There exists a continuous function g on X such that $f(x) = g(x)$ for all x in a closed set F with $\mu(\widetilde{F}) < \varepsilon$. If $|f(x)| \leq M$ for all $x \in X$, we can choose g to satisfy $|g(x)| \leq M$ for all $x \in X$.*

Proof. By Theorem 4.24, there exists a measurable set E such that $\mu(\widetilde{E}) < \varepsilon/2$ and a sequence $\{g_k\}$ of continuous functions on X such that $g_k \to f$ [unif] on E. By condition 4.21, there exists a closed set $F \subset E$ such that $\mu(\widetilde{F}) < \varepsilon$. Since $g_k \to f$ [unif] on E, the restriction $f|F$ of f to F is continuous. By Tietze's theorem, this function can be extended to a function g continuous on all of X, so that g and f have the same bounds on X. ■

Let us return for a moment to Example 4.17. How complicated must a continuous function g be to approximate the function f of that example in the Lusin sense? A theorem in number theory asserts that almost every number in $[0, 1]$ is "normal".[3] This means that for almost all $x \in [0, 1]$ the binary expansion of x has, in the limit, half the bits equaling zero and half equaling one. More precisely, for almost every x in the interval $[0, 1]$ with $x = .a_1a_2a_3\ldots$ the binary expansion of x, it is true that

$$\lim_n \frac{a_1 + \cdots + a_n}{n} = \tfrac{1}{2}.$$

Thus the function f in Example 4.17 satisfies $f(x) = \frac{1}{2}$ a.e. In other words, we can choose $g \equiv \frac{1}{2}$ and conclude that $f = g$ a.e. The approximation was not so difficult in this case! Here we have a much stronger result than Lusin's theorem guarantees. The exceptional set has measure zero.

[3]See Hardy and Wright, *An Introduction to the Theory of Numbers*, Oxford (1938).

When we approximate measurable sets by simpler sets, we get the following results. If we are willing to ignore sets of arbitrarily small measure, we can take the approximating sets to be open or closed. If we are willing to ignore only zero measure sets, we must give up a bit of the regularity of the approximating sets—we can use sets of type $\mathcal{G}_\delta$ on the outside and sets of type $\mathcal{F}_\sigma$ on the inside.

The analogous situation for the approximation of measurable functions would suggest something similar. If we are willing to ignore sets of arbitrarily small measure, we can choose the approximating functions to be continuous. This is Lusin's theorem. Observe that for a continuous function g the associated sets

$$\{x : \alpha < g(x) < \beta\} \text{ and } \{x : \alpha \le g(x) \le \beta\}$$

are open and closed, respectively. One might expect that, if one is willing to ignore only sets of measure zero, we can choose the approximating functions g in the *first Borel class*; that is, one for which the corresponding associated sets are of type $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$, respectively. This is not quite the case. Instead, g can be taken from the *second Borel class* where the associated sets are of type $\mathcal{G}_{\delta\sigma}$ and $\mathcal{F}_{\sigma\delta}$, respectively. Exercise 4:6.2 at the end of the chapter deals with the Borel and Baire classes of functions and with how one can approximate measurable functions by functions from these classes.

Exercises

4:5.1 Complete the proof of Theorem 4.23 for the case f unbounded.

4:5.2 Show that Lusin's theorem is valid on $(\mathbb{R}, \mathcal{M}, \mu_f)$, where μ_f is a Lebesgue–Stieltjes measure, even if $\mu_f(\mathbb{R}) = \infty$.

4:5.3 Let $X = \mathbb{Q} \cap [0,1]$ and $\mathcal{M} = 2^X$.

(a) Let μ be the counting measure on X, let Q_1 and Q_2 be complementary dense subsets of X, and let $f = \chi_{Q_1}$. Show that the conclusion of Lusin's theorem fails. What hypotheses in Lusin's theorem fail here?

(b) Let $r_1, r_2, r_3, \ldots$ be an enumeration of the rationals, and let μ be the measure that assigns value 2^{-i} to the singleton set $\{r_i\}$. Let f be as in (a). Show how to construct the function g called for in the conclusion of Lusin's theorem.

4:5.4 The purpose of this exercise is to show the essential role that the regularity condition 4.21 plays in the hypotheses of Lusin's theorem. Let E be a subset of $[0,1]$ such that both E and $\widetilde{E}$ are totally imperfect (see Section 3.10). Let $f = \chi_E$. Let g be Lebesgue measurable, and suppose that $L = \{x : f(x) = g(x)\} \in \mathcal{L}$.

(a) Show that $\lambda_*(E) = 0$ and $\lambda^*(E) = 1$.

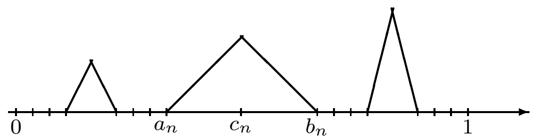

Figure 4.3: Construction of f in Exercise 4:6.1.

(b) Show that
$$E \cap L = \{x : f(x) = 1\} \cap L = \{x : g(x) = 1\} \cap L$$
and hence that $E \cap L \in \mathcal{L}$. Similarly, show that $\widetilde{E} \cap L \in \mathcal{L}$.

(c) Show that $E \cap L \subset E$ and $\lambda_*(E) = 0$, and hence that $\lambda(E \cap L) = 0$. Similarly show that $\lambda(\widetilde{E} \cap L) = 0$ and $\lambda(L) = 0$.

We have shown that if $\lambda_*(E) = 0$ and $\lambda^*(E) = 1$, for $E \subset [0, 1]$, then the function χ_E is not λ-measurable on any set of positive Lebesgue measure. We now use this fact to show that Lusin's theorem can fail dramatically when the condition 4.21 is not hypothesized.

Refer to Exercise 3:11.13. Let $\overline{\overline{\lambda}}$ be the extension of λ to the σ-algebra generated by $\mathcal{L}$ and $\{E\}$. Note that the measure space $([0, 1], \mathcal{M}, \overline{\overline{\lambda}})$ does not satisfy the assertion 4.21.

(d) Show that $\overline{\overline{\lambda}}(L) = \lambda(L) = 0$.

Thus the $\overline{\overline{\lambda}}$-measurable function f does not agree with any function that is λ-measurable even on a set of positive Lebesgue measure. In particular, if g is continuous and $f(x) = g(x)$ for all x in a closed set F, then $\overline{\overline{\lambda}}(F) = \lambda(F) = 0$.

(e) Give an example of a λ-measurable function g (even a continuous one) such that $\overline{\overline{\lambda}}(\{x : f(x) = g(x)\}) = 1$.

4.6 Additional Problems for Chapter 4

4:6.1 Let K be the Cantor ternary set, and let $\{(a_n, b_n)\}$ be the sequence of intervals complementary to K in $(0, 1)$. For each $n \in \mathbb{N}$, let $c_n = (a_n + b_n)/2$. Let $f = 0$ on K be linear and continuous on $[a_n, c_n]$ and on $[c_n, b_n]$, with the values $f(c_n)$ as yet unspecified (see Figure 4.3). What conditions on the values $f(c_n)$ are necessary and sufficient (a) for f to be continuous, (b) for f to be a Baire 1 function, or (c) for f to be of bounded variation? (See Exercise 4:6.2).

4:6.2◇ (Baire functions and Borel functions) For this problem, all functions are assumed finite unless explicitly stated otherwise. Let $\mathcal{B}_0$ consist of the continuous functions on an interval $X \subset \mathbb{R}$. We do not assume X bounded.

(a) For $n \in \mathbb{N}$, let $\mathcal{B}_n$ consist of those functions that are pointwise limits of sequences of functions in $\mathcal{B}_{n-1}$. The class $\mathcal{B}_n$ is called the *Baire functions of class n* or *the Baire-n functions*. Prove that if $f \in \mathcal{B}_1$ then, for all $\alpha \in \mathbb{R}$, the sets $\{x : f(x) > \alpha\}$ and $\{x : f(x) < \alpha\}$ are of type $\mathcal{F}_\sigma$.

(b) If $f \in \mathcal{B}_2$, show that for all $\alpha \in \mathbb{R}$ the sets $\{x : f(x) > \alpha\}$ and $\{x : f(x) < \alpha\}$ are of type $\mathcal{G}_{\delta\sigma}$.

(c) Show that a function $f : X \to \mathbb{R}$ that is continuous except on a countable set is in $\mathcal{B}_1$. (Compare with Exercise 4:1.13.)

(d) Let $f = \chi_{\mathbb{Q}}$. Show that $f \in \mathcal{B}_2 \setminus \mathcal{B}_1$.

(e) Prove that $\mathcal{B}_1$ is closed under addition and multiplication.

(f) Let $\{M_n\}$ be a sequence of positive numbers and suppose that $\sum_{n=1}^\infty M_n < \infty$. Let $\{f_n\} \subset \mathcal{B}_1$ with $|f_n(x)| \le M_n$ for all $n \in \mathbb{N}$ and all $x \in X$. Prove that $\sum_{n=1}^\infty f_n \in \mathcal{B}_1$.

(g) Prove that if $f_n \to f$ [unif] and $f_n \in \mathcal{B}_1$ for all $n \in \mathbb{N}$ then $f \in \mathcal{B}_1$. [Hint: Choose an increasing sequence $\{n_k\}$ of positive integers such that $\lim_k n_k = \infty$ and $|f_{n_k}(x) - f(x)| < 2^{-k}$ on X. Then apply (f) appropriately.]

(h) Prove that the composition of a function $f \in \mathcal{B}_1$ with a continuous function is in $\mathcal{B}_1$.

(i) Prove the converse to part (a): If for every $\alpha \in \mathbb{R}$ the sets $\{x : f(x) > \alpha\}$ and $\{x : f(x) < \alpha\}$ are of type $\mathcal{F}_\sigma$, then $f \in \mathcal{B}_1$.

(j) Prove that if f is differentiable then $f' \in \mathcal{B}_1$.

(k) Prove that if $\{f_n\} \subset \mathcal{B}_1$ then $\sup f_n \in \mathcal{B}_2$.

(l) Prove that if $\{f_n\} \subset \mathcal{B}_0$ then $\limsup_n f_n \in \mathcal{B}_2$.

(m) Prove that if f is finite a.e. and measurable on X then there exists $g \in \mathcal{B}_2$ such that $f = g$ a.e.

(n) Give an example of a finite Lebesgue measurable function on $\mathbb{R}$ that agrees with no $g \in \mathcal{B}_1$ a.e. [Hint: Let $f = \chi_A$ where $\lambda(I \cap A) > 0$ and $\lambda(I \cap \widetilde{A}) > 0$ for every open interval I. Show that if $g \in \mathcal{B}_1$ and $g = f$ a.e. then $\{x : g(x) = 0\}$ and $\{x : g(x) = 1\}$ are disjoint, dense subsets of $\mathbb{R}$ of type $\mathcal{G}_\delta$. This violates the Baire category theorem for $\mathbb{R}$.]

(o) The smallest class of functions that contains $\mathcal{B}_0$ and is closed under the operation of taking pointwise limits is called the class of *Baire functions*. It is true, though difficult to prove, that for each $n \in \mathbb{N}$ there exists $f \in \mathcal{B}_{n+1} \setminus \mathcal{B}_n$. Show that there exists a

Baire function g on $X = [0, \infty)$ that is not in any of the classes $\mathcal{B}_n$. [Hint: Let $g \in \mathcal{B}_{n+1} \setminus \mathcal{B}_n$ on $[n, n + 1)$.]

This function is in the class $\mathcal{B}_\omega$, where ω is the first infinite ordinal. One then defines $\mathcal{B}_{\omega+1}$ as those functions that are limits of sequences of functions in $\mathcal{B}_\omega$. Using transfinite induction, one obtains classes $\mathcal{B}_\gamma$ for every countable ordinal. One can show that for every countable ordinal γ there exist functions $f \in \mathcal{B}_\gamma \setminus \bigcup_{\beta<\gamma} \mathcal{B}_\beta$. One can also show that the class of Baire functions on the interval X is exactly the class of Borel measurable functions.

(p) Use the fact that there are Lebesgue measurable sets that are not Borel sets to show that there are Lebesgue measurable functions that are not Baire functions.

4:6.3 Show that a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous in each variable separately is a Baire 1 function. (This is the original problem that led Baire to this line of research.) [Hint: Define

$$F_n(x, y) = f((i + 1)2^{-n}, y)[x - i2^{-n}] - f(i2^{-n}, y)[x - (i + 1)2^{-n}]$$

if $i2^{-n} \le x < (i + 1)2^{-n}$ for some integer i. Show that F_n is continuous on $\mathbb{R}^2$ and $2^n F_n \to f$ pointwise.]

4:6.4 Construct a function $f : [0, 1] \to [0, 1]$ as follows. Let $\{I_n\}$ be an enumeration of the open intervals in $[0, 1]$ having rational endpoints. For each $n \in \mathbb{N}$, let $K_n \subset I_n$ be a Cantor set of positive Lebesgue measure such that the sequence $\{K_n\}$ is pairwise disjoint and $\sum_{n=1}^{\infty} \lambda(K_n) = 1$. Define f_n on K_n to be continuous on K_n, nondecreasing, and such that $f_n(K_n) = [0, 1]$. Let

$$f(x) = \begin{cases} f_n(x), & \text{if } x \in K_n; \\ 0, & \text{if } x \in [0, 1] \setminus \bigcup_{n=1}^{\infty} K_n. \end{cases}$$

(a) Show that f is Lebesgue measurable.

(b) Show that $f(I) = [0, 1]$ for every open interval $I \subset [0, 1]$.

(c) Using the sets K_n, find continuous functions on $[0, 1]$ that approximate f in the Lusin sense.

(d) Refer to Exercise 4:6.2. Does there exist $g \in \mathcal{B}_1$ such that $g = f$ [a.e.]?

(e) Give an example of a function $g \in \mathcal{B}_2$ for which $f = g$ [a.e.].

4:6.5 Measurability can be expressed as a separation property. Let μ^* be an outer measure on a space X. Show that a function $f : X \to [-\infty, +\infty]$ is measurable with respect to μ^* if and only if

$$\mu^*(T) \ge \mu^*(T \cap \{x \in X : f(x) \le a\}) + \mu^*(T \cap \{x \in X : f(x) \ge b\})$$

for all $T \subset X$ and all $-\infty < a < b < +\infty$.

4:6.6 Let $(X, \mathcal{M}, \mu)$ be a measure space and, for every measurable function $f : X \to [-\infty, +\infty]$, define

$$\|f\|_\mu = \inf \{r : \mu(\{x : |f(x)| > r\}) \leq r\}.$$

(a) Show that $\mu\left(\{x : |f(x)| > \|f\|_\mu\}\right) \leq \|f\|_\mu$.

(b) Check the triangle inequality $\|f+g\|_\mu \leq \|f\|_\mu + \|g\|_\mu$.

(c) Show that $f_n \to f$ in μ–measure if and only if $\|f_n - f\|_\mu \to 0$.

(d) If $f = \chi_A$, then show that $\|cf\|_\mu = \inf\{c, \mu(A)\}$ for any $0 \leq c < \infty$. In particular, it is not true in general that $\|cf\|_\mu = c\|f\|_\mu$.

(e) Show that, for $c > 0$,

$$\|cf\|_\mu \leq \max\left\{\|f\|_\mu, c\|f\|_\mu\right\}$$

and hence that $\|cf\|_\mu \to 0$ as $\|f\|_\mu \to 0$.

(f) Show that if $\mu(\{x : f(x) \neq 0\}) < \infty$ then $\|cf\|_\mu \to 0$ as $c \to 0$.

(g) Show that if

$$\sum_{k=1}^{\infty} \|g_{k+1} - g_k\|_\mu < \infty$$

then $\{g_k\}$ converges to some function g μ–almost everywhere, and $\|g_k - g\|_\mu$ converges to 0.

(h) Show that every Cauchy sequence $\{g_k\}$ in measure has a subsequence that converges both μ–almost everywhere and in measure. [Hint: Pick an increasing sequence $N(k)$ so that

$$\|g_i - g_j\|_\mu \leq 2^{-n}$$

whenever $i \geq j \geq N(k)$.]

Chapter 5

INTEGRATION

We are now ready to develop a theory of the integral based on our studies of measure spaces and measurable functions.

We develop all the basic tools of integration theory in this chapter. Sections 5.2, 5.3, and 5.4 define the integral for measurable nonnegative functions and then for measurable real-valued functions and establish the most immediate properties. The integral can be viewed as a signed measure. This viewpoint is explored in Sections 5.6 and 5.7 and culminates in the important and useful Radon–Nikodym theorem in Section 5.8. A deeper perspective on the Radon–Nikodym theorem will be given in Chapters 7 and 8. The convergence theorems available for the integral appear in Section 5.9.

The integral as defined here is a formidably different object than the simple limit of Riemann sums that one studies in elementary courses. It is by no means obvious from the definitions what relation, if any, this theory has to previous integrals studied when it is placed in the context of Lebesgue measure on the real line. Section 5.5 discusses in detail the relation between the classical Riemann integral and the Lebesgue integral and gives as well a simple version of the fundamental theorem of the calculus for the latter. Section 5.10 continues this theme by comparing the integral here with the improper calculus integral and the generalized Riemann integral. Both Sections 5.5 and 5.10 can be omitted, but there are good cultural reasons for wanting to know such things.

Finally, Section 5.11 gives an account showing how to extend the definition of the integral to complex-valued functions. This is needed for several sections in later chapters where integration of complex-valued functions is used. The related subject of complex-valued measures is developed in exercises at the end of the chapter.

Before proceeding with this program, we shall begin in Section 5.1 with a discussion of the Riemann integral, with special attention to its limitations and how the integral we shall define compares. We shall discover that the class of Riemann integrable functions is not wide enough to include functions that arise from natural limit processes. The reader who feels no need for background and motivation can proceed directly to Section 5.2, where the integral is defined.

5.1 Introduction

Scope of the Concept of Integral

The Riemann integral of a real function f on an interval $[a, b]$ is defined as a limit of sums

$$\int_a^b f(x)\, dx = \lim \sum_{i=1}^{n} f(\xi_i)(x_{i+1} - x_i). \tag{1}$$

The way the limit is taken in (1) restricts the scope of the integral to bounded functions f that are a.e. continuous and restricts the domain to a compact interval $[a, b]$.

It is important to relax these restrictions. The procedures of Cauchy (see Section 1.16) for handling improper integrals allow a modest extension of the integral to accommodate some unbounded functions and unbounded intervals. The domain could be enlarged by defining an integral over sets

$$\int_A f(t)\ dt = \int_a^b f(t)\chi_A(t)\ dt,$$

provided that this exists. Even so, the classes of sets A and functions f for which such a procedure is successful are too small. For example, one might want to integrate a function over the set of its points of differentiability and that set can be too complicated for this method. Moreover, the class of Riemann integrable functions on an interval $[a, b]$ is not closed under the standard limit operations, even when questions of unboundedness do not create problems.

There is also the problem of generalizations. The definition of the Riemann integral extends naturally to functions defined on certain subsets of $\mathbb{R}^n$, but in spaces that do not have this simple geometry a Riemann-type integral would be hard to conceive. There are many other spaces for which a concept of integration is needed. The elements of such spaces need not be points in $\mathbb{R}^n$; they could be other objects such as sequences or functions.

The integral we define in this chapter successfully addresses all these problems. Our framework will involve an arbitrary measure space $(X, \mathcal{M}, \mu)$. Here X can be any set. The integral makes sense for any nonnegative or nonpositive measurable function defined on any measurable set E. For measurable functions f that take both positive and negative values, the integral makes sense unless both its positive and negative parts f^+ and f^- have infinite integrals over the set E. Since the class of measurable sets is a σ-algebra, and the class of measurable functions is closed under diverse operations, the various necessary manipulations with sets and functions will not take us out of our framework.

An entirely different approach to the problem of extending the Riemann integral can be taken. Instead of developing an integral within the context of a measure space, one could seek to reinterpret the limit operation in (1) in some broader sense. Integrals based on Riemann sums have received considerable attention in recent years, because they can solve certain problems in $\mathbb{R}^n$ that the Lebesgue integral cannot. These integrals generalize sufficiently to include the Lebesgue integrals and others when dealing with spaces that have certain partitioning properties. We have already indicated some of the ideas in Section 1.21, and in this chapter we develop them a bit further in Section 5.10.

The Class of Integrable Functions

To fix ideas, we work with functions defined on $[0,1]$. Suppose that $\{f_n\}$ is a sequence of Riemann integrable functions that converges pointwise to a function f on $[0,1]$. We would like to be able to assert that, if $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx$ exists, then f is integrable, and

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx = \lim_{n\to\infty}\int_0^1 f_n(x)\,dx.$$

When there is not sufficient control on the size of the functions f_n, the conclusion can fail for all forms of integration. For example, for each $n = 2, 3, \ldots$, define f_n as follows:

$$f_n(0) = f_n\left(\frac{2}{n}\right) = 0, \qquad f_n\left(\frac{1}{n}\right) = n,$$

f_n continuous and linear on $[0,1/n]$ and on $[1/n, 2/n]$, and $f_n(x) = 0$ for all $x \in [2/n, 1]$. See Figure 5.1. In this example,

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = 1 > 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

But for the Riemann integral, the desired conclusion can fail even when $|f_n(x)| \le 1$ for all $n \in \mathbb{N}$ and all $x \in X$, simply because the limit function f is not integrable.

Example 5.1 Let $q_1, q_2, \ldots$ be an enumeration of the set $\mathbb{Q} \cap [0,1]$. For each $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} 1, & \text{if } x = q_1, \ldots, q_n; \\ 0, & \text{otherwise.} \end{cases}$$

Since $f_n = 0$ except on the finite set $q_1, \ldots, q_n$,

$$\int_0^1 f_n(x)\,dx = 0.$$

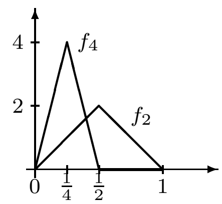

Figure 5.1: Construction of the sequence $\{f_n\}$.

But

$$\lim_{n\to\infty} f_n(x) = \chi_E(x)$$

is a function that is everywhere discontinuous. For any partition P of $[0,1]$ given by $0 = x_0 < x_1 < \cdots < x_n = 1$, the lower and upper Riemann sums of f relative to P are 0 and 1, respectively, so f is not Riemann integrable. Thus $\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = 0$, but

$$\int_0^1 \lim_{n\to\infty} f_n(x)\ dx$$

does not exist as a Riemann integral.

This sort of difficulty disappears when dealing with the integral of this chapter. We shall see that when the sizes of the functions f_n are suitably controlled the limit function will be integrable, and the integral will have the expected value. Furthermore, even convergence in measure will suffice.

We turn now to the fundamental theorem of calculus for Riemann integrals. If f is differentiable on $[a,b]$ *and* f' is Riemann integrable, then

$$f(b) - f(a) = \int_a^b f'(x)\ dx.$$

Within the theory of the Riemann integral this is easy enough to prove, but the hypothesis that f' is Riemann integrable cannot be removed.

The first construction of an everywhere differentiable function with a bounded but nonintegrable derivative was given by Vito Volterra (1860–1940) (see Section 1.18 and Exercise 5:5.5). Here we sketch out an even more interesting example due to D. Pompeiu in 1907 of a strictly increasing differentiable function whose derivative vanishes on a dense set. This derivative cannot be Riemann integrable. (Note that the Cantor function also has a vanishing derivative on a dense set, but it does not offer an example of a Pompeiu type of derivative: it is not differentiable everywhere nor is it strictly increasing.)

Example 5.2 The method employed is due to Cantor and is often described as the "condensation of singularities." The function $f(x) = (x-a)^{\frac{1}{3}}$

has an infinite derivative at $x = a$ and a finite derivative elsewhere. We can construct a function with many more singularities as follows: Let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q} \cap [0,1]$, and for each $n \in \mathbb{N}$, let $f_n(x) = (x - q_n)^{\frac{1}{3}}$. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{10^n}.$$

The series that defines f is uniformly convergent to f, so f is continuous on $[0,1]$. Since each term of the series is strictly increasing, so is f. One would like to assert that f has a derivative at each point of $[0,1]$ and that

$$f'(x) = \sum_{n=1}^{\infty} \frac{f_n'(x)}{10^n} = \sum_{n=1}^{\infty} \frac{(x - q_n)^{-\frac{2}{3}}}{3 \cdot 10^n}, \tag{2}$$

but since the series in (2) does not converge uniformly on $[0,1]$, standard theorems do not apply. Nonetheless, a more delicate argument[1]involving details of the series does verify the validity of (2). In particular, $f'(x) = \infty$ for all $x \in \mathbb{Q} \cap [0,1]$.

The function f maps $[0,1]$ homeomorphically onto an interval $[a,b]$. In particular, $S = f(\mathbb{Q} \cap [0,1])$ is dense in $[a,b]$. Let $h = f^{-1}$. Then h is continuous and strictly increasing on $[a,b]$, and $h' = 0$ on the dense set S. Also, since f has a finite or infinite derivative everywhere and f' is bounded away from zero, h is differentiable and has a bounded derivative.

The fundamental theorem of calculus asserts that if h' is integrable then

$$h(x) - h(a) = \int_a^x h'(t)\, dt$$

for all $x \in [a,b]$. Suppose, if possible, that h' is integrable. Let $a < c \leq b$, and let

$$a = x_0 < x_1 < \cdots < x_n = c$$

be a partition of $[a,c]$. Since $h' = 0$ on a dense subset of $[a,c]$, the lower Riemann sum relative to the partition is zero. It follows that $\int_a^c h'(x)\, dx = 0$. Thus $h(c) - h(a) = 0$. This is true for all $c \in [a,b]$, from which it follows that $h(c) = h(a)$ for all $c \in [a,b]$, and h is constant. It is clear that h is not constant, thus h' is not Riemann integrable.

For the integral developed in this chapter applied to the Lebesgue measure space $([a,b], \mathcal{L}, \lambda)$, we will have

$$h(x) - h(a) = \int_a^x h'(t)\, dt \quad \text{for all} \quad x \in [a,b].$$

We end this section with two remarks. We shall see in Section 5.5 that a function f is Riemann integrable on $[a,b]$ if and only if f is bounded and continuous a.e. with respect to Lebesgue measure. It follows that the

[1]See S. Marcus, *Rend. Circolo Mat. Palermo* 22 (1963), 1–36.

function h' in Example 5.2 is discontinuous on a set of positive measure. One can show that, if a function f is differentiable on $[a,b]$ and $\alpha < \beta$, then $\{x : \alpha < f'(x) < \beta\}$ is either empty or has positive Lebesgue measure. Thus

$$T = \{x : 0 < h'(x) < 1\}$$

has positive measure. Since $h' = 0$ on a dense set, h' is discontinuous at every point of T.

5.2 Integrals of Nonnegative Functions

We shall define an integral for all nonnegative functions f on a measure space $(X, \mathcal{M}, \mu)$. We use the notation

$$\int_X f\,d\mu,$$

which is similar in some ways to the familiar calculus notation. Later we may wish to introduce a dummy variable so that the integral assumes the form

$$\int_X f(x)\,d\mu(x),$$

but, for now, we prefer the simpler notation.

There are many different ways of defining the integral in a measure space. Our definition works immediately for all nonnegative measurable functions. For motivation, let us discuss the ideas behind Lebesgue's definition of the integral for a bounded function defined on an interval $[a,b]$.

Let f be bounded and measurable on $[a,b]$. Let L and U be simple functions such that $L \le f \le U$, say

$$L = \sum_{i=1}^{m} a_i \chi_{E_i} \quad \text{and} \quad U = \sum_{i=1}^{n} b_i \chi_{F_i}.$$

We would like to define an integral $\int_a^b f(x)\,dx$ so that it satisfies

$$\sum_{i=1}^{m} a_i \lambda(E_i) \le \int_a^b f(x)\,dx \le \sum_{i=1}^{n} b_i \lambda(F_i),$$

or, in other notation,

$$\int_a^b L(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b U(x)\,dx.$$

Since these inequalities are to hold whenever $L \le f \le U$, it is natural to define

$$\int_a^b f(x)\,dx = \sup \int_a^b L(x)\,dx = \inf \int_a^b U(x)\,dx,$$

where the supremum is taken over all simple functions $L \leq f$ and the infimum is taken over all simple functions $U \geq f$. It takes only a small argument to show that the integral $\int_a^b f(x)\,dx$ is then well defined (see Exercise 5:2.6), so

$$\sup \int_a^b L(x)\,dx = \inf \int_a^b U(x)\,dx.$$

We then have a definition of the integral similar to Lebesgue's original definition. Such a definition is perfectly adequate when we are dealing with a bounded measurable function and when the underlying measure space $(X, \mathcal{M}, \mu)$ is finite. One could then extend the definition to unbounded functions and to spaces of infinite measure in a variety of ways. (See Exercise 5:12.1 for example.)

Our approach is similar to this but has only two steps. First, we define the integral of an arbitrary nonnegative measurable function. The function need not be bounded, and the space need not have finite measure. We do this in this section. Then, in Section 5.4, we extend the definition to functions that need not be nonnegative.

We begin with the definition of the integral of a nonnegative simple function.

Definition 5.3 Let $(X, \mathcal{M}, \mu)$ be a measure space, and let ϕ be a nonnegative simple function on X. If $\phi = \sum_{k=1}^n a_k \chi_{E_k}$, then

$$\int_X \phi\,d\mu = \sum_{k=1}^n a_k \mu(E_k).$$

If for some k, $a_k = 0$ and $\mu(E_k) = \infty$, we define $a_k\mu(E_k) = 0$.

We leave, as Exercise 5:2.1, the proof that Definition 5.3 does not depend on the representation of ϕ as a simple function.

Theorem 5.4 *Let ϕ and ψ be nonnegative simple functions on X, and let $c \geq 0$.*

1. *If $\phi = \psi$ a.e., then $\int_X \phi\,d\mu = \int_X \psi\,d\mu$.*
2. *$\int_X c\phi\,d\mu = c\int_X \phi\,d\mu$.*
3. *$\int_X (\phi + \psi)\,d\mu = \int_X \phi\,d\mu + \int_X \psi\,d\mu$.*
4. *If $\phi \leq \psi$ on X, then $\int_X \phi\,d\mu \leq \int_X \psi\,d\mu$.*

Proof. The verifications of (1) and (2) are immediate. To verify (3), let

$$\phi = \sum_{k=1}^n a_k \chi_{A_k} \quad \text{and} \quad \psi = \sum_{k=1}^m b_k \chi_{B_k},$$

and we may suppose that $X = \bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{m} B_k$. Then $\phi + \psi$ is a nonnegative simple function. Let

$$C_{ij} = A_i \cap B_j, \quad i = 1, \ldots, n, \; j = 1, \ldots, m.$$

The sets C_{ij} are pairwise disjoint,

$$\bigcup_{i,j} C_{ij} = X,$$

and each of the functions ϕ and ψ is constant on each set C_{ij}. Thus

$$\begin{aligned}
\int_X (\phi + \psi)\, d\mu &= \sum_{i,j} (a_i + b_j)\mu(C_{ij}) \\
&= \sum_{i,j} a_i \mu(C_{ij}) + \sum_{i,j} b_j \mu(C_{ij}) \\
&= \int_X \phi\, d\mu + \int_X \psi\, d\mu.
\end{aligned}$$

This proves (3). To prove (4), we need only note that on the sets $C_{ij} = A_i \cap B_j$, $\phi = a_i \le b_j = \psi$, so

$$\int_X \phi\, d\mu = \sum_{i,j} a_i \mu(C_{ij}) \le \sum_{i,j} b_i \mu(C_{ij}) = \int_X \psi\, d\mu,$$

as required. ∎

Now let f be an arbitrary nonnegative measurable function. Let Φ_f be the family of nonnegative simple functions ϕ such that $\phi(x) \le f(x)$ for all $x \in X$. The family Φ_f contains the zero function, so $\Phi_f \ne \emptyset$.

Definition 5.5 Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be a nonnegative measurable function on X. The *integral of f with respect to μ*, denoted by $\int_X f\, d\mu$, is the quantity

$$\int_X f\, d\mu = \sup\left\{ \int_X \phi\, d\mu : \phi \in \Phi_f \right\}.$$

For $E \in \mathcal{M}$, we write $\int_E f\, d\mu$ for $\int_X f\chi_E\, d\mu$.

We close this section by observing that our concept of integral applies to *every* nonnegative measurable function. For certain functions, the integral will be infinite. Most of the development that follows will deal with functions that have finite integrals.

Definition 5.6 A nonnegative measurable function f defined on a measure space is called *integrable* on a set E if $\int_E f\, d\mu < \infty$.

A few remarks are in order.

Remark 1. It is clear that properties (1), (2), and (4) of Theorem 5.4 hold for integrals of nonnegative measurable functions. Property (3) does too, but is not so easy to prove at this stage.

Remark 2. It is clear that Definitions 5.3 and 5.5 agree when f is a simple function, and so our terminology is consistent.

Remark 3. Our definition of $\int_X f\,d\mu$ does *not* involve approximation of f from above by simple functions. This would have been possible if $\mu(X)$ was assumed finite, but requires modification if $\mu(X) = \infty$. (See Exercise 5:2.6.)

Remark 4. Theorem 4.19 suggests another definition for $\int_X f\,d\mu$ when f is measurable and nonnegative. One could define

$$\int_X f\,d\mu = \lim_{n\to\infty} \int_X \phi_n\,d\mu,$$

where $\{\phi_n\}$ is any nondecreasing sequence of simple functions converging pointwise to f. One would then need to show that the integral does not depend on which sequence of simple functions is chosen. That such a definition is equivalent to ours will be apparent after we prove Theorem 5.8 in the next section.

Exercises

5:2.1 Prove that Definition 5.3 does not depend on the representation of ϕ as a simple function. [Hint: Suppose that $\phi = \sum_{k=1}^m b_k\chi_{B_k} = \sum_{k=1}^n a_k\chi_{A_k}$ and show that

$$\sum_{k=1}^m b_k\mu(B_k) = \sum_{k=1}^n a_k\mu(A_k).$$

5:2.2 Using part (3) of Theorem 5.4 show, for any f, g nonnegative measurable functions on X, that

$$\int_X (f+g)\,d\mu \geq \int_X f\,d\mu + \int_X g\,d\mu.$$

In fact, equality holds, but it is more convenient to prove this later. (See Theorem 5.9 in the next section).

5:2.3$\diamondsuit$ Prove the Tchebychev inequality: Let f be a nonnegative measurable function, E a measurable set, and $\alpha > 0$. Then

$$\mu(\{x \in E : f(x) > \alpha\}) \leq \frac{1}{\alpha}\int_E f\,d\mu.$$

5:2.4 Let f be a nonnegative measurable function. Prove that $\int_X f\,d\mu = 0$ if and only if $f = 0$ a.e.

5:2.5 Check that the theory developed here and in the next section would be unchanged if, in Definition 5.5, the integral were defined for all measurable functions bounded below (rather than nonnegative).

5:2.6 On a finite measure space, we can define *upper and lower integrals* for arbitrary bounded functions. Write

$$\underline{\int} f\,d\mu = \sup\left\{\int_X L\,d\mu : L \leq f, L \text{ simple}\right\},$$

$$\overline{\int} f\,d\mu = \inf\left\{\int_X U\,d\mu : f \leq U, U \text{ simple}\right\},$$

and, if these are equal,

$$\int_X f\,d\mu = \underline{\int} f\,d\mu = \overline{\int} f\,d\mu.$$

(a) Show that this would be well defined and develop the elementary properties of such integrals.

(b) Prove that

$$\underline{\int} f\,d\mu = \overline{\int} f\,d\mu$$

if and only if f is measurable. [Hint: Theorem 5.16 does a special case of this.]

(c) Explain why such a definition is inadequate when $\mu(X) = \infty$. [Hint: Let f be *positive* on X with $\mu(X) = \infty$, and let ϕ be a simple function with $\phi \geq f$ on X. Show that $\int_X \phi\,d\mu = \infty$.]

5.3 Fatou's Lemma

We state and prove a lemma, due to Pierre Fatou (1878–1929), that is basic to all the limit properties of integrals. This allows us to develop the properties of the integral for nonnegative functions.

Lemma 5.7 (Fatou) *Suppose that $\{f_n\}$ is a sequence of nonnegative, measurable functions such that $f = \liminf_{n\to\infty} f_n$ [a.e.]. Then*

$$\int_X f\,d\mu \leq \liminf_{n\to\infty} \int_X f_n\,d\mu. \tag{3}$$

Proof. We may assume without loss of generality that, at each point $x \in X$,

$$f(x) = \liminf_{n\to\infty} f_n(x).$$

We show that, if ϕ is a nonnegative simple function such that $\phi(x) \leq f(x)$ for all $x \in X$, then

$$\int_X \phi \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

The inequality (3) will then follow immediately from the definition of $\int_X f \, d\mu$.

We may suppose that

$$\phi = \sum_{k=1}^{m} a_k \chi_{A_k},$$

where the $\{A_k\}$ are measurable and disjoint and where each a_k is positive. Let $0 < t < 1$. Since $\phi(x) \leq f(x)$, we see that

$$a_k \leq \liminf_{n\to\infty} f_n(x)$$

for each k and each $x \in A_k$. It follows that, for fixed k, the sequence of sets

$$B_{kn} = \{x \in A_k : f_p(x) > ta_k \text{ for all } p \geq n\}$$

increases to A_k. Consequently, $\mu(B_{kn}) \to \mu(A_k)$ as $n \to \infty$. The simple function $\sum_{k=1}^{m} ta_k \chi_{B_{kn}}$ is everywhere less than f_n, and so

$$\int_X f_n \, d\mu \geq \sum_{k=1}^{m} ta_k \mu(B_{kn}).$$

Taking $\liminf$ in this inequality then gives

$$\liminf_{n\to\infty} \int_X f_n \, d\mu \geq \sum_{k=1}^{m} ta_k \mu(A_k) = t \int_X \phi \, d\mu.$$

Finally, then, since t can be chosen arbitrarily close to 1, we have

$$\int_X \phi \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu,$$

as required. ■

From Fatou's lemma we can derive an important convergence theorem. In general, one cannot take limits inside the integral, but if there is some kind of domination, this is possible. Theorem 5.8 can be considered a simple version for nonnegative functions of the Lebesgue dominated convergence theorem (given later as Theorem 5.14), which will become our standard tool in the theory. Applied to the special case where f_n increases to f a.e., Theorem 5.8 is often called the monotone convergence theorem.

Theorem 5.8 *Let $\{f_n\}$ be a sequence of nonnegative measurable functions such that $f_n \to f$ [a.e.] on X. Suppose that $f_n(x) \leq f(x)$ for all $n \in \mathbb{N}$ and $x \in X$. Then*

$$\int_X f\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu.$$

Proof. Since $f_n \leq f$,

$$\int_X f_n\,d\mu \leq \int_X f\,d\mu$$

for all $n \in \mathbb{N}$; thus

$$\limsup{}_{n\to\infty} \int_X f_n\,d\mu \leq \int_X f\,d\mu.$$

On the other hand, it follows from Fatou's lemma that

$$\int_X f\,d\mu \leq \liminf{}_{n\to\infty} \int_X f_n\,d\mu$$

and the theorem is proved. ■

We have already mentioned that three of the four properties of integrals of simple functions in Theorem 5.4 carry over easily to integrals of nonnegative measurable functions. We now verify the missing property, along with two others, with the help of Fatou's lemma.

Theorem 5.9 *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

1. *Let f and g be nonnegative measurable functions on X. Then*

$$\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu.$$

2. *Let $\{f_n\}$ be a sequence of nonnegative measurable functions on X. Then*

$$\int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu.$$

3. *Let f be a nonnegative measurable function on X. Define ν by*

$$\nu(E) = \int_E f\,d\mu \qquad (E \in \mathcal{M}).$$

   Then ν is a measure on $\mathcal{M}$.

Proof. Using Theorem 4.19, we can construct nondecreasing sequences $\{\phi_n\}$ and $\{\psi_n\}$ of simple functions converging pointwise to f and g, respectively. Then the sequence $\{\phi_n + \psi_n\}$ converges to $f + g$. By Theorem 5.8 and Theorem 5.4,

$$\begin{aligned}
\int_X (f+g)\,d\mu &= \lim_{n\to\infty} \int_X (\phi_n + \psi_n)\,d\mu \\
&= \lim_{n\to\infty} \int_X \phi_n\,d\mu + \lim_{n\to\infty} \int_X \psi_n\,d\mu \\
&= \int_X f\,d\mu + \int_X g\,d\mu,
\end{aligned}$$

and we have obtained part (1).

For part (2), let $f = \sum_{n=1}^{\infty} f_n$. For each $k \in \mathbb{N}$, let $S_k = f_1 + \cdots + f_k$. The functions S_k form a nondecreasing sequence of nonnegative measurable functions. Clearly, $\lim_{k\to\infty} S_k(x) = f(x)$ for all $x \in X$, and $S_k \le f$ for all $k \in \mathbb{N}$. By Theorem 5.8, we have

$$\int_X f\,d\mu = \lim_{k\to\infty} \int_X S_k\,d\mu. \tag{4}$$

Now, for all $k \in \mathbb{N}$,

$$\int_X S_k\,d\mu = \int_X f_1\,d\mu + \cdots + \int_X f_k\,d\mu$$

by part (1) and induction; thus, by (4),

$$\int_X f\,d\mu = \lim_{k\to\infty} \int_X S_k\,d\mu = \lim_{k\to\infty} \sum_{n=1}^{k} \int_X f_n\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu,$$

as we wished to prove.

Finally, let us prove part (3). It is clear that ν is nonnegative and that $\nu(\emptyset) = 0$. To show that ν is σ-additive, let $\{E_k\}$ be a sequence of pairwise disjoint measurable sets. Let $f_k = f\chi_{E_k}$. By part (2),

$$\begin{aligned}
\sum_{k=1}^{\infty} \nu(E_k) &= \sum_{k=1}^{\infty} \int_X f_k\,d\mu = \int_X \left(\sum_{k=1}^{\infty} f_k\right) d\mu = \int_X \left(\sum_{k=1}^{\infty} f\chi_{E_k}\right) d\mu \\
&= \int_X \left(f \sum_{k=1}^{\infty} \chi_{E_k}\right) d\mu = \int_X f\chi_{\bigcup E_k}\,d\mu \\
&= \int_{\bigcup E_k} f\,d\mu = \nu\left(\bigcup_{k=1}^{\infty} E_k\right).
\end{aligned}$$

It is clear now that ν is a measure on $\mathcal{M}$. ■

Part (3) of Theorem 5.9 provides a method for obtaining measures on a σ-algebra $\mathcal{M}$. If $(X,\mathcal{M},\mu)$ is a measure space, then each nonnegative measurable function f provides a measure $\nu(E) = \int_E f\,d\mu$. One often uses the terminology "$(X,\mathcal{M})$ is a *measurable* space" to suggest the possibility that there are many measures ν that make $(X,\mathcal{M},\nu)$ into a measure space.

Conversely, one would naturally wish to know when such a representation is possible. That is, if ν and μ are given as measures on a measurable space $(X,\mathcal{M})$, does there exist a nonnegative measurable function f such that $\nu(E) = \int_E f\,d\mu$ for all $E \in \mathcal{M}$? An obvious necessary condition is that $\nu(E) = 0$ for any set E for which $\mu(E) = 0$ (cf. Exercise 3:11.10). We shall see in Section 5.8 that under mild hypotheses on $(X,\mathcal{M},\mu)$ this important condition (called absolute continuity) is also sufficient for ν to be represented as an integral. In general, there are many measures on $(X,\mathcal{M})$ that do not admit such integral representations (Exercises 3:11.10 and 3:11.11).

Exercises

5:3.1 Show by example that the inequality in Fatou's lemma is not in general an equality even if the sequence of functions $\{f_n\}$ converges everywhere.

5:3.2 Show that the hypothesis $f_n \le f$ in the statement of Theorem 5.8 cannot be dropped.

5:3.3 Show that Fatou's lemma can be derived directly from the monotone convergence theorem (Theorem 5.8) thus the latter could have been our starting point in the development of this section.

5:3.4 Let f be the Cantor function (Exercise 1:22.13), and let μ_f be the associated Lebesgue–Stieltjes measure. Show that there is no function g satisfying $\mu_f(E) = \int_E g\,d\lambda$ for each Borel set E.

5.4 Integrable Functions

To this point the integral has been defined and studied only for nonnegative functions. In this section we complete the definition of the integral and give a full description of its properties.

Let $(X,\mathcal{M},\mu)$ be a measure space, and let $E \in \mathcal{M}$. Let f^+ and f^- be the positive and negative parts of the function f defined, as before, by

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \ge 0; \\ 0 & \text{if } f(x) < 0, \end{cases}$$

and

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0; \\ 0 & \text{if } f(x) \ge 0. \end{cases}$$

Then $f = f^+ - f^-$, and if f is measurable, each of f^+ and f^- is measurable and nonnegative.

Definition 5.10 A measurable function f is said to be *integrable* on E if both f^+ and f^- are integrable. In that case we define

$$\int_E f\,d\mu = \int_E f^+\,d\mu - \int_E f^-\,d\mu.$$

We denote the class of integrable functions on X by $L_1(X, \mathcal{M}, \mu)$. This may be shortened to $L_1(X)$ or L_1. Observe that $|f| = f^+ + f^-$. Thus, $|f| \in L_1$ whenever $f \in L_1$. Note that the form of Definition 5.10 forces an absolute integral. We have seen in Chapter 1 that some of the classical integrals of the nineteenth century were nonabsolute. This will play a role in our later comparison of integrals.

Although our definitions require an integrable function to have a finite integral, we can assign a meaning to the expression

$$\int_E f\,d\mu = \int_E f^+\,d\mu - \int_E f^-\,d\mu,$$

even if one (but not both) of the expressions on the right is infinite. Some authors use the term "summable" instead of "integrable" and then employ the term "integrable" to indicate that at least one of the functions f^+ and f^- has a finite integral. Thus, in their terminology, an integrable function may not have a finite integral, but its integral has a well-defined meaning.

Example 5.11 Let $X = \mathbb{N}$, $\mathcal{M} = 2^{\mathbb{N}}$, and let μ be the counting measure on X. Let $f : \mathbb{N} \to \mathbb{R}$. Thus f is a sequence of real numbers. By Definition 5.10, $f \in L_1(\mu)$ if and only if the series $\sum_{n\in\mathbb{N}} f(n)$ converges absolutely. In that case,

$$\int_{\mathbb{N}} f\,d\mu = \sum_{n=1}^{\infty} f(n).$$

Example 5.12 Let $f(0) = 0$, and for $n \in \mathbb{N}$ and $x \in \left(2^{-n}, 2^{-n+1}\right]$, let

$$f(x) = \begin{cases} 2^{n+1}/n & x \in (2^{-n}, 3 \cdot 2^{-n-1}]; \\ -2^{n+1}/n & x \in (3 \cdot 2^{-n-1}, 2^{-n+1}]. \end{cases}$$

Then f^+ and f^- both have infinite integrals, so f is not integrable on $[0, 1]$. The improper Riemann integral

$$\int_0^1 f(x)\,dx = \lim_{\varepsilon\to 0} \int_\varepsilon^1 f(x)\,dx$$

exists and equals 0, because of "cancellations." Such cancellations are not possible within the framework of what we call the integral. This has both advantages and disadvantages. We discuss these in Sections 5.6 and 5.10.

Theorem 5.13 lists some elementary properties of integrable functions. We leave the proofs as Exercise 5:4.2.

Theorem 5.13 *Let $(X, \mathcal{M}, \mu)$ be a measure space, let $\alpha \in \mathbb{R}$, and let $f, g \in L_1$. Then*

1. $$\left| \int_X f \, d\mu \right| \le \int_X |f| \, d\mu.$$
2. $$\int_X \alpha f \, d\mu = \alpha \int_X f \, d\mu.$$
3. $$\int_X (f+g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.$$
4. *If $f(x) \le g(x)$ for all $x \in X$, then* $\displaystyle\int_X f \, d\mu \le \int_X g \, d\mu.$

In the introduction to this chapter we constructed a sequence $\{f_n\}$ of functions on $[0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 f_n(x) \, dx = 1 > 0 = \int_0^1 \lim_{n\to\infty} f_n(x) \, dx.$$

The integrals were Riemann integrals, but we would obtain the same result for any reasonable version of the integral. The reason this sequence behaves this way is that the functions grow large in a way that we cannot control.

Various forms of control on the functions $\{f_n\}$ will lead to the desired conclusion that

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E \lim_{n\to\infty} f_n \, d\mu.$$

One such form of control is provided by our next theorem, called the Lebesgue dominated convergence theorem (LDCT).

Theorem 5.14 (LDCT) *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions such that $f_n \to f$ [a.e.]. If there exists a function $g \in L_1$ such that $|f_n(x)| \le g(x)$ for all $n \in \mathbb{N}$ and $x \in X$, then $f \in L_1$, and*

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu. \tag{5}$$

Proof. Note first that $f \in L_1$, since $|f(x)| \le g(x)$ for almost every $x \in X$. Applying Fatou's lemma to the nonnegative functions $g - f_n$, we obtain

$$\begin{aligned}
\int_X g \, d\mu - \int_X f \, d\mu &= \int_X (g-f) \, d\mu \\
&\le \liminf_{n\to\infty} \int_X (g - f_n) \, d\mu \\
&= \int_X g \, d\mu - \limsup_{n\to\infty} \int_X f_n \, d\mu.
\end{aligned}$$

It now follows that

$$\int_X f\,d\mu \geq \limsup_{n\to\infty} \int_X f_n\,d\mu. \tag{6}$$

Applying a similar argument to the functions $g + f_n$, we infer that

$$\int_X f\,d\mu \leq \liminf_{n\to\infty} \int_X f_n\,d\mu. \tag{7}$$

The desired equality (5) follows from (6) and (7). ■

Corollary 5.15 *The conclusion of the LDCT holds if convergence [a.e.] is replaced by convergence [meas].*

Proof. Apply Theorem 4.14. ■

Exercises

5:4.1 Let ν be a signed measure and ν^+, ν^- its positive and negative variations (see Section 2.5). Define

$$\int_X f\,d\nu = \int_X f\,d\nu^+ + \int_X f\,d\nu^-$$

when the two integrals exist. Explain how this can be used to obtain a notion of a Lebesgue–Stieltjes integral $\int f\,d\mu_g$ when g is of bounded variation on all bounded intervals of $\mathbb{R}$.

5:4.2 Prove Theorem 5.13. [Hint: For part (3), subdivide X into sets where (i) $f \geq 0$ and $g \geq 0$, (ii) $f \geq 0$, $g < 0$, and $f + g \geq 0$, (iii) $f \geq 0$, $g < 0$, and $f + g < 0$, (iv) $f < 0$, $g \geq 0$, and $f + g \geq 0$, (v) $f < 0$, $g \geq 0$, and $f + g < 0$, and (vi) $f < 0$ and $g < 0$.]

5:4.3 (a) Show that if $\mu(E) = 0$ then $\int_E f\,d\mu = 0$ for every measurable f.

(b) Show that if $\int_E f\,d\mu = 0$ for every $E \in \mathcal{M}$ then $f = 0$ a.e.

5:4.4 Prove that Fatou's lemma holds for general measurable functions (not necessarily nonnegative) provided that the sequence of functions $\{f_n\}$ is bounded below by some integrable function.

5:4.5 Suppose that $\mu(X) = 1$, $E_1, E_2, \ldots, E_n$ are measurable subsets of X, and each point of X belongs to at least m of these sets. Show that there exists k such that $\mu(E_k) \geq m/n$.

5:4.6◇ Suppose that $\mu(X) < \infty$. Prove that $f_n \to 0$ [meas] if and only if

$$\int_X \frac{|f_n|}{1 + |f_n|}\,d\mu \to 0.$$

Show that the result fails if the assumption $\mu(X) < \infty$ is dropped.

5:4.7$\diamondsuit$ Suppose that $f \in L_1(X)$, that $f(x) > 0$ for all $x \in X$, and that $0 < \alpha < \mu(X) < \infty$. Prove that

$$\inf \left\{ \int_E f \, d\mu : \mu(E) \geq \alpha \right\} > 0.$$

Give an example to show that the result fails if one drops the hypothesis $\mu(X) < \infty$.

5:4.8 Let $f : X \times [a, b] \to \mathbb{R}$. Find conditions under which you may assert each of the following:

$$\lim_{t \to t_0} \int_X f(x, t) \, d\mu(x) = \int_X \lim_{t \to t_0} f(x, t) \, d\mu(x)$$

$$\frac{d}{dt} \int_X f(x, t) \, d\mu(x) = \int_X \frac{\partial}{\partial t} f(x, t) \, d\mu(x).$$

[Hint: Use sequential limits and the LDC.]

5.5 Riemann and Lebesgue

Some authors have called for the abolition of the Riemann integral, claiming that it offers an integration theory that is technically inadequate and that it serves no useful pedagogic purpose. This extreme position has, fortunately, not been successful, and the reader will have, no doubt, a strong background in the usual integral of the calculus defined by Riemann's methods. It is a natural question then to ask for the relationship between these two integration theories. This section will establish exactly the relation that the Lebesgue integral has to the Riemann integral.

We restrict our attention to bounded functions defined on an interval $[a, b]$. We consider the Lebesgue measure space $([a, b], \mathcal{L}, \lambda)$. The integral we defined in Sections 5.2 and 5.4 is then the Lebesgue integral. By modifying the definition of the integral slightly, we obtain an equivalent form of the Lebesgue integral, which allows us to see at once how this integral generalizes Riemann's integral. We observed in the introduction to this chapter that the Riemann approach to integration has certain flaws, even when we are dealing only with bounded functions on $[a, b]$. We also indicated that these flaws disappear in the setting of Lebesgue's integral. We justify these statements in this section.

To distinguish the two integrals under consideration, we shall use notation such as $\int_a^b f \, d\lambda$ for the Lebesgue integral and $\int_a^b f(t) \, dt$ for the Riemann integral.

Theorem 5.16 *Let f be a bounded measurable function on $[a,b]$. Let*

$$\underline{\int} f \, d\lambda = \sup \left\{ \int_a^b L \, d\lambda : L \leq f, \ L \text{ simple} \right\}$$

and

$$\overline{\int} f\,d\lambda = \inf\left\{\int_a^b U\,d\lambda : f \le U,\ U \textit{ simple}\right\}.$$

Then $\underline{\int} f\,d\lambda = \overline{\int} f\,d\lambda$.

Proof. Let M be an upper bound for $|f|$. Fix $n \in \mathbb{N}$. For every integer k satisfying $-n \le k \le n$, let

$$E_k = \left\{x : \frac{kM}{n} \ge f(x) > \frac{(k-1)M}{n}\right\}.$$

The sets E_k are measurable and pairwise disjoint, and

$$[a,b] = \bigcup_{k=-n}^{n} E_k.$$

Let

$$U_n = \frac{M}{n}\sum_{k=-n}^{n} k\chi_{E_k} \text{ and } L_n = \frac{M}{n}\sum_{k=-n}^{n}(k-1)\chi_{E_k}.$$

The simple functions U_n and L_n satisfy $L_n \le f \le U_n$ on $[a,b]$. Thus

$$\overline{\int} f\,d\lambda \le \int_a^b U_n\,d\lambda = \frac{M}{n}\sum_{k=-n}^{n} k\lambda(E_k)$$

and

$$\underline{\int} f\,d\lambda \ge \int_a^b L_n\,d\lambda = \frac{M}{n}\sum_{k=-n}^{n}(k-1)\lambda(E_k).$$

It follows that

$$0 \le \overline{\int} f\,d\lambda - \underline{\int} f\,d\lambda \le \frac{M}{n}\sum_{k=-n}^{n}\lambda(E_k) = \frac{M}{n}(b-a).$$

Since n is an arbitrary positive integer, we conclude that the upper and lower integrals are identical, as required. ■

Observe that the lower Lebesgue integral in the statement of the theorem is precisely the definition we gave for the integral of a nonnegative measurable function in Section 5.2. We assumed nonnegativity of f for convenience: the definition would have worked equally well for functions bounded below. Exercise 5:2.6 shows that the present assumption that f be defined on a *finite* measure space is essential, however, for Theorem 5.16.

Theorem 5.16 now allows us to give another definition of the Lebesgue integral for bounded functions. A bounded function f (not assumed to be measurable) is Lebesgue integrable on $[a, b]$ if

$$\underline{\int} f\,d\lambda = \overline{\int} f\,d\lambda.$$

Theorem 5.16 establishes that every bounded measurable function on $[a, b]$ is Lebesgue integrable. Let us now formulate a similar definition of the Riemann integral in order to obtain an immediate comparison with Lebesgue's integral. The role of the simple functions is taken by the step functions.

Definition 5.17 Let $I_1, I_2, \ldots, I_n$ be pairwise disjoint intervals with $[a, b] = \bigcup_{k=1}^{n} I_k$, and let $c_1, \ldots, c_n$ be real numbers. Let

$$f = \sum_{k=1}^{n} c_k \chi_{I_k}.$$

Then f is called a *step function*.

Thus a step function is just a special type of simple function.

Definition 5.18 Let f be a function defined on $[a, b]$. Let

$$\underline{\int} f(t)\,dt = \sup\left\{\int_a^b R(t)\ dt : R \leq f,\ R \text{ a step function}\right\}$$

and

$$\overline{\int} f(t)\,dt = \inf\left\{\int_a^b S(t)\ dt : f \leq S,\ S \text{ a step function}\right\}.$$

Then f is Riemann integrable if

$$\underline{\int} f(t)\,dt = \overline{\int} f(t)\,dt.$$

We denote this common value by $\int_a^b f(t)\ dt$.

Definition 5.18 is a standard one for the Riemann integral, but usually stated using the language of lower and upper Darboux sums.

Note that the Lebesgue integral differs from Riemann's in that *all* simple functions figure in the definition of the former integral, while only certain simple functions (the step functions) figure in the definition of the latter. It follows from Theorem 5.16 and the inequalities

$$\underline{\int} f(t)\,dt \leq \underline{\int} f\,d\lambda \leq \overline{\int} f\,d\lambda \leq \overline{\int} f(t)\,dt$$

that every bounded measurable function is Lebesgue integrable and that a function f is Riemann integrable if and only if f is measurable and

$$\underline{\int} f(t)\,dt = \underline{\int} f\,d\lambda \quad \text{and} \quad \overline{\int} f(t)\,dt = \overline{\int} f\,d\lambda.$$

The rigidity of dealing only with step functions can be contrasted with the flexibility of allowing use of all simple functions. Let $f = 0$ on $\mathbb{Q}\cap[a,b]$, $f \geq 1$ elsewhere on $[a,b]$. If R is a step function satisfying $R \leq f$, then $R \leq 0$ on $[a,b]$. In short, one cannot approximate f well from below with step functions: the best lower approximation is $R \equiv 0$. No step function under f can slip through the barrier created by $\mathbb{Q}$ and be a good approximation for f off $\mathbb{Q}$. There are no such barriers for simple functions.

Our next objective is to show that the barrier to good approximations by step functions is related to the set of points of discontinuity of the function. We need some terminology. Let f be a bounded function defined on $[a,b]$, let $x_0 \in [a,b]$, and let $\delta > 0$. Write

$$m_\delta(x_0) = \inf\{f(x) : x \in (x_0-\delta, x_0+\delta)\cap[a,b]\}$$

and

$$M_\delta(x_0) = \sup\{f(x) : x \in (x_0-\delta, x_0+\delta)\cap[a,b]\},$$

and define $m(x_0) = \lim_{\delta\to 0} m_\delta(x_0)$ and $M(x_0) = \lim_{\delta\to 0} M_\delta(x_0)$. The functions m and M are called the *lower* and *upper boundaries* of f. The quantity

$$\omega(x_0) = M(x_0) - m(x_0)$$

is called the *oscillation* of f at x_0.

Note that $m(x_0)$, $M(x_0)$, and $\omega(x_0)$ differ from

$$\liminf_{x\to x_0} f(x), \quad \limsup_{x\to x_0} f(x),$$

and

$$\limsup_{x\to x_0} f(x) - \liminf_{x\to x_0} f(x)$$

only in that the latter three expressions do not take into consideration the value that f takes *at* x_0. It is clear that f is continuous at x_0 if and only if $\omega(x_0) = 0$. We now show that the functions m and M are "barriers" for lower and upper approximations by step functions.

Lemma 5.19 *Let f be bounded on $[a,b]$ and let m be its lower boundary. Then*

1. *m is Lebesgue measurable.*
2. *If R is a step function with $R \leq f$, then $R(x) \leq m(x)$ at each point of continuity of R.*

3. $\underline{\int} f(t)\,dt = \int_a^b m\,d\lambda$.

Proof. If $m(x_0) > \alpha$, then there exists $\beta > \alpha$ such that $f > \beta$ in a neighborhood I of x_0, and hence $m > \alpha$ on I. Thus $\{x : m(x) > \alpha\}$ is open. This proves (1). To verify (2), note that if x_0 is an interior point of an interval of constancy of R then $R(x_0) \le m(x_0)$.

We turn now to the verification of (3). It follows immediately from (2) and Definition 5.18 that

$$\underline{\int} f(t)\,dt \le \int_a^b m\,d\lambda. \tag{8}$$

The reverse inequality requires a bit more work. Let $n \in \mathbb{N}$. Partition $[a, b]$ into 2^n intervals $I_1, \ldots, I_{2^n}$ of equal length, the interval containing a being closed, the others half-open. Let R_n be a function defined on $[a, b]$ that assumes the value $\inf\{f(x) : x \in I_k\}$ on the interval I_k. The function R_n is a step function satisfying $R_n \le f$. Let D_n denote the set of partition points for the nth partition. Then, for each $n \in \mathbb{N}$, D_n is finite, so $D = \bigcup_{n=1}^{\infty} D_n$ is countable. Let $x_0 \in \widetilde{D}$ and let $\alpha < m(x_0)$.

Choose $\delta > 0$ such that $m_\delta(x_0) > \alpha$. For each $n \in \mathbb{N}$, let $I_n(x_0)$ be the interval in the nth partition that contains x_0. It is clear that $I_n(x_0) \subset (x_0 - \delta, x_0 + \delta)$ when n is sufficiently large, say $n \ge N$. Thus

$$m(x_0) \ge R_n(x_0) \ge m_\delta(x_0) > \alpha$$

when $n \ge N$. It follows that

$$\lim_{n\to\infty} R_n(x_0) = m(x_0). \tag{9}$$

Condition (9) is valid for all but countably many values of x_0. In particular, $R_n \to m$ [a.e.]. Since m is a bounded measurable function, m is Lebesgue integrable. By the LDCT,

$$\lim_{n\to\infty} \int_a^b R_n\,d\lambda = \int_a^b m\,d\lambda.$$

But, for step functions, the Riemann and Lebesgue integrals agree; thus

$$\lim_{n\to\infty} \int_a^b R_n\,(t)\ dt = \int_a^b m\,d\lambda.$$

It now follows from Definition 5.18 that

$$\underline{\int} f(t)\,dt \ge \lim_{n\to\infty} \int_a^b R_n\,d\lambda = \int_a^b m\,d\lambda.$$

This, together with (8), completes the verification of (3). ■

We mention that the analog of Lemma 5.19 for the upper boundary M of f is valid, with a similar proof.

Theorem 5.20 *Let f be a function on $[a,b]$. Then f is Riemann integrable if and only if f is bounded and continuous a.e. In that case*

$$\int_a^b f(t)\ dt = \int_a^b f\, d\lambda.$$

Proof. From Lemma 5.19 and its analog for the upper boundary M, we infer that

$$\underline{\int} f(t)\, dt = \int_a^b m\, d\lambda \le \int_a^b f\, d\lambda \le \int_a^b M\, d\lambda = \overline{\int} f(t)\, dt. \tag{10}$$

For f to be Riemann integrable, it is therefore necessary and sufficient that $\int_a^b (M-m)\, d\lambda = 0$. Since $M(x) \ge m(x)$ for each $x \in [a,b]$,

$$\int_a^b (M-m)\, d\lambda = 0$$

if and only if $M = m$ a.e., that is, if and only if f is continuous a.e. When f is Riemann integrable,

$$\int_a^b f(t)\ dt = \int_a^b f\, d\lambda$$

since the five expressions in (10) all represent the same number. ■

In the introduction to this chapter we observed that the fundamental theorem of calculus for Riemann integrals requires the hypothesis that f' be Riemann integrable. Because of Theorem 5.20, this is equivalent to hypothesizing that f' is bounded and continuous a.e. Thus, for example, the derivative h' in Example 5.2 must be discontinuous on a set of positive measure since h' failed to be Riemann integrable. We now show that for functions with bounded derivatives a version of the fundamental theorem of calculus holds for the Lebesgue integral, without further hypotheses. Later, in Chapter 7, we consider the case of unbounded derivatives. Observe first that if f is differentiable on $\mathbb{R}$ then

$$f'(x) = \lim_{n\to\infty} \frac{f(x+1/n) - f(x)}{1/n}.$$

This expresses f' as a pointwise limit of a sequence of continuous functions and hence f' is measurable. In fact, $f' \in \mathcal{B}_1$. [See Exercise 4:6.2(j).]

Theorem 5.21 (Fundamental Theorem of Calculus)
Suppose that f has a bounded derivative on $[a,b]$. Then

$$f(b) - f(a) = \int_a^b f'\, d\lambda.$$

Proof. Extend f to $[a, b+1]$ by setting $f(x) = f(b) + (x-b)f'(b)$ for $b < x \le b+1$. This removes any need to treat the end point b separately. Now f has a bounded derivative on $[a, b+1]$. For $n \in \mathbb{N}$, let $f_n(x) = n(f(x+1/n) - f(x))$. Then $\lim_{n\to\infty} f_n(x) = f'(x)$ for all $x \in [a,b]$. For each $x \in [a,b]$ and $n \in \mathbb{N}$ there exists $\theta \in (0,1)$ such that

$$f_n(x) = f'\left(x + \frac{\theta}{n}\right).$$

Thus the functions f_n are uniformly bounded on $[a,b]$ by the finite number $S = \sup\{|f'(t)| : a \le t \le b\}$. Since the constant function S is integrable, we infer from the LDCT that

$$\int_a^b f'\, d\lambda = \lim_{n\to\infty} \int_a^b f_n\, d\lambda.$$

We have

$$\begin{aligned}
\int_a^b f_n\, d\lambda &= n\int_a^b f\left(x+\frac{1}{n}\right) d\lambda - n\int_a^b f\, d\lambda \\
&= n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f\, d\lambda - n\int_a^b f\, d\lambda \\
&= n\int_b^{b+\frac{1}{n}} f\, d\lambda - n\int_a^{a+\frac{1}{n}} f\, d\lambda.
\end{aligned}$$

By applying the law of the mean to the last two integrals, we obtain constants $\theta_n', \theta_n'' \in (0,1)$ such that

$$\int_a^b f_n\, d\lambda = f\left(b + \frac{\theta_n'}{n}\right) - f\left(a + \frac{\theta_n''}{n}\right).$$

Hence

$$\int_a^b f'\, d\lambda = \lim_{n\to\infty} \int_a^b f_n\, d\lambda = f(b) - f(a),$$

as required. ■

Theorem 5.20 allows us to tighten our discussion of conditions that lead to the conclusion that "a convergent series can be integrated term-by-term," a concern of late nineteenth century mathematics. We formulate our discussion in terms of sequences of functions.

Suppose that $\{f_n\}$ is a uniformly bounded sequence of Riemann integrable functions, and $f_n(x) \to f(x)$ for every $x \in [a,b]$. By Theorem 5.20, each of the functions f_n is Lebesgue integrable. It follows from the LDCT that f is also integrable and that

$$\int_a^b f\, d\lambda = \lim_{n\to\infty} \int_a^b f_n\, d\lambda.$$

If f is Riemann integrable, then

$$\int_a^b f(t)\ dt = \int_a^b f\,d\lambda = \lim_{n\to\infty}\int_a^b f_n\,d\lambda = \lim_{n\to\infty}\int_a^b f_n(t)\ dt.$$

A similar argument shows that any condition that allows the conclusion

$$\int_a^b \lim_{n\to\infty} f_n\,d\lambda = \lim_{n\to\infty}\int_a^b f_n\,d\lambda$$

also allows the conclusion

$$\int_a^b \lim_{n\to\infty} f_n(t)\ dt = \lim_{n\to\infty}\int_a^b f_n(t)\ dt,$$

provided that $\lim_{n\to\infty} f_n$ is Riemann integrable. Thus the limitation of the Riemann integral related to integrating a sequence of functions term by term can be attributed entirely to the fact that the class of Riemann integrable functions is "too small."

Toward the end of the nineteenth century, a number of mathematicians pondered whether uniform boundedness of the sequence $\{f_n\}$ sufficed for the desired conclusion when $\lim_{n\to\infty} f_n$ is Riemann integrable. It was a perplexing problem. Some of the history of the problem can be found in Hawkins.[2] Here we mention only that, with great effort, it was shown that uniform boundedness of the sequence does suffice when the limit function is Riemann integrable.

Exercises

5:5.1 State and prove the analog of Lemma 5.19 for the upper boundary M of f.

5:5.2◇ A function f is called *lower semicontinuous* on $[a,b]$ if for every $\alpha \in \mathbb{R}$ the set $\{x : f(x) > \alpha\}$ is open.

(a) Verify that the lower boundary of a function f is lower semicontinuous.

(b) Prove that a function f is lower semicontinuous if and only if it is its own lower boundary.

(c) Show that the supremum of a sequence of continuous functions is lower semicontinuous.

5:5.3 Prove or disprove that if f is a bounded function and Lebesgue integrable on an interval $[a,b]$, then there exists a Riemann integrable function g so that $f = g$ a.e. and $\int_{[a,b]} f\,d\lambda = \int_a^b g(x)\,dx$.

[2] T. Hawkins, *Lebesgue's Theory of Integration*, Chelsea Publishing Co., (1979).

5:5.4 Suppose that we define for the Riemann integral

$$\int_A f(t)\,dt = \int_a^b f(t)\chi_A(t)\,dt.$$

Over which sets A generally is a Riemann integrable function f now integrable?

5:5.5$\diamondsuit$ (Construction of discontinuous derivatives)

(a) Let $g(x) = x^2 \sin x^{-1}$, for $0 < x \le 1$, $g(0) = 0$. Prove that g is differentiable, with g' bounded and discontinuous only at $x = 0$.

(b) Let P be a Cantor set, $P \subset [0,1]$, $0, 1 \in P$. Let $\{(a_n, b_n)\}$ be the sequence of intervals complementary to P. On each interval $[a_n, b_n]$, construct a differentiable function f_n that satisfies $f_n(a_n) = f_n(b_n) = f_n'(a_n) = f_n'(b_n) = 0$, and so that

$$f_n(x) = (x - a_n)^2 \sin(x - a_n)^{-1}$$

for $a_n < x < a_n + \delta_n < (a_n + b_n)/2$ and

$$f_n(x) = (b_n - x)^2 \sin(b_n - x)^{-1}$$

for $b_n > x > b_n - \delta_n$, with $f'(x) = 0$ on $[a_n + \delta_n, b_n - \delta_n]$.

(c) Let $f = f_n$ on $[a_n, b_n]$, $f = 0$ elsewhere. Prove that f has a bounded derivative on $[0,1]$ with $f' = 0$ on P and f' discontinuous at all points of P.

(d) Show that for every $\varepsilon > 0$ there exists a function h such that h has a bounded derivative on $[0,1]$ and h' is discontinuous on a Cantor set of Lebesgue measure exceeding $1 - \varepsilon$.

(e) Let $\{P_n\}$ be an expanding sequence of Cantor sets in $[0,1]$ with $\lambda(P_n) \to 1$. Use part (d) to construct a differentiable function f on $[0,1]$, with f' bounded, such that f' is discontinuous a.e.

[The derivatives f' that appear in elementary calculus are usually continuous. Part (e) illustrates that derivatives can actually be discontinuous a.e. This goes well beyond the Volterra example in Section 1.18, where a derivative was given whose set of discontinuities had positive measure. In Exercise 10:7.7 we shall see that, in a certain sense, "most" derivatives are discontinuous a.e. Can a derivative be discontinuous *everywhere*? The answer is no. Theorem 1.19 shows that every derivative is continuous except on a set of the first category.]

5.6 Countable Additivity of the Integral

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f \in L_1(\mu)$. For $E \in \mathcal{M}$, let

$$\nu(E) = \int_E f\,d\mu.$$

We have already seen in Section 5.3 that if $f \geq 0$ then ν is a measure on $\mathcal{M}$. We now show that, without the requirement that f be positive, ν is a signed measure.

Theorem 5.22 *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L_1(\mu)$. The set function $\nu(E) = \int_E f\, d\mu$ is a finite signed measure on $\mathcal{M}$.*

Proof. For each $E \in \mathcal{M}$, let

$$\nu^+(E) = \int_E f^+\, d\mu \quad \text{and} \quad \nu^-(E) = \int_E f^-\, d\mu.$$

Then ν^+ and ν^- are measures by Theorem 5.9(3). Since $\nu = \nu^+ - \nu^-$, ν is a signed measure. ■

Observe that ν^+ and $-\nu^-$ are the upper and lower variations of ν. (See Section 2.5 and Exercise 5:4.1.) If f is measurable but not integrable, there are two possibilities. If either f^+ or f^- is integrable, ν is still a signed measure, but not finite. If both f^+ and f^- have infinite integrals, $\nu^+ - \nu^-$ is no longer a signed measure. The integral of f does not exist in that case.

Let us explore this matter a bit further. For the function appearing in Example 5.12,

$$\int_0^1 f^+\, d\lambda = \infty \quad \text{and} \quad \int_0^1 f^-\, d\lambda = \infty.$$

The set functions $\nu^+(E) = \int_E f^+\, d\lambda$ and $\nu^-(E) = \int_E f^-\, d\lambda$ are measures on $\mathcal{L}$, with

$$\nu^+([0,1]) = \nu^-([0,1]) = \infty.$$

Let $0 < \varepsilon < 1$. For $E \subset [\varepsilon, 1]$, $E \in \mathcal{L}$, $\nu(E) = \int_E f\, d\lambda$ is finite, and $\nu(E) = \nu^+(E) - \nu^-(E)$. It is clear that $\lim_{\varepsilon \to 0} \nu([\varepsilon, 1]) = 0$. It is tempting to extend the definition of the integral in such a way that $\nu([0,1]) = 0$. One can do this, and such an approach has certain advantages. But we would no longer have countable additivity of the integral: ν would not be a signed measure. In order for

$$\nu([0,1]) = \sum_{n \in \mathbb{N}} (\nu(L_n) + \nu(R_n)),$$

where L_n is the left open half of the interval $\left(2^{-n}, 2^{-n+1}\right)$ and R_n is the right closed half, we would need every rearrangement of the series

$$1 - 1 + \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{3} + \cdots \tag{11}$$

to converge to 0, which is false.

The integral as we defined it in Section 5.4 is an *absolutely convergent* integral: if f is integrable, so is $|f|$. The Riemann integral, when extended to include (improper) integrals of unbounded functions, is an example of a *nonabsolutely convergent* integral. Theorem 5.22 cannot hold for such integrals, and spaces for which such integrals can be defined need certain partitioning properties. But they provide solutions to various problems for functions defined on $\mathbb{R}$ or on other spaces with appropriate structure. See the discussion in Section 5.10 for more on this topic.

Suppose now that $(X, \mathcal{M}, \mu)$ is a measure space. Each nonnegative $f \in L_1(\mu)$ gives rise to a new measure $\nu(E) = \int_E f\, d\mu$. It is clear that, if $g \in L_1(\mu)$ and $\psi(E) = \int_E g\, d\mu$, then $\nu = \psi$ if and only if $f = g$ a.e. There might therefore be many measures on the measurable space $(X, \mathcal{M})$. Each such measure ν gives rise to yet further measures of the form $\phi(E) = \int_E g\, d\nu$. One might ask how the families of measures that arise by integrating with respect to ν are related to those one obtains by integrating with respect to μ, where $\nu(E) = \int_E f\, d\mu$. The answer is that no additional measures are obtained.

Theorem 5.23 *Let $(X, \mathcal{M}, \mu)$ be a measure space, let f be a nonnegative measurable function, and, for each $E \in \mathcal{M}$, let $\nu(E) = \int_E f\, d\mu$. Let g be a nonnegative measurable function. Then*

$$\int_E g\, d\nu = \int_E g\, f d\mu \qquad (E \in \mathcal{M}). \tag{12}$$

Proof. Let $E \in \mathcal{M}$. Suppose first that $g = \chi_A$ for some $A \in \mathcal{M}$. Then

$$\int_E g\, d\nu = \nu(A \cap E) = \int_{A\cap E} f\, d\mu = \int_E g\, f d\mu.$$

Thus (12) is valid for characteristic functions. Since simple functions are linear combinations of characteristic functions, (12) is valid for all simple functions. Finally, any nonnegative measurable function g is the pointwise limit of a nondecreasing sequence of nonnegative simple functions $\{S_n\}$. The sequence $\{S_n f\}$ increases to gf. By Theorem 5.8,

$$\int_E g\, d\nu = \lim_{n\to\infty} \int_E S_n\, d\nu = \lim_{n\to\infty} \int_E S_n\, f d\mu = \int_E gf\, d\mu.$$

■

The equality (12) suggests the notation $d\nu = f d\mu$, which in turn suggests $\frac{d\nu}{d\mu} = f$. This looks a bit like part of the fundamental theorem of calculus. In our present setting we have no notion of $\frac{d\nu}{d\mu}$ as a derivative. In Section 5.8, we shall see that $f = \frac{d\nu}{d\mu}$ does in fact have some formal resemblance to a derivative. Then, in Chapter 8, we shall see that $f = \frac{d\nu}{d\mu}$ can actually be viewed in the more familiar manner as a limit of a difference quotient.

Exercises

5:6.1 Show that Theorem 5.23 is valid if the nonnegativity of f is replaced by the integrability of f. (Use the definition of integral with respect to a signed measure from Exercise 5:4.1.)

5:6.2 In the statement of Theorem 5.23, suppose that f and g are both μ–integrable (but not necessarily nonnegative). Can you conclude that fg is μ–integrable? What simple condition on g would allow this? (In Section 13.1 we will find some better ideas that can be used to show that certain products are integrable.)

5:6.3 Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be a nonnegative, measurable function. Define the measure $\nu(E) = \int_E f \, d\mu$.

(a) Show that if f is everywhere finite and μ is σ-finite then ν is σ-finite.

(b) Show that if f is everywhere positive and ν is σ-finite then μ is σ-finite.

5.7 Absolute Continuity

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let ν be a signed measure on $\mathcal{M}$. For each $E \in \mathcal{M}$, if $\nu(E) = 0$ whenever $\mu(E) = 0$, we say that ν is *absolutely continuous* with respect to μ, and we write $\nu \ll \mu$.

For example, if $f \in L_1(\mu)$, then by Theorem 5.22 we know that $\nu(E) = \int_E f \, d\mu$ is a finite signed measure. It is clear that ν is absolutely continuous with respect to μ, since if $\mu(E) = 0$ then $\int_E f \, d\mu = 0$.

It is often useful, particularly when dealing with integrals, to use the following ε, δ version of absolute continuity. Expressed this way, it is clearer that we are dealing with a form of continuity.

Theorem 5.24 *Let ν be a finite signed measure on $\mathcal{M}$. Then $\nu \ll \mu$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ for each $E \in \mathcal{M}$ with $\mu(E) < \delta$.*

Proof. In view of Exercise 5:7.1, we may assume that ν is a measure. It is clear that the condition of the theorem implies that $\nu \ll \mu$. To prove the converse, suppose that this condition fails. Then there exists $\varepsilon > 0$ and a sequence $\{E_n\}$ of measurable sets such that, for each n, $\mu(E_n) < 2^{-n}$ and $\nu(E_n) \geq \varepsilon$. Let $E = \limsup_{n \to \infty} E_n$, and let $k \in \mathbb{N}$. Then

$$\mu(E) \leq \sum_{n=k}^{\infty} \mu(E_n) \leq \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}}. \tag{13}$$

Since (13) is valid for each $k \in \mathbb{N}$, $\mu(E) = 0$. But $\nu\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$ by hypothesis, so it follows from Theorem 2.21(2) that

$$\nu(E) = \nu(\limsup_{n \to \infty} E_n) \geq \limsup_{n \to \infty} \nu(E_n) \geq \varepsilon > 0.$$

Thus $\mu(E) = 0$, and yet $\nu(E) > 0$; so ν is not absolutely continuous with respect to μ. ■

To this point we have focused on absolute continuity as it relates to integrals or, more generally, signed measures. The notion of absolute continuity originated in the setting of functions defined on an interval $I \subset \mathbb{R}$ and remains important in this setting for many reasons. We give now the classical definition and show how it relates to the measure-theoretic concept of absolute continuity.

Definition 5.25 Let $f : [a,b] \to \mathbb{R}$. We say that f is *absolutely continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{[a_n, b_n]\}$ is any finite or countable collection of nonoverlapping closed intervals in $[a,b]$, with $\sum_{k=1}^{\infty}(b_k - a_k) < \delta$, then

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \varepsilon.$$

Let us discuss this notion a bit and then relate it to the notion of absolute continuity for integrals or measures. First, let us compare absolute continuity with continuity. If f is continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$. Thus, given $\varepsilon > 0$, we can find a $\delta > 0$ such that, no matter which interval $[a_1, b_1]$ of length less than δ we choose, the total "growth" $|f(b_1) - f(a_1)|$ of f on that interval is less than ε. We can place such an interval anywhere we wish in $[a,b]$ without losing the conclusion. But we cannot split the interval into pieces to be moved around at will. For that we need absolute continuity.

Example 5.26 Let f be the Cantor function and C the Cantor ternary set (Exercise 1:22.13). Let $\varepsilon = \frac{1}{2}$ and $\delta > 0$. Since C has zero Lebesgue measure, we can cover C with a finite number of pairwise disjoint intervals $[a_1, b_1], \dots, [a_n, b_n]$ such that $\sum_{k=1}^{n}(b_k - a_k) < \delta$, but

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| = 1 > \varepsilon.$$

The Cantor function is uniformly continuous on [0,1], but it is clear from this that it is not absolutely continuous.

We now show that every absolutely continuous function is continuous, has bounded variation and maps zero measure sets to zero measure sets. We shall see in Section 7.3 that Theorem 5.27 actually *characterizes* the absolutely continuous functions: a function is absolutely continuous on $[a,b]$ if and only if it satisfies the three stated conditions. Note that the Cantor function satisfies only the first two of these.

Theorem 5.27 *Let f be absolutely continuous on $[a,b]$. Then*

1. *f is continuous on $[a,b]$.*
2. *f is of bounded variation on $[a,b]$.*
3. *For every set E of Lebesgue measure zero in $[a,b]$, $\lambda(f(E)) = 0$.*

Proof. Condition (1) is immediate. To prove (2), choose $\delta > 0$ such that if $[a_1, b_1], \ldots, [a_n, b_n]$ is any finite collection of nonoverlapping closed intervals with

$$\sum_{k=1}^{n}(b_k - a_k) < \delta$$

then

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < 1.$$

If $[c, d]$ is any interval in $[a, b]$ with $d - c < \delta$, then $V(f; [c, d]) \leq 1$. Let $N \in \mathbb{N}$ with $N > (b - a)/\delta$. Partition $[a, b]$ into N intervals $I_1, \ldots, I_N$ of equal length $(b - a)/N < \delta$. The variation of f on each of these intervals is less than 1, so

$$V(f; [a, b]) \leq N < \infty$$

as required.

To prove (3), let $\varepsilon > 0$. Choose $\delta > 0$ such that, if $\{[c_k, d_k]\}$ is any finite or countable collection of nonoverlapping closed intervals in $[a, b]$ with $\sum_{k=1}^{\infty}(d_k - c_k) < \delta$, then

$$\sum_{k=1}^{\infty} |f(d_k) - f(c_k)| < \varepsilon.$$

Let $G = \bigcup_{k=1}^{\infty}(a_k, b_k)$ be an open set containing E with

$$\lambda(G) = \sum_{k=1}^{\infty}(b_k - a_k) < \delta.$$

Now

$$f(E) \subset f(G) \subset f\left(\bigcup_{k=1}^{\infty}[a_k, b_k]\right) \subset \bigcup_{k=1}^{\infty}[f(c_k), f(d_k)],$$

where c_k is a point in $[a_k, b_k]$ at which f assumes its minimum and d_k is a point where f assumes its maximum. Thus

$$\lambda^*(f(E)) \leq \sum_{k=1}^{\infty}(f(d_k) - f(c_k)) < \varepsilon$$

because $\sum_{k=1}^{\infty} |d_k - c_k| \leq \delta$. Since ε is arbitrary, $\lambda(f(E)) = 0$. ∎

We can use Theorem 3.22 to make a connection between the notions of absolute continuity for functions and for Lebesgue–Stieltjes measures.

Theorem 5.28 *A continuous nondecreasing function f is absolutely continuous on $[a, b]$ if and only if its associated Lebesgue–Stieltjes measure μ_f is absolutely continuous with respect to Lebesgue measure λ.*

Proof. Let f be continuous and nondecreasing on $[a, b]$, and let μ_f be the associated Lebesgue–Stieltjes measure. By Theorem 3.22, $\mu_f^*(E) = \lambda^*(f(E))$ for every set $E \subset [a, b]$. If f is absolutely continuous, then f satisfies condition (3) of Theorem 5.27, so $\mu_f \ll \lambda$. On the other hand, suppose that $\mu_f \ll \lambda$. Since μ_f is finite on $[a, b]$, Theorem 5.24 applies. Thus, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu_f(E) < \varepsilon$ if $\lambda(E) < \delta$. If E is a union of nonoverlapping intervals, say $E = \bigcup_{k=1}^{\infty} [a_k, b_k]$, then

$$\sum_{k=1}^{\infty} (f(b_k) - f(a_k)) = \mu_f(E) < \varepsilon$$

whenever $\sum_{k=1}^{\infty} (b_k - a_k) = \lambda(E) < \delta$. ■

The origin of the notion of absolute continuity was in the problem of characterizing those functions that can be represented as integrals. Suppose that f is Lebesgue integrable on $[a, b]$. Let $\nu(E) = \int_E f \, d\lambda$. Then $\nu \ll \lambda$. Let

$$F(x) = \int_a^x f \, d\lambda \ , \ \ a \le x \le b.$$

It follows easily from Theorem 5.24 that F is absolutely continuous. Thus, starting with an integrable function f, we integrate f to obtain an absolutely continuous function F. As a preliminary step toward a result in the reverse direction, consider a function F with a bounded derivative on $[a, b]$. If $|F'(x)| \le M$ for all $x \in [a, b]$, then F satisfies the Lipschitz condition

$$|F(y) - F(x)| \le M|y - x| \ \text{ for all } x, y \in [a, b].$$

This follows from the law of the mean. Thus, for nonoverlapping intervals $[a_1, b_1], \ldots, [a_n, b_n]$, we have

$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| \le M \sum_{k=1}^{n} (b_k - a_k),$$

so F is absolutely continuous (let $\delta = \varepsilon/M$). By Theorem 5.21,

$$F(x) = F(a) + \int_a^x F' \, d\lambda$$

for all $x \in [a, b]$.

This argument shows that certain absolutely continuous functions, namely those with bounded derivatives, can be represented as integrals. We shall see in Section 5.8 that the same is true for *every* absolutely continuous function. We shall also see that a comparable result is available for measures and, in fact, that the integrand is quite reminiscent of a derivative. We can view much of the preceding as a preliminary discussion of the fundamental ways in which integration and differentiation are inverse operations. We will have much more to say on the subject in Section 5.8 and in Chapters 7 and 8.

Exercises

5:7.1 Let $(X, \mathcal{M}, \mu)$ be a measure space, let ν be a signed measure and write $|\nu|$, ν^+, and ν^- for the total variation, positive variation, and negative variation of ν. (See Section 2.5.) Show that these statements are equivalent: (i) $\nu \ll \mu$, (ii) $|\nu| \ll \mu$, and (iii) $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

5:7.2 Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and suppose that ν is a finitely additive set function for which, for all $\varepsilon > 0$, there is a $\delta > 0$ with $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$. Show that ν is a signed measure and $\nu \ll \mu$.

5:7.3 Give an example to show that Theorem 5.24 fails if one drops the requirement that $\nu(X) < \infty$.

5:7.4$\diamondsuit$ Prove that in the definition of absolute continuity of functions one cannot drop the terminology "nonoverlapping." [Hint: Consider $f(x) = \sqrt{x}$.]

5:7.5 In the definition of absolute continuity it is sometimes convenient to replace the increments $|f(d) - f(c)|$ with the oscillation

$$\omega(f, [c, d]) = \sup_{x\in[c,d]} f(x) - \inf_{x\in[c,d]} f(x).$$

Show that a function f is absolutely continuous on $[a, b]$ if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\{[a_k, b_k]\}$ is any finite or countable collection of nonoverlapping closed intervals in $[a, b]$, with $\sum_k (b_k - a_k) < \delta$, then $\sum_k \omega(f, [a_k, b_k]) < \varepsilon$.

5:7.6 Does Theorem 5.28 remain true if "nondecreasing" is replaced with "bounded variation" and "measure" with "signed measure"? What happens if the requirement of continuity of f is dropped?

5:7.7 Show that the class of absolutely continuous functions on $[a, b]$ is closed under addition and multiplication. What can be said about division?

5:7.8 Consider compositions of the form $g \circ f$. Prove each of the following:

(a) If f is absolutely continuous and g satisfies a Lipschitz condition, then $g \circ f$ is absolutely continuous.

(b) If f is absolutely continuous and strictly increasing and g is absolutely continuous, then $g \circ f$ is absolutely continuous.

(c) There exist absolutely continuous functions f and g defined on [0,1] such that $g \circ f$ is not absolutely continuous. [Hint: Choose f appropriately with $f(1/n) = 1/n^2$ and $g(x) = \sqrt{x}$. See Figure 5.2.]

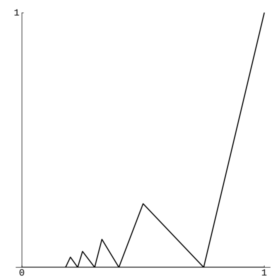

Figure 5.2: Construction of the function f in Exercise 5:7.8.

5:7.9◇ Refer to Exercise 5:7.4. Prove that a function f satisfies a Lipschitz condition on $[a, b]$ if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ for which the following is true: for every finite collection $\{[a_k, b_k]\}_{k=1}^n$ of closed intervals in $[a, b]$ with $\sum_{k=1}^n (b_k - a_k) < \delta$,

$$\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon.$$

Compare with the definition of absolute continuity of a function.

5:7.10 Obtain a partial converse to Theorem 5.27. Let f be continuous and nondecreasing on an interval and suppose that f maps measure zero sets to measure zero sets. Show that f is absolutely continuous. [Hint: Consider the measure μ_f, and use Theorems 3.22 and 5.28.]

5.8 Radon–Nikodym Theorem

We turn now to a development of the material we discussed at the end of Section 5.7. Giuseppe Vitali (1875–1932) and Lebesgue proved that a function F is absolutely continuous on $[a, b]$ if and only if there exists a function f such that

$$F(x) - F(a) = \int_a^x f\, d\lambda$$

for all $x \in [a, b]$. It was Vitali who actually coined the term "absolute continuity." In 1913, Johann Radon (1887–1956) obtained a version for absolutely continuous Lebesgue–Stieltjes measures on $\mathbb{R}^n$. Radon's theorem was then extended to absolutely continuous measures on σ-finite measure spaces by O. Nikodym in 1930.

Theorem 5.29 (Radon–Nikodym) *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let ν be a σ-finite signed measure on $\mathcal{M}$ that is absolutely continuous with respect to μ. Then there exists a function f on X such that*

$$\nu(M) = \int_M f\, d\mu \qquad (M \in \mathcal{M}). \tag{14}$$

This is an important theorem with an interesting proof, but one that can be a bit elusive. We can obtain some insight into this theorem (why it is true and how to prove it) by considering the case $([a,b], \mathcal{L}, \lambda)$ with ν a Lebesgue–Stieltjes measure, $\nu = \mu_F$ where F is an absolutely continuous function on $[a,b]$. In this setting the theorem is more transparent. It follows from material that we now anticipate (from Section 7.5) that such a function F is a.e. differentiable on $[a,b]$ and that, if we define $f(x) = F'(x)$ at points where the derivative exists and arbitrarily on the measure zero set Z where F' does not exist, then

$$\mu_F(E) = \int_E F'\, d\lambda$$

for all measurable subsets E of $[a,b]$.

This suggests that the integrand in (14) might be a derivative. But how does this offer any insight when we are dealing with abstract measure spaces for which we (as yet) have no notion of a derivative of a measure? We need to express the function f in a way that ultimately avoids taking derivatives. For each $x \in [a,b] \setminus Z$, the derivative F' exists, and hence for fixed n the sets

$$A_n^k = \{x : F'(x) < k/n\}, \quad k = 0, 1, 2, 3, \dots$$

expand to cover all of $[a,b] \setminus Z$. Thus, for each $n \in \mathbb{N}$, the sets

$$E_n^k = A_n^k \setminus A_n^{k-1} = \left\{ x \notin Z : \frac{k-1}{n} \leq F'(x) < \frac{k}{n} \right\}, \quad k = 0, \pm 1, \pm 2, \dots$$

partition the set $[a,b] \setminus Z$. Define functions f_n as arbitrary on the measure zero set Z and elsewhere as

$$f_n(x) = \frac{k-1}{n} \quad \text{for all } x \in E_n^k.$$

For each $x \in [a,b] \setminus Z$ we have $f_n(x) \leq F'(x)$ and

$$\lim_{n\to\infty} f_n(x) = F'(x).$$

It follows from Theorem 5.8 that, for each $E \in \mathcal{L}$,

$$\int_E F'\, d\lambda = \lim_{n\to\infty} \int_E f_n\, d\lambda.$$

We can therefore take $f = \lim_{n\to\infty} f_n$ as the integrand in (14).

We need to imitate the argument above *without* having a candidate (F') for f and hence not knowing in advance what sets should play the role of the sets A_n^k. The key tool is the Hahn decomposition theorem (Theorem 2.24). The sets A_n^k can be realized as the negative sets for the signed measure $\nu - \frac{k}{n}\mu$.

Recall that for any signed measure ν on a σ-algebra $\mathcal{M}$ there exists a set $P \in \mathcal{M}$ (called the positive set for ν) such that $\nu(A) \geq 0$ whenever $A \subset P$, $A \in \mathcal{M}$, and for the set $N = \widetilde{P}$ (called the negative set for ν), $\nu(A) \leq 0$ whenever $A \subset N$, $A \in \mathcal{M}$. The pair (P, N) is called a Hahn decomposition for ν. Observe that if $\nu(E) = \int_E f\, d\mu$ we can take $P = \{x : f(x) \geq 0\}$.

The Hahn decomposition theorem provides a connection for carrying out our suggested plan. The connection is this: if $\gamma > 0$ and F is nondecreasing, then the set $E = \{x : F'(x) < \gamma\}$ is a negative set for $\nu - \gamma\lambda$, where $\nu = \mu_F$. To verify this, let $A \subset E$, $A \in \mathcal{L}$. Then

$$(\nu - \gamma\lambda)(A) = \nu(A) - \gamma\lambda(A) = \int_A F'\, d\lambda - \gamma\lambda(A) \leq \gamma\lambda(A) - \gamma\lambda(A) = 0.$$

Thus we can describe sets associated with F' (which we do not know) by Hahn decompositions of signed measures of the form $\nu - \gamma\lambda$ (which we do know).

The set of points Z in this heuristic discussion will appear in the proof as a set of μ-measure zero that must be disposed of somehow. The absolute continuity assumption of the theorem is employed only to ensure that $\nu(Z) = 0$, too.

We return now to the proof of Theorem 5.29. The proof will not depend on any of the heuristic discussion above, but without such discussion it might have appeared "magical."

Proof. Because of the Jordan decomposition theorem, we may assume that ν is a measure. We may also assume that $\mu(X) < \infty$ and $\nu(X) < \infty$. For suppose that we have proved the theorem for finite measures. Since μ and ν are assumed to be σ-finite, we write $X = \bigcup X_i = \bigcup Y_i$ for sequences of disjoint measurable sets, with each $\mu(X_i) < \infty$ and $\nu(Y_i) < \infty$. Order the sets $\{X_i \cap Y_j\}$ into a single sequence $\{Z_k\}$. Since the theorem can be applied for the finite measures μ_k and ν_k, where $\mu_k(E) = \mu(E \cap Z_k)$ and $\nu_k(E) = \nu(E \cap Z_k)$, we can use Theorem 5.9(2) to obtain the theorem for μ and ν.

[As we suggested in our heuristic discussion, the only use we make of our hypothesis that $\nu \ll \mu$ is to assure that a certain troublesome set Z with $\mu(Z) = 0$ also has $\nu(Z) = 0$. Our first task is to identify this set Z that corresponds to the set on which F is not differentiable.]

For the remainder of the proof, μ and ν are finite measures. For each $k, n \in \mathbb{N}$, let A_n^k be a negative set for the signed measure $\nu - \frac{k}{n}\mu$. Let

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_n^k$$

so that

$$Z = \widetilde{E} = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \widetilde{A_n^k}.$$

We show that $\mu(Z) = 0$.

For each $j \in \mathbb{N}$, the set $\widetilde{A_n^j}$ is a positive set for $\nu - \frac{j}{n}\mu$, and $\bigcap_{k=1}^{\infty} \widetilde{A_n^k} \subset \widetilde{A_n^j}$. Thus

$$\nu\left(\bigcap_{k=1}^{\infty} \widetilde{A_n^k}\right) \geq \frac{j}{n}\mu\left(\bigcap_{k=1}^{\infty} \widetilde{A_n^k}\right). \tag{15}$$

Since (15) holds for every j and ν is finite, we infer that

$$\mu\left(\bigcap_{k=1}^{\infty} \widetilde{A_n^k}\right) = 0$$

for every n. Now

$$\mu(Z) = \mu\left(\bigcup_{n=1}^{\infty}\bigcap_{k=1}^{\infty} \widetilde{A_n^k}\right) \leq \sum_{n=1}^{\infty} \mu\left(\bigcap_{k=1}^{\infty} \widetilde{A_n^k}\right),$$

from which it follows that $\mu(Z) = 0$. Since $\nu \ll \mu$, $\nu(Z) = 0$.

Use Theorem 2.17 to replace each system of sets $\{A_n^k\}_{k=1}^{\infty}$ by a pairwise disjoint system of sets $\{E_n^k\}_{k=1}^{\infty}$ from $\mathcal{M}$, with

$$\bigcup_{k=1}^{\infty} E_n^k = \bigcup_{k=1}^{\infty} A_n^k \quad \text{and} \quad E_n^k \subset A_n^k \setminus A_n^{k-1}, \quad \text{for } n, k \in \mathbb{N}.$$

(This corresponds to the sets

$$\left\{x : \frac{k-1}{n} \leq F'(x) < \frac{k}{n}\right\}$$

in the heuristic argument.)

For each $n, k \in \mathbb{N}$, let $g_n = (k-1)/n$ on E_n^k. Since

$$E = \bigcap_{n=1}^{\infty}\bigcup_{k=1}^{\infty} A_n^k = \bigcap_{n=1}^{\infty}\bigcup_{k=1}^{\infty} E_n^k,$$

each function g_n is defined on E. We now replace the functions g_n with functions f_n that form a monotone sequence (which therefore converges pointwise on E).

Fix $n \in \mathbb{N}$. Let

$$f_n(x) = \max_{i \leq n} g_i(x).$$

For $M \in \mathcal{M}$, $M \subset E$, let $B_0 = \emptyset$ and, for $i \leq n$, inductively define

$$B_i = (\{x : f_n(x) = g_i(x)\} \cap M) \setminus B_{i-1}.$$

Then $M = \bigcup_{i=1}^{n} B_i$. This is a disjoint union. Thus

$$\int_M f_n \, d\mu = \sum_{i=1}^{n} \int_{B_i} g_i \, d\mu = \sum_{i=1}^{n}\sum_{k=1}^{\infty} \int_{B_i \cap E_i^k} g_i \, d\mu \tag{16}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{k-1}{i} \mu(B_i \cap E_i^k) \le \sum_{i=1}^{n} \sum_{k=1}^{\infty} \nu(B_i \cap E_i^k) = \nu(M).$$

The inequality follows from the fact that E_i^k is a subset of the set $X \setminus A_i^{k-1}$, which is a positive set for $\nu - \frac{k-1}{i}\mu$. A similar argument using the fact that $E_i^k \subset A_i^k$ leads to the inequalities

$$\int_M f_n \, d\mu \ge \int_M g_n \, d\mu = \sum_{k=1}^{\infty} \left(\frac{k-1}{n} \right) \mu(M \cap E_n^k) \tag{17}$$

$$\ge \sum_{k=1}^{\infty} \left(\nu(M \cap E_n^k) - \frac{\mu(M \cap E_n^k)}{n} \right) = \nu(M) - \frac{\mu(M)}{n}.$$

Comparing (16) with (17), we see that, for every $n \in \mathbb{N}$,

$$\nu(M) - \frac{\mu(M)}{n} \le \int_M f_n \, d\mu \le \nu(M). \tag{18}$$

Since $\{f_n\}$ is a nondecreasing sequence of functions on E, there exists a function f on E such that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in E$. By Theorem 5.8,

$$\int_M f \, d\mu = \lim_{n\to\infty} \int_M f_n \, d\mu$$

for all $M \subset E, M \in \mathcal{M}$. By (18), this limit is $\nu(M)$. Extending f to all of X by defining $f(x) = 0$ if $x \in Z$, we obtain the desired function. ■

Theorem 5.29 implies the theorem of Lebesgue and Vitali that began our discussion in this section.

Corollary 5.30 (Vitali–Lebesgue) *Every function F that is absolutely continuous on $[a,b]$ can be represented as an integral*

$$F(x) - F(a) = \int_a^x f \, d\lambda.$$

Proof. To verify this, we first note that by Theorem 5.28 the signed measure μ_F is absolutely continuous with respect to Lebesgue measure. By Theorem 5.29, there exists a function $f \in L_1(\lambda)$ such that

$$\mu_F(E) = \int_E f \, d\lambda$$

for every $E \in \mathcal{L}$, $E \subset [a,b]$. In particular, for each $x \in [a,b]$,

$$F(x) - F(a) = \mu_F([a,x]) = \int_a^x f \, d\lambda.$$

■

In Chapter 7 we will see that $F' = f$ a.e., so the integrand in the corollary is precisely the derivative of the indefinite integral. By analogy with this fact, the integrand f in (14) is called the Radon–Nikodym derivative of ν with respect to μ and is denoted by $\frac{d\nu}{d\mu}$. This terminology may seem unsatisfying when we are dealing with an abstract measure space, because we are accustomed to thinking of derivatives as represented by limits of difference quotients. We prove in Chapter 8 that such representations are possible in the abstract setting, thereby providing a more satisfying justification for calling

$$f = \frac{d\nu}{d\mu}$$

a derivative. For the moment, we provide a theorem that shows that formally $\frac{d\nu}{d\mu}$ *does* possess some properties reminiscent of derivatives.

Theorem 5.31 *Let $(X, \mathcal{M})$ be a measurable space, let ν, ζ, and μ be measures on $\mathcal{M}$, and suppose that μ is σ-finite. Then*

1. *If $\zeta \ll \mu$ and g is a nonnegative μ–measurable function, then*

$$\int_E g\, d\zeta = \int_E g \frac{d\zeta}{d\mu}\, d\mu$$

   for every $E \in \mathcal{M}$.

2. *If $\nu \ll \mu$ and $\zeta \ll \mu$, then* $\dfrac{d(\nu+\zeta)}{d\mu} = \dfrac{d\nu}{d\mu} + \dfrac{d\zeta}{d\mu}$.

3. *If $\nu \ll \zeta \ll \mu$, then* $\dfrac{d\nu}{d\mu} = \dfrac{d\nu}{d\zeta}\dfrac{d\zeta}{d\mu}$.

4. *If $\nu \ll \mu$ and $\mu \ll \nu$, then* $\dfrac{d\nu}{d\mu} = \left(\dfrac{d\mu}{d\nu}\right)^{-1}$.

Proof. Part (1) is just Theorem 5.23, and part (2) is Theorem 5.13(3). To verify (3), let $E \in \mathcal{M}$. Then

$$\nu(E) = \int_E \frac{d\nu}{d\zeta}\, d\zeta = \int_E \frac{d\nu}{d\zeta}\frac{d\zeta}{d\mu}\, d\mu,$$

the second equality following from (1) with $g = d\nu/d\zeta$. Part (4) now follows from (3) since $1 = d\nu/d\nu = (d\nu/d\mu)(d\mu/d\nu)$. ■

Example 5.32 Let $X = \mathbb{N}$, and let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers, with

$$\sum_{n=1}^{\infty} a_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty.$$

For $E \subset \mathbb{N}$, define

$$\nu(E) = \sum_{n \in E} a_n \quad \text{and} \quad \mu(E) = \sum_{n \in E} b_n.$$

Then ν and μ are measures on $2^{\mathbb{N}}$. Clearly, $\nu \ll \mu$. For f any nonnegative function on $\mathbb{N}$ and $E \subset \mathbb{N}$, we have

$$\int_E f\,d\mu = \sum_{n\in E} f(n)b_n.$$

Thus $f = \frac{d\nu}{d\mu}$ if, for each $E \subset \mathbb{N}$,

$$\sum_{n\in E} a_n = \nu(E) = \sum_{n\in E} f(n)b_n,$$

that is, $f(n) = a_n/b_n$. It is also true that $\mu \ll \nu$ and the derivative $\frac{d\mu}{d\nu}$ is $1/f$.

Example 5.33 We illustrate an interesting decomposition of a measure as a sum of two measures. Theorem 5.34, which follows, shows how to do this in general. Let f be the Cantor function, and let $g(x) = x^2$ on [0,1]. Since $\mu_f(E) = 0$ whenever E is a measurable set disjoint from the zero measure Cantor set, the measures μ_f and λ are, by definition, mutually singular, i.e., $\mu_f \perp \lambda$. (See Section 2.5). The measure μ_{g+f} can therefore be decomposed into a sum

$$\mu_{g+f} = \mu_g + \mu_f$$

of two measures, one absolutely continuous with respect to λ and the other mutually singular with λ.

The next theorem shows that a decomposition such as illustrated in the example always occurs for a σ-finite measure space.

Theorem 5.34 (Lebesgue decomposition) *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let ν be a σ-finite measure on $\mathcal{M}$. Then there exist measures α and β such that $\alpha \ll \mu$ and $\beta \perp \mu$ and for which $\nu = \alpha + \beta$. The measures α and β are unique.*

Proof. Let $\zeta = \mu + \nu$. Then ζ is a σ-finite measure on $\mathcal{M}$, and $\mu \ll \zeta$, $\nu \ll \zeta$. By Theorem 5.29, there exist nonnegative measurable functions f and g such that, for each $E \in \mathcal{M}$,

$$\mu(E) = \int_E f\,d\zeta \quad\text{and}\quad \nu(E) = \int_E g\,d\zeta.$$

Let $A = \{x : f(x) > 0\}$ and $B = \{x : f(x) = 0\}$. Then $X = A \cup B$, $A \cap B = \emptyset$, and

$$\mu(B) = \int_B f\,d\zeta = 0.$$

Define measures α and β on $\mathcal{M}$ by

$$\alpha(E) = \nu(E \cap A) \quad\text{and}\quad \beta(E) = \nu(E \cap B). \tag{19}$$

We infer from (19) that $\nu = \alpha + \beta$. Since $\beta(A) = \nu(A \cap B) = \nu(\emptyset) = 0$, we have $\beta \perp \mu$. To verify that $\alpha \ll \mu$, let E be any member of $\mathcal{M}$ for which $\mu(E) = 0$. We show that $\alpha(E) = 0$. From the equalities

$$0 = \mu(E) = \int_E f \, d\zeta,$$

we infer $f(x) = 0$ for ζ–almost every $x \in E$. Now $f > 0$ on $A \cap E$, so $\zeta(A \cap E) = 0$. Thus, by (19),

$$\alpha(E) = \nu(A \cap E) = \int_{A \cap E} g \, d\zeta = 0,$$

and $\alpha \ll \mu$.

It remains to show the uniqueness of α and β. We leave the verification of this fact as Exercise 5:8.2. ■

Exercises

5:8.1 Show that Theorem 5.29 fails if one drops the requirement that the space be σ-finite. [Hint: Let μ be the counting measure on the subsets of $\mathbb{R}$ and $\nu = \lambda$.]

5:8.2 (a) Prove that if $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$.

(b) Prove that if each of ν_1 and ν_2 is absolutely continuous [singular] with respect to μ then so is any linear combination of ν_1 and ν_2.

(c) Prove the uniqueness part of Theorem 5.34.

5.9 Convergence Theorems

In Section 4.3, we discussed several modes of convergence of a sequence of measurable functions, and we indicated implications that exist among them. We now use our knowledge of the integral to obtain some further convergence theorems. We begin by defining a new notion of convergence for a sequence of integrable functions.

Definition 5.35 Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of integrable functions. If there exists $f \in L_1$ such that

$$\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0,$$

we say that $\{f_n\}$ *converges to f in the mean* and write $f_n \to f$ [mean].

We can put a metric on the space L_1 that expresses mean convergence by writing

$$\rho(f, g) = \int_X |f - g| \, d\mu.$$

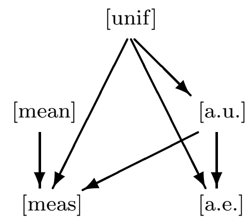

Figure 5.3: Further comparison of modes of convergence in a measure space.

(See also Chapter 13 for a more detailed account of this space.) Since this is the most natural and useful metric on L_1, this convergence is commonly called L_1–convergence or convergence in L_1. One of the most useful consequences of mean convergence is that if $f_n \to f$ [mean] then f_n converges to f *weakly* in the sense that

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu \tag{20}$$

for every measurable set E. This follows immediately from the inequality

$$\left| \int_E f_n \, d\mu - \int_E f \, d\mu \right| \leq \int_E |f_n - f| \, d\mu \leq \int_X |f_n - f| \, d\mu.$$

Mean convergence is easily seen to be stronger than convergence in measure. This is our first theorem. Note immediately, however, that mean convergence is not implied by any other of our forms of convergence. Figure 5.3 illustrates and is a repeat of Figure 4.1 with mean convergence now added. Without some restrictions, even uniform convergence does not imply mean convergence. For example, the sequence of functions

$$f_n = n^{-1} \chi_{[n,2n]}$$

converges uniformly to zero on $\mathbb{R}$, but

$$\int_{\mathbb{R}} |f_n| \, d\lambda = 1$$

for every $n \in \mathbb{N}$. If we assume that the space has finite measure, then clearly uniformly convergent sequences converge in mean, but there are no other new implications. Figure 5.4 illustrates this.

Theorem 5.36 *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ [mean]. Then $f_n \to f$ [meas].*

Proof. The conclusion follows from the inequality

$$\mu\left(\{x : |f_n(x) - f(x)| \geq \eta\}\right) \leq \eta^{-1} \int_X |f_n - f| \, d\mu$$

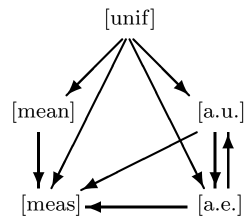

Figure 5.4: Further Comparison of modes of convergence in a *finite* measure space.

(cf. Exercise 5:2.3). ■

The Lebesgue dominated convergence theorem (Theorem 5.14) provides a condition under which mean convergence follows from convergence in measure.

Theorem 5.37 *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions such that $f_n \to f$ [meas]. If there exists $g \in L_1$ such that $|f_n| \le g$ a.e. for every $n \in I\!N$, then $f_n \to f$ [mean].*

Proof. By Theorem 4.14 there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ [a.e.]. Thus $|f| \le g$ a.e., so $|f| \in L_1$. In particular, then, $|f_n - f| \le 2g$ [a.e.] and so, by Corollary 5.15,

$$\int_X |f_n - f| \, d\mu \to 0,$$

as required. ■

The preceding proof is quick, but not revealing. A direct proof that does not invoke the Lebesgue dominated convergence theorem is more illuminating and illustrates a principle that is often the basis for estimates involving integrals. We refer to this technique as the *rectangle principle*. In its crudest form it states that the area of a rectangle, whose dimensions $a \times b$ may vary, can be made arbitrarily small if one of the dimensions is controlled in size and the other can be made sufficiently small. Analogously, in the setting of integrals it states that an integral $\int_E F \, d\mu$, where F and E may vary, can be made arbitrarily small if the size of either F or E can be controlled and the other can be made sufficiently small. In the following proof of Theorem 5.37, observe the roles played by convergence in measure and by absolute continuity to allow use of the rectangle principle. (See also Exercise 5:9.5 for a similar application of this principle.)

Proof. (Alternative proof of Theorem 5.37) Let $\varepsilon > 0$. Since $g \in L_1$, we can choose $\alpha > 0$ so that

$$\int_{\{x:2g(x)\le\alpha\}} 2g \, d\mu < \varepsilon/3.$$

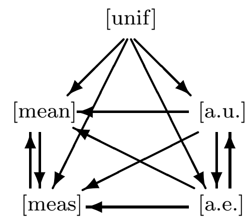

Figure 5.5: Comparison of modes of convergence when there exists $g \in L_1$ such that $|f_n| \le g$ for all n.

Letting
$$A = \{x : 2g(x) > \alpha\},$$
we note that $\mu(A) < \infty$ so there is a $\eta > 0$ with $\eta\mu(A) < \varepsilon/3$. From the absolute continuity of the integral, there is a $\delta > 0$ so that
$$\int_E 2g\,d\mu < \varepsilon/3$$
whenever $\mu(E) < \delta$. Finally, choose N so that $\mu(B_n) < \delta$ for all $n \ge N$, where
$$B_n = \{x \in A : |f_n(x) - f(x)| \ge \eta\}.$$

Now, using the inequalities $|f_n - f| \le 2g$ a.e. and $|f_n - f| < \eta$ on $A \setminus B_n$, we have
$$\int_X |f_n - f|\,d\mu \le \int_{X\setminus A} 2g\,d\mu + \int_{B_n} 2g\,d\mu + \int_{A\setminus B_n} \eta\,d\mu < \varepsilon$$
for all $n \ge N$, as required to prove the theorem. ■

Note how the second and third integrals illustrate the rectangle principle. In the first case B_n is small and $2g$ controlled, while in the other case η is small, $\mu(A \setminus B_n)$ is controlled.

The condition of the theorem, that there is an integrable function g dominating the sequence $\{f_n\}$, gives a number of implications among the types of convergence (uniform, a.e., a.u., measure, and mean). To display these, we now add a further convergence chart (Figure 5.5).

Exercise 5:9.2 calls for the verification of several of these implications that exist among our five notions of convergence. One of these, that convergence [a.e.] implies convergence [a.u.], requires a revision of Egoroff's theorem (Theorem 4.16) to handle the case where the sequence is dominated, in place of the original assumption that the space had finite measure. (Exercise 4:3.4 has already suggested that such a result should be possible.) We shall prove this now. In particular, note that the proof essentially contains the observation that, when the functions $|f_n|$ are dominated by a function $g \in L_1$, then convergence [a.e.] implies convergence [meas]. This result is also an immediate consequence of Theorems 5.37 and 5.36.

Theorem 5.38 (Egoroff) *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of finite a.e. measurable functions for which $f_n \to f$ [a.e.]. If there exists $g \in L_1$ such that, for every $n \in \mathbb{N}$, $|f_n| \leq g$ a.e., then $f_n \to f$ [a.u.].*

Proof. We define sets A_{nk}, $n, k \in \mathbb{N}$, by

$$A_{nk} = \bigcap_{m=n}^{\infty} \left\{ x : |f_m(x) - f(x)| < \frac{1}{k} \right\},$$

and we show that

$$\lim_{n\to\infty} \mu(X \setminus A_{nk}) = 0. \tag{21}$$

Let $k \in \mathbb{N}$, $x \in X$. If $\lim_{n\to\infty} f_n(x) = f(x)$, then

$$x \in \bigcup_{n=1}^{\infty} A_{nk}.$$

Thus our assumption that $f_n \to f$ [a.e.] implies that

$$\mu\left(\bigcap_{n=1}^{\infty} (X \setminus A_{nk}) \right) = 0.$$

The sequence $A_{1k}, A_{2k}, \dots$ is an expanding sequence of measurable sets. We verify (21) by showing that there exists $n \in \mathbb{N}$ such that $\mu(X \setminus A_{nk}) < \infty$ and then applying Theorem 2.20(2). Our hypotheses imply that $|f| \leq g$ a.e. Thus

$$|f_m - f| \leq 2g \ \textit{a.e.} \tag{22}$$

for every $m \in \mathbb{N}$. Now

$$X \setminus A_{nk} = \bigcup_{m=n}^{\infty} \left\{ x : |f_m(x) - f(x)| \geq \frac{1}{k} \right\} \subset S \cup T,$$

where

$$S = \left\{ x : 2g(x) \geq \frac{1}{k} \right\}$$

and

$$T = \bigcup_{m=n}^{\infty} \{ x : |f_m(x) - f(x)| > 2g \}.$$

By (22) we see that $\mu(T) = 0$. From the fact that $g \in L_1$ we obtain that $\mu(S) < \infty$. Thus it follows that $\mu(X \setminus A_{nk}) < \infty$.

We have shown that our present hypotheses imply the validity of (21). Observe that (21) is identical to equation (1) in the proof of Theorem 4.16, and so the proof may be continued by repeating the remainder of that proof without changes. ■

We close with a final remark about the condition $|f_n| \leq g$ that has played such an important role in the convergence theory of the integral here and in earlier sections. One should ask whether there is a weaker hypothesis than this under which Theorem 5.37 can be proved and, indeed whether there is a condition that is both necessary and sufficient. The clue is that the condition $|f_n| \leq g$ ensures that the measures $\nu_n = \int |f_n|\, d\mu$ are uniformly absolutely continuous with respect to μ in a certain sense. This analysis was initiated by Vitali and completed by Lebesgue. Exercise 5:9.5 gives the version for a finite measure space, and Exercise 5:9.8 gives a version valid in general.

Exercises

5:9.1 If $f_n \to f$ [mean] show that $\int_E f_n\, d\mu \to \int_E f\, d\mu$ for every measurable set E. Show that the converse is false. [Hint: a counterexample for the converse will require than f_n not converge to f in measure.]

5:9.2 We have established most of the implications and provided counterexamples for the most of the nonimplications in Figure 5.5 in the text. Verify that the remaining implications are valid and that no implications were omitted.

5:9.3 Show that if $f_n \to f$ [mean] and g is a bounded measurable function then $f_n g \to fg$ [mean].

5:9.4 For every $n \in \mathbb{N}$, let $\{a_{nk}\}$ be a sequence of numbers with $|a_{nk}| \leq 2^{-k}$ for each k. Suppose for each k that the sequence $\{a_{nk}\}$ converges to some number a_k. Prove that the series $\sum_{k=1}^{\infty} a_k$ is convergent and that

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}.$$

5:9.5$\diamondsuit$ Let $\mathcal{V}$ be a family of measures defined on $\mathcal{M}$. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\mu(E) < \delta$, then $\nu(E) < \varepsilon$ for every $\nu \in \mathcal{V}$, we say that the family is *uniformly absolutely continuous* with respect to μ. Prove the theorem and show that this theorem does not necessarily hold on a space of infinite measure.

> **Theorem (Vitali–Lebesgue)** *Let $(X, \mathcal{M}, \mu)$ be a finite measure space, let f be measurable, and let $\{f_n\}$ be a sequence of integrable functions. Then $f_n \to f$ [mean] if and only if $f_n \to f$ [meas] and $\nu_n = \int |f_n|\, d\mu$ are uniformly absolutely continuous.*

[Hint: The hypothesis of uniform absolutely continuity can be used to show that $f \in L_1$. Its use in the remainder of the proof involves an application of the "rectangle principle" used in proving Theorem 5.37. Try to prove the result for $X = [a, b]$ first. In the general case of a space of finite measure, you might wish to use Exercise 2:13.8(a) when μ is nonatomic and observe that, for any $\gamma > 0$, there can be only finitely many atoms whose measures exceed γ.]

5:9.6 Prove the theorem:

> **Theorem (de la Vallée Poussin)** *Let $\mathcal{F}$ be a family of measurable functions defined on a measure space $(X, \mathcal{M}, \mu)$ with $\mu(X) < \infty$. If there exists a positive increasing function $\phi : (0, \infty) \to I\!R$ with $\lim_{t\to\infty} \phi(t) = \infty$ and a constant A such that*
> $$\int_X |f|\, \phi(|f|)\, d\mu < A$$
> *for all $f \in \mathcal{F}$, then the members of $\mathcal{F}$ are in L_1 and the family of measures $\nu_f = \int |f|\, d\mu$ is uniformly absolutely continuous.*

[Hint: For $\varepsilon > 0$ choose K such that $A/\phi(K) < \varepsilon/2$. For $f \in \mathcal{F}$ and $E \in \mathcal{M}$, consider the set $\{x \in E : |f(x)| \leq K\}$. Use that set to show $\int_E |f|\, d\mu \leq A/\phi(K) + K\mu(E)$.]

5:9.7 Let $\mathcal{F}$ be a family of measurable functions defined on a measure space $(X, \mathcal{M}, \mu)$ with $\mu(X) < \infty$ and suppose that $\int_X f^2\, d\mu < A$ for all $f \in \mathcal{F}$. Prove that the integrals $\int |f|\, d\mu$ are uniformly absolutely continuous. Deduce from this that, if $f_n \to f$ [meas] and $f_n \in \mathcal{F}$, then $f_n \to f$ [mean]. [Hint: Apply the de la Vallée Poussin theorem of Exercise 5:9.6.]

5:9.8 Let $\mathcal{V}$ be a family of measures defined on $\mathcal{M}$. We say the family is *equicontinuous* at $\emptyset$ if for every $\varepsilon > 0$ and every decreasing sequence of measurable sets E_n shrinking to $\emptyset$ there exists N such that $\nu(E_n) < \varepsilon$ for every $n \geq N$ and every $\nu \in \mathcal{V}$.

(a) Let $\mathcal{V}$ be equicontinuous at $\emptyset$ and suppose each member of $\mathcal{V}$ is absolutely continuous with respect to μ. Show that $\mathcal{V}$ is uniformly absolutely continuous with respect to μ.

(b) Show that on a finite measure space a uniformly absolutely continuous family of measures must be also equicontinuous at $\emptyset$.

(c) Prove the theorem:

> **Theorem (Vitali–Lebesgue)** *Let $(X, \mathcal{M}, \mu)$ be a measure space, let f be measurable, and let $\{f_n\}$ be a sequence of integrable functions. Then $f_n \to f$ [mean] if and only if $f_n \to f$ [meas] and $\nu_n = \int |f_n|\, d\mu$ are equicontinuous.*

5:9.9 Show that Lebesgue's dominated convergence theorem follows from the Vitali–Lebesgue theorems of the preceding exercises.

5.10 Relations to Other Integrals

The beginning student of integration theory is often left somewhat bewildered by the relation that the Lebesgue integral has to various other integrals previously learned. To be sure, as we have seen in Section 5.5, the Lebesgue integral includes the Riemann integral and is (it should now appear) an entirely natural extension of Riemann's integral. One is easily led to assume incorrectly that the Lebesgue integral, since it is clearly the dominant integral in modern analysis, must be an extension of every other integration method.

We have seen in the introductory chapter a number of other methods for integrating functions. How does the Lebesgue integral compare to the improper Cauchy integrals, against the Newton integral or the generalized Riemann integral? One key notion allows us to see some differences. The Lebesgue integral is an absolute integral: in order for $\int_a^b f(x)\,dx$ to exist in the Lebesgue sense, so also must $\int_a^b |f(x)|\,dx$. This immediately reveals some distinctions. The improper Cauchy integrals, the Newton integral, and the generalized Riemann integral are all nonabsolute integrals. One well-known example illustrates the situation: the derivative of the function $f(x) = x^2 \sin x^{-2}$ is integrable in each of these senses on $[0,1]$, but the integral $\int_0^1 |f'(x)|\,dx$ taken in any sense (including Lebesgue's) must be infinite. Thus the Lebesgue integral does not include any of these integrals. In the other direction, it is easy to give examples of functions that are Lebesgue integrable on the interval $[0,1]$ and yet not integrable as Cauchy or Newton integrals. If an integral exists as both a Newton integral and a Lebesgue integral, then the values must be the same; this follows from the fundamental theorem of calculus for the Lebesgue integral. (Theorem 5.21 does this for bounded derivatives; Section 7.5 will do this for integrable derivatives.) Thus, while distinct, the Newton integral and the Lebesgue integral on an interval are compatible.

In fact, there remain only two questions requiring answers.

1. Is the Cauchy procedure for integrating unbounded functions or integrating over unbounded intervals compatible with that of Lebesgue? Do they produce the same value?

2. How does the Lebesgue integral compare to the generalized Riemann integral?

We shall now address both of these questions.

The first question is easy. The reader should quickly find proofs for the following three assertions. They are enough to see that the Cauchy procedure may be used to compute the value of a Lebesgue integral, provided only that one knows in advance that the Lebesgue integral exists. We use the conventional calculus notation for our Lebesgue integrals here.

Theorem 5.39 *Let f be Lebesgue integrable over an interval $[a,b]$. Then*

$$\int_a^b f(x)\,dx = \lim_{t \searrow a} \int_t^b f(x)\,dx.$$

Theorem 5.40 *Let f be a function bounded below on an interval $[a,b]$, and suppose that f is Lebesgue integrable over each interval $[t,b]$ for $a < t < b$. Then f is Lebesgue integrable over $[a,b]$ if and only if the limit*

$$\lim_{t \searrow a} \int_t^b f(x)\,dx$$

exists.

Theorem 5.41 *Suppose that f is Lebesgue integrable over the interval $(-\infty,+\infty)$. Then*

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{s,t \to +\infty} \int_{-s}^{t} f(x)\,dx.$$

The second problem mentioned, establishing the relation of the Lebesgue integral to the generalized Riemann integral, is far less trivial. On an interval $[a,b]$ it turns out that the generalized Riemann integral strictly contains Lebesgue's integral. This shows that the Lebesgue integral may be expressed as a limit of "Riemann sums," much in the spirit of the origins of integration theory with Cauchy and Riemann. While nowadays this might seem a curiosity, it was considered important enough in Lebesgue's time that he proved (in 1909) that his integral could be so expressed, but his expression of this fact was not so simple as in this theorem.

Theorem 5.42 *Let f be Lebesgue integrable on an interval $[a,b]$. Then, for any $\varepsilon > 0$, there is a positive function δ on $[a,b]$ so that whenever*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

is a partition of $[a,b]$ with associated points $\xi_i \in [x_{i-1}, x_i]$ such that

$$x_i - x_{i-1} < \delta(\xi_i) \qquad (i = 1, 2, \ldots, n),$$

we have

$$\left| \sum_i f(\xi_i)\,(x_i - x_{i-1}) - \int_{[a,b]} f(x)\,dx \right| < \varepsilon.$$

We shall prove this theorem in a metric space for greater generality; this also gives us an opportunity to use some of the techniques we have acquired in our study of the integration theory. The proof we give is due to Davies and Schuss.[3]

[3] R. O. Davies and Z. Schuss, *J. London Math. Soc.* (2) 2 (1970), 561–562.

Theorem 5.43 *Let X be a metric space and μ a Borel regular measure on X. Let f be a real function integrable on a measurable set $E \subset X$ for which $\mu(E) < \infty$. Then for any $\varepsilon > 0$ we can associate with each $x \in E$ an open set $G(x)$ containing x in such a way that the following statement holds:*

> *Whenever B_1, B_2, ... is a finite or infinite sequence of disjoint measurable subsets of E for which*
>
> $$\mu\left(E \setminus \bigcup_i B_i\right) = 0$$
>
> *and $\xi_i \in B_i$ with $B_i \subset G(\xi_i)$, then*
>
> $$\left|\sum_i f(\xi_i)\mu(B_i) - \int_E f(x)\,d\mu(x)\right| < \varepsilon.$$

Proof. Using the absolute continuity of the integral, we can determine $\eta > 0$ so that, whenever A is a measurable subset of E with $\mu(A) < \eta$, then

$$\int_A |f|\,d\mu < \frac{\varepsilon}{3}.$$

Write $\kappa = \frac{1}{3}\varepsilon(\eta + \mu(E))^{-1}$ and partition E into the sequence of measurable sets

$$E_m = \{x \in E : (m-1)\kappa < f(x) \le m\kappa\} \quad (m = 0, \pm 1, \pm 2, \pm 3, \dots).$$

Choose an open set $G_m \supset E_m$ so that

$$\mu(G_m \setminus E_m) < \frac{\eta}{2^{|m|}3(|m|+1)}.$$

We determine our sets $G(x)$ now by writing $G(x) = G_m$, where $x \in E_m$.

Suppose that B_1, B_2, ... is a sequence of disjoint measurable subsets of E for which

$$\mu\left(E \setminus \bigcup_i B_i\right) = 0$$

and that each $\xi_i \in B_i$ with $B_i \subset G(\xi_i)$. Choose $m(i)$ so that $\xi_i \in E_{m(i)}$; this means that $B_i \subset G_{m(i)}$ and

$$B_i \setminus E_{m(i)} \subset G_{m(i)} \setminus E_{m(i)}.$$

Then we compute

$$\left|\sum_i f(\xi_i)\mu(B_i) - \int_E f(x)\,d\mu(x)\right| = \left|\sum_i \int_{B_i} \left(f(\xi_i) - f(x)\right)\,d\mu(x)\right|$$

$$\leq \sum_i \int_{B_i} |f(\xi_i) - f(x)| \, d\mu(x).$$

We split this last sum into three parts and show that each is less than $\varepsilon/3$. The theorem evidently follows.

Write

$$P = \sum_i \int_{B_i \cap E_{m(i)}} |f(\xi_i) - f(x)| \, d\mu(x),$$

$$Q = \sum_i \int_{B_i \setminus E_{m(i)}} |f(\xi_i)| \, d\mu(x),$$

and

$$R = \sum_i \int_{B_i \setminus E_{m(i)}} |f(x)| \, d\mu(x).$$

Since, whenever $x \in B_i \cap E_{m(i)}$, the numbers $f(x)$ and $f(\xi_i)$ can differ by no more than κ we have

$$P \leq \sum_i \int_{B_i \cap E_{m(i)}} \kappa \, d\mu(x) \leq \kappa \sum_i \mu(B_i) = \kappa\mu(E) < \frac{\varepsilon}{3}.$$

We can write Q as

$$Q = \sum_{m=-\infty}^{\infty} \left[\sum_{m(i)=m} \int_{B_i \setminus E_{m(i)}} |f(\xi_i)| \, d\mu(x) \right]$$

$$\leq \sum_{m=-\infty}^{\infty} \left[\sum_{m(i)=m} (|m|+1)\kappa\mu\left(B_i \setminus E_{m(i)}\right) \right]$$

$$\leq \sum_{m=-\infty}^{\infty} (|m|+1)\kappa\mu\left(G_m \setminus E_m\right) < \frac{\varepsilon}{3}.$$

If we define

$$A = \bigcup_i \left(B_i \setminus E_{m(i)}\right),$$

then we see, since this is a disjoint union, that

$$\mu(A) = \sum_{m=-\infty}^{\infty} \left[\sum_{m(i)=m} \mu\left(B_i \setminus E_{m(i)}\right) \right]$$

$$\leq \sum_{m=-\infty}^{\infty} \mu\left(G_m \setminus E_m\right) < \eta.$$

Consequently,

$$R = \int_A |f(x)|\,d\mu(x) < \frac{\varepsilon}{3}$$

by the definition of η. Putting these together, we have

$$\Big|\sum_i f(\xi_i)\mu(B_i) - \int_E f(x)\,d\mu(x)\Big|$$

$$\leq \sum_i \int_{B_i} |f(\xi_i) - f(x)|\; d\mu(x) \leq P + Q + R < \varepsilon$$

and the theorem is proved. ■

Exercises

5:10.1 Prove Theorem 5.39. Show also that the limit can exist for functions f that are not Lebesgue integrable over $[a,b]$ (but are integrable on $[t,b]$ for all $a < t < b$).

5:10.2 Prove Theorem 5.40.

5:10.3 Prove Theorem 5.41. Show also that the limit can exist for functions f that are not Lebesgue integrable over $(-\infty,+\infty)$ (but are integrable on all bounded intervals).

5:10.4 Compute the Lebesgue integral of $\int_0^1 x^p\,dx$ for $p < 0$, making sure to justify the computations.

5:10.5 Show that Theorem 5.42 does follow from Theorem 5.43.

5.11 Integration of Complex Functions

So far the integral has been defined first for measurable functions assuming finitely many real values, then for nonnegative measurable functions, and finally for arbitrary real-valued measurable functions. There is also a need in many applications of integration theory to be able to handle complex-valued functions. Indeed, in parts of Chapters 13, 14, and 15 we will require such a theory. In this section we shall discuss how the integral may be extended in this way.

At the outset, let us avoid a possible misconception. We are not embarking on a study of complex analysis itself. That subject concerns itself with complex-valued analytic functions defined on a subset of $\mathbb{C}$ (or $\mathbb{C}^n$), and the integrals encountered there are usually line integrals of continuous functions. The Lebesgue theory is not commonly required in such a study. Our setting is a measure space $(X,\mathcal{M},\mu)$, and we wish to investigate integrals of functions defined on X, but with complex values. Even as dedicated "real" analysts, we cannot avoid dealing with such functions because they arise in a wide variety of problems.

Our first task is to interpret what we shall mean by a measurable complex-valued function. There are two possible approaches here. A function $f : X \to \mathbb{C}$ can be written as $f(x) = f_1(x) + if_2(x)$ by splitting into real and imaginary parts so that f_1 and f_2 are real-valued functions; measurability of f could be interpreted as measurability of the real and imaginary parts f_1 and f_2. Alternatively, we could directly generalize the definition of measurability (Definition 4.1) for real-valued functions to the situation of functions $f : X \to Y$, where Y is any metric space. Complex-valued functions are then handled as well, since $\mathbb{C}$ is a metric space under its usual modulus metric. We choose the latter definition and then show that this is equivalent to requiring measurability of the real and imaginary parts.

Definition 5.44 Let $(X, \mathcal{M}, \mu)$ be a measure space, let Y be a metric space, and let $f : X \to Y$. The function f is *measurable* if for every open set $G \subset Y$ the set $f^{-1}(G)$ is a measurable set.

Note that for real-valued functions Definitions 4.1 and 5.44 are equivalent; the latter, however, better shows the true nature of measurable functions as mappings from one structure (a measure space) to another (a metric space) that preserves elements of the structures (inverse images of open sets in the metric space are measurable sets in the measure space). Since we are concerned in this section only with complex-valued functions, let us immediately relate this definition to the real case.

Theorem 5.45 *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{C}$ with $f = f_1 + if_2$, where f_1 and f_2 are the real and imaginary parts of f. The following hold.*

1. *If f is measurable, then f_1, f_2, and $|f|$ are real-valued measurable functions.*

2. *If f_1, f_2 are real-valued measurable functions, then f is measurable.*

3. *If f is measurable, then there is a measurable function $h : X \to \mathbb{C}$ with $|h| = 1$ and $f = h|f|$.*

Proof. Let us first prove (1) from an interesting general observation. If Y_1, Y_2 are metric spaces and $g : Y_1 \to Y_2$ is continuous, and if $f : X \to Y_1$ is measurable, then the composed map $h = g \circ f$ taking X to Y_2 must be measurable. This is easy to see since if $G \subset Y_2$ is open then $g^{-1}(G)$ is open (since g is continuous) and $h^{-1}(G) = f^{-1}(g^{-1}(G))$ is measurable (since f is a measurable function).

Item (1) now follows from our observation above. By taking $Y_1 = \mathbb{C}$, $Y_2 = \mathbb{R}$, and $g(x + iy) = x$, we get that $f_1 = g \circ f$ is a measurable function from X to $\mathbb{R}$. Similarly, if $g(x + iy) = y$, we get that $f_2 = g \circ f$ is measurable. Finally, if $g(x + iy) = \sqrt{x^2 + y^2}$, we get that $|f| = g \circ f$ is measurable.

Item (3) can be proved from the same observation. If f never assumes the value zero in the complex plane, then take $Y_1 = \mathbb{C} \setminus \{0\}$, $Y_2 = \mathbb{C}$, and $g(z) = z/|z|$. Note that g is a continuous map of Y_1 to Y_2. We get then that $g = h \circ f$ is a measurable function from X to $\mathbb{C}$, $|g(x)| = |f(x)|/|f(x)| = 1$, and $g(x)|f(x)| = f(x)$, as required. However, f might assume zero values and so some modification in our argument is needed. Since this is easy enough and entertaining, too, it is left as an exercise.

Finally, let us turn to proving (2), again from an interesting general observation. Let u_1 and u_2 be real-valued measurable functions on X and let h be any continuous mapping of $\mathbb{R}^2$ into a metric space Y. Then the function

$$F(x) = h(u_1(x), u_2(x))$$

is a mapping from X to Y. We shall show that F must be measurable. Let $f(x) = (u_1(x), u_2(x))$ so that f is a mapping from X to $\mathbb{R}^2$. Since $F = h \circ f$ we see that the measurability of F follows (by our first observation) from the measurability of f.

To see the measurability of f, consider any open rectangle

$$R = (a, b) \times (c, d)$$

in $\mathbb{R}^2$. The set

$$f^{-1}(R) = u_1^{-1}(a, b) \cap u_2^{-1}(c, d)$$

must be measurable since both functions u_1 and u_2 are measurable. This is just for open rectangles; but any open set in $\mathbb{R}^2$ can be expressed as a countable union of such open rectangles. Suppose that G is an open subset of $\mathbb{R}^2$ and that $G = \bigcup_i R_i$ where R_i are open rectangles. Then

$$f^{-1}(G) = f^{-1}\left(\bigcup_i R_i\right) = \bigcup_i f^{-1}(R_i)$$

must be measurable, since it is a countable union of measurable sets. Because G is an arbitrary open set in $\mathbb{R}^2$, we see that f is measurable by definition. Item (2) now follows from this observation by taking $u_1 = f_1$, $u_2 = f_2$, $Y = \mathbb{C}$, and $h(x, y) = x + iy$. ■

Definition 5.46 Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be a complex-valued measurable function on X with $f(x) = f_1(x) + i f_2(x)$ and f_1, f_2 real. Suppose that $\int_X |f|\, d\mu < \infty$. Then f is said to be *integrable*, and the *integral* of f with respect to μ is defined to be the quantity

$$\int_X f\, d\mu = \int_X f_1\, d\mu + i \int_X f_2\, d\mu.$$

In this definition we have required that $\int_X |f|\, d\mu < \infty$, which implies that the integrals $\int_X f_1\, d\mu$ and $\int_X f_2\, d\mu$ both exist and are finite. For complex-valued functions we do not wish to allow infinite values for the integral.

Does the integration theory change at all now that we have allowed complex values for the functions? The answer is no, but occasionally it takes some care and attention to detail to check this. We state the following theorems and leave it to the reader to manufacture the details of the proofs. There is nothing difficult about this process; the hardest part is to realize what it is that needs to be proved. Remember to use the real versions to get the complex versions.

Theorem 5.47 *Let $(X, \mathcal{M}, \mu)$ be a measure space and let f, $g : X \to \mathbb{C}$, with both f, g integrable. Then*

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu$$

for any complex numbers α, β.

The next theorem is particularly easy to prove for real functions as it follows from the monotonicity of the integral. For complex-valued functions, it requires some different thinking. Here the symbol $|\cdot|$ means a complex modulus, not merely an absolute value.

Theorem 5.48 *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to \mathbb{C}$ with f integrable. Then*

$$\left| \int_X f\, d\mu \right| \le \int_X |f|\, d\mu.$$

Exercises

5:11.1 Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to \mathbb{R}$. Show that f is measurable (according to Definition 4.1) if and only if for every open set $G \subset \mathbb{R}$ the set $f^{-1}(G)$ is a measurable set.

5:11.2 Let $(X, \mathcal{M}, \mu)$ be a measure space and f a function on X to a metric space Y. Show that f is measurable if and only if $f^{-1}(B) \in \mathcal{M}$ for every Borel set $B \subset Y$ (cf. Theorem 4.6 and Exercise 4:1.2).

5:11.3 If f is a measurable complex-valued function, show that there exists a measurable complex-valued function $h : X \to \mathbb{C}$ with $|h| = 1$ and $f = h|f|$. [Hint: This is started in Theorem 5.45. It is only the possibility that the set $E = \{x \in X : f(x) = 0\}$ is nonempty that remains to be handled. Is E measurable? How should $h(x)$ be defined for $x \in E$ and for $x \notin E$?]

5:11.4 Show that

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu$$

for any complex numbers α, β and integrable complex-valued functions f and g. [Hint: Split each of α, β, f, and g into real and imaginary parts and handle the pieces.]

5:11.5 Show that

$$\left|\int_X f\,d\mu\right| \le \int_X |f|\,d\mu$$

for complex-valued functions. [Hint: Let $c = \int_X f\,d\mu$ and choose b so that $bc = |c|$ and $|b| = 1$. Consider the integral of the function bf.]

5:11.6 Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of complex-valued measurable functions such that $f_n \to f$ [a.e.]. Suppose that there exists a function $g \in L_1$ such that $|f_n(x)| \le g(x)$ for all $n \in \mathbb{N}$ and $x \in X$. Show that f is integrable and

$$\int_X f\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu.$$

[Hint: This is a complex version of the Lebesgue dominated convergence theorem (Theorem 5.14).]

5.12 Additional Problems for Chapter 5

5:12.1 This problem presents a collection of different ways in which the integral has been defined[4]. In each case show that the method is equivalent to the one given in the text in an appropriate setting.

(a) (de la Vallée Poussin's method) For an unbounded, nonnegative measurable function f, introduce the "truncates" f_n defined as $f_n(x) = f(x)$ if $f(x) < n$ and $f_n(x) = n$ if $f(x) \ge n$. The integral is extended from bounded to unbounded functions by defining

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

(b) (Hobson's method) For an unbounded, measurable function f, introduce the "double truncates" f_{mn} defined as $f_{mn}(x) = f(x)$ if $-m < f(x) < n$, $f_{mn}(x) = n$ if $f(x) \ge n$, and $f_{mn}(x) = -m$ if $f(x) \le -m$. The integral is extended from bounded to unbounded functions by defining

$$\int_E f = \lim_{m,n\to\infty} \int_E f_{mn}.$$

(c) (Saks's method) For a nonnegative measurable function f, define the integral as

$$\int_E f = \sup \sum_{k=1}^{n} \mu(E_k) \left(\inf_{x\in E_k} f(x) \right),$$

[4]The collection is from Munroe, *Introduction to Measure and Integration*, Addison-Wesley, 2nd ed., 1971, pp. 126–127, Exercises d–g.

where the supremum is taken over all finite partitions of E into disjoint measurable subsets.

(d) (Carathéodory's method) For a nonnegative measurable function f on $E \subset \mathbb{R}^n$, define the Lebesgue integral to be the $(n+1)$-dimensional Lebesgue measure of the ordinate set:

$$\int_E f = \lambda_{n+1}\left(\{(x,y) : x \in E,\ 0 \le y \le f(x)\}\right).$$

5:12.2 Let E be a Lebesgue measurable set of positive measure, and let $\{x_n\}$ be some sequence of points from the interval $[0,1]$. Show that there must exist a point y and a subsequence $\{x_{n_k}\}$ so that $y+x_{n_k} \in E$ for all k. [Hint: Consider the functions $f_n(t) = \chi_E(t - x_n)$ and their integrals.]

5:12.3 Let f be nonnegative and Lebesgue integrable in the interval $[0,1]$, and suppose that, for every integer $n = 1, 2, 3, 4, \dots$,

$$\int_0^1 [f(x)]^n\, dx = \int_0^1 f(x)\, dx.$$

Show that f must be a.e. equal to the characteristic function χ_E of some measurable set $E \subset [0,1]$. [Hint: Apply Fatou's lemma.]

5:12.4 Let f be Lebesgue integrable on the interval $(-\infty,\infty)$ and let $\varepsilon > 0$. Show that there is a continuous function g that vanishes outside some interval and such that $\int_{-\infty}^{\infty} |f(x)-g(x)|\, dx < \varepsilon$. [Hint: Apply Lusin's theorem.]

5:12.5 Let f be Lebesgue integrable on the interval $(-\infty,\infty)$. Show that

$$\lim_{h\to 0} \int_{-\infty}^{\infty} |f(x+h) - f(x)|\, dx = 0.$$

[Hint: Use the approximation result in Exercise 5:12.4.]

5:12.6 Let $\{a_k\}$ be a sequence of real numbers with $\sum_{k=1}^{\infty} |a_k| < +\infty$, and let $\{r_k\}$ be an enumeration of the rationals in $[0,1]$. Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{\sqrt{|x - r_k|}}$$

converges absolutely almost everywhere in $[0,1]$. [Hint: What can you say about the sum $\sum_{k=1}^{\infty} |f_k(x)|$ if the sum $\sum_{k=1}^{\infty} \int_E |f_k(x)|\, dx$ converges?]

5:12.7 Let f be Lebesgue integrable on $[0,1]$ and suppose that $0 < c < 1$. If $\int_E f(t)\, dt = 0$ for every measurable set $E \subset [0,1]$ with $\lambda(E) = c$, then prove that f must vanish almost everywhere.

5:12.8 Prove that if $\mu(X) < \infty$ and $f^2 \in L_1$ then $f \in L_1$. Is this statement true without the hypothesis $\mu(X) < \infty$?

5:12.9◇ The discussion following Theorem 5.23, as well as much of the material in Sections 5.7 and 5.8, suggests that, when $\nu(E) = \int_E f\,d\mu$, the Radon–Nikodym derivative $f = \frac{d\nu}{d\mu}$ behaves very much like a derivative. In preparation for material in Chapter 8, do the following:

(a) Summarize the ways in which $\frac{d\nu}{d\mu}$ behaves formally like an ordinary derivative.

(b) Criticize the following statement. "Since $f = \frac{d\nu}{d\mu}$, the Radon–Nikodym Theorem generalizes both halves of the fundamental theorem of calculus: if we integrate f and differentiate the resulting measure ν, we get $\frac{d\nu}{d\mu} = f$; if we integrate $\frac{d\nu}{d\mu}$, we get ν."

(c) Let F have a bounded derivative on $[a, b]$, $F' = f$, and let $\nu(E) = \int_E f\,d\lambda$ for all $E \in \mathcal{L}$. Let $x \in [a, b]$ and let $I_y = [x, y]$. Show that

$$\lim_{y\to x} \frac{\nu(I_y)}{\lambda(I_y)} = f(x).$$

This suggests the notation $\frac{d\nu}{d\lambda} = f$ and also suggests that

$$\nu(E) = \int_E \frac{d\nu}{d\lambda}\,d\lambda.$$

Does this remark indicate a way of developing a theory that will allow the Radon–Nikodym derivative to "look" like a derivative when dealing with more general measure spaces?

5:12.10 (In this exercise and the next six we develop material on the Banach indicatrix.) Let f be a continuous function on an interval $[a, b]$, and let $\omega(f, I)$ denote the oscillation of f on a subinterval I of $[a, b]$; that is,

$$\omega(f, I) = \max_{x\in I} f(x) - \min_{x\in I} f(x).$$

Show that the variation of f on $[a, b]$ is given as

$$V(f; [a, b]) = \lim_{\|\mathcal{P}\|\to 0} \sum_{i=1}^{n} \omega(f, I_i)$$

where $\mathcal{P} = \{I_i\}$ is a partition of $[a, b]$ and $\|\mathcal{P}\| \to 0$ indicates that the length of the largest interval in the partition shrinks to zero.

5:12.11 Let f be a continuous function on an interval $[a, b]$ and define the function

$$N_f(y) = \operatorname{card}\{x \in [a, b] : f(x) = y\}$$

(called the *Banach indicatrix* for the function f). If f has only a finite number of maxima and minima, prove that

$$V(f; [a, b]) = \int_{-\infty}^{\infty} N_f(y)\,dy.$$

[Hint: Consider intervals on which f is monotone and use Exercise 1:14.9.]

5:12.12 Let f be an arbitrary continuous function on an interval $[a,b]$. Prove that

$$V(f;[a,b]) = \int_{-\infty}^{\infty} N_f(y)\,dy.$$

[Hint: Here is a sketch, whose details should be expanded upon. Partition $[a,b]$ into 2^n subintervals $\{I_k\}$ of equal length. For $1 \le k \le 2^n$, write $L_k(y) = 1$ if f assumes the value y in the interval I_k and $L_k(y) = 0$ if not. Compute $\int L_k(y)\,dy$. Define

$$N_n = \sum_{1 \le k \le 2^n} L_k.$$

Show that N_n is measurable and $N_n \to N_f$ everywhere. Complete the proof by obtaining

$$\lim_{n\to\infty} \int N_n(y)\,dy = V(f;[a,b])$$

using Exercise 5:12.10.]

5:12.13 Show that the result in Exercise 5:12.12 can fail if f is not continuous. Modify the definition of N_f to an appropriate function N_f^* so that the identity

$$V(f;[a,b]) = \int_{-\infty}^{\infty} N_f^*(y)\,dy$$

holds for all functions f of bounded variation, continuous or not. [Hint: Remember that such functions have one-sided limits at all points.]

5:12.14 Let f be a continuous function on an interval $[a,b]$. Show that f has bounded variation if and only if the Banach indicatrix for f is integrable.

5:12.15 Let f be a continuous function of bounded variation on an interval $[a,b]$. Show that the set of values that f takes on infinitely many times has Lebesgue measure zero.

5:12.16 Is it true or false that a continuous function f on an interval $[a,b]$ for which $N_f(y) < \infty$ for all $y \in \mathbb{R}$ must have bounded variation there?

5:12.17$\diamondsuit$ Develop (in this and the next five exercises) a theory for complex-valued measures defined on a σ-algebra of sets of some set X.

What should be meant by a complex measure ν (i.e., the complex-valued analog of a signed measure) and by the total variation measure $|\nu|$?

5:12.18 Show that, for a complex measure ν on a measure space, $|\nu|$ must be finite.

5:12.19 Let ν be a complex measure and μ a positive, real measure. Give a definition for absolute continuity of ν with respect to μ.

5:12.20 Formulate and prove a complex version of the Radon–Nikodym theorem (Theorem 5.29).

5:12.21 Let μ be a complex measure with total variation $|\mu|$. Show that there is a complex-valued measurable function h such that $|h(x)| = 1$ everywhere and

$$\int_E f\,d\mu = \int_E f h d|\mu|.$$

5:12.22 Let μ be a positive real measure and g a complex-valued integrable function. Define the complex measure $\nu(E) = \int_E g\,d\mu$. Show that $|\nu|(E) = \int_E |g|\,d\mu$.

Chapter 6

FUBINI'S THEOREM

If f is a continuous function in a rectangle $[a,b]\times[c,d]$, then the integral can be computed by two one-dimensional integrals:

$$\int\int_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_a^b \left\{\int_c^d f(x,y)\,dy\right\} dx.$$

This has been known since the time of Cauchy. Here one says that the integral in the plane has been computed by "iterated integrals" or "repeated integration," meaning merely a succession of the two ordinary integrals. Certainly, this is the most familiar method of computing integrals in higher dimensions. Indeed, apart from numerical methods, we have almost no other way to obtain the value of such integrals.

To go beyond continuous functions requires some caution, even using nineteenth-century methods. For example, the function

$$f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$$

in the unit square $[0,1]\times[0,1]$ has just the one discontinuity at the origin. We find that

$$\int_0^1 \left\{\int_0^1 f(x,y)\,dx\right\} dy = -\frac{\pi}{4}$$

but that

$$\int_0^1 \left\{\int_0^1 f(x,y)\,dy\right\} dx = \frac{\pi}{4}.$$

(The geometry of such examples can be seen more clearly in Exercise 6:1.2.) Clearly, there are technicalities that need to be addressed in any such study.

Naturally, this technique from the integral calculus has been extended to modern theories of integration. The development of these ideas was carried out first by Lebesgue and then by two Italian mathematicians,

G. C. Fubini (1879–1943) and L. Tonelli. Most of the work was done by Lebesgue. Fubini's proof was incomplete, and Tonelli's work just extended Lebesgue's ideas for bounded functions to the general case. Even so, the name of Fubini is now most firmly attached to the theory in most accounts.

The measure-theoretic ideas underlying the formula above for the iterated integral are simple. The measure (area) in the plane of the rectangle $[a,b] \times [c,d]$ is $(b-a)(d-c)$. Indeed, it is this trivial fact that allows the two-dimensional Riemann integral for continuous functions to be reduced to one-dimensional Riemann integrals. This fact extends to general rectangles by the formula

$$\lambda_2(A \times B) = \lambda(A)\lambda(B),$$

where λ_2 is two-dimensional Lebesgue measure and where λ is one-dimensional Lebesgue measure. Once the measure-theoretic apparatus is established, this product formula for the measures allows one to prove the formula for the integrals in an entirely expected way.

We can place these ideas in their most general measure setting. Many accounts start with a pair of measure spaces and construct a product measure; normally, the product measure is not complete, and so further construction is needed. Here, rather than using a measure space, we have chosen to use outer measures. In Chapter 2 we saw that any complete measure space can be studied by means of outer measures. Let μ^* be an outer measure on a set X and ν^* an outer measure on a set Y. We expect that there should be an outer measure π^* that is somehow the product of these two outer measures in the same way that two-dimensional Lebesgue measure can be considered the product of two copies of one-dimensional Lebesgue measure. Thus we try for an outer measure π^* on $X \times Y$ with the property, if possible, that

$$\pi^*(A \times B) = \mu^*(A)\nu^*(B) \qquad (A \subset X,\, B \subset Y)\ .$$

This leads us to the study of product measures and then finally to the integration formulas of Fubini and Tonelli.

The material of this chapter offers one of the most important and useful tools in measure theory. We have included no specific applications here but the reader will find the Fubini and Tonelli theorems of use in a wide variety of situations in which integration is used. Later in the text we will call on these theorems to prove the Lebesgue density theorem (Section 8.4) and to establish facts about the convolution of a pair of integrable functions (Section 13.9).

6.1 Product Measures

Let μ^* be an outer measure on a set X and ν^* an outer measure on a set Y. As usual, we denote by μ and ν the associated measures. There is an entirely natural way of defining a product measure on the product set $X \times Y$ in such a way that "rectangles" inherit the correct measure:

$$(\mu^* \times \nu^*)(A \times B) = \mu^*(A)\nu^*(B).$$

One must not expect this identity to hold for all rectangles, but we can obtain this for rectangles of the form $A \times B$, where A is μ^*–measurable and B is ν^*–measurable. The method is the very familiar Method I that has served us so well for many constructions of measures. We apply the method to the covering class consisting of all these rectangles $A \times B$ and with the premeasure

$$\tau(A \times B) = \mu^*(A)\nu^*(B).$$

Definition 6.1 Let μ^* be an outer measure on a set X and ν^* an outer measure on a set Y. We define the *product outer measure* $\mu^* \times \nu^*$ on each subset $S \subset X \times Y$ as

$$(\mu^* \times \nu^*)(S) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) \right\}.$$

The infimum is taken over all coverings of S by sequences $\{A_i \times B_i\}$ of rectangles, where A_i is μ^*–measurable and B_i is ν^*–measurable. The outer measure $\mu^* \times \nu^*$ is said to be the product of the outer measures μ^* and ν^*.

We see immediately that $\mu^* \times \nu^*$ is indeed an outer measure; this measure is regular even if the two original outer measures are not. We shall write $\mu \times \nu$ for the associated measure, without meaning to imply that this is a product of the two measures. In fact, one can develop this subject directly from measure spaces without appealing to outer measures; many texts carry out such a development and the interested reader can consult them.

One should not jump to conclusions when interpreting product measures. It is true that for Lebesgue measure $\lambda_m \times \lambda_n = \lambda_{m+n}$, as expected (see Exercise 6:4.1 for $n = m = 1$), but for Hausdorff measures, the formula $\mu^{(s)} \times \mu^{(t)} = \mu^{(s+t)}$ is *not* valid[1] in general.

Our product measure seems designed to handle rectangles in the required manner, but it is by no means immediate that it does so. This is our first theorem.

Theorem 6.2 *Let μ^* be an outer measure on a set X and ν^* an outer measure on a set Y. Then the product outer measure $\mu^* \times \nu^*$ is a regular outer measure on $X \times Y$, and for any μ^*–measurable set $A \subset X$ and any ν^*–measurable set $B \subset Y$, the rectangle $A \times B$ is $\mu^* \times \nu^*$–measurable and*

$$(\mu^* \times \nu^*)(A \times B) = \mu^*(A)\nu^*(B).$$

[1] See, for example, Falconer, *Fractal Geometry*, Wiley (1990), p. 97.

Proof. Let $\mathcal{F}$ denote the collection of all subsets $S \subset X \times Y$ for which the integral

$$\int_X \chi_S(x,y)\,d\mu(x)$$

exists for all $y \in Y$ and also such that the expression

$$\rho(S) = \int_Y \left\{ \int_X \chi_S(x,y)\,d\mu(x) \right\} d\nu(y)$$

exists. (Note that $+\infty$ is allowed here.)

If $S_1, S_2, S_3, \ldots$ are disjoint members of the family $\mathcal{F}$, and $S = \bigcup_{i=1}^{\infty} S_i$, then $S \in \mathcal{F}$. One sees this by noting that

$$\chi_S = \sum_{i=1}^{\infty} \chi_{S_i}.$$

By the usual convergence theorems for integrals we obtain then that

$$\rho(S) = \sum_{i=1}^{\infty} \rho(S_i). \tag{1}$$

Thus we see that $\mathcal{F}$ is closed under disjoint unions.

There is a similar assertion for intersections with some conditions. Let $S_1 \supset S_2 \supset S_3 \supset \cdots$ be members of the family $\mathcal{F}$, and let $S = \bigcap_{i=1}^{\infty} S_i$; then, provided that $\rho(S_1) < +\infty$, $S \in \mathcal{F}$. This follows because

$$\chi_S = \lim_{n\to\infty} \chi_{S_n}.$$

By the dominated convergence theorem for integrals, then

$$\rho(S) = \lim_{n\to\infty} \rho(S_n). \tag{2}$$

It follows that $\mathcal{F}$ is closed under decreasing intersections with some finiteness condition imposed.

Define

$$\mathcal{P}_0 = \{A \times B : \ A \text{ is } \mu^*\text{–measurable, } B \text{ is } \nu^*\text{–measurable}\},$$

$$\mathcal{P}_1 = \left\{ \bigcup_{i=1}^{\infty} S_i : S_i \in \mathcal{P}_0 \right\},$$

and

$$\mathcal{P}_2 = \left\{ \bigcap_{i=1}^{\infty} E_i : E_i \in \mathcal{P}_1 \right\}.$$

The members of $\mathcal{P}_0$ are often called *measurable rectangles* and play a key role in the theory. The class $\mathcal{P}_1$ consists of countable unions of measurable rectangles and $\mathcal{P}_2$ of countable intersections of these, in turn. The latter sets constitute a class relative to which our product measure will be seen to be regular.

Here are some elementary observations on these classes that are needed in the proof. First, $\mathcal{P}_0$ is evidently a subset of $\mathcal{F}$, and whenever $A \times B$ is a member of $\mathcal{P}_0$, we have the identity

$$\rho(A \times B) = \mu^*(A)\nu^*(B). \tag{3}$$

Note, too, that if $A_i \times B_i \in \mathcal{P}_0$ for $i = 1, 2$ then

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$$

is also a member of $\mathcal{P}_0$. Thus $\mathcal{P}_0$ is closed under finite intersections.

Second, any member of $\mathcal{P}_1$ can be expressed as union of a countable *disjointed* family from $\mathcal{P}_0$. But we know that $\mathcal{F}$ contains $\mathcal{P}_0$ and is closed under such unions (see above). Consequently, $\mathcal{F}$ contains $\mathcal{P}_1$ too. Here is how to check that each member of $\mathcal{P}_1$ can be expressed as union of a countable disjointed family from $\mathcal{P}_0$. Write

$$(A_1 \times B_1) \setminus (A_2 \times B_2) = ((A_1 \setminus A_2) \times B_1) \cup ((A_1 \cap A_2) \times (B_1 \setminus B_2))$$

and we have expressed this difference as a disjoint union of elements of $\mathcal{P}_0$. From this it follows that a countable union of members of $\mathcal{P}_0$ can be rewritten as a countable disjoint union of members of $\mathcal{P}_0$ and hence that $\mathcal{P}_1$ is contained in $\mathcal{F}$, as claimed.

Finally, every member of $\mathcal{P}_2$ is, by definition, the intersection of a sequence of members of $\mathcal{P}_1$. But this can be made decreasing since, as we have just seen, the intersection of any two members of $\mathcal{P}_1$ also belongs to $\mathcal{P}_1$.

We are now ready to obtain our main estimate on the value $(\mu^* \times \nu^*)(S)$ for any set $S \subset X \times Y$ using our set function ρ.

6.3 *For any $S \subset X \times Y$,*

$$(\mu^* \times \nu^*)(S) = \inf\{\rho(R) : S \subset R \in \mathcal{P}_1\}.$$

Let us prove assertion 6.3 now. Suppose that $A_i \times B_i \in \mathcal{P}_0$ for $i = 1, 2, \dots$ and that

$$S \subset V = \bigcup_{i=1}^{\infty} A_i \times B_i.$$

Then

$$\chi_V \leq \sum_{i=1}^{\infty} \chi_{A_i \times B_i}.$$

It follows, with the help of (1) and (3), that

$$\rho(V) \leq \sum_{i=1}^{\infty} \rho(A_i \times B_i) = \sum_{i=1}^{\infty} \mu^*(A_i)\nu^*(B_i).$$

Consequently,

$$(\mu^* \times \nu^*)(S) \geq \inf\{\rho(R) : S \subset R \in \mathcal{P}_1\}.$$

Moreover, if $V = \bigcup_{i=1}^{\infty} A_i \times B_i$ is any such set then, as noted above, there is a disjoint sequence $\{A_i' \times B_i'\}$ from $\mathcal{P}_0$ so that

$$V = \bigcup_{i=1}^{\infty} A_i \times B_i = \bigcup_{i=1}^{\infty} A_i' \times B_i',$$

and for this set V we have

$$\rho(V) = \sum_{i=1}^{\infty} \mu^*(A_i')\nu^*(B_i') \geq (\mu^* \times \nu^*)(S).$$

This proves 6.3.

We can now use this fact to obtain the identity that we must prove. If $A \subset X$ is μ^*–measurable and $B \subset Y$ is ν^*–measurable, then, for any $V \in \mathcal{P}_1$ with $A \times B \subset V$, we know that

$$(\mu^* \times \nu^*)(A \times B) \leq \mu^*(A)\nu^*(B) = \rho(A \times B) \leq \rho(V).$$

From this and 6.3, we obtain

$$(\mu^* \times \nu^*)(A \times B) = \mu^*(A)\nu^*(B),$$

as required. We show that $A \times B$ is $(\mu^* \times \nu^*)$–measurable. Let T be any subset of $X \times Y$. Let $R \in \mathcal{P}_1$ with $T \subset R$. Then $R \setminus (A \times B)$ and $R \cap (A \times B)$ are disjoint members of $\mathcal{P}_1$. Thus

$$(\mu^* \times \nu^*)\left(T \setminus (A \times B)\right) + (\mu^* \times \nu^*)\left(T \cap (A \times B)\right)$$
$$\leq \rho\left(R \setminus (A \times B)\right) + \rho\left(R \cap (A \times B)\right) = \rho(R).$$

From this and 6.3, again we must have

$$(\mu^* \times \nu^*)\left(T \setminus (A \times B)\right) + (\mu^* \times \nu^*)\left(T \cap (A \times B)\right) \leq (\mu^* \times \nu^*)(T).$$

Since this holds for all $T \subset X \times Y$, we have proved that $A \times B$ is $(\mu^* \times \nu^*)$–measurable. It follows, too, that each of the classes $\mathcal{P}_0$, $\mathcal{P}_1$, and $\mathcal{P}_2$ consists of $(\mu^* \times \nu^*)$–measurable sets.

The proof is now complete except for checking the regularity of the measure $(\mu^* \times \nu^*)$. We obtain this from the following:

6.4 *For any $S \subset X \times Y$ there is a set $W \in \mathcal{P}_2$ such that $S \subset W$ and*

$$(\mu^* \times \nu^*)(S) = (\mu^* \times \nu^*)(W) = \rho(W).$$

Let us prove assertion 6.4 now. If $(\mu^* \times \nu^*)(S) = +\infty$, there is nothing to prove, since we may take $W = X \times Y$ and everything holds trivially. Suppose that $(\mu^* \times \nu^*)(S) < +\infty$. Then, using 6.3, we can find for each natural number j a set $V_j \in \mathcal{P}_1$ such that $S \subset V_j$ and

$$\rho(V_j) < (\mu^* \times \nu^*)(S) + 1/j.$$

Let

$$W = \bigcap_{j=1}^{\infty} V_j.$$

Since each $V_j \in \mathcal{F}$, we know that $W \in \mathcal{F}$, and by the limit properties of ρ over decreasing sequences of sets from $\mathcal{F}$, we see that

$$(\mu^* \times \nu^*)(S) \le \rho(W) = \lim_{n\to\infty} \rho\left(\bigcap_{j=1}^{n} V_j\right) \le (\mu^* \times \nu^*)(S).$$

This proves 6.4.

We may now complete the proof of Theorem 6.2. The assertion 6.4 shows that $(\mu^* \times \nu^*)$ is $\mathcal{P}_2$–regular. Since every member of $\mathcal{P}_2$ is $(\mu^* \times \nu^*)$–measurable, we have obtained the required regularity of the product measure. ■

Theorem 6.2 shows how to compute the measure $(\mu^* \times \nu^*)(S)$ for rectangles by using the two measures μ^* and ν^*. For a general set S, is it still possible to estimate the measure using the separate lower-dimensional measures? In a natural intuitive way, we can do this by integrating along slices of the set S parallel to the coordinate axes. This is a simple version of the Fubini theorem, given in its full generality in the next section.

Theorem 6.5 *Let μ^* be an outer measure on a set X and ν^* an outer measure on a set Y, and suppose that the set $S \subset X \times Y$ is $\mu^* \times \nu^*$–measurable and σ-finite with respect to this measure. Then the set*

$$S_y = \{x : (x,y) \in S\}$$

is μ^*–measurable for ν^*–almost every $y \in Y$, the set

$$S^x = \{y : (x,y) \in S\}$$

is ν^*–measurable for μ^*–almost every $x \in X$, and

$$(\mu \times \nu)(S) = \int_Y \mu(S_y)\, d\nu(y) = \int_X \nu(S^x)\, d\mu(x).$$

Proof. The proof continues the notation of the preceding proof. If $S \subset X \times Y$ and $(\mu^* \times \nu^*)(S) = 0$, then, by 6.4, there must be a set $R \in \mathcal{P}_2$ so that $S \subset R$ and $\rho(R) = 0$. It follows that $S \in \mathcal{F}$ and $\rho(S) = 0$.

If $S \subset X \times Y$, if S is $\mu^* \times \nu^*$–measurable, and if $(\mu^* \times \nu^*)(S)$ is finite, then, again by 6.4, there must be a set $R \in \mathcal{P}_2$ so that $S \subset R$ and

$$(\mu^* \times \nu^*)(R \setminus S) = 0$$

and, consequently, $\rho(R \setminus S) = 0$. It follows that

$$\mu^* (\{x : (x, y) \in S\}) = \mu^* (\{x : (x, y) \in R\})$$

for ν^*–almost every $y \in Y$ and that

$$(\mu^* \times \nu^*)(S) = \rho(R) = \int_Y \mu (\{x : (x, y) \in S\})\, d\nu(y).$$

This proves the theorem for sets of finite $(\mu^* \times \nu^*)$–measure, since the other formula is symmetric with X replaced by Y and μ^* by ν^*. The extension to σ-finite $(\mu^* \times \nu^*)$–measure is then obvious. ■

Exercises

6:1.1 Check the details for the example given in the introduction. Let $f(x, y) = (x^2 - y^2)(x^2 + y^2)^{-2}$ in the unit square $[0, 1] \times [0, 1]$. Show that

$$\int_0^1 \left\{ \int_0^1 f(x, y)\, dx \right\} dy = -\frac{\pi}{4},$$

but

$$\int_0^1 \left\{ \int_0^1 f(x, y)\, dy \right\} dx = \frac{\pi}{4}.$$

Show that

$$\int_0^1 \left\{ \int_0^1 |f(x, y)|\, dy \right\} dx = +\infty,$$

a fact that will help later to explain the difference between the two iterated integrals.

6:1.2 Consider the function f defined on $[0, 1] \times [0, 1]$ (as in Figure 6.1) by

$$f(x, y) = \begin{cases} 2^{2n} & \text{if } 2^{-n} \le x < 2^{-n+1},\ 2^{-n} \le y < 2^{-n+1}; \\ -2^{2n} & \text{if } 2^{-n-1} \le x < 2^{-n},\ 2^{-n} \le y < 2^{-n+1}; \\ 0 & \text{otherwise.} \end{cases}$$

Show that the integral of f along any horizontal line is zero, that the integral along a vertical line in the left half of the square is also zero, but that the integral along a vertical line in the right half of the square is 2. Verify that

$$\int_0^1 \left(\int_0^1 f(x, y)\, dx \right) dy = 0 \quad \text{but} \quad \int_0^1 \left(\int_0^1 f(x, y)\, dy \right) dx = 1$$

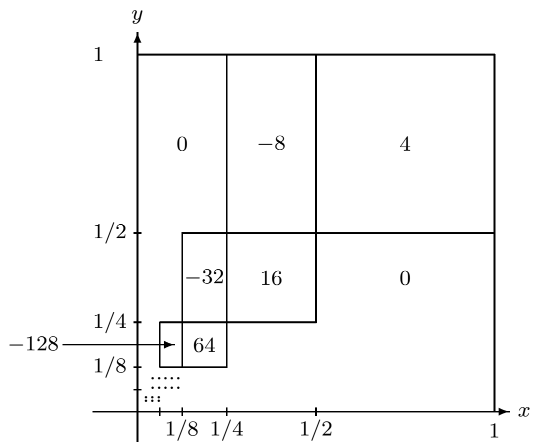

Figure 6.1: Construction of the function f in Exercise 6:1.2.

and that

$$\int_0^1 \left(\int_0^1 |f(x,y)|\,dy \right) dx = +\infty.$$

6:1.3 Is it true that a rectangle $A \times B$ can be $(\mu^* \times \nu^*)$–measurable if and only if A is μ^*–measurable and B is ν^*–measurable?

6:1.4 Show that the class of measurable rectangles is a *semi-algebra* (i.e., it is closed under finite intersections and the complement of any member is expressible as a finite, disjoint union of such sets).

6:1.5 Let X and Y be metric spaces, and let μ^*, ν^* be metric outer measures on X and Y. Show that $(\mu^* \times \nu^*)$ is a metric outer measure on the product space $X \times Y$ (given its usual metric).

6:1.6 Let E be a $(\mu^* \times \nu^*)$–measurable subset of $X \times Y$, σ-finite with respect to this measure. A necessary and sufficient condition that $(\mu^* \times \nu^*)(E) = 0$ is that the "section"

$$E_y = \{x : (x,y) \in E\}$$

have μ^*–measure zero for ν^*–almost every $y \in Y$.

6:1.7 Let A, B be $(\mu^* \times \nu^*)$–measurable subsets of $X \times Y$, both σ-finite with respect to this measure. If the "sections" A_y and B_y have the same μ^*–measure for ν^*–almost every $y \in Y$, then

$$(\mu^* \times \nu^*)(A) = (\mu^* \times \nu^*)(B).$$

6:1.8◇ Let $X = Y = \mathbb{R}$, let μ^* be Lebesgue outer measure on X, and let ν^* be the counting outer measure on Y. Show that $(\mu^* \times \nu^*)$ is Borel regular and that the diagonal of the unit square $S = \{(x,x) : 0 \le x \le 1\}$ is $(\mu^* \times \nu^*)$–measurable, but is not σ-finite with respect to this measure. Show that the measure $(\mu^* \times \nu^*)(S)$ cannot be computed in both of the ways of Theorem 6.5.

6:1.9$\diamondsuit$ In Theorem 6.5 it is essential to assume that the set $S \subset X \times Y$ is $\mu^* \times \nu^*$–measurable. It is not enough merely to have the sections

$$S_y = \{x : (x, y) \in S\}$$

μ^*–measurable for every $y \in Y$, and

$$S^x = \{y : (x, y) \in S\}$$

ν^*–measurable for every $x \in X$.

[Here is a construction, using CH: well-order the interval $[0, 1]$ in such a way that every element has only countably many predecessors. Let S be the set of all pairs (x, y) with x,$y \in [0, 1]$ such that x precedes y in the order. Then each section S_y, S^x is either countable or the complement of a countable subset of $[0, 1]$ and so, in particular, measurable. Check that

$$\int_0^1 \lambda(S_y)\, dy \neq \int_0^1 \lambda(S^x)\, dx,$$

where, as usual, λ is Lebesgue measure.]

6.2 Fubini's Theorem

We can now give the full general version of Fubini's theorem. The simplest setting is that for an integrable function with respect to the measure $\mu^* \times \nu^*$, so that the finite value $\int_{X\times Y} f(x, y)\, d(\mu \times \nu)$ is obtained from two iterated integrations. We can allow infinite values in this statement with some care and attention to the details. Here the phrase "f is a countably $(\mu^* \times \nu^*)$–measurable function on $X \times Y$" signifies that f is measurable with respect to this measure and, moreover, that the set $\{(x, y) \in X \times Y : f(x, y) \neq 0\}$ of points where f does not vanish is σ-finite with respect to the measure. For an integrable function, this is necessarily the case.

Theorem 6.6 (Fubini) *Let μ^* be an outer measure on a set X and ν^* an outer measure on a set Y, and suppose that f is a countably $(\mu^* \times \nu^*)$–measurable function on $X \times Y$ for which the integral*

$$\int_{X\times Y} f(x, y)\, d(\mu^* \times \nu^*)$$

exists (finite or infinite). Then the mapping

$$x \to \int_Y f(x, y)\, d\nu(y)$$

is a μ^*–measurable function on X, the mapping

$$y \to \int_X f(x, y)\, d\mu(x)$$

is a ν^*–measurable function on Y, and

$$\int_{X\times Y} f(x,y)\,d(\mu\times\nu)$$
$$= \int_Y \left\{\int_X f(x,y)\,d\mu(x)\right\} d\nu(y) = \int_X \left\{\int_Y f(x,y)\,d\nu(y)\right\} d\mu(x).$$

Proof. This follows almost immediately from Theorem 6.5 by using the standard tools of integration theory. Certainly, for $f=\chi_S$ the present theorem reduces to Theorem 6.5. This then extends to simple functions by additivity of the integral. For f nonnegative, use approximation by simple functions and an appropriate convergence theorem (Theorem 5.8). Finally, for general functions f, write, as usual, as $f=f^+-f^-$. ■

Exercises

6:2.1 Suppose that f is a $(\mu^*\times\nu^*)$–integrable function on $X\times Y$. (Recall that this means that the integral exists and has a finite value.) Show that f is a countably $(\mu^*\times\nu^*)$–measurable function.

6:2.2◇ In Theorem 6.6 it is essential to assume that the function f is $\mu^*\times\nu^*$–measurable even if the spaces have finite measure. It is not enough merely that each section

$$f_y : x \to f(x,y) \text{ and } f^x : y \to f(x,y)$$

be measurable in the separate spaces. (See Exercise 6:1.9.)

6:2.3 (Cf. Exercise 6:1.8.) Let $X=Y=\mathbb{R}$, let μ^* be Lebesgue outer measure on X, and let ν^* be the counting outer measure on Y. Let f be the characteristic function of S, the diagonal of the unit square ($S=\{(x,x) : 0\le x\le 1\}$). Show that f is $(\mu^*\times\nu^*)$–measurable, but is not countably $(\mu^*\times\nu^*)$–measurable, and that the Fubini theorem fails to compute

$$\int_{X\times Y} f(x,y)\,d(\mu^*\times\nu^*)$$

in this case.

6:2.4 Let $X=Y=\mathbb{R}$ and let μ^* and ν^* be Lebesgue outer measure on X and Y. Let $f(x,y)$ be $\left(4xy-x^2-y^2\right)(x+y)^{-4}$ for x, y both positive, and let $f(x,y)=0$ elsewhere. Show that

$$\int_Y \left\{\int_X f(x,y)d\mu^*(x)\right\} d\nu^*(y) = \int_X \left\{\int_Y f(x,y)d\nu^*(y)\right\} d\mu^*(x) = 0,$$

but that

$$\int_{X\times Y} f(x,y)\,d(\mu^*\times\nu^*)$$

does not exist. Which hypotheses of the Fubini theorem have been violated? [Hint: Check that

$$\int_0^a f(x,y)\,dx = (a^2 - ay)(a+y)^{-3}$$

for each $a > 0$. Use the estimates

$$|f(x,y)| \le (4y + 1 + y^2)y^{-4} \quad \text{for } 0 \le x \le 1,$$
$$|f(x,y)| \le (4y + 1 + y^2)x^{-2} \quad \text{for } x > 1,$$

and

$$\int_1^\infty x^{-2}\,dx = 1$$

to obtain

$$\int_0^\infty f(x,y)\,dx = \lim_{a\to\infty} \int_0^a f(x,y)\,dx = 0.$$

Finally, consider the attempted computation

$$\int_{X\times Y} f(x,y)\,d(\mu^* \times \nu^*) = \lim_{a\to\infty} \int_0^{ma} \int_0^a f(x,y)\,dx\,dy$$

$$= \lim_{a\to\infty} \int_0^{ma} (a^2 - ay)(a+y)^{-3}\,dy = \lim_{a\to\infty} a^2 m(a+ma)^{-2} = m(1+m)^{-2}$$

for positive numbers m.]

6:2.5 Each of the integrals

$$\int_0^1 \left\{ \int_1^\infty \left(e^{-xy} - 2e^{-2xy}\right)\,dx \right\} dy$$

and

$$\int_1^\infty \left\{ \int_0^1 \left(e^{-xy} - 2e^{-2xy}\right)\,dy \right\} dx$$

exists (as absolutely convergent Cauchy integrals and as Lebesgue integrals), but they are unequal. What can you conclude? Compare this with Exercise 6:3.5 and explain the (rather subtle) difference.

6.3 Tonelli's Theorem

Tonelli's theorem is merely a corollary of the Fubini theorem (Theorem 6.6), but it is useful to restate it in this form. Here information about the finiteness of the iterated integral implies integrability of the integral in the product space. Note that the hypothesis that f is nonnegative has been added to the statement of the theorem in this case. Exercise 6:3.3 is a frequently helpful version of this theorem.

Theorem 6.7 (Tonelli) *Let μ^* be an outer measure on a set X and ν^* an outer measure on a set Y, and suppose that both spaces are σ-finite.*

Let f be a nonnegative $(\mu^* \times \nu^*)$–measurable function on $X \times Y$. Then the mapping

$$x \to \int_Y f(x,y)\, d\nu(y)$$

is a μ^*–measurable function on X, the mapping

$$y \to \int_X f(x,y)\, d\mu(x)$$

is a ν^*–measurable function on Y, and

$$\int_{X\times Y} f(x,y)\, d(\mu\times\nu)$$
$$= \int_Y \left\{ \int_X f(x,y)\, d\mu(x) \right\} d\nu(y) = \int_X \left\{ \int_Y f(x,y)\, d\nu(y) \right\} d\mu(x).$$

Exercises

6:3.1 Check all the necessary details to be sure that Theorem 6.7 follows from Theorem 6.6.

6:3.2 In Theorem 6.7 it is essential to assume that the function f is $\mu^*\times\nu^*$–measurable even if the spaces have finite measure. It is not enough merely that each section

$$f_y : x \to f(x,y) \text{ and } f^x : y \to f(x,y)$$

be measurable in the separate spaces. (See Exercises 6:1.9 and 6:2.2.)

6:3.3 Let μ^* be an outer measure on a set X and ν^* an outer measure on a set Y, and suppose that both spaces are σ-finite. Let f be a $(\mu^* \times \nu^*)$–measurable function on $X \times Y$. If any one of the three integrals

$$\int_{X\times Y} |f(x,y)|\, d(\mu\times\nu),$$
$$\int_Y \left\{ \int_X |f(x,y)|\, d\mu(x) \right\} d\nu(y),$$
$$\int_X \left\{ \int_Y |f(x,y)|\, d\nu(y) \right\} d\mu(x)$$

is finite, then so are all three, and the usual conclusion of the Fubini theorem holds.

6:3.4 Use Exercise 6:1.8 to show that the σ-finiteness of the measure spaces (or some such assumption) would be needed for the Tonelli theorem and for Exercise 6:3.3.

6:3.5 Let f be a real function defined on $\mathbb{R}^2$. If the integrals

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy \quad \text{and} \quad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dy\,dx$$

exist as two-dimensional Cauchy integrals and if one of them is absolutely convergent, then the two integrals are equal. Can equality occur in a situation where both integrals are nonabsolutely convergent?

6.4 Additional Problems for Chapter 6

6:4.1◇ We now have two ways of obtaining Lebesgue measure in $\mathbb{R}^2$: as a Lebesgue–Stieltjes measure and as a product measure. Show that the two procedures give the same result. [Hint: Use Exercise 2:13.15.]

6:4.2 The main work of this chapter involves the proof of Theorem 6.2. A development similar to that suggested in Exercise 2:12.4 (see also Section 3.7) is possible, though lengthy. Carry out such a development. That is, take $\mathcal{T}$ to be the class of measurable rectangles, and define τ by $\tau(A \times B) = \mu(A)\nu(B)$. Extend $\mathcal{T}$ and τ appropriately so that Theorems 2.40 and 2.42 apply, obtaining Theorem 6.2. (Observe that the proof in the text actually does much of this in hidden form.)

6:4.3 Let f be a nonnegative function defined on a measurable subset E of $\mathbb{R}^n$. Then f is measurable if the region $\{(x,y) : x \in E,\ f(x) \geq y\}$ is a measurable subset of $\mathbb{R}^{n+1}$.

6:4.4 Let E be a $\mu^* \times \nu^*$–measurable subset of $X \times Y$ such that for μ^*–almost every $x \in X$ the set $\{y : (x,y) \in E\}$ has ν^*–measure zero. Show that $(\mu \times \nu)(E) = 0$ and that for ν^*–almost every $y \in Y$ the set $\{x : (x,y) \in E\}$ has μ^*–measure zero.

6:4.5 Let f be a nonnegative $\mu^* \times \nu^*$–measurable function on $X \times Y$ such that for μ^*–almost every $x \in X$ the value $f(x,y)$ is finite for ν^*–almost every $y \in Y$. Show that for ν^*–almost every $y \in Y$ the value $f(x,y)$ is finite for μ^*–almost every $x \in X$.

6:4.6 What form does the Fubini–Tonelli theorem take if

$$f(x,y) = h(x)g(y)?$$

6:4.7 If g is a measurable real function on the interval $[0,1]$ such that the function $f(x,y) = g(x) - g(y)$ is Lebesgue integrable over the square $[0,1] \times [0,1]$, show that g is integrable over $[0,1]$.

6:4.8 Let f be a measurable function with period 1 on the real line such that

$$\int_0^1 |f(a+t) - f(b+t)|\,dt$$

is bounded uniformly for all a, $b \in \mathbb{R}$. Show that f is integrable on $[0,1]$. [Hint: Use $a = x$, $b = -x$, integrate with respect to x, and change variables to $\xi = x + t$, $\eta = -x + t$.]

6:4.9 Two integrable functions x and y on a measure space $(T, \mathcal{T}, \mu)$ are *comonotone* if

$$(x(t) - x(s))(y(t) - y(s)) \geq 0$$

for all s, t in T. Similarly, x and y are *contramonotone* if

$$(x(t) - x(s))(y(t) - y(s)) \leq 0$$

for all s, t in T. Suppose that μ is a probability measure. Show that

$$\int_T x(t)\, d\mu(t) \int_T y(t)\, d\mu(t) \leq \int_T x(t)y(t)\, d\mu(t)$$

or

$$\int_T x(t)\, d\mu(t) \int_T y(t)\, d\mu(t) \geq \int_T x(t)y(t)\, d\mu(t),$$

depending on whether the functions are co- or contramonotone.

6:4.10 We have seen that the equality of the two iterated integrals is not enough for Fubini's theorem to hold. In fact[2], there exists a function

$$f : [a,b] \times [c,d] \to \mathbb{R}$$

such that

$$\int_A \left\{ \int_B f(x,y)\, dx \right\} dy = \int_B \left\{ \int_A f(x,y)\, dy \right\} dx$$

holds for all measurable sets $A \subset [a,b]$ and $B \subset [c,d]$, and still Fubini's theorem fails.

6:4.11 There is a set $E \subset \mathbb{R}^2$ such that E meets every closed subset of $\mathbb{R}^2$ having positive Lebesgue measure, and no three points of E are collinear. (The construction is sketched in Exercise 6:4.12.) Show that such a set cannot be Lebesgue measurable.

6:4.12 (cf. Exercise 6:4.11.) There is a set $E \subset \mathbb{R}^2$ such that E meets every closed subset of $\mathbb{R}^2$ having positive Lebesgue measure and no three points of E are collinear.

[This is due to Sierpiński. Here is a sketch that uses CH: well-order the class of closed subset of $\mathbb{R}^2$ having positive Lebesgue measure in such a way that each member has only countably many predecessors. Choose points from each member in the sequence in turn in such a way to obtain E. At any stage, remember that there will be only countably many lines to "avoid" and that constitutes only a set of measure zero to stay away from.]

[2]See G. M. Fichtenholz, *Fund. Math.* 6 (1924), 30–36.

6:4.13 Here is a category analog of the Fubini theorem. Let A be a subset of $\mathbb{R}^2$ of the first Baire category. Then the "section"

$$A_y = \{x : (x, y) \in A\}$$

is a first-category set in $\mathbb{R}$ for all y, except possibly in a first-category set.

6:4.14 Show that the graph of a continuous function $f : [0, 1] \to \mathbb{R}$ has measure zero with respect to two-dimensional Lebesgue measure. If f is not continuous, this is not necessarily the case. [Hint: Use CH to construct a function with nonmeasurable graph.]

Chapter 7

DIFFERENTIATION

The great contribution that Lebesgue made was not merely in defining an integration process that would open up new methods for analysts. Indeed, W. H. Young only a few years later defined an integral equivalent to that of Lebesgue; thus a new definition of an integral was inevitable. The greatest contribution of Lebesgue rests in the many studies that he made using this tool. Certainly, his development of differentiation theory using the methods of measure and integration is among his most impressive achievements.

In this chapter we study the differentiation theory of real functions at a depth that would not have been available at an advanced calculus level. The most successful tools in general differentiation theory are supplied by covering arguments. In Section 7.1 we prove the Vitali covering theorem. This will allow us to obtain, in Section 7.2, the differentiation properties of functions of bounded variation that Lebesgue found by different methods. The Banach–Zarecki theorem of Section 7.3 reveals the exact structure of absolutely continuous functions; In Sections 7.4 to 7.7 we study the intimate connections among differentiation, variation, measure, and integration. Finally, the fundamental concepts of approximate continuity, density points, and Lebesgue points, also closely related to differentiation theory, are discussed in Section 7.8.

7.1 The Vitali Covering Theorem

One of the most important theorems related to the "growth" of real functions is the Vitali covering theorem. Before stating and proving Vitali's theorem, let us generalize an elementary growth theorem.

Suppose that f is strictly increasing and differentiable on an interval $I = [a, b]$. Then $f' \geq 0$ on I. If, also, $f' < p$ on I, then

$$f(b) - f(a) \leq p(b - a)$$

by the mean-value theorem. In other notation, $\lambda(f(I)) \leq p\lambda(I)$.

The hypothesis "$0 \le f' < p$" on I can be interpreted as a local growth condition: all sufficiently small intervals containing a point $x_0 \in I$ are magnified by a factor less than p. The conclusion can be interpreted as a global growth condition: the entire interval I maps onto an interval whose length is no more than p times the length of I.

We would like to generalize our elementary growth theorem. Suppose that f is any strictly increasing function on I and $E \subset I$. We do not assume f differentiable, and we do not assume E measurable. We shall replace the local growth condition $f' < p$ by a much weaker one involving derived numbers. Recall that an extended real number α is said to be a *derived number* for a function f at x_0 if there exists a sequence $\{h_k\} \to 0$ $(h_k \neq 0)$ such that

$$\lim_{k\to\infty} \frac{f(x_0 + h_k) - f(x_0)}{h_k} = \alpha.$$

We shall often write $Df(x)$ to indicate a derived number of f at x.

A function must have at least one derived number, finite or infinite, at each point. It might have many derived numbers at a point. For example, the function $f(x) = \sqrt{|x|}\sin x^{-1}$ $(f(0) = 0)$ has every extended real number as a derived number at $x = 0$. It is clear that a function f has a derivative at x_0 if and only if all derived numbers at x_0 agree and are finite. It is also clear that, if f is nondecreasing on an interval I, then all derived numbers are nonnegative at each point $x \in I$. We leave verification of these remarks as Exercises 7:1.3, 7:1.4, and 7:1.5.

Lemma 7.1 *Let f be strictly increasing on an interval $[a, b]$, and let $E \subset [a, b]$. If at each point $x \in E$ there exists a derived number $Df(x) < p$, then $\lambda^*(f(E)) \le p\lambda^*(E)$.*

Thus a very weak local growth condition leads to a strong global growth conclusion. In trying to prove Lemma 7.1, one might begin reasoning roughly as follows:

> Our hypothesis about derived numbers guarantees that each $x \in E$ has an interval $I(x)$ such that $x \in I(x)$ and such that the length of $f(I(x))$ is less than $p\lambda(I(x))$. The intervals $\{f(I(x))\}$, for $x \in E$, cover $f(E)$. Thus the sum of the lengths of these intervals, $\sum_{x\in E} \lambda(f(I(x)))$, is less than p times the sum $\sum_{x\in E} \lambda(I(x))$. There are some problems. The set E may be uncountable, but we can probably reduce our sums to countable ones. Can we also arrange for those sums to approximate $\lambda^*(E)$ and $\lambda^*(f(E))$?

The Vitali covering theorem allows us to select disjoint families of intervals with exactly the approximation properties that we require.

Definition 7.2 Let $\mathcal{I}$ be the family of nondegenerate closed intervals in $\mathbb{R}$. Let $E \subset \mathbb{R}$ and let $\mathcal{V} \subset \mathcal{I}$. If for each $x \in E$ and $\varepsilon > 0$ there exists $V \in \mathcal{V}$ such that $x \in V$ and $\lambda(V) < \varepsilon$, then $\mathcal{V}$ is called a *Vitali cover* for E (or a *Vitali covering* of E).

For example, if f is strictly increasing and

$$E = \{x : \text{ there is a derived number } Df(x) < p \text{ of } f \text{ at } x\}$$

then $\mathcal{V} = \{V \in \mathcal{I} : \lambda(f(V)) < p\lambda(V)\}$ forms a Vitali cover for E. To verify this, simply observe that for $x \in E$ there exists a sequence $\{h_k\} \to 0$ ($h_k \neq 0$) such that, for every $n \in \mathbb{N}$,

$$\frac{f(x+h_n) - f(x)}{h_n} < p.$$

Thus, for $V = [x, x+h_n]$ (or $[x+h_n, x]$ if $h_n < 0$), we have $\lambda(V) = |h_n|$ and

$$\lambda(f(V)) = |f(x+h_n) - f(x)| < p|h_n| = p\lambda(V).$$

Theorem 7.3 (Vitali covering theorem) *Let $\mathcal{V}$ be a Vitali covering of a set $E \subset \mathbb{R}$. Then there exists a countable family $\{V_k\}$ of sets chosen from $\mathcal{V}$ such that*

$$V_i \cap V_j = \emptyset \quad (i \neq j)$$

and

$$\lambda(E \setminus \bigcup_{k=1}^{\infty} V_k) = 0.$$

Theorem 7.3 was first obtained by Vitali in 1907. The standard proof nowadays is due to S. Banach. Banach's proof has the virtue of extending naturally to more general settings. We shall discuss this point in Chapter 8. (See also Exercise 7:1.8.) Before proving Theorem 7.3, let us see how it enables us to provide a proof of Lemma 7.1 along the lines we indicated.

Proof. (Proof of Lemma 7.1) Let $\varepsilon > 0$, and let G be a bounded open set containing E such that

$$\lambda(G) < \lambda^*(E) + \varepsilon. \tag{1}$$

For $x_0 \in E$ there exists a sequence $\{h_k\} \to 0$ ($h_k \neq 0$) such that, for each $n \in \mathbb{N}$, $[x_0, x_0 + h_n] \subset G$ and

$$\frac{f(x_0+h_n) - f(x_0)}{h_n} < p. \tag{2}$$

(For simplicity of notation, we are writing $[x_0, x_0+h_n]$ in place of $[x_0 + h_n, x_0]$ in the event that $h_n < 0$.) For each $n \in \mathbb{N}$, let

$$I_n(x_0) = [x_0, x_0+h_n] \text{ and } J_n(x_0) = [f(x_0), f(x_0+h_n)].$$

Since f is strictly increasing, $f(I_n(x_0)) \subset J_n(x_0)$, and $J_n(x_0)$ is a nondegenerate closed interval. It follows from (2) and the equalities $\lambda(I_n(x_0)) = |h_n|$ and $\lambda(J_n(x_0)) = |f(x_0+h_n) - f(x_0)|$ that

$$\lambda(J_n(x_0)) < p\lambda(I_n(x_0)). \tag{3}$$

Now $\lim_{n\to\infty} h_n = 0$, so $\lim_{n\to\infty} \lambda(I_n(x_0)) = 0$. From (3) we infer that

$$\lim_{n\to\infty} \lambda(J_n(x_0)) = 0.$$

Thus the family of intervals

$$\mathcal{V} = \{J_n(x_0) : x_0 \in E, n \in \mathbb{N}\}$$

forms a Vitali cover of the set $f(E)$. By Theorem 7.3, there exists a countable disjoint family $\{J_{n_i}(x_i)\}$, $i \in \mathbb{N}$, such that

$$\lambda\left(f(E) \setminus \bigcup_{i=1}^{\infty} J_{n_i}(x_i)\right) = 0. \tag{4}$$

Using (4), we find that

$$\lambda^*(f(E)) \le \sum_{i=1}^{\infty} \lambda(J_{n_i}(x_i)) < p\sum_{i=1}^{\infty} \lambda(I_{n_i}(x_i)). \tag{5}$$

Since f is *strictly* increasing, the intervals $I_{n_i}(x_i)$ form a pairwise disjoint family. From (1) we infer that

$$\sum_{i=1}^{\infty} \lambda(I_{n_i}(x_i)) = \lambda\left(\bigcup_{i=1}^{\infty} I_{n_i}(x_i)\right) \le \lambda(G) < \lambda^*(E) + \varepsilon. \tag{6}$$

Combining (5) and (6), we obtain

$$\lambda^*(f(E)) < p(\lambda^*(E) + \varepsilon)$$

for every $\varepsilon > 0$. Thus $\lambda^*(f(E)) \le p\lambda^*(E)$, as was to be shown. ■

Observe the role of Theorem 7.3. First, it allowed us to obtain the family $\{J_{n_i}(x_i)\}$ that almost covers the set $f(E)$ in the equation (4). The fact that this family is a disjoint family allowed us to conclude the same for the family $\{I_{n_i}(x_i)\}$, which we needed for the inequality (6). Observe also the role of the set G. It guarantees that the family $\{I_{n_i}(x_i)\}$ does not cover much more than the set E.

We shall use Lemma 7.1 in Section 7.2. We shall also need a companion lemma with a similar proof (left as Exercise 7:1.6).

Lemma 7.4 *Let f be strictly increasing on $[a, b]$, and let $E \subset [a, b]$. If at each $x \in E$ there exists a derived number $Df(x) > q \ge 0$, then*

$$\lambda^*(f(E)) \ge q\lambda^*(E).$$

We now prove Theorem 7.3. The idea of the proof is very simple: choose intervals from $\mathcal{V}$ one by one. Make sure that, at each stage, we choose a "relatively large interval" from those that are disjoint from the ones already chosen.

Proof. (Proof of Theorem 7.3) We assume E bounded. The extension to unbounded sets is left as Exercise 7:1.7.

Let J be any open interval containing E, and let $\mathcal{V}_0$ consist of those intervals in $\mathcal{V}$ that are contained in J. It is clear that $\mathcal{V}_0$ is also a Vitali cover for E. Let $V_1 \in \mathcal{V}_0$. If $\lambda(E \setminus V_1) = 0$, there is nothing further to prove. If not, we proceed inductively.

Suppose that we have chosen pairwise disjoint intervals

$$V_1, V_2, \dots, V_n$$

from $\mathcal{V}_0$. If

$$\lambda(E \setminus \bigcup_{k=1}^{n} V_k) = 0,$$

we are done. If not, we choose V_{n+1} according to the following procedure. Let

$$F_n = V_1 \cup V_2 \cup \cdots \cup V_n\,, \quad G_n = J \setminus F_n.$$

Note that G_n is open. Let

$$\mathcal{V}_n = \{V \in \mathcal{V}_0 : V \subset G_n\}\,.$$

Since $E \setminus F_n \neq \emptyset$ and $\mathcal{V}_0$ is a Vitali cover for E, the family $\mathcal{V}_n$ is not empty. Let

$$S_n = \sup\{\lambda(V) : V \in \mathcal{V}_n\}\,.$$

Then $0 < S_n$, since members of a Vitali cover are nondegenerate, and $S_n < \infty$, since each $V \in \mathcal{V}_0$ is contained in J. Choose $V_{n+1} \in \mathcal{V}_n$ such that

$$\lambda(V_{n+1}) > \tfrac{1}{2} S_n. \tag{7}$$

Since $V_{n+1} \subset G_n$, we see that $\{V_1, \dots, V_{n+1}\}$ forms a pairwise disjoint system of intervals from $\mathcal{V}_0$.

If this process does not stop after a finite number of steps, we obtain a pairwise disjoint sequence $\{V_k\}$ of intervals from $\mathcal{V}$. We show that

$$\lambda(E \setminus \bigcup_{k=1}^{\infty} V_k) = 0. \tag{8}$$

Let $S = \bigcup_{k=1}^{\infty} V_k$. For every $k \in \mathbb{N}$, let W_k be a closed interval with the same midpoint as V_k and such that $\lambda(W_k) = 5\lambda(V_k)$. Now

$$\sum_{k=1}^{\infty} \lambda(W_k) = 5 \sum_{k=1}^{\infty} \lambda(V_k) \le 5\lambda(J) < \infty. \tag{9}$$

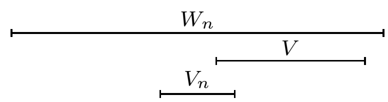

Figure 7.1: An illustration of the fact that $V \subset W_n$.

It therefore suffices to show that

$$E \setminus S \subset \bigcup_{k=i}^{\infty} W_k \tag{10}$$

for every $i \in \mathbb{N}$. This, together with (9), implies (8).

To verify (10), let $x \in E \setminus S$. Then $x \in \bigcap_{i=1}^{\infty} G_i$. Fix $i \in \mathbb{N}$. Since G_i is open, there exists $V \in \mathcal{V}_0$ such that $x \in V \subset G_i$. Consider now this interval V. Since $x \in V$, V is not one of the intervals of our chosen sequence $\{V_k\}$. The intervals V_k are pairwise disjoint and are contained in J, so $\lim_{k\to\infty} \lambda(V_k) = 0$. Thus, by (7), $\lim_{k\to\infty} S_k = 0$. Choose $N \in \mathbb{N}$ such that $S_N < \lambda(V)$. Then $V \notin \mathcal{V}_N$, so V is not contained in G_N, and $V \cap F_N \neq \emptyset$.

Let $n = \min\{j : V \cap F_j \neq \emptyset\}$. Since $V \cap F_i = \emptyset$ and the sequence $\{F_k\}$ is expanding, we infer that $n > i$. Thus $V \cap F_n \neq \emptyset$, but $V \cap F_{n-1} = \emptyset$. This implies that $V \cap V_n \neq \emptyset$, and $V \subset G_{n-1}$. From the latter inclusion, we infer that $\lambda(V) \le S_{n-1} < 2\lambda(V_n)$. Recalling the definition of W_n, we conclude that $V \subset W_n$. See Figure 7.1. Since $n > i$, $V \subset \bigcup_{k=i}^{\infty} W_k$, so $x \in \bigcup_{k=i}^{\infty} W_k$. This inclusion establishes (10), completing the proof of the theorem. ■

Exercises

7:1.1 Show that if $0 \le f' < p$ on $I = [a,b]$ then $\lambda(f(I)) < p\lambda(I)$.

[Hint: You may use the fact (Theorem 1.18) that f' has a point of continuity in I.]

7:1.2 Show that a function must have at least one derived number, finite or infinite, at each point.

7:1.3 Show that the function $f(x) = \sqrt{|x|}\sin x^{-1}$, $f(0) = 0$, has every extended real number as a derived number at $x = 0$.

7:1.4 Show that $f'(x_0)$ exists if and only if all derived numbers are finite and agree at x_0.

7:1.5 Show that all derived numbers are nonnegative at every point if and only if f is nondecreasing.

7:1.6 Prove Lemma 7.4. [Hint: Begin with an appropriate open set G containing $f(E)$. Note that the set of discontinuities of f is countable.]

7:1.7 Prove Vitali's theorem for unbounded sets.

7:1.8$\diamondsuit$ Replace the family of intervals $\mathcal{I}$ with the family $\mathcal{S}$ of closed squares with sides parallel to the coordinate axes in $\mathbb{R}^2$. State and prove the analog to Vitali's theorem in this setting.

7:1.9 Use the Vitali covering theorem to prove that an arbitrary union of nondegenerate closed intervals in $\mathbb{R}$ is measurable. (Note that this also follows from Exercise 1:3.18.)

7:1.10 Use Exercise 7:1.8 to prove that an arbitrary union of nondegenerate closed squares with sides parallel to the coordinate axes in $\mathbb{R}^2$ is Lebesgue measurable, but not necessarily Borel measurable.

7.2 Functions of Bounded Variation

The two growth lemmas, Lemmas 7.1 and 7.4, allow a quick proof that a function of bounded variation has a finite derivative almost everywhere. This result was proved by Lebesgue, but by an entirely different method.

Theorem 7.5 *Let f be of bounded variation on $[a, b]$. Then f has a finite derivative almost everywhere.*

Proof. Since each function of bounded variation is a difference of two nondecreasing functions, it suffices to prove the theorem for f nondecreasing. Assume then that f is nondecreasing on $[a, b]$. By considering $f(x)+x$, if necessary, we may assume that f is strictly increasing.

Let E_∞ consist of those points in $[a, b]$ at which f has an infinite derived number. Using Lemma 7.4 and the fact that

$$f(E_\infty) \subset [f(a), f(b)],$$

we have

$$q\lambda^*(E_\infty) \leq \lambda^*(f(E_\infty)) \leq f(b) - f(a) < \infty$$

for all $q \in \mathbb{N}$. It follows that

$$\lambda^*(E_\infty) = 0. \tag{11}$$

Now let $0 \leq p < q < \infty$, and let

$$E_{pq} = \{x \;:\; \text{there exist derived numbers } D_1 f(x) \text{ and } D_2 f(x) \text{ such that } D_1 f(x) < p < q < D_2 f(x)\}.$$

From Lemmas 7.1 and 7.4, we infer that

$$q\lambda^*(E_{pq}) \leq \lambda^*(f(E_{pq})) \leq p\lambda^*(E_{pq}). \tag{12}$$

Since $p < q$, the inequalities in (12) imply that

$$\lambda^*(E_{pq}) = 0. \tag{13}$$

If f is *not* differentiable at a point x, then either f has ∞ as a derived number at x or f has derived numbers $D_1 f(x) < D_2 f(x)$. In the latter case, there exist rational numbers p and q such that

$$D_1 f(x) < p < q < D_2 f(x),$$

so $x \in E_{pq}$. Thus

$$N = \{x : f \text{ is not differentiable at } x\} \subset E_\infty \cup \bigcup \{E_{pq} : p, q \in \mathbb{Q}\}.$$

Because of (11) and (13), $\lambda(N) = 0$. ■

Theorem 7.5 cannot be improved: given any set E of measure zero, there exists a strictly increasing function f such that f is not differentiable at any point of E, indeed such that $f'(x) = \infty$ at every $x \in E$. [It is also possible to choose an f so that, at each $x \in E$, f has distinct derived numbers $D_1 f(x) \neq D_2 f(x)$: see Exercise 7:9.4.]

Theorem 7.6 *Let $E \subset [a, b]$ with $\lambda(E) = 0$. There exists a continuous, strictly increasing function f such that $f'(x) = \infty$ for all $x \in E$.*

Proof. For each $n \in \mathbb{N}$, let G_n be an open set containing E such that $\lambda(G_n) < 2^{-n}$. Let $f_n(x) = \lambda(G_n \cap [a, x])$. Then f_n is nondecreasing and continuous, and $0 \leq f_n(x) \leq 2^{-n}$ for every $x \in [a, b]$. Let $f = \sum_{n=1}^{\infty} f_n$. The function f is nondecreasing and continuous, and $0 \leq f(x) \leq 1$ for all $x \in [a, b]$. Let $x \in E$. Fix $n \in \mathbb{N}$. If $h > 0$ is sufficiently small, $[x, x + h] \subset G_n$, so

$$\begin{aligned} f_n(x + h) &= \lambda(G_n \cap [a, x + h]) \\ &= \lambda((G_n \cap [a, x]) \cup (G_n \cap [x, x + h])) \\ &= \lambda(G_n \cap [a, x]) + \lambda(G_n \cap [x, x + h]) = f_n(x) + h. \end{aligned}$$

A similar argument shows that $f_n(x + h) = f_n(x) + h$ when $h < 0$ is sufficiently small. Thus, for $|h|$ sufficiently small,

$$\frac{f_n(x + h) - f_n(x)}{h} = 1.$$

It follows that if $N \in \mathbb{N}$ then, for $|h|$ sufficiently small,

$$\frac{f(x + h) - f(x)}{h} \geq \sum_{n=1}^{N} \frac{f_n(x + h) - f_n(x)}{h} = N.$$

Since N is arbitrary,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \infty.$$

This function is as required, but may not be strictly increasing. Take $f(x) + x$ for an example of a continuous, strictly increasing function with an infinite derivative at every point of E. ■

We have already observed that the integral is invariant to changes in the values of a function if these changes occur on a set of measure zero: if $f = g$ a.e. and $g \in L_1$, then $f \in L_1$ and

$$\int_E f\, d\mu = \int_E g\, d\mu$$

for every $E \in \mathcal{M}$. For convenience of notation, we shall write $\int_X f\, d\mu$ even if f is defined only a.e. on X. Thus the expression $\int_a^b f'\, d\lambda$ in Theorem 7.7, which follows, should be taken in the sense that we are integrating the function f', which we know might exist only almost everywhere.

Theorem 7.7 *Let f be nondecreasing on $[a, b]$. Then its derivative f' is measurable and*

$$\int_a^b f'\, d\lambda \leq f(b) - f(a). \tag{14}$$

Proof. Extend f to $[a, b+1]$ by setting $f(x) = f(b)$ if $b < x \leq b+1$. Let

$$f_n(x) = \frac{f(x + 1/n) - f(x)}{1/n}.$$

Then $f_n(x)$ converges to $f'(x)$ at each point of differentiability. It follows that f' is measurable and $f_n \to f'$ [a.e.] on $[a, b]$. By Fatou's lemma (Lemma 5.7)

$$\begin{aligned}
\int_a^b f'\, d\lambda &\leq \liminf_{n\to\infty} \int_a^b f_n\, d\lambda \leq \sup \int_a^b f_n\, d\lambda \\
&= \sup \left\{ n \int_a^b \left[f\left(x + \frac{1}{n}\right) - f(x) \right] dx \right\}.
\end{aligned}$$

The last integrals can be taken in the Riemann sense, since their integrands have only countably many discontinuities and are obviously bounded. Since

$$\int_a^b f\left(x + \frac{1}{n}\right) dx = \int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)\, dx \quad \text{for all} \quad n \in \mathbb{N},$$

we can calculate

$$\begin{aligned}
\int_a^b \left[f\left(x + \frac{1}{n}\right) - f(x) \right] dx &= \int_b^{b+\frac{1}{n}} f(x)\, dx - \int_a^{a+\frac{1}{n}} f(x)\, dx \\
&= \frac{1}{n} f(b) - \int_a^{a+\frac{1}{n}} f(x)\, dx \\
&\leq \frac{1}{n} [f(b) - f(a)].
\end{aligned}$$

Thus

$$\int_a^b f'\,d\lambda \le \sup_n \left\{ n \int_a^b \left[f\left(x+\frac{1}{n}\right) - f(x)\right] dx \right\} \le f(b)-f(a),$$

as required. ■

The inequality in (14) cannot, in general, be replaced by an equality. The Cantor function F illustrates: here $F'=0$ a.e., so

$$\int_0^1 F'\,d\lambda = 0 < 1 = F(1)-F(0).$$

We shall see in Section 7.5 that, when f is absolutely continuous, inequality (14) does become an equality.

Theorem 7.7 gives an upper bound on $\int_a^b f'\,d\lambda$. We can also give an upper bound on $\int_E f'\,d\lambda$ by using Lebesgue–Stieltjes measures.

Theorem 7.8 *Let f be increasing on $[a,b]$, let μ_f be the associated Lebesgue–Stieltjes measure and let $\nu = \int f'\,d\lambda$. Then $\nu(E) \le \mu_f(E)$ for every Borel set $E \subset [a,b]$.*

Proof. Let $[c,d] \subset [a,b]$. By Theorem 7.7,

$$\nu((c,d]) = \int_{(c,d]} f'\,d\lambda = \int_{[c,d]} f'\,d\lambda \le f(d)-f(c) = \mu_f((c,d]).$$

Let $\mathcal{T}$ consist of $\emptyset$ and the half-open intervals contained in $(a,b]$, and use the premeasures $\tau_1 = \nu$ and $\tau_2 = \mu_f$ on $\mathcal{T}$. Applying Method I, we see that $\nu(E) \le \mu_f(E)$ for every Borel set in $(a,b]$. Since $\nu(\{a\}) = 0$, the theorem follows. ■

We shall sharpen Theorem 7.8 in Section 7.5.

Exercises

7:2.1 Show that the function f in Theorem 7.6 is absolutely continuous.

7:2.2 Let F be the Cantor function. Show that, for every Borel set E,

$$\mu_F(E) = \int_E F'\,d\lambda + \mu_F(E \cap K),$$

where K is the Cantor ternary set. This is a special case of the form of the Lebesgue decomposition theorem that we shall consider in Section 7.5.

7:2.3◇ Let f be defined in a neighborhood of x_0. Among the derived numbers of f at x_0, there are four extreme ones, called the *Dini derivates*of f at x_0, denoted by $D^+f(x_0)$, $D_+f(x_0)$, $D^-f(x_0)$, and $D_-f(x_0)$. For example,

$$D^+f(x_0) = \limsup_{h\to 0+} \frac{f(x+h)-f(x)}{h}.$$

(a) Provide definitions of $D_+f(x_0)$, $D^-f(x_0)$, and $D_-f(x_0)$.

(b) Let $f = \chi_{\mathbb{Q}}$, the characteristic function of the rationals. Calculate the four Dini derivates for a point $x_0 \in \mathbb{Q}$.

(c) Must the Dini derivates of a function of bounded variation be finite a.e.?

(d) Prove that for a continuous function f on (a,b) that the four Dini derivates are measurable.

7.3 The Banach–Zarecki Theorem

We now prove the converse of Theorem 5.27, using two growth lemmas that are themselves of interest. Note that the first of these, Lemma 7.9, is similar to but more elementary than the growth lemmas of Section 7.1, since we need not use the Vitali Covering Theorem.

Lemma 7.9 *Let f be a finite function on an interval I, and let $E \subset I$. If there exists $p > 0$ such that, for every $x \in E$, all derived numbers $Df(x)$ satisfy $|Df(x)| < p$, then*

$$\lambda^*(f(E)) \le p\lambda^*(E).$$

Proof. Let $\varepsilon > 0$. For each $n \in \mathbb{N}$ let

$$E_n = \{x \in E : |f(t) - f(x)| < p|t - x| \text{ whenever } |t - x| < 1/n\}.$$

The sequence $\{E_n\}$ is expanding and, by our hypothesis,

$$E = \lim_{n\to\infty} E_n.$$

Since λ^* is regular, we see (from Exercise 2:9.2) that

$$\lambda^*(E) = \lim_{n\to\infty} \lambda^*(E_n) \text{ and } \lambda^*(f(E)) = \lim_{n\to\infty} \lambda^*(f(E_n)). \tag{15}$$

For each $n \in \mathbb{N}$, let $\{I_k^n\}$ be a sequence of intervals each of length less than $\frac{1}{n}$ such that $E_n \subset \bigcup_{k=1}^{\infty} I_k^n$ and so that

$$\sum_{k=1}^{\infty} \lambda(I_k^n) \le \lambda^*(E_n) + \varepsilon. \tag{16}$$

Suppose now that x_1 and x_2 are points in $E_n \cap I_k^n$. Then

$$|f(x_2) - f(x_1)| < p|x_2 - x_1| \le p\lambda(I_k^n).$$

It follows that $\lambda^*(f(E_n \cap I_k^n)) \le p\lambda(I_k^n)$. From (16) we infer for each n that

$$\begin{aligned}\lambda^*(f(E_n)) &\le \sum_{k=1}^{\infty} \lambda^*(f(E_n \cap I_k^n)) \le p\sum_{k=1}^{\infty} \lambda(I_k^n)\\ &\le p(\lambda^*(E_n) + \varepsilon).\end{aligned}$$

Using (15), we see that

$$\lambda^*(f(E)) = \lim_{n\to\infty} \lambda^*(f(E_n)) \le p(\lambda^*(E) + \varepsilon).$$

Since ε is arbitrary, $\lambda^*(f(E)) \le p\lambda^*(E)$. ∎

Lemma 7.10 *Let f be measurable on an interval I, and let E be a measurable subset of I. If f is differentiable at each point of E, then*

$$\lambda^*(f(E)) \le \int_E |f'|\,d\lambda. \tag{17}$$

Proof. We may assume that E is bounded. Let $\varepsilon > 0$, and for each $n \in \mathbb{N}$, let

$$E_n = \{x \in E : (n-1)\varepsilon \le |f'(x)| < n\varepsilon\}.$$

Then $E_n \in \mathcal{L}$ (Exercise 7:3.1). By Lemma 7.9,

$$\begin{aligned}\lambda^*(f(E)) &\le \sum_{n=1}^{\infty} \lambda^*(f(E_n)) \le \sum_{n=1}^{\infty} n\varepsilon\lambda(E_n)\\ &= \sum_{n=1}^{\infty}(n-1)\varepsilon\lambda(E_n) + \sum_{n=1}^{\infty}\varepsilon\lambda(E_n) \le \int_E |f'|\,d\lambda + \varepsilon\lambda(E).\end{aligned}$$

Since ε is arbitrary, $\lambda^*(f(E)) \le \int_E |f'|\,d\lambda$. ∎

We can now prove the main result of this section. Theorem 7.11 was proved independently by S. Banach and M. A. Zarecki.

Theorem 7.11 (Banach–Zarecki) *Let f be defined on $[a,b]$. A necessary and sufficient condition that f be absolutely continuous is that f satisfy the following three conditions:*

1. *f is continuous on $[a,b]$.*
2. *f is of bounded variation on $[a,b]$.*
3. *f satisfies Lusin's condition (N); that is, f maps zero measure sets onto zero measure sets.*

Proof. The necessity of the conditions was established in Theorem 5.27. To prove sufficiency, suppose that f satisfies conditions (1), (2), and (3). We first show that

$$|f(d) - f(c)| \le \int_c^d |f'|\,d\lambda \tag{18}$$

for every subinterval $[c,d]$ of $[a,b]$. Let E denote the set of points of differentiability of f in $[c,d]$, and let $F = [c,d] \setminus E$. Since f is of bounded variation on $[a,b]$, $\lambda(F) = 0$. By condition (3), it follows that $\lambda(f(F)) = 0$.

Since f is continuous, $[f(c), f(d)] \subset f([c,d])$, so by applying Lemma 7.10 we obtain

$$\begin{aligned} |f(d) - f(c)| &\leq \lambda(f([c,d])) \leq \lambda^*(f(E)) + \lambda^*(f(F)) \\ &= \lambda^*(f(E)) \leq \int_c^d |f'|\, d\lambda. \end{aligned}$$

This establishes (18). It is now easy to complete the proof of the theorem. Since f is of bounded variation, f' is integrable on $[a,b]$. Let $\varepsilon > 0$. From the absolute continuity of the integral and Theorem 5.24 there is a $\delta > 0$ so that $\int_A |f'|\, d\lambda < \varepsilon$ if $\lambda(A) < \delta$. Let $\{[a_k, b_k]\}$ be any sequence of nonoverlapping closed intervals in $[a,b]$, with total length less than δ. Then, by (18), with $A = \bigcup_{k=1}^{\infty}[a_k, b_k]$, we have

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| \leq \int_A |f'|\, d\lambda < \varepsilon$$

since $\lambda(A) < \delta$. This establishes the absolute continuity of f. ■

Observe that the hypothesis that f be of bounded variation on $[a,b]$ was used only to establish that f is differentiable a.e. and that f' is integrable. We therefore can state the following corollary to Theorem 7.11.

Corollary 7.12 *Let f be continuous and satisfy Lusin's condition (N) on $[a,b]$. Then f is absolutely continuous if and only if f is differentiable a.e. and f' is integrable.*

Theorem 7.11 also indicates that a composition of two absolutely continuous functions can fail to be absolutely continuous if and only if it is not of bounded variation. To see this, observe that both continuity and Lusin's condition (N) are preserved under composition.

Exercises

7:3.1 Verify that, if f is measurable on an interval I containing a measurable set E, then for $\alpha < \beta \in \mathbb{R}$, $\{x \in E : \alpha \leq f'(x) < \beta\}$ is measurable. (The measure under consideration is λ.)

7:3.2 Let $E \subset \mathbb{R}$, and let $\mathcal{W}$ be a family of intervals. If each $x \in E$ is in arbitrarily small intervals from $\mathcal{W}$, then $\mathcal{W}$ is a Vitali cover for E. If for every $x \in E$ *all* sufficiently small intervals containing x are in $\mathcal{W}$, we say that $\mathcal{W}$ is a *full cover* of E. Observe that Vitali covers figure in the lemmas of Section 7.2, while full covers apply to Lemma 7.9. Verify the following statements.

(a) A full cover is a Vitali cover.

(b) If $f : \mathbb{R} \to \mathbb{R}$ and for each $x \in E$ there exists a derived number $Df(x) < M$, then

$$\mathcal{W} = \left\{ [a,b] : \frac{f(b) - f(a)}{b - a} < M \right\}$$

is a Vitali cover for E.

(c) If $f:\mathbb{R}\to\mathbb{R}$ and for each $x \in E$ *every* derived number satisfies $Df(x) < M$, then $\mathcal{W}$ is a full cover for E.

(d) If $\mathcal{W}$ is a full cover of an interval $[a,b]$, then there exists a finite collection of intervals $W_1, W_2, \dots, W_n$ from $\mathcal{W}$ that forms a *partition* of $[a,b]$: that is, the intervals W_i are pairwise nonoverlapping and $[a,b] = W_1 \cup W_2 \cup \dots \cup W_n$. [Hint: Consider

$$\left\{ x \in [a,b] \ : \ \begin{array}{l} \text{For every } d \in (a,x], \text{ there} \\ \text{exists a partition of } [a,d] \\ \text{using members of } \mathcal{W} \end{array} \right\}.$$

7:3.3 Prove that, if all derived numbers of a nondecreasing function f satisfy the inequality $Df(x) < p$ for every $x \in E$ then the family of intervals V such that $\lambda(f(V)) < p\lambda(V)$ forms a full cover of E.

7:3.4 The result in part (d) of Exercise 7:3.2 can be used to provide simple proofs of a number of theorems. Use it to prove the following.

(a) The Heine–Borel theorem: If $\mathcal{U}$ is a covering of a closed and bounded set in $\mathbb{R}$ by open sets, then $\mathcal{U}$ can be reduced to a finite subcover.

(b) Every infinite bounded set E in $\mathbb{R}$ has a limit point in $\mathbb{R}$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ have the property that to every $x_0 \in \mathbb{R}$ corresponds a $\delta > 0$ such that $f(x) \geq f(x_0)$ on $(x, x_0 + \delta)$ and $f(x) \leq f(x_0)$ on $(x_0 - \delta, x_0)$. Then f is nondecreasing on $\mathbb{R}$.

(d) The intermediate-value property for continuous functions.

7:3.5 Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and let $Z = \{x : f'(x) = 0\}$. Prove that $\lambda(f(Z)) = 0$.

7:3.6 Prove that a differentiable function f must satisfy Lusin's condition (N) and deduce that f is absolutely continuous on an interval $[a,b]$ if and only if f is of bounded variation.

7:3.7 Prove that if f is differentiable on $[a,b]$ and $f' = 0$ a.e. then f is constant. [Hint: Use Exercise 7:3.5 and the fact that f satisfies Lusin's condition (N). Compare with the Cantor function.]

7.4 Determining a Function by Its Derivative

It follows from the mean-value theorem that an everywhere differentiable function is determined by its derivative up to a constant. To see this, suppose that f and g are differentiable functions on $[a,b]$ and $f' = g'$. Let $h = f - g$. Then h is differentiable, and $h' = 0$. Thus h is a constant, so f and g differ by a constant.

We would like to extend this result from elementary calculus to functions that are differentiable almost everywhere. The Cantor function F is continuous and nondecreasing and $F' = 0$ a.e., but F is not a constant. Since F does its rising on a set of measure zero, one might expect that ruling out that possibility for a continuous function f would provide the desired result. This is, in fact, the case.

Theorem 7.13 *Let f be continuous and satisfy Lusin's condition (N) on $[a, b]$. If $f' = 0$ a.e. on $[a, b]$, then f is a constant.*

Proof. Let $E = \{x : f'(x) = 0\}$, and let $Z = [a, b] \setminus E$. Then $\lambda(Z) = 0$, so $\lambda(f(Z)) = 0$. It follows directly from Lemma 7.9 that $\lambda(f(E)) = 0$. Thus $\lambda(f([a, b])) \le \lambda(f(Z)) + \lambda(f(E)) = 0$. But f is continuous, so $f([a, b])$ is an interval J with $\lambda(J) = 0$. That is, J is a single point and so f is constant. ■

Corollary 7.14 *An absolutely continuous function whose derivative vanishes a.e. is a constant.*

Corollary 7.15 *If f and g are absolutely continuous on $[a, b]$ and $f' = g'$ a.e., then $f - g$ is a constant.*

Let us return to the theorem from elementary calculus: If f and g are differentiable on $[a, b]$ with $f'(x) = g'(x)$ for all $x \in [a, b]$, then $f - g$ is a constant. The hypothesis that f be differentiable means that f has a *finite* derivative. It is easy to define two functions f and g with $f' = g'$ everywhere, but with $f - g$ not constant if we are allowed infinite values for the derivatives. For example, let $f(x) = g(x) = 0$ for $x < 0$, $f(0) = g(0) = 1$, $f(x) = 2$ for $x > 0$, and $g(x) = 3$ for $x > 0$. Note that $f'(0) = g'(0) = \infty$ and f and g are discontinuous there. It may be of interest that a similar situation can occur for *continuous* functions.

Example 7.16 Let K be the Cantor ternary set, and let F be the Cantor function. We construct a function G that is absolutely continuous and such that G' is infinite on K and finite on $[0, 1] \setminus K$. It is then easy to verify that for $H = G + F$ we have $H' = G'$ on [0,1], but $H - G = F$ is nonconstant (Exercise 7:4.1).

For each $n \in \mathbb{N}$, let A_n be the union of those intervals complementary to K that have length 3^{-n}. Thus A_n is the union of 2^{n-1} pairwise disjoint intervals, and

$$\lambda(A_n) = \tfrac{1}{2}(\tfrac{2}{3})^n.$$

Let g be any function defined on [0,1] which meets the following conditions:

(i) $g(x) = \infty$ if $x \in K$,

(ii) $\lim_{x \to c} g(x) = \infty$ for all $c \in K$,

(iii) g is continuous on every interval complementary to K,

(iv) for every $n \in \mathbb{N}$ and $x \in A_n$, $g(x) \ge n$, and

(v) for every $n \in \mathbb{N}$,

$$\int_{A_n} g \, d\lambda = (\tfrac{2}{3})^n n.$$

Then

$$\int_0^1 g\,d\lambda = \sum_{n=1}^{\infty} \int_{A_n} g\,d\lambda = \sum_{n=1}^{\infty} (\tfrac{2}{3})^n n < \infty,$$

and it follows that $g \in L_1$. Let

$$G(x) = \int_0^x g\,d\lambda, \qquad (0 \le x \le 1).$$

Then G is absolutely continuous. Moreover, $G'(x) = g(x)$ for all $x \in [0,1]$. To verify this, use (i) and (ii) for $x \in K$ and (iii) for $x \notin K$. The function F has a zero derivative off K. On K, all derived numbers are nonnegative, since F is nondecreasing. Thus $H = F + G$ has an infinite derivative at each point of K. It is now clear that $H' = G'$ on $[0,1]$, and $H - G = F$.

Exercises

7:4.1 Show that the functions H and G in Example 7.16 have equal derivatives everywhere on [0,1], but do not differ by a constant.

7:4.2 Corollary 7.14 is often proved by use of the Vitali covering theorem. Provide such a proof.

7:4.3 Construct a function g that satisfies conditions (i) to (v) of Example 7.16.

7.5 Calculating a Function from Its Derivative

In Section 7.4 we saw that, if $F' = G'$ a.e. for two absolutely continuous functions F and G, then F and G differ by a constant. We now show how to calculate F from F'. This form of the fundamental theorem of calculus extends Theorem 5.21. We shall also obtain several more general representation theorems for continuous functions of bounded variation and for Lebesgue–Stieltjes signed measures. We begin with a lemma. The main theorems of this section follow readily from this lemma.

Lemma 7.17 *Let F be continuous on $[a,b]$, and let A be the set of points of differentiability of F. Then*

1. *A is a Borel set.*
2. *If F is strictly increasing, then $F(A)$ is a Borel set and*

$$\lambda(F(A)) = \int_A F'\,d\lambda = \int_a^b F'\,d\lambda. \tag{19}$$

Proof. The set A consists of all points at which all derived numbers are equal and finite. We show first that, for any $p \in \mathbb{R}$, the set

$$E_p = \{x : \text{ there exists a derived number } DF(x) < p\} \tag{20}$$

is a Borel set. For $n \in \mathbb{N}$, let

$$A_n = \left\{ \begin{array}{lcl} x \in [a,b] & : & \exists y \in [a,b] \text{ such that } |x-y| < 1/n \\ & & \text{and } F(y) - F(x) < p(y-x) \end{array} \right\}.$$

Then $E_p = \bigcap_{n=1}^{\infty} A_n$. Since F is continuous, each of the sets A_n is open, so E_p is of type $\mathcal{G}_\delta$ and hence a Borel set.

A similar argument will show that, for any $q \in \mathbb{R}$, the set

$$E^q = \{x : \text{there exists a derived number } DF(x) > q\}$$

is also a Borel set. It follows that if $p < q$ the set $E_p^q = E_p \cap E^q$ is a Borel set. Now the set of points at which F does not have a derivative, finite or infinite, can be represented as $\bigcup E_p^q$, where the union is taken over all pairs of rational numbers p and q.

Similarly,

$$\{x : F \text{ has } \infty \text{ as a derived number at } x\ \} = \bigcap_{q=1}^{\infty} E^q$$

and

$$\{x :\ F \text{ has } -\infty \text{ as a derived number at } x\ \} = \bigcap_{p=1}^{\infty} E_{-p}.$$

Each of these sets is a Borel set, so the same is true of A. The proof of (1) is thus complete.

Let us now prove assertion (2). If F is strictly increasing, then F is a homeomorphism and therefore maps Borel sets onto Borel sets (Exercise 3:10.4). Thus $F(A)$ is a Borel set. To establish (19), let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that $(b-a)/n < \varepsilon$. For $k \in \mathbb{N}$, let

$$A_k = \left\{ x : \frac{k-1}{n} \le F'(x) < \frac{k}{n} \right\}.$$

Since F is strictly increasing, $A = \bigcup_{k=1}^{\infty} A_k$. By Lemma 7.1,

$$\lambda(F(A_k)) \le \frac{k}{n}\lambda(A_k).$$

By Lemma 7.4, $q\lambda(A_k) \le \lambda(F(A_k))$ for any $q < (k-1)/n$. Thus

$$\frac{k-1}{n}\lambda(A_k) \le \lambda(F(A_k)) \le \frac{k}{n}\lambda(A_k). \tag{21}$$

In addition,

$$\frac{k-1}{n}\lambda(A_k) \leq \int_{A_k} F'\,d\lambda \leq \frac{k}{n}\lambda(A_k). \tag{22}$$

Combining (21) and (22), we find that

$$\left|\lambda(F(A_k)) - \int_{A_k} F'\,d\lambda\right| \leq \frac{1}{n}\lambda(A_k). \tag{23}$$

Now

$$\lambda(F(A)) = \sum_{k=1}^{\infty} \lambda(F(A_k)) \quad \text{and} \quad \int_A F'\,d\lambda = \sum_{k=1}^{\infty} \int_{A_k} F'\,d\lambda.$$

From (23) we infer that

$$\begin{aligned} \left|\lambda(F(A)) - \int_A F'\,d\lambda\right| &= \left|\sum_{k=1}^{\infty}\left(\lambda(F(A_k)) - \int_{A_k} F'\,d\lambda\right)\right| \\ &\leq \sum_{k=1}^{\infty}\left|\lambda(F(A_k)) - \int_{A_k} F'\,d\lambda\right| \\ &\leq \frac{1}{n}\sum_{k=1}^{\infty}\lambda(A_k) = \frac{1}{n}\lambda(A) \leq \frac{b-a}{n} < \varepsilon. \end{aligned}$$

Since ε is arbitrary,

$$\lambda(F(A)) = \int_A F'\,d\lambda,$$

and the proof is complete. ■

Theorem 7.18 *Let F be absolutely continuous on $[a,b]$. Then*

$$F(b) - F(a) = \int_a^b F'\,d\lambda.$$

Proof. Assume first that F is strictly increasing. As before, write A for the set of points of differentiability of F, and let $B = [a,b] \setminus A$. Using Lemma 7.17, we have

$$\begin{aligned} F(b) - F(a) &= \lambda(F([a,b])) = \lambda(F(A)) + \lambda(F(B)) \\ &= \int_A F'\,d\lambda + \lambda(F(B)). \end{aligned}$$

Since F is monotonic, $\lambda(A) = b - a$ and $\lambda(B) = 0$, and since F satisfies Lusin's condition (N), $\lambda(F(B)) = 0$. Thus

$$F(b) - F(a) = \int_a^b F'\,d\lambda.$$

In the general case, let $F = G - H$, where G and H are absolutely continuous strictly increasing functions (Exercise 7:5.2). The theorem follows by observing that

$$\begin{aligned} F(b) - F(a) &= (G(b) - G(a)) - (H(b) - H(a)) \\ &= \int_a^b G'\,d\lambda - \int_a^b H'\,d\lambda = \int_a^b F'\,d\lambda, \end{aligned}$$

as required. ■

Applying Theorem 7.18 to Lebesgue–Stieltjes signed measures, we obtain Theorem 7.19. Thus, for the Lebesgue–Stieltjes measure μ_F on the line, the Radon–Nikodym derivative is the actual derivative of the distribution function F almost everywhere. This is the result that we anticipated in our heuristic discussion preceding Theorem 5.29.

Theorem 7.19 *Let μ_F be a Lebesgue–Stieltjes signed measure with $\mu_F \ll \lambda$. Then*

$$\mu_F(E) = \int_E F'\,d\lambda \quad \textit{for every bounded set } E \in \mathcal{L}.$$

We turn now to generalizations of Theorems 7.18 and 7.19. Suppose that F is continuous and strictly increasing on an interval $[a, b]$. Again write A for the set of points of differentiability of F, and let $B = [a, b] \setminus A$. From Lemma 7.17, we have

$$F(b) - F(a) = \lambda(F([a, b])) = \int_A F'\,d\lambda + \lambda(F(B)).$$

Since F is monotonic, $\lambda(A) = b - a$, so $\int_A F'\,d\lambda = \int_a^b F'\,d\lambda$. Thus

$$F(b) - F(a) = \int_a^b F'\,d\lambda + \lambda(F(B)). \tag{24}$$

Equation (24) shows us how Theorem 7.18 can fail if we do not assume that F is absolutely continuous. The growth of F on $[a, b]$ has two components, one of which vanishes when F is absolutely continuous. Let us examine the quantity $\lambda(F(B))$ in more detail. Recall that the set B consists of those points at which F does not have a finite derivative. For every $n \in \mathbb{N}$, let

$$B_n = \{x \in B : \text{there exists a derived number } DF(x) < n\}.$$

Since $\lambda(B) = 0$, $\lambda(B_n) = 0$. It follows from Lemma 7.1 that $\lambda(F(B_n)) = 0$ for every $n \in \mathbb{N}$. Thus

$$\lambda\left(F\left(\bigcup_{n=1}^{\infty} B_n\right)\right) \le \sum_{n=1}^{\infty} \lambda(F(B_n)) = 0.$$

If F is not absolutely continuous, then $\lambda(F(B)) > 0$ and

$$\lambda(F(B - \bigcup_{n=1}^{\infty} B_n)) > 0.$$

The set $B_\infty = B \setminus \bigcup_{n=1}^{\infty} B_n$ is the set where $F' = \infty$. Thus

$$F(b) - F(a) = \int_a^b F'\, d\lambda + \lambda(F(B_\infty)).$$

For a Lebesgue–Stieltjes measure μ_F, we obtain the equality

$$\mu_F(E) = \int_E F'\, d\lambda + \mu_F(E \cap B_\infty).$$

Theorem 7.20 is the analogous version for Lebesgue–Stieltjes signed measures. The proof depends on other growth lemmas. We shall defer a proof to Section 8.5, where we prove the theorem in a more general setting.

Theorem 7.20 (de la Vallée Poussin) *Suppose that F is a continuous function of bounded variation on $[a, b]$, and let μ_F be the associated Lebesgue–Stieltjes signed measure. Then, for every Borel set E,*

$$\mu_F(E) = \int_E F'\, d\lambda + \mu_F(E \cap B_\infty) + \mu_F(E \cap B_{-\infty}), \tag{25}$$

where $B_\infty = \{x : F'(x) = \infty\}$ and $B_{-\infty} = \{x : F'(x) = -\infty\}$.

From (25) we see that, when $F' = 0$ a.e., then the mass of any set is concentrated in the null set $B_\infty \cup B_{-\infty}$. This happens, for example, with the Cantor measure μ_F (F the Cantor function) whose mass is concentrated in the Cantor ternary set K. Expression (25) also shows that the converse is true. If $\mu_F \perp \lambda$, then $F' = 0$ a.e. To see this, suppose that F' were positive on a set P of positive (Lebesgue) measure. Let $Q = P \setminus (B_\infty \cup B_{-\infty})$. Then $\lambda(Q) > 0$ and

$$\mu_F(Q) = \int_Q F'\, d\lambda > 0,$$

so μ_F has mass outside $B_\infty \cup B_{-\infty}$.

A function F of bounded variation is called *singular* if $F' = 0$ a.e. For continuous nonconstant singular functions F, our discussion shows that F must have an infinite derivative on an uncountable set. For example, the Cantor function F has F' infinite on a set that is uncountable in every open interval containing points of the Cantor set K. It is not true, however, that $F' = \infty$ at all two-sided limit points of K. One can show, in fact, that D_+F (as defined in Exercise 7:2.3) takes all values in $[0, \infty]$ in every open interval containing points of K (See Exercise 7:9.15).

Theorem 7.20 is due to Charles de la Vallée Poussin. Observe that this theorem provides a refinement of the Lebesgue decomposition for Lebesgue–Stieltjes measures. We simply let

$$\alpha(E) = \int_E F'\,d\lambda \ \text{ and } \ \beta(E) = \mu_F(E \cap B),$$

where $B = B_\infty \cup B_{-\infty}$. Then $\mu_F = \alpha + \beta$, $\alpha(B) = 0$ and $\beta(A) = 0$.

Let us return to the fundamental theorem of calculus in its various forms. We now know that if F is differentiable a.e. on $[a, b]$, then

$$F(x) - F(a) = \int_a^x F'\,d\lambda \ \textit{ for all } x \in [a, b] \tag{26}$$

if and only if F is absolutely continuous. We also know that if F is continuous and of bounded variation then F' exists a.e. and is integrable, but (26) need not hold. What can fail is Lusin's condition (N). On the other hand, if F is differentiable *everywhere*, then F *does* satisfy condition (N), but need not be of bounded variation (see Exercise 7:5.7). It follows that, for such a function F, (26) fails. The difficulty is that F' is not integrable (see Exercise 7:5.4).

Theorem 7.21 *If F is differentiable on $[a, b]$ and $F' \in L_1$, then*

$$F(x) - F(a) = \int_a^x F'\,d\lambda \ \textit{ for all } \ x \in [a, b].$$

Proof. Since every differentiable function satisfies Lusin's condition (N), the result is an immediate consequence of Corollary 7.12 and Theorem 7.18. ■

Thus the Lebesgue integral is sufficiently powerful to recapture a differentiable function from its derivative, provided that derivative is Lebesgue integrable. But not every derivative is Lebesgue integrable. One can view this as a flaw in Lebesgue integration. The Lebesgue integral does much better in this regard than the Riemann integral does—at least every bounded derivative is Lebesgue integrable. This is not necessarily true for Riemann integrals, as we saw in Section 5.5.

Other more general integrals have been developed for which any differentiable function can be recaptured from its derivative via integration. We have addressed this question in Sections 1.21 and 5.10.

We can view Theorems 7.18 and 7.21 as versions of half of the fundamental theorem of calculus: differentiate a function, then integrate the derivative to get back the function. The other half, in which we integrate first, is the content of Theorem 7.22.

Theorem 7.22 *Let f be Lebesgue integrable on $[a, b]$, and let*

$$F(x) = \int_a^x f\,d\lambda \ \textit{ for } \ x \in [a, b].$$

Then F is differentiable at almost every point, and $F' = f$ almost everywhere.

Proof. The function F is absolutely continuous and $F(a) = 0$, so

$$F(x) = \int_a^x F'\,d\lambda.$$

It follows that $\int_a^x (F' - f)\,d\lambda = 0$ for all $x \in [a,b]$. But this implies readily that $F' = f$ a.e. (see Exercise 7:5.8). ■

Exercises

7:5.1 Show that the set A in Lemma 7.17 is of type $\mathcal{F}_{\sigma\delta}$. (This is actually true without the assumption that F is continuous, although the proof is then more complicated.)

7:5.2 Prove that if a function F is absolutely continuous on an interval then F is a difference of two strictly increasing absolutely continuous functions.

7:5.3◇ Apply Theorem 7.22 to an appropriately chosen function f to prove that there exists an absolutely continuous function F that is nowhere monotonic. That is, for every $c, d \in \mathbb{R}$ such that $a \le c < d \le b$, F is not monotonic on $[c,d]$.

7:5.4 Let F be continuous and of bounded variation on $[a,b]$, let μ_F be the associated Lebesgue–Stieltjes signed measure, and let $|\mu_F|$ be the variation measure,

$$|\mu_F|(E) \equiv V(\mu_F, E).$$

(See Section 2.2.) Prove, for every Borel set E, that

$$|\mu_F|(E) = \int_E |F'|\,d\lambda + \mu_F(E \cap B_\infty) + |\mu_F(E \cap B_{-\infty})|,$$

where $B_\infty = \{x : F'(x) = \infty\}$ and $B_{-\infty} = \{x : F'(x) = -\infty\}$. In particular, if f is absolutely continuous, then

$$V(f;[a,b]) = \int_a^b |f'|\,d\lambda.$$

7:5.5 Theorem 3.34 provides a sense in which an increasing function F needs Cantor sets to support its rising: If $\lambda(F(E)) > 0$, then E contains a Cantor set. Now we can add this insight: If F rises on a set E of measure zero, then all the rising F does on E can be attributed to the set on which F' is infinite. Make this statement precise.

7:5.6 State and prove a version of Theorem 7.20 applicable to all Lebesgue–Stieltjes signed measures on $[a,b]$ (not necessarily nonatomic).

7:5.7 Show that the function $F(x) = x^2 \sin x^{-2}$, $F(0) = 0$, is differentiable for all $x \in \mathbb{R}$, but is not of bounded variation on any closed interval containing 0. Thus F' is not integrable on [0,1].

7:5.8 Prove that if $f \in L_1$ on $[a, b]$ and $\int_a^x f \, d\lambda = 0$ for all $x \in [a, b]$ then $f = 0$ a.e. on $[a, b]$. [Hint: Suppose that $f > 0$ on a closed set P of positive measure. Show that, on some component interval (c, d) of $(a, b) \setminus P$, the integral $\int_c^d f \, d\lambda$ is nonzero.]

7:5.9 Given next are two theorems related to the Lebesgue decomposition of a function and of a measure. Prove these theorems, giving the necessary definitions for "pure jump function" and "pure atomic measure." Let f be nondecreasing on $[a, b]$, and let μ_f be the associated Lebesgue–Stieltjes measure. Then

(a) $f = a + s + j$, where a, s, and j are nondecreasing functions with a absolutely continuous, s continuous and singular, and j a pure jump function.

(b) $\mu_f = \alpha + \sigma + \kappa$, where α, σ, and κ are Lebesgue–Stieltjes measures with $\alpha \ll \lambda$, $\sigma \perp \lambda$, and κ is a pure atomic measure.

7:5.10 Give examples that illustrate the theorems in Exercise 7:5.9 nontrivially. That is, none of the functions or measures should reduce to the zero function or zero measure on any open subinterval of $[a, b]$.

7:5.11 (Growth lemmas for continuous functions of bounded variation.) Let F be a continuous function of bounded variation on $[a, b]$. Prove:

(a) If $r \in \mathbb{R}$ and $F' > r$ on a set $A \subset [a, b]$, then $\mu_F^*(A) \geq r\lambda^*(A)$.

(b) The statement in (a) remains valid if the direction of both inequalities is reversed.

(c) If $B \subset [a, b]$, $\lambda(B) = 0$, and F is differentiable on B, then $\mu_F^*(B) = 0$.

7.6 Total Variation of a Continuous Function

The methods of measure theory can be used to reveal many aspects about the structure of real functions, particularly the differentiation structure. We have already seen how the Lebesgue-Stieltjes measure associated with any monotonic function shows a close interrelation between measure, integral, and derivative.

These ideas can be extended to functions of bounded variation immediately, since any function of bounded variation is the difference of two monotonic functions. To extend them in greater generality, however, requires an entirely different approach. We wish to associate with an arbitrary continuous function f a measure V_f that carries information about the variation and differentiation properties of f, and that allows a formula

$$V_f(E) = \int_E |f'| \, d\lambda$$

if f has a derivative everywhere on a measurable set E. Recall that, for an absolutely continuous function f, we have already obtained this formula for the total variation on a set E.

To do this, we use Methods III and IV from Section 3.9. Here are the details. We assume that f is a continuous function on the real line. Let $\mathcal{T}$ be the collection of all intervals $(a, b]$ $(a,\ b \in \mathrm{I\!R})$. For any subcollection $\mathcal{C} \subset \mathcal{T}$, we write

$$V(f, \mathcal{C}) = \sup \sum_{n=1}^{m} |f(b_i) - f(a_i)|,$$

where the supremum is taken over all $\{(a_i, b_i]\}$ forming a disjoint sequence of intervals taken from $\mathcal{C}$. We can think of $V(f, \mathcal{C})$ as the "variation" of f on $\mathcal{C}$. If $\mathcal{C}$ is the set of *all* subintervals of $(a, b]$, then certainly $V(f, \mathcal{C})$ is precisely the variation of f on the interval $[a, b]$.

We say that $\mathcal{C} \subset \mathcal{T}$ is a *full cover* of a set $E \subset \mathrm{I\!R}$ if, for every $x \in E$, there is a $\delta > 0$ so that

$$0 < y - x < \delta \Rightarrow (x, y] \in \mathcal{C}.$$

A family $\mathcal{C} \subset \mathcal{T}$ is said to be a *fine cover* of E if, for every $x \in E$ and every $\varepsilon > 0$,

$$\exists (x, y] \in \mathcal{C}, y - x < \varepsilon.$$

Here the geometry, in the language of Section 3.9, is to attach to each interval $(x, y]$ the left-hand endpoint x. The measures V_f and v_f shall be defined to be the Methods III and IV measures constructed using the family $\mathcal{T}$ and the premeasure

$$\tau((a, b]) = |f(b) - f(a)|.$$

Explicitly, this means for every $E \subset \mathrm{I\!R}$ we define

$$V_f(E) = \inf\{V(f, \mathcal{C}) :\ \mathcal{C} \text{ a full cover of } E\}$$

and

$$v_f(E) = \inf\{V(f, \mathcal{C}) :\ \mathcal{C} \text{ a fine cover of } E\}.$$

The outer measures V_f and v_f carry variational information about the function f. Note that we are assuming that f is continuous to keep matters simple, although these measures are defined in general. Note, too, that the particular geometry that we are using here (where we take the left-hand endpoint of the intervals) can be changed to suit the study at hand. It is the *methods* that are of the greatest interest to us at this point.

Theorem 7.23 *For any continuous function f, the set functions V_f and v_f are metric outer measures, and $v_f \leq V_f$.*

Proof. See Theorem 3.29 for a proof that these are metric outer measures and that $v_f \leq V_f$. ■

Theorem 7.24 *For any continuous function f, the outer measure V_f is regular.*

Proof. See Theorem 3.30 for a method that will work here. The details differ a little. ■

That these measures do compute something related to the variation of the function f should be apparent. In particular, we have the following result showing that the variation of a function f on an interval $[a, b]$ is exactly $V_f((a, b])$. Recall that $V(f; [a, b])$ denotes the variation of f on the interval $[a, b]$ and that this is finite if and only if f has bounded variation on that interval.

Theorem 7.25 *For any continuous function f,*

$$V_f((a, b]) = v_f((a, b]) = V(f; [a, b]).$$

Proof. The inequality $V_f((a, b]) \leq V(f; [a, b])$ follows simply from the fact that, for any full cover $\mathcal{C}$ of $(a, b]$, it must be true that

$$V(f, \mathcal{C}) \leq V(f; [a, b]).$$

The other direction is more delicate. We obtain this from the following claim.

7.26 *Let $\mathcal{C}$ be a fine cover of $(c, d]$. Then $|f(d) - f(c)| \leq V(f, \mathcal{C})$ for any continuous f.*

We prove this by transfinite induction. Let $x_0 = c$, and choose $x_1 > x_0$ so that $(x_0, x_1] \in \mathcal{C}$. Since $\mathcal{C}$ is fine at x_0, this is possible. Then we have $|f(x_1) - f(x_0)| \leq V(f, \mathcal{C})$. We continue to define a sequence

$$x_0 < x_1 < x_2 < \cdots x_\alpha \leq d$$

inductively. At limit ordinals λ use $x_\lambda = \sup_{\alpha<\lambda} x_\alpha$, and otherwise ensure that $(x_\alpha, x_{\alpha+1}] \in \mathcal{C}$. We always shall have

$$|f(x_\alpha) - f(x_0)| \leq V(f, \mathcal{C}), \tag{27}$$

as one can see inductively. At limit ordinals the continuity supplies this. The process stops in a countable number of steps when $x_\alpha = d$, and at that point the claim 7.26 is proved because of (27).

We can now complete the proof of the theorem. Consider any sum

$$\sum_{k=1}^{n} |f(d_i) - f(c_i)|$$

for disjoint intervals $\{(c_i, d_i]\}$ contained in $(a, b]$. It follows from 7.26 and an easy argument that

$$\sum_{k=1}^{n} |f(d_i) - f(c_i)| \le V(f, \mathcal{C}),$$

and so the inequality

$$V(f, \mathcal{C}) \ge V(f; [a, b])$$

follows. Since $\mathcal{C}$ is an arbitrary fine cover of $(a, b]$, we have

$$v_f((a, b]) \ge V(f; [a, b])$$

Putting these together, we have

$$V(f; [a, b]) \le v_f((a, b]) \le V_f((a, b]) \le V(f; [a, b]),$$

and the conclusion of the theorem follows. ■

In the case of monotonic functions, the two measures V_f and v_f are identical and recover the Lebesgue-Stieltjes measure associated with f. This also extends to functions of bounded variation.

Theorem 7.27 *For any continuous function f that has bounded variation on each finite interval,*

$$V_f = v_f = \mu_T^*,$$

where the last is the Lebesgue-Stieltjes outer measure associated with T, the total variation of f.

Proof. The proof needs only a Vitali argument. We shall use the Vitali theorem for a Lebesgue-Stieltjes measure associated with a continuous, monotonic function. See Section 7.1 for a proof of the Vitali theorem for Lebesgue measure. For more general Lebesgue-Stieltjes measure, the theorem still holds with a modified proof.

We suppose that f is continuous and nondecreasing. Let $E \subset \mathbb{R}$. Let $\mathcal{C}$ be a full cover of E, and suppose that $G \supset E$ is open. Then

$$\mathcal{C}_1 = \{C \in \mathcal{C} : C \subset G\}$$

is also a full cover of E, and so

$$V_f(E) \le V(f, \mathcal{C}_1) \le \mu_f(G).$$

Taking an infimum over all such open sets, we see that $V_f(E) \le \mu_f^*(E)$.

Let $\mathcal{C}$ be any fine cover of E. By the Vitali theorem, there is a collection of disjoint intervals $\{(a_i, b_i]\} \subset \mathcal{C}$ so that

$$\mu_f\left(E \setminus \bigcup_{i=1}^{\infty} (a_i, b_i]\right) = 0,$$

and hence

$$\mu_f^*(E) \le \sum_{i=1}^{\infty} \mu_f\left((a_i, b_i]\right) = \sum_{i=1}^{\infty} \left(f(b_i) - f(a_i)\right) \le V(f, \mathcal{C}).$$

Since $\mathcal{C}$ is arbitrary, we have $\mu_f^*(E) \le v_f(E)$. Thus we have

$$\mu_f^* \le v_f \le V_f \le \mu_f^*,$$

and the identity is proved for f continuous and monotonic. This extends to functions of bounded variation by using the total variation function. ■

A useful equivalence relation is now introduced.

Definition 7.28 Let f and g be continuous functions and let $E \subset \mathbb{R}$. We shall write

$$f \sim g \text{ on } E$$

provided that $V_{f-g}(E) = 0$.

Theorem 7.29 *Let $E \subset \mathbb{R}$, and suppose that $f \sim g$ on E. Then $V_f(E) = V_g(E)$ and $v_f(E) = v_g(E)$.*

The interplay between variation and differentiation is particularly easy to establish. Each of the following computations relates a growth condition on the measure to the derivative in the same spirit as we have already seen in Section 7.1. The proofs are particularly immediate because the Vitali argument is incorporated into the definitions of the measures themselves. The focus on right-hand derivatives is dictated to us by our choice of full and fine covers, using as attached points the left-hand endpoints.

Lemma 7.30 *Let f be a continuous function. Then*

1. *If $\limsup_{y\to x+} \left|\frac{f(y)-f(x)}{y-x}\right| \ge c$ at each $x \in E$, then*
$$c\lambda^*(E) \le V_f(E).$$
2. *If $\liminf_{y\to x+} \left|\frac{f(y)-f(x)}{y-x}\right| \ge c$ at each $x \in E$, then*
$$c\lambda^*(E) \le v_f(E).$$
3. *If $\limsup_{y\to x+} \left|\frac{f(y)-f(x)}{y-x}\right| \le c$ at each $x \in E$, then*
$$V_f(E) \le c\lambda^*(E).$$
4. *If $\liminf_{y\to x+} \left|\frac{f(y)-f(x)}{y-x}\right| \le c$ at each $x \in E$, then*
$$v_f(E) \le c\lambda^*(E).$$

Proof. Each of these is proved the same way with the obvious changes in interpretation of the covers. One of the main tools is the fact, from Theorem 7.27, that the Lebesgue outer measure $\lambda^*(E)$ of any set E can be written as

$$\lambda^*(E) = V_h(E) = \inf\{V(h,\mathcal{C}) : \mathcal{C} \text{ a full cover of } E\}$$

and also as

$$\lambda^*(E) = v_h(E) = \inf\{V(h,\mathcal{C}) : \mathcal{C} \text{ a fine cover of } E\},$$

where here and in the following $h(x) = x$.

For example, we prove the first assertion of the lemma. Let $c' < c$, and let $\mathcal{C}$ be any full cover of E. Let $\mathcal{C}_1$ denote the collection of all intervals $(x,y]$ with $x \in E$ and

$$|f(y) - f(x)| > c'(y-x) = c'(h(y) - h(x)).$$

Then $\mathcal{C}_1$ is a fine cover of E (see Exercise 3:9.3). Hence $\mathcal{C}_1 \cap \mathcal{C}$ is also a fine cover of E (see Exercise 3:9.7). Consequently,

$$c'\lambda^*(E) \le c'V(h,\mathcal{C}_1) \le V(f,\mathcal{C}).$$

Since $\mathcal{C}$ is an arbitrary full cover of E, we have $c'\lambda^*(E) \le V_f(E)$. Let $c' \to c$, and the required inequality is proved. ■

Exercises

7:6.1 For any continuous function f show that $V_f(\{x\}) = 0$ for each $x \in \mathbb{R}$. If f is not assumed continuous, what precisely are $V_f(\{x\})$ and $v_f(\{x\})$?

7:6.2 Prove Theorem 7.24 (using the proof of Theorem 3.30 as a model if necessary).

7:6.3 Let f be a continuous function on $[a,b]$ that has a zero right-hand derived number at every point of $(a,b]$. Show that $v_f((a,b]) = 0$. Use Theorem 7.25 to conclude that f is constant. (Find another, more elementary, proof of this fact.)

7:6.4 Verify the inequality (27) by transfinite induction and show that the process stops in a countable number of steps.

7:6.5 Show that the relation $f \sim g$ on E is an equivalence relation.

7:6.6 Show that if $f \sim g$ on E then $f \sim g$ on E' for every $E' \subset E$.

7:6.7 Show that if $f \sim g$ on E_n for $n = 1, 2, \ldots$ then $f \sim g$ on $\bigcup_{n=1}^{\infty} E_n$.

7:6.8 Prove Theorem 7.29: Suppose that $f \sim g$ on E. Then $V_f(E) = V_g(E)$ and $v_f(E) = v_g(E)$.

7:6.9 Prove the remaining three parts of Lemma 7.30.

7.7 VBG$_*$ Functions

A continuous function f is said to be VBG$_*$ on a set E if the outer measure V_f is σ-finite on E. If V_f is finite on $[a,b]$, then we know that f has bounded variation, so this terminology can be considered an extension of that language.

This is classical terminology, although the classical definition is different (see Exercise 7:7.6). Some such extension of the class of functions of bounded variation is evidently needed in a study of differentiation. A function may be everywhere differentiable and yet have unbounded variation on some intervals, but not on all intervals (see Exercise 7:9.7). The variational ideas needed to discuss such functions were developed by A. Denjoy and S. Saks.

Our main theorem relates the variational properties of a function to its differentiation structure. We can consider it an extension of the Lebesgue differentiation theorem for functions of bounded variation. We have stated it for continuous functions only so that we can avoid extra details that would have to be handled to take care of the discontinuities in our development of the variational measures. The theorem is stated for right-hand derivatives because the measures V_f and v_f have been defined using this special left-hand geometry. (In fact, though, if a right-hand derivative exists almost everywhere on a set, then the derivative itself exists almost everywhere on that set; this follows from the Denjoy–Young–Saks theorem, Exercise 7:9.5.)

Theorem 7.31, together with Exercises 7:7.6 and 7:7.7, relate the concepts of differentiability, variation, and measure.

Theorem 7.31 *The following conditions are equivalent for a continuous function f and a set E.*

1. *f is VBG$_*$ on E.*
2. *The outer measure V_f is σ-finite on E.*
3. *The outer measures V_f and v_f are identical on E.*
4. *f has a finite right-hand derivative a.e. on E and a finite or infinite right-hand derivative V_f–a.e. on E.*

Proof. The second statement is the one we have adopted as our definition of VBG$_*$. Let us show that (2) $\Rightarrow$ (3). We assume that $V_f(E) < +\infty$ and show that this implies that $V_f(E) = v_f(E)$. Pick a full cover $\mathcal{C}$ of E so that

$$V(f,\mathcal{C}) < +\infty.$$

There must be a $\delta(x) > 0$ for each $x \in E$ such that

$$y - x < \delta(x) \Rightarrow (x,y] \in \mathcal{C}.$$

Define

$$E_n = \left\{ x \in E : \delta(x) < \frac{1}{n} \right\}.$$

Then the sets E_n expand to E. The function f is of bounded variation relative to each set $\overline{E_n}$ in the following sense: if $\{[a_i, b_i]\}$ are nonoverlapping intervals with endpoints in $\overline{E_n}$ and each $b_i - a_i < 1/n$, then the sum

$$\sum |f(b_i) - f(a_i)| \tag{28}$$

remains bounded. To see this, one can adjust the intervals slightly without altering the sum (28) by more than a specified amount so that the intervals have a left endpoint in E_n and still remain shorter than $1/n$. The resulting sum (28) would have to be bounded by $2V(f, \mathcal{C})$ since it can be split into two disjoint sequences.

This allows us to define a continuous function g_n to be f on $\overline{E_n}$ and linear on the complementary intervals. This function g_n is continuous, has bounded variation, and agrees with f on $\overline{E_n}$.

We shall prove the following claim. The equivalence relation used here is defined in Definition 7.28.

7.32 *$f \sim g_n$ on $\overline{E_n}$.*

Let $\varepsilon > 0$. Let $\{I_i\}$ be the intervals complementary to $\overline{E_n}$. Since

$$\sum_{i=1}^{\infty} \omega(f, I_i) < +\infty,$$

there is an integer N so that

$$\sum_{i=N+1}^{\infty} \omega(f, I_i) < \varepsilon/2.$$

Inside each interval I_i $(i = 1, 2, \dots, N)$, choose a centered interval J_i so that the oscillation of $f - g$ on the two components of $I_i \setminus J_i$ is less than $\varepsilon/4N$. Since both f and g are continuous and there are only a finite number of intervals to handle, this is easily done. Now choose a full cover $\mathcal{C}$ of $\overline{E_n}$ as follows: we allow all intervals $(x, y]$, with $x \in \overline{E_n}$ and $y - x < 1/n$, that meet no interval J_i for $i = 1, 2, \dots N$. Consider any collection $\{(a_k, b_k]\}$ of disjoint intervals from $\mathcal{C}$, and estimate the sum

$$\sum_k |f(b_k) - g(b_k) - f(a_k) + g(a_k)|\,. \tag{29}$$

We can increase the sum (29), by adding further points if necessary, and we assume that each a_k, $b_k \in \overline{E_n}$ or else that (a_k, b_k) misses $\overline{E_n}$. If a_k, $b_k \in \overline{E_n}$, then $f(a_k) = g(a_k)$ and $f(b_k) = g(b_k)$. If the interval (a_k, b_k) misses $\overline{E_n}$, then it either lies in some $I_i \setminus J_i$ $(i = 1, 2 \dots N)$ or else in I_i for $i > N$. In either case, we see that the sum (29) must be smaller than

$$\sum_{i=N+1}^{\infty} \omega(f, I_i) + 2N(\varepsilon/4N) < \varepsilon.$$

Consequently,

$$V_{f-g_n}(E_n) \le V(f-g_n, \mathcal{C}) \le \varepsilon,$$

and 7.32 is proved.

From 7.32 and Theorem 7.29, we have

$$v_f(E_n) = v_{g_n}(E_n)$$

and

$$V_f(E_n) = V_{g_n}(E_n).$$

But g_n is a continuous function of bounded variation, and so

$$V_{g_n}(E_n) = v_{g_n}(E_n).$$

From these identities and the regularity of the measure V_f, we get

$$v_f(E) \ge \lim_{n\to\infty} v_f(E_n) = \lim_{n\to\infty} V_f(E_n) = V_f(E),$$

and the identity $V_f(E) = v_f(E)$ is proved.

The converse, (3) $\Rightarrow$ (2), follows from the fact that v_f is always σ-finite (see Exercise 7:7.3).

Let us now prove that (2) $\Rightarrow$ (4). We can use (3) to help obtain this. Again we can assume that $V_f(E) < +\infty$. We shall use the notation

$$D(x) = \limsup_{y\to x+} \left|\frac{f(y)-f(x)}{y-x}\right| \text{ and } d(x) = \liminf_{y\to x+} \left|\frac{f(y)-f(x)}{y-x}\right|.$$

The set of points

$$E_1 = \{x \in E : D(x) = \infty\}$$

can be shown to have Lebesgue measure zero. Write this set as the intersection of the sets $\{x \in E : D(x) \ge n\}$ and apply Lemma 7.30. The set of points

$$E_2 = \{x \in E : d(x) < D(x) < \infty\}$$

can be shown to have Lebesgue measure zero and V_f–measure zero. The set of points

$$E_3 = \{x \in E : d(x) < D(x) \le \infty\}$$

can be shown to have V_f–measure zero. See Exercise 7:7.1 for hints on how to accomplish the proof of these statements. There remains to consider only the following sets:

$$E_4 = \{x \in E : d(x) = D(x) < \infty\},$$

$$E_5 = \{x \in E : d(x) = D(x) = \infty\}.$$

The set E_4 is precisely the set where f has a right-hand derivative (finite) and, since f is continuous, the set E_5 is exactly the set where $f'_+(x) = \pm\infty$. From these observations, we obtain the proof that (2) $\Rightarrow$ (4).

To complete the proof of the theorem, we must show that (4) $\Rightarrow$ (1). The set D_1 of points in E where f has a finite right-hand derivative has σ-finite V_f–measure as an application of Lemma 7.30 will show. Let D_2 and D_3 be the sets of points where $f'_+(x) = +\infty$ and $f'_+(x) = -\infty$, respectively. We have left it as an exercise (Exercise 7:7.4) to show that each of the sets D_2 and D_3 has σ-finite V_f–measure. One concludes that V_f is σ-finite on E, since E is the union of D_1, D_2, D_3 and a set of V_f–measure zero. This completes the proof. ∎

Exercises

7:7.1 Let f be continuous and write

$$D(x) = \limsup_{y \to x+} \left| \frac{f(y) - f(x)}{y - x} \right|$$

and

$$d(x) = \liminf_{y \to x+} \left| \frac{f(y) - f(x)}{y - x} \right|$$

.

(a) Show that $D(x) = d(x)$ if and only if f has a right-hand derivative $f'_+(x) = D(x) = d(x)$ at the point x.

(b) Let E be a set of points such that $0 \leq \alpha < D(x) < \beta$ for $x \in E$. Show that $\alpha\lambda^*(E) \leq V_f(E) \leq \beta\lambda^*(E)$.

(c) Let E be a set of points such that $0 \leq \alpha < d(x) < \beta$ for $x \in E$. Show that $\alpha\lambda^*(E) \leq v_f(E) \leq \beta\lambda^*(E)$.

(d) Let E be a measurable set of points such that $0 < D(x) < +\infty$ for $x \in E$. Show that $v_f(E) \leq \int_E D\, d\lambda$.

(e) Let E be a measurable set of points such that $0 < d(x) < +\infty$ for $x \in E$. Show that $v_f(E) \leq \int_E d\, d\lambda$.

(f) Let E be a measurable set of points such that $0 < d(x) \leq D(x) < +\infty$ for $x \in E$. Show that

$$\int_E (D - d))\; d\lambda = V_f(E) - v_f(E).$$

What can you conclude?

7:7.2 Using Exercise 7:7.1 formulate an economical proof of the Lebesgue differentiation theorem for continuous, monotonic functions f given the identity $V_f = v_f$ for such functions.

7:7.3 Show that the measure v_f is σ-finite for any continuous function. [Hint: Let E_1 denote the set of points x for which there is a sequence $x_n \searrow x$ with $f(x_n) = f(x)$, let E_2 denote the set of points x for which there is a $\delta(x) > 0$ so that $f(y) > f(x)$ if $x < y < x + \delta(x)$, and let E_3 denote the set of points x for which there is a $\delta(x) > 0$ so that $f(y) < f(x)$ if $x < y < x + \delta(x)$. Show that v_f vanishes on E_1 and is σ-finite on E_2 and E_3.]

7:7.4 Suppose that f is a continuous function such that $f'(x) = +\infty$ for each $x \in E$. Show that E has σ-finite V_f–measure. [Hint: Split E into a sequence of bounded sets on each of which f is increasing.]

7:7.5 Prove the following version of the de la Vallée Poussin theorem. Let f be a continuous function and E a Borel set, and suppose that $V_f(E) < +\infty$. Then f' exists a.e. on E, and

$$V_f(E) = \int_E |f'|\, d\lambda + V_f\left(\{x \in E : f'(x) = \pm\infty\}\right).$$

7:7.6 This definition is due to S. Saks. A function F is Saks-VB$_*$ on a set $E \subset \mathbb{R}$ if, for any sequence of nonoverlapping intervals $\{[a_k, b_k]\}$ with endpoints in E, the sum of the oscillations $\sum_{k=1}^{\infty} \omega(F, [a_k, b_k])$ converges. A function F is Saks-VBG$_*$ on a set $E \subset \mathbb{R}$ if $E = \bigcup_{n=1}^{\infty} E_n$ with F Saks-VB$_*$ on each set E_n. Show that a continuous function is Saks-VBG$_*$ on a set if and only if it is VBG$_*$ on that set in our sense.

7:7.7 Characterize the class of continuous functions that are almost everywhere differentiable in terms of the concepts VBG$_*$ and Saks-VBG$_*$.

7.8 Approximate Continuity, Lebesgue Points

Let f be a Lebesgue integrable function defined on $[a, b]$. Then the function

$$F(x) = \int_a^x f\, d\lambda$$

is differentiable a.e., and $F'(x) = f(x)$ almost everywhere.

In this section we obtain some information about the set on which $F'(x) = f(x)$ holds; this is true at every point of continuity of f, but f can be discontinuous everywhere on $[a, b]$. In the process, we obtain an important theorem of Lebesgue. Consider first the case of characteristic functions. Let A be measurable. Then χ_A is integrable, and for $F(x) = \int_a^x \chi_A\, d\lambda$ we have

$$F'(x) = \begin{cases} 1, & a.e.\ on\ \ A; \\ 0, & a.e.\ on\ \ \widetilde{A}. \end{cases} \tag{30}$$

Let us analyze this derivative further. For $h \neq 0$, we have

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} \chi_A\, d\lambda = \frac{\lambda(A \cap [x, x+h])}{h}.$$

Thus

$$\lim_{h\to 0} \frac{\lambda(A\cap[x,x+h])}{h} = \begin{cases} 1, & \textit{a.e. on } A; \\ 0, & \textit{a.e. on } \widetilde{A}. \end{cases} \tag{31}$$

The argument leading to (31) is easily modified to give the following result.

Theorem 7.33 *Let A be a measurable set in $\mathbb{R}$. Then*

$$\lim_{h\to 0,\, k\to 0,\, h\geq 0,\, k\geq 0} \frac{\lambda(A\cap[x-h,x+k])}{h+k} = \begin{cases} 1, & \text{a.e. on } A; \\ 0, & \text{a.e. on } \widetilde{A}. \end{cases}$$

Theorem 7.33 is called the *Lebesgue density theorem.* Intuitively, it states that, for almost all $x \in A$, small intervals containing x consist predominantly of points of A. Consider, for example, the set E called for in Exercise 2:13.9. That set and its complement have positive measure in every interval contained in [0,1]. Theorem 7.33 tells us that some intervals consist predominantly of points of E, others of $\widetilde{E}$.

Definition 7.34 Let A be a measurable set, and let $x \in A$. Let

$$d(A,x) = \lim_{h\to 0,\, k\to 0,\, h\geq 0,\, k\geq 0} \frac{\lambda(A\cap[x-h,x+k])}{h+k}$$

if this limit exists. Then $d(A,x)$ is called the *density* of A at x. If $d(A,x) = 1$, x is called a *density point* of A. If $d(A,x) = 0$, x is called a *dispersion point* of A.

From Theorem 7.33, we see that almost all points in a measurable set A are density points of A; almost all points in $\widetilde{A}$ are dispersion points of A. We should mention that it is possible that $0 < d(A,x) < 1$ or that $d(A,x)$ does not exist (Exercise 7:8.2).

Returning to the main topic of this section, we see from Theorem 7.33 that, for $F(x) = \int_a^x \chi_A\, d\lambda$, the derivative $F'(x)$ is the integrand at all density points of A and all density points of $\widetilde{A}$. (Clearly, the density points of $\widetilde{A}$ are the same as the dispersion points of A.) Let us now replace χ_A by any bounded measurable function f. We shall see how the notion of density allows us to obtain a generalization of continuity, called *approximate continuity*, that allows $F'(x) = f(x)$ to hold at each point of approximate continuity. We then show that a measurable function is approximately continuous almost everywhere.

Definition 7.35 Let f be a function defined in a neighborhood of x_0. If there exists a set E such that

$$d(E,x_0) = 1 \quad\text{and}\quad \lim_{x\to x_0,\, x\in E} f(x) = f(x_0),$$

we say that f is *approximately continuous* at x_0. If f is approximately continuous at all points of its domain, we simply say that f is *approximately continuous.*

If a function is defined on a closed interval $[a, b]$, then approximate continuity at the end points is defined in the obvious way, invoking one-sided densities. Note that f is approximately continuous at x_0 if there exists a set E having x_0 as a density point, such that $f|E$ is continuous at x_0. In short, we can ignore the behavior of f on a set (in this case $\widetilde{E}$) having x_0 as a dispersion point. For example, if $A \subset \mathbb{R}$ is Lebesgue measurable, then the function χ_A is approximately continuous at every point that is either a point of density or a point of dispersion of A.

Theorem 7.36 *Let f be a bounded measurable function on $[a, b]$. If f is approximately continuous at $x_0 \in [a, b]$ and*

$$F(x) = \int_a^x f \, d\lambda \quad \textit{for all} \quad x \in [a, b],$$

then $F'(x_0) = f(x_0)$.

Proof. Choose a set E such that $d(E, x_0) = 1$ and $f|E$ is continuous at x_0. Let M be an upper bound for $|f|$, and let $h > 0$. Then

$$\begin{aligned}
&\left| \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) \right| \\
&\quad = \left| \frac{1}{h} \int_{x_0}^{x_0+h} f \, d\lambda - f(x_0) \right| = \left| \frac{1}{h} \int_{x_0}^{x_0+h} (f - f(x_0)) \, d\lambda \right| \\
&\quad \leq \frac{1}{h} \int_{x_0}^{x_0+h} |f - f(x_0)| \, d\lambda \\
&\quad = \frac{1}{h} \int_{[x_0, x_0+h] \cap E} |f - f(x_0)| \, d\lambda + \frac{1}{h} \int_{[x_0, x_0+h] \setminus E} |f - f(x_0)| \, d\lambda.
\end{aligned}$$

We apply the "rectangle principle" we mentioned in Section 5.9. Let $\varepsilon > 0$, and choose $\delta > 0$ such that (i) if $t \in E$ and $|t - x_0| < \delta$ then $|f(t) - f(x_0)| < \varepsilon/2$, and (ii) if $h < \delta$, then

$$\frac{\lambda([x_0, x_0 + h] \setminus E)}{h} < \frac{\varepsilon}{4M}.$$

For $h < \delta$, we calculate

$$\begin{aligned}
&\left| \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) \right| \\
&\quad \leq \frac{\varepsilon}{2h} \lambda([x_0, x_0 + h] \cap E) + \frac{2M}{h} \lambda([x_0, x_0 + h] \setminus E) \\
&\quad \leq \varepsilon \frac{h}{2h} + 2M \frac{\varepsilon}{4M} = \varepsilon.
\end{aligned}$$

A similar calculation holds if $h < 0$. Since ε is arbitrary, we conclude that

$$\lim_{h \to 0} \frac{F(x_0 + h) - F(x_0)}{h} = f(x_0).$$

That is, $F'(x_0) = f(x_0)$. ■

We next show that a measurable, finite a.e. function must be approximately continuous a.e. This can be viewed as an extension of Theorem 7.33, when the latter is interpreted in terms of characteristic functions of measurable sets. (In fact, the converse of Theorem 7.37 is also true, but a bit more difficult to prove. Thus measurable, finite a.e. functions can be characterized in terms of a type of continuity.)

Theorem 7.37 *A measurable, finite a.e. function is approximately continuous at almost every point.*

Proof. Let $\varepsilon > 0$. By Lusin's theorem (Theorem 4.25), there exists a continuous function g such that

$$\lambda(\{x : g(x) \neq f(x)\}) < \varepsilon. \tag{32}$$

Let $E = \{x : g(x) = f(x)\}$. By Theorem 7.33, almost every point of E is a density point of E. If $x_0 \in E$ and x_0 is a density point of E, we have

$$\lim_{x \to x_0,\ x \in E} f(x) = \lim_{x \to x_0} g(x) = g(x_0) = f(x_0).$$

Thus f is approximately continuous at x_0. Since x_0 is an arbitrary density point of E, f is approximately continuous at each density point of E. From (32), we infer that f is approximately continuous except perhaps on a set of measure less than ε. Since ε is arbitrary, f is approximately continuous a.e. ■

In Theorem 7.36, we required f to be bounded. We cannot drop this part of the hypotheses in the statement of the theorem (Exercise 7:8.4). For unbounded functions, a stronger condition on a point x_0 suffices.

Definition 7.38 Let f be Lebesgue integrable on a neighborhood of a point x_0. If

$$\lim_{h \to 0} \frac{1}{h} \int_{x_0}^{x_0+h} |f - f(x_0)|\, d\lambda = 0,$$

we say that x_0 is a *Lebesgue point* of f.

Theorem 7.39 *Let x_0 be a Lebesgue point for a function f integrable on $[a, b]$, and let $F(x) = \int_a^x f\, d\lambda$. Then $F'(x_0) = f(x_0)$.*

Proof. As in the proof of Theorem 7.36, we calculate

$$\left| \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) \right| \leq \frac{1}{|h|} \int_{x_0}^{x_0+h} |f - f(x_0)|\, d\lambda.$$

The result follows directly from Definition 7.38. ■

Actually, a Lebesgue point is a special kind of point of approximate continuity [Exercise 7:8.4(a)], and for bounded measurable functions, the two notions coincide [Exercise 7:8.4(c)]. We next show that Theorem 7.36 extends to Lebesgue points.

Theorem 7.40 *Let f be integrable on $[a,b]$. Then almost every point of $[a,b]$ is a Lebesgue point of f.*

Proof. Let $r \in \mathbb{Q}$. Then $f - r \in L_1$, and thus

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} |f - r|\, d\lambda = |f(x) - r| \tag{33}$$

a.e. on $[a,b]$. Let $E(r) = \{x \in [a,b] : (33) \text{ fails}\}$. Then $\lambda(E(r)) = 0$. Let

$$E = \bigcup_{r\in\mathbb{Q}} E(r) \cup \{x \in [a,b] : |f(x)| = \infty\}.$$

Then $\lambda(E) = 0$. We show that every point x_0 in $[a,b]\setminus E$ is a Lebesgue point for f. Let $x_0 \in [a,b]\setminus E$, and let $\varepsilon > 0$. Choose $r_n \in \mathbb{Q}$ such that

$$|f(x_0) - r_n| < \tfrac{1}{3}\varepsilon. \tag{34}$$

We then have

$$||f - r_n| - |f - f(x_0)|| < \tfrac{1}{3}\varepsilon.$$

on $[a,b]$ so that

$$\left|\frac{1}{h}\int_{x_0}^{x_0+h} |f - r_n|\, d\lambda - \frac{1}{h}\int_{x_0}^{x_0+h} |f - f(x_0)|\, d\lambda\right| \le \frac{\varepsilon}{3} \tag{35}$$

whenever $x_0 + h \in [a,b]$. Since $x_0 \notin E$, (33) applies, so there exists $\delta > 0$ such that

$$\left|\frac{1}{h}\int_{x_0}^{x_0+h} |f - r_n|\, d\lambda - |f(x_0) - r_n|\right| < \frac{\varepsilon}{3}$$

if $|h| < \delta$. From (34), we infer that, for $|h| < \delta$,

$$\frac{1}{h}\int_{x_0}^{x_0+h} |f - r_n|\, d\lambda < \frac{2\varepsilon}{3}$$

so

$$\frac{1}{h}\int_{x_0}^{x_0+h} |f - f(x_0)|\, d\lambda < \varepsilon \tag{36}$$

by (35).

We have shown that for all $x_0 \notin E$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that (36) holds whenever $|h| < \delta$. Since $\lambda(E) = 0$, we conclude that almost every $x \in [a,b]$ is a Lebesgue point of f. ■

It is clear that every point of continuity of a function $f \in L_1$ is a Lebesgue point. Note that a difference between x_0 being a Lebesgue point for f and x_0 being a point at which it is the derivative of its integral is that, in the former case, "cancellations" are not possible. See Exercise 7:8.5 in conjunction with Exercise 7:8.4(c).

Exercises

7:8.1 Prove Theorem 7.33.

7:8.2 Construct measurable sets $A, B \subset [0,1]$ such that $d(A,0) = \frac{1}{2}$ and $d(B,0)$ does not exist. One-sided notions of density apply here.

7:8.3 Define $d^+(A,x)$, $d_+(A,x)$, $d^-(A,x)$, and $d_-(A,x)$, the unilateral extreme densities of A at x. Give an example of a set A for which

$$d^+(A,0) = 1 > 0 = d_+(A,0).$$

Relate this to the Dini derivates defined in Exercise 7:2.3.

7:8.4 (a) Prove that an integrable function f is approximately continuous at each Lebesgue point.

(b) Show the converse of (a) fails by giving an example that shows that Theorem 7.36 fails if f is not assumed bounded.

(c) Prove that if f is bounded and measurable then x_0 is a Lebesgue point for f if and only if f is approximately continuous at x_0.

7:8.5◇ Give an example of a function f such that for $F(x) = \int_0^x f\,d\lambda$, $F'(0) = f(0)$, but f is not approximately continuous at 0. [Hint: Use the set A called for in Exercise 7:8.2.]

7:8.6 Show that if f and g are approximately continuous at x_0 so are $f+g$ and fg.

7:8.7 Let f be approximately continuous on an interval I, and let g be a continuous function defined on $f(I)$. Prove that $g \circ f$ is approximately continuous.

7:8.8 Show that the composition of two approximately continuous functions need not be approximately continuous.

7:8.9 Prove that a function that is approximately continuous must have the intermediate-value property and must belong to $\mathcal{B}_1$ (the first class of Baire). [Hint: Use Theorem 7.36, Exercise 7:8.7, and parts of Exercise 4:6.2.]

7:8.10 Prove that a function f is approximately continuous on $\mathbb{R}$ if and only if, for every $\alpha < \beta$, the set

$$E_\alpha^\beta = \{x : \alpha < f(x) < \beta\}$$

is of type $\mathcal{F}_\sigma$ and satisfies $d(E_\alpha^\beta, x) = 1$ for all $x \in E_\alpha^\beta$; that is, every point in E_α^β is a point of density of E_α^β.

7:8.11 Prove that if $f_n \to f$ [unif] on $\mathbb{R}$ and f_n is approximately continuous for all $n \in \mathbb{N}$ then f is also approximately continuous. [Hint: Use Exercises 7:8.9 and 7:8.10.]

7:8.12 Prove the converse of Theorem 7.37.

7.9 Additional Problems for Chapter 7

7:9.1 Let f be absolutely continuous on an interval $[a,b]$ and g continuous there. Show that

$$\int_a^b g(x)\,df(x) = \int_a^b g(x)f'(x)\,dx,$$

where the first integral is interpreted as a Riemann–Stieltjes integral.

7:9.2$\diamondsuit$ (Integration by parts) Let f, g be absolutely continuous on an interval $[a,b]$. Show that

$$\int_a^b g(x)f'(x)\,dx = g(b)f(b) - g(a)f(a) - \int_a^b g'(x)f(x)\,dx.$$

7:9.3 Let f be continuously differentiable on $[a,b]$, and let $E \in \mathcal{L}$. Prove that $\lambda(f(E)) = 0$ if and only if $f' = 0$ a.e. on E. (This result is actually true under much weaker hypotheses. It holds, for example, if f is measurable and differentiable only on E.)

7:9.4 (Differentiability of Lipschitz functions) According to Theorem 7.5, a function f of bounded variation on $[a,b]$ is differentiable a.e. Thus the set N of points of nondifferentiability of f is small in the sense of measure. The set N can be large in the sense of category. Carry out the following steps:

(a) (Converse to the Lebesgue density theorem.) Let $Z \subset [a,b]$ be any set of measure zero. Then there exists a measurable set S such that, for every $z \in Z$,

$$\limsup_{h\to 0, k\to 0, h+k>0} \frac{\lambda(S \cap [z-h, z+k])}{h+k} = 1$$

and

$$\liminf_{h\to 0, k\to 0, h+k>0} \frac{\lambda(S \cap [z-h, z+k])}{h+k} = 0.$$

[Hint: Let $\{G_n\}$ be a decreasing sequence of open sets such that the set $H = \bigcap_{n=1}^\infty G_n$ is a measurable cover for Z. Choose the sets G_n in such a way that the relative measure of G_{n+1} is $1/n$ in each component interval of G_n. Let

$$S = (G_1 \setminus G_2) \cup (G_3 \setminus G_4) \cup (G_5 \setminus G_6) \cup \dots]$$

(b) Let Z and S be as in (a). Let $F(x) = \int_a^x \chi_S\,d\lambda$. Then F is a Lipschitz function with all Dini derivates bounded by 0 and 1 on $[a,b]$, and F is not differentiable at any point of Z.

(c) There exists a Lipschitz function for which the set of points of differentiability is first category.

7:9.5 (Denjoy–Young–Saks theorem) The theorem with this name is a far-reaching theorem relating the four Dini derivates D^+f, D_+f, D^-f, and D_-f (see Exercise 7:2.3). It was proved independently by Grace Chisolm Young and Arnaud Denjoy for continuous functions in 1916 and 1915, respectively. Young then extended the result to measurable functions. Finally, S. Saks removed the hypothesis of measurability in 1924. Here is their theorem.

> **Theorem (Denjoy–Young–Saks)** *Let f be an arbitrary finite function defined on $[a, b]$. Then almost every point $x \in [a, b]$ is in one of four sets:*
>
> (1) *A_1 on which f has a finite derivative;*
>
> (2) *A_2 on which $D^+f = D_-f$ (finite), $D^-f = \infty$, and $D_+f = -\infty$;*
>
> (3) *A_3 on which $D^-f = D_+f$ (finite), $D^+f = \infty$, and $D_-f = -\infty$;*
>
> (4) *A_4 on which $D^-f = D^+f = \infty$ and $D_-f = D_+f = -\infty$.*

(a) Sketch a picture illustrating points in the sets A_2, A_3, and A_4. To which set does $x = 0$ belong when $f(x) = \sqrt{|x|} \sin x^{-1}$, $f(0) = 0$?

(b) Give examples showing that it is possible that $\lambda(A_1) = b - a$. Do the same for A_2 and A_3.

(c) Use DYS to prove that an increasing function f has a finite derivative a.e.

(d) Use DYS to show that if all derived numbers of f are finite a.e. then f is differentiable a.e.

(e) Use DYS to show that, for every finite function f,
$$\lambda(\{x : f'(x) = \infty\}) = 0.$$

7:9.6 Theorem 7.20 and the discussion preceding it might suggest the following formula for a continuous function F of bounded variation:
$$F(b) - F(a) = \int_a^b F' \, d\lambda + \lambda(F(B_\infty)) - \lambda(F(B_{-\infty})).$$

(a) Show that such a formula fails.

(b) Partitioning B_∞ and $B_{-\infty}$ into sets $\{C_n\}$ and $\{D_n\}$ appropriately, we can arrive at a formula of the form
$$F(b) - F(a) = \int_a^b F' \, d\lambda + \sum_{n=1}^{\infty} \lambda(F(C_n)) - \sum_{n=1}^{\infty} \lambda(F(D_n)).$$
Show how to obtain the necessary partitions of B_∞ and $B_{-\infty}$. [Hint: Use Theorem 3.22.]

7:9.7 A differentiable function f need not be of bounded variation on an interval $[a, b]$. The interval $[a, b]$ can be decomposed into countably many sets A_k such that "f is of bounded variation on each of these sets." Provide a definition for the statement in quotes, and prove that the statement correct. Then show that there exists a sequence of intervals $\{I_k\}$ with $\bigcup I_k$ dense in $[a, b]$ such that f is of bounded variation on each interval I_k. (These intervals need not be the components of $\bigcup I_k$.)

7:9.8 (a) Construct a function f that satisfies the following conditions on [0,1]:

(i) f is continuous except at 0,

(ii) $f(0) = 0$, $-1 \leq f(x) \leq 1$ for all $x \in [0, 1]$ and

(iii) $d(\{x : f(x) = 1\}, 0) = d(\{x : f(x) = -1\}, 0) = \frac{1}{2}$.

(b) Let $F(x) = \int_0^x f\, d\lambda$. Prove that $F'(x) = f(x)$ for all $x \in [0, 1]$.

(c) Prove that f^2 is not the derivative of any function G everywhere on [0,1]. [Hint: What is $H'(0)$ if $H(x) = \int_0^x f^2\, d\lambda$?]

(d) Prove that if $g \in \triangle'$ and $g^2 \in \triangle'$ then $g \in L_1$. [Hint: Use an appropriate theorem from Section 7.2.]

[Part (c) shows that the class $\triangle'$ of derivatives on $[0, 1]$, i.e., the class

$$\triangle' = \{f : \exists F : [0, 1] \to \mathbb{R} \text{ so that } F'(x) = f(x) \text{ for all } x \in [0, 1]\},$$

is not closed under multiplication or under composition on the outside with continuous functions. Observe that f is *not* approximately continuous at 0.]

7:9.9 Suppose that F and G are differentiable on [0,1]. Can we conclude that $FG' \in \triangle'$? (See Problem 7:9.8.) Since one of the factors, F, is very well behaved (it is differentiable, not just a derivative), one might suspect that $H' = FG' \in \triangle'$ where

$$H(x) = \int_0^x FG'\, d\lambda.$$

But FG' need not be integrable. What if we assume that $FG' \in L_1$?

(a) Let $F(x) = x^2 \sin x^{-3}$ and $G(x) = x^2 \cos x^{-3}$ with $F(0) = G(0) = 0$. Show that FG' and GF' are bounded and therefore integrable on [0,1]. Then verify that

$$F(x)G'(x) - F'(x)G(x) = \begin{cases} 3, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

If $FG' \in \triangle'$, then $F'G \in \triangle'$ and vice versa, since

$$FG' + GF' = (FG)' \in \triangle'.$$

But then $FG' - GF' \in \triangle'$, which is impossible, because this function does not even have the intermediate-value property.

(b) *(A positive result.)* Show that if F' is continuous then $FG' \in \triangle'$. [Hint: $FG' = (FG)' - F'G$.]

7:9.10 In the early part of this century, relatively little was known about derivatives. The only sufficient condition that was known is that the function be continuous. Not many necessary conditions were known either. Lamenting the state of knowledge, W. H. Young wrote in 1911:

> The necessary conditions ...are of considerable importance and interest. ...[A derivative] must be pointwise discontinuous with respect to every perfect set; it can have no discontinuities of the first kind; it assumes in every interval all values between its upper and lower bounds in that interval, ..., its upper and lower bounds, when finite, are unaltered if we omit the values on any countable set of points; the points at which it is infinite form an inner limiting set of content zero (i.e., is a $\mathcal{G}_\delta$ of measure zero) ....

(a) Verify each of the statements made by Young. [Hint: See Exercises 7:9.5 and 4:6.2(a). The condition involving "pointwise discontinuity" is the content of the comment at the end of Section 1.6 or of the comment following the proof of Theorem 1.19. (See also Theorem 10.13.)]

(b) Which theorem in Chapter 7 gives another sufficient condition for a function to be a derivative?

Most important classes $\mathcal{F}$ of functions have many known characterizations, that is theorems of the form $f \in \mathcal{F}$ if and only if some condition is met. For example, F is an integral of some function on $[a, b]$ if and only if F is absolutely continuous.

(c) State theorems that provide characterizations for each of the following classes of functions:

(*i*) Integrals of functions on $[a, b]$. (There are other characterizations than the one mentioned above.)

(*ii*) $\mathcal{C}[a, b]$.

(*iii*) The measurable functions on $[a, b]$.

(*iv*) $\mathrm{BV}[a, b]$.

(*v*) Complex analytic functions on the disk $\{z : |z| < 1\}$.

Useful characterizations of each of these classes were already known at the time Young commented about the lack of knowledge of derivatives. The problem of characterizing derivatives, however, has not been solved satisfactorily to this day.

7:9.11◇ (For readers with a background in topology.) Show that the class of subsets of $\mathbb{R}$ that are measurable and have density 1 at each point forms a topology on $\mathbb{R}$ (called the *density topology*). Show that the functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous (with the density topology on the domain and ordinary topology on the range) are precisely the approximately continuous functions.

7:9.12 (Set porosity) A number of theorems we have encountered state that some property holds except on a "small" set. We have interpreted the term small in various ways: A is small in the sense of cardinality (measure, category) if A is countable (of zero measure, first category). There are other notions of smallness. One of these has assumed importance in various parts of analysis, such as differentiation theory, cluster set theory, and trigonometric series. The notion of porosity originates in the work of Denjoy; the concept of σ-porosity was introduced by E. P. Dolzhenko (1934– .)

> **Definition.** Let $A \subset \mathbb{R}$, and let $x \in A$. We define the *porosity of A at x* as
> $$p(A, x) = \limsup_{h \to 0} \frac{\ell(x, h, A)}{h},$$
> where $\ell(x, h, A)$ is the length of the longest interval in $(x - h, x + h) \setminus A$.

When $p(A, x) > 0$, we say that A *is porous at x*. If $p(A, x) > 0$ for all $x \in A$, we say A is a *porous set.* A countable union of porous sets is called *σ-porous.*

(a) Let
$$A = \{0\} \cup \bigcup_{n=1}^{\infty} \{(-1)^n n^{-1}\}$$
and
$$B = \{0\} \cup \bigcup_{n=1}^{\infty} \{(-1)^n 2^{-n}\}.$$
Calculate $p(A, 0)$ and $p(B, 0)$.

(b) Prove that no point of a porous set is a point of density and that a porous set is nowhere dense.

(c) Prove that a σ-porous set has measure zero and is of the first-category.

(d) Give an example of a first-category set of measure zero that is not σ-porous. (This is not easy.)

(e) Give an example of a Cantor set C for which $p(C, x) = 1$ for all $x \in C$.

(f) Show, for each Cantor set C, that the set $\{x : p(C,x) = 1\}$ is of type $\mathcal{G}_\delta$ and is dense in C.

(g) It can be proved from the Denjoy–Young–Saks theorem (see Exercise 7:9.5) that, for a Lipschitz function f defined on $[a,b]$, the set
$$\big\{x : D^+f(x) > D^-f(x)\big\}$$
has measure zero. Show that this set is actually σ-porous.

(h) Prove the following porous version of the Vitali covering theorem, due to Y. A. Shevchenko (1989): If $\mathcal{V}$ is a Vitali covering of a set $E \subset \mathbb{R}$, then there is a countable disjoint collection $\{V_k\}$ of sets chosen from $\mathcal{V}$ so that $E \setminus \bigcup_{k=1}^\infty V_k$ is porous.

7:9.13 Let F be continuous on an interval I. Prove that the bounds of the difference quotient
$$\frac{F(y) - F(x)}{y - x} \quad (x, y \in I,\ y \neq x)$$
are the same as the bounds of each of the four Dini derivates on I.

7:9.14 (a) Review and contrast the definitions of Vitali cover, fine cover, and full cover.

(b) Give examples that illustrate how such covers can arise naturally in a study of sets on which some or all derived numbers are bounded.

(c) State some theorems or lemmas that relate global "growth" conditions to local conditions on the derived numbers.

(d) In Exercise 7:3.5 we noted that, if f is measurable and all derived numbers of f vanish at all points of a measurable set E, then $\lambda(f(E)) = 0$. Give an example of a continuous function $f : [0,1] \to [0,1]$ such that, for each $x \in [0,1]$, there exists a derived number $Df(x) = 0$, and yet f maps [0,1] onto [0,1]. [Hint: See Exercise 3:11.7.]

(We shall see in Section 10.6 that "most" continuous functions on [a,b] have the property expressed in (d).)

7:9.15 The following theorem, due to A. P. Morse, can be used to provide insights into the differentiability structure of certain continuous functions.

> **Theorem (Morse).** *Let F be continuous on $\mathbb{R}$, and let $-\infty < \alpha < \infty$. If the set $\{x : D^+F(x) \geq \alpha\}$ is dense in $\mathbb{R}$, and there exists $x_0 \in \mathbb{R}$ such that $D^+F(x_0) < \alpha$, then the set $\{x : D^+F(x) = \alpha\}$ has cardinality c.*

(a) Prove that if F is continuous on $\mathbb{R}$ and a Dini derivate is unbounded both from above and below on every interval then

D^+F takes on every value on every interval. In fact, for every $\alpha \in \mathbb{R}$, the set $\{x : D^+F(x) = \alpha\}$ has cardinality c in every interval. [Hint: Use Exercise 7:9.13.]

(b) Let F be continuous and nowhere differentiable on $\mathbb{R}$. Prove that D^+F takes on every real value in every interval. In fact, for every $\alpha \in \mathbb{R}$, the set $\{x : D^+F(x) = \alpha\}$ has cardinality c in every interval.

(c) Let E be a set of real numbers with the property that, for every open interval I, $\lambda(I \cap E) > 0$ and $\lambda(I \setminus E) > 0$. Let $f = \chi_E$, and let $F(x) = \int_0^x f\,d\lambda$. Prove that, for every $\alpha \in [0, 1]$, $\{x : D^+F(x) = \alpha\}$ has cardinality c in every interval.

(d) Let F be the Cantor function and let I be any open interval containing points of the Cantor set. Prove that, for every $\alpha > 0$, the set $I \cap \{x : D_+F(x) = \alpha\}$ has cardinality c. [Hint: Apply Morse's theorem to $-F$.]

7:9.16 Prove Smítal's lemma. (This is also true in $\mathbb{R}^n$.)

Lemma (Smítal) *Let B, $D \subset \mathbb{R}$ so that B has positive outer Lebesgue measure and D is dense. Then*

$$\lambda^*((B + D) \cap (a, b)) = b - a$$

for any interval (a, b).

[Hint: Let $c < 1$, and choose $x_0 \in B$ and $\delta > 0$ so that

$$\lambda^*(B \cap [x_0 - h, x_0 + h]) > c\lambda^*([x_0 - h, x_0 + h])$$

for all $h < \delta$. Show that

$$\lambda^*((B + D) \cap [x - h, x + h]) > c\lambda^*([x - h, x + h])$$

for all $x \in D + x_0$ and $h < \delta$. Construct a Vitali cover of (a, b) from these intervals.]

Chapter 8

DIFFERENTIATION OF MEASURES

The differentiation theory of real functions can be extended to a theory of differentiation for measures that has many similar features and many intriguing problems. The first problem to address is how to find an appropriate way to differentiate a measure. In Section 8.1 we discuss an approach that is appropriate for Lebesgue–Stieltjes measures in $\mathbb{R}^n$. We develop this in Sections 8.2 to 8.5. Then in Section 8.6 we extend the method to abstract measure spaces.

Even for Lebesgue-Stieltjes measures in $\mathbb{R}^2$ it is not clear how to begin, and it is less clear which of the many possibilities is the correct one to pursue. Motivation for this is given in Section 8.1. We shall discuss differentiation in $\mathbb{R}^n$ based on cubes in Section 8.2, intervals in Section 8.4, and net structures in Section 8.5.

One of our main concerns is to reconsider the Radon-Nikodym theorem as a genuine differentiation theorem. We recall that we have defined a Radon–Nikodym derivative of a measure ν with respect to a measure μ, and have denoted it by $\frac{d\nu}{d\mu}$. This function was not, however, obtained by any process even remotely similar to a differentiation process. It may appear a bit of a fraud to label it as a derivative. This chapter will show how to resolve this problem. In particular, we find in Section 8.6 that $\frac{d\nu}{d\mu}$ can be viewed as a "genuine" derivative whenever the hypotheses of the Radon-Nikodym theorem (Theorem 5.29) are satisfied.

Our concern throughout is the differentiation of measures, and we do not touch upon differentiation of other types of set functions. Some references that deal with that subject appear in Section 8.7.

8.1 Differentiation of Lebesgue–Stieltjes Measures

It is not immediately clear how one might try to extend the familiar derivative of a real function of one real variable to more general structures. We can motivate an approach by reconsidering the ordinary derivative.

Let f be integrable on $[a, b]$, and let $F(x) = \int_a^x f\,d\lambda$. Then, because of Theorem 7.22,

$$F'(x) = f(x) \quad a.e. \tag{1}$$

We rewrite (1) in a way that suggests a route for generalization. Let $\nu = \int f\,d\lambda$. Then, for $x \in [a, b]$,

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f\,d\lambda = \frac{\nu([x, x+h])}{\lambda([x, x+h])}.$$

Expression (1) then takes the form

$$\lim_{h\to 0} \frac{\nu([x, x+h])}{\lambda([x, x+h])} = f(x) \quad a.e. \tag{2}$$

To this point we have been dealing with intervals that have x as an endpoint. We wish to be less restrictive by allowing any closed nondegenerate intervals that contain x. It is easy to verify (Exercise 8:1.1) that

$$\lim_{h\to 0+, k\to 0+, h+k>0} \frac{\nu[x-h, x+k]}{\lambda[x-h, x+k]} = f(x) \quad a.e. \tag{3}$$

Finally, we simplify the notation. We write

$$\lim_{I \Longrightarrow x} \frac{\nu(I)}{\lambda(I)} = f(x) \quad a.e. \tag{4}$$

The understanding of the symbol $I \Longrightarrow x$ (read "I contracts to x") is that I is an arbitrary closed interval, $x \in I$ and the diameters $\delta(I) \to 0$. [Here and elsewhere in this chapter, for any set $I \subset \mathbb{R}^n$, we write $\delta(I)$ to denote its diameter.]

When dealing with more general spaces $(X, \mathcal{M}, \mu)$, we seek a family $\mathcal{I}$ of sets of positive measure and a notion $\Longrightarrow$ of "contraction" of sets in $\mathcal{I}$ to points of X such that (4) is valid. This can often be done in many ways. A pair $(\mathcal{I}, \Longrightarrow)$, where $\mathcal{I}$ is a family of sets of positive measure and "$\Longrightarrow$" is a notion of contraction, is called a *differentiation basis*.

Consider first the case $X = \mathbb{R}^n$ with μ equal to Lebesgue measure. As an example of a differentiation basis, we take $\mathcal{I}$ to be the family of closed nondegenerate cubes having edges parallel to the coordinate axes in $\mathbb{R}^n$, and we write $I \Longrightarrow x$ if $x \in I$ and the diameters $\delta(I) \to 0$. This will provide a relatively simple theory of differentiation of Lebesgue–Stieltjes signed measures in $\mathbb{R}^n$. For simplicity, we shall usually denote n-dimensional Lebesgue measure by λ (instead of λ_n) and the class of measurable sets by $\mathcal{L}$. No confusion should arise from this practice, since the dimension will usually be fixed in any part of our development.

Let ν be a Lebesgue–Stieltjes signed measure on $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. Let $\{I_k\}$ be a sequence from $\mathcal{I}$ such that $I_k \Longrightarrow x$; that is, $x \in I_k$, for all $k \in \mathbb{N}$ and the diameters $\delta(I_k)$ tend to 0. If

$$\lim_{k\to\infty} \frac{\nu(I_k)}{\lambda(I_k)}$$

exists or is infinite, this limit is called an *ordinary derived number* of ν at x. The supremum of all ordinary derived numbers at x (taken over all sequences $\{I_k\}$ contracting to x) is called the *upper ordinary derivative of ν at x*, denoted as $\overline{D}\nu(x)$. The *lower ordinary derivative* $\underline{D}\nu(x)$ is defined similarly. Thus

$$\overline{D}\nu(x) = \sup \limsup_{k\to\infty} \frac{\nu(I_k)}{\lambda(I_k)}$$

and

$$\underline{D}\nu(x) = \inf \liminf_{k\to\infty} \frac{\nu(I_k)}{\lambda(I_k)},$$

the sup and inf being taken over all sequences $\{I_k\}$ contracting to x. If $\overline{D}\nu(x) = \underline{D}\nu(x)$ we say that ν has a derivative $D\nu(x)$. If $D\nu(x)$ is finite, we say that ν is *differentiable at x* or has an ordinary derivative there.

The following example illustrates the computations involved and will prove useful to us several times in this chapter.

Example 8.1 Let L be the line with equation $y = x$ in $\mathbb{R}^2$, and let $\nu(E) = \lambda_1(E \cap L)$, where λ_1 is one-dimensional Lebesgue measure on L. Let λ_2 denote two-dimensional Lebesgue measure in $\mathbb{R}^2$. Note that $\nu \perp \lambda_2$, since $\nu(\mathbb{R}^2 \setminus L) = 0$ and $\lambda_2(L) = 0$. Let $x \in L$. By choosing $\{I_k\} \subset \mathcal{I}$ such that $I_k \Longrightarrow x$ and x is the lower-right corner of I_k, we find that

$$\frac{\nu(I_k)}{\lambda_2(I_k)} = 0$$

for all $k \in \mathbb{N}$; thus $\underline{D}\nu(x) = 0$. If, instead, x is the lower-left corner of I_k, we find that

$$\frac{\nu(I_k)}{\lambda_2(I_k)} = \frac{\sqrt{2}S_k}{{S_k}^2},$$

where S_k is the side length of I_k, so $\overline{D}\nu(x) = \infty$. Thus $D\nu = 0$ on $\mathbb{R}^2 \setminus L$, and $\overline{D}\nu(x) = \infty > 0 = \underline{D}\nu(x)$ on L.

The cube basis and the ordinary derivative are not powerful enough to describe all ideas in multivariable differentiation. As an example, let us look at the details involved in computing mixed partial derivatives for functions in $\mathbb{R}^2$. We shall use this example as a basis for some applications of the differentiation theory proved in Section 8.4.

Example 8.2 In elementary calculus, one usually has enough regularity on a function $F:\mathbb{R}^2 \to \mathbb{R}$ to imply that

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y},$$

so that the order of computing mixed partials does not affect the outcome. (Sometimes, however, the order *does* matter: see Exercise 8:1.2.)

Let us try to interpret this as a derivative, in an appropriate sense, when F is an integral. Suppose that f is integrable on $S = [0,1] \times [0,1]$, and define F on S by

$$F(\xi, \eta) = \int_{[0,\xi]\times[0,\eta]} f \, d\lambda.$$

The function F determines a Lebesgue–Stieltjes measure ν on the Lebesgue measurable sets in S. For $I = [\xi, \xi + h] \times [\eta, \eta + k] \subset S$,

$$\nu(I) = F(\xi + h, \eta + k) - F(\xi, \eta + k) - F(\xi + h, \eta) + F(\xi, \eta).$$

Thus the quotient $\nu(I)/\lambda(I)$ can be written as

$$\frac{1}{k}\left[\frac{F(\xi + h, \eta + k) - F(\xi, \eta + k)}{h} - \frac{F(\xi + h, \eta) - F(\xi, \eta)}{h}\right] \tag{5}$$

or as

$$\frac{1}{h}\left[\frac{F(\xi + h, \eta + k) - F(\xi + h, \eta)}{k} - \frac{F(\xi, \eta + k) - F(\xi, \eta)}{k}\right]. \tag{6}$$

Suppose now that F possesses second partial derivatives in a neighborhood of $(\xi, \eta) \in S$. Letting first h and then k approach zero in (5), we obtain the mixed partial

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right).$$

On the other hand, letting first k and then h approach zero in (6), we obtain the other mixed partial

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right).$$

A stronger kind of limit that will express both of these computations and require them to be equal is to ask for the limit as h, $k \to 0$ together. We can express this as a derivative by letting $\mathcal{I}$ denote the family of *all* intervals in $\mathbb{R}^2$ and by requiring that "$I \Longrightarrow (\xi,\eta)$" mean $(\xi,\eta) \in I \in \mathcal{I}$ with diameters $\delta(I) \to 0$. If

$$\lim_{I \Longrightarrow (\xi,\eta)} \frac{\nu(I)}{\lambda(I)} = f(\xi,\eta)$$

for some $(\xi,\eta) \in \mathbb{R}^2$, then the *double* limit appearing in (5) or (6) exists and converges to $f(\xi,\eta)$. In that case

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$$

at (ξ,η).

This example suggests that we should investigate a stronger version of the derivative, one that uses arbitrary intervals rather than cubes. Let $\mathcal{I}$ denote the family of closed *intervals* in $\mathbb{R}^n$. Each element I of $\mathcal{I}$ is a Cartesian product of nondegenerate closed intervals in $\mathbb{R}^1$:

$$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

Let $x \in \mathbb{R}^n$. Write "$I \Longrightarrow x$" if $x \in I \in \mathcal{I}$ and the diameters $\delta(I) \to 0$. Let ν be a Lebesgue–Stieltjes signed measure on $\mathbb{R}^n$. If

$$\lim_{I \Longrightarrow x} \frac{\nu(I)}{\lambda(I)}$$

exists, we denote this limit by $D_s\nu(x)$ and call it the *strong derivative* of ν at x. When $D_s\nu$ does not exist at x, we can still define the *strong upper derivative* $\overline{D}_s\nu(x)$ and *strong lower derivative* $\underline{D}_s\nu(x)$ via lim sups and lim infs, as we just did for the ordinary derivative. We thus have a framework for studying strong differentiation of a measure, that is, a theory in which the family of intervals replaces the family of cubes.

There is an immediate relation between ordinary differentiation and strong differentiation. It is clear that the inequalities

$$\underline{D}_s\nu \le \underline{D}\nu \le \overline{D}\nu \le \overline{D}_s\nu$$

are valid at every point. They can be strict, as the following example shows.

Example 8.3 Let

$$A = \left\{(\xi,\eta) \in \mathbb{R}^2 : |\eta| \ge |\xi|\right\},$$

and let $\nu(E) = \lambda(E \cap A)$ for all $E \in \mathcal{L}$. Then

$$\underline{D}_s\nu(0) = 0 < \tfrac{1}{4} = \underline{D}\nu(0) < \overline{D}\nu(0) = \tfrac{3}{4} < 1 = \overline{D}_s\nu(0).$$

Exercises

8:1.1 Verify that (3) is valid at each x for which (2) is valid.

8:1.2 Let
$$f(x,y) = xy\left(\frac{x^2-y^2}{x^2+y^2}\right), \quad f(0,0)=0.$$
Show that
$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$
[Hint: Note that
$$\frac{\partial f}{\partial x}(0,y) = -y \text{ and } \frac{\partial f}{\partial y}(x,0) = x.$$

8:1.3 Let F be defined as in Example 8.2. One can show that $\partial F/\partial x$ and $\partial F/\partial y$ exist a.e. If $\partial F/\partial y$ fails to exist *everywhere* on a segment of the form $x_1 < x < x_2$, $y = y_0$, then one cannot even discuss
$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial x \partial y}$$
at some point (x_0, y_0) of that segment. Interpret the meaning of the statement
$$D_s\nu(x_0,y_0) = \frac{\partial^2 F}{\partial x \partial y}$$
at such a point.

8:1.4 Show that the lower and upper ordinary derivatives of a measure ν, $\underline{D}\nu$, and $\overline{D}\nu$ are themselves lower and upper ordinary derived numbers. Thus these derivatives are the minimum and maximum of the ordinary derived numbers. (The same is true for strong derivatives.)

8.2 The Cube Basis; Ordinary Differentiation

We begin by studying the ordinary derivative of a Lebesgue-Stieltjes measure in $\mathbb{R}^n$. Throughout this section, $\mathcal{I}$ will denote the family of all cubes in $\mathbb{R}^n$ having edges parallel to the coordinate axes, and we write $I \Longrightarrow x$ if $x \in I$ and the diameters $\delta(I) \to 0$. By λ we mean Lebesgue measure on $\mathbb{R}^n$.

Because the Vitali theorem is valid for the family $\mathcal{I}$, we can make use of the methods in Sections 7.1 to 7.8. We shall do this with some modifications and streamlining. We begin with a rephrasing of the definition of a Vitali cover and with the statement of the Vitali covering theorem in our present notation.

Definition 8.4 Let $\mathcal{V} \subset \mathcal{I}$, and let $E \subset \mathbb{R}^n$. If, for every $x \in E$, there exists a sequence $\{V_k\}$ from $\mathcal{V}$ such that $V_k \Longrightarrow x$, we say that $\mathcal{V}$ is a *Vitali cover* of E.

Theorem 8.5 *If $\mathcal{V}$ is a Vitali cover of a set $E \subset \mathbb{R}^n$, then there exists a sequence of sets $\{V_k\}$ from $\mathcal{V}$ such that*

1. $V_i \cap V_j = \emptyset$ *if* $i \neq j$.
2. $\lambda\left(E \setminus \bigcup_{k=1}^{\infty} V_k\right) = 0$.

Proof. Exercise 7:1.8 addressed the case when $n = 2$. The general case is similar; see the references at the end of this chapter for details if needed. ■

To begin our development, we obtain a lemma reminiscent of Lemma 7.4.

Lemma 8.6 *Let ν be a Lebesgue–Stieltjes measure on $\mathbb{R}^n$, and let $E \subset \mathbb{R}^n$. If, for all $x \in E$, $\overline{D}\nu(x) \geq q > 0$, then*

$$\nu^*(E) \geq q\lambda^*(E). \tag{7}$$

Proof. We establish (7) on the assumption that E is bounded, the extension to unbounded sets being left as Exercise 8:2.1.

Let $\varepsilon > 0$, and let $0 < q_0 < q$. Choose a bounded open set G such that $E \subset G$ and $\nu^*(E) > \nu(G) - \varepsilon$. Let

$$\mathcal{V} = \{V \in \mathcal{I} : V \subset G \ \text{ and } \ \nu(V) \geq q_0\lambda(V)\}.$$

Since, by hypothesis $\overline{D}\nu(x) \geq q > q_0$ for all $x \in E$, the family $\mathcal{V}$ forms a Vitali cover of E. By Theorem 8.5, there exists a pairwise disjoint sequence $\{V_k\}$ of sets from $\mathcal{V}$ such that

$$\lambda\left(E \setminus \bigcup_{k=1}^{\infty} V_k\right) = 0.$$

Thus

$$\nu^*(E) > \nu(G) - \varepsilon \geq \sum_{k=1}^{\infty} \nu(V_k) - \varepsilon \geq q_0 \sum_{k=1}^{\infty} \lambda(V_k) - \varepsilon \geq q_0\lambda^*(E) - \varepsilon.$$

We obtain (7) by letting $\varepsilon \to 0$ and $q_0 \to q$. ■

The reader may have observed that Lemma 8.6 provides an analog to Lemma 7.4. What about an analog for Lemma 7.1? For $n = 1$, we can provide an analog simply by rephrasing Lemma 7.1 in terms of the Lebesgue–Stieltjes measure μ_f. But for $n > 1$, such an analog is no longer available. This can be seen from the measure ν constructed in Example 8.1. Let $S = [0,1] \times [0,1]$ denote the unit square. We see that $\underline{D}\nu = 0$ on S. Thus, for $0 < p < \sqrt{2}$, $\underline{D}\nu < p$ on S, yet

$$\nu(S) = \sqrt{2} > p\lambda_2(S).$$

We can also use this example to see where an attempt to prove an analog of Lemma 7.1 along the lines of the proof of Lemma 8.6 would fail.

We could take $\mathcal{V} = \mathcal{I}$, select a pairwise disjoint sequence $\{V_k\}$ from $\mathcal{V}$ that covers almost all of S except $L \cap S$, and obtain $\nu(\bigcup_{k=1}^{\infty} V_k) < p\lambda_2(S)$. Now $\lambda_2(S \setminus \bigcup_{k=1}^{\infty} V_k) = 0$, but

$$\nu\left(S \setminus \bigcup_{k=1}^{\infty} V_k\right) = \nu(L) = \sqrt{2} \neq 0.$$

Observe that in one dimension a Lebesgue–Stieltjes measure ν for which $\underline{D}\nu < p$ on $[a, b]$ implies, by Theorem 7.20, that $\nu \ll \lambda$. Example 8.1 shows that this is not the case in higher dimensions. Nonetheless, we can use Lemma 8.6, together with some of the ideas in the proof that functions of bounded variation are differentiable a.e., to prove that Lebesgue–Stieltjes measures on $\mathbb{R}^n$ are differentiable a.e.

Theorem 8.7 *Let ν be a signed Lebesgue–Stieltjes measure on $\mathbb{R}^n$. Then ν is differentiable a.e.*

Proof. Because of the Jordan decomposition theorem (Theorem 2.22), we may assume that ν is a measure. Let

$$A = \left\{x \in \mathbb{R}^n : \overline{D}\nu(x) > \underline{D}\nu(x)\right\},$$

and for each pair (p, q) of rational numbers satisfying $0 < p < q$, let

$$A_{pq} = \left\{x : \underline{D}\nu(x) < p < q < \overline{D}\nu(x)\right\}.$$

Then $A = \bigcup_{p,q} A_{pq}$.

Suppose that $\lambda^*(A) > 0$. Then there must exist p and q such that $\lambda^*(A_{pq}) > 0$. Let B be a bounded subset of A_{pq} such that $\lambda^*(B) > 0$. Let $\varepsilon > 0$, and let G be a bounded open set such that $B \subset G$ and $\lambda(G) < \lambda^*(B) + \varepsilon$. Now let

$$\mathcal{V} = \{V \in \mathcal{I} : V \subset G \;\text{ and }\; \nu(V) \leq p\lambda(V)\}.$$

Then $\mathcal{V}$ is a Vitali cover for B. Thus there exists a pairwise disjoint sequence $\{V_k\}$ from $\mathcal{V}$ such that

$$\lambda\left(B \setminus \bigcup_{k=1}^{\infty} V_k\right) = 0,$$

so

$$\lambda^*\left(\bigcup_{k=1}^{\infty} (V_k \cap B)\right) = \lambda^*(B). \tag{8}$$

It follows that

$$\nu\left(\bigcup_{k=1}^{\infty} V_k\right) = \sum_{k=1}^{\infty} \nu(V_k) \leq p \sum_{k=1}^{\infty} \lambda(V_k) \leq p\lambda(G) < p(\lambda^*(B) + \varepsilon). \tag{9}$$

Now, since $B \subset A_{pq}$, we have $\overline{D}\nu(x) > q$ at each point of B. Applying Lemma 8.6 and noting (8), we obtain the inequalities

$$\nu\left(\bigcup_{k=1}^{\infty} V_k\right) \geq \nu^*\left(\bigcup_{k=1}^{\infty}(V_k \cap B)\right) \geq q\lambda^*\left(\bigcup_{k=1}^{\infty}(V_k \cap B)\right) = q\lambda^*(B). \quad (10)$$

Comparing (9) with (10), we find that

$$q\lambda^*(B) < p(\lambda^*(B) + \varepsilon). \quad (11)$$

The inequality (11) is valid for every $\varepsilon > 0$, since ε was not chosen until after p, q, and B had been determined. Thus $q\lambda^*(B) \leq p\lambda^*(B)$. Since $p < q$ and $\lambda^*(B) < \infty$, we conclude that $\lambda(B) = 0$. But this contradicts our choice of B.

We have shown that $\underline{D}\nu = \overline{D}\nu$ a.e. It remains to show that the set $A_\infty = \{x : D\nu(x) = \infty\}$ has measure zero. If $\lambda(A_\infty) > 0$, there exists a bounded set B such that $\lambda^*(B \cap A_\infty) > 0$. From Lemma 8.6, we infer that

$$\nu^*(B \cap A_\infty) \geq q\lambda^*(B \cap A_\infty)$$

for every $q \in \mathbb{N}$. But this would imply that $\nu^*(B \cap A_\infty) = \infty$, which is impossible, since a Lebesgue–Stieltjes outer measure is finite on bounded sets. ■

Lebesgue obtained Theorem 8.7 in a slightly more general form in 1910. We mention that the sets A and A_{pq} are actually measurable (see Exercise 8:2.3). Our proof could have been given using only measurable sets, but doing so would not have simplified matters.

In 1915, G. Fubini proved that if $\{F_k\}$ is a convergent series of nondecreasing functions on $[a, b]$ and $F = \sum_{k=1}^{\infty} F_k$, then

$$F' = \sum_{k=1}^{\infty} F_k' \quad \text{a.e.}$$

We next obtain the analog for Lebesgue–Stieltjes measures in $\mathbb{R}^n$. We shall use this result to obtain a version of the fundamental theorem of calculus for the ordinary derivative of integrals.

Theorem 8.8 *Suppose that $\{\nu_j\}$ is a monotone sequence of Lebesgue Stieltjes measures on $\mathbb{R}^n$ such that, for every $E \in \mathcal{L}$, $\nu(E) = \lim_{j\to\infty} \nu_j(E)$ is also a Lebesgue–Stieltjes measure. Then*

$$D\nu = \lim_{j\to\infty} D\nu_j \quad \textit{a.e.}$$

Proof. Assume without loss of generality that $\{\nu_j\}$ is nondecreasing. Let $\eta_j = \nu - \nu_j$. It suffices to show that the set

$$A = \left\{x : \lim_{j\to\infty} \overline{D}\eta_j(x) = 0 \text{ does not hold}\right\}$$

has measure zero. For $k \in \mathbb{N}$, let

$$A_k = \left\{ x : \lim_{j\to\infty} \overline{D}\eta_j(x) \geq \frac{1}{k} \right\}.$$

Then $A = \bigcup_{k=1}^{\infty} A_k$. Let B be a bounded subset of A_k. The sequence $\{\nu_j\}$ is nondecreasing by hypothesis, so the sequence $\{\eta_j\}$ is nonincreasing. Therefore, the sequence $\{\overline{D}\eta_j\}$ is also nonincreasing. From this it follows that $\overline{D}\eta_j \geq 1/k$ for all $j \in \mathbb{N}$ and all $x \in B \subset A_k$. Applying Lemma 8.6, we find that

$$k\eta_j^*(B) \geq \lambda^*(B)$$

for every $j \in \mathbb{N}$.

Let $K \in \mathcal{I}$, $K \supset B$. Then, for all $j \in \mathbb{N}$,

$$k\eta_j(K) \geq k\eta_j^*(B) \geq \lambda^*(B). \tag{12}$$

From (12) we infer that

$$k \lim_{j\to\infty} \eta_j(K) \geq \lambda^*(B).$$

But, from the definition of η_j, we infer that

$$k \lim_{j\to\infty} \eta_j(K) = k \lim_{j\to\infty} (\nu(K) - \nu_j(K)) = 0.$$

Thus $\lambda^*(B) = 0$.

We have shown that, for each $k \in \mathbb{N}$, every bounded subset of A_k is of measure zero. It follows that $\lambda(A_k) = 0$. Thus $\lambda(A) = 0$. From the definition of the set A, we see that

$$\lim_{j\to\infty} \overline{D}\eta_j = 0$$

holds a.e. ■

We can now state and prove half of the fundamental theorem of calculus for our present setting. Theorem 8.9 provides an analog to Theorem 7.22.

Theorem 8.9 *Let f be integrable on $\mathbb{R}^n$, and let*

$$\nu = \int f \, d\lambda.$$

Then $f = D\nu$ a.e.

Proof. As usual, we may assume that f is nonnegative. Let us suppose first that $f = \chi_A$, where $A \subset \mathbb{R}^n$ is measurable, and let $\nu(E) = \int_E \chi_A \, d\lambda$. We show that $D\nu = \chi_A$ a.e. Since computation of a derivative at a point $x \in \mathbb{R}^n$ involves only local behavior, we may assume that A is bounded. Let

$\{G_k\}$ be a descending sequence of open sets such that $A \subset H = \bigcap_{k=1}^{\infty} G_k$ and $\lambda(H) = \lambda(A)$. For each $k \in \mathbb{N}$, let

$$\nu_k = \int \chi_{G_k}\, d\lambda.$$

Then $\{\nu_k\}$ is a nonincreasing sequence of Lebesgue–Stieltjes measures on $\mathbb{R}^n$.

Now $\chi_{G_k} \to \chi_H$ everywhere and $\chi_H = \chi_A$ a.e., so

$$\int_E \chi_{G_k}\, d\lambda \to \int_E \chi_A\, d\lambda$$

for every bounded measurable set E; that is, $\lim_{k\to\infty} \nu_k = \nu$. It follows from Theorem 8.8 that $D\nu = \lim_{k\to\infty} D\nu_k = 1$ a.e. on A. A similar argument shows that $D\nu = 0$ a.e. on $\widetilde{A}$, and we have $D\nu = \chi_A$ a.e., as required.

It follows easily now that the result of the theorem is valid for integrable simple functions. For an arbitrary nonnegative integrable function f, let $\{f_k\}$ be a nondecreasing sequence of simple functions converging pointwise to f, and let $\nu_k = \int f_k\, d\lambda$. Then $\nu = \lim_{k\to\infty} \nu_k$. An application of Theorem 8.8 results in the equalities

$$D\nu = \lim_{k\to\infty} D\nu_k = \lim_{k\to\infty} f_k = f \text{ a.e.},$$

as required. ■

In Section 5.8 we defined the Radon–Nikodym derivative of ν as that function f such that $\nu = \int f\, d\mu$. We used the notation $f = \frac{d\nu}{d\mu}$ and provided some explanation for the notation. We can now see that the notation is indeed appropriate, at least in the setting of this section. If ν is a Lebesgue–Stieltjes signed measure on $\mathbb{R}^n$ and $\nu \ll \lambda$, then the Radon–Nikodym derivative $\frac{d\nu}{d\lambda}$ is the ordinary derivative $D\nu$. That is,

$$\frac{d\nu}{d\lambda} = \lim_{I \Longrightarrow x} \frac{\nu(I)}{\lambda(I)} \quad \text{a.e. on } \mathbb{R}^n.$$

Exercises

8:2.1 Verify that Lemma 8.6 is valid for unbounded sets $E \subset \mathbb{R}^n$.

8:2.2 Prove that an arbitrary union of nondegenerate closed cubes in $\mathbb{R}^n$ for $n \geq 2$ is Lebesgue measurable, but not necessarily Borel measurable. [Hint: Use the Vitali covering theorem for the first statement. For the second statement, consider a subset S of the line $y = x$ that is not a Borel subset of $\mathbb{R}^2$. Show that a linear set is a Borel set when viewed as a subset of the line if and only if it is a Borel set when considered as a subset of the plane.]

8:2.3 Let ν be a signed measure on $\mathbb{R}^n$. Prove that $\overline{D}\nu$ and $\underline{D}\nu$ are Lebesgue measurable functions. [Hint: For $\alpha \in \mathbb{R}$, let

$$A_{jk} = \bigcup \left\{ I \in \mathcal{I} : \delta(I) \leq 1/k \text{ and } \frac{\nu(I)}{\lambda(I)} > \alpha + 1/j \right\}.$$

Show that

$$\left\{ x \in \mathbb{R}^n : \overline{D}\nu(x) > \alpha \right\} = \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} A_{jk}.$$

Use Exercise 8:2.2.]

8.3 The Lebesgue Decomposition Theorem

As an application of our methods we now show that the ordinary derivative allows a version of the Lebesgue decomposition theorem in $\mathbb{R}^n$ and clarifies the nature of Lebesgue-Stieltjes measures that are singular or absolutely continuous with respect to Lebesgue measure. This is similar to the one-dimensional theory.

Recall that the Cantor function F is *singular* because F is nondecreasing and $F' = 0$ a.e. On the other hand, the Cantor measure μ_F and Lebesgue measure λ are *mutually singular*, $\mu_F \perp \lambda$, because μ_F and λ are concentrated on disjoint sets. Theorem 8.10 relates singularity of a measure to ordinary differentiation of the measure.

Theorem 8.10 *Let ν be a Lebesgue–Stieltjes signed measure on $\mathbb{R}^n$. Then $\nu \perp \lambda$ if and only if $D\nu = 0$ a.e.*

Proof. We may assume that $\nu \geq 0$. Suppose first that $\nu \perp \lambda$. By definition there exist Borel sets A and B such that $\mathbb{R}^n = A \cup B$, $A \cap B = \emptyset$, $\lambda(B) = 0$ and $\nu(A) = 0$. For $k \in \mathbb{N}$, let

$$P_k = \{x \in A : D\nu(x) \geq 1/k\}.$$

Then $0 = \nu(A) = \nu(P_k) \geq \lambda^*(P_k)/k$, the inequality following from Lemma 8.6. Let $P = \bigcup_{k=1}^{\infty} P_k$. Then $\lambda(P) = 0$. Now

$$\{x : D\nu(x) > 0\} \subset P \cup B.$$

Since $\lambda(P) = 0$ and $\lambda(B) = 0$, we conclude that $D\nu = 0$ a.e.

Conversely, suppose that $D\nu = 0$ a.e. By Theorem 5.34, there exist measures α and β such that $\alpha \ll \lambda$, $\beta \perp \lambda$, and $\nu = \alpha + \beta$. It follows from Theorem 8.9 that $\alpha = \int D\alpha \, d\lambda$. Since $\alpha = \nu - \beta$, we have $D\alpha = D\nu - D\beta$. Since $\beta \perp \lambda$, it follows, from the first paragraph of this proof, that $D\beta = 0$ a.e. But $D\nu = 0$ a.e. by hypothesis, and so $D\alpha = 0$ a.e., from which we obtain $\alpha = \int D\alpha \, d\lambda = 0$. We have shown that $\nu = \alpha + \beta = \beta$, so $\nu \perp \lambda$ as required. ■

We can now obtain a form of the Lebesgue decomposition theorem that displays derivatives explicitly.

Theorem 8.11 *Let ν be a signed Lebesgue–Stieltjes measure on $\mathbb{R}^n$. Then, for all bounded Borel sets E,*

$$\nu(E) = \int_E D\nu\, d\lambda + \beta(E),$$

where β is a signed Lebesgue–Stieltjes measure on $\mathbb{R}^n$ for which $D\beta = 0$ a.e.

Proof. Again, we may assume that $\nu \geq 0$. By Theorem 5.34, there exist Lebesgue–Stieltjes measures α and β such that $\alpha \ll \lambda$, $\beta \perp \lambda$, and $\nu = \alpha + \beta$. By Theorem 8.9,

$$\alpha = \int D\alpha\, d\lambda.$$

Now $D\nu = D\alpha + D\beta$ a.e. By Theorem 8.10, $D\beta = 0$ a.e. Thus $D\nu = D\alpha$ a.e., so that $\alpha = \int D\nu\, d\lambda$ and

$$\nu = \int D\nu\, d\lambda + \beta,$$

as required. ■

As an immediate corollary, we obtain the other half of the fundamental theorem of calculus. Corollary 8.12 extends Theorem 7.19 to $\mathbb{R}^n$.

Corollary 8.12 *A Lebesgue–Stieltjes signed measure ν is absolutely continuous with respect to λ if and only if*

$$\nu(E) = \int_E D\nu\, d\lambda$$

for all bounded measurable sets E.

Proof. See Exercise 8:3.3. ■

We have seen that most of the results in Section 7.5 involving μ_f carry over to $\mathbb{R}^n$. A notable exception is de la Vallée Poussin's result Theorem 7.20. Example 8.1 shows that no such theorem is available in the setting of this section. In Section 8.5, we provide a setting in which an analog of Theorem 7.20 is valid.

Exercises

8:3.1 Show that the analog of Theorem 7.20 is not valid in dimensions greater than 1 when $\mathcal{I}$ and $\Longrightarrow$ have the meanings given in this section. (In Section 8.5, we provide a setting in which that analog is available.)

8:3.2◇ Let $\{P_n\}$ be a sequence of pairwise disjoint Cantor sets of measure zero in [0,1] with $\bigcup_{n=1}^{\infty} P_n$ dense in [0,1]. For each $n \in \mathbb{N}$, let F_n be a Cantor-like function that maps P_n onto $[0, 2^{-n}]$, let $G_n = \sum_{k=1}^{n} F_n$, and let $\nu_n = \mu_{G_n}$.

(a) Show that $\{\nu_n\}$ forms a nondecreasing sequence of Lebesgue–Stieltjes measures.

(b) Show that $\nu = \lim_{n\to\infty} \nu_n$ is a nonatomic Lebesgue–Stieltjes measure by showing that $\nu = \mu_F$, where $F = \sum_{n=1}^{\infty} F_n$.

(c) Show that $\nu(I) > 0$ for every open interval $I \subset [0, 1]$.

(d) Show that F is strictly increasing and continuous on [0,1].

(e) Show that $\nu \perp \lambda$.

(f) Show that $F' = 0$ a.e.

Thus F is a continuous *strictly* increasing singular function.

8:3.3 (a) Show that the conclusion of Theorem 8.11 does not hold for every bounded Lebesgue measurable set . [Hint: Let F be the Cantor function, and let $\nu = \mu_F$. Show that the Cantor set has a subset E that is not ν-measurable.]

(b) Prove Corollary 8.12. [Hint: Prove that $\mu_f \ll \lambda$ if and only if f is continuous and every λ-measurable set is ν-measurable.]

8.4 The Interval Basis; Strong Differentiation

We turn now to a study of the strong derivative of a Lebesgue-Stieltjes measure in $\mathbb{R}^n$. Throughout this section $\mathcal{I}$ denotes the family of all intervals in $\mathbb{R}^n$; that is, rectangles having edges parallel to the coordinate axes. We write $I \Longrightarrow x$ if $x \in I$ and the diameters $\delta(I) \to 0$. Again, λ is Lebesgue measure in $\mathbb{R}^n$.

A difficulty in dealing with strong differentiation is that the family $\mathcal{I}$ of intervals does not have the Vitali covering property; that is, the Vitali covering theorem is not valid[1] for this family $\mathcal{I}$. This means that the methods of the preceding sections that worked for the ordinary derivative are not available here to apply to the strong derivative. Indeed, it turns out that we cannot always assert that if $\nu = \int f\,d\lambda$ then $D_s\nu = f$ a.e. We can, however, prove that if f is *bounded* then $D_s\nu = f$ a.e.

The tool needed is the analog of Lebesgue's density theorem, which we now prove is valid in any dimension. Note that this theorem is already proved to be true for the weaker notion of ordinary convergence using cubes (it was the first step in the proof of Theorem 8.9). Here we must prove it for strong convergence using intervals.

Theorem 8.13 *Let A be a measurable subset of $\mathbb{R}^n$, and let $\mathcal{I}$ be the family of intervals in $\mathbb{R}^n$. Then*

$$\lim_{I\Longrightarrow x} \frac{\lambda(I\cap A)}{\lambda(I)} = \begin{cases} 1, & \textit{a.e. on } A; \\ 0, & \textit{a.e. on } \widetilde{A}. \end{cases}$$

[1] This is proved, for example, in M. de Guzmán, *Differentiation of Integrals in $\mathbb{R}^n$*, Lecture Notes in Mathematics, vol. 481, Springer, Berlin (1975).

Proof. For simplicity of notation, we present the proof for sets in $\mathbb{R}^2$. We use λ_2 for Lebesgue's two-dimensional measure and λ_1 for one-dimensional measure. Using Theorem 3.12, one verifies easily that we may assume that A is closed and bounded. We leave this verification as Exercise 8:4.1.

The proof continues in two steps. We first obtain certain one-dimensional density estimates. We then apply the pre-Fubini theorem (Theorem 6.5) to obtain the desired two-dimensional density estimate.

For $S \subset \mathbb{R}^2$ and $\eta \in \mathbb{R}$, let $S^{[\eta]} = \{x : (x, \eta) \in S\}$. Let $\varepsilon > 0$. For $n \in \mathbb{N}$, let E_n denote the set of points $(\xi, \eta) \in A$ for which

$$\lambda_1(A^{[\eta]} \cap I) \geq (1 - \varepsilon)\lambda_1(I)$$

whenever I is a linear interval containing ξ and $\lambda_1(I) \leq 1/n$.

The sequence E_n is an expanding sequence of sets on each of which a certain one-dimensional density estimate is satisfied. Let $N = A \setminus \lim_{n\to\infty} E_n$. We show that $\lambda_2(N) = 0$. To verify this, observe first that, if $\xi \in N^{[\eta]}$, then for each $n \in \mathbb{N}$ there exists a linear interval I such that $\xi \in I$, $\lambda_1(I) < 1/n$ and

$$|N^{[\eta]} \cap I| < |A^{[\eta]} \cap I| < (1 - \varepsilon)\lambda_1(I).$$

From the one-dimensional Lebesgue density theorem (Theorem 7.33), it follows that

$$\lambda_1(N^{[\eta]}) = 0 \text{ for all } \eta \in \mathbb{R}. \tag{13}$$

In order to apply Theorem 6.5 and thereby claim that $\lambda_2(N) = 0$, we must show that N is measurable. To do this, we note that each of the sets E_n is closed. To see this, fix $n \in \mathbb{N}$ and let $\{(\xi_k, \eta_k)\}$ be a sequence of points in E_n converging to $\{\xi_0, \eta_0\}$. Let I be a linear interval containing I_0 in its interior with $\lambda_1(I) < 1/n$. For k sufficiently large, $\xi_k \in I$, so

$$\lambda_1(A^{[\eta_k]} \cap I) \geq (1 - \varepsilon)\lambda_1(I).$$

But A is closed, so

$$A^{[\eta_0]} \supset \limsup_{k\to\infty} A^{[\eta_k]}.$$

Thus

$$\lambda_1(A^{[\eta_0]} \cap I) \geq \limsup_{k\to\infty} \lambda_1(A^{[\eta_k]} \cap I) \geq (1 - \varepsilon)\lambda_1(I).$$

Letting $I \to I_0$, we find that

$$\lambda_1(A^{[\eta_0]} \cap I_0) \geq (1 - \varepsilon)\lambda_1(I_0),$$

so $(\xi_0, \eta_0) \in E_n$ and E_n is closed. It follows that $N = A \setminus \lim_{n\to\infty} E_n$ is measurable. We can now apply Theorem 6.5 and, noting (13), conclude that $\lambda_2(N) = 0$.

From this it follows that the sequence $\lambda_2(A \setminus E_n) \to 0$. Consequently, for each $\varepsilon > 0$ there exists $\sigma > 0$ and a closed set $E \subset A$ such that $\lambda_2(A \setminus E) < \varepsilon$ and such that

$$\lambda_1(\{x : (x, \eta) \in A \text{ and } a \leq x \leq b\}) \geq (1 - \varepsilon)(b - a) \tag{14}$$

whenever $(\xi, \eta) \in E$, $a \leq \xi \leq b$, and $b - a < \sigma$.

Interchanging the roles of x and y and applying the above argument to E, we obtain $\tau > 0$ and a closed set $F \subset E$ such that $\tau < \sigma$, $\lambda_2(E \setminus F) < \varepsilon$, and

$$\lambda_1(\{y : (\xi, y) \in E \text{ and } a \leq y \leq b\}) \geq (1 - \varepsilon)(b - a) \tag{15}$$

whenever $(\xi, \eta) \in F$, $a \leq \eta \leq b$ and $b - a < \tau$.

On the set F, we have one-dimensional density estimates in both directions. We now apply Theorem 6.5 once again to obtain a two-dimensional density estimate.

Let $(\xi_0, \eta_0) \in F$. Let $J = [a_1, b_1] \times [a_2, b_2]$ be any interval in $\mathbb{R}^2$ having diameter less than τ and containing (ξ_0, η_0). From Theorem 6.5 we infer that

$$\lambda_2(A \cap J) = \int_{a_2}^{b_2} \lambda_1(\{x : (x, y) \in A, a_1 \leq x \leq b_1\})\, dy.$$

It follows from (15) and (14) that

$$\lambda_2(A \cap J) \geq (1 - \varepsilon)(b_2 - a_2)(b_1 - a_1) = (1 - \varepsilon)\lambda_2(J).$$

From this it now follows that

$$\lim_{J \Longrightarrow (\xi_0, \eta_0)} \inf \frac{\lambda_2(A \cap J)}{\lambda_2(J)} \geq (1 - \varepsilon)$$

for all $(\xi_0, \eta_0) \in F$. But $\lambda_2(A \setminus F) \leq 2\varepsilon$ and ε is arbitrary. We can thus conclude that, for almost every point (ξ_0, η_0) in A,

$$\lim_{J \Longrightarrow (\xi_0, \eta_0)} \frac{\lambda_2(A \cap J)}{\lambda_2(J)} = 1.$$

Thus almost every point of A is a point of density of A. It is now clear that almost every point of $\widetilde{A}$ is a point of dispersion of A. ■

As before, if

$$\lim_{I \Longrightarrow x} \frac{\lambda(I \cap A)}{\lambda(I)} = 1,$$

we call x a *density point* of A. Theorem 8.13 thus states that almost all points of a measurable set A are density points of A. We shall obtain analogs to Theorems 7.36 and 7.37 with the help of Theorem 8.13. We then use these theorems to obtain an analog to Theorem 7.22 for *bounded* measurable functions.

As in Section 7.8, we say a function f is *approximately continuous* at $x_0 \in \mathbb{R}^n$ if there exists a measurable set E that contains x_0 and has x_0 as a density point and such that $f|E$ is continuous at x_0.

Theorem 8.14 *A measurable, finite a.e. function is approximately continuous a.e.*

Proof. Because of Theorem 8.13, the proof of Theorem 8.14 is identical to that of Theorem 7.37. ■

Theorem 8.15 *Let f be a bounded integrable function on $\mathbb{R}^n$, and let $\nu = \int f\,d\lambda$. Then $D_s\nu(x) = f(x)$ at each point of approximate continuity of f. In particular, $D_s\nu = f$ a.e.*

Proof. Let x_0 be a point of approximate continuity of f. Let E be a measurable set having x_0 as a density point such that $f|E$ is continuous at x_0. Without loss of generality, assume that $f(x_0) = 0$. Let $\varepsilon > 0$. There exists $\gamma > 0$ such that if $x_0 \in I \in \mathcal{I}$ and $\delta(I) < \gamma$ then (i) $\lambda(I \cap \widetilde{E}) < \varepsilon\lambda(I)$ and (ii) $|f(x)| < \varepsilon$ for each $x \in I \cap E$. Let M be an upper bound for $|f|$. Let $x_0 \in I \in \mathcal{I}$ with $\delta(I) < \gamma$. Then, from (i) and (ii), we infer that

$$\begin{aligned} |\nu(I)| &\leq |\nu(I \cap \widetilde{E})| + |\nu(I \cap E)| \\ &\leq M\varepsilon\lambda(I) + \varepsilon\lambda(I) = \varepsilon(M+1)\lambda(I). \end{aligned}$$

Thus

$$\frac{|\nu(I)|}{\lambda(I)} \leq \varepsilon(M+1).$$

It now follows that $D_s\nu(x_0) = 0 = f(x_0)$. ■

Theorem 8.15 sheds some light on Example 8.2. Let f be a bounded measurable function on the square S, and let $\nu = \int f\,d\lambda$. Then

$$D_s\nu = f \quad \text{a.e.} \tag{16}$$

Recalling our discussion in Example 8.2, we find that

$$D_s\nu = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$$

wherever $D_s\nu$ exists. We thus see from (16) that

$$f = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} \quad \text{a.e.}$$

We summarize with a theorem.

Theorem 8.16 *Let f be a bounded measurable function defined on the square $S = [0,1] \times [0,1]$, and let*

$$F(\xi, \eta) = \int_{[0,\xi]\times[0,\eta]} f\,d\lambda.$$

If F has first partials on S, then a.e. on S the second mixed partials

$$\frac{\partial^2 F}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 F}{\partial x \partial y}$$

exist and are equal. Furthermore, they are equal to f at each point of approximate continuity of f.

A version of the other half of the fundamental theorem of calculus is also available.

Theorem 8.17 *Let ν be a Lebesgue–Stieltjes signed measure on $I\!R^n$. If there exists a number $M > 0$ such that $|\nu(I)| \le M\lambda(I)$ for all intervals $I \subset I\!R^n$, then*

$$\nu(E) = \int_E D_s\nu \, d\lambda$$

for all $E \in \mathcal{L}$.

Proof. We show that $\nu \ll \lambda$. To see this, let $E \in \mathcal{L}$ with $\lambda(E) = 0$. We need to prove that $\nu(E) = 0$. Let $\varepsilon > 0$, and let $\{I_k\}$ be a sequence of intervals whose interiors cover E and such that

$$\sum_{k=1}^{\infty} \lambda(I_k) < \lambda(E) + \varepsilon.$$

Then

$$\begin{aligned} |\nu(E)| &\le \left| \nu\left(\bigcup_{k=1}^{\infty} I_k \right) \right| \le \sum_{k=1}^{\infty} |\nu(I_k)| \\ &\le M \sum_{k=1}^{\infty} \lambda(I_k) < M(\lambda(E) + \varepsilon). \end{aligned}$$

Since ε is arbitrary, $\nu(E) = 0$. Thus $\nu \ll \lambda$ and, consequently, there exists $f \in L_1$ such that $\nu = \int f \, d\lambda$.

We may apply Theorem 8.15, provided we show that f is bounded off a set of measure zero. We verify that $|f| \le M$ a.e. It is enough to show that the set $A = \{x : f(x) > M\}$ has measure zero, since a similar argument applies to the $\{x : f(x) < -M\}$. If $\lambda(A) > 0$, then, by Theorem 8.9, $D\nu(x) > M$ a.e. on A. But since $\overline{D}_s\nu \ge D\nu$, this implies the existence of a point $x \in A$ such that $\overline{D}_s\nu(x) > M$. In view of the assumed inequality $|\nu(I)| \le M\lambda(I)$, this is impossible. Thus $\lambda(A) = 0$. By redefining f on a set of measure zero if necessary, we can take $|f| \le M$ everywhere.

It now follows from Theorem 8.15 that $D_s\nu = f$ a.e. Thus

$$\nu = \int D_s\nu \, d\lambda$$

as required. ■

We conclude this section with several remarks offering the reader further insight into some aspects of these ideas.

Remark 1. We can compare Theorem 8.17 with Corollary 8.12. In the latter, we assumed only that $\nu \ll \lambda$ and were able to conclude that

$\nu = \int D\nu\, d\lambda$. For Theorem 8.17, we assumed more and obtained the stronger conclusion $\nu = \int D_s\nu\, d\lambda$. In other language, Corollary 8.12 required only that $D\nu \in L_1$, while our hypothesis in Theorem 8.17 required $D_s\nu$ to be bounded. It is Theorem 8.17 that applies to Example 8.2. Under appropriate hypotheses on F, we obtain the conclusion

$$F(\xi, \eta) = \int_{[0,\xi]\times[0,\eta]} \frac{\partial^2 F}{\partial y \partial x}\, d\lambda = \int_{[0,\xi]\times[0,\eta]} \frac{\partial^2 F}{\partial x \partial y}\, d\lambda.$$

Remark 2. Observe that the inequality $|\nu(I)| \leq M\lambda(I)$ of Theorem 8.17 is reminiscent of a Lipschitz condition. The analogy with a Lipschitz condition can be reinforced. Note that the intervals I_k that appear in the proof of Theorem 8.17 need not be pairwise disjoint. Compare this with Exercises 5:7.4 and 5:7.9.

Remark 3. If we strengthen the requirements on differentiability, we might expect to obtain fewer theorems related to the fundamental theorem of calculus. We saw this when we passed from the system of cubes to the system of intervals. What happens if, for example, we let $\mathcal{I}$ consist of *all* nondegenerate closed rectangles in $\mathbb{R}^2$? (In contrast to intervals, a rectangle need not have sides parallel to the coordinate axes in $\mathbb{R}^2$.) In that setting it is no longer true that an analog of the Lebesgue density theorem is available. In fact, there exists a closed set $K \subset \mathbb{R}^2$ such that, for

$$\nu = \int \chi_K\, d\lambda,$$

the equality

$$\lim_{I \Longrightarrow x} \frac{\nu(I)}{\lambda(I)} = \chi_K(x)$$

fails a.e. on K. See Exercise 8:4.2. No pleasing theory of differentiation is possible[2] with this choice of $\mathcal{I}$.

In the other direction, weakening the requirements for differentiability can produce additional results. Suppose, for example, that we let $\mathcal{I}$ consist of the nondegenerate closed disks in $\mathbb{R}^2$ and write $I \Longrightarrow x$ if $I \in \mathcal{I}$, $\delta(I) \to 0$ *and* x is the *center* of I. In that case, a version of de la Vallée Poussin's theorem (Theorem 7.20) is available. Denoting the resulting derivative by $D_{\mathrm{sym}}\nu$, we obtain, for a Lebesgue–Stieltjes signed measure, the identity

$$\nu(E) = \int_E D_{\mathrm{sym}}\nu\, d\lambda + \nu(E \cap S_\infty), \tag{17}$$

where S_∞ consists of those points at which ν has an infinite symmetric derived number. Consider Example 8.1 once again. Here $D_{\mathrm{sym}}\nu = \infty$ on

[2] See M. de Guzmán, *Differentiation of Integrals in* $\mathbb{R}^n$, Lecture Notes in Mathematics, vol. 481, Springer, Berlin (1975), for a discussion of differentiation with respect to this system.

L and $D_{\text{sym}}\nu = 0$ on $\widetilde{L}$, so (17) clearly applies. We shall not prove (17). Instead, we shall study another less restrictive form of differentiation in Section 8.5. We shall prove a version of de la Vallée Poussin's theorem in that setting.

Remark 4. Applications and interpretation of the type of differentiation theory that we developed in Section 8.2 and are discussing in this section are plentiful.[3] The family of cubes or intervals in $\mathbb{R}^n$ can be replaced with other families of sets, and the notion $\Longrightarrow$ of contraction can vary. We mention some examples.

A number of important concepts in vector analysis can be viewed as derivatives. This is true of the concepts of *circulation, curl* and *divergence.* The same is true of the Jacobian of a differentiable transformation T defined on an open subset of $\mathbb{R}^n$. The *Jacobian* $J_T(x)$ of T at x is usually defined as a determinant involving partial derivatives. One can show that

$$|J_T(x)| = \lim_{I \Longrightarrow x} \frac{\lambda(T(I))}{\lambda(I)} = D\nu(x),$$

where $\nu(E) = \lambda(T(E))$.

Here is a quick heuristic treatment in $\mathbb{R}^2$. Suppose that I is a square in $\mathbb{R}^2$ with sides parallel to the coordinate axes, and let $T = (f, g)$ be a continuously differentiable surjection of I onto a set S. By use of line integrals, one verifies in elementary calculus that

$$\nu(I) = \lambda(T(I)) = \lambda(S) = \int_I \left| \frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial u} \right| d\lambda = \int_I |J_T|\, d\lambda.$$

Thus, by Theorem 8.9,

$$D\nu(x) = \lim_{I \Longrightarrow x} \frac{\nu(I)}{\lambda(I)} = |J_T| \ \text{ a.e.}$$

The Jacobian applies to "change of variable" theorems. For example, if T is a differentiable homeomorphism mapping a bounded open set $V \subset \mathbb{R}^n$ onto another bounded open set $W \subset \mathbb{R}^n$, then, for each integrable function f on W,

$$\int_W f\, d\lambda = \int_V (f \circ T)|J_T|\, d\lambda = \int_V (f \circ T) \left| \frac{d\nu}{d\lambda} \right| d\lambda,$$

where $\nu(E) = \lambda(T(E))$ for every measurable set $E \subset V$.

Exercises

8:4.1 In the proof of Theorem 8.13, show that Theorem 3.12 can be used to reduce the argument to the case where the set A is closed.

[3] See A. M. Bruckner, "Differentiation of Integrals," *Amer. Math. Monthly* **78** (1971), no. 9, Part II.

8:4.2 In 1927, Nikodym gave an example of a closed set $S \subset \mathbb{R}^2$ of positive Lebesgue measure such that to almost every $x \in S$ corresponds a line segment L_x such that $S \cap L_x = \{x\}$. That is, almost all points of S are linearly accessible from $\widetilde{S}$. Use this to show that the family $\mathcal{R}$ of closed nondegenerate rectangles in $\mathbb{R}^2$ does not have Lebesgue's density property. Here "$I \Longrightarrow x$" means $I \in \mathcal{R}$, $x \in I$, $\delta(I) \to 0$. Show that, for almost all $x \in S$,

$$\lim_{I \Longrightarrow x} \inf \frac{\nu(I)}{\lambda(I)} = 0,$$

where $\nu = \int \chi_S \, d\lambda$.

8.5 Net Structures

In Section 7.5 we discussed relationships holding between integrals and derivatives in the one-dimensional setting. We saw in Section 8.2 that much of our development carried over to n dimensions if we used cubes for the family $\mathcal{I}$ in our differentiation basis. No analog of de la Vallée Poussin's theorem 7.20 was available, however, as Example 8.1 showed. Then, in Section 8.4, we discussed the differentiation basis of intervals in $\mathbb{R}^n$. We found that some theorems of Section 8.2 were no longer valid without additional assumptions. The class of intervals in $\mathbb{R}^n$ $(n > 1)$ is larger than the class of cubes. This made it sufficiently more difficult for $D_s\nu$ to exist that even the analog of Theorem 8.9 required a stronger hypothesis than absolute continuity of ν with respect to λ.

In this section we study a certain type of differentiation basis called a *net structure*. Here, the requirements for differentiability of a measure are less demanding. We shall see that an analog of de la Vallée Poussin's theorem is available in this setting. We present a development in $\mathbb{R}^n$, but mention that virtually the same development is possible in any σ-finite measure space $(X, \mathcal{M}, \mu)$ for which X is a separable metric space.

We begin with an example of a net structure in $\mathbb{R}^2$. Partition $\mathbb{R}^2$ into half-open squares of side length 1, and denote the resulting family by $\mathcal{I}_1$. Now partition each member I of $\mathcal{I}_1$ into four congruent half-open squares of side length $\frac{1}{2}$, and let $\mathcal{I}_2$ be the resulting family of squares. Continue the process, obtaining a sequence $\{\mathcal{I}_k\}$ of partitions of $\mathbb{R}^2$ into half-open squares. Each family $\mathcal{I}_k$ is called a *net*, and the sequence $\{\mathcal{I}_k\}$ is called a *net structure*. The members of $\mathcal{I}_k$ are called *cells*.

We list the important features of this net structure.

8.18 (Net structure features)

1. *Each family $\mathcal{I}_k$ consists of Borel sets of finite positive measure and partitions $\mathbb{R}^n$ (here $n = 2$).*
2. *Each family $\mathcal{I}_{k+1}$ refines $\mathcal{I}_k$: that is, if $I \in \mathcal{I}_{k+1}$, then there exists $J \in \mathcal{I}_k$ such that $I \subset J$.*

3. *Let* $\delta_k = \sup\{\delta(I) : I \in \mathcal{I}_k\}$. *Then* $\lim_{k\to\infty} \delta_k = 0$.

We use the three assertions of 8.18 to define nets and net structures in $\mathbb{R}^n$. Thus a *net* is any family $\mathcal{I}_k$ satisfying the first condition. A *net structure* is a sequence $\{\mathcal{I}_k\}$ of nets that satisfies conditions (2) and (3). A member of $\mathcal{I}_k$ is called a *cell* of $\mathcal{I}_k$.

In order to discuss differentiation with respect to a net structure, we need to determine a family $\mathcal{I}$ of sets and a notion $\Longrightarrow$ of contraction. For $\mathcal{I}$, we simply take

$$\{I : \text{There exists } k \in \mathbb{N} \text{ such that } I \in \mathcal{I}_k\}.$$

For contraction, we note that for all $x \in \mathbb{R}^n$ and every $k \in \mathbb{N}$ there is a unique $I_k \in \mathcal{I}_k$ such that $x \in I_k$. This follows from condition (1) of the net structure. From conditions (2) and (3), we see that the resulting sequence $\{I_k\}$ is a decreasing sequence whose intersection is $\{x\}$. We shall write $I \Longrightarrow x$ or $I_k \Longrightarrow x$ to indicate that the sequence contracts to x. We call the resulting differentiation basis $(\mathcal{I}, \Longrightarrow)$ the basis associated with the net structure $\{\mathcal{I}_k\}$. As before, we define upper and lower derivatives of a Lebesgue–Stieltjes measure ν on $\mathbb{R}^n$ by

$$\overline{D}_{\mathcal{I}}\nu(x) = \limsup_{I \Longrightarrow x} \frac{\nu(I)}{\lambda(I)} \text{ and } \underline{D}_{\mathcal{I}}\nu(x) = \liminf_{I \Longrightarrow x} \frac{\nu(I)}{\lambda(I)} \tag{18}$$

and write $D_{\mathcal{I}}\nu(x)$ if $\overline{D}_{\mathcal{I}}\nu(x) = \underline{D}_{\mathcal{I}}\nu(x)$. When $D_{\mathcal{I}}\nu(x)$ is finite, we say ν is differentiable at x.

Lemma 8.19 *Let* ν *be a Lebesgue–Stieltjes signed measure on* $\mathbb{R}^n$, *and let* $\{\mathcal{I}_k\}$ *be a net structure with associated differentiation basis* $(\mathcal{I}, \Longrightarrow)$. *The functions* $\overline{D}_{\mathcal{I}}\nu(x)$ *and* $\underline{D}_{\mathcal{I}}\nu(x)$ *are Borel measurable.*

Proof. To see this, let

$$d_k(x) = \frac{\nu(I_k)}{\lambda(I_k)}$$

if $x \in I_k \in \mathcal{I}_k$. Since each $I_k \in \mathcal{I}_k$ is a Borel set, d_k takes on only countably many values, each on a Borel set. Thus d_k is Borel measurable, so the same is true of $\overline{D}_{\mathcal{I}}\nu$ and $\underline{D}_{\mathcal{I}}\nu$, by (18). ■

We could now attempt to follow the development in Section 7.1 for functions and Section 8.2 for measures. This would involve establishing the Vitali property, followed by certain growth lemmas. The structure here is much simpler, however. There are only countably many cells in our family $\mathcal{I}$, and disjointedness is given as one of the features. The Vitali property is clearly satisfied, but it is not needed for our development. We prove the relevant growth lemma directly.

Lemma 8.20 *Let* ν *be a Lebesgue–Stieltjes signed measure on a cube* X *in* $\mathbb{R}^n$, *and let* $\{\mathcal{I}_k\}$ *be a net structure with associated differentiation basis* $(\mathcal{I}, \Longrightarrow)$.

1. *If $A \subset X$ is a Borel set, $q \in \mathbb{R}$, and $\overline{D}_{\mathcal{I}}\nu \geq q$ on A, then $\nu(A) \geq q\lambda(A)$.*
2. *If $B \subset X$ is a Borel set, $\lambda(B) = 0$, and ν does not have an infinite derivative at any point of B, then $\nu(B) = 0$.*

Proof. Without loss of generality, we assume that $q = 0$. (See Exercise 8:5.3.) Let $\varepsilon > 0$. Using Corollary 3.14 and applying the Jordan decomposition to ν, we obtain an open set $G \supset A$ such that $\lambda(G) < \infty$ and $|\nu(E)| < \varepsilon$ for every Borel set $E \subset G \setminus A$. Let $x \in A$. By hypothesis, $\overline{D}_{\mathcal{I}}\nu(x) \geq 0$. Thus, for each $k \in \mathbb{N}$, there exists $j \geq k$ and $I \in \mathcal{I}_j$ such that

$$x \in I\ ,\ I \subset G, \quad \text{and} \quad \nu(I) > -\varepsilon\lambda(I). \tag{19}$$

Let $\mathcal{J}_1$ consist of those cells $I \in \mathcal{I}_1$ that satisfy (19). Inductively, for $k > 1$, let $\mathcal{J}_{k+1}$ consist of those cells I in $\mathcal{I}_{k+1}$ that satisfy (19) and are not contained in any cells of $\mathcal{J}_1 \cup \cdots \cup \mathcal{J}_k$. Our construction guarantees that the cells of $\bigcup_{k=1}^{\infty} \mathcal{J}_k$ form a disjoint sequence $\{J_j\}$. From our construction and (19), we see that

$$\nu(J_j) \geq -\varepsilon\lambda(J_j) \ \text{ for each } \ j = 1, 2, 3, \ldots$$

and that

$$A \subset \bigcup_{j=1}^{\infty} J_j \subset G.$$

Our choice of G guarantees that $|\nu(G \setminus \bigcup_k J_k)| < \varepsilon$, so

$$\nu(G) \geq \nu\left(\bigcup_j J_j\right) - \varepsilon.$$

Thus,

$$\begin{aligned}
\nu(A) + \varepsilon \quad &> \quad \nu(G) \geq \nu\left(\bigcup_j J_j\right) - \varepsilon = \sum_j \nu(J_j) - \varepsilon \\
&\geq \quad -\varepsilon \sum_j \lambda(J_j) - \varepsilon \geq -\varepsilon\lambda(G) - \varepsilon.
\end{aligned}$$

Since these inequalities are valid for every $\varepsilon > 0$, we conclude that $\nu(A) \geq 0$, establishing part (1).

Let $B_n = \{x \in B : \overline{D}_{\mathcal{I}}\nu(x) \geq -n\}$. By the definition of B, $\overline{D}_{\mathcal{I}}\nu(x) > -\infty$ for all $x \in B$, since $D_{\mathcal{I}}\nu(x) = -\infty$ whenever $\overline{D}_{\mathcal{I}}\nu(x) = -\infty$. Thus $B = \bigcup_{n=1}^{\infty} B_n$. By part (1) of the lemma, for all $n \in \mathbb{N}$,

$$\nu(B_n) \geq -n\lambda(B_n) = 0.$$

This implies that $\nu(B) \geq 0$. By applying the same argument to the signed measure $-\nu$, we find that $-\nu(B) \geq 0$; that is, $\nu(B) \leq 0$. Thus $\nu(B) = 0$. ■

We can now prove the main result of this section, an analog of de la Vallée Poussin's theorem (Theorem 7.20).

Theorem 8.21 *Let ν be a Lebesgue–Stieltjes measure on a cube X in $I\!R^n$. Let $\{\mathcal{I}_k\}$ be a net structure on $I\!R^n$ with associated differentiation basis $(\mathcal{I}, \Longrightarrow)$. Then $D_{\mathcal{I}}\nu$ exists a.e. on X and is integrable on X. Furthermore,*

$$\nu(E) = \int_E D_{\mathcal{I}}\nu \, d\lambda + \nu(E \cap B_\infty) + \nu(E \cap B_{-\infty}) \tag{20}$$

for every Borel set $E \subset X$.

Proof. By Theorem 5.34 and the Jordan decomposition, there exist signed measures α, β such that

$$\alpha \ll \lambda \ , \ \beta \perp \lambda, \quad \text{and} \quad \nu = \alpha + \beta.$$

There exists $f \in L_1$ such that $\alpha = \int f \, d\lambda$. Let A and B be complementary Borel sets in X with $\lambda(B) = 0 = |\beta|(A)$. For real numbers $p < q$, let

$$E_{pq} = \left\{x : \overline{D}_{\mathcal{I}}\nu(x) \geq q > p \geq f(x)\right\}.$$

Then, noting that $\beta(E_{pq} \cap A) = 0$, we calculate

$$\begin{aligned} \nu(E_{pq} \cap A) &\geq q\lambda(E_{pq} \cap A) \geq p\lambda(E_{pq} \cap A) \\ &\geq \int_{E_{pq} \cap A} f \, d\lambda = \nu(E_{pq} \cap A). \end{aligned}$$

Since the first and last terms in the preceding inequalities are the same, all the inequalities are, in fact, equalities. Thus

$$q\lambda(E_{pq}) = p\lambda(E_{pq}).$$

But $q > p$, and $\lambda(E_{pq}) \leq \lambda(X) < \infty$. Thus $\lambda(E_{pq}) = 0$.

Now let $M = \bigcup\{E_{pq} : p, q \in \mathbb{Q}\}$. Then

$$M = \left\{x : \overline{D}_{\mathcal{I}}\nu(x) > f(x)\right\},$$

and $\lambda(M) = 0$. Therefore, $\overline{D}_{\mathcal{I}}\nu(x) \leq f(x)$ a.e. on X. The same argument shows that

$$\overline{D}_{\mathcal{I}}(-\nu(x)) \leq -f(x)$$

a.e. on X, that is $\underline{D}_{\mathcal{I}}\nu(x) \geq f(x)$ a.e. on X. Thus $D_{\mathcal{I}}\nu = f$ a.e. on X. We have shown that, for every Borel set $E \subset X$,

$$\nu(E) = \alpha(E) + \beta(E) = \int_E f \, d\lambda + \beta(E \cap B) = \int_E D_{\mathcal{I}}\nu \, d\lambda + \beta(E \cap B).$$

Thus, for every Borel set $E \subset X$,

$$\nu(E) = \int_E D_{\mathcal{I}}\nu \, d\lambda + \beta(E \cap B). \tag{21}$$

To complete the proof, we study the role of the sets B_∞ and $B_{-\infty}$. The function f is integrable, so f is finite a.e. on X. Thus the same is true of $D_{\mathcal{I}}\nu$, so $\lambda(B_\infty \cup B_{-\infty}) = 0$. If E is a Borel set contained in $(B_\infty \cup B_{-\infty}) \cap A$, then $\lambda(E) = 0$, and we see from (21) that

$$\nu(E) = \int_E D_{\mathcal{I}}\nu \, d\lambda = 0.$$

Thus only the parts of B_∞ and $B_{-\infty}$ that are contained in B contribute to the calculation of ν.

We next show that B_∞ and $B_{-\infty}$ are the only parts of B that contribute to ν. Let

$$S = B \setminus (B_\infty \cup B_{-\infty}).$$

Since $S \subset B$, $\lambda(S) = 0$. Applying Lemma 8.20 to S, we find that $\nu(E) = 0$ for every Borel set $E \subset S$. It follows that

$$\beta(E \cap B) = \nu(E \cap B_\infty) + \nu(E \cap B_{-\infty}). \tag{22}$$

Substituting (22) into (21), we obtain the desired form (20). ∎

We conclude with several further remarks.

Remark 1. The assumption that ν be defined only on subsets of X with $\lambda(X) < \infty$ was needed only to assure that the sets E_{pq} have finite measure. By partitioning $\mathbb{R}^n$ into cubes and obtaining (20) for each cube, we can drop this assumption and assume only that ν be finite on $\mathbb{R}^n$.

Remark 2. Since $D_{\mathcal{I}}\nu = f$ a.e., we see that two different sequences of nets will give rise to the same derivatives a.e. It is, perhaps, easiest to visualize $\mathcal{I}$ as half-open cubes, as de la Vallée Poussin did in 1915, but the cells of $\mathcal{I}$ can be any Borel sets of positive measure satisfying the three conditions in 8.18.

Remark 3. When $\nu \ll \lambda$, we see from (20) that

$$\nu(E) = \int_E D_{\mathcal{I}}\nu \, d\lambda$$

as expected. When $\nu \perp \lambda$, all of the mass of ν is concentrated on the set on which $D_{\mathcal{I}}\nu$ is infinite. Thus it follows from (20) that

$$\int_E D_{\mathcal{I}}\nu \, d\lambda = 0$$

for every Borel set E. This implies $D_{\mathcal{I}}\nu = 0$ a.e..

Conversely, if $D_{\mathcal{I}}\nu = 0$ a.e., then it follows once again from (20) that ν is concentrated on $B_\infty \cup B_{-\infty}$, so that $\nu \perp \lambda$. These remarks show $\nu \perp \lambda$ if and only if $D_{\mathcal{I}}\nu = 0$ a.e.

Remark 4. Let us compare the Lebesgue and de la Vallée Poussin decompositions. For a Lebesgue–Stieltjes signed measure on $\mathbb{R}^n$ we have this situation. When differentiating with respect to a net structure, (20) is valid for each Borel set E,

$$\nu(E) = \int_E D_{\mathcal{I}}\nu \, d\lambda + \nu(E \cap (B_\infty \cup B_{-\infty})). \tag{23}$$

When differentiating with respect to the cubes in $\mathbb{R}^n$, we obtain

$$\nu(E) = \int_E D\nu \, d\lambda + \nu(E \cap (B_\infty \cup B_{-\infty})). \tag{24}$$

The set $B_\infty \cup B_{-\infty}$ is the same set in (23) as in (24). The difference is that in (23),

$$B_\infty = \left\{x : D_{\mathcal{I}}\nu(x) = \infty\right\} \quad \text{and} \quad B_{-\infty} = \left\{x : D_{\mathcal{I}}\nu(x) = -\infty\right\},$$

while in (24) no such interpretation is possible as Example 8.1 shows. De la Vallée Poussin's decomposition is simply a more delicate one than Lebesgue's when it applies.

Observe that many of the theorems related to the fundamental theorem of calculus are special cases of (23) and (24).

Exercises

8:5.1 Define a Vitali cover in the setting of this section. Then state and prove a Vitali covering theorem for net structures.

8:5.2 Show that Lemma 8.20 does not hold for the basis of cubes in $\mathbb{R}^n$. [Hint: Use Example 8.1 and take $\nu(E) = -\lambda(E \cap L)$.]

8:5.3 Show that there is no loss of generality in taking $q = 0$ in the proof of Lemma 8.20, part (1). [Hint: Consider $\nu - q\lambda$.]

8:5.4 Prove that if F is continuous and of bounded variation on $[a, b]$ and N is the set on which F' does not exist, finite or infinite, then $\lambda(F(N)) = 0$.

8:5.5 Refer to Example 8.1. Study the behavior of $D_{\mathcal{I}}\nu$ on L with particular focus on Lemma 8.20, part (2).

8.6 Radon–Nikodym Derivative in a Measure Space

In Sections 7.1 to 7.5 we developed enough differentiation theory to understand the inverse relationship that exists between the operations of differentiation and integration on $\mathbb{R}$. Because of the intimate connection between functions of bounded variation and Lebesgue–Stieltjes measures, we were able to interpret many of the results that we obtained for functions in terms of measures. Then, in Sections 8.2 to 8.5, we tried to extend the results to Lebesgue–Stieltjes measures on $\mathbb{R}^n$. We found that the extent to which the material in Sections 7.1 to 7.5 generalized depended on the differentiation basis $(\mathcal{I}, \Longrightarrow)$ under consideration. When this basis has the Vitali property, the Radon–Nikodym derivative can be expressed as a familiar pointwise limit of the form

$$\lim_{I \Longrightarrow x} \frac{\nu(I)}{\lambda(I)} \ a.e.$$

Suppose now that $(X, \mathcal{M}, \mu)$ is a σ-finite, complete measure space. The Radon–Nikodym theorem guarantees that if $\nu \ll \mu$ then there exists $f \in L_1$ such that

$$\nu(E) = \int_E f \, d\mu \quad \text{for each} \quad E \in \mathcal{M}.$$

We shall obtain a differentiation basis $(\mathcal{I}, \Longrightarrow)$ such that

$$\lim_{I \Longrightarrow x} \frac{\nu(I)}{\mu(I)} = f(x) \ a.e. \tag{25}$$

This will provide a sense of how the Radon–Nikodym derivative f behaves like a “genuine” derivative a.e.

In the setting of $\mathbb{R}^n$, a number of bases $(\mathcal{I}, \Longrightarrow)$ come to mind naturally. The family $\mathcal{I}$ can be chosen in many ways, and we were able to obtain a notion of contraction using diameters of the members of $\mathcal{I}$. In the abstract setting, we have no metric to aid us in obtaining a notion of contraction, nor do we have a natural class of subsets, such as the cubes or the intervals, to use for the differentiation basis.

Some considerations will lead us to the right idea of contraction. If the ratio $\nu(I)/\mu(I)$ is to approximate $f(x)$, then I must in some sense be close to x. Thus writing $I \Longrightarrow x$ if $x \in I \in \mathcal{I}$ and $\mu(I) \to 0$ is not likely to provide satisfying results. If, for example, we had chosen this notion of contraction for $\mathcal{I}$, the collection of intervals in $\mathbb{R}^2$, we would not have been able to obtain (25) even for bounded measurable functions. The reason is clear: $\mu(I)$ can be small, but if I is a sufficiently thin interval, then much of I can sample values $f(y)$ for y far from x.

We can obtain a sense of “nearness” of $I \in \mathcal{I}$ to x as follows. We take $\mathcal{I}$ to be a family of sets of positive measure. For each $x \in X$, we let $\mathcal{I}_x = \{I \in \mathcal{I} : x \in I\}$. We require that $\mathcal{I}_x$ be *directed by downward* *inclusion.* This means that there exists an index set $\mathcal{A}$ for $\mathcal{I}_x$ such that $\mathcal{I}_x = \{I_\alpha : \alpha \in \mathcal{A}\}$, and for each pair $\alpha, \beta \in \mathcal{A}$, there exists $\gamma \in \mathcal{A}$ for which $I_\gamma \subset I_\alpha \cap I_\beta$. For example, in $\mathbb{R}^2$ we could take $\mathcal{I}_x$ to consist of all open intervals containing x and index $I \in \mathcal{I}_x$ by its lower-left and upper-right corners. We then write

$$\lim_{I \Longrightarrow x} \frac{\nu(I)}{\mu(I)} = \ell,$$

provided that for each $\varepsilon > 0$ there exists $\alpha \in \mathcal{A}$ such that

$$\left| \frac{\nu(I)}{\mu(I)} - \ell \right| < \varepsilon$$

if $I \in \mathcal{I}_x$ and $I \subset I_\alpha$. Taking $\mathcal{I}$ to be the open intervals in $\mathbb{R}^2$, we find that this notion of contraction agrees with the notions that we considered in Section 8.4. (It does not agree with the notion of contraction relative to closed intervals, however.)

It remains to determine which family we should select for $\mathcal{I}$. As a first attempt, we might try the family of *all* sets of positive measure. We see an immediate difficulty: the families $\mathcal{I}_x$ are not directed by downward inclusion, since two sets of positive measure containing x can intersect in a set of zero measure. A clue for proceeding can be obtained from the Lebesgue density theorem (Theorem 7.33).

Example 8.22 Consider the space $(X, \mathcal{M}, \mu) = ([0,1], \mathcal{L}, \lambda)$. For $A, B \subset X$, write

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

For each $A \in \mathcal{M}$, let $L(A)$ be the set of all density points of A. Then for $A, B \in \mathcal{M}$:

1. $\mu(L(A) \triangle A) = 0$.
2. If $\mu(A \triangle B) = 0$, then $L(A) = L(B)$.
3. $L(\emptyset) = \emptyset$ and $L(X) = X$.
4. $L(A \cap B) = L(A) \cap L(B)$.
5. If $A \subset B$, then $L(A) \subset L(B)$.

We leave verification of these facts as Exercise 8:6.1.

It is easy to verify that the nonempty members of

$$\{I \in \mathcal{M} : \text{There exists } A \in \mathcal{M} \text{ such that } I = L(A)\}$$

can serve as a differentiation basis under our present notion of contraction. If $I_\alpha \in \mathcal{I}_x$ and $I_\beta \in \mathcal{I}_x$, then $I_\gamma = I_\alpha \cap I_\beta \in \mathcal{I}_x$.

Now, in a general σ-finite measure space, we do not have a Lebesgue density theorem. In fact, in order to have such a theorem, we first need a differentiation basis $(\mathcal{I}, \Longrightarrow)$ and then need to determine whether the basis has the Lebesgue density property. (Recall that the rectangle basis in $\mathbb{R}^2$ does not have this property.)

In 1931, J. von Neumann proved a theorem that can serve as a suitable substitute for the density property. He showed that in every complete finite measure space $(X, \mathcal{M}, \mu)$ there is a mapping $L : \mathcal{M} \to \mathcal{M}$ that satisfies conditions (1) to (5) of Example 8.22. We call L a *lifting,* and we call

$$\{I \in \mathcal{M} : \text{There exists } A \in \mathcal{M} \text{ such that } I = L(A)\}$$

the family of *lifted sets.* Let $\mathcal{I}$ denote the nonempty members of this family. In 1968, D. Kölzow[4] showed that von Neumann's theorem is valid in any complete measure space for which the Radon–Nikodym theorem holds. In particular, von Neumann's theorem holds in a complete σ-finite space.

The term *lifting* derives from the following interpretation. The relation $\mu(A \triangle B) = 0$ partitions $\mathcal{M}$ into equivalence classes. The mapping $L : \mathcal{M} \to \mathcal{M}$ *lifts* one member from each equivalence class. Observe that, for each $M \in \mathcal{M}$, $L(L(M)) = L(M)$.

We shall make frequent use of the following:

8.23 *If A and B are measurable and $\mu(A \cap B) = 0$, then*

$$L(A) \cap L(B) = \emptyset.$$

To verify (8.23), note that

$$\emptyset = L(\emptyset) = L(A \cap B) = L(A) \cap L(B)$$

by conditions (3) and (4).

We can now begin our formal development. For the rest of this section we shall make the following five assumptions about the measure space.

(a) $(X, \mathcal{M}, \mu)$ is a complete σ-finite measure space.

(b) $L : \mathcal{M} \to \mathcal{M}$ is a lifting on $\mathcal{M}$.

(c) $\mathcal{I}$ consists of all nonempty lifted sets.

(d) For each $x \in X$, $\mathcal{I}_x = \{I \in \mathcal{I} : x \in I\}$.

(e) $\{\mathcal{I}_x\}$ is directed by downward inclusion.

Definition 8.24 Let $\mathcal{V} \subset \mathcal{I}$, and let $E \in \mathcal{M}$. If for all $x \in E$ and $J \in \mathcal{I}_x$, there exists $I \in \mathcal{I}_x \cap \mathcal{V}$ such that $I \subset J$, we say that $\mathcal{V}$ is a *Vitali cover* for E.

Theorem 8.25 (Vitali covering property) *Suppose that $\mathcal{V}$ is a Vitali cover for $E \in \mathcal{M}$. Then there exists a sequence $\{I_k\}$ from $\mathcal{V}$ such that*

1. $I_i \cap I_j = \emptyset$ *if* $i \neq j$,

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[4] D. Kölzow, *Differentiation von Massen*, Lecture Notes in Mathematics, vol. 65, Springer, Berlin (1968).

2. $\mu\left(E \setminus \bigcup_k I_k\right) = 0$, *and*
3. $\mu\left(\bigcup_k I_k \setminus E\right) = 0$.

Observe that condition (3) indicates that the sequence $\{I_k\}$ has "zero overflow." In our earlier settings, we were able to achieve "ε-overflow" by enclosing E in an appropriate open set G. Here we do not have open sets to use, but (3) more than overcomes this deficiency. In the proof of Theorem 8.25, we make use of Zorn's lemma (a statement of which can be found in Section 1.11).

Proof. Let $\mathcal{V}$ be a Vitali cover for $E \in \mathcal{M}$. Suppose that $\mu(E) > 0$; otherwise, the empty subfamily of $\mathcal{V}$ does the job. Let $B = L(E)$, and let

$$\mathcal{V}^* = \{I \in \mathcal{V} : I \subset B\}.$$

We first verify that $\mathcal{V}^* \neq \emptyset$. Let $x \in E \cap B$. Then $B \in \mathcal{I}_x$. Since $\mathcal{V}$ is a Vitali cover for E, there exists $I \in \mathcal{V}$ such that $I \subset B$. Thus $I \in \mathcal{V}^*$ and $\mathcal{V}^*$ is nonempty.

A subfamily $\mathcal{V}_1$ of $\mathcal{V}^*$ is called *admissible* if each pair of its members is disjoint. Partially order the admissible subfamilies of $\mathcal{V}^*$ by upward inclusion: $\mathcal{V}_1$ is beyond $\mathcal{V}_2$ if $\mathcal{V}_1 \supset \mathcal{V}_2$. Since $(X, \mathcal{M}, \mu)$ is σ-finite, each admissible family is at most countably infinite. Now each chain of admissible families has an upper bound (its union), which is also an admissible family.

By Zorn's lemma, there exists a maximal admissible family. Denote its members by $I_1, I_2, \ldots$. We show that the family $\{I_k\}$ has the desired properties. That the members of $\{I_k\}$ are pairwise disjoint is clear.

We next show that $\mu\left(B \setminus \bigcup_k I_k\right) = 0$. Since $\bigcup_k I_k$ is a finite or countably infinite union of measurable sets, $\bigcup_k I_k$ is also measurable. Suppose that $\mu\left(B \setminus \bigcup_k I_k\right) > 0$. Let

$$M = L\left(B \setminus \bigcup_k I_k\right).$$

Then $\mu(M) > 0$ and, by (8.23), $M \cap I_k = \emptyset$ for every k. Let $y \in M \cap E$. Then $M \in \mathcal{I}_y$, and

$$M = L(M) \subset B.$$

Since $\mathcal{V}$ is a Vitali cover for E, there exists $I_0 \in \mathcal{V}$ such that $I_0 \in \mathcal{I}_y$ and

$$I_0 \subset L(M) \subset B.$$

The family $I_0, I_1, I_2, \ldots$ is thus an admissible family, contradicting our assumption that the family $I_1, I_2, \ldots$ is a *maximal* admissible family. Thus

$$\mu\left(B \setminus \bigcup_k I_k\right) = 0.$$

Since $B = L(E)$, we conclude that

$$\mu\left(E \setminus \bigcup_k I_k\right) = 0,$$

establishing (2). Finally, to verify (3), we need only observe that $\bigcup_k I_k \subset B$ and that $\mu(B \triangle E) = 0$. ■

We can now obtain growth lemmas analogous to Lemmas 7.1 and 7.4. The reader will notice two differences. We restrict our attention to *absolutely continuous* measures, and the definitions of upper and lower derivatives appear more complicated. Exercises 8:6.2, 8:6.3, and 8:6.4 offer some explanations for these differences.

Definition 8.26 Let $x \in X$, and let ν be a signed measure on $\mathcal{M}$ with $\nu \ll \mu$. We define the *lower derivative* $\underline{\boldsymbol{D}}\nu(x)$ as

$$\inf\{p \in \mathbb{R} : \ \forall I \in \mathcal{I}_x \ \exists J \in \mathcal{I}_x \text{ so that } J \subset I \text{ and } \nu(J) < p\mu(J)\ \}.$$

Similarly, we define the *upper derivative* $\overline{\boldsymbol{D}}\nu(x)$ as

$$\sup\{q \in \mathbb{R} : \ \forall I \in \mathcal{I}_x \ \exists J \in \mathcal{I}_x \text{ so that } J \subset I \text{ and } \nu(J) > q\mu(J)\}.$$

If $\underline{\boldsymbol{D}}\nu(x) = \overline{\boldsymbol{D}}\nu(x)$, we say that ν has a *derivative* at x and write $\boldsymbol{D}\nu(x)$ for the common value of $\underline{\boldsymbol{D}}\nu(x)$ and $\overline{\boldsymbol{D}}\nu(x)$. When $\boldsymbol{D}\nu(x)$ is finite, we say that ν is *differentiable* at x.

It is easy to verify (Exercise 8:6.5) that $\boldsymbol{D}\nu(x) = s \in \mathbb{R}$ if and only if for every $\varepsilon > 0$ there exists $I \in \mathcal{I}_x$ such that

$$\left|\frac{\nu(J)}{\mu(J)} - s\right| < \varepsilon$$

for each $J \in \mathcal{I}_x$ such that $J \subset I$.

Lemma 8.27 *Let $E \in \mathcal{M}$, and let ν be a measure on $\mathcal{M}$ with $\nu \ll \mu$.*

1. *If for each $x \in E$, $\underline{\boldsymbol{D}}\nu(x) < p$, then $\nu(E) \le p\mu(E)$.*
2. *If for each $x \in E$, $\overline{\boldsymbol{D}}\nu(x) > q$, then $\nu(E) \ge q\mu(E)$.*

Proof. Assume that $\mu(E) > 0$; otherwise, there is nothing to prove in either assertion. Let

$$\mathcal{V} = \left\{I \in \mathcal{I} : \frac{\nu(I)}{\mu(I)} < p\right\}.$$

Then $\mathcal{V}$ is a Vitali cover for E. By Theorem 8.25, there exists a pairwise disjoint sequence $\{I_k\}$ from $\mathcal{V}$ such that

$$\mu\left(E \setminus \bigcup_k I_k\right) = \mu\left(\bigcup_k I_k \setminus E\right) = 0.$$

Since $\nu \ll \mu$, we also have

$$\nu\left(E \setminus \bigcup_k I_k\right) = \nu\left(\bigcup_k I_k \setminus E\right) = 0.$$

Thus

$$\nu(E) = \sum_k \nu(I_k) \le \sum_k p\mu(I_k) = p\mu\left(\bigcup_k I_k\right) = p\mu(E),$$

establishing assertion (1). The proof of assertion (2) is similar. ■

We can now establish the main result of this section, that the Radon–Nikodym derivative is a genuine derivative.

Theorem 8.28 *Let $(X, \mathcal{M}, \mu)$ be a complete σ-finite measure space, let $L : \mathcal{M} \to \mathcal{M}$ be a lifting, and let $\mathcal{I}$ denote the family of nonempty lifted sets. Let ν be a signed measure on $\mathcal{M}$ with $\nu \ll \mu$, and let $\boldsymbol{D}\nu$ be as in Definition 8.26. Then*

$$\nu(E) = \int_E \boldsymbol{D}\nu \, d\mu \tag{26}$$

for every $E \in \mathcal{M}$.

Proof. We may assume that ν is a measure and that $\mu(X) < \infty$. Since $\nu \ll \mu$, there exists $f \in L_1$ such that $\nu(E) = \int_E f \, d\mu$ for all $E \in \mathcal{M}$. For $0 < p < q$, let $E_{pq} = \{x : f(x) < p < q < \overline{\boldsymbol{D}}\nu(x)\}$. By Lemma 8.27(2), we have

$$\nu(E_{pq}) \ge q\mu(E_{pq}) \ge p\mu(E_{pq}) \ge \int_{E_{pq}} f \, d\mu = \nu(E_{pq}). \tag{27}$$

Since the first and last terms in (27) are the same, it follows that all terms in (27) are equal. But $p < q$ and $\mu(E_{pq}) < \infty$. Thus $\mu(E_{pq}) = 0$.

Let $A = \{x : \overline{\boldsymbol{D}}\nu(x) > f(x)\}$. Then

$$A = \bigcup \{E_{pq} : p, q \in \mathbb{Q}\},$$

so $\mu(A) = 0$. It follows that $\overline{\boldsymbol{D}}\nu \le f$ a.e. In a similar way, using Lemma 8.27(1), we find that $\underline{\boldsymbol{D}}\nu \ge f$ a.e. From these two inequalities we obtain $\boldsymbol{D}\nu = f$ a.e. The desired equality (26) is now apparent. ■

We end this section by mentioning that there are many other differentiation bases that lead to satisfactory developments in abstract spaces. The results one finds are similar to those we obtained in Sections 8.2 to 8.5. If the basis has the Vitali property, the Radon–Nikodym derivative will be an actual derivative whenever $\nu \ll \mu$. Vitali properties come in various strengths: the stronger the property, the better the theorem. The density property is actually a weak form of the Vitali property in which a certain amount of overlap is allowed in the resulting sequence $\{I_k\}$. It is a necessary and sufficient condition for the Radon–Nikodym derivative f to be a genuine derivative when f is bounded and measurable.[5]

[5]We refer the interested reader to C. A. Hayes and C. Y. Pauc, *Derivation*

Exercises

8:6.1 Verify (1) to (5) in Example 8.22.

8:6.2 In Sections 8.2 to 8.5, the families $\mathcal{I}$ consisted of Borel sets. Thus every Lebesgue–Stieltjes measure was defined on $\mathcal{I}$, and we could discuss

$$\lim_{I \Longrightarrow x} \frac{\nu(I)}{\mu(I)}.$$

In the present setting, we cannot make such assumptions in general. Show, however, that if $\nu \ll \mu$ then ν is defined for all $I \in \mathcal{I}$.

8:6.3 The definitions of $\overline{\boldsymbol{D}}\nu(x)$ and $\underline{\boldsymbol{D}}\nu(x)$ seem more complicated than the definitions of $\overline{D}_s\nu(x)$ and $\underline{D}_s\nu(x)$ in Section 8.4.

(a) Show that if, in Section 8.4, we had chosen $\mathcal{I}$ to consist of *open* intervals then the definitions of $\overline{D}_s\nu(x)$ and $\underline{D}_s\nu(x)$ could have been given as in Definition 8.26 while obtaining the same values.

(b) In the setting of part (a), show that there exists a sequence $\{I_k\} \subset \mathcal{I}_x$ such that, if $I \in \mathcal{I}_x$, then there exists k such that $I_k \subset I$.

(c) Refer to Example 8.22. Show that, if $\{I_k\}$ is any sequence of lifted sets containing x, then there exists $I \in \mathcal{I}_x$ such that, for every k, $I_k \setminus I \neq \emptyset$. Thus no sequence has enough members to get some member inside every member of $\mathcal{I}_x$. [Hint: Remove a bit from each I_k in such a way that a lifted set remains.]

(d) Explain why we do not define $\overline{\boldsymbol{D}}\nu(x)$ as

$$\sup\left(\limsup_{n\to\infty} \nu(I_n)/\mu(I_n)\right),$$

where the sup is taken over all sequences $\{I_n\}$ contracting to x [as we defined $\overline{D}\nu(x)$ in Section 8.2].

8:6.4 Refer to Example 8.22. Let $\mathcal{I}$ denote the lifted sets. Let F be the Cantor function, let $\beta = \mu_F$, and let K be the Cantor set. Part (a) shows that β is defined on $\mathcal{I}$. We can now define $\boldsymbol{D}\beta$ as we did for absolutely continuous measures. Part (d) shows that $\boldsymbol{D}\beta = 0$ at every point (not just that $\boldsymbol{D}\beta = 0$ a.e.). This illustrates that a notion of differentiation for measures that are not absolutely continuous may be of limited use in our present setting.

(a) Show that each $I \in \mathcal{I}$ is Borel measurable.

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and Martingales, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 49, Springer, Berlin (1970).

(b) Let x_0 be an arbitrary point in K. Construct an open set $G \subset [0,1]$ such that $H = G \cup \{x_0\} \in \mathcal{I}_{x_0}$. [Hint: Let

$$J_n = \left[x_0 + \frac{1}{2^{n+1}}, x_0 + \frac{1}{2^n}\right] \cup \left[x_0 - \frac{1}{2^n}, x_0 - \frac{1}{2^{n+1}}\right].$$

Remove from J_n a finite number of closed intervals whose union contains $K \cap J_n$. Do this for each $n \in \mathbb{N}$. Make the removed intervals so short that the remaining set H has x_0 as a density point.]

(c) Show that if $I \in \mathcal{I}_{x_0}$ and $I \subset H$ then $\beta(I) = 0$.

(d) Prove that $\boldsymbol{D}\beta(x) = 0$ for all $x \in K$ and, hence, that $\boldsymbol{D}\beta(x) = 0$ for all $x \in [0,1]$.

(e) Show that Lemma 8.27 fails for β.

8:6.5 Verify that $\boldsymbol{D}\nu(x) = s \in \mathbb{R}$ if and only if for every $\varepsilon > 0$ there exists $I \in \mathcal{I}_x$ such that, if $J \in \mathcal{I}_x$ and $J \subset I$, then $|\nu(J)/\mu(J) - s| < \varepsilon$.

8:6.6 Let X be an uncountable set, and let

$$\mathcal{M} = \left\{E \subset X : E \text{ countable or } \widetilde{E} \text{ countable}\right\}.$$

Let $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if $\widetilde{E}$ is countable.

(a) Determine a lifting L for $\mathcal{M}$. That is, indicate for every set $M \in \mathcal{M}$ what the set $L(M)$ should be.

(b) Let $\nu \ll \mu$ with $\nu(X) = 1$. Calculate $\boldsymbol{D}\nu$.

(c) Let $Y = y_1, y_2, \ldots$ be a countable subset of X. Define a measure β on $\mathcal{M}$ by

$$\beta(\{y_i\}) = \frac{1}{2^i} \quad \text{and} \quad \beta\left(X \setminus \bigcup_{i=1}^{\infty}\{y_i\}\right) = 0.$$

Calculate $\boldsymbol{D}\beta$. Observe that $\beta \perp \mu$, yet β never takes the values 0 or ∞ on the set Y.

8:6.7 The Vitali covering theorem fails for the family $\mathcal{I}_1$ of open intervals in $\mathbb{R}^2$ with the notion of contraction of Sections 8.2 and 8.4. If one instead gives contraction the meaning of the present section, we find that the two notions of contraction agree for this family $\mathcal{I}_1$. By Theorem 8.13, the family $\mathcal{I}_1$ has the Lebesgue density property. Thus the mapping $L : \mathcal{M} \to \mathcal{M}$ defined as in Example 8.22 is a lifting. Let $\mathcal{I}_2$ be the family of nonempty lifted sets.

By Theorem 8.25, $(\mathcal{I}_2, \Longrightarrow)$ *does* have the Vitali property if $\Longrightarrow$ has the meaning of this section. Since, for the family of open intervals, the two notions of contraction agree, this seems to be a contradiction. Explain why there is no contradiction. [Hint: Does $\mathcal{I}_1$ contain any Vitali covers when $\Longrightarrow$ has the meaning of this section?]

8.7 Summary, Comments, and References

The unifying theme of Chapters 7 and 8 has been the study of the inverse relationship that exists between integration and differentiation. The starting point may have been the Radon–Nikodym Theorem. See Section 5.8, where we saw that, under suitable hypotheses, if $\nu \ll \mu$, then there exists $f \in L_1$ such that $\nu = \int f\, d\mu$. We called the function f the Radon–Nikodym derivative of ν with respect to μ and wrote

$$f = \frac{d\nu}{d\mu}.$$

While we were able to show that $\frac{d\nu}{d\mu}$ has some properties reminiscent of derivatives of functions (Theorem 5.31), it did not really "look" like a derivative as a limit of an appropriate difference quotient.

In these two chapters we saw that such a realization of $\frac{d\nu}{d\mu}$ is possible, even when dealing with abstract measure spaces. We now know it is essentially correct to say that, when the Radon–Nikodym theorem holds for a measure space $(X, \mathcal{M}, \mu)$, then $\frac{d\nu}{d\mu}$ can actually be expressed as

$$\frac{d\nu}{d\mu} = \lim_{I \Longrightarrow x} \frac{\nu(I)}{\mu(I)} \text{ a.e.}$$

by choosing an appropriate differentiation basis $(\mathcal{I}, \Longrightarrow)$.

Let us review some of the features of this theory.

1. In Sections 7.1 to 7.8 we dealt with differentiation of functions of bounded variation and interpreted some of the results in terms of Lebesgue–Stieltjes signed measures on $\mathbb{R}^1$. The main tools were the Vitali covering theorem and several growth lemmas.

2. A principal objective was to determine when a real function F can be recaptured from its derivative; that is, for F defined on $[a, b]$, when can we write

$$F(x) - F(a) = \int_a^x F'\, d\lambda \quad \text{for all } x \in [a, b]? \tag{28}$$

   For F of bounded variation on $[a, b]$, we found F is differentiable a.e., and $F' \in L_1$; but (28) need not hold, even for F continuous. What can be lacking is Lusin's condition (N): the function F could do some rising and falling on sets of measure zero, as the Cantor function does.

   The Banach-Zarecki theorem showed that this is all that could go wrong. If F is continuous, of bounded variation, and satisfies condition (N), then F is absolutely continuous and (28) holds.

3. For Lebesgue–Stieltjes signed measures, (28) takes the form

$$\mu_F(E) = \int_E F'\, d\lambda. \tag{29}$$

Once again, (29) will hold for all Borel sets E if and only if $\mu_F \ll \lambda$. This is equivalent to F being absolutely continuous (Theorem 5.28).

4. For F continuous and increasing, we obtained formulas that contained many of the other results as special cases:

$$F(b) - F(a) = \int_a^b F' \, d\lambda + \lambda(F(B_\infty)) \tag{30}$$

where $B_\infty = \{x : F'(x) = \infty\}$, and

$$\mu_F(E) = \int_E F' \, d\lambda + \mu_F(E \cap B_\infty). \tag{31}$$

We had already noted in Section 7.2 that $\lambda(B_\infty) = 0$. The proofs of (30) and (31) depended on many earlier results, but now that we have these formulas, we can use them to clarify a number of matters related to continuous functions F of bounded variation and to nonatomic Lebesgue–Stieltjes measures μ_F.

From (30), we see that the growth of F on $[a, b]$ has two components, one related to the absolutely continuous component of F, the other to the singular component. When F is absolutely continuous, $\lambda(F(B_\infty)) = 0$, so (28) is valid. When F is singular, $F' = 0$ a.e., so

$$F(b) - F(a) = \lambda(F(B_\infty)).$$

Thus F does all its rising on the zero measure set on which $F' = \infty$. The formula itself can remind us of several facts: the a.e. differentiability of F, the integrability of F', the measurability of $F(B_\infty)$, the uncountability of B_∞ when F is not absolutely continuous, and others.

From (31), we obtain similar information about μ_F. It also provides a refinement of the Lebesgue decomposition because it shows that

$$\frac{d\mu_F}{d\nu} = F' \text{ a.e.}$$

and that all the mass of the singular component of μ_F is concentrated on the set $B_\infty = \{x : F'(x) = \infty\}$. One also sees from (31) that $\mu_F \perp \lambda$ if and only if $F' = 0$ a.e.

For signed Lebesgue–Stieltjes measures, the equation (31) generalizes to Theorem 7.20, a form of de la Vallée Poussin's theorem.

5. Let us return to (28). It is valid if and only if F is absolutely continuous. We saw that if F is differentiable then F is continuous and satisfies Lusin's condition (N). Thus, because of the Banach-Zarecki theorem, F will be absolutely continuous if and only if F is of bounded variation. And we saw that happens if and only if $F' \in L_1$. As a result, we obtained this form of the fundamental theorem of calculus:

Space	**Basis**	**(32) holds for:**	**Comments**
$(\mathbb{R}^n, \mathcal{L}, \lambda)$	Cubes	All functions in L_1	Vitali valid, LDT valid
$(\mathbb{R}^n, \mathcal{L}, \lambda)$	Intervals	All bounded, measurable functions	Vitali fails, LDT valid
$(\mathbb{R}^n, \mathcal{L}, \lambda)$	Rectangular parallelepipeds	Fails even for characteristic functions of closed sets	Vitali fails, LDT fails
$(\mathbb{R}^n, \mathcal{L}, \lambda)$ (or any separable metric space of finite measure)	Net structure	All functions in L_1	Vitali valid, LDT valid
$(X, \mathcal{M}, \mu)$ σ-finite, complete	Lifted sets	All functions in L_1	Vitali valid, LDT valid

Table 8.1: Fundamental theorem of calculus in various spaces.

If F is differentiable on $[a,b]$ and $F' \in L_1$, then

$$F(b) - F(a) = \int_a^b F'\, d\lambda.$$

The function $F(x) = x^2 \sin x^{-2}$, $F(0) = 0$, provides an example of a differentiable function not of bounded variation. We had already mentioned earlier that, in order for this form of the fundamental theorem of calculus to be valid for *every* differentiable function F, we need a more general form of integration, as, for example, the integral discussed in Sections 1.21 and 5.10.

6. In Sections 8.2 to 8.6 we discussed ways in which the development of differentiation of measures can be extended to spaces more general than $(\mathbb{R}, \mathcal{L}, \lambda)$. The basic idea was to obtain a system $\mathcal{I}$ of sets of positive measure and a notion $\Longrightarrow$ of contraction such that, if $\nu = \int f\, d\mu$, then

$$\lim_{I \Longrightarrow x} \frac{\nu(I)}{\mu(I)} = f(x) \ a.e. \tag{32}$$

When this happens, the Radon–Nikodym derivative f takes the appearance of a derivative. We saw that analogs of tools that we used in Sections 7.1 to 7.8 played important roles in developing the theory. Table 8.7 summarizes some of our findings.

The analog of de la Vallée Poussin's theorem is not valid in these settings except for the case of net structures. In each case but the last, contraction had the usual meaning involving the diameters $\delta(I)$ tending to zero. In general, if $(\mathcal{I}, \Longrightarrow)$ possesses the Vitali covering property, then (32) holds for all $f \in L_1$. The Lebesgue density property is necessary and sufficient for (32) to hold for all bounded functions in L_1.

We end this section by mentioning that most of the material in Sections 8.1 to 8.3 is treated, in some form or other, by many texts on the subject. The material in Sections 8.4 to 8.6 is less standard. We list some works that deal with various aspects of this material in some detail. Several have been mentioned already in footnotes in the chapter.

1. Bruckner, A. M., "Differentiation of Integrals," *Amer. Math. Monthly* **78** (1971), no. 9, Part II.
2. de Guzmán, M., *Differentiation of Integrals in* $\mathbb{R}^n$, Lecture Notes in Mathematics, vol. 481, Springer, Berlin (1975).
3. Hayes, C. A., and Pauc, C. Y., *Derivation and Martingales*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 49, Springer, Berlin (1970).
4. Kölzow, D., *Differentiation von Massen*, Lecture Notes in Mathematics, vol. 65, Springer, Berlin (1968).
5. Munroe, M. E., *Introduction to Measure and Integration*, Addison-Wesley, Reading, MA (1953).
6. Saks, S., *Theory of the Integral*, second revised ed., Monographie Matematyczne, vol. 7, Hafner, New York (1937).

8.8 Additional Problems for Chapter 8

8:8.1 List the various growth lemmas or theorems of Chapter 7 that were based on the Vitali covering theorem. Which were needed for various forms of the fundamental theorem of calculus? Reconsider Example 8.1, noting how the differentiation schemes we studied in Sections 8.2 to 8.6 relate to differentiation on the set L. Which of the relevant growth lemmas will detect that $\nu(L) > 0$?

8:8.2 Let $\mathcal{I}$ be an arbitrary family of measurable sets in $\mathbb{R}^n$ of positive Lebesgue measure, and let $\Longrightarrow$ have the usual meaning. Suppose that $\mathcal{I}$ has the Lebesgue density property: that is, if $A \in \mathcal{L}$, then

$$\lim_{I \Longrightarrow x} \frac{\lambda(A \cap I)}{\lambda(I)} = 1 \text{ a.e. on } A.$$

Prove that if f is a bounded measurable function and $\nu = \int f\, d\lambda$ then

$$\lim_{I \Longrightarrow x} \frac{\nu(I)}{\lambda(I)} = f \text{ a.e.}$$

Thus, for any differentiation basis possessing the Lebesgue density property, half of the fundamental theorem of calculus is valid, at least for bounded measurable functions f: the derivative (relative to $\mathcal{I}$) of the integral of f equals f a.e.

8:8.3 A family $\mathcal{I}$ of bounded closed sets in $\mathbb{R}^n$ is said to have the *Morse halo property* if the "halo"

$$H(I) = \bigcup \{J \in \mathcal{I} : I \cap J \neq \emptyset, \delta(J) \leq 2\delta(I)\}$$

satisfies the inequality $\lambda^*(H(I)) \leq M\lambda(I)$ for some $M > 0$. Let $\mathcal{I}$ be a family of closed sets in $\mathbb{R}^n$, and let $\Longrightarrow$ have the usual meaning. A. Morse showed in 1947 that if $\mathcal{I}$ has the Morse halo property then $\mathcal{I}$ also has the Vitali property. Show that the family of intervals in $\mathbb{R}^n$ does not have the Morse halo property, but the family of cubes in $\mathbb{R}^n$ does.

8:8.4 Let $(X, \mathcal{M}, \mu)$ be a measure space and assume $\mu(X) < \infty$. Let L be a lifting on $\mathcal{M}$ (as defined in Section 8.6).

(a) Show that the statement $L(A \cup B) = L(A) \cup L(B)$ is not necessarily true. [Hint: Use Example 8.22 and take $A = (0, 1)$, $B = (1, 2)$.]

(b) Let $\mathcal{T} = \{A \in \mathcal{M} : A \subset L(A)\}$. Show $\mathcal{T}$ is closed under arbitrary (not necessarily countable) unions. In particular, an arbitrary union of members of $\mathcal{T}$ is measurable. What does this say when applied to Example 8.22?

(c) Show that $\mathcal{T}$ is a topology on X. (See ahead to Definition 9.69.)

In the setting of Example 8.22, $\mathcal{T}$ is called the density topology; see also Exercise 7:9.11. We mention that if

$$\mathcal{T}_1 = \{L(A) \setminus Z : A \in \mathcal{M}, \mu(Z) = 0\}$$

then $\mathcal{T}_1$ is also a topology on X. This topology[6] has interesting properties: the nowhere dense sets are exactly the zero measure sets and the measurable sets are exactly those with the property of Baire (defined in Exercise 11:10.5). The definitions of nowhere dense and Baire property are the same in topological spaces as in metric spaces.

[6] A development of this topology can be found in J. C. Oxtoby, *Measure and Category*, 2nd edition, Springer (1980), Chapter 22.

Chapter 9

METRIC SPACES

We have encountered a number of ways in which a notion of convergence plays a fundamental role. A sequence $\{x_n\}$ of numbers can converge to a number x, and a sequence of functions $\{f_n\}$ can converge in several different senses to a function f. There are, however, many other situations in which various sorts of sequences can converge.

In this chapter we study general notions of convergence in the setting of a *metric space*. We have used, in earlier chapters, some of the more rudimentary ideas in metric space theory. In this chapter and the next we present a self-contained account of the basic theory and its applications. In the first three sections, we present a development of the elementary concepts related to metric spaces and provide some examples that illustrate the scope of the concepts. The most important of metric space concepts—separability, completeness, and compactness—are investigated then. We obtain a few significant theorems for spaces possessing these properties and provide applications to several areas of mathematics.

The Baire category theorem and its applications are the subjects of the Chapter 10.. The special topics of Banach spaces and Hilbert spaces can be found in Chapters 12 and 14.

9.1 Definitions and Examples

We begin by recalling the definition of a metric space.

Definition 9.1 Let X be a set and let $\rho : X \times X \to \mathbb{R}$. If ρ satisfies the following conditions, then we say that ρ is a *metric* on X and call the pair (X, ρ) a *metric space*.

1. $\rho(x, y) \geq 0$ for all $x, y \in X$.
2. $\rho(x, y) = 0$ if and only if $x = y$.
3. $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.

4. $\rho(x,z) \le \rho(x,y) + \rho(y,z)$ for all $x, y, z \in X$ (triangle inequality).

In some situations the metric ρ is understood from the context or does not appear explicitly in the discussion. In that case we sometimes write X for the metric space, suppressing ρ from the notation. For example, when we talk about the metric space $(\mathbb{R}, \rho)$, we shall often write $\mathbb{R}$, omitting mention of the metric ρ. This is not to suggest that $\mathbb{R}$ cannot be equipped with other interesting metrics, just that the majority of studies of $\mathbb{R}$ are done with this metric and that it can be taken for granted.

If (X, ρ) is a metric space and $Y \subset X$, then the restriction of ρ to $Y \times Y$ induces a metric on Y. We shall designate this metric by ρ, as well, and call (Y, ρ) a *subspace* of (X, ρ) or Y a subspace of X. For example, the interval $[a, b]$ is a subspace of $\mathbb{R}$.

Observe that X can be any nonempty set equipped with a metric; sets of numbers, vectors, sequences, functions, or sets can have interesting and important metrics. In the remainder of this section, we provide a few examples that will reappear in later sections. For these examples we will use notation that is in common usage. The verification that the supplied metric ρ has all the properties required of a metric is left, in most cases, to the exercises.

Euclidean Space

The space $\mathbb{R}^n$ of all n-tuples of real numbers is the basic example that should be used to orient ourselves. In this space we use the metric

$$\rho_2(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}.$$

To verify that this is a metric requires some classical elementary inequalities, in particular, the familiar Cauchy–Schwarz inequality,

$$\sum_{i=1}^{n} |a_i b_i| \le \left(\sum_{i=1}^{n} |a_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |b_i|^2 \right)^{1/2}. \tag{1}$$

In this space, there is a wealth of geometric and linear structure as well that is not available in a general metric space. In an abstract metric space, spheres are not "round," there are no lines and planes and no orthogonal directions. Many of the examples we shall now give do have natural algebraic structures: they are linear spaces. We shall exploit this algebraic structure in Chapters 12 and 14; here we consider only the metric structure and ignore any other features that might be present.

The Discrete Space

Let X be any nonempty set with the metric $\rho(x, y) = 1$ for all $x, y \in X$, with $x \ne y$. To verify that this function, called the *discrete metric*, satisfies

Definition 9.1 is entirely trivial. It is useful to test one's intuition for general metric space principles by considering all concepts and theorems as they apply to this extreme example.

The Minkowski Metrics

On the set $\mathbb{R}^n$, a variety of natural metrics were introduced by Hermann Minkowski (1864–1909) in a study having applications in number theory. These metrics will also help to motivate a number of later considerations.

For any points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ and for any $1 \le p < \infty$, we define a distance

$$\rho_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p},$$

and for $p = \infty$,

$$\rho_\infty(x, y) = \max_{1 \le i \le n} |x_i - y_i|.$$

The case $p = 2$ is the usual Euclidean metric. For the cases $p = 1$ and $p = \infty$, it is easy to check that $\rho_1(x, y)$ and $\rho_\infty(x, y)$ are metrics. It is much less immediate that for other values of p we do indeed have a genuine metric.

The triangle inequality is the real challenge. To show that

$$\rho_p(x, y) \le \rho_p(x, z) + \rho_p(z, y)$$

for $1 < p < \infty$, write $a = x - y$ and $b = y - z$. Then the triangle inequality assumes the form

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \le \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p \right)^{1/p} \tag{2}$$

and is known as Minkowski's inequality. A proof is most easily obtained from a related inequality of Otto Hölder (1860–1937):

$$\sum_{i=1}^{n} |a_i b_i| \le \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q \right)^{1/q} \tag{3}$$

where $p > 1$, $q > 1$, and $p^{-1} + q^{-1} = 1$. (Note that for $p = q = 2$ this inequality reduces to that of Cauchy–Schwarz.)

To prove (3), observe that, should it hold for a, $b \in \mathbb{R}^n$, then it holds for any linear combination $\alpha a + \beta b$. Thus we can reduce our proof to the case where $\sum_{i=1}^{n} |a_i|^p = \sum_{i=1}^{n} |b_i|^q = 1$; that is, we show that $\sum_{i=1}^{n} |a_i b_i| \le 1$.

Consider the curves $u = t^{p-1}$ and the inverse $t = u^{q-1}$, and compute

$$\int_0^\alpha t^{p-1}\, dt = p^{-1}\alpha^p \text{ and } \int_0^\beta u^{q-1}\, du = q^{-1}\beta^q.$$

By considering the areas under the curves that these integrals measure, we find that

$$\alpha\beta \leq p^{-1}\alpha^p + q^{-1}\beta^q. \tag{4}$$

(Exercise 9:1.1 shows how to obtain this more analytically.) Apply (4) with $\alpha = |a_i|$ and $\beta = |b_i|$ to get

$$\begin{aligned}\sum_{i=1}^{n} |a_i b_i| &\leq \sum_{i=1}^{n} \left(p^{-1}|a_i|^p + q^{-1}|b_i|^q\right) \\ &= p^{-1}\sum_{i=1}^{n}|a_i|^p + q^{-1}\sum_{i=1}^{n}|b_i|^q = p^{-1} + q^{-1} = 1,\end{aligned}$$

and we have proved (3).

Now (2) follows from some elementary manipulations. Note first that

$$\sum_{i=1}^{n} (|a_i| + |b_i|)^p = \sum_{i=1}^{n} |a_i| \left(|a_i| + |b_i|\right)^{p-1} + \sum_{i=1}^{n} |b_i| \left(|a_i| + |b_i|\right)^{p-1}.$$

The first sum on the right of this equality can be estimated by using Hölder's inequality with p, q as before, so that $(p-1)q = p$ to obtain

$$\begin{aligned}&\sum_{i=1}^{n} |a_i| \left(|a_i| + |b_i|\right)^{p-1} \\ &\qquad \leq \left(\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right)^{1/q} \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}.\end{aligned}$$

The second sum in the equality has a similar estimate, and so

$$\begin{aligned}&\sum_{i=1}^{n} (|a_i| + |b_i|)^p \\ &\qquad \leq \left(\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right)^{1/q} \left[\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}\right].\end{aligned}$$

Finally, dividing both sides of this inequality by the first expression on the right gives

$$\left(\sum_{i=1}^{n}(|a_i| + |b_i|)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p},$$

from which (2) immediately follows. (If we have divided by zero, then the inequality holds trivially.)

Sequence Spaces

All our examples in this next collection are metric spaces formed of sequences of real numbers.

Example 9.2 We write s for the set of all sequences of real numbers equipped with the metric

$$\rho(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1+|x_i-y_i|)}.$$

Example 9.3 (Baire space) By $\mathbb{N}^{\mathbb{N}}$ we denote the space of all sequences $\mathbf{n} = (n_1, n_2, n_3, \dots)$ of natural numbers. The metric on this space is defined as

$$\rho(\mathbf{m},\mathbf{n}) = \sum_{i=1}^{\infty} \frac{|m_i - n_i|}{2^i\,(1+|m_i-n_i|)}.$$

This is a subspace of s of the preceding example and will be studied extensively in Chapter 11.

Example 9.4 (Cantor space) We denote by $2^{\mathbb{N}}$ the set of all sequences of 0's and 1's equipped with the metric

$$\rho(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}.$$

This space is closely related to the Cantor ternary set, hence its name.

Example 9.5 By ℓ_p $(1 \le p < \infty)$, we denote the set of all sequences $x = (x_1, x_2, x_3 \dots)$ of real numbers such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$ and we write

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}.$$

The metric that we furnish on ℓ_p is defined by $\rho(x,y) = \|x-y\|_p$. Checking that this is indeed a metric requires the following version of Minkowski's inequality, which follows directly from (2):

$$\left(\sum_{i=1}^{\infty} |a_i+b_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |b_i|^p\right)^{1/p}. \tag{5}$$

[The ℓ_p spaces $(1 \le p < \infty)$ are particular cases of the general L_p spaces studied in Chapter 13. The space ℓ_2 is a concrete realization of a Hilbert space as studied in Chapter 14.]

Example 9.6 We denote by ℓ_∞ the set of all bounded sequences of real numbers. The notation is chosen to indicate that this space is a natural extension of the ℓ_p spaces ($1 \le p < \infty$). For $x, y \in \ell_\infty$, $x = \{x_i\}$, $y = \{y_i\}$, define the metric

$$\rho(x, y) = \sup_i |x_i - y_i|.$$

It is easy to check that this is a metric. We verify only the triangle inequality. Let $x, y, z \in \ell_\infty$. For each $i \in \mathbb{N}$,

$$\begin{aligned} |x_i - z_i| &\le |x_i - y_i| + |y_i - z_i| \\ &\le \sup_i |x_i - y_i| + \sup_i |y_i - z_i| = \rho(x, y) + \rho(y, z). \end{aligned}$$

These inequalities are valid for all $i \in \mathbb{N}$, so

$$\rho(x, z) = \sup_i |x_i - z_i| \le \rho(x, y) + \rho(y, z).$$

Important subspaces of ℓ_∞ are c, the space of convergent sequences, and c_0, the space of sequences converging to zero.

Function Spaces

All our examples in this collection are metric spaces formed of real-valued functions.

Example 9.7 We denote by $M[a, b]$ the set of all bounded real-valued functions on the closed interval $[a, b]$. For $f, g \in M$, define ρ by

$$\rho(f, g) = \sup_{a \le t \le b} |f(t) - g(t)|.$$

This is often called the *sup metric* or *uniform metric*, since convergence in this metric is exactly uniform convergence. To verify that this is a metric is easy enough. The triangle inequality in the space follows quickly from the triangle inequality for real numbers.

Some important subspaces of $M[a, b]$ that we have encountered in earlier chapters are

1. $\mathcal{C}[a, b]$, the space of continuous functions,
2. $\triangle[a, b]$, the space of differentiable functions,
3. $\mathcal{P}[a, b]$, the space of polynomials, and
4. $\mathcal{R}[a, b]$, the space of Riemann integrable functions.

The bounded members of various other families of functions also form subspaces of $M[a, b]$.

Example 9.8 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f, g \in L_1$. A natural candidate for a metric ρ on L_1 is given by

$$\rho(f, g) = \int_X |f - g| \, d\mu.$$

One sees immediately that, with this definition, $\rho(f,g) = 0$ if and only if $f = g$ a.e., so condition 2 of Definition 9.1 fails. All the other properties of a metric do hold.

We can address this single deficiency by identifying equivalent functions. If $f = g$ a.e., we consider f and g to be the same element of the space. To avoid additional notation, we shall still use the designation L_1 for the resulting space. Properly speaking, now L_1 does not consist of functions, but equivalence classes of functions defined by the relation $f \sim g$ if $f = g$ a.e. In a more formal treatment, we would be obliged now to show that the metric $\rho(f,g)$ remains unchanged if f and g are replaced by any other equivalent functions.

This is a common feature in the study of metric spaces of functions that arise in integration theory. Functions that are identical almost everywhere must be considered to be the "same" function in order for the metric space definitions to work. While this does not often cause any difficulties, one must be cautious on occasion. Suppose that a function $f \in L_1$ has been given and x is a point in X. What is $f(x)$? The answer is that we do not know! For most applications, however, we do not need specific values: we need integrated or averaged values.

Example 9.9 Let $\mathcal{S}$ denote the measurable, finite a.e. functions on [0,1], and let

$$\rho(f,g) = \int_0^1 \frac{|f-g|}{1+|f-g|}\,d\lambda.$$

Again, as in Example 9.8, we shall identify members of $\mathcal{S}$ that agree almost everywhere.

To verify that this is a metric on $\mathcal{S}$ is easy except for the triangle inequality. To prove this, note first that the function $t/(1+t)$ is an increasing function. Thus, if $h(t)$ is between $f(t)$ and $g(t)$, then

$$\frac{|f(t)-h(t)|}{1+|f(t)-h(t)|} \leq \frac{|f(t)-g(t)|}{1+|f(t)-g(t)|}.$$

If $h(t)$ is not between $f(t)$ and $g(t)$, then either

$$|f(t)-h(t)| \leq |g(t)-h(t)|$$

or

$$|f(t)-h(t)| = |f(t)-g(t)| + |g(t)-h(t)|.$$

The first possibility leads to the inequality

$$\frac{|f(t)-h(t)|}{1+|f(t)-h(t)|} \leq \frac{|g(t)-h(t)|}{1+|g(t)-h(t)|}.$$

The second implies that

$$\frac{|f(t)-h(t)|}{1+|f(t)-h(t)|} \leq \frac{|f(t)-g(t)|}{1+|f(t)-g(t)|} + \frac{|g(t)-h(t)|}{1+|g(t)-h(t)|}. \tag{6}$$

Thus, in all cases, (6) holds for all $t \in [0,1]$. The triangle inequality now follows by integrating both sides of (6).

Example 9.10 Denote by BV$[a,b]$, the set of functions of bounded variation on $[a,b]$. Define ρ by

$$\rho(f,g) = |f(a) - g(a)| + V(f-g;[a,b]).$$

(The variation of a function has been defined in Section 1.14.) To verify that this is a metric, one needs to know basic properties of the variation. Note that, if the first part of the definition had been omitted and the metric taken as $\rho(f,g) = V(f-g;[a,b])$, we could have $\rho(f,g)=0$, and yet f and g may not coincide.

A special subspace of this space will be used in Section 12.8. By NBV$[a,b]$ we denote the space of those functions f of bounded variation on $[a,b]$ that are right continuous on (a,b) and that satisfy $f(a)=0$. The metric is that inherited as a subspace and so is evidently given by $\rho(f,g) = V(f-g;[a,b])$. The "N" in the name is meant to indicate that the functions have been "normalized" by selecting a right continuous member that vanishes at the left end of the interval.

Example 9.11 Let $\mathcal{C}'[a,b]$ denote the set of continuously differentiable functions on $[a,b]$. Define ρ by

$$\rho(f,g) = \max_{a\le t\le b} |f(t)-g(t)| + \max_{a\le t\le b} |f'(t)-g'(t)|.$$

To verify that this is a metric is similar to checking that the sup metric has the correct properties in $M[a,b]$.

Spaces of Sets

Both of the examples in this collection are metric spaces whose elements are sets.

Example 9.12 Let $(X,\mathcal{M},\mu)$ be a measure space with $\mu(X)<\infty$. We seek a metric on $\mathcal{M}$ that measures the size of the set on which two sets differ. If, for $A,B\in\mathcal{M}$, we define

$$\rho(A,B) = \mu(A\triangle B),$$

we find that $\rho(A,B)=0$ if and only if A and B agree except on a set of measure zero. In order to have ρ be a metric, we must identify A and B if $\mu(A\triangle B)=0$. We can do this, for example, by restricting our attention to lifted sets. (See Example 8.22.) We have more flexibility, however, by restricting our attention to the equivalence classes; that is, by identifying A and B if $\mu(A\triangle B)=0$.

Example 9.13 Let $\mathcal{K}$ denote the family of nonempty closed subsets of $[0,1]\times[0,1]$. We would like to capture the idea that the distance between two sets A and B in $\mathcal{K}$ is smaller than δ if every point of A is within δ of some point of B, and vice versa.

For $A \in \mathcal{K}$ and $\delta > 0$, let A_δ denote the union of all closed disks of radius δ centered at points of A. Define ρ by

$$\rho(A, B) = \inf \{\delta > 0 : A \subset B_\delta \text{ and } B \subset A_\delta\}.$$

Using the notation of Section 3.2, we also find that

$$\rho(A, B) = \max\left(\max_{x \in A} \operatorname{dist}(x, B), \max_{y \in B} \operatorname{dist}(y, A)\right). \tag{7}$$

In short, $\rho(A, B)$ measures the greatest distance that a point in A can be from the set B or a point in B from the set A.

To verify the triangle inequality, let $A, B, C \in \mathcal{K}$, let $r = \rho(A, B)$, and let $s = \rho(B, C)$. Then $A_{r+s} = (A_r)_s \supset B_s \supset C$ and also $C_{r+s} = (C_s)_r \supset B_r \supset A$. Thus $\rho(A, C) \leq r + s = \rho(A, B) + \rho(B, C)$.

This metric ρ is called the *Hausdorff* metric on the space of closed subsets of $[0, 1] \times [0, 1]$.

Exercises

9:1.1◊ Give an analytic proof for the inequality (4) as follows: Let $p > 1$ and

$$f(t) = t^{1/p} - t/p + 1/p - 1 \quad (t \geq 0).$$

Since $f(1) = f'(1) = 0$ and f' is positive on $(0, 1)$ and negative on $(1, \infty)$, it follows that $f(t) \leq 0$ for all $t \geq 0$. In particular, $f(\alpha^p \beta^{-q}) \leq 0$, and this leads to (4).

9:1.2 Verify that all the examples in this section are actually metric spaces. (In some cases the triangle inequality, usually the hardest part to check, has been proved.)

9:1.3 Verify that in Example 9.13 the alternative expression (7) for ρ is valid.

9:1.4 Describe, informally, what it means for two functions in $M[a, b]$ to be "close" to one another. Do the same for Example 9.8.

9:1.5 Let g be a function defined on $[0, \infty)$ such that $g(0) = 0$ and g is strictly increasing and satisfies $g(x+y) \leq g(x)+g(y)$ for all $x, y \geq 0$.

(a) Prove that if ρ is a metric for a set X then $\sigma = g \circ \rho$ is also a metric for X.

(b) Use (a) to verify that if ρ is a metric on X then so is $\sigma = \rho(1 + \rho)^{-1}$ and that $\sigma(x, y) < 1$ for all $x, y \in X$.

9.2 Convergence and Related Notions

Let (X, ρ) be a metric space. A sequence $\{x_n\}$ of members of X *converges* to $x \in X$ if $\lim_{n\to\infty} \rho(x_n, x) = 0$. When $\{x_n\}$ converges to x, we write

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x.$$

For each of our examples in the previous section, it is an important exercise to determine what convergence of a sequence means relative to the stated metric. For example, applying this definition of convergence to the space $M[a, b]$ or its subspaces, we find that $f_n \to f$ if and only if $\{f_n\}$ converges uniformly to f. In Example 9.8, convergence is our familiar notion of mean convergence. In Example 9.9, convergence is convergence in measure (see Exercise 5:4.6).

A number of familiar concepts from $\mathbb{R}^n$ carry over to arbitrary metric spaces (X, ρ).

- For $x_0 \in X$ and $r > 0$, the set

  $$B(x_0, r) = \{x \in X : \rho(x_0, x) < r\}$$

  is called the *open ball* with center x_0 and radius r.

- The set

  $$B[x_0, r] = \{x \in X : \rho(x_0, x) \leq r\}$$

  is called the *closed ball* with center x_0 and radius r.

- A set $G \subset X$ is called *open* if for each $x_0 \in G$ there exists $r > 0$ such that $B(x_0, r) \subset G$.

- A set F is called *closed* if its complement $\widetilde{F}$ is open.

- A set E is *bounded* if $\sup\{\rho(x, y) : x, y \in E\}$ is finite.[1]

- A *neighborhood* of x_0 is any open set G containing x_0.

- If $G = B(x_0, \varepsilon)$, we call G the ε*-neighborhood* of x_0.

- The point x_0 is called an *interior point* of a set A if x_0 has a neighborhood contained in A.

- The *interior* of A consists of all interior points of A and is denoted by A^o or, occasionally, $\text{int}(A)$.

- A point $x_0 \in X$ is a *limit point* or *point of accumulation* of a set A if every neighborhood of x_0 contains points of A distinct from x_0.

[1] This is the definition of boundedness appropriate to metric space theory. In the setting of a metric linear space, a different (not equivalent) definition is used.

- The *closure* $\overline{A}$ of a set A consists of all points that are either in A or limit points of A. [It is the smallest closed set containing A. That there exists such a set follows from Exercise 9:2.5(c). One verifies easily that $x_0 \in \overline{A}$ if and only if there exists a sequence $\{x_n\}$ of points in A such that $x_n \to x$.]

- A *boundary point* of A is a point x_0 such that every neighborhood of x_0 contains points of A as well as points of $\widetilde{A}$.

- Let A and B be subsets of X. If $\overline{A} \supset B$ or, equivalently, if every open ball centered at a point of B contains a point of A, we say that A is *dense* in B. (Note that this does not require A to be a subset of B.) If $\overline{A} = X$, we simply say that A is *dense*.

- The *distance* between a point $x \in X$ and a nonempty set $A \subset X$ is defined as
$$\operatorname{dist}(x, A) = \inf\{\rho(x, y) : y \in A\}.$$

We illustrate some of these concepts with examples.

Example 9.14 Consider the space $\mathcal{C}[a,b]$ furnished with its supremum norm. Let $f_0 \in \mathcal{C}[a,b]$, and let $\varepsilon > 0$. Then $B(f_0, \varepsilon)$ consists of all continuous functions f that satisfy $|f(t) - f_0(t)| < \varepsilon$ for all $t \in [a,b]$. A continuous function f is a boundary point of $B(f_0, \varepsilon)$ if and only if $|f(t) - f_0(t)| \leq \varepsilon$ for all $t \in [a,b]$ and there exists t_0 such that $|f(t_0) - f_0(t_0)| = \varepsilon$. Geometrically, $f \in B(f_0, \varepsilon)$ if and only if the graph of f lies strictly between the graphs of $f_0 - \varepsilon$ and $f_0 + \varepsilon$. Similarly, f is a boundary point of $B(f_0, \varepsilon)$ if and only if the graph of f lies between the graphs of $f_0 - \varepsilon$ and $f_0 + \varepsilon$ and there exists t_0 such that $f(t_0) = f_0(t_0) + \varepsilon$ or $f(t_0) = f_0(t_0) - \varepsilon$.

The subspace $\triangle[a,b]$ of differentiable functions on $[a,b]$ is neither open nor closed in $\mathcal{C}[a,b]$. To see that $\triangle$ is not open, observe that every neighborhood of $f_0 \in \triangle$ contains a polygonal function that is not differentiable. Thus $\triangle$ is not only not open, it has an empty interior. Since the uniform limit of a sequence of differentiable functions need not be differentiable, $\triangle$ is not closed. (See Exercise 9:2.4.)

Example 9.15 Let K be the Cantor set, and let $\{(a_n, b_n)\}$ be the sequence of complementary intervals. Let $X = K \cup C$, where C consists of the midpoints of the intervals (a_n, b_n). Take $\rho(x, y) = |x - y|$. Then K is closed, C is open and $\overline{C} = X$. Observe that, for $c \in C$, $\{c\}$ is both open and closed. For $c = (a_n + b_n)/2$ and $\varepsilon < (b_n - a_n)/2$,
$$B(c, \varepsilon) = B[c, \varepsilon] = \{c\}.$$

Example 9.16 Let $\mathcal{K}$ be the family of all closed subsets of the square $[0,1] \times [0,1]$ equipped with the Hausdorff metric (see Example 9.13). We shall show that all nonempty members of $\mathcal{K}$ can be approximated by finite subsets of $[0,1] \times [0,1]$ so that the collection of all finite subsets forms a set dense in $\mathcal{K}$.

Let $\varepsilon > 0$, and let K be any nonempty closed set in $\mathcal{K}$. The union of all open disks of radius ε centered at points of K is an open set in $\mathbb{R}^2$. By the Heine-Borel Theorem, there exist points $x_1, x_2, \dots, x_n \in K$ such that

$$K \subset S(x_1, \varepsilon) \cup \cdots \cup S(x_n, \varepsilon),$$

where $S(x, \varepsilon)$ is the open disk of radius ε centered at x. Let E be the finite collection $\{x_1, \dots, x_n\}$. Note that $\rho(E, K) < \varepsilon$, since $E_\varepsilon \supset K$ and $K_\varepsilon \supset K \supset E$. Thus K has been approximated by a finite subset of $[0, 1] \times [0, 1]$.

Exercises

9:2.1 (a) Prove that if $x_n \to x$ and $x_n \to y$ then $x = y$.

(b) Prove that $x_n \to x$ if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in B(x, \varepsilon)$ for all $n \geq N$.

9:2.2 Characterize convergence in Example 9.2 and in Example 9.6.

9:2.3 Show, in a general metric space, that the open ball is open and that the closed ball is closed, but that (contrary to what one finds in Euclidean space) the closed ball $B[x_0, \varepsilon]$ is not necessarily the closure of the open ball $B(x_0, \varepsilon)$. [Hint: Let $X = \mathbb{N}$, $\rho(x, y) = |x - y|$.]

9:2.4 Show that A is closed if and only if A contains all its limit points (i.e., if $A = \overline{A}$).

9:2.5 Let (X, ρ) be a metric space.

(a) Prove that X and $\emptyset$ are both open and closed.

(b) Prove that a finite union of closed sets is closed and a finite intersection of open sets is open.

(c) Prove that an arbitrary union of open sets is open, and an arbitrary intersection of closed sets is closed.

9:2.6 Refer to Example 9.7. Prove that $\mathcal{C}[a, b]$ and $\mathcal{R}$ are closed subspaces of $M[a, b]$, but $\mathcal{P}$ and $\triangle$ are not closed. Let $\mathcal{P}_n$ denote the polynomials of degree $\leq n$. Is $\mathcal{P}_n$ closed?

9:2.7$\diamondsuit$ Refer to Example 9.6. Show that c and c_0 are closed subspaces of ℓ_∞.

9:2.8 Describe the 1/10 (base ten) neighborhood of a point in $2^{\mathbb{N}}$.

9:2.9 Let X be an arbitrary set furnished with the discrete metric. Show that every subset of X is both open and closed.

9:2.10 Consider the set $\mathcal{C}$ of continuous functions on $[0, 1]$ with two different metrics, both of interest: the sup metric

$$\rho_1(f, g) = \sup |f(t) - g(t)|,$$

and the L_1 metric

$$\rho_2 = \int_X |f - g| \, d\lambda$$

from Example 9.8. Let B_1 and B_2 be the open balls centered at the zero function with respect to the two metrics ρ_1 and ρ_2. Is B_1 open in $(\mathcal{C}, \rho_2)$? Is B_2 open in $(\mathcal{C}, \rho_1)$?

9:2.11 The space $\mathcal{C}[a, b]$ of Example 9.7 is a closed subspace of $M[a, b]$. Show that the collections of bounded functions from each of the Baire classes on $[a, b]$ are also closed subspaces of $M[a, b]$. (See Exercise 4:6.2.)

9.3 Continuity

Let (X, ρ) and (Y, σ) be metric spaces, and let $T : X \to Y$. We say that T is *continuous at* $x \in X$ if, for every sequence $\{x_n\}$ converging to x, $\{T(x_n)\}$ converges to $T(x)$. If T is continuous at every $x \in X$, we say T is *continuous*. One verifies, just as for real functions, that T is continuous at x if and only if, for every $\varepsilon > 0$, there is a $\delta > 0$ so that $\sigma(T(x), T(y)) < \varepsilon$, whenever $\rho(x, y) < \delta$. Also T is continuous at every point in X if and only if, for every open set $G \subset Y$, the set

$$T^{-1}(G) = \{x \in X : T(x) \in G\}$$

is open. Proofs of some of the properties of continuous functions are virtually identical to the corresponding proofs for real functions of a real variable. We leave these as Exercises 9:3.1, 9:3.2, and 9:3.3.

We present a few examples of continuous functions on some of the metric spaces we mentioned in Section 9.1.

Example 9.17 Let $X = Y = \mathcal{C}[a, b]$. Define $T : X \to Y$ by

$$(T(f))(t) = \int_a^t f(s) \, ds.$$

To check the continuity of T at $f \in X$, let $f_n \to f$ in (X, ρ). This means that $\rho(f_n, f) = \max_t |f_n(t) - f(t)| \to 0$ as $n \to \infty$. We calculate

$$\begin{aligned}
\rho(T(f_n), T(f)) &= \max_t |(T(f_n))(t) - (T(f))(t)| \\
&= \max_t \left| \int_a^t (f_n(s) - f(s)) \, ds \right| \\
&\leq \max_t \int_a^t |f_n(s) - f(s)| \, ds = \int_a^b |f_n(s) - f(s)| \, ds \\
&\leq (b - a) \max_t |f_n(t) - f(t)| = (b - a)\rho(f_n, f).
\end{aligned}$$

Since $\lim_{n\to\infty} \rho(f_n, f) = 0$ by hypothesis, we conclude that

$$\lim_{n\to\infty} \rho(T(f_n), T(f)) = 0.$$

That is, $T(f_n) \to T(f)$, and T is continuous.

Example 9.18 Let $X = \mathcal{C}[a,b]$, $Y = \mathbb{R}$. Define $T : X \to Y$ by

$$T(f) = \int_a^b f(t)\,dt.$$

We verify easily that if $f_n \to f$ in X then $T(f_n) \to T(f)$ in $\mathbb{R}$, so T is continuous at f.

Observe that the functions in Examples 9.17 and 9.18 are defined by integrals. Such functions are often continuous. When functions are defined by differentiation, continuity is likely to fail, as illustrated by the next example.

Example 9.19 Let $X \subset M[0,1]$ consist of those functions on $[0,1]$ with bounded derivatives, and let $Y \subset M[0,1]$ consist of the derivatives of functions in X. Define $D : X \to Y$ by $D(f) = f'$. In the space $M[0,1]$, $f_n \to f$ if and only if $f_n \to f$ [unif] on $[0,1]$. The sequence $\{f_n\}$ from $M[0,1]$ defined by $f_n(t) = n^{-1}t^n$ furnishes an example such that $f_n \to 0$ in X, but for every $n \in \mathbb{N}$, $f_n'(1) = 1$, and the sequence $\{D(f_n)\} = \{f_n'\}$ does not converge in Y. (See also Example 9.11 and Exercise 9:3.6.)

Several other examples of continuous or discontinuous functions can be found in the exercises. Observe that, when X consists of a space of functions, we are using uppercase symbols such as T or D (rather than f or g). This is a common practice, particularly when X is a linear space other than $\mathbb{R}$ and the functions involved are linear functions. One often emphasizes this by calling the function a *linear transformation* or *operator*. We shall encounter examples of integral or differential operators in what follows.

Example 9.20 Let (X, ρ) be a metric space and A a nonempty subset of X. Let

$$f(x) = \operatorname{dist}(x, A) = \inf\{\rho(x,y) : y \in A\}.$$

Then $f : X \to \mathbb{R}$, and f is continuous. To verify this, let $\varepsilon > 0$, and let x, $y \in X$ with $\rho(x,y) < \varepsilon/2$. Choose $a \in A$ such that

$$\rho(x,a) < \operatorname{dist}(x,A) + \tfrac{1}{2}\varepsilon.$$

Then

$$\begin{aligned} \operatorname{dist}(y,A) &\le \rho(y,a) \le \rho(y,x) + \rho(x,a) \\ &< \tfrac{1}{2}\varepsilon + \operatorname{dist}(x,A) + \tfrac{1}{2}\varepsilon = \operatorname{dist}(x,A) + \varepsilon. \end{aligned}$$

Similarly,

$$\operatorname{dist}(x,A) < \operatorname{dist}(y,A) + \varepsilon.$$

It follows that $|f(y) - f(x)| < \varepsilon$ if $\rho(x,y) < \varepsilon/2$, so f is continuous at x.

One verifies, as in elementary analysis, that the class of real-valued continuous functions on a metric space is closed under uniform limits and under the standard algebraic operations. This allows an immediate proof of a theorem of P. Urysohn, proved in 1925.

Theorem 9.21 (Urysohn) *Let X be a metric space, and let A and B be disjoint nonempty closed subsets of X. Then there exists a continuous function $g : X \to I\!R$ such that $g(x) = 0$ for all $x \in A$, $g(x) = 1$ for all $x \in B$, and $0 < g(x) < 1$ for all $x \in \widetilde{A} \cap \widetilde{B}$.*

Proof. Let

$$g(x) = \frac{\operatorname{dist}(x, A)}{\operatorname{dist}(x, A) + \operatorname{dist}(x, B)}.$$

It is clear that g has all the required properties. ■

Theorem 9.21 is a special case of the Tietze extension theorem that we mentioned in Section 4.5. We needed Tietze's theorem to prove Lusin's theorem, but proved it only for $X = [a, b]$. We can now prove the version of Tietze's theorem that we needed for Theorems 4.23 and 4.25. We begin with a lemma.

Lemma 9.22 *Let X be a metric space, F a closed subset of X and f a real-valued function defined on F and let $M > 0$. Suppose that f is continuous on F and $|f(x)| \leq M$ for all $x \in F$. Then there exists a continuous function $g : X \to I\!R$ such that*

1. *$|g(x)| \leq \frac{1}{3}M$ for all $x \in F$.*
2. *$|g(x)| < \frac{1}{3}M$ for all $x \in \widetilde{F}$.*
3. *$|f(x) - g(x)| \leq \frac{2}{3}M$ for all $x \in F$.*

Proof. Define the sets

$$A = \{x \in F : f(x) \leq -\tfrac{1}{3}M\} \quad \text{and} \quad B = \{x \in F : f(x) \geq \tfrac{1}{3}M\}.$$

Both A and B are closed (see Exercise 9:3.1). It is clear that A and B are disjoint. If A and B are nonempty let

$$g(x) = \tfrac{1}{3}M \frac{\operatorname{dist}(x, A) - \operatorname{dist}(x, B)}{\operatorname{dist}(x, A) + \operatorname{dist}(x, B)}.$$

One verifies routinely that g has the required properties. If A and/or B is empty, the function g must be defined differently. See Exercise 9:3.9. ■

Theorem 9.23 *Let f be a continuous real-valued function defined on a closed subset F of a metric space X. Then there exists a continuous extension $\overline{f}$ of f to all of X. If $|f(x)| \leq M$ for all $x \in F$, where $M > 0$, then $\overline{f}$ can be chosen so that $|\overline{f}(x)| \leq M$ for all $x \in X$ and $|\overline{f}(x)| < M$ for $x \in \widetilde{F}$.*

Proof. Suppose first that f is bounded on F and $|f(x)| \le M$ for all $x \in F$. We shall use Lemma 9.22 to obtain a sequence $\{g_n\}$ of continuous functions on X so that

$$\overline{f} = \sum_{n=0}^{\infty} g_n$$

is the desired function. We obtain the sequence $\{g_n\}$ inductively.

Let $g_0(x) = 0$ for all $x \in X$. Suppose for $n \ge 0$ that we have continuous functions $g_0, \dots, g_n$ defined on X such that

$$\left| f(x) - \sum_{i=0}^{n} g_i(x) \right| \le \left(\tfrac{2}{3}\right)^n M \tag{8}$$

for all $x \in F$. Applying Lemma 9.22 to the functions

$$f - \sum_{i=0}^{n} g_i$$

with respect to the constants $(\frac{2}{3})^n M$, we obtain a continuous function g_{n+1} defined on X such that

$$|g_{n+1}(x)| \le \tfrac{1}{3}(\tfrac{2}{3})^n M \qquad (x \in F), \tag{9}$$

$$|g_{n+1}(x)| < \tfrac{1}{3}(\tfrac{2}{3})^n M \qquad (x \in \widetilde{F}), \tag{10}$$

and

$$\left| f(x) - \sum_{i=0}^{n+1} g_i(x) \right| < \left(\tfrac{2}{3}\right)^{n+1} M \qquad (x \in F). \tag{11}$$

Because of (9), the series $\sum_{n=0}^{\infty} g_n$ converges uniformly on X to some continuous function $\overline{f}$ on X. (See Exercise 9:3.3.) From (8), we infer that $f = \sum_{n=0}^{\infty} g_n$ on F, so $\overline{f} = f$ on F.

It remains to verify that $|\overline{f}| < M$ on $\widetilde{F}$. Let $x \in \widetilde{F}$. Then

$$\begin{aligned} |\overline{f}(x)| &= \left| \sum_{n=0}^{\infty} g_n(x) \right| = \left| \sum_{n=0}^{\infty} g_{n+1}(x) \right| \\ &\le \sum_{n=0}^{\infty} |g_{n+1}(x)| < M \sum_{n=0}^{\infty} \tfrac{1}{3}(\tfrac{2}{3})^n = M, \end{aligned}$$

the last inequality following from (10). This completes the proof of the theorem when f is bounded on F.

We leave the verification of the theorem for unbounded continuous functions on F as Exercise 9:3.7. ∎

Exercises

9:3.1 Prove that $T : X \to Y$ is continuous if and only if $T^{-1}(E)$ is closed (open) for every closed (open) set $E \subset Y$.

9:3.2 (a) Prove that the class of continuous real-valued functions on a metric space is closed under the arithmetic operations of addition, subtraction, and multiplication. (How about division?)

(b) State precisely and prove a theorem that asserts under what conditions the composition $f \circ g$ of two continuous functions is continuous.

9:3.3 Prove that if $\{f_n\}$ is a sequence of continuous real-valued functions on (X, ρ) and $f_n \to f$ [unif] then f is continuous.

9:3.4 (Refer to Example 9.12.) Define $T : \mathcal{M} \to \mathbb{R}$ by $T(A) = \mu(A)$. Is T continuous?

9:3.5 (Refer to Example 9.4.) For each $s = s_1 s_2 s_3 \cdots \in 2^{\mathbb{N}}$, define $T(s) = s_2 s_3 s_4 \ldots$. Then $T : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. Is T continuous?

9:3.6 (Refer to Example 9.11.) Is the mapping $D : \mathcal{C}'[a, b] \to \mathcal{C}[a, b]$, where $D(f) = f'$, continuous?

9:3.7 Complete the proof of Tietze's theorem for unbounded functions. [Hint: Let h be a strictly increasing continuous function mapping $\mathbb{R}$ onto $(-1, 1)$. Consider the function $h \circ f$ and note Exercise 9:3.2.]

9:3.8 In the space of Example 9.12, let $f(A) = \widetilde{A}$. Is f continuous?

9:3.9 In the proof of Lemma 9.22 show how to define g if A and/or B is empty. [Hint: For example, if $A = \emptyset$ and $B \neq \emptyset$, then try using $g(x) = \frac{1}{3}M(1 - \min(1, \operatorname{dist}(x, B)))$.]

9.4 Homeomorphisms and Isometries

Given two metric spaces (X, ρ) and (Y, σ), we shall often need to know if there is a close relation between them. Do the two spaces have identical or nearly identical structures? There are two important ways to describe this.

A bijection $h : X \to Y$ is called a *homeomorphism* if h and h^{-1} are both continuous. The condition that h^{-1} be continuous is equivalent to the condition that h map open sets onto open sets. Two spaces are said to be *homeomorphic*, or *topologically equivalent*, if there is a homeomorphism between them. A property that is preserved under homeomorphisms is called a *topological property*.

A homeomorphism $h : X \to Y$ that also preserves distances is called an *isometry*. This means that

$$\sigma(h(x_1), h(x_2)) = \rho(x_1, x_2) \quad (x_1, x_2 \in X). \tag{12}$$

In fact, this condition alone characterizes isometries: if the mapping $h : X \to Y$ is onto and satisfies (12), then it is a homeomorphism that preserves distances and hence is an isometry. If there exists an isometry between X and Y, we say that X and Y are *isometric*. Two metric spaces that are isometric are, from the metric point of view, the same except for such things as labeling and notation.

A special case should be noted. Suppose that we are given (as we often are) two different metrics ρ and d on the same space X. When are they equivalent? That is, when is the identity mapping a homeomorphism from (X, ρ) to (X, d)? The proof of Theorem 9.24 is left as Exercise 9:4.4.

Theorem 9.24 *Let ρ and d be metrics on a nonempty set X. Then the identity mapping is a homeomorphism from (X, ρ) to (X, d) if and only if, for every $x \in X$ and $\varepsilon > 0$, there is a $\delta > 0$ such that, for all $y \in X$,*

$$\rho(x, y) < \delta \Rightarrow d(x, y) < \varepsilon \quad \textit{and} \quad d(x, y) < \delta \Rightarrow \rho(x, y) < \varepsilon.$$

The following examples will help to illustrate the ideas of this section. Example 9.26 is particularly illuminating, since one can sketch pictures that show how the topological equivalence of the Minkowski metrics can occur. In this example, the spaces compared involve a single set X with two or more different metrics on it. Example 9.27 illustrates that two spaces involving entirely different sorts of objects can be isometric.

Example 9.25 For a simple example, consider any two subsets X and Y of the real numbers, both equipped with the usual metric. When are they topologically equivalent or isometric?

Any two open intervals in $\mathbb{R}$ are topologically equivalent under an obvious mapping, but the homeomorphism between them cannot be an isometry (cannot preserve distances) unless they have the same length. Thus two open subintervals of $\mathbb{R}$ are isometric if and only if they have the same length. Further questions can be asked. For example, are any two Cantor sets homeomorphic (Exercise 4:1.10)? When is there an isometry between two Cantor sets?

Example 9.26 Recall that on the set $\mathbb{R}^n$ we have defined a family of metrics

$$\rho_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p} \qquad (1 \le p < \infty),$$

$$\rho_\infty(x, y) = \max_{1 \le i \le n} |x_i - y_i|,$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Let us compare the spaces $(\mathbb{R}^n, \rho_p)$ with the help of Theorem 9.24.

A picture for the special case $n = 2$ tells it all. Consider the open unit balls centered at the origin in each of the metrics

$$B_p(0, 1) = \{y \in \mathbb{R}^2 : \rho_p(0, y) < 1\}$$

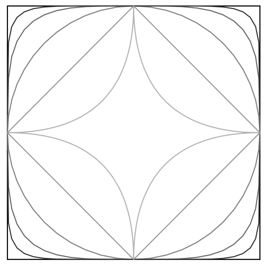

Figure 9.1: The unit balls $B_p(0,1)$ in $\mathbb{R}^2$ ($p = \frac{1}{2}$, 1, 2, 4, 8, and ∞).

for $1 \le p \le \infty$. In Figure 9.1, these are drawn for $p = \frac{1}{2}$, $p = 1$, 2, 4, 8, and $p = \infty$. (The case $p = \frac{1}{2}$ is included for contrast—it does not define a metric.) We see that, as $p \to \infty$, the balls $B_p(0,1)$ become increasingly flatter and approach $B_\infty(0,1)$ from below. In general, we also see that $B_p(0,1) \subset B_q(0,1)$ if $p < q$.

It is easy to see geometrically that, for any fixed $1 \le p, q \le \infty$ and for every $\varepsilon > 0$, there is a $\delta > 0$ so that $B_p(0,\delta) \subset B_q(0,\varepsilon)$. (This can also be verified analytically, as Exercise 9:4.8 demands.) This is true at any point of the space (not just at the origin), and so Theorem 9.24 shows that the identity map is a homeomorphism between $(\mathbb{R}^2, \rho_p)$ and $(\mathbb{R}^2, \rho_q)$. Indeed, the spaces $(\mathbb{R}^n, \rho_p)$ $(1 \le p \le \infty)$ are all, from the topological point of view, the same.

Let us look closer at the metric spaces $(\mathbb{R}^2, \rho_1)$ and $(\mathbb{R}^2, \rho_\infty)$. As we have observed, the function $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x) = x$ is a homeomorphism, so these spaces are topologically equivalent, but h is not an isometry. Nonetheless, these spaces are isometric. For an isometry, we need to find some other homeomorphism that does preserve distances. (This is left as Exercise 9:4.5.)

Example 9.27 Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let (L_1, ρ_1) be the metric space of Example 9.8 with metric

$$\rho_1(f,g) = \int_X |f - g|\, d\mu \quad \text{for } f, g \in L_1,$$

and let $(\mathcal{M}, \rho_2)$ be the metric space of Example 9.12 with metric

$$\rho_2(A,B) = \mu(A \triangle B) \quad \text{for } A, B \in \mathcal{M}.$$

(Here we allow ourselves the usual convenience of writing, for example, f for an equivalence class of functions and A for an equivalence class of sets.) Let

$$K = \left\{ f \in L_1 : f = \chi_A \quad \text{for some } A \in \mathcal{M} \right\}.$$

Then (K, ρ_1) and $(\mathcal{M}, \rho_2)$ are isometric (Exercise 9:4.6).

Exercises

9:4.1 Find a homeomorphism between $[0,1)$ and $[0,\infty)$. Thus an unbounded set can be homeomorphic to a bounded one.

9:4.2 If the mapping h is onto and satisfies (12), then it is an isometry.

9:4.3 Is the curve $y = 1/x$, $x > 0$, in the plane homeomorphic to the interval $(0,\infty)$? Are the two sets isometric? (Assume that $\mathbb{R}$ and $\mathbb{R}^2$ have the usual metrics.)

9:4.4 The identity mapping is a homeomorphism from (X,ρ) to (X,d) if and only if for every $x \in X$ and $\varepsilon > 0$ there is a $\delta > 0$ such that, for all $y \in X$, $\rho(x,y) < \delta \Rightarrow d(x,y) < \varepsilon$ and $d(x,y) < \delta \Rightarrow \rho(x,y) < \varepsilon$.

9:4.5 Show that the spaces $(\mathbb{R}^2,\rho_1)$ and $(\mathbb{R}^2,\rho_\infty)$ are isometric by showing that

$$f(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$$

is an isometry from $(\mathbb{R}^2,\rho_\infty)$ to $(\mathbb{R}^2,\rho_1)$. (Explain the geometry of this mapping.)

9:4.6◇ Prove that the two spaces of Example 9.27 are isometric. [Hint: Let $T(A) = \chi_A$.]

9:4.7◇ Let X be a set and ρ a metric on it. Show that $d = \rho/(1+\rho)$ is also a metric on X and that the function $h : X \to X$ defined by $h(x) = x$ is a homeomorphism, so the spaces (X,ρ) and (X,d) are topologically equivalent. Note, in particular, that a bounded metric can be equivalent to an unbounded metric.

9:4.8 Verify analytically that the identity map is a homeomorphism between $(\mathbb{R}^2,\rho_p)$ and $(\mathbb{R}^2,\rho_q)$ for any $1 \le p, q \le \infty$.

9:4.9 Show that $\lim_{p\to\infty} \rho_p(x,y) = \rho_\infty(x,y)$.

9:4.10 Sketch the "unit balls" $B_p(0,1)$ for $0 < p < 1$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ and note that they are not convex. (See Figure 9.1 for $p = \frac{1}{2}$ and $n = 2$.) Is

$$\rho_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$$

a metric on $\mathbb{R}^n$ for $0 < p < 1$.

9:4.11 On the Cantor space $2^{\mathbb{N}}$ of Example 9.4, consider the two metrics

$$\rho(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

and

$$d(x,y) = 2^{-n},$$

where n is the first index for which $x_n \ne y_n$. Show that these metrics are equivalent.

9:4.12 Show that Cantor space (Example 9.4) is homeomorphic to the Cantor ternary set.

9.5 Separable Spaces

Many metric spaces possess special properties of some importance. Arguments on the real line can often be carried out by using the fact that the rationals form a dense subset. The only feature here that matters is that there is some countable dense subset. Similar arguments are available in general metric spaces that have a countable dense subset.

Definition 9.28 Let X be a metric space. If X possesses a countable dense subset, then X is called a *separable* metric space.

For example, $\mathbb{R}^n$ with the usual metric is separable since

$$\{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{Q}\}$$

is a countable dense subset of $\mathbb{R}^n$. Let us check some of the spaces from Section 9.1 for separability.

Example 9.29 The space ℓ_∞ (of Example 9.6) is not separable. To see this, observe that the set

$$A = \{\{x_i\} : x_i = 0 \;\text{ or }\; x_i = 1\}$$

is an uncountable subset of ℓ_∞. If x and y are distinct elements of A, then $\rho(x, y) = 1$. Thus the family $\{B(x, 1/2) : x \in A\}$ is an uncountable pairwise disjoint family of balls in ℓ_∞. Any dense subset of ℓ_∞ must contain points of each ball in this family and so must be uncountable.

Example 9.30 The subspace c of ℓ_∞ is separable. To see this, let $\{r_j\}$ be an enumeration of $\mathbb{Q}$. For every triple $i, j, n \in \mathbb{N}$, let

$$A_{ijn} = \{x \in c : \; \forall i,\, x_i \in \mathbb{Q} \text{ and } \forall i \geq n,\, x_i = r_j\;\},$$

and let $A = \bigcup_{i,j,n} A_{ijn}$. One verifies easily that A is dense in c. Since each of the sets A_{ijn} is countable, so is A.

Example 9.31 The space $\mathcal{C}[a, b]$ is separable. This can be based on Weierstrass's approximation theorem, which states that every $f \in \mathcal{C}[a, b]$ is a uniform limit of a sequence of polynomials. Since each polynomial can be approximated uniformly by polynomials with rational coefficients, we see that $\mathcal{C}[a, b]$ is separable. (For proofs of the Weierstrass approximation theorem, see Section 9.13 or Section 15.6.)

Example 9.32 The space $M[a, b]$ is not separable. If f and g are the characteristic functions of distinct sets, then $\rho(f, g) = 1$. There are uncountably many distinct subsets of $[a, b]$ and thus uncountably many distinct elements of $M[a, b]$, each at distance 1 from the other. No countable set can be dense in this space.

Example 9.33 The space $\mathcal{S}$ of Example 9.9 is separable. To see this, recall that to every measurable function f corresponds a sequence $\{f_n\}$ of continuous functions such that $f_n \to f$ [meas]. Each of the functions f_n can be approximated uniformly by polynomials with rational coefficients. It follows that the set of polynomials with rational coefficients is a countable dense subset of $\mathcal{S}$.

Example 9.34 Let $([0,1], \mathcal{L}, \lambda)$ be the Lebesgue measure space, and let ρ be the metric of Example 9.12 on the equivalence classes of $\mathcal{L}$. Then $\mathcal{L}$ is separable. Let A consist of all sets that are finite unions of open intervals with rational endpoints. Then A is a countable and dense subset of this space.

Example 9.35 The space $\mathcal{K}$ of Example 9.13 is separable. We observed in Example 9.16 that the family of finite sets is dense in $\mathcal{K}$. A slight variation in the argument shows that the family of finite sets whose members have rational coordinates is also dense in $\mathcal{K}$.

In Exercise 9:5.1, we indicate the separability or nonseparability of the other spaces appearing in Section 9.1.

Exercises

9:5.1 Verify, or complete the verifications, that each of the spaces s, $2^{\mathbb{N}}$, c, c_0, $\mathcal{C}[a,b]$, $\mathcal{S}$, $\mathcal{C}'[a,b]$, and $\mathcal{K}$ is separable, while the spaces ℓ_∞, $M[a,b]$ and $\mathrm{BV}[a,b]$ are not.

9:5.2 Let $\mathcal{K}_c$ denote the subspace of compact, convex members of $\mathcal{K}$ (the space of Example 9.13) . Prove that $\mathcal{K}_c$ is separable.

9:5.3 Prove that a metric space X is separable if and only if there exists a countable collection $\mathcal{U}$ of open sets such that each open set in X can be expressed as a union of members of $\mathcal{U}$.

9:5.4 Prove that in a separable metric space every uncountable set contains a convergent sequence of distinct points.

9:5.5$\diamondsuit$ Prove Lindelöf's theorem: Every open cover of a separable metric space has a countable subcover.

9:5.6 Prove that a subspace of a separable metric space is itself separable.

9:5.7$\diamondsuit$ Prove that the following spaces are separable:

(a) The spaces ℓ_p for $1 \le p < \infty$. [Also explain how ℓ_1 can be considered a special case of L_1 (Example 9.8).]

(b) The space $L_1([0,1], \mathcal{L}, \lambda)$. [Hint: Show first that the class of continuous functions is dense.]

9.6 Complete Spaces

We turn now to a discussion of one of the most important properties that can be possessed by a metric space—completeness. All the deep properties of real sequences and real functions depend on the fact that $\mathbb{R}$ is complete. Many of these properties can be carried over to general metric spaces.

A sequence $\{x_n\}$ in a metric space is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that, if $m \geq N$ and $n \geq N$, then $\rho(x_m, x_n) < \varepsilon$. This is equivalent to the requirement that

$$\lim_{m,n\to\infty} \rho(x_m, x_n) = 0.$$

Some elementary observations are immediate. A Cauchy sequence must be bounded, since all but a finite number of members of the sequence must lie in some ball of radius 1. Every convergent sequence is a Cauchy sequence. To verify this, observe that if $x_n \to x$ and $\varepsilon > 0$ then there exists $N \in \mathbb{N}$ such that $\rho(x, x_n) < \frac{1}{2}\varepsilon$ for all $n \geq N$. If $m, n \geq N$, then

$$\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x, x_m) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

The converse is not true in general: there can be Cauchy sequences that are not convergent. For example, the sequence $\{1/n\}$ is a Cauchy sequence in $X = (0, 1)$, but does not converge in X.

Definition 9.36 A metric space is said to be *complete* if every Cauchy sequence in X converges.

A useful equivalent definition is that every Cauchy sequence has a convergent subsequence, since this implies (Exercise 9:6.4) that every Cauchy sequence converges. In many completeness proofs it is convenient to stop once we have established this fact. We leave the proof of the next theorem as an exercise. In $\mathbb{R}$, this theorem is just the familiar Cantor intersection theorem (see Theorem 1.2). Observe that, if we do not assume that the radii approach zero, the intersection may be empty (see Exercise 9:6.1).

Theorem 9.37 *A metric space (X, ρ) is complete if and only if the intersection of every descending sequence of closed balls whose radii approach zero consists of a single point.*

Theorem 9.38 *A subspace Y of a complete metric space is complete if and only if Y is closed.*

Proof. Suppose that Y is closed and $\{y_n\}$ is a Cauchy sequence in Y. Since X is complete, $\{y_n\}$ converges to some point $x \in X$. Since Y is closed, $x \in Y$. Thus Y is complete.

Conversely, suppose that Y is complete and x is a limit point of Y. Then there exists a sequence $\{y_n\}$ in Y such that $y_n \to x$. The sequence $\{y_n\}$ is a Cauchy sequence in Y. Since Y is complete, $\{y_n\}$ converges to a point $y \in Y$. But limits are unique, so $y = x$. Thus $x \in Y$, and Y is closed. ■

It is often important in analysis to establish that a given space X is complete. We must show that every Cauchy sequence $\{x_n\}$ in X converges. Unless we have some theorem, such as Theorem 9.38, to apply, this must be done directly. In many cases the method can be described by the following three steps applied to an arbitrary Cauchy sequence $\{x_n\}$:

1. Often there is a natural "candidate" x_0 for the limit of the sequence.
2. The "candidate" x_0 must be shown to be in the space X.
3. We verify that $x_n \to x_0$.

Here is an explanation of the second step. The sequence $\{1/n\}$ is Cauchy in the metric space $X = (0, 1]$. One expects the sequence to converge to 0, so that is our candidate. Unfortunately, $0 \notin X$, so the process collapses. If, instead, X is the space $X = [0, 1]$ then all steps can be carried through.

We now check some of the spaces in Section 9.1 for completeness.

Example 9.39 The space $M[a, b]$ is complete.

Proof. (This space is defined in Example 9.7.) Let $\{f_n\}$ be a Cauchy sequence in $M[a, b]$. For each $t \in [a, b]$, $\{f_n(t)\}$ is a Cauchy sequence of real numbers. This follows immediately from the inequality

$$|f_n(t) - f_m(t)| \le \sup_{a \le s \le b} |f_n(s) - f_m(s)| = \rho(f_n, f_m).$$

Since $\mathbb{R}$ is complete, $\lim_{n\to\infty} f_n(t)$ exists for each $t \in [a, b]$. This limit defines a function f on $[a, b]$. The function f is our candidate for the limit of the sequence.

The second step requires us to check that f is in $M[a, b]$. (The reader should check this. Simply show that f is bounded.)

For the final step, we must show that $f_n \to f$ in the space $M[a, b]$; that is, $f_n \to f$ [unif]. Let $\varepsilon > 0$. Since $\{f_n\}$ is a Cauchy sequence in $M[a, b]$, there exists N such that $n \ge N$ implies that $\rho(f_n, f_N) < \frac{1}{2}\varepsilon$, and so

$$|f_N(t) - f_n(t)| < \tfrac{1}{2}\varepsilon \text{ for all } t \in [a, b].$$

Thus, for all $t \in [a, b]$,

$$|f_N(t) - f(t)| = \lim_{m\to\infty} |f_N(t) - f_m(t)| \le \tfrac{1}{2}\varepsilon.$$

It follows that, for $n \ge N$,

$$|f_n(t) - f(t)| \le |f_n(t) - f_N(t)| + |f_N(t) - f(t)| < \varepsilon$$

for all $t \in [a, b]$. Thus $f_n \to f$ [unif], as required. ∎

We see from Theorem 9.38 and this example that all closed subspaces of $M[a, b]$ are complete. For example, since a uniform limit of continuous functions is continuous, $\mathcal{C}[a, b]$ is a closed subspace of $M[a, b]$. Hence $\mathcal{C}[a, b]$ is a complete metric space.

We next consider Example 9.8. Here L_1 consists of the integrable functions on a complete measure space $(X, \mathcal{M}, \mu)$, with

$$\rho(f, g) = \int_X |f - g| \, d\mu,$$

and our usual understanding that the functions in the space are identical if they are a.e. equal.

Example 9.40 The space L_1 is complete.

Proof. Let $\{f_n\}$ be a Cauchy sequence in L_1. We find a function $f \in L_1$ such that $f_n \to f$. Since $\{f_n\}$ is a Cauchy sequence, there exists an increasing sequence $\{n_k\}$ from $\mathbb{N}$ such that, for every $k \in \mathbb{N}$, $\rho(f_n, f_{n_k}) \leq 2^{-k}$ for all $n \geq n_k$. Thus

$$\begin{aligned} \int_X \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}| \, d\mu &= \sum_{k=1}^{\infty} \int_X |f_{n_{k+1}} - f_{n_k}| \, d\mu \\ &= \sum_{k=1}^{\infty} \rho(f_{n_{k+1}}, f_{n_k}) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1. \end{aligned}$$

It follows that $\displaystyle\sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$ is in L_1 and therefore finite a.e. Let

$$\begin{aligned} g &= \sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k}) = \lim_{m \to \infty} \sum_{k=1}^{m} (f_{n_{k+1}} - f_{n_k}) \\ &= \lim_{m \to \infty} (f_{n_{m+1}} - f_{n_1}) = \lim_{m \to \infty} f_{n_{m+1}} - f_{n_1}. \end{aligned}$$

Let

$$f = \lim_{m \to \infty} f_{n_{m+1}} = f_{n_1} + g.$$

It is clear that $f \in L_1$, and that $f_{n_k} \to f$ [a.e.]. We show that

$$f_{n_k} \to f \quad \text{[mean]}. \tag{13}$$

Fix $k \in \mathbb{N}$. Then

$$\begin{aligned} |f_{n_k}| &= \left| \sum_{m=1}^{k-1} (f_{n_{m+1}} - f_{n_m}) + f_{n_1} \right| \\ &\leq \sum_{m=1}^{k-1} |f_{n_{m+1}} - f_{n_m}| + |f_{n_1}| \\ &\leq \sum_{m=1}^{\infty} |f_{n_{m+1}} - f_{n_m}| + |f_{n_1}|. \end{aligned}$$

Thus all the functions $|f_{n_k}|$ are dominated by a single integrable function, so the same is true of the functions $|f_{n_k} - f|$. Since $|f_{n_k} - f| \to 0$ [a.e.], we infer from the Lebesgue dominated convergence theorem that

$$\lim_{k\to\infty} \rho(f_{n_k}, f) = \lim_{k\to\infty} \int_X |f_{n_k} - f|\, d\mu = 0,$$

and we have proved (13).

We have shown that every Cauchy sequence has a convergent subsequence. But this implies (Exercise 9:6.4) that every Cauchy sequence converges. Thus L_1 is complete. ∎

Example 9.41 The space $\mathcal{K}$ is complete.

Proof. (This space is defined in Example 9.13 and we use the notation A_ε introduced there.) We first observe that, if $\{H_n\}$ is a decreasing sequence of nonempty closed sets in $[0,1]\times[0,1]$ and $H = \bigcap_{n=1}^\infty H_n$, then $H_n \to H$ in $\mathcal{K}$. (Verify this.) Now let $\{A_n\}$ be a Cauchy sequence in $\mathcal{K}$. For each $n \in \mathbb{N}$, let H_n be the closure of the set $\bigcup_{k=n}^\infty A_k$. Then $\{H_n\}$ is a decreasing sequence of closed sets, $H = \bigcap_{n=1}^\infty H_n$ is a nonempty closed set, and $H_n \to H$.

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $\rho(A_n, A_m) < \varepsilon$ if $n, m \geq N$. Thus, for $n, m \geq N$, $(A_n)_\varepsilon \supset A_m$, so

$$(A_n)_\varepsilon \supset \bigcup_{k=n}^\infty A_k.$$

Since $(A_n)_\varepsilon$ is closed,

$$(A_n)_\varepsilon \supset \overline{\bigcup_{k=n}^\infty A_k} = H_n \supset H, \quad \text{if } n \geq N.$$

On the other hand, since $H_n \to H$, there exists $M \in \mathbb{N}$ such that $H_n \subset H_\varepsilon$ if $n \geq M$. But $A_n \subset H_n$, so $A_n \subset H_\varepsilon$ if $n \geq N$. It follows that if $n \geq N$ and $n \geq M$ then

$$(A_n)_\varepsilon \supset H \quad \text{and} \quad H_\varepsilon \supset A_n,$$

that is, $\rho(A_n, H) < \varepsilon$. Thus $A_n \to H$, and $\mathcal{K}$ is complete. ∎

In Chapter 2 we saw that to each measure space $(X, \mathcal{M}, \mu)$ corresponds a complete measure space, the completion of $(X, \mathcal{M}, \mu)$. Something similar is true for metric spaces, although the terminology has different meaning in the two contexts. Consider, for example, the subspace $\mathbb{Q}$ of $\mathbb{R}$. A Cauchy sequence in $\mathbb{Q}$ might not converge in $\mathbb{Q}$, but it will converge in $\mathbb{R}$. We need all of $\mathbb{R}$ to be sure that each Cauchy sequence in $\mathbb{Q}$ converges, and one can then show that $\mathbb{R}$ is complete. Here we are dealing with familiar objects, $\mathbb{Q}$ and $\mathbb{R}$, but how does one obtain a completion of an arbitrary metric space? We begin with a precise formulation of the problem.

Suppose that (X, ρ) and (Y, σ) are metric spaces, and $h : X \to Y$ is an isometry of X and $h(X)$. We say that h *embeds* (X, ρ) in (Y, σ). For example, $h(x) = (x, 0)$ embeds $\mathbb{R}^1$ in $\mathbb{R}^2$.

Theorem 9.42 *Every metric space (X, ρ) can be embedded, as a dense subset, in a complete metric space $(\overline{X}, \overline{\rho})$. The space $(\overline{X}, \overline{\rho})$ is unique up to isometry.*

We outline a proof of Theorem 9.42 in Exercise 9:6.7.

Exercises

9:6.1 Prove Theorem 9.37. Show that, if we do not assume that the radii approach zero, then the intersection may be empty. [Hint: For the counterexample, find a metric on $\mathbb{N}$ so that some sequence of closed balls $B[n, r_n]$, $n = 1, 2, 3, \dots$ is descending, but has an empty intersection.]

9:6.2 Verify that the spaces c, s, and ℓ_∞ are complete. Is the subspace c_0 of c complete?

9:6.3 Let (X, ρ) and (Y, σ) be metric spaces, and let f be a continuous function mapping X onto Y.

(a) If X is separable, must Y be separable?

(b) If X is complete, must Y be complete?

(c) Is separability a topological property? Is completeness?

(d) Do the answers to (a) and/or (b) change if f is an isometry?

9:6.4$\diamondsuit$ Prove that if a Cauchy sequence in a metric space has a convergent subsequence then the full sequence itself converges to the same limit.

9:6.5 Show that $\mathcal{C}'[a, b]$ (Example 9.11) is complete.

9:6.6$\diamondsuit$ Show that the space of Example 9.12 is complete. [Hint: Use Exercise 9:4.6 and Theorem 9.38.]

9:6.7 Provide the details in the following outline of a proof for Theorem 9.42.

(a) *Construction of* $(\overline{X}, \overline{\rho})$: Let C denote the set of Cauchy sequences in X. If $\{x_n\}$ and $\{y_n\}$ are in C, write

$$\{x_n\} \sim \{y_n\}$$

if $\rho(x_n, y_n) \to 0$. Then $\sim$ is an equivalence relation in C. Let $\overline{X}$ consist of the equivalence classes relative to $\sim$. We next define a metric $\overline{\rho}$ on $\overline{X}$. If $\{x_n\}, \{y_n\} \in C$, then $\{\rho(x_n, y_n)\}$ is a Cauchy sequence of real numbers that converges, since $\mathbb{R}$ is complete. We define

$$\overline{\rho}(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} \rho(x_n, y_n).$$

The value of $\overline{\rho}$ is independent of the choice of representatives from an equivalence class, so $\overline{\rho}(\overline{x}, \overline{y})$ is well defined for $\overline{x}, \overline{y} \in \overline{X}$. Show that $(\overline{X}, \overline{\rho})$ is complete.

(b) *Embedding*: For $x \in X$, let $h(x)$ be the equivalence class in $\overline{X}$ containing $\{x, x, x, \dots\}$. Then h is an isometry of X onto a subspace of $\overline{X}$.

(c) *Dense*: Let $\overline{x} \in \overline{X}$, and let $\{x_n\} \in \overline{x}$. Then $\{h(x_n)\} \to \overline{x}$.

From parts (a), (b), and (c) we see that $(\overline{X}, \overline{\rho})$ is a completion of (X, ρ). It remains to verify uniqueness.

(d) *Uniqueness*: We must show that, if $(\overline{X}, \overline{\rho})$ is a completion of (X, ρ) via an isometry h and $(\overline{Y}, \overline{\sigma})$ is another completion via g, then $(\overline{X}, \overline{\rho})$ and $(\overline{Y}, \overline{\sigma})$ are isometric. The function $g \circ h^{-1}$ is an isometry between $h(X)$ and $g(X)$. We extend $g \circ h^{-1}$ to an isometry f between $\overline{X}$ and $\overline{Y}$. Let $\overline{x} \in \overline{X}$, and choose a sequence $\{h(x_n)\}$ in $h(X)$ converging to $\overline{x}$. Then

$$\{g(x_n)\} = \{(g \circ h^{-1} \circ h)(x_n)\}$$

is a Cauchy sequence in $\overline{Y}$. Since $\overline{Y}$ is complete, this sequence converges to a limit $f(\overline{x})$. This defines a function f. It is an isometry of $\overline{X}$ onto $\overline{Y}$.

9.7 Contraction Maps

Let (X, ρ) be a metric space, and let $A : X \to X$. If there exists a number $\alpha \in (0, 1)$ such that

$$\rho(A(x), A(y)) \leq \alpha\rho(x, y) \quad \text{for all} \quad x, y \in X,$$

we say that A is a *contraction map*. It follows immediately from the definition that a contraction map is continuous. Our purpose is to obtain a very simple theorem about contraction maps on complete metric spaces and to show ways in which this theorem can be applied to various problems in analysis. For simplicity of notation, we shall write Ax for $A(x)$, A^2x for $A(A(x))$, and, in general, $A^{n+1}x$ for $A(A^n(x))$.

If $x \in X$ and $Ax = x$, we say that x is a *fixed point* of A. Often the solution of a differential or integral equation can be phrased in the language of fixed points, so it is particularly useful to know when a fixed point exists and if it is unique. The theorem we prove is due to S. Banach. The techniques here evolved from the method of successive approximations used by Émile Picard (1856–1941) to solve differential equations. In the next section we shall use the contraction mapping theorem of Banach to solve such equations.

Theorem 9.43 (Banach) *A contraction map A defined on a complete metric space (X, ρ) has a unique fixed point.*

Proof. Let $x_0 \in X$. Let $x_1 = Ax_0$, $x_2 = Ax_1 = A^2x_0$, and, in general,

$$x_n = Ax_{n-1} = A^n x_0 \qquad (n = 1, 2, 3, \dots).$$

We first show that the sequence $\{x_n\}$ is a Cauchy sequence. Let $n \le m$. Then

$$\begin{aligned}
\rho(x_n, x_m) &= \rho(A^n x_0, A^m x_0) \le \alpha^n \rho(x_0, x_{m-n}) \\
&\le \alpha^n [\rho(x_0, x_1) + \rho(x_1, x_2) + \cdots + \rho(x_{m-n-1}, x_{m-n})] \\
&\le \alpha^n \rho(x_0, x_1)[1 + \alpha + \cdots + \alpha^{m-n-1}] \\
&\le \alpha^n \rho(x_0, x_1) \frac{1}{1-\alpha}.
\end{aligned}$$

Since $\alpha < 1$, this last term can be made arbitrarily small by making n sufficiently large. It follows that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $x \in X$ such that $x_n \to x$. From the continuity of A, we infer that

$$Ax = A(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = x.$$

This shows that x is a fixed point of A. To prove that x is unique, observe that if $Ax = x$ and $Ay = y$ then

$$\rho(Ax, Ay) \le \alpha\rho(x, y) = \alpha\rho(Ax, Ay).$$

Since $\alpha < 1$, this implies that $\rho(Ax, Ay) = 0$, so $\rho(x, y) = 0$ and $x = y$. ■

Observe that the proof of Theorem 9.43 provides a practical method for finding the solution of an equation of the form $Ax = x$. This method is often called the method of *successive approximations*. One can choose x_0 to be *any* point in X. Then the sequence $\{A^n x_0\}$ converges to the unique solution of the equation $Ax = x$.

There is an interesting and useful extension of this theorem. On occasion, a mapping is not itself contractive, but some power of it is contractive. One expects that this should be enough.

Theorem 9.44 *A map A defined on a complete metric space (X, ρ) for which one of the powers of A is a contraction has a unique fixed point.*

Proof. Let us suppose that A^m is a contraction. By Theorem 9.43, there is a unique fixed point of A^m, say $A^m(x) = x$. But then $A(x)$ is also a fixed point of A^m, since $(A^m)(A(x)) = A((A^m)(x)) = A(x)$. Because fixed points are unique, this means that $x = A(x)$ which is exactly the conclusion that we wanted. ■

Exercises

9:7.1 Show that if $f : \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition

$$|f(x) - f(y)| \le M|x - y|$$

for all $x, y \in \mathbb{R}$, and if $M < 1$, then f is a contraction map on $\mathbb{R}$.

9:7.2 Show that one cannot drop the requirement that X is complete in Theorem 9.43.

9:7.3 Give an example of a complete metric space (X, ρ) and a mapping $A : X \to X$ such that $\rho(Ax, Ay) < \rho(x, y)$ for all $x, y \in X$, but A has no fixed point.

9:7.4 Show that the mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A(x_1, x_2) = (x_1, x_2/2)$ has infinitely many fixed points. Is it a contraction? Show that

$$\rho(A(x), A(y)) \leq \rho(x, y) \quad (x, y \in \mathbb{R}^2).$$

9:7.5 Let T be the mapping from $\mathcal{C}[0, 1]$ to itself defined by

$$T(f)(t) = \int_0^t f(s)\, ds.$$

Is this a contraction? Is any power of T a contraction? Show that there is a fixed point.

9.8 Applications of Contraction Mappings

In this section we collect some concrete applications of the contraction mapping theorem. In each case, one solves a problem by constructing a mapping associated with the problem, checking that it is a contraction, and then applying Theorem 9.43 to obtain the existence of a fixed point, which is precisely the solution to the problem posed.

Example 9.45 (**Systems of linear equations**) Consider a system of linear equations

$$x_i = \sum_{j=1}^{n} a_{ij} x_j + b_i, \quad (i = 1, 2, \ldots, n). \tag{14}$$

To solve this system of equations, we can try to use the map defined as follows: If $x = (x_1, \ldots, x_n)$, let $y = Ax$, where $y = (y_1, \ldots, y_n)$ with

$$y_i = \sum_{j=1}^{n} a_{ij} x_j + b_i.$$

Thus $A : \mathbb{R}^n \to \mathbb{R}^n$. We are not obliged to use the Euclidean metric on $\mathbb{R}^n$. Whether A is a contraction map depends on the that metric we choose to use. We consider two cases.
(a) Use the ρ_∞ metric:

$$\rho(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

In this case, with $y = Ax$ and $y^* = Ax^*$, we have

$$\begin{aligned}
\rho(Ax, Ax^*) &= \rho(y, y^*) = \max_i |y_i - y_i^*| \\
&= \max_i |\sum_j a_{ij}(x_j - x_j^*)| \leq \max_i \sum_j |a_{ij}||x_j - x_j^*| \\
&\leq (\max_i \sum_j |a_{ij}|)(\max_j |x_j - x_j^*|) \\
&\leq \max_i \sum_j |a_{ij}| \rho(x, x^*).
\end{aligned}$$

Thus A will be a contraction map if

$$\sum_j |a_{ij}| \leq \alpha < 1 \quad \text{for all} \quad i = 1, \ldots, n.$$

(b) Use the ρ_1 metric:

$$\rho(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

Here we calculate

$$\begin{aligned}
\rho(y, y^*) &= \sum_i |y_i - y_i^*| = \sum_i |\sum_j a_{ij}(x_j - x_j^*)| \\
&\leq \sum_i \sum_j |a_{ij}|(x_j - x_j^*)| \leq (\max_j \sum_i |a_{ij}|)\rho(x, x^*),
\end{aligned}$$

so the condition is

$$\sum_i |a_{ij}| \leq \alpha < 1 \quad \text{for all} \quad j = 1, \ldots, n.$$

Thus, in either case (a) or (b), we have a contraction map and hence a unique solution.

Example 9.46 (**Infinite systems of linear equations**) The preceding ideas can be applied to infinite systems of linear equations. In the late nineteenth century, a number of authors considered such systems arising, for example, in studies of algebraic equations and celestial mechanics. Curiously, the first person to encounter an infinite system of linear equations was Joseph Fourier (1768–1830). In his classic 1822 study of the partial differential equations associated with heat flow, he "solved" such a system by some simple, but unjustified, methods. After that, the subject received no more attention for another half-century.

Suppose that we have a system of equations of the form

$$x_i = \sum_{j=1}^{\infty} a_{ij} x_j + b_i, \ \ i = 1, 2, 3, \ldots. \tag{15}$$

We seek a sequence $x = \{x_i\}$ that satisfies (15). To apply Theorem 9.43, we should first decide what sequence space we wish to consider. Suppose that we want the sequence to be bounded, so that x is a member of ℓ_∞ (Example 9.6). We thus consider ℓ_∞ as the domain of a map $y = Ax$, where

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j + b_i. \tag{16}$$

Since we wish y to be a member of ℓ_∞, we impose the requirement that $b \in \ell_\infty$, too; that is,

$$\textit{There exist } B < \infty \textit{ such that } |b_i| \le B \textit{ for all } i \in \mathbb{N}. \tag{17}$$

Our work with Example 9.45(a) suggests the limitation

$$\sum_{j=1}^{\infty} |a_{ij}| \le \alpha < 1 \ \text{ for } \ i = 1, 2, \ldots. \tag{18}$$

Suppose, then, that the system (15) satisfies (17) and (18) and that A is defined by (16). We wish to show that A is a contraction map on ℓ_∞. It will follow by Theorem 9.43 that the system (15) has a unique solution in ℓ_∞.

We first verify that A maps ℓ_∞ into ℓ_∞. For $x = \{x_1, x_2, \ldots\}$, an element of the space ℓ_∞, write $\|x\|_\infty = \sup_j |x_j|$. From (16), (17), and (18) we find that

$$|y_i| \le \sum_{j=1}^{\infty} |a_{ij}| \|x\|_\infty + |b_i| \le \alpha \|x\|_\infty + B. \tag{19}$$

Since (19) is valid for all $i \in \mathbb{N}$, we see that

$$\|Ax\|_\infty = \|y\|_\infty = \sup_i |y_i| \le \alpha \sup_j |x_j| + B,$$

so $Ax \in \ell_\infty$. Thus A maps ℓ_∞ into ℓ_∞.

We next show that A is a contraction map. Let $x, x^* \in \ell_\infty$, $y = Ax$, and $y^* = Ax^*$. Then

$$y_i^* - y_i = \sum_{j=1}^{\infty} a_{ij} (x_j^* - x_j).$$

Using (18), we conclude that $|y_i^* - y_i| \le \alpha \|x^* - x\|_\infty$, so

$$\|Ax^* - Ax\|_\infty = \sup_i |y_i^* - y_i| \le \alpha \|x^* - x\|_\infty.$$

But this means that

$$\rho_\infty(Ax^*, Ax) \leq \alpha\rho_\infty(x^*, x)$$

and we see that A is a contraction map on ℓ_∞.

We summarize this discussion as a theorem.

Theorem 9.47 *If the system of equations*

$$x_i = \sum_{j=1}^{\infty} a_{ij}x_j + b_i,\ \ i = 1, 2, 3, \ldots$$

satisfies the two conditions

1. *There exist $B < \infty$ such that $|b_i| \leq B$ for $i = 1, 2, \ldots,$ and*
2. *$\sum_{j=1}^{\infty} |a_{ij}| \leq \alpha < 1$ for $i = 1, 2, \ldots,$*

then this system has a unique solution in ℓ_∞.

We next show how Theorem 9.43 can be used to prove existence and uniqueness theorems involving integral equations.

Example 9.48 (Fredholm equation) Consider the equation

$$f(x) = \lambda \int_a^b K(x, y)f(y)\,dy + \phi(x), \lambda \in \mathbb{R}, \tag{20}$$

where ϕ is continuous on $[a, b]$, and K is continuous on $[a, b] \times [a, b]$.

We wish to use Theorem 9.43 to prove that there exists a unique $f \in \mathcal{C}[a, b]$ satisfying (20). To do so, we define $A : \mathcal{C}[a, b] \to \mathcal{C}[a, b]$ by $Af = g$, where

$$(Af)(x) = g(x) = \lambda \int_a^b K(x, y)f(y)\,dy + \phi(x). \tag{21}$$

It is clear that $A : \mathcal{C}[a, b] \to \mathcal{C}[a, b]$. If A is a contraction map, then A has a unique fixed point f, and (21) becomes (20); so f is the unique function in $\mathcal{C}[a, b]$ satisfying (20).

Let $f_1, f_2 \in \mathcal{C}[a, b]$, and let $g_1 = Af_1$ and $g_2 = Af_2$. Then

$$\begin{aligned}
\rho(g_1, g_2) &= \max_x |g_1(x) - g_2(x)| \\
&\leq |\lambda| M \max_x |f_1(x) - f_2(x)|(b - a) \\
&= |\lambda| M(b - a)\rho(f_1, f_2),
\end{aligned}$$

where

$$M = \max\{|K(x, y)| : a \leq x \leq b,\ a \leq y \leq b\}.$$

It follows that A is a contraction map if $|\lambda| \leq M^{-1}(b - a)^{-1}$. Thus the method of successive approximations can be applied provided that $|\lambda|$ is sufficiently small. We shall revisit the Fredholm operator in later chapters.[2]

[2]For applications of these operators to various boundary-value problems associated with the Dirichelet and Neumann problems, see F. Riesz and B. Sz.-Nagy, *Functional Analysis,* Ungar (1955).

Example 9.49 (Volterra equation) Now consider the integral equation

$$f(x) = \lambda \int_a^x K(x,y) f(y)\, dy + \phi(x) \quad (\lambda \in \mathbb{R}).$$

Here we define $A : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ by

$$(Af)(x) = \lambda \int_a^x K(x,y) f(y)\, dy + \phi(x).$$

For $f_1, f_2 \in \mathcal{C}[a,b]$, we calculate (Exercise 9:8.3)

$$\rho(A^n f_1, A^n f_2) \leq |\lambda|^n M^n \frac{(b-a)^n}{n!} \rho(f_1, f_2).$$

Thus, for each $\lambda \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that, if $n \geq N$,

$$|\lambda|^n M^n \frac{(b-a)^n}{n!} < 1.$$

Therefore, A^n is a contraction map. Theorem 9.44 shows that A has a unique fixed point f. This function f provides the unique continuous solution to the integral equation. Observe that in this case λ can be any real number.

As our final illustration of the contraction mapping principle, we prove a standard theorem in differential equations. Let D be an open set in $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$. We say that f satisfies a *Lipschitz condition* in y on D, with *Lipschitz constant* M, if

$$|f(x, y_2) - f(x, y_1)| \leq M|y_2 - y_1|$$

whenever (x, y_1) and (x, y_2) are in D. Under such a condition the differential equation $\frac{dy}{dx} = f(x,y)$ can be proved to have a unique solution by interpreting the problem as a fixed-point problem. Here we find conditions so that a differential equation has a unique local solution "passing through" a given point. Later, in Section 9.12, we shall use a weaker hypothesis and a compactness argument to prove a similar theorem.

Theorem 9.50 (Picard) *Let f be a continuous function on D and satisfying a Lipschitz condition in y on D with Lipschitz constant M, and let $(x_0, y_0) \in D$. Then there exists $\delta > 0$ such that the differential equation*

$$\frac{dy}{dx} = f(x,y) \tag{22}$$

has a unique solution $y = \phi(x)$, $\phi(x_0) = y_0$, for the interval $[x_0 - \delta, x_0 + \delta]$.

Proof. We can reformulate the problem in terms of an integral equation. We seek a function ϕ that satisfies the equation

$$\phi(x) = y_0 + \int_{x_0}^x f(t, \phi(t))\, dt \tag{23}$$

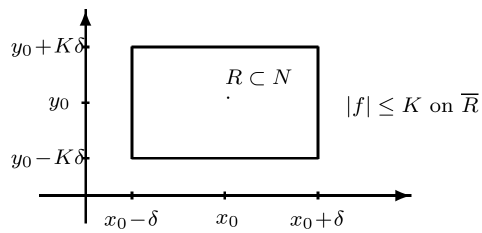

Figure 9.2: Choice of δ in the proof of Picard's Theorem.

for all $x \in [x_0 - \delta, x_0 + \delta]$. Since f is continuous on D, there exists a neighborhood N of (x_0, y_0) and $K > 0$ such that $N \subset D$ and $|f| \leq K$ on N. Choose $\delta > 0$ such that $\delta < M^{-1}$ and so that every point (x, y) with $|x - x_0| \leq \delta$ and $|y - y_0| \leq K\delta$ belongs to N. We arrive at the picture in Figure 9.2.

Let $\mathcal{C}_1$ consist of those members of $\mathcal{C}[x_0 - \delta, x_0 + \delta]$ that satisfy

$$|\phi(x) - y_0| \leq K\delta$$

for all $x \in [x_0 - \delta, x_0 + \delta]$. Then $\mathcal{C}_1$ is a closed subspace of the space $\mathcal{C}[x_0 - \delta, x_0 + \delta]$ and is therefore complete by Theorem 9.38.

Consider now the mapping A on $\mathcal{C}_1$ defined so that

$$(A\phi)(x) = \psi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt$$

for $x_0 - \delta \leq x \leq x_0 + \delta$. We show that A maps $\mathcal{C}_1$ into itself. Let $x \in [x_0 - \delta, x_0 + \delta]$ and suppose $\phi \in \mathcal{C}_1$. Then

$$\begin{aligned} |\psi(x) - y_0| &= \left|\int_{x_0}^{x} f(t, \phi(t))\, dt\right| \leq \int_{x_0}^{x} |f(t, \phi(t))|\, dt \\ &\leq K|x - x_0| \leq K\delta, \end{aligned}$$

so $\psi = A\phi \in \mathcal{C}_1$ and $A : \mathcal{C}_1 \to \mathcal{C}_1$.

We show that A is a contraction map on $\mathcal{C}_1$. To verify the contraction condition, let $\phi_1, \phi_2 \in \mathcal{C}_1$, and let $\psi_1 = A\phi_1$ and $\psi_2 = A\phi_2$. Then, for all $x \in [x_0 - \delta, x_0 + \delta]$,

$$\begin{aligned} |\psi_1(x) - \psi_2(x)| &\leq \int_{x_0}^{x} |f(t, \phi_1(t)) - f(t, \phi_2(y))|\, dt \qquad (24) \\ &\leq M\delta \max_x |\phi_1(x) - \phi_2(x)|. \end{aligned}$$

The last inequality is a consequence of the Lipschitz condition on f and the inequality $|x - x_0| \leq \delta$. Now (24) is valid for all x in the interval $[x_0 - \delta, x_0 + \delta]$, so

$$\rho(\psi_1, \psi_2) \leq M\delta\rho(\phi_1, \phi_2).$$

Since $M\delta < 1$, A is a contraction map, so the equation $\phi = A\phi$ has a unique solution in $\mathcal{C}_1$. In other words, the equation (23) and the equivalent equation (22) have unique local solutions. ■

Exercises

9:8.1 Consider the system of equations

$$x_1 = \frac{1}{2}x_2,\ x_2 = \frac{1}{2}x_3,\ x_3 = \frac{1}{2}x_4, \dots .$$

Show that for each $c \in \mathbb{R}$ the sequence

$$(c, 2c, 4c, \dots)$$

is a solution to this system. Explain why this does not contradict Theorem 9.47.

9:8.2 Consider the system of equations (15). For each integer i, let

$$\alpha_i = \sup_j |a_{ij}|.$$

Prove that the system has a unique solution in the space ℓ_1 provided that $\sum_{i=1}^{\infty} \alpha_i < 1$ and $\sum_{i=1}^{\infty} |b_i| < \infty$.

9:8.3 Fill in the detailed calculations in Example 9.49.

9:8.4 Use Theorem 9.43 to prove the following form of the implicit function theorem.

> **Theorem** *Let $D = [a, b] \times \mathbb{R}$, and let $F : D \to \mathbb{R}$. Suppose that F is continuous on D and $\partial F/\partial y$ exists on D. If there exist positive real numbers α and β such that*
>
> $$\alpha \le \frac{\partial F}{\partial y} \le \beta$$
>
> *on D, then there exists a unique function $f \in \mathcal{C}[a, b]$ such that*
>
> $$F(x, f(x)) = 0 \ \text{ for all } \ x \in [a, b].$$
>
> *That is, the equation $F(x, y) = 0$ can be solved uniquely for y as a continuous function of x on $[a, b]$.*

[Hint: Let $(Ag)(x) = g(x) - cF(x, g(x))$, $c \in \mathbb{R}$, $c \ne 0$. Note that a fixed point of A solves the problem. Find c so that A becomes a contraction map.]

9.9 Compactness

In Section 9.8, we saw how certain theorems, valid for complete metric spaces, could be applied to various parts of mathematics. In the present section, we consider another important property of some metric spaces—compactness. We shall discuss applications of some theorems that are valid for compact spaces in Sections 9.12 and 9.14.

There are actually a number of notions of compactness that agree in our setting of metric spaces. We choose one of these notions as our definition and then show that the other notions are equivalent to the one that we chose. In the more general setting of a topological space these may not be equivalent.

Let X be a metric space, and let $K \subset X$. A collection $\mathcal{U}$ of open sets is called an *open cover* of K if

$$K \subset \bigcup_{U \in \mathcal{U}} U.$$

Definition 9.51 A metric space (X, ρ) is *compact* if every open cover of X has a finite subcover. A subset K of X is compact if (K, ρ) is compact.

The defining property in 9.51 is often called the Heine-Borel property. Theorem 9.52 involves other properties that we can also identify using familiar names.

Theorem 9.52 *The following conditions on a metric space X are equivalent.*

1. *(Heine-Borel) X is compact.*
2. *(Bolzano–Weierstrass I) Every sequence $\{x_n\}$ in X has a cluster point; that is, there is a point $x_0 \in X$ such that, for all $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $n \geq N$ such that $\rho(x_n, x_0) < \varepsilon$.*
3. *(Sequential compactness) Every sequence in X has a convergent subsequence.*
4. *(Bolzano–Weierstrass II) Every infinite set in X has a limit point.*

Proof. It suffices to verify the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$.

$(1) \Rightarrow (2)$. Let X satisfy (1), and let $\{x_n\}$ be a sequence in X. For each $N \in \mathbb{N}$, let $A_N = \{x_n : n \geq N\}$ and let $U_N = X \setminus \overline{A}_n$. One verifies easily that each of the sets U_N is open and that no finite collection of the sets U_N covers X. Since X satisfies condition (1), $\bigcup_{N=1}^{\infty} U_N \neq X$; that is

$$\bigcap_{N=1}^{\infty} \overline{A}_N \neq \emptyset.$$

Let $x_0 \in \bigcap_{N=1}^{\infty} \overline{A}_N$. It follows directly from the definition of the sets $\overline{A}_N$ that x_0 is a cluster point of the sequence $\{x_n\}$.

The implications $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are immediate consequences of the relevant definitions.

(4) $\Rightarrow$ (1). Suppose that X satisfies condition (4). We show that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and open balls

$$B(x_1, \varepsilon), \ldots, B(x_n, \varepsilon)$$

such that $X = \bigcup_{i=1}^{n} B(x_i, \varepsilon)$. If this were false, we could inductively choose a sequence $\{x_n\}$ from X such that $\rho(x_n, x_k) \geq \varepsilon$ for all $k < n$. The set $\{x_n\}$ would have no limit point, contradicting our assumption that X satisfies condition (4).

The set $\{x_1, x_2, \ldots, x_n\}$ is called an ε-net for X. It has the property that if $x \in X$ there exists i such that $\rho(x_i, x) < \varepsilon$. If for every $k \in \mathbb{N}$ we choose a $\frac{1}{k}$-net for X, we arrive at a countable dense subset for X, so X is separable.

Now let $\mathcal{U}$ be an open cover of X. It follows from Lindelöf's theorem (Exercise 9:5.5) that $\mathcal{U}$ can be reduced to a countable subcover $U_1, U_2, \ldots$. We now show that this subcover can be further reduced to a finite subcover. If this were not the case, then for each $N \in \mathbb{N}$ there exists $x_N \in X \setminus \bigcup_{i=1}^{N} U_i$. Since X satisfies condition (4), the set $\{x_1, x_2, \ldots\}$ has a limit point x_0. But $X = \bigcup_{i=1}^{\infty} U_i$, so there exists $j \in \mathbb{N}$ such that $x_0 \in U_j$. This implies that $x_i \in U_j$ for infinitely many $i \in \mathbb{N}$. This is impossible because our choice of the points x_N implies that $x_N \in X \setminus U_j$ when $N \geq j$. This contradiction implies that the collection $U_1, U_2, \ldots$ can be reduced to a finite subcover, completing the proof of (4) $\Rightarrow$ (1). ■

Theorem 9.52 applies to subsets of X, as well as to X itself. If one wishes to use conditions (2), (3), or (4) to verify that a subset K of a space X is compact, one must verify that the cluster point, limit of the convergent subsequence, or limit point is in K. Thus a compact subspace of X must be closed. It is also clear that, if X is compact and $K \subset X$ is closed, then K is compact.

Standard theorems about continuous functions on compact subsets of $\mathbb{R}^n$ carry over to general metric spaces.

Definition 9.53 If $f : (X, \rho) \to (Y, \sigma)$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sigma(f(x), f(x')) < \varepsilon$ whenever $\rho(x, x') < \delta$, we say f is *uniformly continuous* on X.

One proves, as for continuous functions defined on a compact subset of $\mathbb{R}^n$, that continuous functions on compact spaces are uniformly continuous.

Theorem 9.54 *If X is compact and $f : X \to Y$ is continuous, then f is uniformly continuous.*

The elementary theorem that asserts that a continuous real-valued function on a compact interval I achieves absolute extrema on I takes the following form for general metric spaces.

Theorem 9.55 *If $f : X \to Y$ is continuous and X is compact, then the set $f(X)$ is compact in Y.*

Proof. Let $\mathcal{U}$ be an open cover of $f(X)$. Then the family

$$\mathcal{V} = \{V : \text{There exists } U \in \mathcal{U} \text{ such that } V = f^{-1}(U)\}$$

is an open cover of X. Since X is compact, $\mathcal{V}$ has a finite subcover $V_1, V_2, \ldots, V_n$. The sets

$$U_1 = f(V_1), \ldots, U_n = f(V_n)$$

form the required subcover of Y. ■

Exercises

9:9.1 Prove that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

9:9.2 Let X be an arbitrary set furnished with the discrete metric. Characterize the compact subsets of X.

9:9.3 Show that a compact subset of a metric space is closed and bounded, but that the converse is not true in general. [Hint: Every subset of a discrete space is both closed and bounded.]

9:9.4 Show that $\{x \in \ell_1 : \rho(x,0) = 1\}$ is closed and bounded in ℓ_1, but not compact.

9:9.5 Show that if A and B are compact subsets of a metric space then there exist $a \in A$ and $b \in B$ such that $\rho(a,b) = \operatorname{dist}(A,B)$.

9:9.6 Show that closed balls in $\mathcal{C}[a,b]$, $M[a,b]$ and ℓ_∞ are not compact by using Theorem 9.52.

9:9.7 Show that the set $I^\infty = \{x \in \ell_2 : |x_n| \leq n^{-1}\}$, called the Hilbert cube, is compact and nowhere dense in ℓ_2.

9:9.8 Show that if $f : X \to Y$ is uniformly continuous and $\{x_n\}$ is a Cauchy sequence in X then $\{f(x_n)\}$ is a Cauchy sequence in Y.

9:9.9 Let X and Y be metric spaces with X compact. Prove that a continuous, one-one mapping of X onto Y is necessarily a homeomorphism.

9:9.10 Let (X,ρ) be a compact metric space and suppose $T : X \to X$ has the property that $\rho(T(x),T(y)) < \rho(x,y)$ for all x, $y \in X$, $x \neq y$. Show that T has a unique fixed point. How does this compare with Exercise 9:7.3? [Hint: Consider $\min_{x\in X} \rho(x,T(x))$.]

9:9.11 If K is a compact subset of a metric space (X,ρ) and $x_0 \in X \setminus K$ then show that there must exist a point $y \in K$ so that

$$\operatorname{dist}(x_0,K) = \rho(x_0,y).$$

Give an example to show that it is not enough merely for K to be complete.

9.10 Totally Bounded Spaces

Observe that we have *not* stated that a closed and bounded set in a metric space is compact. That statement is valid in $\mathbb{R}^n$, but not in general. In a metric space a closed and bounded set may have no special properties and need not be compact. Indeed every metric space is closed and has an equivalent metric that makes it bounded (Exercise 9:4.7).

A characterization of compactness that reduces to "closed and bounded" in $\mathbb{R}^n$ is available. The key is in the proof of the implication (4) $\Rightarrow$ (1) in Theorem 9.52. There we showed that if X is compact via condition (4) then, for every $\varepsilon > 0$, there is an ε–net, that is, a finite set

$$\{x_1, x_2, \dots, x_n\} \subset X$$

such that the finite collection of balls $\{B(x_i, \varepsilon)\}$ covers X. When a space X has, for every $\varepsilon > 0$, an ε-net, we say that X is *totally bounded.* We express this formally.

Definition 9.56 Let X be a metric space. We say that X is *totally bounded* if for every $\varepsilon > 0$ there is a finite set

$$\{x_1, x_2, \dots, x_n\} \subset X$$

such that

$$B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon) = X.$$

The proof of (4) $\Rightarrow$ (1) in Theorem 9.52 shows that a compact space is totally bounded. It is clear that a totally bounded space must be separable. One can also characterize total boundedness in terms of Cauchy sequences; we leave the straightforward proof as Exercise 9:10.1.

Theorem 9.57 *A metric space X is totally bounded if and only if every sequence has a Cauchy subsequence.*

We can now show that, if one replaces "closed and bounded" as a characterization of compactness in $\mathbb{R}^n$ by "complete and totally bounded," we obtain a characterization of compactness that is valid for arbitrary metric spaces.

Theorem 9.58 *A metric space is compact if and only if it is complete and totally bounded.*

Proof. Suppose that X is compact. Let $\{x_n\}$ be a Cauchy sequence in X. By condition (3) of Theorem 9.52, $\{x_n\}$ has a convergent subsequence. But a Cauchy sequence with a convergent subsequence is itself convergent; thus X is complete. That X is totally bounded follows immediately from condition (3) of Theorem 9.52 and Theorem 9.57.

Conversely, suppose that X is complete and totally bounded. If $\{x_n\}$ is an arbitrary sequence from X, then $\{x_n\}$ has a Cauchy subsequence, by Theorem 9.57. This subsequence converges, since X is complete. Thus X is compact by Theorem 9.52 (3). ■

Exercises

9:10.1 Prove Theorem 9.57.

9:10.2 Show that the space $\mathcal{S}$ of Example 9.9 is bounded but not totally bounded. [Hint: Let $f_n(x) = n$. Compute $\rho(f_n, f_m)$, and verify that $\mathcal{S}$ has no $\frac{1}{4}$-net or that $\{f_n\}$ has no Cauchy subsequence.]

9:10.3 Show that the space of Example 9.12 with respect to $([0,1], \mathcal{L}, \lambda)$ is not totally bounded. [Hint: Let

$$A_n = \left[0, \frac{1}{2^n}\right] \cup \left[\frac{2}{2^n}, \frac{3}{2^n}\right] \cup \cdots \cup \left[\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n}\right].$$

Verify that $\{A_n\}$ has no Cauchy subsequence.]

9:10.4 Show that a closed ball in $L_1([0,1], \mathcal{L}, \lambda)$ is not totally bounded. [Hint: See Exercise 9:10.3.]

9:10.5 Show that the space $2^{\mathbb{N}}$ from Example 9.4 is compact by verifying that it is complete and totally bounded.

9:10.6 Show that closed balls in $\mathcal{C}[a,b]$, $M[a,b]$, and ℓ_∞ are not compact by using Theorem 9.58.

9:10.7 Prove that a totally bounded metric space must be separable.

9.11 Compact Sets in $\mathcal{C}(X)$

Let X be a compact metric space, and let f and g be continuous real-valued functions on X. In view of Theorem 9.55, we can define

$$d(f,g) = \max_{x \in X} |f(x) - g(x)|.$$

It follows readily that d is a metric on the set of continuous real-valued functions on X. We denote the resulting metric space by $\mathcal{C}(X)$. We have already encountered the particular case $\mathcal{C}[a,b]$. As in that case, one verifies easily that $\mathcal{C}(X)$ is complete.

Our purpose here is to obtain a useful characterization of the compact subsets of $\mathcal{C}(X)$. This characterization involves two properties that a family of functions on X may or may not possess. For the first property, let us ask what characterizes the bounded subsets of $\mathcal{C}(X)$, since every compact set must also be bounded.

Definition 9.59 A family $\mathcal{F}$ of functions on a set X is said to be *uniformly bounded* on X if there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in X$ and $f \in \mathcal{F}$.

It is easy to see that, if X is a compact metric space and $\mathcal{F}$ consists of continuous functions on X, then the family $\mathcal{F}$ is uniformly bounded if and only if $\mathcal{F}$ is a bounded subset of $\mathcal{C}(X)$.

The other relevant notion concerns the uniformity of the continuity behavior of continuous functions in a compact subset of $\mathcal{C}(X)$. Let $f \in \mathcal{C}(X)$, let $x_0 \in X$, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, if $\rho(x, x_0) < \delta$, $|f(x) - f(x_0)| < \varepsilon$. The number δ depends on x_0, ε, and f and should perhaps be written $\delta = \delta(x_0, \varepsilon, f)$. Since X is assumed compact, each $f \in \mathcal{C}(X)$ is uniformly continuous (see the discussion preceding Theorem 9.55), so δ is independent of x_0 for a given ε and f. If $\mathcal{F} \subset \mathcal{C}(X)$ and we can choose δ so as also to be independent of $f \in \mathcal{F}$, we say that $\mathcal{F}$ is an equicontinuous family. The concept is due to Giulio Ascoli (1843–1896).

Definition 9.60 A family $\mathcal{F}$ of functions on a metric space (X, ρ) is *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x, y \in X$ and $\rho(x, y) < \delta$, then $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{F}$.

For an easy example, note that a collection of functions that satisfies a uniform Lipschitz condition is equicontinuous.

Example 9.61 Let $X = [a, b]$, let $M > 0$, $C > 0$, and let

$$\mathcal{F} = \{f : X \to \mathbb{R} : |f(x) - f(y)| \leq M|x - y| \text{ for all } x, y \in [a, b]\}.$$

Then $\mathcal{F}$ is an equicontinuous family. If we require in addition that $|f(x)| \leq C$ for all $x \in X$ and $f \in \mathcal{F}$, then $\mathcal{F}$ is also uniformly bounded. Under these two conditions, we see from the next theorem, usually attributed to both Ascoli and Cesare Arzelà (1847–1912), that the closure of $\mathcal{F}$ will be a compact subset of $\mathcal{C}[a, b]$.

Theorem 9.62 (Arzelà–Ascoli) *Let (X, ρ) be a compact metric space, and let K be a closed subset of $\mathcal{C}(X)$. Then K is compact if and only if K is uniformly bounded and equicontinuous.*

Proof. Since K is assumed closed in the complete space $\mathcal{C}(X)$, K is complete. In view of Theorem 9.58, it suffices to show that the stated conditions taken together are equivalent to K being totally bounded.

Suppose first that K is totally bounded in $\mathcal{C}(X)$. Then K is bounded in $\mathcal{C}(X)$ and is therefore a uniformly bounded family of functions. We show that K is equicontinuous. Let $\varepsilon > 0$, and let $f_1, f_2, \ldots, f_n$ be an $(\varepsilon/3)$-net in K. Let $f \in K$. There exists $j \leq n$ such that

$$\max_{z \in X} |f(z) - f_j(z)| < \tfrac{1}{3}\varepsilon. \tag{25}$$

Then, for $x, y \in X$,

$$|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)|. \tag{26}$$

Since X is compact, the functions f_i are uniformly continuous on X. Thus there exists $\delta > 0$ such that

$$\rho(x, y) < \delta,\ 1 \leq i \leq n \Rightarrow\ |f_i(x) - f_i(y)| < \varepsilon/3. \tag{27}$$

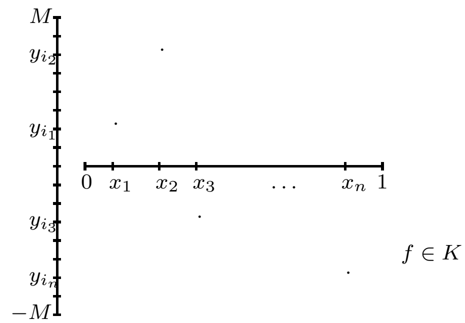

Figure 9.3: An illustration for $X = [0, 1]$.

It now follows from (25), (26), and (27) that $|f(x) - f(y)| < \varepsilon$ for all x, $y \in X$ with $\rho(x, y) < \delta$ and all $f \in K$. This shows that K is equicontinuous.

To prove the converse, suppose that K is uniformly bounded and equicontinuous. We show that K is totally bounded. Choose $M \in \mathbb{N}$ such that $|g(x)| \leq M$ for all $x \in X$ and $g \in K$. Let $\varepsilon > 0$. Since K is equicontinuous, there exists $\delta > 0$ such that

$$\rho(x, y) < \delta,\ g \in K \ \Rightarrow\ |g(x) - g(y)| < \varepsilon/4. \tag{28}$$

Since X is compact, there is a δ-net $x_1, x_2, \ldots, x_n$ for X. Choose $m \in \mathbb{N}$ such that $1/m < \varepsilon/4$, and partition the interval $[-M, M]$ into $2Mm$ congruent intervals:

$$-M = y_0 < y_1 < \cdots < y_{2Mm} = M.$$

Consider now all n-tuples

$$(y_{i_1}, y_{i_2}, \ldots, y_{i_n})$$

of the numbers y_i, $i \leq 2Mm$. There are finitely many such n-tuples. Some such n-tuples can be approximated within $\varepsilon/4$ by a function $f \in K$ on the set $x_1, x_2, \ldots, x_n$. We shall use these n-tuples to obtain an ε-net for K. Figure 9.3 illustrates the situation for $X = [0, 1]$.

To be precise, if for a particular n-tuple $(y_{i_1}, \ldots, y_{i_n})$ there exists $f \in K$ such that

$$|f(x_j) - y_{i_j}| < \varepsilon/4 \quad \text{for all } j \leq n, \tag{29}$$

associate one such f with that n-tuple. Let N be the collection of functions in K associated with such n-tuples. The set N is finite. We show that N is an ε-net for K.

Let $g \in K$. There exists an n-tuple $(y_{i_1}, y_{i_2}, \ldots, y_{i_n})$ such that

$$|g(x_j) - y_{i_j}| < \varepsilon/4 \text{ for all } j \leq n. \tag{30}$$

Let f be that function in N associated with $(y_{i_1}, y_{i_2}, \ldots, y_{i_n})$. For $x \in X$, there exists $j \leq n$ such that $\rho(x, x_j) < \delta$. Using (28) and (29), we see that

$$\begin{aligned} |g(x) - f(x)| \leq |g(x) - g(x_j)| \quad &+ \quad |g(x_j) - y_{i_j}| \\ + |y_{i_j} - f(x_j)| \quad &+ \quad |f(x_j) - f(x)| < \varepsilon. \end{aligned}$$

These inequalities imply that

$$\max_x |f(x) - g(x)| < \varepsilon.$$

We have shown that N is an ε-net, so K is totally bounded, as was to be proved. ■

Exercises

9:11.1 Verify that $\mathcal{C}(X)$ is a complete metric space.

9:11.2 Let A be a bounded subspace of $\mathcal{C}[a, b]$. Prove that the set of all functions of the form

$$F(x) = \int_a^x f(t)\, dt$$

for $f \in A$ is an equicontinuous family.

9:11.3 Let σ be continuous and nondecreasing on $[0, \infty)$, with $\sigma(0) = 0$. A function $f \in \mathcal{C}[a, b]$ has *modulus of continuity* σ if

$$|f(x) - f(y)| \leq \sigma(|x - y|)$$

for all $x, y \in [a, b]$. Let $\boldsymbol{C}(\sigma)$ denote

$$\{f : \sigma \text{ is a modulus of continuity for } f\}.$$

(a) Show that every $f \in \mathcal{C}[a, b]$ has a modulus of continuity.

(b) Let σ be a modulus of continuity. Show that $\boldsymbol{C}(\sigma)$ is an equicontinuous family.

(c) Exhibit a modulus of continuity for the class of Lipschitz functions with constant M.

(d) Let σ be a modulus of continuity. Is it necessarily true that $\sigma \in \boldsymbol{C}(\sigma)$ on $[a, b]$? What if σ is concave down?

(e) Prove that the set

$$K = \left\{f \in \mathcal{C}[0, 1] : |f(x) - f(y)| \leq \sqrt{|x - y|} \text{ and } f(0) = 0\right\}$$

is a compact subset of $\mathcal{C}[0, 1]$. Is $\sqrt{x} \in K$? What about x^2?

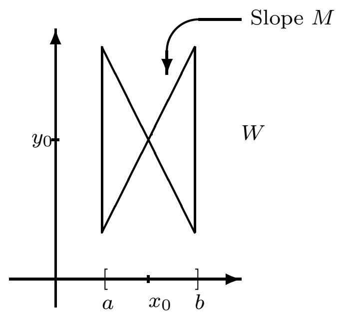

Figure 9.4: The set W and its projection to $I = [a, b]$.

9.12 An Application of the Arzelà–Ascoli Theorem

In Section 9.8, we saw how the contraction mapping principle can be used to prove an existence and uniqueness theorem for solutions to the differential equation $y' = f(x, y)$. We now use the Arzelà–Ascoli theorem to obtain an existence theorem under much weaker hypotheses on the function f. Exercise 9:12.1 shows that this may be, however, without uniqueness.

Theorem 9.63 (Peano) *Let f be continuous on an open subset D of $\mathbb{R}^2$, and let $(x_0, y_0) \in D$. Then the differential equation $y' = f(x, y)$ has a local solution passing through the point (x_0, y_0).*

Proof. We shall obtain an interval I containing x_0 and a family K of approximate solutions through (x_0, y_0) on I. We then show that the set $\overline{K}$ is compact in $\mathcal{C}(I)$, and use compactness to show the existence of an exact solution, that is, a differentiable function k defined on I such that

$$k(x_0) = y_0 \ \text{ and } \ k'(x) = f(x, k(x)) \ \text{ for all } x \in I. \tag{31}$$

Let R be a closed rectangle contained in D having sides parallel to the coordinate axes and having (x_0, y_0) as center. Let $M \geq 1$ be an upper bound for $|f|$ on R. Let

$$W = \{(x, y) \in R : |y - y_0| \leq M|x - x_0|\},$$

and let $I = [a, b]$ be the projection of W onto the x-axis, as in Figure 9.4.

We next obtain a family K of approximate solutions to (31). Since W is compact in $\mathbb{R}^2$, f is uniformly continuous on W. Thus, for every $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that, if $(x, y) \in W$ and $(\overline{x}, \overline{y}) \in W$ with $|x - \overline{x}| < \delta$ and $|y - \overline{y}| < \delta$, then $|f(\overline{x}, \overline{y}) - f(x, y)| < \varepsilon$.

Choose points $x_1, x_2, \ldots, x_n$ such that

$$x_0 < x_1 < x_2 < \cdots < x_n = b \ \text{ and } \ |x_i - x_{i-1}| < \delta/M$$

for all $i = 1, \dots, n$. Define a function k_ε on $[x_0, b]$ as follows: $k_\varepsilon(x_0) = y_0$ and, on $[x_0, x_1]$, k_ε is linear with slope $f(x_0, y_0)$; on $[x_1, x_2]$, take k_ε to be linear with slope $f(x_1, k_\varepsilon(x_1))$; continuing in this way, we extend the definition of k_ε to all of $[x_0, b]$.

We have arrived at a function k_ε defined on $[x_0, b]$ whose graph is a polygonal arc through the point (x_0, y_0) and is contained in W. Since the slopes of the line segments composing the graph of k_ε are determined by values of the function f in W, we see that

$$|k_\varepsilon(x) - k_\varepsilon(\overline{x})| \le M|x - \overline{x}| \tag{32}$$

for all $x, \overline{x} \in [x_0, b]$. Now let $x \in [x_0, b]$, $x \ne x_i$, $i = 0, 1, \dots, n$. Then there exists $j \in \{1, 2, \dots, n\}$ such that $x_{j-1} < x < x_j$. Noting that $|x_j - x_{j-1}| < \delta/M$ and using (32), we see that

$$|k_\varepsilon(x) - k_\varepsilon(x_{j-1})| \le M|x - x_{j-1}| < \delta.$$

This implies that

$$|f(x_{j-1}, k_\varepsilon(x_{j-1})) - f(x, k_\varepsilon(x))| < \varepsilon.$$

But $k_\varepsilon'(x) = f(x_{j-1}, k_\varepsilon(x_{j-1}))$, so

$$|k_\varepsilon'(x) - f(x, k_\varepsilon(x))| < \varepsilon. \tag{33}$$

The inequality (33) is valid for all $x \in [x_0, b]$ except at points x in the finite set $\{x_0, \dots, x_n\}$, at which k_ε need not be differentiable. By (33), we see that the functions k_ε are approximate solutions to (31).

We have constructed a family K of functions, one function corresponding to every $\varepsilon > 0$. The family K is uniformly bounded on $[x_0, b]$, since the graph of each of the functions k_ε is contained in W. It follows from (32) that K is an equicontinuous family, since (32) does not depend on ε. The Arzelà–Ascoli theorem now implies that $\overline{K}$ is compact in $\mathcal{C}[x_0, b]$.

We can now complete the proof of the theorem. For all $x \in [x_0, b]$, we have

$$\begin{aligned} k_\varepsilon(x) &= y_0 + \int_{x_0}^{x} k_\varepsilon'(t)\,dt \\ &= y_0 + \int_{x_0}^{x} \left(f(t, k_\varepsilon(t)) + (k_\varepsilon'(t) - f(t, k_\varepsilon(t)))\right) dt. \end{aligned} \tag{34}$$

The fact that k_ε' may fail to exist on the set $\{x_0, x_1, \dots, x_n\}$ does not affect the integral.

Thus the sequence $\{k_{(1/n)}\}$ contains a subsequence $\{k_{1/n_i)}\}$ that converges uniformly to some function k that is continuous on $[x_0, b]$. Since f is uniformly continuous on W, the functions $f(t, k_{(1/n_i)}(t))$ converge uniformly to the function $f(t, k(t))$ on $[x_0, b]$. Noting (33), we now infer from (34) that

$$k(x) = y_0 + \int_{x_0}^{x} f(t, k(t))\,dt$$

for all $x \in [x_0, b]$. It follows that k is a solution to (31) on $[x_0, b]$.

In a similar manner, we obtain a solution $\overline{k}$ to (31) on $[a, x_0]$. The function y given by

$$y(x) = \begin{cases} k(x) & \text{for } x \in [x_0, b]; \\ \overline{k}(x) & \text{for } x \in [a, x_0], \end{cases}$$

satisfies (31) on all of $I = [a, b]$, as required. ∎

Exercises

9:12.1 Show that the hypotheses of Theorem 9.63 are not sufficient to guarantee uniqueness of solutions to the equation $y' = f(x, y)$ by taking, for example, the equation $y' = 3y^{2/3}$, $y(0) = 0$. Does this example conflict with the uniqueness assertion of Theorem 9.50?

9.13 The Stone–Weierstrass Theorem

In this section we prove one of the most famous and enduring of the modern theorems of analysis. The clever blend of compactness arguments with algebraic ones both in the statement and in the proof of the theorem makes this a typical example of the methods and viewpoint that analysts have taken in this century.

The starting point is the approximation theorem of Karl Weierstrass asserting that the polynomials form a dense subset of the metric space $\mathcal{C}[a, b]$. This theorem has numerous applications and equally numerous proofs. It was Marshall Stone (1903–1989) who first viewed this theorem in a different light. The special feature that the polynomials have is an algebraic one: linear combinations and products of polynomials are themselves polynomials. The metric space $\mathcal{C}[a, b]$ forms an algebra, that is a linear space in which a product is also defined. The polynomials form a subalgebra. To this we just add some analytic arguments and the theorem takes on a more powerful form. The setting is generalized to the space $\mathcal{C}(X)$, where X is a compact metric space. (A compact topological space would do as well here, for those readers with the appropriate background.)

Theorem 9.64 (Stone–Weierstrass) *Let X be a compact metric space, and let A be a closed subalgebra of $\mathcal{C}(X)$ such that $1 \in A$ and A separates points of X. Then $A = \mathcal{C}(X)$.*

Proof. A word about the language: "$1 \in A$" means that the function identically equal to 1 is in the subalgebra A and that "A separates points of X" means that, for distinct x, $y \in X$, some element $f \in A$ exists for which $f(x) \neq f(y)$. A subalgebra is just a subset closed under linear combinations and products.

Our proof takes as a starting point an idea due to Lebesgue: we use the fact that the function $h(t) = |t|$ on $[-1, 1]$ can itself be approximated uniformly by a polynomial on $[-1, 1]$. We take this for granted (see Exercise 9:13.1 or Section 15.6).

The first step is to show that, if $|f(x)| \leq 1$ for all $x \in X$ and $f \in A$, then $|f| \in A$. Using Lebesgue's idea let $\varepsilon > 0$ and choose a polynomial so that

$$|a_0 + a_1 t \dots a_n t^n - |t|| < \varepsilon \quad (t \in [-1, 1]).$$

Then, certainly,

$$|a_0 + a_1 f(x) \dots a_n (f(x))^n - |f(x)|| < \varepsilon \quad (x \in X).$$

But $a_0 + a_1 f(x) \dots a_n (f(x))^n$ belongs to A since A is an algebra. As such a choice is possible for every ε, and the function $|f|$ is in the closure of A, that is A itself. From this we see, in fact, that $f \in A$ implies that $|f| \in A$. Choose c positive so that $c|f(x)| \leq 1$; then $cf \in A$ and so also $|cf| \in A$, and hence $|f| = c^{-1}|cf| \in A$ as required.

For the second step, we claim that if f, g are members of A then so too are both $\max\{f, g\}$ and $\min\{f, g\}$. This is immediate since

$$\max\{f, g\} = \tfrac{1}{2}(f + g) + \tfrac{1}{2}|f - g|$$

and

$$\min\{f, g\} = \tfrac{1}{2}(f + g) - \tfrac{1}{2}|f - g|,$$

and both $(f + g)$ and $|f - g|$ belong to A. By induction then, it follows that if f_1, f_2, $\dots f_n \in A$ then $\max\{f_1, f_2, \dots f_n\}$ and $\min\{f_1, f_2, \dots f_n\}$ are in A.

Now, finally, fix $f \in \mathcal{C}(X)$, and let $\varepsilon > 0$. The proof is completed if we can show that there is a function F in A so that everywhere in X the inequality $|F(z) - f(z)| < \varepsilon$ must hold.

Consider any two distinct points x, $y \in X$. Let g_x be the function on X that assumes the constant value $f(x)$ (this belongs to A by hypothesis), and choose some other $h_{xy} \in A$, so that $h_{xy}(x) \neq h_{xy}(y)$ (again possible by hypothesis); by subtracting a suitable function in A we can suppose that $h_{xy}(x) = 0$. We can find a constant a so that the function $f_{xy} = g_x + a h_{xy}$ satisfies $f_{xy}(x) = f(x)$ and $f_{xy}(y) = f(y)$. Clearly, f_{xy} is also in A. Thus far we have shown only that for any two given points x, $y \in X$ we can find a function f_{xy} in A that agrees with our function f at the two given points. Two compactness arguments are needed to complete the proof.

Hold x fixed. For each $y \in X$, there is an open ball B_y containing y so that $|f_{xy}(z) - f_{xy}(y)| < \varepsilon/2$ and $|f(y) - f(z)| < \varepsilon/2$ for all $z \in B_y$. This just uses the continuity of the functions at the point y. In particular, since $f_{xy}(y) = f(y)$, we have

$$f_{xy}(z) - f(z) \leq |f_{xy}(z) - f_{xy}(y)| + |f(y) - f(z)| < \varepsilon$$

for all $z \in B_y$. As X is compact, we can reduce the open covering $\{B_y : y \in X\}$ to a finite subcovering, say B_{y_1},B_{y_2},$B_{y_3} \dots B_{y_m}$. Define

$$F_x = \min\{f_{xy_1}, f_{xy_2}, \dots f_{xy_m}\}$$

and observe that F_x is in A, that $F_x(x) = f(x)$, and everywhere in X the inequality $F_x(z) < f(z)+\varepsilon$ must hold. Thus far, to keep track of how far we have come, we know that for any given point $x \in X$ we can find a function F_x in A that agrees with our function f at the point x and remains below $f + \varepsilon$ everywhere. One more compactness argument is needed to complete the proof.

For each $x \in X$, there is an open ball A_x containing x so that

$$|F_x(z) - F_x(x)| < \varepsilon/2 \quad \text{and} \quad |f(x) - f(z)| < \varepsilon/2$$

for all $z \in A_x$. This just uses the continuity of the functions at the point x. In particular, since $F_x(x) = f(x)$, we have

$$F_x(z) - f(z) \geq -|F_x(z) - F_x(x)| - |f(x) - f(z)| > -\varepsilon$$

for all $z \in A_x$. Since X is compact, the open covering

$$\{A_y : y \in X\}$$

can be reduced to a finite subcovering say, A_{y_1},A_{y_2},$A_{y_3} \dots A_{y_p}$.

Define

$$F = \max\{F_{x_1}, F_{x_2}, \dots, F_{x_p}\},$$

and observe that F is in A and that everywhere in X the inequality $|F(z) - f(z)| < \varepsilon$ must hold, as required to complete the proof. ■

The classical Weierstrass approximation theorem follows from this as a corollary.

Corollary 9.65 *Every continuous function on a compact subset K of $\mathbb{R}^n$ can be uniformly approximated on K by a polynomial in the coordinates.*

Proof. The polynomials in the coordinates form a subalgebra and can be considered as continuous functions on K and hence as elements of $\mathcal{C}(K)$. Polynomials separate points and contain the function identically 1, and so the theorem applies. ■

Many classes of functions form dense subalgebras in appropriate function spaces; Exercise 9:13.2 gives another instance. We shall return to these ideas in Section 15.6, but from an entirely different perspective.

Exercises

9:13.1 Show that the function $h(t) = |t|$ can be approximated uniformly by a polynomial on $[-1, 1]$. [Hint: The function $g(t) = \sqrt{t + a^2}$ can be approximated uniformly by a Taylor polynomial p on $[0, 1]$. If $|g(t) - p(t)| < \varepsilon/2$ for all $t \in [0, 1]$, then

$$|\sqrt{x^2 + a^2} - p(x^2)| < \varepsilon/2 \ \ (x \in [-1, 1]).$$

Use $a = \varepsilon/2$, and then

$$||x| - p(x^2)| \leq ||x| - \sqrt{x^2 + a^2}| + |\sqrt{x^2 + a^2} - p(x^2)| < \varepsilon.]$$

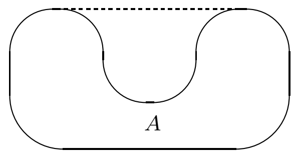

Figure 9.5: A solution to the isoperimetric problem must be convex.

9:13.2 Show that every continuous, 2π–periodic function on $\mathbb{R}$ can be uniformly approximated by a trigonometric polynomial

$$\tfrac{1}{2}a_0 + \sum_{j=1}^{n} (a_j \cos jt + b_j \sin jt)\,.$$

[Hint: Let $T = [-\pi, \pi]$, but considered as the unit circle (with $-\pi$ and π identified) in $\mathbb{R}^2$. Then every continuous, 2π–periodic function on $\mathbb{R}$ can be considered an element of $\mathcal{C}(T)$.]

9:13.3 Let X be the set of complex numbers $\{z : |z| \le 1\}$, and let $C(X, \mathbb{C})$ be the metric space of continuous complex-valued functions on X with the sup metric. Show that the complex polynomials are not dense in $C(X, \mathbb{C})$.

9:13.4 Give a complex version of the Stone–Weierstrass theorem. (In view of Exercise 9:13.3 the hypotheses must be strengthened; the additional assumption is that the subalgebra is closed also under complex conjugation.)

9.14 The Isoperimetric Problem

In this section we present another application of a compactness argument to verify that the circle is the solution of the isoperimetric problem.

Consider the family $\mathcal{G}$ of open sets in the plane that are bounded by a simple closed curve of length 1. Which of these sets has the largest area? This problem is called the *isoperimetric problem,* the length of the bounding curve being called the perimeter of the set. Some simple experimentation may lead one to believe that the answer is an open disk, bounded by a circle. J. Steiner was the first to "prove" this, in several different ways. We use quotation marks because Steiner's arguments are subject to criticism. Here is one of his arguments; it is simple and appealing, but not a proof!

First, observe that if a set $A \in \mathcal{G}$ is a solution then A must be convex. Otherwise, one could replace an arc of the bounding curve for A with a line segment to arrive at a set B with a smaller perimeter and larger area, as in Figure 9.5.

Next we note that if a chord of a convex set $A \in \mathcal{G}$ bisects the perimeter it must also bisect the area. If not, there is a set $B \in \mathcal{G}$ with the same perimeter, but larger area. As a third elementary observation, we note that, among all triangles with two given sides, the triangle for which these sides are perpendicular encloses the largest area.

We can now complete Steiner's argument. Suppose that A is a convex set bounded by the curve C of length 1. A simple continuity argument shows that there exists a chord L that bisects the length of C. Our second observation shows that if A solves the isoperimetric problem, then L also bisects the area of A. Let p be any point of C other than the endpoints of L, and consider the triangle T whose vertices are p and the endpoints of L. Then T must be a right triangle (Exercise 9:14.1). Thus every such triangle must be a right triangle. It follows from elementary geometry that C must be a circle: all inscribed angles determined by a diameter are right angles.

The flaw in Steiner's argument is easy to detect. His argument shows that if C is not a circle then there exists a convex curve C_1 of the same perimeter, but bounding a set $A \in \mathcal{G}$ of larger area than that of the set bounded by C. But this is not to say that C does the job. There may be *no* solution to the problem. Steiner's argument would work equally well to solve a similar problem: among all sets bounded by simple closed curves of length *less than* 1, which bounds the largest area? Steiner's argument would simply show that if C is not a circle it does not solve the problem. But there is no solution.

To solve the isoperimetric problem, we show that there is a solution. Steiner's argument then shows that the solution must be bounded by a circle. Our proof of existence will be based on the fact that a continuous real-valued function on a compact space achieves a maximum. The continuous function will be the "area" function $\lambda = \lambda_2$. The space will be the space of convex sets.

Let $(\mathcal{K}, \rho)$ be the metric space consisting of compact subsets of the square $[0,1] \times [0,1]$ and furnished with the Hausdorff metric (see Example 9.13). In Section 9.6, we saw how to prove that $\mathcal{K}$ is complete. We now show that it is compact.

Theorem 9.66 *The space $\mathcal{K}$ is compact.*

Proof. Since $\mathcal{K}$ is complete, it suffices, by Theorem 9.58, to show that $\mathcal{K}$ is totally bounded. Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $2^{-n}\sqrt{2} < \varepsilon$ and partition the square $[0,1] \times [0,1]$ into 4^n nonoverlapping closed squares, each of side length 2^{-n}. Let $\mathcal{S}$ denote the family of these squares, and let $\mathcal{T}$ denote the family of nonempty finite unions of members of $\mathcal{S}$. Thus $\mathcal{T}$ has

$$2^{4^n} - 1$$

members. We show that $\mathcal{T}$ is an ε-net for $\mathcal{K}$.

Let $K \in \mathcal{K}$. Let $\mathcal{S}_K$ denote those members of $\mathcal{S}$ that K intersects, and let

$$T = \bigcup_{S \in \mathcal{S}_K} S.$$

Then $T \in \mathcal{T}$. Now $K \subset T$, so $K \subset T_\varepsilon$. To see that $T \subset K_\varepsilon$, we need only observe that the diameter of each member of $\mathcal{S}$ is $\sqrt{2}/2^n < \varepsilon$ and that K and T intersect exactly the same members of $\mathcal{S}$. Thus $\rho(K,T) < \varepsilon$, and $\mathcal{K}$ is totally bounded, as was to be shown. ■

The space $(\mathcal{K}, \rho)$ is compact, but the context of the isoperimetric problem requires us to deal with a certain subspace of $\mathcal{K}$: the space of those sets in $\mathcal{K}$ with nonempty interior that are convex and bounded by a convex curve of length 1. Our next objective is to show that this space is closed in $\mathcal{K}$ and therefore compact. We need some elementary lemmas whose proofs we leave as exercises.

Lemma 9.67 *Let $\mathcal{K}^* = \{K \in \mathcal{K} : \ K \text{ is convex}\}$. Then $\mathcal{K}^*$ is closed in $\mathcal{K}$ and therefore compact.*

For $K \in \mathcal{K}^*$, let $\lambda(K)$ be the Lebesgue measure of K. If K has interior, let $\alpha(K)$ be the length of the boundary curve C of K. That λ is defined on $\mathcal{K}^*$ follows immediately from the fact that Lebesgue measure is defined for all closed sets. In connection with the function α, we note that the curve C can be decomposed into the union of the graphs of two functions, one concave up and the other concave down. Such functions have one-sided derivatives everywhere, and these derivatives are monotonic. It follows that C has finite length. We shall not prove any of these statements.

Lemma 9.68 *Let $\varepsilon > 0$, let $K \in \mathcal{K}^*$, and let K_ε be the union of all closed disks of radius ε centered at points of K. If K has a nonempty interior, then*

$$\alpha(K_\varepsilon) = \alpha(K) + 2\pi\varepsilon$$

and

$$\lambda(K_\varepsilon) = \lambda(K) + \varepsilon\alpha(K) + \pi\varepsilon^2.$$

It follows readily from Lemma 9.68 that, if $K \in \mathcal{K}^*$ and K has interior points, then α and λ are continuous at K. Exercises 9:14.2 and 9:14.3 show that λ is not continuous on all of $\mathcal{K}$ and that α is not continuous on all of $\mathcal{K}^*$.

Now let $\mathcal{K}^{**}$ consist of those members of $\mathcal{K}^*$ such that $\alpha(K) = 1$ and $\lambda(K) \geq 1/(4\pi)$. The set $\mathcal{K}^{**}$ is not empty, since any disk K inside the square $[0,1] \times [0,1]$ and having radius $1/(2\pi)$ is a member of $\mathcal{K}^{**}$. It follows from Lemma 9.68 that $\mathcal{K}^{**}$ is closed in the compact space K^* and is therefore compact. (See Exercise 9:14.5.) It now follows from Theorem 9.55 that the function λ achieves a maximum on $\mathcal{K}^{**}$. Steiner's argument shows that this maximum can be achieved only for K a disk. Thus a disk of radius $1/(2\pi)$ provides a solution to the isoperimetric problem.

We mention that elementary proofs that the disk provides a solution to the isoperimetric problem are available.[3]

[3] See, for example, I. M. Yaglom and V. G. Boltyanski, *Convex Figures*,

Exercises

9:14.1 Refer to Steiner's argument. Prove that T must be a right triangle.

9:14.2 Show that λ is upper semicontinuous, but not continuous, on $\mathcal{K}$. [Hint: An arbitrary $K \in \mathcal{K}$ can be approximated by finite sets.]

9:14.3 Show that α is not continuous on all of $\mathcal{K}^*$. [Hint: A line segment can be approached by simple closed curves.]

9:14.4 (The problem of Dido.) Dido, the mythical founder and queen of Carthage, was given an ox and told she would be given as much land as she could surround with its hide. She cut the skin into strips and used the straight seashore together with the strips to enclose a much larger tract of land than had been anticipated.

(a) Given a line segment L of length ℓ, which convex set bounded by L and a curve of length $s > \ell$ has the largest area?

(b) Given a line segment L of length ℓ, which convex set bounded by a subsegment of L and a curve of length $s < \ell$ has the largest area?

9:14.5 Prove that the set $\mathcal{K}^{**}$ defined after the statement of Lemma 9.68 is closed in $\mathcal{K}^*$. [Hint: Observe that if $A, B \in \mathcal{K}^*$ and $A \subset B$, then $\lambda(A) \leq \lambda(B)$ and $\alpha(A) \leq \alpha(B)$. Use Lemma 9.68.]

9.15 More on Convergence

Most of the notions of convergence that we have encountered can be expressed within the setting of a metric space; most, but not all. The more general notion of a topological space captures those ideas of convergence that cannot be expressed by a metric. This section contains a discussion that leads to and introduces the concept of a topological space. We shall not, however, assume any familiarity with topological ideas in the sequel, and this section may easily be omitted.

We have already noticed how the structure of a metric space provides a unified framework for studying many familiar forms of convergence. Consider, for example, the chart in Table 9.1. Each of the spaces can be viewed as a function space. Sequence spaces also allow this interpretation, since a sequence can be viewed as a function on $\mathbb{N}$. In each example, the connection between convergence in the metric and the familiar notion of convergence is clear. A sequence $\{f_n\}$ converges with respect to the given metric ρ, that is, $\rho(f_n, f) \to 0$, if and only if the sequence converges in the familiar sense.

Holt, Rinehart and Winston (1961). This reference also provides a proof of Lemma 9.68, as well as a discussion of the isoperimetric problem and related topics.

Example	**Space**	**Metric** $\rho(f,g)$	**Familiar Name**
9.7	$M[a,b]$	$\sup_{a\leq x\leq b} \|f(x)-g(x)\|$	Uniform convergence
9.8	$L_1(X)$	$\int_X \|f-g\|\,d\mu$	Mean convergence
9.9	$\mathcal{S}$	$\int_0^1 \frac{\|f-g\|}{1+\|f-g\|}\,d\mu$	Convergence in measure
9.2	s	$\sum_{i=1}^{\infty} \frac{\|f_i-g_i\|}{1+\|f_i-g_i\|}$	Pointwise convergence

Table 9.1: Convergence in function spaces.

Let us look at pointwise convergence a bit more closely. We might wish to obtain a metric ρ on the set $\mathcal{F}$ of real-valued functions on $[a,b]$ such that $\rho(f_n,f)\to 0$ if and only if $\{f_n\}$ converges pointwise to f. What must be true about the metric ρ?

Suppose that ρ is such a metric. For $x_0\in[a,b]$, let

$$U(x_0)=\{f\in\mathcal{F}:|f(x_0)|<1\}\,.$$

First note that $U(x_0)$ must be open in $\mathcal{F}$. To see this, we verify that $\widetilde{U}(x_0)$ is closed. Let $\{f_k\}$ be a sequence of functions in $\widetilde{U}(x_0)$ such that $\rho(f_k,f)\to 0$ for some $f\in\mathcal{F}$. Then $f_k\to f$ pointwise, so $|f(x_0)|\geq 1$. We thus have $f\in\widetilde{U}(x_0)$. This shows that $\widetilde{U}(x_0)$ is closed, so $U(x_0)$ is open. It follows that $U(x_0)$ is a neighborhood of the function $f\equiv 0$, so there exists $n\in\mathbb{N}$ such that $B(0,1/n)\subset U(x_0)$.

Now let

$$A_n=\{x\in[a,b]:B\left(0,1/n\right)\subset U(x)\}\,.$$

Since $[a,b]$ is uncountable, there exists n such that the set A_n is infinite. Let $X=\{x_1,x_2,\dots\}$ be a countable subset of A_n. Consider now the sequence $\{f_k\}$, where $f_k=\chi_{\{x_k\}}$. It is clear that, for every $x\in[a,b]$, $f_k(x)\to 0$, so $f_k\to 0$ pointwise. But $f_k\in\widetilde{B}(0,1/n)$ for all $n\in\mathbb{N}$, so $\rho(f_k,0)\geq 1/n$ for all $k\in\mathbb{N}$. Thus $\{f_k\}$ does not converge to zero with respect to the metric ρ. This shows that no metric can describe pointwise convergence on $\mathcal{F}$.

Let us try to obtain a different scheme for describing pointwise convergence on $[a,b]$ by defining what is meant by a topology.

Definition 9.69 A *topology* for a set X is a family $\mathcal{T}$ of subsets of X satisfying the following conditions:

1. $X\in\mathcal{T}$, $\emptyset\in\mathcal{T}$.
2. If $U_1\in\mathcal{T}$ and $U_2\in\mathcal{T}$, then $U_1\cap U_2\in\mathcal{T}$.

3. If $U_\alpha \in \mathcal{T}$ for all $\alpha \in A$, then $\bigcup_{\alpha\in A} U_\alpha \in \mathcal{T}$.

In (3), the set A is an arbitrary index set; it need not be countable. A *topological space* is a pair $(X, \mathcal{T})$ with X a set and $\mathcal{T}$ a topology on X.

For example, the open sets in a metric space X form a topology for X merely because they satisfy these properties: they are closed under finite intersections and arbitrary unions. In general, one calls the members of $\mathcal{T}$ *open sets* and the complements of open sets *closed sets*.

Let us return to our set $\mathcal{F}$ of real-valued functions on $[a, b]$. For $x \in [a, b]$ and G open in $\mathbb{R}$, $G \neq \emptyset$, let

$$U(x, G) = \{f \in \mathcal{F} : f(x) \in G\}.$$

We obtain a topology $\mathcal{T}$ for $\mathcal{F}$ as follows: First, we consider all sets of the form

$$V = U(x_1, G_1) \cap U(x_2, G_2) \cap \cdots \cap U(x_n, G_n). \tag{35}$$

We denote the family of sets of the form (35) by $\mathcal{B}$. The family $\mathcal{B}$ forms a *basis* for $\mathcal{T}$. This means that $\mathcal{T}$ consists of all sets that are unions of sets of $\mathcal{B}$. One verifies easily that $\mathcal{T}$ satisfies the conditions of Definition 9.69.

Observe that if $U \in \mathcal{T}$ and $f \in U$ there exists a set $V \in \mathcal{B}$ such that $f \in V \subset U$, since U is a union of sets in $\mathcal{B}$. Let $V \in \mathcal{B}$

$$V = U(x_1, G_1) \cap U(x_2, G_2) \cap \cdots \cap U(x_n, G_n).$$

Then $f \in V$ if and only if $f(x_i) \in G_i$ for all $i = 1, \ldots, n$. It follows that a sequence $\{f_n\}$ from $\mathcal{F}$ converges pointwise to $f \in \mathcal{F}$ if and only if, for all $V \in \mathcal{B}$ that contain f, there exists $N \in \mathbb{N}$ such that $f_n \in V$ for all $n \geq N$ (Exercise 9:15.2).

Let us summarize the preceding discussion. We have seen that no metric can describe pointwise convergence for sequences from $\mathcal{F}$. But a more general notion than metric space, that of topological space, can.

Let us look deeper into the situation. Let (X, ρ) be a metric space. Starting with the notion of metric convergence, we can define closed sets: a set A is closed if and only if $x \in A$ whenever x is a limit of a convergent sequence from A. We can then define a set to be open if its complement is closed. Thus we can obtain the metric topology by taking sequential convergence as a primitive notion. Can we do the same for topological spaces?

Consider once again the space $\mathcal{F}$. Let

$$A = \{f \in \mathcal{F} : f \geq 0 \text{ except on a countable set}\}.$$

If $\{f_n\}$ is a sequence from A, and $f_n \to f$ pointwise, then $f \in A$. Thus A is closed under pointwise convergence. But A is not a closed set, since

$$\widetilde{A} = \{f \in \mathcal{F} : f(x) < 0 \text{ on an uncountable set}\}$$

is not a member of $\mathcal{T}$. To see this, let $f(x) \equiv -1$. Then $f \in \widetilde{A}$. Choose $V \in \mathcal{B}$ such that $f \in V$, say,

$$V = U(x_1, G_1) \cap U(x_2, G_2) \cap \cdots \cap U(x_n, G_n).$$

Define $g \in \mathcal{F}$ by

$$g(x) = \begin{cases} -1, & \text{if } x = x_1, \ldots, x_n; \\ 1, & \text{otherwise.} \end{cases}$$

Then $g \in V \cap A$. It follows that no open set containing f is contained in $\widetilde{A}$, so $\widetilde{A}$ is not open and A is not closed.

What the preceding discussion shows is that, in the general setting of a topological space, one cannot take sequential convergence as a primitive notion and obtain the topology from convergence. It turns out that a notion of convergence more general than sequential convergence can be taken as primitive. It is beyond our purposes to develop such a notion. We mention only that it can be made to include certain convergencelike concepts that we have already encountered. For example "contraction by inclusion," Section 8.6, fits into the framework of generalized convergence. Recall that no sequence had enough members to describe convergence adequately in that setting [Exercise 8:6.3(c)].

Exercises

9:15.1 Show that $\mathcal{T}$ as determined by the basis of sets of the form (35) is a topology on $\mathcal{F}$.

9:15.2 Show that $f_n \to f$ pointwise if and only if, for every $V \in \mathcal{B}$, $f \in V$, there exists $N \in \mathbb{N}$ such that $f_n \in V$ for all $n \geq N$.

9:15.3 Let X be a countable set, and let $\mathcal{F}$ denote the real-valued functions on X. Provide a metric for $\mathcal{F}$ such that $\rho(f_n, f) \to 0$ if and only if $f_n \to f$ pointwise in X. Determine where the argument we gave to show that no such metric basis exists when $X = [a, b]$ breaks down when X is countable.

9:15.4 Refer to our discussion of the family $\mathcal{F}$ of real-valued functions on $[a, b]$. The family of sets $\mathcal{B} \subset \mathcal{T}$ forms a basis for $\mathcal{T}$. This means that each $U \in \mathcal{T}$ is a union of sets from $\mathcal{B}$. If we denote by $\mathcal{B}(f)$ those members of $\mathcal{B}$ that contain f, we find that $\mathcal{B}(f)$ is uncountable. Show that, if $\mathcal{V}$ is any collection of sets in $\mathcal{B}$ satisfying the conditions (i) $0 \in V$ for all $V \in \mathcal{V}$ and (ii) if $0 \in U \in \mathcal{T}$, there exists $V \in \mathcal{V}$ such that $0 \in V \subset U$, then $\mathcal{V}$ must be uncountable. Use this to show that there is no metric ρ on $\mathcal{F}$ for which a set S is open relative to ρ if and only if $S \in \mathcal{T}$.

9.16 Additional Problems for Chapter 9

9:16.1 Let f be defined on a subset E of a metric space X and have values in a complete metric space Y. Prove that if f is continuous on E then f can be extended to a continuous function defined on a set H of type $\mathcal{G}_\delta$ such that $H \supset E$. (For example, any real-valued function defined and continuous on $\mathbb{Q}$ can be extended to a function continuous on some set H of type $\mathcal{G}_\delta$ that contains $\mathbb{Q}$.)

9:16.2 Let E be a subset of a metric space X. If every continuous function on E is uniformly continuous on E, then show that E is closed but not necessarily compact. [Hint: If x is a limit point of E, but $x \notin E$, consider the function $f(x) = [\mathrm{dist}(x, E)]^{-1}$. Regarding compactness, consider $E = X = \mathbb{N}$.]

9:16.3 Let $(X, \mathcal{M}, \mu)$ be a complete measure space with $\mu(X) = 1$. Define an equivalence relation on $\mathcal{M}$ by saying that $A \equiv B$ if $\mu(A \triangle B) = 0$, and let $\mathcal{M}(\mu)$ be the family of equivalence classes. Let $P_n = (A_n, B_n)$ be a sequence of partitions of X, that is, the sets A_n, $B_n \in \mathcal{M}$, $A_n \cap B_n = \emptyset$, and $\mu(A_n \cup B_n) = 1$. Define

$$|P_{n+1} - P_n| = \mu(A_{n+1} \triangle A_n) + \mu(B_{n+1} \triangle B_n).$$

(a) Show that, with the metric $\rho(A, B) = \mu(A \triangle B)$, $\mathcal{M}(\mu)$ is a complete metric space.

(b) Show that if $|P_{n+1} - P_n| \le 2^{-n}$ then there is a partition $P = (A, B)$ so that $|P_n - P| \to 0$.

(c) If, in addition, $\mu(A_n)\mu(B_n) > 0$ for all n, can you conclude that $\mu(A)\mu(B) > 0$?

9:16.4$\diamondsuit$ (Scattered sets) A set E in a metric space X is called *dense-in-itself* if E has no isolated points. A set $S \subset X$ is called *scattered* if the only subset of E that is dense-in-itself is the empty set.

(a) Prove that a set each of whose points is isolated is scattered, but that its closure need not be. [Hint: Consider the midpoints of the intervals contiguous to the Cantor set.]

(b) Prove that if X is dense-in-itself every scattered subset S of X is nowhere dense. Thus $X \setminus S$ is dense-in-itself.

(c) Prove that the union of two scattered sets is scattered.

(d) Prove that every metric space X can be expressed in the form $X = P \cup S$, where P is perfect and S is scattered. [Hint: Let P be the union of all sets in X that are dense-in-themselves.]

(e) Prove that the boundary of a scattered set is nowhere dense.

(f) Prove that a necessary and sufficient condition that $S \subset X$ be scattered is that, for every perfect set $P \subset X$, $S \cap P$ is nowhere dense in P.

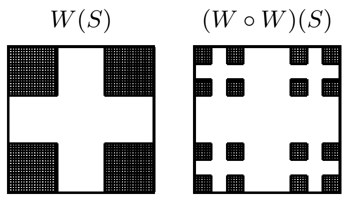

Figure 9.6: Illustration for Exercise 9:16.6(c).

(g) Suppose that X is separable and $S \subset X$ is scattered. Prove that S is denumerable. Show that the statement is false without the assumption that X is separable.

9:16.5 (Cf. Corollary 3.14) Let μ be a finite, metric outer measure on a complete, separable metric space X. Show that, for every μ–measurable set $E \subset X$,

$$\mu(E) = \sup\{\mu(K) : K \subset E, \quad K \text{ compact}\}.$$

[Hint: It is enough to show that

$$\mu(X) = \sup\{\mu(K) : K \subset X,\, K \text{ compact}\}.$$

For each n, pick a sequence of closed balls B_{in} covering X with diameters smaller than 2^{-n}. Choose $j(n)$ so that

$$\mu\left(X \setminus \bigcup_{i \le j(n)} B_{in}\right) < \varepsilon 2^{-n-1},$$

and set $K = \bigcap_n \bigcup_{i \le j(n)} B_{in}$. Show that K is totally bounded.]

9:16.6 (Collage theorem) The purpose of this exercise is to use the theory of contraction maps to lead to the *collage theorem.* This theorem figures in the technique of "fractal image compression" that is used to encode and store graphic images in computers.[4] Let $w_1, w_2, \dots, w_n$ be contraction maps on the square $S = [0,1] \times [0,1]$. For example, for $n = 2^m$, each w_i might map S onto the ith square in a "tiling" of S by 2^m smaller squares. Let $(\mathcal{K}, h)$ denote the space of nonempty compact subsets of S, with h the Hausdorff metric (see Example 9.13 and Theorem 9.66). Let $W : \mathcal{K} \to \mathcal{K}$ be defined by $W(K) = \bigcup_{i=1}^{n} w_i(K)$. Let α be the maximum of the contraction factors of the maps w_i, $i = 1, 2, \dots, n$.

[4] An interesting recent discussion of the technique can be found in M. F. Barnsley, "Fractal image compression," *Notices Amer. Math. Soc.* **43**(6) June 1996, 657–662. That discussion also includes some pictures that illustrate how faithfully the method reproduces an original image.

(a) Prove that W is a contraction map with factor α on $\mathcal{K}$. Thus W has a unique fixed point in $\mathcal{K}$. This means there exists a unique nonempty compact subset A of S such that $W(A) = A$. The set A is called the *attractor* of the *iterated function system* (IFS) $\{w_1, \ldots, w_n\}$.

(b) Verify that for the system involving tilings above $A = S$. Thus S is a *collage* of n smaller copies of itself.

(c) Let $w_1(x, y) = \left(\frac{1}{3}x, \frac{1}{3}y\right)$, and choose w_2, w_3,and w_4 as appropriate modifications of w_1 so that $W(S)$ is a union of squares located in the corners of S. Iteration of W leads to the limit set $A = C \times C$, where C is the Cantor ternary set. See Figure 9.6 for illustrations of the first two stages of the iteration. Verify analytically that $W(A) = A$. Observe that, if one replaces the $\frac{1}{3}$ in w_1 by $\frac{1}{2}$ and defines appropriate modified functions w_2, w_3, and w_4, one obtains the tiling system of part (b).

The collage theorem below is useful in solving the following problem: Given $K \in \mathcal{K}$, find an IFS that has K as its attractor.

(d) Prove the collage theorem:

Theorem (Collage theorem) *Let $(w_1, \ldots, w_n)$ be an IFS for S with contraction factor α, let A be its attractor, and let $K \in \mathcal{K}$. Then*

$$h(K, A) \leq (1 - \alpha)^{-1} h(K, W(K)).$$

[Hint: The proof is easy. Prove the analogous result for any contraction mapping on a complete metric space.]

This theorem tells us that, if K is near $W(K)$, then K is also near A. The problem thus reduces to finding the maps w_i, $i = 1, \ldots, n$, such that, for an original "picture" K, $h(K, W(K))$ is small. (The Barnsley article cited in the footnote discusses how this can be done.) Once one has W so that K is its attractor, we have

$$K = W(K) = \bigcup_{i=1}^{n} w_i(K).$$

Thus K is a collage. The technique and variants have been used in a variety of ways, including pattern recognition (e.g., comparison of fingerprints).

Chapter 10

BAIRE CATEGORY

In this chapter we study the Baire category theorem in complete (or topologically complete) metric spaces. This theorem offers one of the most basic and useful methods for proving existence theorems. Our emphasis is often on applications to illustrate this.

We have seen category notions already in the setting of the real line, which is where Baire originated his ideas. In our first section we introduce the ideas from a new perspective, that of the Banach–Mazur game. In Section 10.2 we show that the Banach–Mazur game can be used to characterize category notions and to obtain proofs of category assertions. Sections 10.3 and 10.4 study the concept of a Baire 1 function and give some applications.

Although the setting is mainly that of a complete metric space we see in Section 10.5 that category arguments can be conducted in more general metric spaces, those that are topologically complete. Finally, we conclude with some applications to function spaces.

10.1 The Baire Category Theorem

We introduce the theorem of this section via a game between two players (A) and (B).

Player (A) is given a subset A of $I_0 = [0,1]$, and player (B) is given the complementary set $B = \widetilde{A}$. Player (A) selects a closed interval $I_1 \subset I_0$; then (B) chooses a closed interval $I_2 \subset I_1$. The players alternate moves, a move consisting of selecting a closed interval inside the previously chosen interval. The players determine a nested sequences of closed intervals, (A) choosing those with odd index, (B) those with even index. If

$$A \cap \bigcap_{n=1}^{\infty} I_n \neq \emptyset,$$

then player (A) wins; otherwise, (B) wins. The goal of player (A) is to make sure that the intersection contains a point of A; the goal of (B) is for the intersection to be empty or to contain only points of B.

One expects that player (A) should win if his set A is "large," while player (B) should win if his set is "large." It is not, however, immediately clear what large and small might mean for this game.

It is easy to see that, if the set A given to (A) contains an interval J, then (A) can win by choosing $I_1 \subset J$. Let us consider a more interesting example. Let A consist of the irrational numbers in $[0,1]$. Player (A) can win by following the strategy that we now describe. Let $q_0, q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Let I_1 be any closed interval such that $q_0 \notin I_1$. Inductively, suppose that $I_1, I_2, \ldots, I_{2n}$ have been chosen according to the rules of the game. It is now time for (A) to choose I_{2n+1}. The set $\{q_0, q_1, q_2, \ldots, q_n\}$ is finite, so there exists a closed interval $I_{2n+1} \subset I_{2n}$ such that

$$I_{2n+1} \cap \{q_0, q_1, q_2, \ldots, q_n\}$$

is empty. Player (A) chooses such an interval. Since, for each $n \in \mathbb{N}$, $q_n \notin I_{2n+1}$, the set $\bigcap_{n=1}^{\infty} I_n$ contains no rational numbers, but, as a nested sequence of closed intervals, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Thus

$$A \cap \bigcap_{n=1}^{\infty} I_n \neq \emptyset,$$

and (A) wins.

Using informal language, we can say that player (A) has a strategy to win: no matter how (B) plays, (A) can "answer" each move (B) makes in such a way that

$$A \cap \bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Player (A) has an advantage. The set A is larger than the set B. But in what sense is it larger? It is not the fact that $\lambda(A) = 1$ while $\lambda(B) = 0$ that matters here. It is something else. It is the fact that, given an interval I_{2n}, player (A) can choose I_{2n+1} inside I_{2n} in such a way that I_{2n+1} misses the set $\{q_0, q_1, q_2, \ldots, q_n\}$.

Let us elaborate a bit. Suppose that for each $n \in \mathbb{N}$ we replace $\{q_n\}$ with a set Q_n such that, given any interval $J \subset [0,1]$ and any $n \in \mathbb{N}$, there exists an interval $I \subset J$ such that

$$I \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_n) = \emptyset.$$

Then the same "strategy" will prevail: we see that the set $\bigcap_{n=1}^{\infty} I_n$ will be nonempty and will miss the set $\bigcup_{n=1}^{\infty} Q_n$. Thus, if

$$B = \bigcup_{n=1}^{\infty} Q_n,$$

player (A) has a winning strategy. It is in this sense that the set B is "small." The set A is "large" because the set B is "small."

Let us make the preceding discussion precise. Let (X, ρ) be a metric space. A set $S \subset X$ is called *nowhere dense* if, given any open ball $B(x, \varepsilon)$ in X, there exists an open ball $B(y, \delta) \subset B(x, \varepsilon)$ such that $S \cap B(y, \delta) = \emptyset$. In other words, S fails to be dense in any open ball. It is easy to check that S is nowhere dense if and only if $\overline{S}$ has empty interior. It is likewise easy to verify (Exercise 10:1.1) that a finite union of nowhere dense sets in X is also nowhere dense.

Thus, if $B = \bigcup_{n=1}^{\infty} Q_n$ in the game described, and each of the sets Q_n is nowhere dense, player (A) can use the strategy that we indicated. It will then follow that $\bigcap_{n=1}^{\infty} I_n$ contains no points of B. For (A) to win, however, $\bigcap_{n=1}^{\infty} I_n$ must contain a point in A; that is, $\bigcap_{n=1}^{\infty} I_n$ must be nonempty. (For our game on [0,1], that $\bigcap_{n=1}^{\infty} I_n$ is nonempty follows from a version of the Cantor intersection theorem.) The statement that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ implies that $\bigcup_{n=1}^{\infty} Q_n$ is not all of [0,1]. Thus [0,1] cannot be expressed as a countable union of nowhere dense sets.

The preceding motivational discussion provides the essence of a proof of the theorem of this section.

Theorem 10.1 (Baire category) *Let (X, ρ) be a complete metric space, and let S be a countable union of nowhere dense sets in X. Then $\widetilde{S}$ is dense in X.*

Proof. Let $S = \bigcup_{n=1}^{\infty} S_n$, where each of the sets S_n is nowhere dense, and let B_0 be a nonempty open ball in X. We show that $\widetilde{S} \cap B_0 \neq \emptyset$. Choose, inductively, a nested sequence of balls $B_n = B_n(x_n, r_n)$ with $r_n < 1/n$ such that

$$\overline{B}_{n+1} \subset B_n \setminus \overline{S}_{n+1}.$$

To see that this is possible, note first that $B_n \setminus \overline{S}_{n+1} \neq \emptyset$, since S_{n+1}, and therefore $\overline{S}_{n+1}$ is nowhere dense. Thus we can choose

$$x_{n+1} \in B_n \setminus \overline{S}_{n+1}.$$

Since $\overline{S}_{n+1}$ is closed,

$$\operatorname{dist}(x_{n+1}, S_{n+1}) > 0,$$

so we can choose B_{n+1} as required. The sequence $\{x_n\}$ is a Cauchy sequence since, for $n, m \geq N$,

$$\rho(x_n, x_m) \leq \rho(x_n, x_N) + \rho(x_N, x_m) < 2N^{-1}.$$

Because X is complete, there exists $x \in X$ such that $x_n \to x$. But $x_{n+1} \in \overline{B}_n$ for all n, so

$$x \in \bigcap_{n=1}^{\infty} \overline{B}_n \subset B_0 \cap \widetilde{S},$$

as was to be proved. ■

The following terminology is standard:

- A set $A \subset X$ is called *first category* if A is a countable union of nowhere dense sets.
- A set that is not of the first category is called a set of the *second category.*
- The complement of a first-category set is called a *residual* set.

For complete metric spaces, first category sets are the "small" sets and residual sets are the "large" sets in the sense of category. Second-category sets are merely "not small." For spaces that are not complete, a residual set can be empty (e.g., the entire space $\mathbb{Q}$ is of the first category). On the other hand, consider the subspace $\mathbb{N}$ of $\mathbb{R}$. As a subset of $\mathbb{R}$, $\mathbb{N}$ is of the first category, since $\{n\}$ is nowhere dense in $\mathbb{R}$ for each $n \in \mathbb{N}$. But as a space in itself, $\mathbb{N}$ cannot be expressed as a countable union of nowhere dense sets, since each set $\{n\}$ is dense in $B(n, \frac{1}{2})$. In fact, the only residual set in $\mathbb{N}$ is $\mathbb{N}$ itself.

Let us illustrate some of the concepts of this section.

Example 10.2 We show that the space c of convergent sequences is nowhere dense in the space ℓ_∞ of all bounded sequences.

Proof. It suffices to show that c is closed in ℓ_∞ and that $\ell_\infty \setminus c$ is dense in ℓ_∞ (See Exercise 10:1.4). That c is closed follows from Exercise 9:2.7. To show that the complement of c is dense, let $B(x, \varepsilon)$ be an open ball in ℓ_∞. If $x \notin c$, there is nothing further to prove, so assume that $x \in c$. Let $x = \{x_k\}$ with $\lim_{k\to\infty} x_k = \alpha$. There exists $N \in \mathbb{N}$ such that $|x_k - \alpha| < \varepsilon/2$ if $k \geq N$.

Choose $y = \{y_k\}$ in ℓ_∞ such that $y_k = x_k$ if $k < N$ and

$$y_k = \begin{cases} \alpha + \varepsilon/2, & \text{if } k \geq N,\ k \text{ odd;} \\ \alpha - \varepsilon/2, & \text{if } k \geq N,\ k \text{ even.} \end{cases}$$

Then $\rho(x, y) = \sup_k |x_k - y_k| < \varepsilon$, so $y \in B(x, \varepsilon)$. Since

$$\limsup y_k = \alpha + \tfrac{1}{2}\varepsilon \quad \text{and} \quad \liminf y_k = \alpha - \tfrac{1}{2}\varepsilon,$$

it follows that $y \notin c$. This shows that $\ell_\infty \setminus c$ is dense in ℓ_∞ and hence c is nowhere dense. ■

Recall that when a property is valid for all points in a measure space, except for a set of measure zero, we say that the property holds almost everywhere, abbreviated a.e. Let us introduce similar language when dealing with a complete metric space. If a property is valid for all points in a complete metric space except for a set of the first category, we shall say that the property holds *typically.* Other terms in common usage are *generically* and *residually.*

Thus, in connection with Example 10.2, we can say that, typically, elements of ℓ_∞ are divergent sequences or that the typical element in ℓ_∞ is divergent. To use such language, one must have a specific complete metric space in mind, just as in the setting of measure spaces the term "almost everywhere" pertains to a specific measure. For example, the statement "the typical real number is irrational" is correct when we assume the usual metric on $\mathbb{R}$. It would be false relative to the metric $\rho(x,y) = 1$ for all $x \neq y$ in $\mathbb{R}$. With this latter metric, a property is typical if and only if it holds for all real numbers.

Example 10.3 The typical $f \in \mathcal{C}[a,b]$ is nowhere monotonic; that is, it is monotonic on no open subinterval of $[a,b]$.

Proof. Let I denote an open subinterval of $[a,b]$, and let

$$A(I) = \{f \in \mathcal{C}[a,b] : f \quad \text{is nondecreasing on } I\}\,.$$

We show that $A(I)$ is nowhere dense in $\mathcal{C}[a,b]$ by showing that $A(I)$ is closed and has a dense complement in $\mathcal{C}[a,b]$.

Since a uniform limit of a sequence of functions that are nondecreasing on an open interval is also nondecreasing on that interval, $A(I)$ is closed. Let $B(f,\varepsilon)$ be an open ball in $\mathcal{C}[a,b]$. As in Example 10.2, if $f \notin A(I)$, there is nothing to prove, so assume that f is nondecreasing on I. Using the continuity of f, choose $x_1 < x_2$ in I such that $f(x_2) - f(x_1) < \varepsilon/3$. Choose $g \in B(f,\varepsilon)$ such that $g(x_1) = f(x_1)$ and $g(x_2) = f(x_2) - \varepsilon/3$. For example, g can be chosen to equal f except on a small neighborhood of x_2. Then

$$g \in \widetilde{A(I)} \cap B(f,\varepsilon),$$

so $\widetilde{A(I)}$ is dense. Thus $A(I)$ is nowhere dense.

Now let $\{I_k\}$ be an enumeration of those open subintervals of $[a,b]$ having rational endpoints. If $f \in \mathcal{C}[a,b]$ is nondecreasing on some interval $I \subset [a,b]$, then there exists $k \in \mathbb{N}$ such that f is nondecreasing on I_k. Thus $f \in \bigcup_{k=1}^{\infty} A(I_k)$. But this set is first category. Similarly, we show that

$$\{f \in \mathcal{C}[a,b] : f \quad \text{is nonincreasing on some open subinterval of } [a,b]\}$$

is of the first category in $\mathcal{C}[a,b]$. Since a union of two first category sets is itself of the first category, we have shown that the set of functions that are monotonic on some open interval in $[a,b]$ is a first-category subset of $\mathcal{C}[a,b]$. We infer that the typical $f \in \mathcal{C}[a,b]$ is nowhere monotonic. ■

We shall make frequent use of the Baire category theorem. In particular, we devote Sections 10.3 and 10.6 to specific applications. See also the exercises for this section.

Exercises

10:1.1 Show that in a metric space X a finite union of nowhere dense sets is nowhere dense.

10:1.2 Recall that a set in a metric space X is said to be of type $\mathcal{F}_\sigma$ if it is a countable union of closed sets. It is of type $\mathcal{G}_\delta$ if it is a countable intersection of open sets.

(a) Show that A is of type $\mathcal{F}_\sigma$ if and only if $\widetilde{A}$ is of type $\mathcal{G}_\delta$.

(b) Show that a dense set of type $\mathcal{G}_\delta$ in a complete metric space is residual.

(c) Show that every residual subset of a complete metric space contains a dense set of type $\mathcal{G}_\delta$.

10:1.3 Give an example of a set $A \subset \mathbb{R}$ such that A is residual in $\mathbb{R}$ and $\lambda(A) = 0$.

10:1.4 Show that a closed set A in a metric space X is nowhere dense if and only if $\widetilde{A}$ is dense.

10:1.5 Show that c_0 is nowhere dense in c and that $\mathcal{C}[a,b]$ is nowhere dense in $M[a,b]$.

10:1.6 Let $\mathcal{P}$ denote the polynomials on $[a,b]$, and let $\mathcal{P}_n \subset \mathcal{P}$ denote the polynomials of degree at most n. Show that $\mathcal{P}_n$ is nowhere dense in $\mathcal{C}[a,b]$; thus $\mathcal{P}$ is a first-category subset of $\mathcal{C}[a,b]$.

10:1.7 Prove that in a complete metric space X, a countable union of first-category sets is of the first category, and a countable intersection of residual sets is residual.

10:1.8 Show that a closed interval cannot be the union of a countable number of pairwise disjoint closed sets unless all but one of these sets is empty.

10:1.9 Let f have derivatives of all orders on $I = [0,1]$. Prove that if, for every $x \in I$, there exists $n = n(x)$ such that $f^{(n)}(x) = 0$ then f is a polynomial on I.

10:1.10 Let $\{f_n\}$ be a sequence of continuous functions on $I = [a,b]$. Prove that if, for every $x \in I$, there exists $M(x) \in \mathbb{R}$ such that $|f_n(x)| \le M(x)$ then there exists $M \in \mathbb{R}$ and an interval $(c,d) \subset [a,b]$ such that $|f_n(x)| \le M$ for all $n \in \mathbb{N}$ and $x \in (c,d)$. Thus the family $\{f_n\}$ is uniformly bounded in some open interval.

10:1.11 Prove that if the oscillation $\omega(f,x)$ (see Section 5.5) is positive for all $x \in [a,b]$ then there exists $\varepsilon > 0$ and an interval $(c,d) \subset [a,b]$ such that $\omega(f,x) \ge \varepsilon$ for all $x \in (c,d)$.

10:1.12 Show that for the metric space of Example 9.12, with respect to the measure space $([0,1], \mathcal{L}, \lambda)$, the typical $A \in \mathcal{L}$ has the property that, for every open interval $I \subset [0,1]$, $\lambda(A \cap I) > 0$ and $\lambda(\widetilde{A} \cap I) > 0$.

10:1.13 Show that in Example 9.13 the typical $K \in \mathcal{K}$ has no isolated points. [Hint: Let $\mathcal{K}_n$ be the set of all sets K for which there exists an isolated point $x \in K$ such that $\operatorname{dist}(x, K \setminus \{x\}) > 1/n$. Show that $\mathcal{K}_n$ is nowhere dense in $\mathcal{K}$.]

10:1.14 Let $\mathcal{K}$ consist of the nonempty compact subsets of [0,1] furnished with the Hausdorff metric (see Example 9.13).

(a) Show that the typical $K \in \mathcal{K}$ is a Cantor set.

(b) Show that the typical $K \in \mathcal{K}$ contains only irrational numbers.

(c) Show that the typical $K \in \mathcal{K}$ has Lebesgue measure zero.

(d) Show that the typical $K \in \mathcal{K}$ has Hausdorff dimension zero (see Section 3.8).

(e) Show that the typical $K \in \mathcal{K}$ is porous (see Exercise 7:9.12).

10.2 The Banach–Mazur Game

Let us return to our game of Section 10.1 for a moment. It was invented by Stanislaw Mazur (1905–1981) around 1928. We have seen that player (A) can win if his set A is residual in some interval. By the same reasoning, player (B), who plays by the same rules but starts the game after player (A), will win if A is first category. Mazur conjectured that (B) has a winning strategy *only* if A is of the first category. This conjecture was proved to be true by Banach (who never did publish the proof). The game is accordingly called the Banach–Mazur game. To present a proof that (B) has a winning strategy if and only if A is of the first category involves a precise statement of what one means by a "winning strategy."

Let X be an arbitrary metric space. We suppose that there is given a class $\mathcal{E}$ of subsets of X that the players of the game are required to use. Each member of $\mathcal{E}$ must have a nonempty interior, and every open set in X contains some member of $\mathcal{E}$.

The players are given two sets $A \subset X$ and $B = X \setminus A$. Then the game $\ll A, B \gg$ is played according to the following rules: two players (A) and (B) alternately choose sets

$$U_1 \supset V_1 \supset U_2 \supset V_2 \supset U_3 \supset V_3 \supset \dots U_n \supset V_n \dots \tag{1}$$

from the class $\mathcal{E}$. Player (A) starts the game and chooses $U_1 \in \mathcal{E}$, then player (B) chooses a subset $V_1 \in \mathcal{E}$, and so on, with player (A) choosing the U_i and player (B) choosing the V_i. Player (A) is declared the winner if

$$\bigcap_{i=1}^{\infty} V_i \cap A \neq \emptyset$$

and player (B) is the winner otherwise, that is, if

$$\bigcap_{i=1}^{\infty} V_i \subset B.$$

Player (A) evidently hopes that his set A is "large" enough that he can arrange for this; player (B) would have the same hope for the set B.

The ideas sketched in Section 10.1 suggest that if A is first category then player (B) has a method of winning no matter how player (A) chooses his sets $\{U_i\}$. What is most interesting here and useful, too, is that this is the only situation in which player (B) can be assured a win. But to explore this we need some terminology from the theory of games.

Any nested sequence as in (1) of sets from $\mathcal{E}$ is called *a play of the game.* A *strategy* for player (B) is a sequence of functions $\beta = \{\beta_n\}$, where

$$V = \beta_n(U_1, V_1, U_2, V_2, \ldots V_{n-1}, U_n)$$

is defined for any nested sequence $(U_1, V_1, U_2, V_2, \ldots V_{n-1}, U_n)$ of sets from $\mathcal{E}$, and V is a member of $\mathcal{E}$ contained in U_n. A play of the game (1) is said to be *consistent* with the strategy β if at each stage

$$V_n = \beta_n(U_1, V_1, U_2, V_2, \ldots V_{n-1}, U_n).$$

Thus a strategy $\beta = \{\beta_n\}$ is just a well-defined method for choosing the next play in the game for player (B). We say this is a *winning strategy* for player (B) if he is assured a win using it. Thus, if β is a winning strategy, then every play of the game consistent with the strategy β results in a win for player (B). The game is said to be *determined* in favor of (B) if there is a winning strategy for (B).

It was Mazur who conjectured the following theorem and Banach who found a proof. The version given here is more general in that it is set in full generality (rather than the narrow case where the players play intervals of real numbers). The proof we present is due to Oxtoby.[1] The proof for the real line is rather easier.[2] Remember that a set S is residual in a metric space if there is a sequence of dense open sets G_k so that $S \supset \bigcap_{k=1}^{\infty} G_k$. (The theorem is stated in a metric space, but is valid in any topological space.)

Theorem 10.4 (Banach–Mazur) *Let X be an arbitrary metric space. Then the game $\ll A, B \gg$ is determined in favor of player (B) if and only if the set B is residual in X.*

Proof. The first part of the proof is just to exhibit the "strategy" suggested in Section 10.1. Write $B \supset \bigcap_{i=1}^{\infty} G_i$, where each G_i is dense and open. Then if the sequence

$$U_1, V_1, U_2, V_2, \ldots V_{n-1}, U_n$$

has been played, we instruct player (B) to play a set

$$V_n \subset U_n \cap G_n$$

[1] In *Contributions to a Theory of Games*, Vol. III, Ann. of Math. Stud., 39 (1957), pp. 159–163.

[2] See J. C. Oxtoby, *Measure and Category*, Graduate Texts in Mathematics, Springer (1980), p. 28.

from $\mathcal{E}$, which can be done since G_n is open and dense.

We should perhaps make this a little more explicit. Let $\mathcal{E}_0$ be a well-ordered subclass of $\mathcal{E}$ such that each member of $\mathcal{E}$ contains a member of $\mathcal{E}_0$. If X is a separable metric space, then we can choose $\mathcal{E}_0$ countable and so we have an ordinary sequence; in general, we can just well order $\mathcal{E}$. Then our strategy can be explicitly stated by requiring that

$$\beta_n(U_1, V_1, U_2, V_2, \dots V_{n-1}, U_n)$$

be the first member of $\mathcal{E}_0$ that is contained in the set $U_n \cap G_n$. It is easy to see that any play of game consistent with β has

$$\bigcap_{i=1}^{\infty} V_i \subset B,$$

and so we have devised a winning strategy.

Conversely, we suppose that there does exist a winning strategy $\beta = \{\beta_n\}$ for player (B). Let us call (just for the purposes of the proof) any nested sequence of sets in $\mathcal{E}$

$$U_1 \supset V_1 \supset U_2 \supset V_2 \supset U_3 \supset V_3 \supset \dots U_n \supset V_n \tag{2}$$

such that

$$V_i = \beta_i(U_1, V_1, U_2, V_2, \dots, U_i) \quad (1 \le i \le n)$$

a *β-chain of order n*. The interior of the set V_n will be called the *interior of the chain.* A β-chain of order $n+k$ is a *continuation* of a β-chain of order n if the first $2n$ sets of the chains are the same. The class of all β-chains is ordered by this relation of continuation.

We wish to show that B contains the intersection of some sequence of dense open sets $\{G_n\}$. We construct the sequence inductively.

Among all β-chains of order 1, let F_1 denote a maximal family with the property that the interiors of any two members of F_1 are disjoint. Let G_1 be the union of the interiors of the members of F_1. Certainly, G_1 is open; it is also dense since F_1 is maximal.

Proceeding by induction, we suppose that, among all β-chains of order n, we have chosen a family F_n with the property that the interiors of any two members of F_n are disjoint and so that the set G_n, defined as the union of the interiors of the members of F_n, is open and dense. We shall describe how to select F_{n+1}. Among all β-chains of order $n+1$ that are continuations of members of the family F_n, we let F_{n+1} be a maximal family with the property that the interiors of any two members of F_{n+1} are disjoint. Such a maximal family must exist by Zorn's lemma (Section 1.11). If G_{n+1} denotes the union of the interiors of the members of F_{n+1}, then we see that G_{n+1} is open; it is also dense, since F_{n+1} is maximal.

This defines our sequence of families $\{F_n\}$ and associated dense, open sets $\{G_n\}$. Recall that each member of F_{n+1} is a β-chain of order $n+1$

that is a continuation of some member of F_n. We show now that

$$B \supset \bigcap_{n=1}^{\infty} G_n \tag{3}$$

and the proof is complete. Let x be a point in this intersection. There is a unique sequence $\{C_n\}$ of β-chains so that $C_n \in F_n$ and such that x is in the interior of the chain C_n for each n. This sequence of β-chains is linearly ordered by continuation and defines an infinite nested sequence of sets belonging to $\mathcal{E}$ whose intersection contains x. This sequence is a play consistent with the strategy β and so must win for player (B) by our assumptions. Accordingly, $x \in B$. This applies to every point in the set $\bigcap_{n=1}^{\infty} G_n$, and so the inclusion (3) has been established. This proves that B is residual and the theorem is proved. ■

We repeat Example 10.3 with a proof now using a game argument, but designed so that essentially it follows the same arithmetic. (The direct proof given in Section 10.1 also established that somewhere monotonic functions formed a first-category set of type $\mathcal{F}_\sigma$; the methods here do not provide this refinement.)

Example 10.5 The typical $f \in \mathcal{C}[a,b]$ is nowhere monotonic; that is, it is monotonic on no open subinterval of $[a,b]$.

Proof. Let B denote the set of functions $f \in \mathcal{C}[a,b]$ that are monotonic on no open subinterval of $[a,b]$. We play a Banach–Mazur game in which the players must choose closed balls $B(f,r)$ in $\mathcal{C}[a,b]$, where the function f is continuous and piecewise linear and where $r > 0$. We show that player (B) has a winning strategy in this game, and we can conclude, by Theorem 10.4, that B is residual in $\mathcal{C}[a,b]$.

Suppose that at the nth stage the players have already played the sets

$$U_1 \supset V_1 \supset U_2 \supset V_2 \supset U_3 \supset V_3 \supset \cdots \supset U_n$$

according to the rules of the game. [Thus $U_n = B(g_n, \delta_n)$ for some piecewise linear g_n.] How may we advise player (B) to make his next move? He is merely to play a closed ball $B(f_n, \varepsilon_n)$ centered at a continuous, piecewise linear function f_n and with radius ε_n by the following device (commented for convenience):

1. Partition the interval into points

   $$a = x_0 < x_1 < \cdots < x_k = b$$

   so that the points are closer together than n^{-1} and so that g_n varies by no more than $\delta_n/3$ on each interval $[x_i, x_{i+1}]$. (This makes sure that the partitions are getting finer as the game progresses. Note that the uniform continuity of the function g_n allows this.)

2. Choose a piecewise linear function f_n so that, at each of the points of the partition $f_n(x_i) = g_n(x_i)$ and at the further subdivided points,

$$f_n(x_i + \tfrac{1}{3}(x_{i+1} - x_i)) = g_n(x_i) - \tfrac{1}{3}\delta_n$$

and

$$f_n(x_i + \tfrac{2}{3}(x_{i+1} - x_i)) = g_n(x_i) + \tfrac{1}{3}\delta_n,$$

and make f_n linear elsewhere. (This way f_n is close to g_n and rises and falls inside every interval of the partition.)

3. Make sure that $\varepsilon_n < \delta_n/9$ and $\varepsilon_n < n^{-1}$. [This keeps $B(f_n, \varepsilon_n)$ inside $B(g_n, \delta_n)$ and also ensures that no function this close to g_n can be monotonic on large intervals, larger than n^{-1}, for example.]

By these criteria, we see that the closed ball $B(f_n, \varepsilon_n)$ is contained in $B(g_n, \delta_n)$. Also, we see that any function $h \in B(f_n, \varepsilon_n)$ is not monotonic on any interval of the partition $\{[x_i, x_{i+1}]\}$. Thus the intersection of these sets cannot contain a function that is monotonic on any interval. Hence (B) wins by following this strategy. ■

Exercises

10:2.1 In the game described for Example 10.5, a picture would be better than all these words. Give a presentation of and justification for the winning strategy that uses a minimum of words and formulas.

10:2.2 Suppose that we were to play the Banach–Mazur game on $\mathbb{Q} \cap [0, 1]$, rather than on [0,1]. Devise a strategy for (B) that will allow (B) to win regardless of the set A given to (A).

10:2.3 In the proof of Theorem 10.4 the definition of F_{n+1} required an appeal to Zorn's lemma. Show that if $\mathcal{E}_0$ is a sequence then this can be done without such an appeal. [Hint: Let $\mathcal{E}_1$ be that subsequence of $\mathcal{E}_0$ consisting of those sets that are contained in the last term of some chain belonging to F_n. Each member of $\mathcal{E}_1$ determines a β-chain of order $n + 1$ of which it is the $(2n + 1)$th term. Arrange these chains in a sequence. Taking these in order, select those whose interior is disjoint to the interiors of the chains already selected.]

10:2.4 Use the Banach–Mazur game to prove the following theorem, valid in any metric space.

> **Theorem (Banach category theorem)** *For any set A of second category in X there exists a nonempty open set G such that A is second category at every point of G.*

(A set A is first category at a point x if there is some neighborhood U of x so that $U \cap A$ is first category. Otherwise, A is second category at x.)

10:2.5 Explain how a winning strategy for player (A) should be defined. [Hint: Player (A) needs to be told what set to play first.]

10:2.6 Show that there are sets $A \subset \mathbb{R}$ and $B = \mathbb{R} \setminus A$ so that the game $\ll A, B \gg$ is not determined for either player (A) or for player (B). [Hint: Let A and B intersect every perfect set. (This requires the axiom of choice.)]

10:2.7 Prove the following theorem.

> **Theorem (Oxtoby)** *Let X be a complete metric space. The game $\ll A, B \gg$ is determined in favor of player (A) if (and only if) the set B is first category at some point of X.*

[The "if" part should certainly be attempted. For the "only if," perhaps see the article of Oxtoby (1957) cited earlier in this section.]

10.3 The First Classes of Baire and Borel

In Exercise 4:6.2 we discussed a bit of the Borel and Baire classifications of real-valued functions defined on an interval of $\mathbb{R}$. In this section we consider the important case of real-valued functions in the first classes of Borel and Baire whose domain is a metric space. Such classifications carry over also to mappings between metric spaces.[3]

Let (X, ρ) be a metric space, and let $f : X \to \mathbb{R}$. The function f is said to be in the *first class of Baire* or a *Baire-1 function*, if f is the pointwise limit of a sequence of continuous functions. We denote this class by $\mathcal{B}_1$.

If for every $\alpha \in \mathbb{R}$ the sets

$$\{x : f(x) < \alpha\} \text{ and } \{x : f(x) > \alpha\}$$

are of type $\mathcal{F}_\sigma$ in X, we say that f is in the *first class of Borel* or a *Borel-1 function.* We denote this class by $\mathcal{B}or_1$. It is clear that $f \in \mathcal{B}or_1$ if and only if $f^{-1}(G)$ is of type $\mathcal{F}_\sigma$ in X for every open set $G \subset \mathbb{R}$ and, equivalently, if and only if $f^{-1}(F)$ is of type $\mathcal{G}_\delta$ for every closed set $F \subset \mathbb{R}$.

We shall show in Theorem 10.12 that $\mathcal{B}or_1$ and $\mathcal{B}_1$ are identical for real-valued functions defined on a metric space. This is not the case in a general topological space.

Example 10.6 Let $X = \mathbb{R}$, and let A be a finite subset of $\mathbb{R}$, and let $f = \chi_A$. For every $\alpha \in \mathbb{R}$, the sets

$$\{x : f(x) < \alpha\} \quad \text{and} \quad \{x : f(x) > \alpha\}$$

are finite or have finite complements and are therefore of type $\mathcal{F}_\sigma$. It follows that $f \in \mathcal{B}or_1$. (It is also true that the function f is in $\mathcal{B}_1$; this is left as Exercise 10:3.1.)

[3]See C. Kuratowski, *Topology*, Academic Press (1966).

Example 10.7 The function $\chi_{\mathbb{Q}}$ is not Borel-1 on $\mathbb{R}$, because

$$\mathbb{R} \setminus \mathbb{Q} = \left\{x : \chi_{\mathbb{Q}} < \tfrac{1}{2}\right\}$$

is not of type $\mathcal{F}_\sigma$. To see this, observe first that a closed subset of $\mathbb{R} \setminus \mathbb{Q}$ is nowhere dense in $\mathbb{R}$. If

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{k=1}^{\infty} F_k$$

with each of the sets F_k closed, then we would have

$$\mathbb{R} = \mathbb{Q} \cup \bigcup_{k=1}^{\infty} F_k.$$

But this would imply that $\mathbb{R}$ is a countable union of nowhere dense sets. This is impossible, since $\mathbb{R}$ is complete.

Neither $\mathcal{B}_1$ nor $\mathcal{B}or_1$ is closed under pointwise limits. Let

$$\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$$

be an enumeration of the rationals. For $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} 1, & \text{if } x = q_1, q_2, \dots, q_n; \\ 0, & \text{otherwise.} \end{cases}$$

From Example 10.6 we see that $f_n \in \mathcal{B}_1$ and $f_n \in \mathcal{B}or_1$, for all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} f_n(x) = \chi_{\mathbb{Q}}(x)$$

for all $x \in \mathbb{R}$, we see from Example 10.6 that $\mathcal{B}_1$ and $\mathcal{B}or_1$ fail to be closed under pointwise limits.

Both $\mathcal{B}_1$ and $\mathcal{B}or_1$ are, however, closed under uniform limits. We now verify this for $\mathcal{B}or_1$. We shall prove presently that $\mathcal{B}_1 = \mathcal{B}or_1$, so $\mathcal{B}_1$ is also closed under uniform limits.

Theorem 10.8 *Let X be a metric space. Then the class $\mathcal{B}or_1$ on X is closed under uniform limits.*

Proof. Let $\{f_n\}$ be a sequence of functions in $\mathcal{B}or_1$ converging uniformly to f. Let $\{m_n\}$ be an increasing sequence of positive integers such that

$$|f(x) - f_{m_n+k}(x)| < \tfrac{1}{n}$$

for all $x \in X$ and $k = 0, 1, 2, \dots$. Let $\alpha \in \mathbb{R}$. We show that $\{x : f(x) \ge \alpha\}$ is of type $\mathcal{G}_\delta$. Now

$$\{x : f(x) \ge \alpha\} = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \left\{x : f_{m_n+k}(x) \ge \alpha - \tfrac{1}{n}\right\}. \tag{4}$$

We leave verification of (4) as Exercise 10:3.2. Each of the functions f_{m_n+k} is in $\mathcal{B}or_1$ by hypothesis. Thus each of the sets

$$\left\{x : f_{m_n+k}(x) \geq \alpha - \tfrac{1}{n}\right\}$$

is of type $\mathcal{G}_\delta$. Thus

$$\{x : f(x) \geq \alpha\}$$

is also of type $\mathcal{G}_\delta$. One shows similarly that the set

$$\{x : f(x) \leq \alpha\}$$

is of type $\mathcal{G}_\delta$. ■

Our next objective is to show that the classes $\mathcal{B}_1$ and $\mathcal{B}or_1$ on a metric space X coincide. We do this via a sequence of lemmas. We begin with a simple lemma and leave its proof as an exercise.

Lemma 10.9 *Let X be a metric space, and let F, F_1, and F_2 be subsets of X. Then:*

1. *If F is closed, then F is of type $\mathcal{G}_\delta$.*
2. *If F is open, then F is of type $\mathcal{F}_\sigma$.*
3. *If F_1 and F_2 are closed, then $F_1 \setminus F_2$ is both of type $\mathcal{G}_\delta$ and of type $\mathcal{F}_\sigma$.*

Lemma 10.10 *Let X be a metric space, and let $A_1, A_2, \ldots, A_n$ be sets of type $\mathcal{F}_\sigma$. Let $A = A_1 \cup \cdots \cup A_n$. Then there exist sets $B_1, B_2, \ldots, B_n$ such that*

1. *B_i is of type $\mathcal{F}_\sigma$ for all $i = 1, \ldots, n$,*
2. *$B_i \subset A_i$ for all $i = 1, \ldots, n$,*
3. *the sets B_i are pairwise disjoint, and*
4. *$A = B_1 \cup \cdots \cup B_n$.*

Proof. Each of the sets A_i is a countable union of closed sets. We can therefore express A in the form $A = \bigcup_{k=1}^\infty F_k$, where each F_k is closed and contained in one of the sets A_i. Let

$$E_1 = F_1,\ E_k = F_k \setminus (F_1 \cup \cdots \cup F_{k-1}).$$

Then each of the sets E_k is a difference of closed sets and is therefore of type $\mathcal{F}_\sigma$, by Lemma 10.9. Furthermore, the sets E_k are pairwise disjoint, and $A = \bigcup_{k=1}^\infty E_k$. For each $i = 1, \ldots, n$, let

$$N_i = \{k \in \mathbb{N} : E_k \subset A_i\} \quad \text{and} \quad B_i = \bigcup_{k \in N_i} E_k.$$

One verifies routinely that the sets B_i satisfy conditions (1) through (4). ■

Our next lemma shows that a Borel-1 simple function is also a Baire-1 function.

Lemma 10.11 *Let X be a metric space, and let $f : X \to \mathbb{R}$. Suppose that f has finite range $c_1 < c_2 < \cdots < c_n$ and that each of the sets*

$$E_k = \{x : f(x) = c_k\}, \quad k = 1, \ldots, n,$$

is of type $\mathcal{F}_\sigma$. Then $f \in \mathcal{B}_1$.

Proof. For each $k = 1, \ldots, n$, let

$$E_k = \bigcup_{i=1}^{\infty} F_{ki}, \tag{5}$$

where the sets F_{ki} are closed. For each $m \in \mathbb{N}$, let

$$S_m = \bigcup_{k=1}^{n} \bigcup_{i=1}^{m} F_{ki}.$$

The set S_m is closed. It consists of the first m sets appearing in the representation (5) for each of the sets E_k, $k = 1, \ldots, n$.

Now we define functions s_m on S_m by

$$s_m(x) = c_k \quad \text{if} \quad x \in \bigcup_{i=1}^{m} F_{ki}.$$

Each function s_m is continuous on S_m. By the Tietze extension theorem, for each $m \in \mathbb{N}$ there exists a continuous extension $\overline{s}_m$ of s_m to all of X. If $x \in X$, then there exists k such that $f(x) = c_k$; thus $x \in E_k$. It follows that there exists $m \in \mathbb{N}$ such that $x \in \bigcup_{i=1}^{m} F_{ki}$ and that $s_m(x) = c_k$. Since the sequence $\{S_m\}$ is an expanding sequence of sets, $s_j(x) = c_k$ for all $j \geq m$. Thus, for $j \geq m$,

$$f(x) = s_j(x) = \overline{s}_j(x) = c_k,$$

so $\lim_{j\to\infty} \overline{s}_j(x) = f(x)$. This is true for all $x \in X$, from which we infer that f is the pointwise limit of the sequence $\{\overline{s}_j\}$ of continuous functions. This shows that $f \in \mathcal{B}_1$. ■

Theorem 10.12 *Let X be a metric space, and let $f : X \to \mathbb{R}$. A necessary and sufficient condition that f be in the first class of Baire is that f be in the first class of Borel.*

Proof. Suppose that $f \in \mathcal{B}_1$. Let $\{f_n\}$ be a sequence of continuous functions on X such that

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for all } x \in X.$$

Let $\alpha \in \mathbb{R}$. We show that $\{x : f(x) < \alpha\}$ is of type $\mathcal{F}_\sigma$. The proof that $\{x : f(x) > \alpha\}$ is also of type $\mathcal{F}_\sigma$ is similar.

Consider the set

$$S = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ x : f_n(x) \le \alpha - \frac{1}{k} \right\}.$$

One verifies routinely that $S = \{x : f(x) < \alpha\}$ (Exercise 10:3.5). Since each f_n is continuous, the sets

$$\left\{ x : f_n(x) \le \alpha - \frac{1}{k} \right\}$$

are closed in X. An intersection of closed sets is closed, so the set

$$\bigcap_{n=m}^{\infty} \left\{ x : f_n(x) \le \alpha - \frac{1}{k} \right\}$$

is also closed. Thus S is a countable union of closed sets and is therefore of type $\mathcal{F}_\sigma$, as was to be proved.

To prove the converse, suppose first that f is a bounded Borel-1 function, say $|f(x)| < M$ for all $x \in X$. Let $n \in \mathbb{N}$. Choose numbers $c_0, c_1, \dots, c_n$ such that

$$-M = c_0 < c_1 < \cdots < c_n = M$$

and $c_{k+1} - c_k = 2M/n$. Let

$$A_0 = \{x : f(x) < c_1\} \text{ and } A_n = \{x : f(x) > c_{n-1}\}$$

and for, $k = 1, \dots, n-1$, let

$$A_k = \{x : c_{k-1} < f(x) < c_{k+1}\}.$$

Then $X = A_0 \cup \cdots \cup A_n$. Each of these sets is of type $\mathcal{F}_\sigma$, but the sets need not be pairwise disjoint. We now apply Lemma 10.10 to obtain sets $B_0, \dots, B_n$ of type $\mathcal{F}_\sigma$ and pairwise disjoint such that $X = B_1 \cup \cdots \cup B_n$ and $B_k \subset A_k$ for all $k = 0, 1, \dots, n$.

For each $n \in \mathbb{N}$, define a function f_n by $f_n(x) = c_k$ if $x \in B_k$ and $k = 0, 1, \dots, n$. According to Lemma 10.11, each of these functions is a Baire-1 function. We show that $f_n \to f$ [unif] and then apply Exercise 4:6.2(g). (Exercise 4:6.2 deals with functions defined on intervals in $\mathbb{R}$, but the same proof works in general.)

Let $x \in X$. Then there exists k such that $x \in B_k \subset A_k$. Since $f_n(x) = c_k$ and $c_{k-1} < f(x) < c_{k+1}$, we have

$$|f_n(x) - f(x)| < \frac{2M}{n}.$$

This is true for all $x \in X$; thus $f_n \to f$ [unif], so $f \in \mathcal{B}_1$. This proves that a bounded function $f \in \mathcal{B}or_1$ is in $\mathcal{B}_1$.

It remains to verify the conclusion $\mathcal{B}or_1 \subset \mathcal{B}_1$ without the assumption that f is bounded. We leave the verification as Exercise 10:3.8. ■

Exercises

10:3.1 Prove that $\chi_A \in \mathcal{B}_1$, where $A \subset \mathbb{R}$ is a finite set.

10:3.2 Verify the identity (4).

10:3.3 One can define the classes $\mathcal{B}_1$ and $\mathcal{B}or_1$ for mappings between metric spaces in the obvious manner. Give an example to show that one cannot in general conclude that $\mathcal{B}_1 = \mathcal{B}or_1$. [Hint: Let $X = [0,1]$, and let $Y = \{0,1\}$. Show that $\mathcal{B}_1$ consists of the two functions $f_0 \equiv 0$ and $f_1 \equiv 1$.]

10:3.4 Prove Lemma 10.9.

10:3.5 Verify that $S = \{x : f(x) < \alpha\}$ in the proof of Theorem 10.12.

10:3.6$\diamondsuit$ Verify that the classes $\mathcal{B}_1$ and $\mathcal{B}or_1$ are closed under the usual arithmetic operations and under composition with continuous functions.

10:3.7 Show that, if f_1 and f_2 are Baire-1 or Borel-1 functions on a metric space X, then so are the functions $g(x) = \min\{f_1(x), f_2(x)\}$ and $h(x) = \max\{f_1(x), f_2(x)\}$.

10:3.8 Complete the proof of Theorem 10.12. [Hint: Work with the function $g = \arctan f$.]

10.4 Properties of Baire-1 Functions

One reason that the class $\mathcal{B}_1$ is important is that it includes a number of classes of functions that arise naturally in analysis. For functions on $\mathbb{R}$, for example, the class $\mathcal{B}_1$ contains the class $\triangle'$ of derivatives, as well as the class $\mathcal{C}_{ap}$ of approximately continuous functions and the class of semicontinuous functions (see Exercises 10:4.1 and 10:4.2). Each of these classes contains discontinuous members. Our next theorem shows that functions in these classes can be discontinuous only on first-category sets; we have already seen this theorem (Theorem 1.19) for functions on $\mathbb{R}$.

Theorem 10.13 *Let f be a Baire-1 function on the complete metric space X. Then f is continuous on a residual subset of X.*

It is convenient to prove Theorem 10.13 by using the notion of oscillation of a function at a point. (See Section 5.5.) Here we need that notion for functions on metric spaces. Let (X, ρ) be a metric space, let A be a nonempty subset of X, and let $f : X \to \mathbb{R}$. The extended real number

$$\omega(A) = \sup\{|f(x) - f(y)| : x, y \in A\}$$

is called the *oscillation of f on A.* For $x_0 \in X$, we define the *oscillation of f at x_0* by

$$\omega(x_0) = \lim_{\delta \to 0} \omega(B(x_0, \delta)).$$

Using these definitions, f is continuous at x_0 if and only if $\omega(x_0) = 0$.

Lemma 10.14 *Let $f : X \to \mathbb{R}$, and let $\varepsilon > 0$. Let*

$$W_\varepsilon = \{x : \omega(x) < \varepsilon\}.$$

Then W_ε is an open set. Thus the set of points of continuity of X is of type $\mathcal{G}_\delta$.

Proof. Let $x_0 \in W_\varepsilon$, so $\omega(x_0) < \varepsilon$. Thus there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in B(x_0, \delta)$. Let $z \in B(x_0, \delta/2)$. If $z_1, z_2 \in B(z, \delta/2)$, then $z_1, z_2 \in B(x_0, \delta)$. Thus

$$|f(z_1) - f(z_2)| < \varepsilon.$$

This shows that $\omega(B(z, \delta/2)) < \varepsilon$. It follows that $\omega(z) < \varepsilon$ and that W_ε is open.

To verify the second conclusion of Lemma 10.14, we need only observe that the set

$$\{x : \omega(x) = 0\} = \bigcap_{n=1}^{\infty} W_{(1/n)}$$

consists precisely of those points at which f is continuous. ■

Proof. (Proof of Theorem 10.13) Let $\{f_n\}$ be a sequence of continuous functions on X such that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in X$. Let B_0 be an open ball in X. It suffices to show that B_0 contains a point of continuity of f.

We show first that for every $\varepsilon > 0$ there exists an open ball $B_1 = B(x_1, \delta_1)$ with $\overline{B}_1 \subset B_0$ such that $\omega(\overline{B_1}) \le \varepsilon$.

For $m, n \in \mathbb{N}$, let

$$A_{nm} = \left\{x \in \overline{B}_0 : |f_n(x) - f_{n+m}(x)| \le \frac{\varepsilon}{3}\right\}.$$

Since each of the functions f_n is continuous, each of the sets A_{nm} is closed; thus the set

$$D_n = \bigcap_{m=1}^{\infty} A_{nm}$$

is also closed. Now $\overline{B}_0 = \bigcup_{n=1}^{\infty} D_n$. To see this, let $x_0 \in \overline{B}_0$. Since $\{f_n(x_0)\}$ converges, we have for sufficiently large n and all m that

$$|f_n(x_0) - f_{n+m}(x_0)| \le \frac{\varepsilon}{3},$$

so $x_0 \in D_n$. Thus $\overline{B}_0 \subset D_n$. The reverse conclusion is obvious. Thus, by the Baire category theorem, there exists $n \in \mathbb{N}$ for which D_n is dense in some ball $B(z, \delta)$. Since D_n is closed, $D_n \supset \overline{B}(z, \delta)$.

For $x \in B(z,\delta)$, we have $|f_n(x) - f_{n+m}(x)| \leq \varepsilon/3$ for all $m \in \mathbb{N}$. Letting $m \to \infty$, we see that

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{3}. \tag{6}$$

Now choose $\delta_1 < \delta$ such that the oscillation of f_n on $B(z,\delta_1)$ is less than $\varepsilon/3$. This is possible since f_n is continuous. We show that for $x_1 = z$ the ball $B_1 = B(x_1,\delta_1)$ has the required property.

Let $x, y \in B_1$. Then $|f_n(x) - f_n(y)| < \varepsilon/3$, as we have just shown. By (6),

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{3} \text{ and } |f_n(y) - f(y)| \leq \frac{\varepsilon}{3}.$$

Thus

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon.$$

To this point we have established that for every $\varepsilon > 0$ every open ball B_0 contains a ball B_1 on which the oscillation of f is less than ε. We can obviously choose B_1 to be closed. Proceeding inductively, we can obtain a nested sequence $\{B_k\}$ of balls, with $\overline{B}_{k+1} \subset B_k$ for every k and such that the oscillation of f on $\overline{B}_k$ is less than $1/k$. We may choose these balls in such a way that their radii approach zero. Since X is complete, it follows from Theorem 9.37 that $\bigcap_{k=1}^{\infty} \overline{B}_k$ consists of a single point x_0. Since, for every $k \in \mathbb{N}$, $x_0 \in B_k$, we have $\omega(x_0) < 1/k$, so $\omega(x_0) = 0$. Thus f is continuous at x_0.

Since B_0 was an arbitrary ball in X, we have shown that the set E of points of continuity of f is dense. By Lemma 10.14, E is of type $\mathcal{G}_\delta$. But a dense set of type $\mathcal{G}_\delta$ in a complete metric space is residual. ■

Corollary 10.15 *Let F be a closed nonempty subset of a complete metric space X, and let f be a Baire-1 function on X. Then $f|F$ has a point of continuity.*

Proof. The space F is complete, since F is closed in a complete space. It is clear that $f|F$ is a Baire-1 function on F. The conclusion follows from Theorem 10.13. ■

In Exercise 5:5.5, we indicated some examples of differentiable functions f whose derivatives are badly discontinuous. Part (f) of that exercise shows how to construct f so that f' is bounded but discontinuous a.e. Thus f' can be discontinuous on a set that is large in measure. Theorem 10.13 shows, however, that f' must be continuous on a set that is large in category: the set of points of discontinuity must be a first-category set. We shall discuss continuity of a derivative a bit more in Section 10.6.

A converse of Corollary 10.15 is also true, but more difficult to prove.[4] The function f is in $\mathcal{B}_1$ if and only if, for every closed set F, $f|F$ has a

[4]For a proof when $X = [a,b]$, see I. Natanson, *Theory of Functions of a Real Variable*, vol. II, Ungar (1955). A proof in a more general setting can be found in C. Kuratowski, Topology, Academic Press (1966).

point of continuity.

Example 10.16 We consider functions from $\mathbb{R}$ to $\mathbb{R}$.

1. Let $f = \chi_K$, where K is a Cantor set. Then $f \in \mathcal{B}_1$. (Use Theorem 10.12 or the converse of Corollary 10.15 to verify this.)

2. Let
$$g(x) = \begin{cases} 1, & \text{if } x \text{ is a two-sided limit point of } K; \\ 0, & \text{elsewhere.} \end{cases}$$
Then $g \notin \mathcal{B}_1$ since $g|K$ has no points of continuity. (Note that f and g agree except on a countable set, yet $f \in \mathcal{B}_1$ and $g \notin \mathcal{B}_1$.)

3. Let h be continuous except on a countable set. Then $h \in \mathcal{B}_1$. This is proved most easily by using the converse to Corollary 10.15. If F has an isolated point x_0, then $h|F$ is continuous at x_0. If F is perfect, then F is uncountable and therefore contains a point of continuity of f. Clearly, $f|F$ is continuous at this point. One can also verify that f is a Baire-1 function using Theorem 10.12. See Exercise 10:4.3. Thus functions of bounded variation are members of $\mathcal{B}_1$.

In Section 3.2 we obtained an outer measure μ^* as a limit of a sequence $\{\mu_n^*\}$ of outer measures. We next use Theorem 10.13 to outline a proof that a convergent sequence of finite measures on a common σ-algebra converges to a measure. We leave verification of details as Exercise 10:4.4.

Theorem 10.17 *Let $\{\mu_n\}$ be a sequence of finite measures on a σ-algebra $\mathcal{M}$ of subsets of a set X. If, for all $E \in \mathcal{M}$, $\lim_{n\to\infty} \mu_n(E)$ exists, then the set function σ defined by*

$$\sigma(E) = \lim_{n\to\infty} \mu_n(E)$$

is a measure on $\mathcal{M}$.

Proof. We first obtain a measure μ such that, for all $n \in \mathbb{N}$, μ_n is continuous on the metric space of μ-equivalent sets in $\mathcal{M}$ with the metric $\rho(A, B) = \mu(A \triangle B)$. Thus σ is a Baire-1 function in this complete metric space. We then apply Theorem 10.13. Define a measure μ on $\mathcal{M}$ by

$$\mu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n(1 + \mu_n(X))}. \tag{7}$$

Let $(\mathcal{M}, \rho)$ be the metric space of Example 9.12, with $\rho(A, B) = \mu(A \triangle B)$. Then each of the functions μ_n is continuous on $(\mathcal{M}, \rho)$. Now $(\mathcal{M}, \rho)$ is complete by Exercise 9:6.6. Thus the Baire-1 function σ has a point of continuity $A \in \mathcal{M}$.

To show that σ is a measure, note first that σ is additive. Let $\emptyset$ denote the equivalence class of zero-measure sets. Then σ is continuous at $\emptyset$. If

$\{E_n\}$ is a sequence of pairwise disjoint measurable sets and $E = \bigcup_{n=1}^{\infty} E_n$, then

$$\lim_{n\to\infty} \sigma\Big(\bigcup_{k=n}^{\infty} E_k\Big) = 0.$$

It follows that σ is countably additive. The other requirements for σ to be a measure are obviously met. ■

Exercises

10:4.1 A function $f : X \to \mathbb{R}$ is called *lower semicontinuous at* $x_0 \in X$ if

$$\liminf_{x\to x_0} f(x) \geq f(x_0).$$

If f is lower semicontinuous at every point of X, we say that f is *lower semicontinuous.*

(a) Show that every lower semicontinuous function is a Baire-1 function.

(b) Show that a lower semicontinuous function on an interval $[a, b]$ achieves a minimum value.

(c) Show that a pointwise limit of an *increasing* sequence of continuous functions on $[a, b]$ is lower semicontinuous.

(d) Define upper semicontinuity of a function at x_0 and show that f is continuous at x_0 if and only if f is upper semicontinuous at x_0 and lower semicontinuous at x_0.

(e) Prove that a bounded lower semicontinuous function f on $[a, b]$ is a derivative if and only if f is approximately continuous. Compare this result with Theorem 7.36 and Exercise 7:8.5.

10:4.2 Prove that an approximately continuous function $f : \mathbb{R} \to \mathbb{R}$ is in $\mathcal{B}_1$. [Hint: For f bounded, use an appropriate theorem from Chapter 7. Then use Exercise 10:3.6 for the general case.]

10:4.3 Refer to Example 10.16(3). Verify that $f \in \mathcal{B}_1$ by using Theorem 10.12. [Hint: $\{x : f(x) > \alpha\}$ is a union of an open set and a countable set.]

10:4.4 Complete the details in the proof of Theorem 10.17.

10.5 Topologically Complete Spaces

Consider the interval $X = (0, \infty)$. This space is not complete when furnished with the usual metric $\rho(x, y) = |x - y|$. Suppose that we wished to make every Cauchy sequence in X converge. We can do that in two ways. We could add points to X appropriately, as we did in Theorem 9.42. This results in the completion $(\overline{X}, \overline{\rho})$ of (X, ρ). Or we could simply strip the title of "Cauchy sequence" from every offending (nonconverging) Cauchy sequence. We do this by obtaining another metric σ for X so that (X,ρ) and (X,σ) are topologically equivalent and (X,σ) is complete. We wish to satisfy the condition that $\rho(x_n,x)\to 0$ if and only if $\sigma(x_n,x)\to 0$; that is, the two spaces (X,ρ) and (X,σ) have exactly the same convergent sequences with exactly the same limits. We also wish to accomplish the following: if $\{x_n\}$ is a nonconvergent Cauchy sequence with respect to ρ, it will simply not be a Cauchy sequence with respect to σ.

Here is one way to accomplish this. For $x,y\in(0,\infty)$, let

$$\sigma(x,y)=|x-y|+\left|\frac{1}{x}-\frac{1}{y}\right|.$$

Then σ is a metric on $(0,\infty)$, and $\rho(x_n,x)\to 0$ if and only if $\sigma(x_n,x)\to 0$. Thus ρ and σ are equivalent metrics: $(X,\rho)$and (X,σ) are topologically equivalent. Suppose that $\{x_n\}$ is a Cauchy sequence with respect to σ. Then both $\{x_n\}$ and $\{\frac{1}{x_n}\}$ are Cauchy sequences, and one verifies easily that there exists $x>0$ such that

$$\rho(x_n,x)\to 0 \quad\text{and}\quad \rho\left(\frac{1}{x_n},\frac{1}{x}\right)\to 0.$$

It follows that $\sigma(x_n,x)\to 0$, so $\{x_n\}$ is σ-convergent. Thus (X,σ) is complete. Offending sequences, such as the sequence $\{1/n\}$, are simply not σ-Cauchy!

How did we come up with the metric σ? Consider the curve Y with equation $y=1/x\ \ (x>0)$ in $\mathbb{R}^2$. Furnish Y with the ℓ_1 metric

$$\gamma\left(\left(x_1,\frac{1}{x_1}\right),\left(x_2,\frac{1}{x_2}\right)\right)=|x_1-x_2|+\left|\frac{1}{x_1}-\frac{1}{x_2}\right|.$$

Then Y is a closed subspace of $\mathbb{R}^2$ and is therefore complete. The function $f:X\to Y$ defined by $f(x)=(x,1/x)$ is a homeomorphism of X onto Y. We can define σ by

$$\sigma(x_1,x_2)=\gamma(f(x_1),f(x_2)).$$

This simple idea can be extended to a number of metric spaces. For example, it can be applied to $X=\mathbb{R}\setminus\mathbb{Q}$. The reader may wish to use this space X as a model while reading the proof of the main theorem of this section, the theorem of Alexandroff, which is presented as Theorem 10.18.

To state Alexandroff's theorem as it was proved in 1924, we need a bit of terminology. The metric space (X,ρ) is *topologically complete* if it is homeomorphic via h to some complete metric space (Y,γ). In that case,

$$\sigma(x,y)=\gamma(h(x),h(y))$$

is a metric on X that is topologically equivalent to ρ, and (X,σ) is complete. Thus (X,ρ) is topologically complete if X can be remetrized with a topologically equivalent metric (i.e., one which gives rise to the same open sets as ρ) so as to be complete. In such spaces the Baire category theorem is valid (Exercise 10:5.1).

We already know that a closed subset of a complete metric space is complete without any change in metric. Alexandroff's theorem, together with the converse that follows, gives an indication of the importance of sets of type $\mathcal{G}_\delta$.

Theorem 10.18 (Alexandroff) *Let X be a nonempty set of type $\mathcal{G}_\delta$ contained in a complete metric space (Y, ρ). Then X can be remetrized so as to be complete.*

Proof. Since X is of type $\mathcal{G}_\delta$, there exists a sequence $\{G_i\}$ of open sets in Y such that $X = \bigcap_{i=1}^\infty G_i$. If $X = Y$, there is nothing to prove, so assume that $X \neq Y$. In that case, we may assume that for every $i \in \mathbb{N}$ the set $F_i = \widetilde{G_i}$ is nonempty. For every $i \in \mathbb{N}$, define a function d_i by

$$d_i(x) = \operatorname{dist}(x, F_i) = \inf \{\rho(x, y) : y \in F_i\}.$$

Then d_i is real valued and continuous on Y and $d_i(x) > 0$ for all $x \in X$.

Consider now the function σ on $X \times X$ defined by

$$\sigma(x, y) = \rho(x, y) + \sum_{i=1}^\infty \frac{1}{2^i} \min\left(1, \left|\frac{1}{d_i(x)} - \frac{1}{d_i(y)}\right|\right).$$

(The reader may observe that this definition of σ is just an adaptation to our present setting of the metric that we obtained for $X = (0, \infty)$.)

We show that σ is a metric on X, that σ and ρ are equivalent metrics on X, and that (X, σ) is complete. That σ is a metric is clear, the triangle inequality being satisfied by each term of the series defining σ.

We first verify that σ and ρ are equivalent metrics on X. We do this by showing that $\rho(x_n, x) \to 0$ if and only if $\sigma(x_n, x) \to 0$. Since $\rho(x, y) \leq \sigma(x, y)$ for all $x, y \in X$, $\rho(x_n, x) \to 0$ whenever $\sigma(x_n, x) \to 0$. To prove the converse, let $\varepsilon > 0$, and let $x \in X$. Choose $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon/3$. Now choose δ such that $0 < \delta < \varepsilon/3$ and

$$\left|\frac{1}{d_i(x)} - \frac{1}{d_i(y)}\right| < \frac{\varepsilon}{3} \tag{8}$$

whenever $\rho(x, y) < \delta$ and $i = 1, \ldots, N$. This is possible since d_i is positive on X and continuous everywhere. If $\rho(x, y) < \delta$, then it follows from (8) and the definitions of σ and N that

$$\sigma(x, y) < \frac{\varepsilon}{3} + \sum_{i=1}^N \frac{1}{2^i} \left|\frac{1}{d_i(x)} - \frac{1}{d_i(y)}\right| + \frac{1}{2^N} < \varepsilon.$$

Therefore, $\sigma(x, x_n) \to 0$ whenever $\rho(x, x_n) \to 0$. This proves that ρ and σ are equivalent metrics on X.

It remains to verify that (X, σ) is complete. Let $\{x_n\}$ be a Cauchy sequence in X relative to σ. Let $i \in \mathbb{N}$. Then there exists $N \in \mathbb{N}$ such that

$$\sigma(x_N, x_m) < \frac{1}{2^i} \quad \text{for all } m \geq N.$$

Thus, if $m \geq N$,

$$1 > 2^i \sigma(x_N, x_m) \geq \min\left(1, \left|\frac{1}{d_i(x_N)} - \frac{1}{d_i(x_m)}\right|\right),$$

so

$$\left|\frac{1}{d_i(x_N)} - \frac{1}{d_i(x_m)}\right| < 1.$$

It follows that the sequence

$$\left\{\frac{1}{d_i(x_n)}\right\} \tag{9}$$

is bounded for all $i \in \mathbb{N}$, so that $\operatorname{dist}(x_n, F_i)$ is bounded away from zero. Observe that this means the sequence $\{x_n\}$ does not get close to the set F_i in the ρ metric.

Now $\rho(x, y) \leq \sigma(x, y)$ for all $x, y \in X$. Thus the sequence $\{x_n\}$ is a Cauchy sequence with respect to ρ (as well as with respect to σ). Since Y is complete, there exists $y \in Y$ such that $\lim_{n\to\infty} \rho(x_n, y) = 0$. The point y cannot belong to any set F_i because the points $\{x_n\}$ are bounded away from F_i in the ρ metric. Thus, for all $i \in \mathbb{N}$, $y \in G_i$, so that $y \in X$. Since the two metrics σ and ρ are equivalent on X, $\lim \sigma(x_n, y) = 0$ and, hence, (X, σ) is complete. ■

Applying Theorem 10.18 to the set $\widetilde{\mathbb{Q}} \subset \mathbb{R}$, we see that $\widetilde{\mathbb{Q}}$ is not complete, but is topologically complete. A converse of Theorem 10.18, first proved by Stefan Mazurkiewicz (1888–1945) in 1916, is also available.

Theorem 10.19 *Let (Z, ρ) be a metric space, and let $X \subset Z$. If X is homeomorphic to a complete space (Y, γ), then X is of type $\mathcal{G}_\delta$ in Z.*

Proof. Let h be a homeomorphism of X onto Y. For each $x \in X$ and $n \in \mathbb{N}$ there exists $\delta(x, n)$ such that $0 < \delta(x, n) < 1/n$ and

$$\gamma(h(x), h(x')) < \frac{1}{n}$$

for all $x' \in X \cap B(x, \delta(x, n))$. Let

$$G_n = \bigcup_{x\in X} B\left(x, \tfrac{1}{2}\delta(x, n)\right).$$

Then G_n is open in Z. We show that

$$X = \bigcap_{n=1}^{\infty} G_n. \tag{10}$$

It is clear that $X \subset \bigcap_{n=1}^{\infty} G_n$. To prove the reverse inclusion, let $z \in \bigcap_{n=1}^{\infty} G_n$. For each $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$\rho(z, x_n) < \tfrac{1}{2}\delta(x_n, n).$$

Since $\delta(x_n, n) < 1/n$, it is clear that $\rho(z, x_n) \to 0$, so $x_n \to z$.

Now, if $m > n$, then

$$\begin{aligned}
\rho(x_n, x_m) &\leq \rho(x_n, z) + \rho(z, x_m) \\
&< \tfrac{1}{2}\delta(x_n, n) + \tfrac{1}{2}\delta(x_m, m) \\
&\leq \max\{\delta(x_n, n), \delta(x_m, m)\}.
\end{aligned}$$

Therefore,

$$\gamma(h(x_n), h(x_m)) < \frac{1}{n}$$

for all $m > n$. Let $y_n = h(x_n)$. Then $\{y_n\}$ is a Cauchy sequence in (Y, γ). Since (Y, γ) is complete, $\{y_n\}$ converges to a point $y \in Y$. Let $x = h^{-1}(y)$. Then $x \in X$, and $x_n \to x$ since h^{-1} is continuous. We have arrived at the situation $x_n \to z$ and $x_n \to x$. Thus $x = z$, so $z \in X$.

We have shown that $\bigcap_{n=1}^{\infty} G_n \subset X$, completing the verification of (10). This completes the proof that X is of type $\mathcal{G}_\delta$ in Z. ■

Exercises

10:5.1 Show that the Baire category theorem is valid in every topologically complete space.

10:5.2 Let K be the Cantor set, and let T consist of the two-sided limit points of K. Is T complete? Is T topologically complete?

10.6 Applications to Function Spaces

In Section 10.1 we saw a number of applications of the Baire category theorem. (See Example 10.3 and several of the exercises.) One way in which the Baire category theorem is often used is to prove the existence of objects that might be difficult to imagine or to construct. In this section we provide three examples that illustrate this point. Our objects will be functions. We shall view these functions as members of a complete or topologically complete metric space, and shall show that "most" members of that space exhibit properties that are difficult to envision.

Continuous Nowhere Differentiable Functions

Some nineteenth century mathematicians were aware of the existence of continuous functions that had no point of differentiability. Constructions of such functions involved summations of infinite series whose successive terms contributed increasingly to the nondifferentiability of their sum. Perhaps the first such construction was given by Weierstrass around 1875. Use of the Baire category theorem to prove the existence of such functions had to wait until 1931, at which time S. Banach and S. Mazurkiewicz, in separate papers in the journal *Studia Mathematica,* provided such proofs. Their proofs have now become part of the standard material in a first course in real analysis. We shall use a somewhat different approach that provides some additional insights into the way a typical $f \in \mathcal{C}[a, b]$ is nowhere differentiable.

Our approach is based on a simple idea. Suppose that $f \in \mathcal{C}[a, b]$, and all derived numbers of f at $x_0 \in (a, b)$ are less than M. Let L be the line with slope M through $(x_0, f(x_0))$. Then there exists an open interval $I \subset (a, b)$ such that, over I, the graph of f lies below L to the right of x_0 and lies above L to the left of x_0. Intuitively, the line L "crosses" the graph of f at $(x_0, f(x_0))$. A similar argument shows that, if all derived numbers exceed M, some line crosses the graph of f at $(x_0, f(x_0))$. It follows that if *no* line crosses the graph of f at x_0, then both $+\infty$ and $-\infty$ are derived numbers of f at x_0, and thus f is not differentiable at x_0. Our first objective is to make precise and then prove that the typical f in $\mathcal{C}[a, b]$ "crosses no lines."

We need some terminology. We say f is *nondecreasing at* $x \in [a, b]$ if there exists $\delta > 0$ such that

$$f(t) \leq f(x) \quad \text{for all} \quad t \in (x - \delta, x) \cap [a, b]$$

and

$$f(t) \geq f(x) \quad \text{for all} \quad t \in (x, x + \delta) \cap [a, b].$$

The function f is *nonincreasing at* x if $-f$ is nondecreasing at x, and f is *monotonic at* x if f is either nondecreasing or nonincreasing at x. Loosely speaking, f is monotonic at x if the graph of f "crosses" a horizontal line at $(x, f(x))$. (There are degenerate possibilities; e.g., f might be constant on one or both sides of x.)

Now let $\gamma \in \mathbb{R}$, and define a function $f_{-\gamma}$ by $f_{-\gamma}(x) = f(x) - \gamma x$. If there exists $\gamma \in \mathbb{R}$ and $x \in [a, b]$ such that $f_{-\gamma}$ is monotonic at x, we say that f is of *monotonic type at* x. If f is not of monotonic type at any $x \in [a, b]$ we say that f is of *nonmonotonic type.*

It might not be easy to visualize the graph of a function that is of nonmonotonic type. Intuitively, the graph of f cannot cross any lines. As we mentioned before, if f is of nonmonotonic type, then f has ∞ and $-\infty$ as derived numbers at every point and is therefore nowhere differentiable. We prove existence of such functions by use of the Baire category theorem.

Theorem 10.20 *The functions of nonmonotonic type form a dense subset of type $\mathcal{G}_\delta$ in $\mathcal{C}[a, b]$.*

Proof. Let

$$\begin{aligned} A = \{f \in \mathcal{C}[a, b] \quad : \quad & \text{there exists } \gamma \in \mathbb{R} \text{ and } x \in [a, b] \\ & \text{such that } f_{-\gamma} \text{ is nondecreasing at } x\}. \end{aligned}$$

For each $n \in \mathbb{N}$, let A_n denote those functions $f \in \mathcal{C}[a,b]$ for which there exists $\gamma \in [-n,n]$ and $x \in [a,b]$ such that

$$f_{-\gamma}(t) \le f_{-\gamma}(x) \quad \text{when} \quad t \in [a,b] \cap \left(x - \tfrac{1}{n}, x\right)$$

and

$$f_{-\gamma}(t) \ge f_{-\gamma}(x) \quad \text{when} \quad t \in [a,b] \cap \left(x, x + \tfrac{1}{n}\right).$$

Then $A = \bigcup_{n=1}^{\infty} A_n$. We show that for $n \in \mathbb{N}$ the set A_n is closed and the set $\widetilde{A_n}$ is dense; thus A_n is nowhere dense.

To verify that A_n is closed, let $\{f_k\}$ be a sequence of functions in A_n such that $f_k \to f$ [unif]. Then $f \in \mathcal{C}[a,b]$, and we must show that $f \in A_n$. For each $k \in \mathbb{N}$, there exists $\gamma_k \in [-n,n]$ and $x_k \in [a,b]$ such that

$$f_{-\gamma_k}(t) \le f_{-\gamma_k}(x_k) \quad \text{when} \quad t \in [a,b] \cap \left(x_k - \tfrac{1}{n}, x_k\right)$$

and

$$f_{-\gamma_k}(t) \ge f_{-\gamma_k}(x_k) \quad \text{when} \quad t \in [a,b] \cap \left(x_k, x_k + \tfrac{1}{n}\right).$$

There exists an increasing sequence $\{k_i\}$ from $\mathbb{N}$ such that $\{\gamma_{k_i}\}$ converges to some $\gamma \in [-n,n]$ and $\{x_{k_i}\}$ converges to some $x \in [a,b]$. It is easy to verify that

$$f_{-\gamma}(t) \le f_{-\gamma}(x) \quad \text{when} \quad t \in [a,b] \cap \left(x - \tfrac{1}{n}, x\right)$$

and

$$f_{-\gamma}(t) \ge f_{-\gamma}(x) \quad \text{when} \quad t \in [a,b] \cap \left(x, x + \tfrac{1}{n}\right).$$

Thus $f \in A_n$, and A_n is closed in $\mathcal{C}[a,b]$.

To show that A_n is nowhere dense, we verify that every ball in $\mathcal{C}[a,b]$ contains points of $\widetilde{A_n}$. Let B be an open ball in $\mathcal{C}[a,b]$. It is easy to visualize (though tedious to verify analytically) that we can choose an appropriate sawtooth function, with many steep teeth, such that $f \in B$, but $f \in \widetilde{A_n}$. Intuitively, the line segments that make up the graph of f have such steep slopes and there are so many segments that no line whose slope is bounded by $-n$ and n can cross the graph of f as required for f to be in A_n.

Thus A_n is nowhere dense, and A is first category and of type $\mathcal{F}_\sigma$ in $\mathcal{C}[a,b]$. The same is true of the set $B = \{f \in \mathcal{C}[a,b] : -f \in A\}$. It follows that the set $\mathcal{C}[a,b] \setminus (A \cup B)$ is a residual set of type $\mathcal{G}_\delta$. This set consists of the functions of nonmonotonic type. ■

Corollary 10.21 *The typical $f \in \mathcal{C}[a,b]$ has both ∞ and $-\infty$ as derived numbers at every point and is therefore nowhere differentiable.*

Let us obtain a bit more insight into the behavior of the typical $f \in \mathcal{C}[a,b]$. We need one more term. A function is called *nonangular at x* if $D_-f(x) \le D^+f(x)$ and $D_+f(x) \le D^-f(x)$. Inspection of an appropriate diagram suggests the geometric content of the term "nonangular." A function that is nonangular at each point of $[a,b]$ is called *nonangular.*

It is easy to verify that, for $f \in \mathcal{C}[a,b]$, f is nonangular if and only if for every $x \in (a,b)$ there exists γ such that γ is simultaneously a left-derived number and a right-derived number at x.

Suppose now that f is both nonangular and of nonmonotonic type at a point $x_0 \in (a,b)$. Let $\gamma \in \mathbb{R}$. If the set

$$\{x : f_{-\gamma}(x) = f_{-\gamma}(x_0)\}$$

has x_0 as a limit point, then γ is clearly a derived number for f. If not, then, since f is of nonmonotonic type at x_0, $f_{-\gamma}(t) < f_{-\gamma}(x_0)$ [or $f_{-\gamma}(t) > f_{-\gamma}(x_0)$] for all t in some deleted neighborhood of x_0. But this implies that γ is a derived number for f at x_0, because f is nonangular at x_0. If $\gamma = \pm\infty$, then, once again, we can show that γ is a derived number for f at x_0. Thus we have established the following result.

Theorem 10.22 *Let $f \in \mathcal{C}[a,b]$. If f is both nonangular and of nonmonotonic type at a point $x_0 \in (a,b)$, then every extended real number γ is a derived number for f at x_0.*

Functions of nonmonotonic type must be very badly behaved, but nonangular functions need not be. For example, every differentiable function is nonangular. As Corollary 10.21 shows, however, the class of differentiable functions is only a first-category subset of $\mathcal{C}[a,b]$. Our next result shows that the set of nonangular functions is large in $\mathcal{C}[a,b]$.

Theorem 10.23 *The set of nonangular functions forms a dense set of type $\mathcal{G}_\delta$ in $\mathcal{C}[a,b]$.*

The proof is similar to the proof of Theorem 10.20. We leave it as Exercise 10:6.5. Theorems 10.20, 10.22, and 10.23 give us the following theorem, first proved by V. Jarník in 1933.

Theorem 10.24 *A typical continuous function has every extended real number as a derived number at every point.*

To have a sense of the graph of a typical $f \in \mathcal{C}[a,b]$ near a point x, refer to the Denjoy–Young–Saks theorem (Exercise 7:9.5). When a function $f_{-\gamma}$ assumes a local extremum at x_0, x_0 is in one of the sets A_2 or A_3 of that theorem. Otherwise, x_0 is in the set A_4 [see Exercise 7:9.5(a)] or in an exceptional set where all derived numbers are achieved from one side, and $\pm\infty$ is the only derived number from the other side.

We end this discussion with a remark. Jarník's theorem shows that a typical $f \in \mathcal{C}[a,b]$ does not have a finite or infinite derivative at any point. Nor does f have a one-sided finite derivative at any point. S. Saks proved in 1932, however, that the typical f *does* have *infinite* one-sided derivatives on an uncountable set. Does there exist a continuous f that has no one-sided derivative, finite or infinite, at any point? The answer is yes! A construction was given by A. Besicovitch in 1925. The theorem of Saks shows that the Baire category theorem *cannot* be applied to $\mathcal{C}[a,b]$ to prove this result. It has, however, been applied to an appropriate closed subspace of $\mathcal{C}[a,b]$ in which "Besicovitch functions" form a residual subspace. This was accomplished by J. Maly in 1984.

Differentiable, Nowhere Monotonic Functions

Over 100 years ago, du Bois-Reymond expressed the view that a nowhere monotonic function cannot be differentiable. Dini, on the other hand, believed the existence of such functions to be highly probable. Example 10.3 shows that there are continuous nowhere monotonic functions, and Exercise 7:5.3 shows us that a nowhere monotonic function can be absolutely continuous. But in order for a *differentiable* function in $\mathcal{C}[a, b]$ to be nowhere monotonic, it must be true that both sets

$$\{x : f'(x) > 0\} \quad \text{and} \quad \{x : f'(x) < 0\}$$

are dense in $[a, b]$. Such functions may be difficult to visualize.

In 1887, Köpcke provided a construction of such a function. In discussing Köpcke's work, Denjoy[5] wrote in 1915:

> In 1887, Köpcke gave in Math. Annalen an example of a function possessing at each point (or so he thought) a derivative which vanished and took both signs in every interval in its domain. [He] returned to this subject on several occasions [references], correcting each time the errors contained in the previous proofs. This question of differentiable, nowhere monotonic functions has also provoked many other works [references].

The various constructions Denjoy referenced were quite complicated. At this point in Denjoy's paper, he had already given three separate constructions of his own. He was about to give a Köpcke-type construction, but, before doing so, he alerted the reader to the "clarity and simplicity" of his own construction made possible by borrowing ideas from "Borel and Lebesgue on measure of sets." To Denjoy, his constructions were simple and clear. We would probably find them horrendous.

Hobson modified Pereno's modification of Köpcke's construction in his book.[6] This was published about 40 years after Köpcke's first correction, 30 years after Pereno's modification, and 12 years after Denjoy's "simple and clear" development. It required ten pages!

Today a number of proofs are available. Some are constructive, but simpler than those early proofs. We shall provide a simple proof based on the Baire category theorem. It was advanced by C. Weil[7] in 1976 and required two pages. To present Weil's proof, we need to know that a uniform limit of bounded derivatives is a derivative. Since Weil's proof uses several ideas developed in the last two sections, we provide the necessary details.

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[5]A. Denjoy, "Sur les fonctions dérivées sommables," *Bull. Soc. Math. France* **43** (1915) pp. 161–248.

[6]E. Hobson, *Theory of Functions of a Real Variable*, II, Dover, New York, (1957) (reprinting of original book published in 1927).

[7]C. E. Weil, "On nowhere monotonic functions," Proc. Amer. Math. Soc., 56 (1976), 388–389.

Let $M\triangle'$ denote the space of bounded derivatives on [0,1] with

$$\rho(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

Then $M\triangle'$ is a closed subspace of $M[0,1]$ and is therefore complete. Now let

$$M\triangle'_0 = \{f \in M\triangle' : f = 0 \text{ on a dense set}\}.$$

We first show that $M\triangle'_0$ is closed under addition and is complete.

Let $f, g \in M\triangle'_0$. Since $f, g \in \mathcal{B}_1$, it follows that the sets $\{x : f(x) = 0\}$ and $\{x : g(x) = 0\}$ are of type $\mathcal{G}_\delta$. But the intersection of two dense sets of type $\mathcal{G}_\delta$ is also dense, so

$$\{x : f(x) + g(x) = 0\}$$

is dense. Thus $f + g \in M\triangle'_0$. Similarly, if $\{f_n\}$ is a sequence from $M\triangle'_0$ and $f_n \to f$ [unif], then $f \in M\triangle'_0$. To see this, let

$$A_n = \{x : f_n(x) = 0\},$$

and let $A = \bigcap_{n=1}^\infty A_n$. Each of the sets is dense and of type $\mathcal{G}_\delta$, so the same is true of A. But

$$A \subset \{x : f(x) = 0\}.$$

It follows that $f \in M\triangle'_0$. Thus $M\triangle'_0$ is closed in $M\triangle'$ and therefore complete.

Refer now to Example 5.2. There we constructed a strictly increasing differentiable function h such that $h' \in M\triangle'_0$. Thus $M\triangle'_0 \neq \emptyset$. Let I be an interval in [0,1], and let

$$P = \{p \in M\triangle'_0 : p \geq 0 \text{ on } I\}.$$

Then P is closed in $M\triangle'_0$. We show that P is nowhere dense in $M\triangle'_0$.

Let $B(f,\varepsilon)$ be an open ball in $M\triangle'_0$. If $f \notin P$, we have shown that $\widetilde{P} \cap B(f,\varepsilon) \neq \emptyset$. If $f \in P$, let x_0 be a point of continuity of f in the interval I. The existence of such a point follows from Theorem 10.13. It is clear that $f(x_0) = 0$, since $\{x : f(x) = 0\}$ is dense in [0,1]. Choose an open interval $J \subset I$ such that $f(x) < \varepsilon/2$ on J. Now choose $g \in M\triangle'_0$ such that $-g \in P$ and

$$\sup_{x\in[0,1]} (-g(x)) = \sup_{x\in J}(-g(x)) = \varepsilon.$$

(We can, for example, take $-g$ to be an appropriate modification of the function h' of Example 5.2.) Then $f + g \in M\triangle'_0$ and

$$\rho(f, f+g) = \rho(g, 0) = \varepsilon.$$

On J we have $0 \leq f(x) < \varepsilon/2$. In addition, there exists $x_1 \in J$ such that $-g(x_1) > \varepsilon/2$, so $g(x_1) < -\varepsilon/2$. Thus $f(x_1) + g(x_1) < 0$. It follows that

$$f + g \in B(f, 2\varepsilon) \setminus P$$

and that P is nowhere dense in $M\triangle'_0$.

In a similar way, we show that

$$N = \{f \in M\triangle'_0 : f \leq 0 \ \text{ on } \ I\}$$

is closed and nowhere dense in $M\triangle'_0$.

We have shown that, given any open interval $I \subset [0,1]$, the set

$$A(I) = \{f \in M\triangle'_0 : \exists x_1, x_2 \in I \text{ so that } f(x_1) > 0 \text{ and } f(x_2) < 0\}$$

is a dense open subset of $M\triangle'_0$. Let $\{I_k\}$ be an enumeration of the open intervals in [0,1] with rational endpoints. Let $A_k = A(I_k)$. Then $A = \bigcap_{k=1}^{\infty} A_k$ is a dense subset of type $\mathcal{G}_\delta$ in $M\triangle'_0$. If $f \in A$, then both sets

$$\{x : f(x) > 0\} \ \text{ and } \ \{x : f(x) < 0\}$$

are dense in [0,1]. Thus f is a bounded derivative that takes both signs in every open interval contained in [0,1].

Let $F(x) = \int_0^x f\,d\lambda$. Then $F' = f$ on [0,1]. Thus F is a nowhere monotonic differentiable function on [0,1]. We have proved the following theorem:

Theorem 10.25 *Let $\triangle_0$ denote the set of differentiable functions F on $[0,1]$ such that $F(0) = 0$ and $F' \in M\triangle'_0$. For $F, G \in \triangle_0$, let*

$$\rho(F,G) = \sup_{x\in[0,1]} |F'(x) - G'(x)|.$$

Then $(\triangle_0, \rho)$ is a complete metric space in which the typical member is a differentiable nowhere monotonic function.

Observe that if F is differentiable and nowhere monotonic then F' can be continuous at a point x_0 only if $F'(x_0) = 0$. The set of points of continuity of F' must be large in the category sense: it must be a dense $\mathcal{G}_\delta$. But it can be small in measure (see Exercise 5:5.5). One can, in fact, show that the typical derivative f in $M\triangle'$ is discontinuous a.e. (see Exercise 10:7.7).

The Space of Automorphisms

Let H denote the family of strictly increasing continuous functions on [0,1] that leave the endpoints fixed. Since a uniform limit of functions on H need not be *strictly* increasing, H is not closed in $\mathcal{C}[0,1]$. Thus H is not complete with respect to the metric

$$\rho(f,g) = \max_x |f(x) - g(x)|.$$

It would appear that our usual methods would fail in a study of this space. We show that H is of type $\mathcal{G}_\delta$ and therefore topologically complete. Consequently, Baire category arguments can still be applied.

Let I be a closed nondegenerate interval in [0,1], and let

$$A(I) = \left\{h \in \overline{H} : h \text{ is constant on } I\right\}.$$

Then $A(I)$ is a closed subspace of $\mathcal{C}[a,b]$. Let $\{I_n\}$ be an enumeration of the closed nondegenerate subintervals of [0,1] with rational endpoints. The set

$$A = \bigcup_{n=1}^{\infty} A(I_n)$$

is of type $\mathcal{F}_\sigma$ in $\mathcal{C}[a,b]$ and therefore also of type $\mathcal{F}_\sigma$ in the closed set $\overline{H}$. The set $\overline{H} \setminus A$ is therefore of type $\mathcal{G}_\delta$ in the complete space $\overline{H}$. Now $\overline{H}$ consists of the nondecreasing continuous functions on [0,1] that leave the endpoints fixed, and A consists of those members of $\overline{H}$ that are not *strictly* increasing. Thus $\overline{H} \setminus A$ consists of the strictly increasing members of $\overline{H}$; that is, $\overline{H} \setminus A = H$.

Definition 10.26 A homeomorphism h of an interval $[a,b]$ onto $[a,b]$ that satisfies $h(a) = a$ and $h(b) = b$ is called an *automorphism* of $[a,b]$.

Our discussion above shows that the set H of automorphisms of [0,1] is of type $\mathcal{G}_\delta$ in the complete space $\overline{H}$ and is therefore topologically complete.

Exercise 10:6.12 provides a simple example of an automorphism of $[0,1]$ that maps a given Cantor set onto a set of measure zero. As an illustration of the way in which Theorem 10.18 can be applied, we next show that a given first-category subset A of [0,1] can also be mapped onto a zero measure set by an automorphism. We do this by applying the Baire category theorem to the topologically complete space H of automorphisms of [0,1]. The reader may wish to try to construct such an automorphism if A is a first-category set of measure 1.

Theorem 10.27 *Let A be a first-category subset of $[0,1]$. Let*

$$H_1 = \{h \in H : \lambda(h(A)) = 0\}.$$

Then $H \setminus H_1$ is first category in H; that is, H_1 is residual in the topologically complete space H.

Proof. Let $A = \bigcup_{n=1}^{\infty} A_n$ with each set A_n nowhere dense in [0,1]. For $n, k \in \mathbb{N}$, let

$$H_{nk} = \left\{h \in H : \lambda(h(\overline{A_n})) < 1/k\right\}.$$

We show that each of the sets H_{nk} is open and dense in H. It will then follow that $\bigcap_{n,k=1}^{\infty} H_{nk}$ is residual in H.

If $h \in \bigcap_{n,k=1}^{\infty} H_{nk}$, then $\lambda(h(\overline{A_n})) = 0$ for all $n \in \mathbb{N}$, so

$$\lambda(h(A)) \leq \lambda\left(\bigcup_{n=1}^{\infty} h(\overline{A_n})\right) = 0;$$

that is,

$$\bigcap_{n,k=1}^{\infty} H_{nk} \subset H_1,$$

and H_1 is residual in H, as was to be proved.

To show that H_{nk} is open, let $h \in H_{nk}$ and let G be an open set in $\mathbb{R}$ such that

$$h(\overline{A_n}) \subset G$$

and $\lambda(G) < 1/k$. The set $h(\overline{A_n})$ is closed, as is the set $\widetilde{G}$. Thus

$$\operatorname{dist}(h(\overline{A_n}), \widetilde{G}) > 0.$$

Let

$$\delta < \operatorname{dist}(h(\overline{A_n}), \widetilde{G}).$$

If $\rho(g,h) < \delta$, then $g(\overline{A_n}) \subset G$, so $g \in H_{nk}$. This shows that H_{nk} is open in H.

It remains to show that H_{nk} is dense in H. Let $g \in H$, and let $\varepsilon > 0$. By perturbing g a bit, we shall arrive at a function in $H_{nk} \cap B(g, \varepsilon)$. This will show that H_{nk} is dense in H.

Let $I_1, \dots, I_N$ be nonoverlapping closed intervals, each of length less than ε, with

$$[0,1] = I_1 \cup \cdots \cup I_N.$$

Since g is a homeomorphism and A_n is nowhere dense in [0,1], the set $g(\overline{A_n})$ is also nowhere dense in [0,1]. Thus there exist closed intervals $J_1, \dots, J_N$ such that, for each $i = 1, 2, \dots, N$,

$$J_i \subset {I_i}^o \setminus g(\overline{A_n}).$$

Let h_i be an automorphism of I_i such that

$$\lambda(h_i(J_i)) > \lambda(I_i) - \frac{1}{kN} .$$

See Figure 10.1. Define h on [0,1] by $h(x) = h_i(x)$ if $x \in I_i$. Then $h \in H$ and

$$h(g(\overline{A_n})) \subset [0,1] \setminus (h(J_1) \cup \cdots \cup h(J_N)).$$

Thus

$$\lambda(h(g(\overline{A_n}))) < \frac{1}{k},$$

and $h \circ g \in H_{nk}$. Since h maps each of the intervals I_i onto itself, and each such interval has length less than ε,

$$|h(g(x)) - g(x)| < \varepsilon$$

for all $x \in [0,1]$. That is, $\rho(h \circ g, g) < \varepsilon$, so $h \circ g \in B(g, \varepsilon)$.

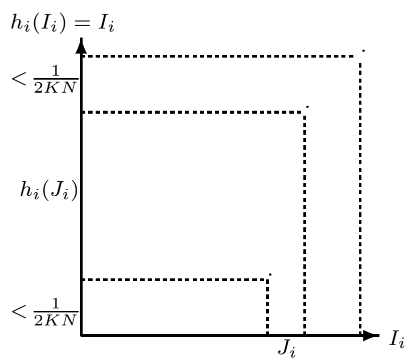

Figure 10.1: Construction of the automorphism h_i of I_i.

We have shown that

$$h \circ g \in H_{nk} \cap B(g, \varepsilon).$$

It follows that H_{nk} is dense in H, completing the proof of the theorem. ∎

In Exercise 8:3.2 we constructed a strictly increasing singular function on [0,1]. Theorem 10.27 provides an immediate proof of the existence of such functions. To see this, let A be a first-category subset of [0,1] with $\lambda(A) = 1$. For $h \in H_1$, $\lambda(h(A)) = 0$, so that $\lambda(h(\widetilde{A})) = 1$. Thus each $h \in H_1$ maps the zero measure set $\widetilde{A}$ onto a set of full measure. It follows that every $h \in H_1$ is a strictly increasing continuous singular function.

When dealing with topologically complete spaces, we shall use terms such as "residual" or "typical" in the same way as we use them for complete spaces.

Exercises

10:6.1 Prove in detail that a function $f \in \mathcal{C}[a, b]$ of nonmonotonic type has both ∞ and $-\infty$ as derived numbers at every $x \in [a, b]$.

10:6.2 Is $f(x) = |x|$ nonangular? What about $g(x) = x \sin x^{-1}$, $(g(0) = 0)$?

10:6.3 Show that a differentiable function is nonangular. Can a differentiable function be of nonmonotonic type?

10:6.4 Prove that there exist continuous nowhere monotonic functions that are not of nonmonotonic type.

10:6.5 Prove Theorem 10.23. [Hint: Consider the sets A_{pqn} of all functions $f \in \mathcal{C}[a,b]$ so that there exists $x \in [a+1/n, b-1/n]$ such that
$$f(t) - f(x) \leq p(t-x)$$
for $0 < t - x < 1/n$ and
$$f(t) - f(x) \geq q(t-x)$$
for $0 < x - t < 1/n$.]

10:6.6 In his famous centennial lecture at the International Congress of Mathematicians in Paris in 1900, Hilbert posed a number of important problems for mathematicians of the twentieth century to attack. The thirteenth problem[8] involves the representation of continuous functions of several variables in terms of continuous functions of one variable via a finite number of sums and compositions.

In 1957, A. N. Kolmogorov showed that every continuous real-valued function defined on the n-dimensional cube I^n admits such a representation. His constructions were very complicated. Among other things, he constructed continuous functions on I^n that are one to one almost everywhere. From the Baire category theorem, we can deduce that there are many such functions with this property.

(a) Prove that the typical $f \in \mathcal{C}[a,b]$ is one to one a.e. That is, there exists $A \subset [a,b]$ such that $\lambda([a,b] \setminus A) = 0$ and f is one to one on A. [Hint: Let K be a Cantor set. Show first that the typical $f \in \mathcal{C}[a,b]$ is one to one on K.]

(b) Extend part (a) to continuous real-valued functions defined on a closed square $[a,b] \times [a,b]$.

(c) A curve in $\mathbb{R}^2$ is a continuous function $f : [0,1] \to \mathbb{R}^2$. Make precise the following statement and determine whether or not it is true: The typical curve in $\mathbb{R}^2$ is one to one on $[0,1]$.

10:6.7 Let $\mathcal{B} \subset M[a,b]$ consist of the bounded Borel measurable functions on $[a,b]$. Then $\mathcal{B}$ is complete (why?). Prove that the typical $f \in \mathcal{B}$ is one to one on $[a,b]$.

10:6.8 Let X be a closed subspace of $M[a,b]$, and let R be a residual subset of X. Show that for every $h \in X$ there exists $f, g \in R$ such that $h = f + g$. [Hint: Let $A = \{h - f : f \in R\}$. Show that $A \cap R$ is residual in X. Choose $g \in A \cap R$.]

10:6.9 Use Exercise 10:6.8 to verify the following statements:

(a) Every $f \in \mathcal{C}[a,b]$ is a sum of two continuous functions that are one to one a.e. (see Exercise 10:6.6).

[8] An interesting discussion concerning Hilbert's problem and related topics can be found in D. Sprecher, "On functional complexity and superpositions of functions," *Real Analysis Exchange* **9** (1983-84), no. 2, 417–431.

(b) Every $f \in \mathcal{C}[a,b]$ is a sum of two continuous nowhere differentiable functions.

(c) Every $f \in \mathcal{B}$ (Exercise 10:6.7) is a sum of two one to one functions in $\mathcal{B}$.

10:6.10 Show that there exists a set $Z \subset \mathbb{R}$ such that $\lambda(Z) = 0$ and such that, for every $x \in \mathbb{R}$, there exists $z_1, z_2 \in Z$ such that $x = z_1 + z_2$. [Hint: Prove and then apply an analog of Exercise 10:6.8 for $\mathbb{R}$ in place of $M[a,b]$.]

10:6.11 Prove that the typical derivative in $M\triangle'$ is discontinuous on a dense set. [Hint: Show that the set

$$A(I) = \{f \in M\triangle' : f \text{ continuous on } I\}$$

is nowhere dense in $M\triangle'$. For a stronger result see Exercise 10:7.7.]

10:6.12 Let K be a nowhere dense closed subset of [0,1]. Let

$$h(x) = \frac{\lambda([0,x] \cap \widetilde{K})}{\lambda(\widetilde{K})}.$$

Prove that h is an automorphism of [0,1] and that $\lambda(h(K)) = 0$.

10:6.13 *(Riemann integrability under changes of variable)* Let f be a bounded function on $[a,b]$, let H be the family of automorphisms of [0,1], let $\mathcal{R}$ be the family of Riemann integrable functions on $[a,b]$, and let D be the set of points of discontinuity of f.

(a) Prove that $f \circ h \in \mathcal{R}$ for all $h \in H$ if and only if D is countable.

(b) Prove that $f \circ h \in \mathcal{R}$ for some $h \in H$ if and only if D is first category.

(c) Prove that $f \circ h \in \mathcal{R}$ for the identity function h if and only if $\lambda(D) = 0$.

Thus the three types of "small" sets that we have encountered all figure into questions concerning Riemann integrability of a function under homeomorphic changes of variable.

10:6.14 Let $\mathcal{S}$ denote the space of Lebesgue–Stieltjes measures μ on the unit interval $X = [0,1]$ that satisfy (i) $\mu(X) = 1$, (ii) $\mu(G) > 0$, if G is nonempty and open, and (iii) μ is nonatomic. Make precise the statement that the typical $\mu \in \mathcal{S}$ is singular with respect to λ. Prove your statement.

10.7 Additional Problems for Chapter 10

10:7.1 (Refer to Exercise 3:11.6.) Show that the typical $f \in \mathcal{C}[0,1]$ gives rise to the zero measure when Method II is applied to $\tau([a,b]) = |f(b) - f(a)|$.

10:7.2 (a) In Exercise 4:4.1, show that the function f is not in the first Baire class and that the function g has the Darboux property, but its graph is not a connected subset of $\mathbb{R}^2$.

(b) Show that if a function $h:[0,1]\to\mathbb{R}$ has the Darboux property and is also in the first class of Baire then the graph of h must be a connected subset of $\mathbb{R}^2$. [Hint: If not, there exist disjoint nonempty open subsets G_1 and G_2 of $\mathbb{R}^2$ such that $G_1 \cap G_2 = \emptyset$ and $G_1 \cup G_2$ contains the graph of h. Let

$$A_1 = \{x : (x, f(x)) \in G_1\} \quad \text{and} \quad A_2 = \{x : (x, f(x)) \in G_2\}.$$

Let K be the boundary of A_1. Show that K is perfect and that both A_1 and A_2 are dense in K. Obtain a contradiction by considering a point of continuity of $h|K$.]

10:7.3 A real number z is called a *Liouville number* if z is irrational and has the property that for each n there exist integers p and q such that

$$\left| z - \frac{p}{q} \right| < \frac{1}{q^n}.$$

Prove the following statements about the set L of Liouville numbers.

(a) $L = \widetilde{\mathbb{Q}} \cap \bigcap_{n=1}^{\infty} G_n$, where $G_n = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right)$.

(b) L is a dense set of type $\mathcal{G}_\delta$, so L is large in the sense of category.

(c) $\lambda(L) = 0$, so L is small in the sense of measure.

(d) L has Hausdorff dimension zero.

10:7.4 Let A consist of those continuous functions f on [0,1] for which each level set contains no more than one point at which f achieves a relative extremum. [A level set for f is a set of the form $\{x : f(x) = \alpha\}$, where $\alpha \in \mathbb{R}$.] Show that A is a dense set of type $\mathcal{G}_\delta$ in $\mathcal{C}[0,1]$. One can actually show that, for the typical $f \in \mathcal{C}[0,1]$,

(i) the top and bottom level sets are singletons,

(ii) for a countable dense set of levels, the level set consists of a Cantor set with one added isolated point, and

(iii) all other level sets are Cantor sets.

A similar pattern of intersections of the graph of f with nonvertical lines occurs in all directions, except for a countable dense set of directions. For each of those exceptional directions, the intersection pattern is the same, except that there will be a single line that intersects the graph of f in a set that contains *two* isolated points instead of one.

Thus, the typical $f \in \mathcal{C}[a,b]$ exhibits some "pathological" behavior, but the graphs of such functions all "look alike." The only way in which one such function differs from another is in *which* lines and directions are exceptional. The Baire category theorem was useful in describing the intersection patterns of the graph of a typical $f \in \mathcal{C}[a,b]$ with the family of lines. No function that exhibits the stated intersection pattern has yet been constructed!

10:7.5 Let $\mathcal{C}* = \{f \in \mathcal{C}[0,1] : f([0,1]) \subset [0,1]\}$ and for $f \in \mathcal{C}*$, let $\operatorname{Fix}(f)$ be the set of fixed points of f; that is, $x \in \operatorname{Fix}(f)$ if and only if $f(x) = x$.

(a) Prove that $\mathcal{C}*$ is complete, but is not compact.

(b) Show that, for the typical $f \in \mathcal{C}*$, $\operatorname{Fix}(f)$ is a Cantor set.

10:7.6 The only nonempty compact, connected subsets of $\mathbb{R}$ are singleton sets and closed intervals. For $\mathbb{R}^2$, the situation is more complicated, and rather strange compact connected subsets of $\mathbb{R}^2$ exist. To obtain a bit of insight, work the several parts of this problem, read the paragraphs that follow, and, if so inclined, the references we provide. In particular, part (e) shows that in the sense of category, most compact connected sets in the plane are "strange."

> **Definition** A metric space X is *connected* if it cannot be expressed as a disjoint union of two nonempty open sets. A subset S of X is connected if S is a connected metric space.

(a) Prove that the only connected sets in $\mathbb{R}$ are intervals (possibly degenerate), $\mathbb{R}$, and the empty set.

> **Definition** A compact connected set in a metric space is called a *continuum*. (The plural is *continua*.)

(b) Show that the set

$$\left\{(x,y) \in \mathbb{R}^2 : y = \sin 1/x, 0 < x \leq 1\right\} \cup \{(0,y) : -1 \leq y \leq 1\}$$

is a continuum.

Can you visualize a nonempty continuum in $\mathbb{R}^2$ that consists of more than one point, but contains no arcs? (An *arc* is a homeomorphic image of [0,1]). The purpose of the following parts is to use the Baire category theorem to demonstrate that the typical continuum in $\mathbb{R}^2$ contains no arcs.

Let $\mathcal{K}$ be the metric space of nonempty compact subsets of the unit square $U = [0,1] \times [0,1]$ furnished with the Hausdorff metric

(Example 9.13). This space is complete (Example 9.41) and, in fact, compact (Theorem 9.66). Let

$$\mathcal{C} = \{C \in \mathcal{K} : C \text{ is connected}\}.$$

(c) Prove that $\mathcal{C}$ is a compact metric space. [Hint: Prove that $\mathcal{C}$ is closed in $\mathcal{K}$.]

A set $S \subset U$ is called *snakelike* if for each $\varepsilon > 0$ there exists a finite collection $G_1, \ldots, G_n$ of open sets with diameters less than ε such that $S \subset \bigcup_{j=1}^{n} G_j$ and $G_i \cap G_j = \emptyset$ unless $|i - j| \leq 1$.

(d) Prove that an arc in U is snakelike.

A continuum is *indecomposable* if it cannot be expressed as a union of two proper subcontinua. (These subcontinua will not be disjoint). A continuum is *hereditarily indecomposable* if each of its subcontinua is indecomposable. Do you think there exists a hereditarily indecomposable continuum in U containing more than one point? Read on.

Let $\mathcal{N}$ denote the set of all continua in U that contain more than one point, $\mathcal{S}$ denote the snakelike continua in U, and $\mathcal{H}$ the hereditarily indecomposable continua in U. Let $\mathcal{P} = \mathcal{N} \cap \mathcal{S} \cap \mathcal{H}$. The elements of $\mathcal{P}$ are called *pseudo arcs*. Observe that a member of $\mathcal{P}$ is a continuum containing more than one point, but not containing an arc (since an arc can be decomposed).

(e) Prove that $\mathcal{P}$ is not empty. [Hint: See J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, (1961), p. 142. Part (e) is not easy to conceive, but the argument in the reference is not difficult.]

(f) Prove that $\mathcal{P}$ is dense in $\mathcal{C}$. [Hint: First show that each $C \in \mathcal{C}$ can be approximated by a polygonal arc L, which in turn can be approximated by an element of $\mathcal{P}$. You do not have to understand the details of the construction in (e) to do this.]

(g) Conclude that $\mathcal{P}$ is a dense $\mathcal{G}_\delta$ in $\mathcal{C}$ and therefore residual. Thus the typical continuum in U is a pseudo arc and, in particular, contains no arcs.

Understanding pseudo arcs and related types of continua is of importance in many parts of mathematics. For example, the Riemann mapping theorem in complex analysis states that any open, simply connected, proper subset W of the complex plane can be mapped conformally onto the unit disk. On encountering this theorem, students will first visualize W as the inside of a simple closed curve C and then perhaps more generally as such a domain with some arcs that have an endpoint on C removed from the interior. See the Figure 10.2.

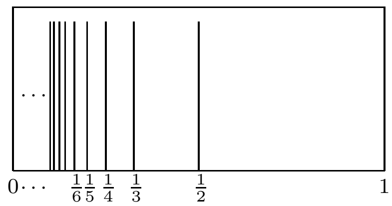

Figure 10.2: The boundary of an open, simply connected set.

But what if the boundary of W is something like a "closed pseudo arc"? The student might not appreciate that the proof of the Riemann Mapping Theorem must allow such a possibility. And, also, it must allow other strange possibilities, for example, that W is only one of many simply connected regions that have C as a common boundary.

Such continua have received new interest in recent years.[9] A reason is that many dynamical systems, even smooth ones, have attractors most (in the sense of category) of whose connected components are hereditarily indecomposable and therefore contain no arcs.

10:7.7 (Continuity of the typical bounded derivative.) In Exercise 5:5.5 we constructed a bounded derivative on [0,1] that is discontinuous a.e. and alerted the reader that 'most' derivatives are discontinuous a.e. The purpose of the present exercise is to verify that statement.

Let $M\triangle'$ denote the space of bounded derivatives on [0,1] with

$$\rho(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Thus $f \in M\triangle'$ if and only if there exists $F : [0,1] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [0,1]$. For $f \in M\triangle'$, let C_f denote the set of points of continuity of f.

(a) Let $0 < \delta \le 1$ and let $A_\delta = \{f \in M\triangle' : \lambda(C_f) \ge \delta\}$. Show that A_δ is closed in $M\triangle'$. [Hint: Let f_n be a sequence in A_δ converging to f. Let

$$C = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} C_{f_n}.$$

Show $C \subset C_f$.]

(b) Let P be a Cantor set in [0,1]. Show that there exists $f \in M\triangle'$ such that $f(x) = 0$ on P, $\max|f(x)| = 1$ on each interval

[9] An interesting discussion of recent work in this area, together with a new construction of a "strange attractor," can be found in Kennedy and Yorke, *Bull. Amer. Math. Soc.* **32** (1995), no. 3, 309–316.

complementary to P, and $C_f = [0,1] \setminus P$. [Hint: Construct f to be approximately continuous and use Theorem 7.36.]

(c) Show that for each $\delta \in (0,1)$ the set A_δ in part (a) is nowhere dense in $M\triangle'$. [Hint: Choose a ball $B(g,\varepsilon)$ in $M\triangle'$ with center $g \in A_\delta$ and use part (b) to obtain a function h such that $g + h \in B(g,\varepsilon) \setminus A_\delta$.]

(d) Prove that the typical bounded derivative is discontinuous a.e.

Chapter 11

ANALYTIC SETS

The Borel sets in a metric space are closed under the operations of complementation, countable unions, and countable intersections. They are not, however, closed under continuous images. To discuss the continuous images of Borel sets, we need a larger class – the analytic sets. We shall see that the class of analytic sets in a complete, separable metric space can be described as the class of sets obtainable as continuous images of some Borel set. We develop the basic properties of analytic sets, showing how such sets can be approximated and separated by Borel sets. The analytic sets are proved to be measurable for any metric outer measure on the space. Many important examples of sets in functions spaces turn out to be analytic (or complements of analytic sets), but not Borel sets. We complete the chapter with several instances.

In Chapter 1 we have reviewed the origins of the theory of analytic sets in a famous error of Lebesgue. Here we develop the machinery for proving most of the assertions given in Section 1.13. Everything is set in a complete, separable metric space. For most purposes of analysis, this is more than enough to be able to use these ideas. Topologists and descriptive set theorists would need more refined and subtle techniques. For those, the reader might consult Kuratowski[1] and Moschovakis.[2]

We conclude with an application of the theory to the study of differentiable functions by presenting some material of Stefan Mazurkiewicz (1888–1945). Further applications can be found in probability theory, the theory of capacities, the theory of Hausdorff measures (fractals), and in many other aspects of real and functional analysis.

[1] C. Kuratowski, *Topology,* Academic Press (1966).
[2] Y. N. Moschovakis, *Descriptive Set Theory,* North-Holland (1980).

11.1 Products of Metric Spaces

The material in this chapter depends on some familiarity with metric spaces and mappings on metric spaces. We begin with some preliminary material on products of metric spaces. The reader with a good background in topology who happens to remember the product topology can probably scan through most of this quickly.

Let (X, d) be a metric space. We first recall that the following provides an equivalent metric:

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)},$$

which is bounded by 1. This metric changes distances and diameters, but it preserves the open sets and the convergence properties. In many applications this is what matters, and so whenever an unbounded metric is troublesome we can consider making this replacement.

We now review how to provide a metric on a product space. Let (X_i, d_i) be a sequence of metric spaces. By the set

$$X = \prod_{i=1}^{\infty} X_i$$

we mean the set of all sequences $x = (x_1, x_2, x_3, \dots)$ of elements $x_i \in X_i$ for $i \in \mathbb{N}$. We provide X with a metric ρ by writing, for any pair of points $x = (x_1, x_2, x_3, \dots)$, $y = (y_1, y_2, y_3, \dots)$ in X,

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d(x_i, y_i)}.$$

(Compare this with Example 9.2.) This metric ρ makes X a metric space furnished with what is known as the *product topology*. Convergence of a sequence of points in X amounts to "coordinate-wise convergence." More precisely, suppose that we are given a sequence of points

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, x_4^{(n)}, \dots)$$

in the space X. Then $x^{(n)}$ converges to a point $(c_1, c_2, c_3, \dots) \in X$ if and only if, in each coordinate, we have the convergence of the sequence $x_i^{(n)}$ to c_i in the space X_i. (See Exercise 11:1.2.) This fact makes checking statements in the space X quite simple: one merely checks separately what is happening in each coordinate.

The following two facts are fundamental and easy enough to prove, since convergence need only be checked coordinate by coordinate.

11.1 *Let (X_i, d_i) be a sequence of complete metric spaces. Then the product space*

$$X = \prod_{i=1}^{\infty} X_i$$

is also a complete metric space.

11.2 *Let (X_i, d_i) be a sequence of separable metric spaces. Then the product space*

$$X = \prod_{i=1}^{\infty} X_i$$

is also a separable metric space.

Exercises

11:1.1 Let (X_i, d_i) be a sequence of metric spaces and $X = \prod_{i=1}^{\infty} X_i$ be the product. Show that ρ as defined is a metric on X and that it is bounded.

11:1.2 Show that a sequence of points

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, x_4^{(n)}, \dots)$$

in the space $X = \prod_{i=1}^{\infty} X_i$ converges to a point $(c_1, c_2, c_3, \dots) \in X$ if and only if $x_i^{(n)} \to c_i$ in the space X_i for each $i \in \mathbb{N}$.

11:1.3 Prove 11.1: a product of a sequence of complete spaces is complete.

11:1.4 Prove 11.2: a product of a sequence of separable spaces is separable.

11:1.5 Prove that a product of a sequence of compact spaces is compact.

11:1.6 Show that the infinite product $\prod_{i=1}^{\infty}[0,1]$ of countably many copies of the unit interval is homeomorphic to the Hilbert cube

$$I^{\infty} = \{x \in \ell_2 : |x_n| \leq n^{-1}\}.$$

11.2 Baire Space

The space of natural numbers $\mathbb{N}$ is commonly given a metric that makes convergence trivial. With the metric

$$\rho(m,n) = |m - n|,$$

a sequence $\{n_k\}$ evidently converges to a number c in $\mathbb{N}$ if and only if the sequence has $n_k = c$ for all sufficiently large k. We consider $\mathbb{N}$ as a metric space furnished with this metric. Without much trouble, one sees that $\mathbb{N}$ is a complete, separable metric space.

The space $\mathbb{N}^{\mathbb{N}}$, sometimes called *Baire space*, is the product of countably many copies of $\mathbb{N}$. To make this explicit, let $X_i = \mathbb{N}$ for each $i = 1, 2, 3 \dots$ and form the product space

$$\prod_{i=1}^{\infty} X_i$$

equipped with the product metric introduced in Section 11.1. This is again a complete, separable metric space. Thus $\mathbb{N}^{\mathbb{N}}$ is the space of all sequences $\mathbf{n} = (n_1, n_2, n_3, \dots)$ of natural numbers, and the metric on this space is defined as

$$\rho(\mathbf{m}, \mathbf{n}) = \sum_{i=1}^{\infty} \frac{|m_i - n_i|}{2^i\,(1 + |m_i - n_i|)}.$$

Convergence is merely coordinate wise convergence which, as we have noted, just means that each sequence of coordinates becomes constant at some stage.

The space $\mathbb{N}^{\mathbb{N}}$ permits various manipulations that are convenient in the study of products and projections. The exercises show how to exploit this structure. The basic idea is that $\mathbb{N}^{\mathbb{N}}$ contains many copies of itself and one can put together a product of a sequence of copies of $\mathbb{N}^{\mathbb{N}}$ and obtain a space identical to the space $\mathbb{N}^{\mathbb{N}}$ itself.

The Baire space $\mathbb{N}^{\mathbb{N}}$ can be used to study the class of complete, separable metric spaces. The following theorems show the intimate connections between these spaces.

Theorem 11.3 *Every complete, separable metric space is a continuous image of* $\mathbb{N}^{\mathbb{N}}$.

Proof. Let X be a complete, separable metric space. Since separable, it is possible to express X as a union of a sequence of closed sets of small diameter, say

$$X = \bigcup_{i=1}^{\infty} E(i),$$

where each $E(i)$ is closed and has diameter smaller than 2^{-3}. Note that no claim is made that the sets are disjoint; they are just closed sets of small diameter that cover the space.

We can continue this process on each $E(i)$, expressing it as a union of even smaller sets. Inductively, for each $k \in \mathbb{N}$ we define nonempty, closed sets

$$E(n_1, n_2, n_3, \dots, n_k)$$

with diameter smaller than 2^{-k-2} so that

$$X = \bigcup_{i=1}^{\infty} E(i)$$

and

$$E(n_1, n_2, n_3, \dots, n_{k-1}) = \bigcup_{i=1}^{\infty} E(n_1, n_2, n_3, \dots, n_{k-1}, i).$$

For each $\mathbf{n} = (n_1, n_2, n_3, \dots) \in \mathbb{N}^{\mathbb{N}}$, we observe that

$$\bigcap_{k=1}^{\infty} E(n_1, n_2, n_3, \dots, n_k),$$

being a nested sequence of closed sets with diameters shrinking to zero in a complete space, must contain a unique point. We call this $g(\mathbf{n})$. Clearly, the function g is a mapping of $\mathbb{N}^{\mathbb{N}}$ onto X [although it is not one-one: a point may belong to different $E(n_1, n_2, n_3, \ldots, n_k)$].

We have only to check that the mapping is continuous. For this we obtain a local Lipschitz condition. If $\mathbf{m}, \mathbf{n} \in \mathbb{N}^{\mathbb{N}}$ with $\rho(\mathbf{m}, \mathbf{n})$ smaller than $\frac{1}{4}$, we can choose a natural number k so that

$$2^{-k-2} \leq \rho(\mathbf{m}, \mathbf{n}) < 2^{-k-1},$$

where ρ is the metric in $\mathbb{N}^{\mathbb{N}}$. This means, directly from the definition of the metric ρ, that $m_i = n_i$ for all $i \leq k$. Hence

$$\begin{aligned} \rho_X(g(\mathbf{m}), g(\mathbf{n})) &\leq \text{diameter}(E(n_1, n_2, n_3, \ldots, n_k) \\ &< 2^{-k-2} \leq \rho(\mathbf{m}, \mathbf{n}), \end{aligned}$$

where ρ_X is the metric in our space X. This inequality shows that g is continuous. ■

In the theorem just proved, we have shown that the space X can be written as the image $g(\mathbb{N}^{\mathbb{N}})$ for some continuous function g. The function g need not be one-one nor, even if it is one-one, need the inverse g^{-1} be continuous. That is, we do not have a *homeomorphic* copy. If we wish a copy of $\mathbb{N}^{\mathbb{N}}$ that is more faithful (i.e., the one-one image of a continuous function such that g^{-1} is also continuous), we must look inside X for appropriate subsets. The next theorem follows from the methods of the proof above by making suitable refinements.

Theorem 11.4 *Every perfect, complete, separable metric space has a Borel subset that is a homeomorphic image of* $\mathbb{N}^{\mathbb{N}}$.

Proof. We refine the construction of the last proof. Choose disjoint, nonempty open sets $U(n_1, n_2, n_3, \ldots, n_k)$ with diameters smaller than 2^{-k-2}, but do not attempt to cover the whole space. In the induction step, make sure that

$$U(n_1, n_2, n_3, \ldots, n_k) \supset \bigcup_{j=1}^{\infty} \overline{U(n_1, n_2, n_3, \ldots, n_k, j)}.$$

Then, defining the mapping g exactly as before, we can check that g is a homeomorphism between $\mathbb{N}^{\mathbb{N}}$ and the set

$$\bigcap_{k=1}^{\infty} \bigcup_{(n_1, n_2, n_3, \ldots, n_k)} U(n_1, n_2, n_3, \ldots, n_k).$$

Since this set is a countable intersection of open sets we have the required Borel set. ■

Exercises

11:2.1 Characterize the compact subsets of $\mathbb{N}$.

11:2.2 Check that $\mathbb{N}^{\mathbb{N}}$ is a complete, separable metric space using the metric as given.

11:2.3 Characterize the compact subsets of $\mathbb{N}^{\mathbb{N}}$.

11:2.4$\diamondsuit$ Show that $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ itself.

11:2.5 Define Z_j for each $j = 1, 2, 3, \dots$ as the set of all points in $\mathbb{N}^{\mathbb{N}}$ whose first coordinate is j. Show that Z_j is a clopen (i.e., closed and open) subset of $\mathbb{N}^{\mathbb{N}}$ that is in fact homeomorphic to $\mathbb{N}^{\mathbb{N}}$ itself.

11:2.6 Define $X_j = \mathbb{N}^{\mathbb{N}}$ for each $j = 1, 2, 3, \dots$ and $X = \prod_{j=1}^{\infty} X_j$. Show that X is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ itself.

11:2.7 Fill in the details in the proof of Theorem 11.4 and check that g is a homeomorphism.

11:2.8 If $X = \mathbb{R}$ in Theorem 11.4, then the sets $U(n_1, n_2, n_3, \dots, n_k)$ can be chosen at each stage so that only a countable set of points in $\mathbb{R}$ is not covered. Conclude that there is a countable set $Z \subset \mathbb{R}$ so that $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{R} \setminus Z$.

(The next exercise shows that the countable set Z can be taken as the rationals. For this reason, one can call the space $\mathbb{N}^{\mathbb{N}}$ the space of *irrationals*.)

11:2.9 Show that the continued fraction mapping f from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{R} \setminus \mathbb{Q}$ defined for $\mathbf{n} = (n_1, n_2, n_3, \dots)$ as

$$f(\mathbf{n}) = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \dots}}}}$$

is a homeomorphism.

11:2.10 Find a closed set $B \subset \mathbb{N}^{\mathbb{N}}$ that maps continuously one-one onto to $\mathbb{R}$. [Hint: Map $\{1\} \times \mathbb{N}^{\mathbb{N}}$ to $\mathbb{R} \setminus Z$ for a countable set of reals $Z = \{z_2, z_3, z_4, \dots\}$ and then map $(k, 1, 1, 1, 1, 1, \dots)$ to z_k for $k \geq 2$. This gives a closed subset of $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ mapping to $\mathbb{R}$.]

11.3 Analytic Sets

We work throughout in complete, separable metric spaces. A space that is homeomorphic to a complete, separable metric space is often called a *Polish space* because of the extensive analysis of these spaces conducted by the Polish mathematicians of the 1920s and 1930s. The following three definitions are equivalent. The first exhibits analytic sets as images of a standard space, the second as images of any complete, separable metric space and the third, returning to an original idea of Suslin, as sets arising from projections. See also Theorem 11.19, where analytic sets are described using the Suslin operation. Recall that by a projection mapping from a space $X \times Y$ to X we mean the function $p_X(x, y) = x$.

Definition 11.5 Let X be a complete, separable metric space. A set $A \subset X$ is said to be *analytic* provided that

1. A is the continuous image of $\mathbb{N}^{\mathbb{N}}$, or
2. A is the continuous image of some complete, separable metric space, or
3. A is the projection of a closed subset C of $X \times \mathbb{N}^{\mathbb{N}}$ to X.

We check the equivalence of these three. For (3)$\Rightarrow$(2), note that $X \times \mathbb{N}^{\mathbb{N}}$ and C are both complete, separable metric spaces. For (2)$\Rightarrow$(1), assume that $A = f(Y)$ for some complete, separable metric space Y and then invoke Theorem 11.3 to obtain Y as a continuous image of $\mathbb{N}^{\mathbb{N}}$, say $Y = g(\mathbb{N}^{\mathbb{N}})$. Then $A = f \circ g(\mathbb{N}^{\mathbb{N}})$. Finally, to obtain (1)$\Rightarrow$(3), let $A = f(\mathbb{N}^{\mathbb{N}})$. Then A is the projection of the closed set

$$\{(\mathbf{n}, f(\mathbf{n})) : \mathbf{n} \in \mathbb{N}^{\mathbb{N}}\}.$$

We now proceed immediately to obtain the basic properties of analytic sets. We show that these sets are preserved under many basic operations. In all the statements that follow, all the spaces X, X_i, and Y that appear are complete, separable metric spaces. Each assertion has been left as an exercise with a hint that should be adequate.

11.6 *If $f : X \to Y$ is continuous and $A \subset X$ is analytic, then so too is the image $f(A)$.*

11.7 *If $f : X \to Y$ is continuous and $A \subset Y$ is analytic, then so too is the inverse image $f^{-1}(A)$.*

11.8 *If $A_i \subset X_i$ are each analytic ($i = 1, 2, 3, \dots$), then the product set $\prod_{i=1}^{\infty} A_i$ is an analytic subset of $\prod_{i=1}^{\infty} X_i$.*

11.9 *If $A_i \subset X$ are each analytic ($i = 1, 2, 3, \dots$) then so too are the intersection $\bigcap_{i=1}^{\infty} A_i$ and the union $\bigcup_{i=1}^{\infty} A_i$.*

With regard to this last assertion, do not jump to the conclusion that the analytic sets form a σ–algebra. While this class is closed under countable unions and intersections, it is definitely not closed under complementation. In fact, the only sets that are both analytic and have an analytic complement will turn out to be the Borel sets.

We mentioned in Section 3.10 that Cantor sets were not part of the mathematical repertoire until late in the nineteenth century. Nowadays, Cantor sets are commonplace, and it is difficult to visualize an uncountable subset of $\mathbb{R}$ that contains no Cantor set (see the construction in statement 3.35). Our next theorem indicates why such visualization might be difficult: such a set cannot be analytic. To state this theorem, we need to say what we mean by a Cantor set when we are dealing with a general metric space. The definition that the set be nonempty, nowhere dense, and perfect suffices in $\mathbb{R}$, but not in general. For example, we would not want to say that a line $L \subset \mathbb{R}^2$ is a Cantor set.

Definition 11.10 Let X be a metric space. A set $K \subset X$ is called a *Cantor set* if K is topologically equivalent to the classical Cantor ternary set.

Theorem 11.11 *Let U be an uncountable analytic subset of a complete, separable metric space Y. Then U contains a Cantor set.*

Proof. By Definition 11.5, there exists a complete, separable metric space X and a continuous function $f : X \to Y$ such that $f(X) = U$. For each $u \in U$, choose a point $x_u \in X$ such that $f(x_u) = u$, and let $X_U = \{x_u : u \in U\}$. Since X is separable and X_U is uncountable, there exists a set $D \subset X_U$ such that D is nonempty and dense-in-itself (Exercise 9:16.4). We now undertake a process on D that is reminiscent of the construction of the Cantor set. This construction will give rise to a set A that f maps homeomorphically onto a subset of U.

Let x_0 and x_2 be distinct points in D. Since f is continuous and is one to one on D, there exist closed balls B_0 and B_2, centered at x_0 and x_2, having diameters less than 1 and such that $f(B_0) \cap f(B_2) = \emptyset$. This completes the first step of our inductive process.

We know that D is dense-in-itself. Thus there exist distinct points x_{00} and x_{02} in $D \cap {B_0}^o$ and two closed balls B_{00} and B_{02} centered at x_{00} and x_{02} with diameters less than $1/2$ such that $B_{00} \cup B_{02} \subset B_0$ and $f(B_{00}) \cap f(B_{02}) = \emptyset$. We can follow the same procedure in B_2, obtaining closed balls B_{20} and B_{22} with the analogous properties.

Continuing inductively, we obtain a system $\{B_{c_1 c_2 \dots c_k}\}$ of closed balls, one for each finite sequence $c_1, \dots, c_k$ of 0's and 2's, such that

1. For each $k+1$-tuple $(c_1, c_2, \dots, c_k, c_{k+1})$ of 0's and 2's,
$$B_{c_1 c_2 \dots c_k c_{k+1}} \subset B_{c_1 c_2 \dots c_k}.$$
2. For each k-tuple $(c_1, c_2, \dots, c_k)$, the diameter of $B_{c_1 c_2 \dots c_k}$ is less than $1/k$.
3. If $(c_1, c_2, \dots, c_k) \neq (d_1, d_2, \dots, d_k)$, then
$$f(B_{c_1 c_2 \dots c_k}) \cap f(B_{d_1 d_2 \dots d_k}) = \emptyset.$$

Let
$$A = \bigcap_{k=1}^{\infty} \bigcup B_{c_1 c_2 \dots c_k},$$
where the union is taken over all k-tuples of 0's and 2's.

Now let C be the classical Cantor set. Define $g : C \to A$ by

$$g(c) = B_{c_1} \cap B_{c_1 c_2} \cap B_{c_1 c_2 c_3} \cap \dots,$$

where

$$c = \frac{c_1}{3} + \frac{c_2}{3^2} + \frac{c_3}{3^3} \dots.$$

It follows readily from (1) and (2) that g maps C homeomorphically onto A. (We leave verification of this fact to Exercise 11:3.5.)

To complete the proof that U contains a Cantor set, we must show that f is one to one on A, for in that case $f(A)$ will be the one to one continuous image of the Cantor set A and, hence, itself a Cantor set. To show this, let z_1 and z_2 be distinct points of A. There exist $k \in \mathbb{N}$ and distinct k-tuples $(c_1, c_2, \dots, c_k)$ and $(d_1, d_2, \dots, d_k)$ such that $z_1 \in B_{c_1 c_2 \dots c_k}$ and $z_2 \in B_{d_1 d_2 \dots d_k}$. It follows from (3) that $f(z_1) \neq f(z_2)$. Thus f is one to one, and the set $f(A)$ is a Cantor set contained in U. ■

Exercises

11:3.1 Prove assertion 11.6. [Hint: There is a map $h : \mathbb{N}^{\mathbb{N}} \to X$ so that $h(\mathbb{N}^{\mathbb{N}}) = A$. Consider $f \circ h$.]

11:3.2 Prove assertion 11.7. [Hint: Let p_Y be the projection onto Y from the space $Y \times \mathbb{N}^{\mathbb{N}}$; that is, $p_Y(y, \mathbf{n}) = y$ for $y \in Y$ and $\mathbf{n} \in \mathbb{N}^{\mathbb{N}}$. Then there is a closed set $C \subset Y \times \mathbb{N}^{\mathbb{N}}$ so that $p_Y(C) = A$. Consider the set $D = \{(x, \mathbf{n}) : (f(x), \mathbf{n}) \in C\}$ and $p_X(D)$.]

11:3.3 Prove assertion 11.8. [Hint: There are continuous functions f_i so $f_i(\mathbb{N}^{\mathbb{N}}) = A_i$. Consider the function f on $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ defined by

$$f(n_1, n_2, n_3, \dots) = (f_1(n_1), f_2(n_2), f_3(n_3) \dots).$$

11:3.4 Prove assertion 11.9. [Hint: Extract the appropriate method from the proof of assertion 11.13 in the next section.]

11:3.5 Show that the function g defined in Theorem 11.11 by

$$g(c) = B_{c_1} \cap B_{c_1 c_2} \cap B_{c_1 c_s c_3} \cap \dots$$

where $c = c_1/3 + c_2/3^2 + c_3/3^3 \dots$, is a homeomorphism between the Cantor set and the set A appearing in the proof of Theorem 11.11.

11:3.6 Prove that every complete metric space, separable or not, that is dense-in-itself contains a Cantor set. [Hint: Apply the reasoning of Theorem 11.11 to $D = X$, $f(x) = x$.]

11:3.7 Give an example of an uncountable complete space X that contains no Cantor set.

11:3.8 Show that there exists an uncountable separable metric space that contains no Cantor set.

11.4 Borel Sets

We now show that all Borel sets are analytic. In fact, the characterization can be made very precise. We recall that analytic sets have been described as projections of closed sets; here we see that the Borel sets are the one-one projections. We have seen that continuous images of analytic sets are analytic; here we see that one-one continuous images of Borel sets are Borel sets. Finally, the Borel sets are precisely those sets that are analytic and also are complements of analytic sets. Definition 11.5 should be compared with the first three characterizations of the following theorem.

Theorem 11.12 *Let X be a complete, separable metric space. The class of Borel subsets of X may be described as:*

1. *The class of all sets that are* one–one *continuous images of $\mathbb{N}^{\mathbb{N}}$.*
2. *The class of all sets that are* one–one *continuous images of a Borel subset of a complete, separable metric space.*
3. *The class of all sets that are* one–one *projections onto X of closed subsets of $X \times \mathbb{N}^{\mathbb{N}}$.*
4. *The class of all analytic sets whose complements are also analytic.*

The proof is obtained from a series of observations, which we prove individually.

11.13 *Let X be a complete, separable metric space. Every Borel set in X is the one-one projective image of a closed set in $X \times \mathbb{N}^{\mathbb{N}}$.*

Proof. Let $\mathcal{B}$ be the class of all subsets of X that are the one-one projective image of a closed set in $X \times \mathbb{N}^{\mathbb{N}}$. We show first that $\mathcal{B}$ contains the open sets. If G is open in X, then

$$E = \{(x,t) : \ x \in X, \, t \in \mathbb{R}, \, \operatorname{dist}(x, X \setminus G) = 1/t\}$$

is a closed subset of $X \times \mathbb{R}$, and G is its one-one projection. We are almost done, but we require a closed subset of $X \times \mathbb{N}^{\mathbb{N}}$. For this we can use Exercise 11:2.10 to obtain a closed set $B \subset \mathbb{N}^{\mathbb{N}}$ and a continuous map $g : B \to \mathbb{R}$ that is one-one and onto. Then

$$C = \{(x, \mathbf{n}) : \ x \in X, \, \mathbf{n} \in B, \, (x, g(\mathbf{n})) \in E\}$$

projects in a one-one manner to G as required.

We show next that $\mathcal{B}$ is closed under countable, disjoint unions. If $\{A_i\}$ are each in $\mathcal{B}$ and disjoint, then select closed sets

$$C_j \subset X \times \{j\} \times \mathbb{N}^{\mathbb{N}}$$

that project in a one-one manner to A_j. The set

$$F = \bigcup_{j=1}^{\infty} C_j$$

is a closed subset of $X \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, and it projects in a one-one manner to $\bigcup_{j=1}^{\infty} A_j$. Since $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ (see Exercise 11:2.4) we are done.

For the final step, we show that $\mathcal{B}$ is closed under countable intersections. If $\{A_i\}$ are each in $\mathcal{B}$, select closed sets $C_j \subset X \times \mathbb{N}^{\mathbb{N}}$ that project in a one-one manner to A_j. Consider the set

$$\{(x, \mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}, \dots) : \ x \in X,\ \mathbf{n}^{(k)} \in \mathbb{N}^{\mathbb{N}},\ (x, \mathbf{n}^{(k)}) \in C_k,\ k \in \mathbb{N}\}.$$

This is a closed subset of $X \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, and it projects in a one-one manner to $\bigcap_{j=1}^{\infty} A_j$. We are done when we recall that $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

Now we put these three facts together. The class $\mathcal{B}$ contains all open sets, is closed under countable disjoint unions, and is closed under countable intersections. Consequently (Exercise 3:1.4), $\mathcal{B}$ contains all Borel sets and we are done. ■

This next assertion is known as the Lusin separation theorem. Two disjoint sets P and Q are said to be *Borel separated* if there are disjoint Borel sets B_1 and B_2 with $P \subset B_1$ and $Q \subset B_2$.

11.14 *Let X and Y be complete, separable metric spaces, let $f : X \to Y$ be continuous, and let C and D be closed disjoint subsets of X. Then if the image sets $f(C)$ and $f(D)$ are disjoint, these sets are Borel separated.*

Proof. We obtain a contradiction by supposing that $f(C)$ and $f(D)$ are not Borel separated. We carve up the space $X = \bigcup_{i=1}^{\infty} E(i)$ into sets of small diameter (using the notation and ideas of the proof of Theorem 11.3) and observe that there must be $n_1, m_1 \in \mathbb{N}$, so $f(C \cap E(n_1))$ and $f(D \cap E(m_1))$ are not Borel separated. Otherwise, using the results of Exercise 11:4.3, we could show that $f(C)$ and $f(D)$ are Borel separated.

By induction on k we then obtain (continuing the notation of Theorem 11.3) $(n_1, n_2, n_3, \dots), (m_1, m_2, m_3, \dots) \in \mathbb{N}^{\mathbb{N}}$ so that the sets

$$f(C \cap E(n_1, n_2, n_3, \dots, n_k)),\ f(D \cap E(m_1, m_2, m_3, \dots, m_k)) \tag{1}$$

are not Borel separated for each $k \in \mathbb{N}$. But the sets

$$C \cap E(n_1, n_2, n_3, \dots, n_k),\ D \cap E(m_1, m_2, m_3, \dots, m_k)$$

each shrink down to, at most, a single point in C and D, respectively. A simple argument using the continuity of f shows that, for large enough k, the sets in (1) must then be able to be separated by disjoint open sets. This contradicts our statement that they are not Borel separated. ■

11.15 *Let X and Y be complete, separable metric spaces, let $f : X \to Y$ be continuous, and let $\{A_i\}$ be a sequence of disjoint analytic subsets of X. If the image sets $\{f(A_i)\}$ are disjoint, then there exist disjoint Borel sets B_1, B_2, B_3, $\dots$ so that $f(A_i) \subset B_i$ for each natural number i.*

This is left as Exercise 11:4.4.

11.16 *Let X and Y be complete, separable metric spaces, let $f : X \to Y$ be continuous and one-one, and let $C \subset X$ be closed. Then $f(C)$ is a Borel subset of Y.*

Proof. We shall express $f(C)$ as a Borel set explicitly. We partition X into a sequence of disjoint Borel sets of small diameter. Using the sets $E(n_1, n_2, n_3, \dots, n_k)$ once again, we have

$$X = \bigcup_{(n_1,n_2,n_3,\dots,n_k)\in\mathbb{N}^k} E(n_1, n_2, n_3, \dots, n_k),$$

and so also

$$X = \bigcup_{(n_1,n_2,n_3,\dots,n_k)\in\mathbb{N}^k} A(n_1, n_2, n_3, \dots, n_k),$$

where

$$A(n_1, n_2, n_3, \dots, n_k) = [A(n_1, n_2, n_3, \dots, n_{k-1}) \cap E(n_1, n_2, n_3, \dots, n_k)] \setminus \bigcup_{i<n_k} E(n_1, n_2, n_3, \dots, n_{k-1}, i)$$

just re-expresses our union as a disjoint union. The elements are certainly Borel sets, since we started with closed sets.

Since f is one-one, the sets $f(C \cap A(n_1, n_2, n_3, \dots, n_k))$ are disjoint for distinct $(n_1, n_2, n_3, \dots, n_k) \in \mathbb{N}^k$. Consequently, using assertion 11.15 just proved, we can select disjoint Borel sets

$$B(n_1, n_2, n_3, \dots, n_k)$$

so that

$$f(C \cap A(n_1, n_2, n_3, \dots, n_k)) \subset B(n_1, n_2, n_3, \dots, n_k)$$

$$\subset B(n_1, n_2, n_3, \dots, n_{k-1}) \cap \overline{f(C \cap E(n_1, n_2, n_3, \dots, n_k))}.$$

Now define

$$T = \bigcup_{(n_1,n_2,n_3,\dots)} \bigcap_{k=1}^{\infty} B(n_1, n_2, n_3, \dots, n_k). \tag{2}$$

This is a Borel set since, using the fact that the sets are disjoint, we can express it also as

$$T = \bigcap_{k=1}^{\infty} \bigcup_{(n_1,n_2,n_3,\dots,n_k)} B(n_1, n_2, n_3, \dots, n_k).$$

It remains only to see that $T = f(C)$. The inclusion $f(C) \subset T$ follows immediately from the first expression (2) for T and the definitions of the sets. In the opposite direction, we can check that

$$T \subset \bigcup_{(n_1,n_2,n_3,\dots)} \bigcap_{k=1}^{\infty} \overline{f(C \cap E(n_1,n_2,n_3,\dots,n_k))} \subset f(C),$$

because f is continuous and $C \cap E(n_1,n_2,n_3,\dots,n_k)$ shrinks to, at most, a single point u of C. Here we use the fact that any neighborhood of u must contain the set $f(C \cap E(n_1,n_2,n_3,\dots,n_k))$ and its closure for large enough values of k. This completes the proof of assertion 11.16. ■

The main theorem, Theorem 11.12, now follows from our four observations. Putting these together is left as an exercise.

Exercises

11:4.1 Show that P and Q are Borel separated if $\operatorname{dist}(P,Q) > 0$.

11:4.2 Suppose that Q and P_i are Borel separated for each $i \in \mathbb{N}$. Show that Q and $\bigcup_{i=1}^{\infty} P_i$ are Borel separated.

11:4.3 Suppose that Q_j and P_i are Borel separated for each $i, j \in \mathbb{N}$. Show that $\bigcup_{j=1}^{\infty} Q_j$ and $\bigcup_{i=1}^{\infty} P_i$ are Borel separated and that $\bigcap_{j=1}^{\infty} Q_j$ and $\bigcup_{i=1}^{\infty} P_i$ are Borel separated.

11:4.4 Let X and Y be complete, separable metric spaces, let $f : X \to Y$ be continuous, and let A_i be a sequence of disjoint analytic subsets of X. If the image sets $\{f(A_i)\}$ are disjoint, then there exist disjoint Borel sets B_1, B_2, $B_3, \dots$ so that $f(A_i) \subset B_i$ for each natural number i. [Hint: Use the fact that if A_1 and A_2 are disjoint analytic sets then they are the projections of disjoint closed sets C_1 and C_2 in $X \times \mathbb{N}^{\mathbb{N}}$. The earlier exercises give ideas on how to handle sequences.]

11:4.5 Give the necessary arguments that deduce Theorem 11.12 from the statements proved in this section.

11:4.6 Show that assertion 11.16 is not valid in general if X is a *nonseparable,* complete metric space. [Hint: Use $X = Y = \mathbb{R}$ and suppose that f is the identity map, but use the discrete metric on X.]

11:4.7 Let $I = [0,1]$ and $A = [0,1] \setminus \mathbb{Q}$. Verify each of the following statements:

(*a*) If $f : A \to \mathbb{R}$ is continuous, then $f(A)$ is analytic.

(*b*) If $f : A \to \mathbb{R}$ is continuous and one-one, then $f(A)$ is a Borel set, but need not be of type $\mathcal{G}_\delta$.

(*c*) If $f : A \to \mathbb{R}$ is continuous and can be extended to a continuous function f on I, then $f(A)$ is residual in $f(I)$.

(*d*) If $f : A \to \mathbb{R}$ is continuous and can be extended to a one-one continuous function f on I, then $f(A)$ is of type $\mathcal{G}_\delta$.

11:4.8 Let $\mathcal{C}(I, I)$ denote the space of all continuous mappings from $I = [0, 1]$ into itself, supplied with the supremum metric. Let

$$\mathcal{C}_2 = \{f \circ f : f \in \mathcal{C}(I, I)\}.$$

Prove that $\mathcal{C}_2$ is an analytic subset[3] of $\mathcal{C}(I, I)$. [Hint: Consider the map $f \to f \circ f$.]

11.5 An Analytic Set That Is Not Borel

It is essential that we prove the existence of analytic sets that are not also Borel sets or else the theory we have developed has no substance beyond finding extra properties of Borel sets. Suslin was the first to obtain this.

Theorem 11.17 *Suppose that X is a perfect, complete, separable metric space. Then X contains an analytic set that is not a Borel set.*

Proof. Observe first that it is enough to do this in the space $\mathbb{N}^{\mathbb{N}}$. Suppose that we can find such a set in $\mathbb{N}^{\mathbb{N}}$. By Theorem 11.4, X contains a Borel set B that is a homeomorphic copy of $\mathbb{N}^{\mathbb{N}}$. Inside B is then a copy of that analytic, non-Borel set.

For the first step, suppose that we are given a complete, separable metric space Z. We claim that we can find a closed set $C \subset Z \times \mathbb{N}^{\mathbb{N}}$ so that every closed set in Z occurs as a "slice":

$$C_{\mathbf{n}} = \{z : (z, \mathbf{n}) \in C\} \quad (\mathbf{n} \in \mathbb{N}^{\mathbb{N}}). \tag{3}$$

Let $\{U(i)\}$ be any countable base for the topology of Z. An arbitrary closed set $F \subset Z$ can be described by announcing for which integers i the sets $U(i)$ are complementary to F, say for $i = n_1, n_2, n_3, \ldots$. Thus, to each $z \in Z$, we can associate those sequences $\mathbf{n} = (n_1, n_2, n_3, \ldots)$ such that, for all i, z is not in $U(n_i)$. This leads us to define the set

$$C = \{(z, n_1, n_2, n_3, \ldots) : z \in Z, i, n_i \in \mathbb{N}, z \notin U(n_i)\}$$

which is a subset of $Z \times \mathbb{N}^{\mathbb{N}}$. The set C is closed, since convergence in the product metric is just coordinatewise convergence. It is clear that every closed set in Z appears as one of the slices (3) for a particular choice of $\mathbf{n} \in \mathbb{N}^{\mathbb{N}}$.

Suppose now that we are given a complete, separable metric space Y. For the second step, we claim that we can find an analytic set $A \subset Y \times \mathbb{N}^{\mathbb{N}}$ so that every analytic set in Y occurs as a slice:

$$A_{\mathbf{m}} = \{z : (z, \mathbf{m}) \in A\} \quad (\mathbf{m} \in \mathbb{N}^{\mathbb{N}}).$$

[3] It is also true that $\mathcal{C}_2$ is a nowhere dense subset of $\mathcal{C}(I, I)$ so that very few members of $\mathcal{C}(I, I)$ are themselves iterates of a continuous function. See A. M. Blokh, *Trans. Amer. Math. Soc.* 333 (1992), 787–798.

We do this by applying the first step to the space $Z = Y \times \mathbb{N}^{\mathbb{N}}$ to get a closed set
$$C \subset Y \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$$
so that every closed set in $Y \times \mathbb{N}^{\mathbb{N}}$ appears as a slice:
$$C_{\mathbf{m}} = \{(y, \mathbf{n}) : (y, \mathbf{n}, \mathbf{m}) \in C\} \quad (\mathbf{m} \in \mathbb{N}^{\mathbb{N}}).$$
Define
$$A = \{(y, \mathbf{t}) : (y, \mathbf{s}, \mathbf{t}) \in C \text{ for some } \mathbf{s} \in \mathbb{N}^{\mathbb{N}}\}.$$
Certainly, A is an analytic subset of $Y \times \mathbb{N}^{\mathbb{N}}$, since A is a projection of a closed subset of
$$Y \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}.$$
Fix any $\mathbf{m} \in \mathbb{N}^{\mathbb{N}}$. Then
$$p_Y(C_{\mathbf{m}}) = \{y : (y, \mathbf{m}, \mathbf{n}) \in C\} = \{y : (y, \mathbf{m}) \in A\} = A_{\mathbf{m}},$$
and every analytic subset of Y can be represented this way.

For the final step, we can now claim, because of what we have just done (i.e., apply step 2 to $Y = \mathbb{N}^{\mathbb{N}}$), that there is an analytic set
$$A \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$$
so that every analytic subset of $\mathbb{N}^{\mathbb{N}}$ appears as a slice:
$$A_{\mathbf{m}} = \{\mathbf{n} : (\mathbf{n}, \mathbf{m}) \in A\}.$$
Having before us a copy of each analytic subset of $\mathbb{N}^{\mathbb{N}}$, we can use a familiar diagonal argument to find a set that is not analytic. Consider the set
$$T = \{\mathbf{n} : (\mathbf{n}, \mathbf{n}) \in A\}$$
and its complement
$$\mathbb{N}^{\mathbb{N}} \setminus T = \{\mathbf{n} : (\mathbf{n}, \mathbf{n}) \notin A\}.$$
The former is analytic, and the latter cannot be the same as any set $A_{\mathbf{m}}$ and so is not analytic. In particular, by Theorem 11.12, neither is a Borel set. ■

Exercises

11:5.1 In the proof of Theorem 11.17, explain why
$$T = \{\mathbf{n} : (\mathbf{n}, \mathbf{n}) \in A\}$$
is analytic.

11:5.2 In the proof of Theorem 11.17, explain why there is no $\mathbf{m} \in \mathbb{N}^{\mathbb{N}}$ for which
$$\mathbb{N}^{\mathbb{N}} \setminus T = \{\mathbf{n} : (\mathbf{n}, \mathbf{n}) \notin A\} = \{\mathbf{n} : (\mathbf{n}, \mathbf{m}) \in A\}.$$

11.6 Measurability of Analytic Sets

We are now in a position to show that this new class of sets, the analytic sets, can be handled by any reasonable measure. In a metric space the Borel sets are all measurable with respect to any metric outer measure. So, too, are the analytic sets. (The theorem is stated for a complete, separable metric space, but does not require all these hypotheses.)

Theorem 11.18 *Let X be a complete, separable metric space, and let μ^* be a metric outer measure on X. Then every analytic subset of X is μ^*-measurable.*

Proof. Let $A \subset X$ be analytic, and let T be an arbitrary subset of X with $\mu^*(T) < +\infty$. We wish to verify that

$$\mu^*(T) \geq \mu^*(T \cap A) + \mu^*(T \setminus A).$$

From this, by definition, the measurability of A follows. With no loss of generality, we can obtain this under the extra assumption that μ^* is finite and regular (Exercise 11:6.1). Thus, to prove that A is measurable, we can reduce our arguments to the case where μ^* is finite and regular and show that

$$\mu^*(X) \geq \mu^*(A) + \mu^*(X \setminus A) \tag{4}$$

since (by Theorem 2.35) this establishes the measurability of A.

We first show the existence, for any $\varepsilon > 0$, of a closed set $C \subset A$ so that

$$\mu^*(C) > \mu^*(A) - \varepsilon. \tag{5}$$

Since A is analytic, there is a continuous map $f : \mathbb{N}^{\mathbb{N}} \to X$ so that $f(\mathbb{N}^{\mathbb{N}}) = A$. Define $E(k)$ for each $k \in \mathbb{N}$ as $\{\mathbf{n} \in \mathbb{N}^{\mathbb{N}} : n_1 \leq k\}$. This expresses $\mathbb{N}^{\mathbb{N}}$ as the limit of an increasing sequence of closed sets

$$E(1) \subset E(2) \subset E(3) \subset \cdots \nearrow \mathbb{N}^{\mathbb{N}}.$$

Thus, by Exercise 2:9.2,

$$\mu^*(f(E(k))) \nearrow \mu^*(A)$$

since μ^* is regular. Consequently, we can choose $m_1 \in \mathbb{N}$ so that

$$\mu^*(f(E(m_1))) > \mu^*(A) - \varepsilon/2.$$

This process can be continued inductively to produce an element

$$\mathbf{m} = (m_1, m_2, m_3, \dots) \in \mathbb{N}^{\mathbb{N}}$$

so that if we write

$$E(m_1, m_2, m_3, \dots, m_k)$$
$$= \{\mathbf{n} \in \mathbb{N}^{\mathbb{N}} : n_1 \leq m_1,\ n_2 \leq m_2,\ \dots\ n_k \leq m_k\}$$

then

$$\mu^*(f(E(m_1, m_2, m_3, \ldots, m_k)))$$
$$> \mu^*(f(E(m_1, m_2, m_3, \ldots, m_{k-1}))) - \varepsilon/2^k. \tag{6}$$

From (6) we obtain immediately that

$$\mu^*(f(E(m_1, m_2, m_3, \ldots, m_k))) > \mu^*(A) - \varepsilon. \tag{7}$$

Now define

$$C = \bigcap_{k=1}^{\infty} \overline{f(E(m_1, m_2, m_3, \ldots, m_k))}.$$

Certainly, C is closed and, using (7), we compute

$$\mu^*(C) = \lim_{k\to\infty} \mu^*(\overline{f(E(m_1, m_2, m_3, \ldots, m_k))}) \geq \mu^*(A) - \varepsilon.$$

This proves (5), except that we do not yet know that C is a subset of A. We check this now.

Define

$$K = \{\mathbf{n} \in \mathbb{N}^{\mathbb{N}} : \mathbf{n} \leq \mathbf{m}\},$$

where $\mathbf{m} = (m_1, m_2, m_3, \ldots)$ is as constructed above and $\mathbf{n} \leq \mathbf{m}$ means that, for each k, $n_k \leq m_k$. The set K is a compact subset of $\mathbb{N}^{\mathbb{N}}$. We show $C = f(K)$. Certainly, $C \supset f(K)$ by the manner in which C was constructed. Conversely, let a be a point not in $f(K)$; we shall show that $a \notin C$. The sets $\{a\}$ and $f(K)$ are disjoint and compact and so are a positive distance apart. Thus there must be open sets W and V with $K \subset W \subset \mathbb{N}^{\mathbb{N}}$ and $\{a\} \subset V \subset X$ so that $f(W) \cap V = \emptyset$. The distance from K to $\mathbb{N}^{\mathbb{N}} \setminus W$ is positive, say greater than 2^{-i} for some i. Consider a point

$$\mathbf{n} = (n_1, n_2, n_3, \ldots) \in E(m_1, m_2, m_3, \ldots, m_i).$$

We know that

$$(n_1, n_2, n_3, \ldots, n_i, 1, 1, 1, 1, 1, 1, \ldots)$$

must belong to K and hence, since $\operatorname{dist}(\mathbf{n}, K) \leq 2^{-i}$, the point $\mathbf{n}$ is in W. This shows that

$$E(m_1, m_2, m_3, \ldots, m_i) \subset W$$

and hence that

$$f(E(m_1, m_2, m_3, \ldots, m_i)) \subset X \setminus V.$$

In particular, since V is a neighborhood of a, the point a is not in the set

$$\overline{f(E(m_1, m_2, m_3, \ldots, m_i))},$$

and so $a \notin C$, as required.

Thus we have established (5). Now we obtain (4). Using the fact that C is measurable (since it is closed) and (5), we have

$$\begin{aligned}\mu^*(X) &\geq \mu^*(C) + \mu^*(X \setminus C)\\ &\geq \mu^*(A) - \varepsilon + \mu^*(X \setminus A).\end{aligned}$$

Since ε is arbitrary, (4) follows. ■

Exercises

11:6.1 Show that in the proof of Theorem 11.18 the argument can reduce to the case where μ^* is finite and regular. [Hint: If not finite, use the metric outer measure $\mu_1^*(E) = \mu^*(T \cap E)$. If not regular, use the metric outer measure

$$\mu_2^*(E) = \inf\{\mu_1^*(B) : E \subset B \text{ and } B \text{ measurable}\}.$$

11:6.2 Let $g : X \to Y$, where X is a complete, separable metric space, Y is a metric space, and g is continuous. If μ^* is a metric outer measure on X and $A \subset X$ is a Borel set or is analytic, show that $g(A)$ is measurable. (The proof here did not use the completeness or the separability of the space.)

11.7 The Suslin Operation

We recall (Section 1.13) that Suslin originally defined a set $E \subset \mathbb{R}$ to be analytic if it can be expressed in the form

$$E = \bigcup_{(n_1,n_2,n_3,\dots)} \bigcap_{k=1}^{\infty} I_{n_1,n_2,n_3,\dots,n_k}$$

where each $I_{n_1,n_2,n_3,\dots,n_k}$ is a nonempty, closed interval for each

$$(n_1, n_2, n_3, \dots, n_k) \in \mathbb{N}^k$$

and each $k \in \mathbb{N}$ and where the union is taken over all possible sequences $(n_1, n_2, n_3, \dots)$ of natural numbers. In this section we relate this operation to the notion of an analytic set as we have given it in Definition 11.5.

Let $\mathcal{E}$ be any family of sets. By the family Suslin-$\mathcal{E}$ we mean the collection of sets E that can be written in the form

$$E = \bigcup_{(n_1,n_2,n_3,\dots)} \bigcap_{k=1}^{\infty} I_{n_1,n_2,n_3,\dots,n_k}, \tag{8}$$

where each $I_{n_1,n_2,n_3,\dots,n_k} \in \mathcal{E}$ for each

$$(n_1, n_2, n_3, \dots, n_k) \in \mathbb{N}^k$$

and each $k \in \mathbb{N}$. If the family $\mathcal{E}$ is closed under intersections, then we can insist that the representation in (8) be descending in the sense that

$$I_{n_1,n_2,n_3,\dots,n_k} \subset I_{n_1,n_2,n_3,\dots,n_{k-1}}.$$

We can always accomplish this by replacing $I_{n_1,n_2,n_3,\dots,n_k}$ by

$$\bigcap_{i=1}^{k} I_{n_1,n_2,n_3,\dots,n_i}.$$

Theorem 11.19 *Let X be a complete, separable metric space, and let $\mathcal{F}$ denote the class of closed subsets. Then every analytic set can be obtained as a Suslin-$\mathcal{F}$ set.*

Proof. Let $A \subset X$ be analytic. Then there is a continuous mapping f so that $A = f(\mathbb{N}^{\mathbb{N}})$. Define the closed sets

$$C_{n_1,n_2,n_3,\dots,n_k} = \overline{f(S_{n_1,n_2,n_3,\dots,n_k})},$$

where

$$S_{n_1,n_2,n_3,\dots,n_k} = \{\mathbf{m} \in \mathbb{N}^{\mathbb{N}} : m_i = n_i,\ i = 1, 2, \dots, k\}.$$

We claim that

$$A = \bigcup_{(n_1,n_2,n_3,\dots)} \bigcap_{k=1}^{\infty} C_{n_1,n_2,n_3,\dots,n_k},$$

which expresses A as a set obtained from the Suslin operation performed on a family of closed sets, as required to prove the theorem.

Fix an $\mathbf{n} = (n_1, n_2, n_3, \dots) \in \mathbb{N}^{\mathbb{N}}$. Then, for each $k \in \mathbb{N}$, $\mathbf{n} \in S_{n_1,n_2,n_3,\dots,n_k}$, and so $f(\mathbf{n}) \in C_{n_1,n_2,n_3,\dots,n_k}$. Consequently,

$$f(\mathbf{n}) \in \bigcap_{k=1}^{\infty} C_{n_1,n_2,n_3,\dots,n_k}. \tag{9}$$

But diameter$(S_{n_1,n_2,n_3,\dots,n_k}) \searrow 0$, and this implies that

$$\text{diameter}(f(S_{n_1,n_2,n_3,\dots,n_k})) = \text{diameter}(C_{n_1,n_2,n_3,\dots,n_k}) \searrow 0,$$

and so the intersection in (9) is a single point. Consequently,

$$\begin{aligned} A = f(\mathbb{N}^{\mathbb{N}}) &= \bigcup_{(n_1,n_2,n_3,\dots)} \{f((n_1, n_2, n_3, \dots))\} \\ &= \bigcup_{(n_1,n_2,n_3,\dots)} \bigcap_{k=1}^{\infty} C_{n_1,n_2,n_3,\dots,n_k}, \end{aligned}$$

as required. ■

We see that the analytic sets are obtained from the Suslin operation applied to the closed sets. One should know, although we omit the proofs, that the analytic sets and the measurable sets are each closed under any application of the Suslin operation. The texts cited in the introduction to the chapter contain proofs.

11.8 A Method to Show a Set Is Not Borel

In many instances, one can see easily that a certain set under consideration is analytic or *co-analytic* (i.e., the complement of an analytic set), but it is not clear that the set is not also a Borel set. We describe in this section a method (which the logicians call a "completeness argument") for proving that a given co-analytic set is not Borel.

Definition 11.20 A co-analytic set Q in a complete, separable, metric space X is said to be a *complete co-analytic set* if for every co-analytic subset P of $\mathbb{N}^{\mathbb{N}}$ there is a Borel function $f : \mathbb{N}^{\mathbb{N}} \to X$ such that

$$x \in P \Longleftrightarrow f(x) \in Q.$$

We might say that the function f reduces the set P to Q since it reduces the question of membership in the set P, by way of the function f, to membership in the set Q. We have seen in Theorem 11.17 that the space $\mathbb{N}^{\mathbb{N}}$ contains an analytic set that is not a Borel set. The complement of that set is a co-analytic set that is also not a Borel set. Consequently, a complete co-analytic set cannot be a Borel set (Exercise 11:8.4). Thus we have a beginning of a strategy for determining that certain sets are not Borel sets.

To polish off the strategy requires us to have in our possession one example of a complete co-analytic set that can be used to compare to a given set. The set of *well-founded trees* is perhaps the most used example. For the rest of this section we describe these notions and show that the set of well-founded trees is indeed a complete co-analytic set in an appropriate space.

We define and describe the following:

- By $\mathbb{N}^*$ we denote the set of all finite sequences

$$\mathbf{n} = (n_1, n_2, n_3, \ldots, n_k)$$

  of natural numbers. We include the empty sequence, denoted ( ).

- The metric space $2^{\mathbb{N}^*}$ is the set of all subsets of $\mathbb{N}^*$ and is isometric as a metric space to $2^{\mathbb{N}}$ (the Cantor set).

- A *tree* is a set $T \subset \mathbb{N}^*$ that includes the empty sequence ( ) and with the property that

$$(n_1, n_2, n_3, \ldots, n_k) \in T \Longrightarrow (n_1, n_2, n_3, \ldots, n_p) \in T \quad (\forall p < k).$$

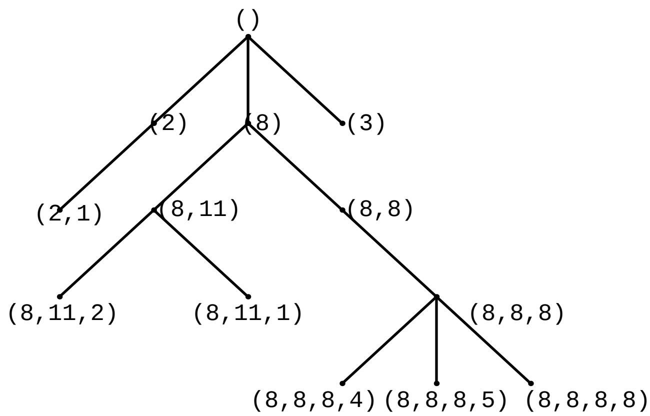

Figure 11.1: A tree.

- For any tree T, by $[T]$ we denote the set of all elements
$$(n_1, n_2, n_3, \dots) \in \mathbb{N}^{\mathbb{N}}$$
such that
$$(n_1, n_2, n_3, \dots, n_k) \in T$$
for all k.
- A tree T is said to be *well founded* if $[T] = \emptyset$.
- The set of all trees is a closed subset of $2^{\mathbb{N}^*}$.
- The set $\mathcal{WF}$ denotes the set of all well-founded trees in the metric space $2^{\mathbb{N}^*}$.

There is a natural partial order on any tree: a rough sketch of what this partially ordered set looks like will reveal a treelike structure and hence the name. (See Figure 11.1, which, viewed upside down, has the form of a tree.) The collection $[T]$ of infinite sequences would be considered the infinite branches of the tree T. A tree is well founded if it has no infinite branches; the terminology has some special meaning in descriptive set theory. The collection of all trees resides quite conveniently inside the complete, separable metric space $2^{\mathbb{N}^*}$, where it is a closed set; then $\mathcal{WF}$ will turn out to be a complete co-analytic subset that is useful in many instances.

Let us begin with some preliminary observations that connect the study of trees with our study so far of analytic sets. The proofs are left as Exercises 11:8.6 and 11:8.7.

Lemma 11.21 *If C is a closed subset of $\mathbb{N}^{\mathbb{N}}$, then there is a tree T such that $[T] = C$.*

Lemma 11.22 *If A is an analytic subset of $\mathbb{N}^{\mathbb{N}}$, then there is a tree S such that A can be represented in the form*

$$(n_1, n_2, n_3, \dots) \in A \iff \begin{array}{l}(n_1, m_1, n_2, m_2, n_3, m_3, \dots) \in [S] \\ \textit{for some } (m_1, m_2, m_3, \dots) \in \mathbb{N}^{\mathbb{N}}.\end{array}$$

Theorem 11.23 *$\mathcal{WF}$ is a complete co-analytic subset of $2^{\mathbb{N}^*}$.*

Proof. Let P be a co-analytic subset of $\mathbb{N}^{\mathbb{N}}$. Then P', the complement of P, is analytic. By Lemma 11.22, there is a tree S such that P' can be represented in the form

$$(n_1, n_2, n_3, \dots) \in P' \iff \begin{array}{l}(n_1, m_1, n_2, m_2, n_3, m_3, \dots) \in [S] \\ \text{for some } (m_1, m_2, m_3, \dots) \in \mathbb{N}^{\mathbb{N}}.\end{array}$$

Using S, we construct a mapping from $\mathbb{N}^{\mathbb{N}}$ into $2^{\mathbb{N}^*}$ by associating with each

$$\mathbf{n} = (n_1, n_2, n_3, \dots) \in \mathbb{N}^{\mathbb{N}}$$

the tree $T(\mathbf{n})$ defined as the collection of all sequences

$$(b_1, b_2, b_3, \dots, b_k)$$

for which

$$(n_1, b_1, n_2, b_2, n_3, b_3, \dots, n_k, b_k) \in S.$$

[We include the empty sequence in $T(\mathbf{n})$, too.] Clearly, $T(\mathbf{n})$ so defined is a tree, and the map $\mathbf{n} \to T(\mathbf{n})$ is a Borel mapping from $\mathbb{N}^{\mathbb{N}}$ into $2^{\mathbb{N}^*}$. We analyze membership in the co-analytic set P in terms of this tree:

$$\begin{array}{rcl}\mathbf{n} \in P \iff \mathbf{n} \notin P' & \iff & (n_1, m_1, n_2, m_2, n_3, m_3, \dots) \notin [S] \\ & & \text{for all } (m_1, m_2, m_3, \dots) \in \mathbb{N}^{\mathbb{N}} \\ & \iff & T(\mathbf{n}) \in \mathcal{WF}.\end{array}$$

By definition, since P was an arbitrary co-analytic set in $\mathbb{N}^{\mathbb{N}}$, we have proved that $\mathcal{WF}$ is a complete co-analytic set in $2^{\mathbb{N}^*}$. ■

Exercises

11:8.1 For $\mathbf{n}$, $\mathbf{m} \in \mathbb{N}^*$, write $\mathbf{n} \preceq \mathbf{m}$ if $\mathbf{n}$ is an *initial segment* of $\mathbf{m}$, that is, if $\mathbf{n} = (\ )$ or if $\mathbf{n} = (n_1, n_2, n_3, \dots, n_k)$ and

$$\mathbf{m} = (n_1, n_2, n_3, \dots, n_k, c_{k+1}, \dots, c_p)$$

for some $c_{k+1}, \dots, c_p$. Show that this is a partial order on any tree. Show that a tree is simply a subset of $\mathbb{N}^*$ that is closed under initial segments.

11:8.2$\diamondsuit$ Show that $\mathbb{N}^*$ is denumerable. (Hence there is a way of listing the elements and we can write, for each $\mathbf{n} \in \mathbb{N}^*$, $\langle \mathbf{n} \rangle = j$ if $\mathbf{n}$ is the jth element in the listing.)

11:8.3 Explain how a tree can be considered as a point in the space $2^{\mathbb{N}^*}$. What point represents the tree containing only the empty sequence $(\ )$?

11:8.4 Prove that a complete co-analytic set in a complete, separable, metric space X is not a Borel set.

11:8.5 Let Y, Y' be complete, separable metric spaces and $Q \subset Y$, $Q' \subset Y'$ co-analytic subsets. Suppose that there is a Borel function $g : Y \to Y'$ such that

$$y \in Q \iff g(y) \in Q'.$$

Show that Q' must be a complete co-analytic set if Q is.

11:8.6 Prove Lemma 11.21: If C is a closed subset of $\mathbb{N}^{\mathbb{N}}$, then there is a tree T such that $[T] = C$. [Hint: Let T be the set of all initial segments of C. The fact that C is closed is needed to show that if $\mathbf{n} \in [T]$ then $\mathbf{n} \in C$.]

11:8.7 Prove Lemma 11.22: If A is an analytic subset of $\mathbb{N}^{\mathbb{N}}$, then there is a tree S such that A can be represented in the form

$$\begin{aligned}(n_1, n_2, n_3, \dots) \in A \quad \iff \qquad & (n_1, m_1, n_2, m_2, n_3, m_3, \dots) \in [S] \\ & \text{for some } (m_1, m_2, m_3, \dots) \in \mathbb{N}^{\mathbb{N}}.\end{aligned}$$

[Hint: Use Exercise 11:8.6 and Definition 11.5.]

11:8.8 Check that the mapping $\mathbf{n} \to T(\mathbf{n})$ (in the proof of Theorem 11.23) is a Borel mapping from $\mathbb{N}^{\mathbb{N}}$ into $2^{\mathbb{N}^*}$.

11.9 Differentiable Functions

The theory of analytic sets is playing an increasingly important role in analysis. We shall complete our investigations by proving one classical result as an application of these methods to an interesting example.

In 1936 S. Mazurkiewicz[4]showed that the set of differentiable functions in the metric space $C[0,1]$ forms a set that is not Borel. Banach had asked this question, pointing out that the set could be seen with little difficulty to be at least co-analytic. We shall give a proof of this result in Theorem 11.24. We have already seen (Corollary 10.21) that this set is first category; it is an analysis of a different kind we do now.

There have been further investigations of this type that appear in the analysis literature. By now it is becoming natural to ask for the complexity

[4]S. Mazurkiewicz. "Über die Menge der differenzierbaren Funktionen," *Fund. Math.* 27 (1936), 244–249.

of certain sets that arise. For example, it is also true[5] that the set of nowhere differentiable functions in the metric space $C[0,1]$ forms a set that is co-analytic and not Borel. Recall that this set is residual.

Theorem 11.24 (Mazurkiewicz) *The set of differentiable functions is a co-analytic subset of $C[0,1]$ that is not a Borel set.*

Proof. Our proof is essentially the original of Mazurkiewicz, but follows a more recent treatment.[6] We show that the set D of differentiable functions is a complete co-analytic set in $C[0,1]$ by reducing it to the set $\mathcal{WF}$ of Theorem 11.23.

We use the fact that $\mathbb{N}^*$ is denumerable to list its elements and so write for each $\mathbf{n} \in \mathbb{N}^*$, $\langle \mathbf{n} \rangle = j$ if $\mathbf{n}$ is the jth element in the listing. (See Exercise 11:8.2.) For every element

$$\mathbf{n} = (n_1, n_2, n_3, \dots, n_k) \in \mathbb{N}^*,$$

we define (by induction on k) an open interval $J_{\mathbf{n}} \subset [0,1]$ and a much smaller closed subinterval $K_{\mathbf{n}}$ such that:

1. The interval $K_{\mathbf{n}}$ is concentric with $J_{\mathbf{n}}$, and its length is smaller than
$$2^{-\langle \mathbf{n} \rangle}|J_{\mathbf{n}}|.$$
2. For any n,
$$J_{(n_1,n_2,n_3,\dots,n_k,n)} \subset L_{\mathbf{n}},$$
where $L_{\mathbf{n}}$ denotes the left half of $K_{\mathbf{n}}$.
3. For any distinct n, m,
$$J_{(n_1,n_2,n_3,\dots,n_k,n)} \cap J_{(n_1,n_2,n_3,\dots,n_k,m)} = \emptyset.$$

Let $R_{\mathbf{n}}$ denote the right half of $K_{\mathbf{n}}$. Then, because of the way we have chosen in (2) the intervals always in the left half, the collection of these right half-intervals $\{R_{\mathbf{n}}\}$ is disjoint.

For any $\mathbf{n} = (n_1, n_2, n_3, \dots) \in \mathbb{N}^{\mathbb{N}}$, we write $x(\mathbf{n})$ for the unique point that lies in the intersection

$$\bigcap_{k=1}^{\infty} J_{(n_1,n_2,n_3,\dots,n_k)} = \bigcap_{k=1}^{\infty} K_{(n_1,n_2,n_3,\dots,n_k)} = \bigcap_{k=1}^{\infty} L_{(n_1,n_2,n_3,\dots,n_k)}.$$

For each tree T, we can use this system to construct a continuous function on $[0,1]$. For any interval $[a,b] \subset [0,1]$, write $\phi(x,[a,b])$ as

$$\phi(x,[a,b]) = \begin{cases} 16(x-a)^2(x-b)^2(b-a)^{-3} & \text{if } a \le x \le b; \\ 0 & \text{if } x < a \text{ or } b < x. \end{cases}$$

[5] D. Mauldin, *Pacific J. Math.* 121 (1986), 119-120.
[6] A. S. Kechris and W. H. Woodin, *Mathematika* 33 (1986), 252–278.

Then, if T is a tree, we can define

$$F_T(x) = \sum_{\mathbf{n} \in T} \phi(x, R_{\mathbf{n}}).$$

The maximum of the nonnegative function $\phi(x, [a, b])$ occurs at the midpoint of $[a, b]$ and has the value $(b - a)$. (See Exercise 11:9.1 and Figure 11.2.) Consequently,

$$|\phi(x, R_{\mathbf{n}})| \leq |R_{\mathbf{n}}| \leq 2^{-\langle \mathbf{n} \rangle},$$

and so it is clear that the series defining F_T is uniformly convergent to a continuous function on $[0, 1]$.

The mapping $T \to F_T$, considered as a mapping from the metric space $2^{\mathbb{N}^*}$ into $C[0, 1]$, is continuous. We claim that

$$T \in \mathcal{WF} \iff F_T \text{ is everywhere differentiable,} \tag{10}$$

and this, because of Theorem 11.23, will prove that the set D of differentiable functions is a complete co-analytic set in $C[0, 1]$.

The rest of the proof is now a real variables argument. To check (10) we show two things: (1) if $\mathbf{n}$ is an element of $[T]$ and $x_0 = x(\mathbf{n})$ (the corresponding point), then $F_T'(x_0)$ does not exist, and (2) if x_0 is a point of $[0, 1]$ such that for no element $\mathbf{n}$ of $[T]$ does $x_0 = x(\mathbf{n})$, then $F_T'(x_0)$ does exist.

To prove (1) suppose that there is an element $\mathbf{n} = (n_1, n_2, \dots) \in [T]$ and let $x_0 = x(\mathbf{n})$. Then x_0 belongs to $L_{(n_1, n_2, \dots, n_k)}$ for all k. Let ξ_k denote the midpoint of the interval $R_{(n_1, n_2, \dots, n_k)}$, and let η_k be half the length of that interval. Since x_0 belongs to none of the right-hand intervals of the system, $F_T(x_0) = 0$. Since $\xi_k + \eta_k$ is an endpoint of a right-hand interval, $F_T(\xi_k + \eta_k) = 0$. Also, since x_0 is in the corresponding left half $L_{(n_1, n_2, \dots, n_k)}$ of the interval $K_{(n_1, n_2, \dots, n_k)}$, we know that

$$|x_0 - \xi_k| < 3\eta_k,$$

and hence

$$F_T(\xi_k) = 2\eta_k \geq \tfrac{2}{3}|x_0 - \xi_k|.$$

This is enough to see that $F_T'(x_0)$ does not exist. For as $k \to \infty$,

$$\xi_k \to x_0 \text{ and } \xi_k + \eta_k \to x_0,$$

and yet

$$\frac{F_T(\xi_k + \eta_k) - F_T(x_0)}{\xi_k + \eta_k - x_0} = 0,$$

while

$$\left| \frac{F_T(\xi_k) - F_T(x_0)}{\xi_k - x_0} \right| \geq \tfrac{2}{3}.$$

Let us now prove (2). Suppose that x_0 is a point of $[0,1]$ such that for no element $\mathbf{n}$ of $[T]$ does $x_0 = x(\mathbf{n})$. Then there must be an integer N so that, whenever $\mathbf{n} \in T$ and $\langle \mathbf{n} \rangle \geq N$, x_0 does not belong to $J_{\mathbf{n}}$.

Let us first establish the estimate

$$\left| \frac{\phi(x_0 + t, R_{\mathbf{n}}) - \phi(x_0, R_{\mathbf{n}})}{t} \right| \leq 2^{-\langle \mathbf{n} \rangle} \tag{11}$$

for $\langle \mathbf{n} \rangle \geq N$. Since x_0 does not belong to $J_{\mathbf{n}}$, it must follow that $\phi(x_0, R_{\mathbf{n}}) = 0$. If also $x_0 + t$ does not belong to $R_{\mathbf{n}}$, it follows that

$$\phi(x_0 + t, R_{\mathbf{n}}) = 0$$

and (11) holds. On the other hand, if $x_0 + t$ does belong to $R_{\mathbf{n}}$ it follows that

$$\phi(x_0 + t, R_{\mathbf{n}}) \leq |R_{\mathbf{n}}| \leq |R_{\mathbf{n}}|(\tfrac{1}{2}|J_{\mathbf{n}}|)^{-1} t \leq 2^{-\langle \mathbf{n} \rangle} t,$$

and again (11) holds.

The function F_T can be expressed as a uniform limit of a sequence of continuous functions by writing

$$F_T(x) = \lim_{k \to \infty} F_T^k(x) = \lim_{k \to \infty} \sum_{\mathbf{n} \in T,\ \langle \mathbf{n} \rangle \leq k} \phi(x, R_{\mathbf{n}}).$$

Then, for $k \geq N$, we obtain from (11) that

$$\left| \frac{F_T(x_0 + t) - F_T(x_0)}{t} - \frac{F_T^k(x_0 + t) - F_T^k(x_0)}{t} \right|$$

$$\leq \sum_{m=k+1}^{\infty} 2^{-m} \leq 2^{-k}. \tag{12}$$

Thus we can obtain upper and lower estimates on the fraction

$$\left| \frac{F_T(x_0 + t) - F_T(x_0)}{t} \right|$$

by considering the fractions

$$\left| \frac{F_T^k(x_0 + t) - F_T^k(x_0)}{t} \right|.$$

As $t \to 0$, this latter fraction approaches a limit for each k, and we can conclude from this and (12) that $F_T'(x_0)$ does indeed exist.

Since (1) and (2) are now proved, the theorem is proved. ∎

Exercises

11:9.1 Prove that the function

$$\phi(x,[a,b]) = \begin{cases} 16(x-a)^2(x-b)^2(b-a)^{-3} & \text{if } a \le x \le b; \\ 0 & \text{if } x < a \text{ or } b < x. \end{cases}$$

is nonnegative, continuous, and differentiable on $[0,1]$ and has a maximum value of $(b-a)$. Show that the graph looks much like that given in Figure 11.2.

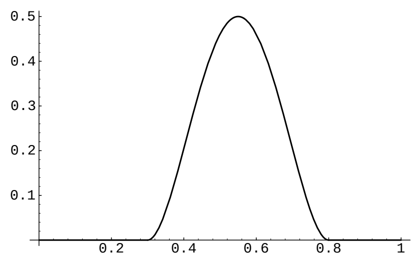

Figure 11.2: Graph of $\phi(x,[a,b])$ with $[a,b] = [0.3, 0.8]$ in Exercise 11:9.1

11.10 Additional Problems for Chapter 11

11:10.1 Let X and Y be complete, separable metric spaces. If $f : X \to Y$ is Borel measurable, then the graph

$$\text{graph}\,(f) = \{(x, f(x)) : x \in X\}$$

is a Borel subset of $X \times Y$. [Hint: Consider $F : X \times Y \to Y \times Y$ defined by $F(x,y) = (f(x), y)$ and $F^{-1}(\{(y,y) : y \in Y\})$.]

11:10.2 Let X and Y be complete, separable metric spaces. If $f : X \to Y$ and the graph

$$\text{graph}\,(f) = \{(x, f(x)) : x \in X\}$$

is a Borel subset of $X \times Y$, then f is Borel measurable. [Hint: For any Borel set $B \subset Y$, note that the sets graph $(f) \cap (X \times B)$ and graph $(f) \cap (X \times \widetilde{B})$ are Borel subsets of $X \times Y$ that project to $f^{-1}(B)$ and $f^{-1}(\widetilde{B})$.]

11:10.3 Let X and Y be complete, separable metric spaces. If $f : X \to Y$ and the graph

$$\text{graph}\,(f) = \{(x, f(x)) : x \in X\}$$

is an analytic subset of $X \times Y$, then f is Borel measurable.

11:10.4 Let X and Y be complete, separable metric spaces. If $f : X \to Y$ is a bijection and f is Borel measurable then f^{-1} is also Borel measurable. [Hint: Compare $\operatorname{graph}(f) \subset X \times Y$ and $\operatorname{graph}(f^{-1}) \subset Y \times X$.]

11:10.5 (Sets with the property of Baire) A subset S of a metric space X is said to have the *property of Baire* provided that it can be represented in the form

$$S = G \triangle E = (G \setminus E) \cup (E \setminus G),$$

where G is open and E is first category in X. Establish the following:

(a) S has the property of Baire if and only if it can be represented in the form $S = F \triangle E$, where F is closed and E is first category in X.

(b) The class of subsets of X having the property of Baire is a σ–algebra. This σ–algebra is precisely that σ–algebra generated by the class of all sets that are open or are of the first category.

(c) If X is a complete, separable metric space and S is analytic, then S has the property of Baire.

(d) Give an example of a set in a complete, separable metric space that is not analytic, but does have the property of Baire.

11:10.6 (Functions with the property of Baire) Let X and Y be metric spaces. A function $f : X \to Y$ is said to have the *property of Baire* provided that for every closed set $F \subset Y$ the set $f^{-1}(F)$ has the property of Baire as a subset of X. Assume that X and Y are complete, separable metric spaces and prove the following:

(a) Every Borel measurable function $f : X \to Y$ has the property of Baire.

(b) A necessary and sufficient condition that a function $f : X \to Y$ has the property of Baire is that there is a set $E \subset X$ of first category so that the restriction of f to $X \setminus E$ is continuous.

(c) A set $S \subset X$ has the property of Baire if and only if the characteristic function χ_S has the property of Baire.

(d) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is not Borel measurable, but does have the property of Baire.

(e) Prove the following theorem. Explain why it can be interpreted as a category analog of Lusin's theorem (Theorem 4.25).

> **Theorem** *Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then there exists a set H residual in $\mathbb{R}$ so that the restriction of f to H is continuous.*

Show that this statement is not true for an arbitrary Lebesgue measurable function.

(f) Show that a Bernstein subset of $\mathbb{R}$ does not have the property of Baire. (See Exercise 1:22.7.)

Chapter 12

BANACH SPACES

Many of the methods of Banach space theory can be found in the early years of this century in works of M. Fréchet, F. Riesz, Lebesgue, and others. The formal study of these spaces began with the axiomatic treatment given by Stefan Banach (1892–1945) in his doctor's thesis of June 1920, and his name has been attached to this study every since. Norbert Wiener (1894–1964) published an identical study only a few months later, but conceded priority to Banach; he then withdrew from the subject as he was not of a temperament to face competition. The fact that two mathematicians of their stature would independently develop the same structure at the same time shows that the time for these ideas had very much come.

In Chapter 9 we saw a great many examples of important metric spaces, many of which had a feature that we did not much comment upon but did play a role in some of the applications. In these spaces, natural notions of addition of two members of the space and of multiplication of members by real numbers endow them with the structure of a *linear* space. The metric was in all cases invariant under this addition, and the balls in the space had a special geometric property: they were convex. By the time that Banach and Wiener were ready to axiomatize the theory, there were many studies in the analysis of the years 1900 to 1920 that exploited an interconnection between metric space arguments and the geometry of linear spaces. These studies are now considered the beginnings of a major area of modern mathematics, functional analysis. In this chapter we begin the study of some of the most important concepts of functional analysis: normed linear spaces, Banach spaces, linear operators, and linear functionals. This is just a brief introduction to a field that is now highly developed.

12.1 Normed Linear Spaces

We assume that the reader is familiar with the concept of a linear space (vector space). The important examples of linear spaces in analysis have the real or complex numbers as the scalar field. For our short introduction to the subject, we shall assume this field to be $\mathbb{R}$ unless we specifically ask for complex scalars; later we emphasize the complex case. We shall denote the origin of X by $\mathbf{0}$.

Definition 12.1 Let X be a linear space. A *norm* on X is a nonnegative real-valued function, written $\|x\|$, such that

1. $\|x\| = 0$ if and only if $x = \mathbf{0}$,
2. $\|ax\| = |a|\|x\|$ for all $a \in \mathbb{R}$ and $x \in X$, and
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Condition (3) is called the triangle inequality.

A linear space X furnished with a norm $\|x\|$ is called a *normed linear space*. Every normed linear space gives rise to the metric

$$\rho(x, y) = \|x - y\|.$$

The norm properties easily show that this is a metric on X. It is called the *metric induced by the norm*. One verifies easily that ρ is also an *invariant metric*; that is, it is invariant under translation:

$$\rho(x, y) = \rho(x + z, y + z) \quad \text{for all } x, y, z \in X.$$

There is a close interplay between metric properties and linear properties. The mappings $(a, x) \to ax$ and $(x, y) \to x + y$ are linear and continuous; the norm is a continuous real-valued function on X. The most important examples of normed linear spaces are complete, and these are referred to as Banach spaces.

Definition 12.2 A *Banach* space is a complete normed linear space.

A linear space X furnished with a metric is not necessarily a normed linear space. As an example, the space $\mathcal{S}[0, 1]$ (Example 9.9 and repeated in Example 12.11) is a metric linear space, but the metric is not induced by a norm since, in general,

$$\int_0^1 \frac{|af|}{1 + |af|}\, d\lambda \neq |a| \int_0^1 \frac{|f|}{1 + |f|}\, d\lambda.$$

Observe that the unit ball $B(\mathbf{0},1)$ is not convex. (The unit ball in a normed linear space must be convex; see Exercise 12:1.3.) This is an example of a metric linear space that is not, however, a normed linear space.

Definition 12.3 A *metric linear space* is a linear space furnished with a metric $\rho(x, y)$ that is invariant under translation,

$$\rho(x, y) = \rho(x + z, y + z) \quad \text{for all } x, y, z \in X,$$

and such that $a_n \to a$ and $x_n \to x$ imply that $a_n x_n \to ax$.

Note that in a metric linear space the mappings $(a, x) \to ax$, $(x, y) \to x + y$, and $(x, y) \to \rho(x, y)$ are continuous. The major fundamental difference between a general metric linear space and a normed linear space is that the balls in the former need not be convex. This small geometric detail is behind almost all the results that we obtain in this chapter. Our attention to general metric linear spaces is minimal: we indicate occasionally that some space that we are studying has this structure, but is not a normed linear space. While the literature of normed linear spaces greatly exceeds that for metric linear spaces, the latter have an important role to play, too, in many applications of analysis (e.g., in probability theory, integral operators, analytic functions, and Fourier series).

We list some familiar metric linear spaces and Banach spaces, many of which will play a role in the applications that we develop. In most cases we will leave the verifications to the reader. For those spaces discussed in earlier chapters, we just point out that the metric given before yields a metric linear space or, better, in most cases a Banach space.

Sequence spaces

Nearly all the examples of sequence spaces presented in Section 9.1 as metric spaces are metric linear spaces; most are Banach spaces.

Example 12.4 We denote by s the set of all sequences of real numbers equipped with the metric

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}.$$

This is a metric linear space, but the metric is not defined by a norm and so the space is not a normed linear space. It is complete, but not a Banach space. Recall that to verify that this is a metric linear space one must show that the metric is invariant (obvious) and that scalar multiplication is continuous.

The space $2^{\mathbb{N}}$, the set of all sequences of 0's and 1's, was equipped with the same metric, but is not a linear space and so does not enter into our discussion here.

Example 12.5 The sequence space ℓ_p $(1 \leq p < \infty)$ is the collection of all sequences $x = (x_1, x_2, x_3 \dots)$ of real numbers such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$ and furnished with the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}.$$

Both to check that ℓ_p is a linear space and to verify that $\|x\|_p$ is a norm requires Minkowski's inequality. Since these spaces are complete, they are Banach spaces. We verify these statements in Chapter 13.

Example 12.6 The sequence space ℓ_∞ is the set of all bounded sequences of real numbers and furnished with the norm

$$\|x\|_\infty = \sup_i |x_i|.$$

It is easy to see that this is a linear space and that $\|x\|_\infty$ is a norm. Since this space is complete, it too is a Banach space.

We have previously discussed some important subspaces of ℓ_∞: c (the space of convergent sequences) and c_0 (the space of sequences converging to zero). Since these are linear subspaces and closed, they too are Banach spaces.

Function Spaces

We have seen all the examples in this collection before: they are metric linear spaces formed of real-valued functions; some are Banach spaces.

Example 12.7 By $M[a, b]$, we mean the set of all bounded real-valued functions on the closed interval $[a, b]$ with norm

$$\|f\|_\infty = \sup_{a \le t \le b} |f(t)|.$$

This is often called the *sup norm.* It is immediate that $M[a, b]$ is a linear space and that $\|f\|_\infty$ is a norm. Since we already know that $M[a, b]$ is complete, it is a Banach space.

Some important subspaces of $M[a, b]$ that we have encountered already are

1. $\mathcal{C}[a, b]$, the space of continuous functions,
2. $\triangle[a, b]$, the space of differentiable functions,
3. $\mathcal{P}[a, b]$, the space of polynomials, and
4. $\mathcal{R}[a, b]$, the space of Riemann integrable functions.

Each of these is a linear subspace of $M[a, b]$ and so all are normed linear spaces; only the subspaces $\mathcal{C}[a, b]$ and $\mathcal{R}[a, b]$ are closed, and so they are the only Banach spaces here.

There is nothing special about the domain of these functions being an interval $[a, b]$: any nonempty set A can be the domain for a set of bounded functions, and the space $M(A)$ of bounded functions on A becomes a Banach space under the norm

$$\|f\|_\infty = \sup_{t \in A} |f(t)|.$$

Example 12.8 The space $L_1(X, \mathcal{M}, \mu)$ is furnished with the norm

$$\|f\|_1 = \int_X |f|\, d\mu.$$

With this norm, $\|f\|_1 = 0$ if and only if $f = 0$ a.e., so condition (1) of Definition 12.1 for a norm fails. All the other properties of a norm do hold. We have already encountered this problem in the setting of metric spaces, and we know what to do: we identify equivalent functions. If $f = g$ a.e., we consider f and g to be the same element of the space. Thus, as before, L_1 does not consist of functions, but equivalence classes of functions defined by the relation $f \sim g$ if $f = g$ a.e. In a more formal treatment we would be obliged now to show that the norm $\|f\|_1$ remains unchanged if f is replaced by any other equivalent function, and this is obvious.

Example 12.9 Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $0 < p < \infty$. By $L_p(X, \mathcal{M}, \mu)$ or merely $L_p(\mu)$, we denote those measurable functions defined on X such that

$$\int_X |f|^p\, d\mu < \infty.$$

Once again we identify equivalent functions.

That this is a linear space follows from the inequality

$$(a + b)^p \leq 2^p(a^p + b^p)$$

for a, $b \geq 0$. If f, $g \in L_p(\mu)$ then, integrating the inequality

$$|f(x) + g(x)|^p \leq 2^p \left(|f(x)|^p + |g(x)|^p\right),$$

we see that $f + g \in L_p(\mu)$.

For $1 \leq p < \infty$, we use the p norm

$$\|f\|_p = \left(\int_X |f|^p\, d\mu\right)^{1/p},$$

and we shall see in Chapter 13 that these are Banach spaces.

The expression $\|f\|_p$ is not a norm if $0 < p < 1$, and these spaces need special treatment: we use the metrics

$$\rho_p(f - g) = \int_X |f - g|^p\, d\mu$$

and, so equipped, these are complete metric linear spaces, but not Banach spaces.

A treatment of the $L_p(X, \mathcal{M}, \mu)$ spaces can be found in Chapter 13. They have been presented in this chapter just for early reference, but we shall not develop any of the special properties here.

Example 12.10 Let $(X, \mathcal{M}, \mu)$ be a measure space. For any measurable function write

$$\|f\|_\infty = \operatorname{ess\,sup} |f(x)| = \inf \{t > 0 : \mu(\{x : |f(x)| > t\}) = 0\} \qquad (1)$$

and refer to this as the *essential supremum* or ∞ *norm* of the function f. By $L_\infty(X, \mathcal{M}, \mu)$ or merely $L_\infty(\mu)$, we denote those measurable functions defined on X such that $\|f\|_\infty < \infty$. Such functions are said to be *essentially bounded*; they are bounded if a set of measure zero can be ignored. Again, as usual for function spaces associated with measure theory, we identify functions that are equal almost everywhere with respect to the underlying measure. Then $\|f\|_\infty = 0$ only for the zero function. Note too, that every essentially bounded function can be identified in this space with an equivalent bounded function.

It is easy to check that $L_\infty(\mu)$ is a linear space and that $\|f\|_\infty$ is a norm; like the other L_p spaces, this too is a Banach space (see Chapter 13).

Example 12.11 Let $\mathcal{S}$ denote the measurable functions on [0,1] furnished with the metric

$$\rho(f, g) = \int_0^1 \frac{|f - g|}{1 + |f - g|}\, d\lambda.$$

Provided that we identify functions that are a.e. equal, as we normally do in function spaces associated with measures, this becomes a complete metric linear space. It is not a Banach space.

Example 12.12 We denote by BV=BV$[a, b]$, the set of functions of bounded variation on $[a, b]$, furnished with the norm

$$\|f\| = |f(a)| + V(f; [a, b]).$$

(The variation $V(f; [a, b])$ of a function f on an interval $[a, b]$ has been defined in Section 1.14.) This can be proved to be a Banach space and plays an important role in many investigations.

A special subspace of this space will be used in Section 12.8. By NBV$[a, b]$, we denote the space of those functions f of bounded variation on $[a, b]$ that are right continuous on (a, b) and that satisfy $f(a) = 0$. The norm is that inherited as a subspace, and so it is evidently given by

$$\|f\| = V(f; [a, b]).$$

This too is a Banach space. The N in the name is meant to indicate that the functions have been "normalized" by selecting a right continuous member that vanishes at the left end of the interval.

Example 12.13 Let $\mathcal{C}' = \mathcal{C}'[a, b]$ denote the set of continuously differentiable functions on $[a, b]$. We furnish the space with the norm

$$\|f\| = \max_{a \le t \le b} |f(t)| + \max_{a \le t \le b} |f'(t)|.$$

To verify that this is a norm is similar to checking that the sup norm has the correct properties in $M[a, b]$. It is an instructive exercise to check that this space is complete.

Exercises

12:1.1 Verify that a metric induced by a norm is invariant and that the norm is a continuous function on X.

12:1.2 Verify that the mapping $(x, y) \to x + y$ is continuous (i.e., $x_n \to x$ and $y_n \to y$ imply that $x_n + y_n \to x + y$) both in a normed linear space and in a metric linear space.

12:1.3 Prove that the unit ball of a normed linear space is convex.

12:1.4 Check the assertions made in this section about the examples. Are they linear spaces? Do the norms satisfy the conditions of Definition 12.1? Are the metrics invariant? Are the spaces claimed to be Banach spaces complete? (Not all the verifications are elementary; for the L_p–spaces the tools developed in Chapter 13 are needed.)

12.2 Compactness

One of our most successful tools in the study of metric spaces has been the notion of compactness. Naturally, we would expect that compact sets play an important role in the study of Banach spaces, too. Indeed they do, but there are a few surprises here. In contrast to the Euclidean spaces $\mathbb{R}^n$, closed and bounded sets are not compact in an arbitrary Banach space. Compact sets are quite a bit "smaller" than one might have expected. The unit ball in an infinite-dimensional Banach space has a great deal of room for moving about.

Here we shall collect a few interesting observations that arise from questions related to the notion of compactness as it appears in Banach space theory. Let Y be a proper subspace of a normed linear space, and let $x \in X$ be a point not in Y.

> **Problem 1.** Does there exist a point in Y that is nearest to the point x?
>
> **Problem 2.** Does there exist a point on the unit sphere $\{z : \|z\| = 1\}$ that is farthest from all points in Y?

In the finite-dimensional spaces $\mathbb{R}^n$, the geometry is transparent enough to see how to answer these. There is a point in Y that is nearest to the point x; take the orthogonal projection of x on the subspace Y. There is a point on the unit sphere $\{z : \|z\| = 1\}$ that is farthest from all points in Y; just head in an orthogonal direction from the origin to find a point at distance 1. In an infinite-dimensional space, what happens?

Let us look at problem (2) first. An example shows that there is no such point, in general, even if we make the natural assumption that the subspace is closed (all subspaces of finite-dimensional spaces are closed anyway).

Example 12.14 Consider the subspaces X and Y of $\mathcal{C}([0, 1])$, where X is that proper subspace consisting of those functions vanishing at the left endpoint of $[0,1]$, and Y is that proper subspace of X consisting of those functions f with $f(0)=0$ and $\int_0^1 f(t)\,dt = 0$. These are closed subspaces. Is there a function $f_1 \in X$ with $\|f_1\| = 1$ such that $\inf\{\|f - f_1\| : f \in Y\} = 1$? Suppose that such a function f_1 has been chosen; since $f_1(0) = 0$, $|f_1(t)| \leq 1$, we note that

$$-1 < \alpha = \int_0^1 f_1(t)\,dt < 1. \tag{2}$$

Now for any $\delta < 1$ we can choose $f \in X$ with $\|f\| = 1$ and $\beta_f = \int_0^1 f(t)\,dt > \delta$. The function $f_1 - (\alpha/\beta_f)f$ is in Y (check the integral), and hence

$$1 \leq \|f_1 - (f_1 - (\alpha/\beta_f)f)\| = |\alpha/\beta_f|$$

so that $|\alpha| > \beta_f > \delta$. Since $\delta < 1$ is arbitrary, this contradicts (2).

From this example we see that we cannot find a point on the sphere quite as distant from the subspace as we might have hoped. A simple computation of F. Riesz shows that we can get close.

Lemma 12.15 (Riesz) *Let X be a normed linear space and Y a proper closed subspace. Then for every $0 < \delta < 1$ there is an element $x_\delta \in X$ with $\|x_\delta\| = 1$ and such that dist$(x_\delta, Y) \geq \delta$.*

Proof. There is a point x_1 not in Y and at a positive distance from Y, say $d = \text{dist}(x_1, Y)$. There must be a point $x_0 \in Y$ close enough to x_1 so that $\|x_1 - x_0\| \leq d\delta^{-1}$. We merely take

$$x_\delta = \frac{1}{\|x_1 - x_0\|}(x_1 - x_0).$$

To check that this works, we use an arbitrary $y \in Y$ and obtain

$$\|x_\delta - y\| = \frac{1}{\|x_1 - x_0\|}\,\|x_1 - (x_0 + \|x_1 - x_0\|\,y)\| \geq \frac{d}{d\delta^{-1}} = \delta,$$

since $x_0 + \|x_1 - x_0\|\,y$ is an element of the subspace Y. Hence $\text{dist}(x_\delta, Y) \geq \delta$, as required. ∎

Theorem 12.16 (Riesz) *A necessary and sufficient condition that a normed linear space be finite-dimensional is that the the closed unit ball is compact.*

Proof. If X is a finite-dimensional normed linear space, then it is easy to check that the closed unit ball is compact (for the same reason that the closed unit ball in $\mathbb{R}^n$ is compact).

Conversely, suppose that X is infinite-dimensional. Choose an element x_1 with $\|x_1\| = 1$. By applying Lemma 12.15 to the subspace Y_1 spanned by x_1, there is an element x_2 with $\|x_2\| = 1$ and $\text{dist}(x_2, Y_1) \geq \frac{1}{2}$. Once again applying the lemma to the subspace Y_2 spanned by $\{x_1, x_2\}$, there is an element x_3 with $\|x_3\| = 1$ and $\text{dist}(x_3, Y_2,) \geq \frac{1}{2}$. Continuing inductively, we obtain a sequence such that each pair of its members is at a distance apart of at least $\frac{1}{2}$. As X is infinite-dimensional, this process cannot stop. Since such a sequence can have no convergent subsequence, the unit ball cannot be compact. ■

A corollary expresses the theorem in a different way.

Corollary 12.17 *A compact set in an infinite-dimensional normed linear space is nowhere dense.*

Proof. If K is compact, then it is closed. If it has an interior point, then it contains a closed ball. But the proof of the theorem applies to a ball of any radius and center, not just the unit ball, and so K contains a noncompact closed subset, which is impossible. ■

We turn now to problem (1) at the beginning of this section. We can use compactness arguments to answer best-approximation problems. We have seen though that compactness arguments in infinite dimensions will require some caution, since closed, bounded sets are not necessarily compact. Our best-approximation problem will avoid this by seeking a best-approximation from a finite-dimensional subspace. The problem is that we are given an element x in a Banach space X and a finite-dimensional subspace $Y \subset X$. We know that $\operatorname{dist}(x, Y)$ will be finite, and we hope to find an element $y_0 \in Y$ for which the distance is realized:

$$\operatorname{dist}(x, Y) = \|x - y_0\| \ .$$

Theorem 12.18 *Let Y be a finite-dimensional subspace of a normed linear space X. Then for every element $x_0 \in X$ not in Y there is $y_0 \in Y$ with $dist(x_0, Y) = \|x_0 - y_0\|$.*

Proof. Let $d = \operatorname{dist}(x_0, Y)$. We look for a nearest point to x_0 that is in Y and also somewhere in the closed ball $B[x_0, 2d]$ (the nearest point cannot be anywhere else). But the set

$$K = B[x_0, 2d] \cap Y$$

is a closed, bounded set in the finite-dimensional space Y and hence is compact. Ordinary metric space arguments (Exercise 9:9.11) supply a point $y_0 \in K$ with $\operatorname{dist}(x_0, K) = \|x_0 - y_0\|$, and this is exactly what we wanted. ■

Example 12.19 Let f be a continuous function on the interval $[a, b]$ and determine

$$\min_{\lambda_0,\lambda_1,\lambda_2,\lambda_3} \max_{t\in[a,b]} \left|f(t) - \lambda_0 - \lambda_1 t - \lambda_2 t^2 - \lambda_3 t^3\right|$$

and

$$\min_{\lambda_0,\lambda_1,\lambda_2,\lambda_3} \int_a^b \left|f(t) - \lambda_0 - \lambda_1 t - \lambda_2 t^2 - \lambda_3 t^3\right| \, dt.$$

Without our current (admittedly elementary) tools, we might be in doubt as to whether these minima exist. It is easy to interpret these as best-approximation problems for an element of the space $\mathcal{C}[a,b]$ (and the space $L_1[a,b]$ relative to a four-dimensional subspace and so apply Theorem 12.18 to show that these exist. Note, however, that the two answers may differ, and there is no guarantee that there is a *unique* cubic polynomial that expresses the approximation.

In many cases, one would like uniqueness and not just existence of a best approximation. For that the geometry of the norm comes into play. Here are two conditions that would allow us to claim uniqueness. A norm is said to be *rotund* if for every distinct x, y with $\|x\| = \|y\| = 1$ there is the strict inequality

$$\|x + y\| < 2.$$

A norm is said to satisfy the *parallelogram rule* if

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for every distinct x, y in the space. Under either of these conditions the existence of a best approximation guarantees the uniqueness. We leave this as Exercise 12:2.1. The parallelogram rule holds only in normed linear spaces in which the norm is supplied by an inner product. This is the subject of Chapter 14.

Exercises

12:2.1 Let C be a convex set in a normed linear space and suppose that $x_0 \notin C$ and that there is a point $y_0 \in C$ with

$$\operatorname{dist}(x_0, C) = \|x_0 - y_0\|.$$

If the norm is rotund, show that such a point is unique. If the norm satisfies the parallelogram rule, show that such a point is unique.

12:2.2 The set

$$I^\infty = \{(x_1, x_2, x_3, \dots) : |x_i| \le i^{-1}\}$$

is called the Hilbert cube. Show that this is a compact subset of ℓ_2 and verify directly that it is nowhere dense.

12:2.3 Let X be a normed linear space and A and B two closed subsets. The "sum" is the set

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

Show that $A + B$ need not be closed. Show that $A + B$ is closed if A is compact.

12.3 Linear Operators

When dealing with a linear space, one is often interested in linear operators defined on that space. Several of the contraction maps we mentioned in Section 9.7 provide examples of linear operators. We give here an introduction to the theory of linear operators on a normed linear space. The most important and deep theorems about linear operators are collected in later sections.

Definition 12.20 Let X and Y be linear spaces. Let $T : X \to Y$. If

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$$

for all $\alpha_1, \alpha_2 \in \mathbb{R}$ and $x_1, x_2 \in X$, we say T is a *linear operator* or a *linear transformation* from X into Y.

When $Y = \mathbb{R}$, we say that T is a *linear functional* on X. We might prefer a lowercase letter to indicate linear functionals. Thus a mapping $h : X \to \mathbb{R}$ is a linear functional if

$$h(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 h(x_1) + \alpha_2 h(x_2)$$

for all $\alpha_1, \alpha_2 \in \mathbb{R}$ and $x_1, x_2 \in X$. Linear functionals are quite special, and their investigation can proceed differently from just general linear operators. For the rudiments of the theory appearing in this section, the same arguments work for both. A deeper study of continuous linear functionals is given in Sections 12.5, 12.6, and 12.7.

A number of familiar operators involving integration and differentiation are linear. For example, $Df = f'$ is a linear operator from $\triangle$ to $\triangle'$,

$$(Sf)(x) = \int_a^x f \, d\lambda$$

is linear from $\mathcal{C}[a, b]$ to $\mathcal{C}[a, b]$ or from $L_1[a, b]$ to $\mathcal{C}[a, b]$, and

$$I(f) = \int_a^b f \, d\lambda$$

is a linear functional on $L_1[a, b]$.

If X and Y are normed linear spaces, we can discuss continuity of a linear operator. The integral operators S and I, above, are continuous everywhere, while the differential operator D is discontinuous everywhere on the space of differentiable functions with bounded derivatives when both the domain and range of D are furnished with the sup norm. (We leave verification of these statements as Exercise 12:3.1.) Our first observation is that continuity for linear operators is already a very special thing: continuity at a single point ensures continuity everywhere and, in fact, it ensures uniform continuity on the space.

Theorem 12.21 *Let X and Y be normed linear spaces, and let $T : X \to Y$ be a linear operator. If T is continuous at a point, then T is continuous everywhere, and the continuity is uniform.*

Proof. For $x_1, x_2 \in X$ we have

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\|.$$

Now T is continuous at some point, which by translation invariance we may take to be $\mathbf{0}$. Thus for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x\| < \delta$ then $\|Tx\| < \varepsilon$. It follows that if $\|x_1 - x_2\| < \delta$ then

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| < \varepsilon.$$

■

The requirement that T be linear allows a characterization of continuity in terms of how T maps bounded sets.

Definition 12.22 A linear operator $T : X \to Y$ is *bounded* if there exists $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in X$. The *operator norm* for a bounded linear operator T is defined as

$$\|T\| = \inf \{M : \|Tx\| \leq M\|x\| \ \text{ for all } x \in X\}.$$

For all $x \neq \mathbf{0}$, we have

$$Tx = T\left(\frac{x}{\|x\|}\right)\|x\|.$$

It follows that T is bounded if and only if T is a bounded function on the closed unit ball $\{x : \|x\| \leq 1\}$ or, equivalently, T is a bounded function on the unit sphere $\{x : \|x\| = 1\}$. Thus a bounded linear operator is one that maps bounded sets to bounded sets. The operator norm also expresses this boundedness:

$$\|Tx\| \leq \|T\| \, \|x\| \quad (\text{for all } x \in X).$$

We have referred to this as the "operator norm," but we have yet to see that it is a genuine norm on some linear space. We shall develop these ideas after we have proved that boundedness and continuity are equivalent for linear operators.

Theorem 12.23 *A linear operator is bounded if and only if it is continuous.*

Proof. Suppose that T is a bounded linear operator with bound M. Let $\varepsilon > 0$, and let $\delta = \varepsilon/M$. If $\|x\| < \delta$, then

$$\|Tx\| \leq M\|x\| < M\delta = \varepsilon,$$

so T is continuous at $x = \mathbf{0}$. It follows from Theorem 12.21 that T is continuous on X.

To prove the converse, suppose that T is continuous. Choose $\delta > 0$ such that $\|Tx\| < 1$ if $\|x\| = \delta$. For $x \in X$, $x \neq \mathbf{0}$, we have

$$\left\| \frac{\delta x}{\|x\|} \right\| = \delta.$$

Thus

$$\|Tx\| = \left\| T\left(\frac{\delta x}{\|x\|}\right) \right\| \frac{\|x\|}{\delta} < \frac{1}{\delta}\|x\|,$$

so T is bounded. ■

Now let X and Y be normed linear spaces, and let $B(X, Y)$ be the set of bounded linear operators from X to Y. We shall interpret $B(X, Y)$ as a normed linear space using the operator norm of Definition 12.22. That $B(X, Y)$ is a linear space under the usual interpretations is straightforward: for $T_1, T_2 \in B(X, Y)$ define $T_1 + T_2$ by

$$(T_1 + T_2)(x) = T_1(x) + T_2(x) \quad \text{for all} \quad x \in X,$$

and for $\alpha \in \mathbb{R}$ define αT_1 by

$$(\alpha T_1)(x) = \alpha T_1(x) \quad \text{for all} \quad x \in X.$$

To show that $B(X, Y)$ is linear, we must show that it is closed under addition and multiplication by scalars: linear combinations of bounded operators must themselves be bounded. Let T, T_1, $T_2 \in B(X, Y)$, and let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \|T_1(x) + T_2(x)\| &\leq \|T_1(x)\| + \|T_2(x)\| \\ &\leq \|T_1\|\,\|x\| + \|T_2\|\,\|x\| \\ &= (\|T_1\| + \|T_2\|)\|x\| \end{aligned}$$

for all $x \in X$. Thus

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|, \tag{3}$$

and so the sum $T_1 + T_2$ is bounded. It is clear that

$$\|\alpha T\| = |\alpha|\|T\|, \tag{4}$$

so the scalar multiple αT is bounded.

We have shown that $B(X, Y)$ is a linear space; it is also a normed linear space when furnished with the operator norm. It is clear that $\|T\| = 0$ if and only if $T = \mathbf{0}$. The remaining properties of a norm can be seen from (3) and (4).

We next show that $B(X, Y)$ is a Banach space if Y is complete.

Theorem 12.24 *Let $B(X, Y)$ be the normed linear space of bounded linear operators from a normed linear space X to a Banach space Y. Then $B(X, Y)$ is a Banach space.*

Proof. Let $\{T_n\}$ be a Cauchy sequence in $B(X,Y)$. Then, for all $x \in X$, $\{T_n x\}$ is a Cauchy sequence in Y, since

$$\|T_m x - T_n x\| \leq \|T_m - T_n\| \, \|x\|.$$

Define T by

$$Tx = \lim_{n\to\infty} T_n x.$$

We show that $T \in B(X,Y)$ and $\{T_n\}$ converges to T in $B(X,Y)$.

First, it is clear that T is a linear operator. To see that T is bounded, observe that the sequence $\{\|T_n\|\}$ is a bounded sequence of real numbers. This follows immediately from the fact that every Cauchy sequence in a metric space is bounded. Thus there exists M such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$. It follows that, for all $x \in X$, $\|T_n x\| \leq M\|x\|$, so $\|Tx\| \leq M\|x\|$, and $\|T\| \leq M$. This shows that T is bounded.

Finally, we show that $\|T_m - T\| \to 0$ as $m \to \infty$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\|T_m - T_n\| < \varepsilon$ if $m, n \geq N$. Then

$$\|T_m x - T_n x\| < \varepsilon\|x\| \quad \text{for } m, n \geq N \quad \text{and } x \in X.$$

Fixing $m \geq N$ and letting $n \to \infty$, we infer that

$$\|T_m x - Tx\| \leq \varepsilon\|x\| \quad \text{for all } x \in X.$$

This implies that $\|T_m - T\| \leq \varepsilon$ for all $m \geq N$.

We have shown that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|T_m - T\| \leq \varepsilon$ for all $m \geq N$; that is, $T_m \to T$ in $B(X,Y)$. Thus every Cauchy sequence in $B(X,Y)$ converges, so $B(X,Y)$ is complete and hence a Banach space. ■

We end this section with two remarks. First, we have used the same symbol $\| \, . \, \|$ for norms in the spaces X, Y and $B(X,Y)$. This standard practice should cause no confusion. In constructing proofs the reader may wish to label the norms in some way so as to keep oriented.

Our second remark relates to the scalar field. We have chosen $\mathbb{R}$ rather than $\mathbb{C}$, the set of complex numbers. Most of the development so far would be essentially the same had we chosen $\mathbb{C}$.

Exercises

12:3.1 Define the linear operators D, S, and I on the appropriate spaces of functions by $Df = f'$, $(Sf)(x) = \int_a^x f\,d\lambda$ and $I(f) = \int_a^b f\,d\lambda$. Show that S and I are continuous operators, while D is discontinuous.

12:3.2 Let X and Y be normed linear spaces with X finite-dimensional. Show that every linear mapping $T : X \to Y$ must belong to $B(X,Y)$.

12:3.3 Let X and Y be normed linear spaces with X infinite-dimensional. Show that there must exist a linear mapping $T : X \to Y$ that does not belong to $B(X,Y)$.

12.4 Banach Algebras

Let X be a Banach space. According to Theorem 12.24, the space $B(X, X)$ of bounded linear operators on X is also a Banach space. But there is more structure here: we can also define a multiplication operation. For $S, T \in B(X, X)$, let $ST = S \circ T$. It is clear that ST is a linear operator. To show that $ST \in B(X, X)$ observe that, for each $x \in X$,

$$\|ST(x)\| = \|S(Tx)\| \leq \|S\| \, \|Tx\| \leq \|S\| \, \|T\| \, \|x\|,$$

so

$$\|ST\| \leq \|S\| \, \|T\|. \tag{5}$$

Thus ST is a bounded operator on X. It is also clear that $\|I\| = 1$, where I is the identity operator defined by $Ix = x$ for all $x \in X$. It now follows readily that $B(X, X)$ is an algebra with unit I. In fact, $B(X, X)$ is a Banach algebra under the following definition.

Definition 12.25 A *Banach algebra* is a Banach space B on which is defined a multiplication operation that satisfies the following conditions:

1. The multiplication operation is associative; that is,
$$S(TU) = (ST)U \qquad (S, T, U \in B).$$
2. The multiplication operation is distributive; that is,
$$S(T + U) = ST + SU\,, \quad (S + T)U = SU + TU \qquad (S, T, U \in B).$$
3. Scalar multiplication associates with the multiplication operation, that is
$$(\lambda S)T = \lambda(ST) = S(\lambda T)$$
for every S, $T, \in B$, and every real (or complex) scalar λ.
4. The norm satisfies
$$\|ST\| \leq \|S\| \, \|T\| \qquad (S, T \in B).$$

If there is an element I of B such that $IS = SI = S$ for all $S \in B$ we say I is a *unit* for B. There is a highly developed field devoted to the study and applications of Banach algebras. We pause here only to prove one simple fact and apply it to a solution of an integral equation in Example 12.27. We shall return to these ideas in Section 13.9, where we shall see that with an appropriate multiplication operation the Banach space $L_1(\mathbb{R})$ is also a Banach algebra.

The reader is familiar with the fact that if a is a complex number with $|a| < 1$ then $(1 - a)^{-1} = 1 + a + a^2 + \cdots$. It is of interest that a similar representation is valid in the setting of a Banach algebra.

Theorem 12.26 *Let B be a Banach algebra with unit element I, and let $T \in B$. Suppose that $\|T\| < 1$. Then*

1. $(I-T)^{-1}$ *exists,*
2. $(I-T)^{-1} = I + T + T^2 + \cdots$, *and*
3. $\|(I-T)^{-1}\| \le (1-\|T\|)^{-1}$. *Thus* $(I-T)^{-1} \in B$.

Proof. Observe first that the series $\sum_{j=0}^{\infty} \|T\|^j$ converges, since $\|T\| < 1$. By equation (5), $\|T^n\| \le \|T\|^n$, so the sequence $\left\{\sum_{j=0}^{n} T^j\right\}$ is Cauchy and hence the series in (2) must converge. Let

$$S = I + T + T^2 + T^3 + \cdots .$$

Then $TS = ST = T + T^2 + \cdots$, so $(I-T)S = S(I-T) = I$. This shows that $S = (I-T)^{-1}$, establishing (1) and (2).

Finally, to verify (3) we calculate

$$\|(I-T)^{-1}\| \le \sum_{n=0}^{\infty} \|T^n\| \le \sum_{n=0}^{\infty} \|T\|^n = \frac{1}{1-\|T\|},$$

as required. ■

We shall apply this theorem to the Banach algebra $B(X,X)$, where X is the Banach space $\mathcal{C}[a,b]$ of continuous functions on the interval $[a,b]$, furnished with the usual sup norm. This example is in the same setting as Example 9.48 involving Fredholm equations. Here, however, we shall use material of this section instead of contraction mappings to prove the existence of a solution and exhibit how it can be obtained.

Example 12.27 Let $X = \mathcal{C}[a,b]$, and let K be continuous on the square $[a,b] \times [a,b]$. Define an operator A on X by $Af = g$, where

$$g(x) = \int_a^b K(x,y) f(y)\, dy.$$

It is clear that $A \in B(X,X)$. Consider the equation

$$\phi(x) = f(x) - \int_a^b K(x,y) f(y)\, dy, \tag{6}$$

for a fixed $\phi \in \mathcal{C}[a,b]$. We can write this in the more suggestive form

$$\phi = f - Af = (I-A)f \quad \text{or} \quad (I-A)^{-1}\phi = f.$$

If $\|A\| < 1$, we can infer from Theorem 12.26 that this equation has the unique solution

$$f = \phi + A\phi + A^2\phi + \cdots . \tag{7}$$

Let
$$M = \sup\{|K(x,y)| : x, y \in [a,b]\}.$$
From the definition of the operator A, we see that $\|A\| \le M(b-a)$; thus $\|A\| < 1$ if $M < (b-a)^{-1}$. Observe that we have obtained precisely the same sufficient condition for a unique solution to (6) here as we obtained in Example 9.48, but by different methods.

The operator A in Example 12.27 has another property that we should mention, and this property holds independently of the norm of $\|A\|$. Let $\{f_n\}$ be any bounded sequence in $\mathcal{C}[a,b]$, and let $g_n = Af_n$. Since $\|g_n\| \le \|A\|\,\|f_n\|$ for all $n \in \mathbb{N}$, the sequence $\{g_n\}$ is also bounded in $\mathcal{C}[a,b]$. We next show that the set $\{g_n\}$ is equicontinuous, from which it will follow from Ascoli's theorem (see Section 9.11) that the set $\{g_n\}$ has compact closure in $\mathcal{C}[a,b]$.

Let $S = \sup_n \|f_n\|$. Now K is uniformly continuous on the square $[a,b] \times [a,b]$ and so, for each $\varepsilon > 0$, there exists $\delta > 0$ such that
$$|K(x_1,y) - K(x_2,y)| < \frac{\varepsilon}{S(b-a)}$$
for every $y \in [a,b]$, provided that $|x_2 - x_1| < \delta$. Thus, if $|x_2 - x_1| < \delta$, then, for each $n \in \mathbb{N}$,
$$\begin{aligned} |g_n(x_1) - g_n(x_2)| &= \left| \int_a^b [K(x_1,y) - K(x_2,y)] f_n(y)\,dy \right| \\ &\le (b-a)S\left(\frac{\varepsilon}{S(b-a)}\right) = \varepsilon. \end{aligned}$$
It follows that $\{g_n\}$ is equicontinuous.

We have shown that the operator A has the property that if $\{f_n\}$ is bounded in X then the sequence $\{Af_n\}$ has a convergent subsequence in X. An operator with this property is said to be a *compact* operator. We shall study such operators in Chapter 14 in the setting of Hilbert spaces. Here we mention only that every compact operator is clearly bounded, but the converse is false. In fact, the identity operator on any infinite-dimensional Banach space cannot be a compact operator, since the unit ball in such a space contains a sequence $\{x_n\}$ that has no convergent subsequence. (See Theorem 12.16.)

Exercises

12:4.1 Show that $B(X,X)$ is an algebra with unit I; that is, for R, S, $T \in B(X,X)$ and $a, b \in \mathbb{R}$ (or $\mathbb{C}$)

(a) $(RS)T = R(ST)$.

(b) $R(S+T) = RS + RT$.

(c) $(R+S)T = RT + ST$.

(d) $(aR)(bS) = (ab)(RS)$.

(e) $IR = RI = R$.

12:4.2 Show that in Theorem 12.26 it is also true that

$$\|(I-T)^{-1} - I - T\| \le \frac{\|T\|}{1-\|T\|}.$$

12.5 The Hahn–Banach Theorem

One of the most indispensable tools of modern analysis is the theorem of Hahn and Banach that we shall now present. Hans Hahn (1879–1934) first showed in 1927 that a continuous linear functional defined on a subspace of a normed linear space could be extended to the whole of the space. He showed how to extend it up one dimension at a time and then used transfinite induction on the dimension; in our version, this is replaced by an appeal to Zorn's lemma. Banach in 1929, apparently unfamiliar with Hahn's paper, proved his version stated for linear functionals dominated by a subadditive functional. This is the version we give first. Note that it is entirely an algebraic theorem, making no mention of normed linear spaces or continuous functionals. We obtain Hahn's version as a consequence of this. A further more geometric version is given in Section 12.9 due apparently to Jean Dieudonné (1906–1992) in 1941.

These theorems are all set in real linear spaces, which is where we wish to remain for most of our considerations. For complex linear spaces and complex Banach spaces, a different idea is needed. This was supplied by H. F. Bohnenblust and A. Sobczyk in 1938. Any modern functional analysis text will supply the details.

We will have several opportunities to apply the Hahn–Banach theorem. In particular, applications of this theorem appear in Sections 12.6 and 12.8 and the exercises to this section.

Let X be a linear space, and let $p : X \to \mathbb{R}$. If

$$p(ax) = ap(x) \text{ for all } x \in X \text{ and } a \ge 0,$$

we say that p is *positively homogeneous*. If

$$p(x+y) \le p(x) + p(y) \text{ for all } x, y \in X,$$

we say that p is *subadditive*.

For example, a norm for a linear space is both positively homogeneous and subadditive. Some examples that are not norms are also important. See the exercises and Section 12.6. Observe that a subadditive, positively homogeneous functional p on X is a *convex function*; that is, it satisfies the inequality

$$p(ax + (1-a)y) \le ap(x) + (1-a)p(y)$$

for all $x, y \in X$ and all $a \in [0,1]$.

Our main result is Banach's version of the Hahn–Banach theorem.

Theorem 12.28 (Hahn–Banach) *Let X be a linear space, and let Y be a linear subspace of X. Let p be a subadditive, positively homogeneous functional defined on X. If f is a linear functional defined on Y and*

$$f(x) \leq p(x) \quad \textit{for all } x \in Y,$$

then there exists an extension F of f to all of X such that F is linear, and

$$F(x) \leq p(x) \quad \textit{for all } x \in X.$$

Proof. We present the proof in two parts. First we show that if Y is not all of X, f can be extended to a linear functional f_1 defined on a linear subspace Y_1 of X such that Y_1 properly contains Y and such that

$$f_1(x) \leq p(x) \quad \text{for all } x \in Y_1.$$

We then use Zorn's lemma to show that the required extension of f to *all* of X is possible.

If $Y \neq X$, let $z \in X \setminus Y$, and let

$$Y_1 = \{x \in X : \exists a \in \mathbb{R} \text{ and } \exists y \in Y \quad \text{such that } x = y + az\}.$$

Observe that the given representation for members of Y_1 is unique: if

$$x = y_1 + a_1 z = y_2 + a_2 z,$$

then

$$(a_1 - a_2)z = y_2 - y_1 \in Y.$$

Thus $a_1 - a_2 = 0$, since $z \notin Y$, and therefore $y_2 - y_1 = \mathbf{0}$. It is clear that Y_1 is a linear space and that Y_1 contains Y properly.

Now we can define f_1 on Y_1 by

$$f_1(x) = f_1(y + az) = f(y) + af_1(z)$$

for an appropriate choice of the value $af_1(z)$. Any choice for the value of $f_1(z)$ will result in the linearity of f_1. Our problem is to choose this value in such a way that $f_1(x) \leq p(x)$ for all $x \in Y_1$. Let us write u for $f_1(z)$. Our requirement then becomes

$$f(y) + au \leq p(y + az) \quad \textit{for all } y \in Y \quad \textit{and } a \in \mathbb{R}. \tag{8}$$

The inequality (8) is satisfied for $a = 0$, by hypothesis.

Let us see what is required of the number u for that inequality to hold for $a > 0$ and for $a < 0$. For $a \neq 0$, write $y = av$. For $a > 0$, we can then write (8) in the form

$$f(v) + u \leq p(v + z). \tag{9}$$

This is possible because p is positively homogeneous. For $a < 0$, $-a > 0$, so we can write (8) in the form

$$-f(v) - u \leq p(-v - z). \tag{10}$$

The effect of the substitution $y = av$ is to replace the requirement (8) by the tractable inequalities

$$-p(-v_1 - z) - f(v_1) \leq u \leq p(v_2 + z) - f(v_2) \tag{11}$$

for v_1, $v_2 \in Y$ that follow from (9) and (10). But for $v_1, v_2 \in Y$,

$$\begin{aligned} f(v_2) - f(v_1) &= f(v_2 - v_1) \leq p(v_2 - v_1) \\ &= p((v_2 + z) + (-v_1 - z)) \\ &\leq p(v_2 + z) + p(-v_1 - z) \end{aligned}$$

so

$$-p(-v_1 - z) - f(v_1) \leq p(v_2 + z) - f(v_2)$$

for all $v_1, v_2 \in Y$. Thus

$$A = \sup_{v_1}(-p(-v_1 - z) - f(v_1)) \leq \inf_{v_2}(p(v_2 + z) - f(v_2)) = B.$$

From (11) we see that if

$$A \leq u \leq B$$

then (11) and therefore (8) are satisfied. For such a value of $u = f_1(z)$, the functional

$$f_1(y + az) = f(y) + af_1(z)$$

has the required property. This completes the first part of the proof.

Now, let L denote the family of all linear extensions of f that are dominated by p on their domains. Partially order L by writing

$$f_1 \preceq f_2$$

for any pair of elements f_1, $f_2 \in L$ if f_2 is an extension of f_1. This means that the domain Y_2 of f_2 contains Y_1, the domain of f_1, and $f_2 = f_1$ on Y_1. The domains of linear functionals are linear spaces.

To apply Zorn's lemma we must verify that every chain in L has an upper bound. Let C be a chain in L. Thus C is a subset of L such that if $f_1, f_2 \in C$ either

$$f_1 \preceq f_2 \quad \text{or} \quad f_2 \preceq f_1.$$

Let U be the union of the domains of functions in C. Define F on U by $F(x) = g(x)$ if $g \in C$ and x is in the domain of g. Since C is a chain, the definition of F is consistent. For $x, y \in U$ there exists $g \in C$ such that x and y are both in the domain of g. Thus U and F are linear, and

$$F(x) \leq p(x) \quad \text{for all } x \in U.$$

We have shown that F is an upper bound for the chain C. By Zorn's lemma, L has a maximal element, F_0. This linear functional F_0 must have all of X as its domain; otherwise, one could extend F_0 by the first part of the proof, and this would contradict the maximality of F_0. ■

If we apply this theorem to a normed linear space, we obtain the following version, due to Hahn. One useful outcome of this next theorem is that there are always an abundance of continuous linear functionals in the setting of a normed linear space or a Banach space (see Exercise 12:7.1). This is not the case for general metric linear spaces (as Section 13.7 shows).

Theorem 12.29 *Let X be a normed linear space, Y a subspace of X, and let f be a bounded linear functional on the space Y. Then there exists an extension of f to a bounded linear functional $\hat{f}$ on the entire space X with the same norm, that is, so that $\|f\| = \|\hat{f}\|$.*

Proof. Let $p(x) = \|f\| \, \|x\|$. Then, for every $y \in Y$

$$|f(y)| \leq \|f\| \, \|y\|.$$

By Theorem 12.28, f can be extended to a linear functional F on X so that

$$|F(x)| \leq \|f\| \, \|x\| \quad \text{for all } x \in X.$$

But this last inequality implies that F is a bounded linear functional on X. Writing $\hat{f} = F$, we see that

$$|\hat{f}(x)| \leq \|f\| \, \|x\| \quad \text{for all } x \in X.$$

Thus $\|\hat{f}\| \leq \|f\|$. The reverse inequality is obvious. ■

Exercises

12:5.1◊ (Extending the Riemann integral) Let $p(f) = \overline{\int} f(t)\, dt$ denote the upper Riemann integral of a bounded function f on the interval $[a, b]$. Let $\mathcal{R}$ denote the linear space of Riemann integrable functions on $[a, b]$. For $f \in \mathcal{R}$, let

$$R(f) = \int_a^b f(t)\; dt.$$

(a) Show that $p(f)$ is positively homogeneous and subadditive on the set of bounded functions on $[a, b]$.

(b) Show that R is a linear functional on $\mathcal{R}$ and that $R(f) \leq p(f)$ for all $f \in \mathcal{R}$.

(c) What conclusion can one now draw from Theorem 12.28?

12:5.2 (Banach limits) Define a function p on the sequence space ℓ_∞ by

$$p(x) = \limsup_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n},$$

and define a linear functional l on the subspace c by

$$l(x) = \lim_{n\to\infty} x_n.$$

(a) Show that p is subadditive and positively homogeneous on ℓ_∞.

(b) Apply the Hahn–Banach theorem to obtain a linear functional L on ℓ_∞ such that, for $x = \{x_n\}$,

   (i) $L(x) \geq 0$ if $x_n \geq 0$ for all $n \in \mathbb{N}$.

   (ii) $L(\{x_1, x_2, x_3, \dots\}) = L(\{x_2, x_3, x_4, \dots\})$ for all $x \in \ell_\infty$.

   (iii) $\liminf x_n \leq L(x) \leq \limsup x_n$ for all $x \in \ell_\infty$.

   (iv) $L(x) = \lim_{n\to\infty} x_n$ for all $x \in c$.

   Thus L provides a notion of limit applied to all bounded sequences. The four properties (i) through (iv) are ones that we would expect of a generalized limit. One calls L a *Banach limit.*

(c) Calculate $L(\{0, 1, 0, 1, \dots\})$.

12:5.3 In most cases the extension of ideas in the text from real to complex Banach spaces is routine. The complex version of Theorem 12.28 is not routine. Find and present a proof of this theorem (attributed, usually, to Bohnenblust and Sobczyk).

12.6 Improving Lebesgue Measure

In this section we illustrate how the Hahn–Banach theorem can be used to address some of the measure theory problems on extensions of Lebesgue measure that arise from concerns in our earlier chapters.

In Section 2.1 we began with the primitive notion of the length of an interval and then extended this notion to apply it to a σ-algebra of sets, the class $\mathcal{L}$ of Lebesgue measurable sets. The result was Lebesgue measure λ. As a measure, λ is countably additive on $\mathcal{L}$ and therefore satisfies our intuitive requirement that "the whole should be the sum of its parts." But λ satisfies another intuitive requirement as well, that length should be invariant under translations. (In fact, the only Lebesgue–Stieltjes measures on $\mathbb{R}$ that are translation invariant are multiples of λ. Other Lebesgue–Stieltjes measures are not meant to model length.)

Thus λ seems to be an optimal generalization of the concept of lengths for subsets of $\mathbb{R}$. But, as we saw in Section 2.1, not all subsets of an interval are Lebesgue measurable. This leads to a natural question:

12.30 Can one extend Lebesgue measure to a measure $\overline{\lambda}$ defined for all subsets of $[0,1]$ such that $\overline{\lambda}$ is translation invariant?[1]

We saw in Section 3.10 that the answer is no. In fact, there is no finite nontrivial nonatomic measure defined on all subsets of [0,1]. This leads to the following questions:

12.31 Can one obtain an affirmative answer to question 12.30 if we weaken our requirement on $\overline{\lambda}$ to that of being a *finitely* additive measure?

12.32 Can one extend λ to a genuine measure $\overline{\lambda}$, defined on a "large" σ-algebra $\overline{\mathcal{L}}$ containing $\mathcal{L}$, such that $\overline{\lambda}$ is still translation invariant?

We discuss these two questions and some analogs in higher dimension in this section.

The reader may recall that we have already mentioned in Section 2.10 that the question 12.31 has an affirmative answer. We can prove this now. The key tool is the Hahn–Banach theorem. We seek a subadditive, positively homogeneous function p defined on the linear space of bounded functions on [0,1] such that

$$\int_0^1 f\,d\lambda \le p(f)$$

for all bounded measurable functions f. This functional p should be such that the linear extension $\overline{I}$ of

$$I(f) = \int_0^1 f\,d\lambda$$

to all bounded functions on [0,1] has a certain translation invariant property. By considering characteristic functions, we should then be able to interpret $\overline{\lambda}$ in terms of the extended linear functional $\overline{I}$ applied to characteristic functions.

Exercise 12:5.1 suggests using the upper Riemann integral

$$p(f) = \overline{\int} f(t)\,dt$$

of an arbitrary bounded function f on the interval $[a,b]$. This functional p is subadditive and positively homogeneous, and as part of the proof of Theorem 5.20 [see the inequalities (10) in that proof], we have obtained that $I(f) \le p(f)$ for all bounded measurable functions f. The extended linear functional $\overline{I}$, whose existence follows from the Hahn–Banach theorem, will

[1] Since we state our question for subsets of [0,1], rather than for all bounded subsets of $\mathbb{R}$, translation must be understood as "translation modulo $[0,1)$." That means that the part of a translated set that lies outside the interval $[0,1)$ must be moved to the corresponding part of $[0,1)$.

then provide an extension of λ by the interpretation we described. However, translation invariance becomes a problem (see Exercise 12:6.1). Another choice of p, one advanced by Banach, does the job, as we now show.

Let X denote the family of bounded real-valued functions on $[0,1)$. For purposes of calculations, extend each $f \in X$ to all of $\mathbb{R}$ by periodicity. Thus each $f \in X$ is extended to all of $\mathbb{R}$ so that

$$f(t+1) = f(t) \quad \text{for all } t \in \mathbb{R}.$$

For $f \in X$ and for each finite sequence $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of real numbers, let

$$M(f : \alpha_1, \alpha_2, \dots, \alpha_n) = \sup_t \frac{1}{n} \sum_{i=1}^{n} f(t + \alpha_i).$$

Finally, let

$$p(f) = \inf M(f : \alpha_1, \alpha_2, \dots, \alpha_n),$$

the infimum being taken over all finite sequences $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of real numbers. Loosely speaking, each n-tuple provides a mean value of f over a certain set of translates of t. The supremum of these means over $t \in \mathbb{R}$ provides a single number, $M(f : \alpha_1, \alpha_2, \dots, \alpha_n)$. The functional p provides "efficient" choices of translates.

It is clear that p is positively homogeneous. We now show that p is subadditive.

Let $f, g \in X$, and let $\varepsilon > 0$. Choose $(\alpha_1, \dots, \alpha_m)$ and $(\beta_1, \dots, \beta_m)$ such that

$$M(f : \alpha_1, \dots, \alpha_m) < p(f) + \varepsilon \quad \text{and} \quad M(g : \beta_1, \dots, \beta_m) < p(g) + \varepsilon.$$

The set of mn numbers $\alpha_i + \beta_j$ can be arranged into a single sequence

$\gamma_1, \gamma_2, \ldots, \gamma_{mn}$. Then

$$\begin{aligned}
p(f+g) &\leq M(f+g:\gamma_1,\ldots,\gamma_{mn}) \\
&= \sup_t \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(f+g)(t+\alpha_i+\beta_j) \\
&\leq \sup_t \left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{m}\sum_{i=1}^{m} f(t+\beta_j+\alpha_i)\right) \\
&\quad + \sup_t \left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{n}\sum_{j=1}^{n} g(t+\alpha_i+\beta_j)\right) \\
&\leq \sup_t \left(\frac{1}{n}\sum_{j=1}^{n} M(f:\alpha_1,\ldots,\alpha_m)\right) \\
&\quad + \sup_t \left(\frac{1}{m}\sum_{i=1}^{m} M(g:\beta_1,\ldots,\beta_n)\right) \\
&= M(f:\alpha_1,\ldots,\alpha_m) + M(g:\beta_1,\ldots,\beta_n) \\
&\leq p(f)+p(g)+2\varepsilon.
\end{aligned}$$

Since ε is arbitrary, $p(f+g) \leq p(f)+p(g)$. This proves that p is subadditive.

We next show that p dominates the Lebesgue integral

$$I(f) = \int_0^1 f\,d\lambda$$

defined for bounded measurable functions on $[0,1)$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any finite sequence of real numbers. For $\alpha \in \mathbb{R}$, let $f_\alpha(t) = f(t+\alpha)$. Since f has been extended periodically to $\mathbb{R}$, we see that, for each $\alpha \in \mathbb{R}$, $I(f) = I(f_\alpha)$. Thus

$$I(f) = \frac{1}{n}\int_0^1 (f(t+\alpha_1)+\cdots+f(t+\alpha_n))\,d\lambda \leq M(f:\alpha_1,\ldots,\alpha_n).$$

Since this is valid for every finite sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, we infer that $I(f) \leq p(f)$ on the linear space Y of bounded measurable functions on $[0,1)$.

We now apply the Hahn–Banach theorem, obtaining a linear functional $\overline{I}$ defined on X such that $\overline{I}(f) \leq p(f)$ for all $f \in X$. We show that, for each $\alpha \in \mathbb{R}$, $\overline{I}(f) = \overline{I}(f_\alpha)$.

To see this, let $g = f_\alpha - f$, and let

$$\alpha_1 = 0,\ \alpha_2 = \alpha,\ \alpha_3 = 2\alpha, \ldots,\ \alpha_n = (n-1)\alpha.$$

Then
$$p(g) \le M(g : \alpha_1, \ldots, \alpha_n) = \frac{1}{n} \sup_t (f(t + n\alpha) - f(t)). \tag{12}$$
Since (12) is valid for all $n \in \mathbb{N}$ and f is bounded we infer that $p(g) \le 0$, and therefore that $\overline{I}(g) \le 0$.

In a similar way, we find that $p(-g) \le 0$. Thus
$$\overline{I}(g) = -\overline{I}(-g) \ge -p(-g) \ge 0.$$
Therefore, $\overline{I}(g) = 0$; that is, $\overline{I}(f) = \overline{I}(f_\alpha)$, as was to be shown.

We summarize this discussion as a theorem.

Theorem 12.33 *There exists a linear functional $\overline{I}$ on the space of bounded real-valued functions on $[0,1)$ such that*

1. *If f is a bounded measurable function on $[0,1)$, then*
$$\overline{I}(f) = \int_0^1 f\, d\lambda.$$
2. *For each $\alpha \in \mathbb{R}$, $\overline{I}(f) = \overline{I}(f_\alpha)$.*

As an immediate corollary, we have an answer to question 12.31.

Theorem 12.34 *There exists a finitely additive measure $\overline{\lambda}$ defined on all subsets of $[0,1)$ such that*

1. *If $E \in \mathcal{L}$, then $\overline{\lambda}(E) = \lambda(E)$.*
2. *For each $\alpha \in \mathbb{R}$, $\overline{\lambda}(E) = \overline{\lambda}(E_\alpha)$, where*

$$E_\alpha = \{t \in [0,1) : \textit{There exists } x \in E \textit{ such that } t = x + \alpha (\text{mod } 1)\}.$$

Proof. Let $\overline{\lambda}(E) = \overline{I}(\chi_E)$, and apply Theorem 12.33. ■

Exercise 12:6.2 shows that we can obtain a bit more. Condition (2) of Theorem 12.34 is a form of translation invariance of $\overline{\lambda}$. One can also obtain reflection invariance of $\overline{\lambda}$.

We turn now to a brief discussion of question 12.32. We saw in Exercise 3:11.13 that we can extend Lebesgue measure on $[0,1)$ to a σ-algebra $\overline{\mathcal{L}}$ of sets properly containing $\mathcal{L}$. We made no provision to guarantee that the extended measure $\overline{\lambda}$ is translation invariant. In fact, if $\overline{\mathcal{L}}$ contains the nonmeasurable set A from Section 2.1, $\overline{\lambda}$ cannot be translation invariant. S. Kakutani and J. Oxtoby[2] have shown that λ *can* be extended to a σ-algebra $\overline{\mathcal{L}}$ much larger than $\mathcal{L}$ in such a way that the resulting measure $\overline{\lambda}$ is translation invariant. What does "much larger" mean here?

[2] S. Kakutani and J. Oxtoby, "Construction of a non-separable invariant extension of the Lebesgue measure space," *Ann. of Math.* 52 (1950), 580–590.

To answer this question, we recall that the metric space $(\mathcal{L}, \rho)$ with metric given by

$$\rho(A, B) = \lambda(A \triangle B)$$

is separable. The Kakutani–Oxtoby σ-algebra $\overline{\mathcal{L}}$ has the property that any dense subset of $(\overline{\mathcal{L}}, \rho)$ must have cardinality 2^c. A similar result holds in $\mathbb{R}^n$ for all $n \in \mathbb{N}$. There one obtains an extension of n-dimensional Lebesgue measure λ_n to a much larger σ-algebra on which the extended measure $\overline{\lambda}_n$ is invariant with respect to isometries: if A and B are congruent, then $\overline{\lambda}(A) = \overline{\lambda}(B)$.

The analog of Theorem 12.34 is valid in $\mathbb{R}^2$. There is an isometry-invariant extension of λ_2 to all subsets of the unit square (or, more generally, to all bounded subsets of $\mathbb{R}^2$). But the result fails for $n \geq 3$. This is part of the content of the Banach–Tarski[3] paradox. It is possible to decompose a ball in $\mathbb{R}^3$ into five pairwise disjoint pieces and then reassemble these pieces to form *two* disjoint balls of the same size. More precisely, there exists sets $A_1, \ldots, A_5$ and $E_1, \ldots, E_5$ such that

1. $A_i \cap A_j = \emptyset$ if $i \neq j$.
2. A_i is congruent to E_i for $i = 1, \ldots, 5$.
3. $\bigcup_{i=1}^5 A_i = B(0, 1)$.
4. $\bigcup_{i=1}^5 E_i = B(0, 1) \cup B(2, 1)$.

It is therefore clear that no congruent-invariant finitely additive extension of λ_3 to all bounded subsets of $\mathbb{R}^3$ can exist. The difference between dimension 2 and dimension $n \geq 3$ is that in the latter case there are simply too many isometries to allow such an extension.

It is beyond our purpose to develop the Banach–Tarski paradoxes, but we can give an example of a related "paradox" in $\mathbb{R}^2$. We construct two disjoint sets R and T, situated in $\mathbb{R}^2$, such that $R \cong T \cong R \cup T$, with $\cong$ being the congruence relation.

For each complex number z, let

$$t(z) = z + 1\ ,\ r(z) = e^i z.$$

Thus t is just a right translation by one unit and r is a rotation by one radian. Let S consist of those points that can be obtained from the origin by a finite number of applications of t and r. Thus each member of S can be represented as a polynomial in e^i with integer coefficients. Since e^i is transcendental, the representation is unique. Let R consist of those points of S whose representations have no constant term, and let $T = S \setminus R$. Then

$$t(S) = T \quad \text{and} \quad r(S) = R$$

[3]See S. Wagon, *The Banach–Tarski Paradox,* Cambridge University Press (1985).

from which it follows that the sets R, T, and $S = R \cup T$ are congruent.

This simple example provides no measure-theoretic paradox. It simply shows that a certain set S allows a rotation r and a translation t such that the sets S, $R = r(S)$ and $T = t(S)$ are pairwise congruent, with $R \cup T = S$ and $R \cap T = \emptyset$.

Exercises

12:6.1 Apply the reasoning leading to Theorem 12.33, but using instead the upper Riemann integral

$$p(f) = \overline{\int} f(t)\, dt.$$

Let $\overline{I}$ be the resulting linear functional. Let $g = f - f_{\sqrt{2}}$. Show that $-1 \leq \overline{I}(g) \leq 1$. Can one conclude that $\overline{I}(g) = 0$, as desired?

12:6.2 (Refer to the discussion leading to Theorem 12.33.) Let

$$\overline{\overline{I}}(f) = \frac{1}{2}\left(\overline{I}(f) + \overline{I}(f(1-t))\right).$$

Show that $\overline{\overline{I}} = I$ on Y and $\overline{\overline{I}}(f) = \overline{\overline{I}}(f(1-t))$ on X. Interpret this result for characteristic functions to obtain an extension $\overline{\overline{\lambda}}$ of λ to all bounded functions on $[0, 1)$. Show that this extension is invariant under isometries.

12:6.3 (Refer to the discussion ending this section.) Verify that $t(S) = T$ and that $r(S) = R$.

12.7 The Dual Space

The material in the preceding two sections suggests the importance of continuous linear functionals on normed linear spaces. Many concepts of analysis can be expressed in this language.

In fact, the importance lies even deeper. The structure of a normed linear space is revealed to a great extent in the collection of continuous linear functionals. It is customary to denote $B(X, \mathbb{R})$ by X^*, the space of *continuous linear functionals* on X. This space is called the *dual* of X (or perhaps *conjugate* or *adjoint*.).

This space can be furnished with a norm, too, so that X^* is also a normed linear space. The norm of a linear functional $x^* \in X^*$ can be described by

$$\|x^*\| = \sup\left\{|x^*(x)| : \|x\| = 1\right\}, \tag{13}$$

which is the supremum of the values of $|x^*(x)|$ on the unit sphere $\{x : \|x\| = 1\}$. Note that there are two spaces here, X and X^*, each equipped with a norm, which one commonly writes using the same symbol. Be careful to make a distinction between the norms as needed.

There is a true "duality" between a space X and its dual X^* that can only be fully exploited in the language of topological vector spaces. Nonetheless, even restricting ourselves to the language of normed linear spaces, we can still discover a great deal about the interplay between a space and its dual. The function notation $x^*(x)$ for the value of the functional x^* at the point x is often abandoned for a notation that encourages the duality notions:

$$\langle x, x^* \rangle = x^*(x).$$

We point out first that the dual of a normed linear space, even an incomplete space, is a Banach space.

Theorem 12.35 *If X is a normed linear space, then its dual X^* is a Banach space.*

Proof. This follows directly from Theorem 12.24. ∎

One of our main tools in embarking on the study of the dual space is the Hahn–Banach theorem. We restate is here. This is just a rewording of Theorem 12.29 in the language of the dual space and in the form in which it is frequently applied.

Theorem 12.36 (Hahn–Banach) *Let X be a normed linear space, Y a subspace of X and $y^* \in Y^*$. Then there exists an extension of y^* to a functional $x^* \in X^*$, with $\|x^*\| = \|y^*\|$.*

One of our first observations is that, because of the Hahn–Banach theorem, there is an abundance of continuous linear functionals. That is, the space X^* is supplied with enough elements for most applications. The first theorem shows that we can find elements of X^* to "pick off" any element x_0 of X. The second theorem shows that we can use continuous linear functionals to distinguish between such an element x_0 and a closed subspace Y at a positive distance from x_0.

Theorem 12.37 *Let X be a normed linear space and x_0 a nonzero element of X. Then there exists a functional $x^* \in X^*$ with $\|x^*\| = 1$ and $x^*(x_0) = \|x_0\|$.*

Proof. Let Y be the subspace spanned by the single element x_0. Every element y of Y can be written uniquely in the form $y = \alpha x_0$ for some real α. Define an element y^* of Y^* by $y^*(y) = y^*(\alpha x_0) = \alpha\|x_0\|$. It is easy to check that y^* has the required properties, that $\|y^*\| = 1$ and $y^*(x_0) = \|x_0\|$. The proof is completed by invoking Theorem 12.36 to obtain an extension of y^* to an element $x^* \in X^*$ with the same norm. ∎

Theorem 12.38 *Let X be a normed linear space and x_0 a nonzero element of X. Suppose that Y is a closed subspace of X and that dist$(x_0, Y) = h_0 > 0$. Then there exists a functional $x^* \in X^*$ such that $\|x^*\| = 1$, $x^*(x_0) = h_0$, and $x^*(y) = 0$ for each $y \in Y$.*

Proof. Let Y_1 be the subspace spanned by Y and x_0. Every element y_1 of Y_1 can be written uniquely as $y_1 = y + (\alpha/h_0)x_0$ for some $y \in Y$ and some real α. An easy computation shows that $\|y_1\| \geq |\alpha|$.

We define an element y^* of Y_1^* by

$$y^*(y_1) = y^*(y + (\alpha h_0^{-1})x_0) = \alpha$$

using the representation above. It is routine to verify that y^* has the required properties, that $\|y^*\| = 1$, $y^*(x_0) = h_0$, and $y^*(y) = 0$ for each $y \in Y$. The proof is completed by invoking Theorem 12.36 to obtain an extension of y^* to an element $x^* \in X^*$ with the same norm. ■

The study of the dual will play an important role in the study of any normed linear space. As a simple application, let us use the material developed so far to show that a certain important property of the dual space is reflected in the space.

Theorem 12.39 *Let X be a normed linear space. If X^* is separable, then so too is X.*

Proof. Let x_n^* be a sequence of elements of X^* forming a dense set. Then, for each n, we may find an element $x_n \in X$ so that $\|x_n\| = 1$ and $|x_n^*(x_n)| > \frac{3}{4}\|x_n^*\|$. Let Y be the closure of the linear space spanned by the set x_n in X. If $Y = X$, then we are done, since the set of rational linear combinations of the set of all x_n forms a countable dense subset of X.

Suppose, contrary to this, that Y is a proper subspace. Then there is a point of X at positive distance from Y. Applying Theorem 12.38, we find then an element x^* with $\|x^*\| = 1$ and $x^*(y) = 0$ for all $y \in Y$. In particular $x^*(x_n) = 0$ for all n. There must be an element x_m^* with $\|x^* - x_m^*\| < \frac{1}{4}$, since the sequence x_n^* forms a dense set in X^*. Since $\|x^*\| = 1$, we see that $\|x_m^*\| \geq \frac{3}{4}$. But this is impossible since

$$\tfrac{3}{4}\|x_m^*\| \leq |x_m^*(x_m)| = |x_m^*(x_m) - x^*(x_m)| \leq \|x_m^* - x^*\| < \tfrac{1}{4}.$$

From this contradiction the theorem follows. ■

Exercises

12:7.1 If x, y are distinct points in a normed linear space X, show that there is a member of X^* that separates x and y (i.e., $\langle x, x^*\rangle \neq \langle y, x^*\rangle$ for some $x^* \in X^*$).

12:7.2 Let X be a normed linear space. Prove that, for any $x \in X$,

$$\|x\| = \sup\{|x^*(x)| : x^* \in X^*,\ \|x^*\| = 1\},$$

which can be considered a dual assertion to (13). [Hint: Use Theorem 12.37.]

12:7.3 Prove that the converse of Theorem 12.39 does not hold. [Hint: You may assume that the dual of the space ℓ_1 can be taken as ℓ_∞, a fact that is proved in Section 13.6.]

12:7.4 Show that if $T \in B(X,Y)$ then

$$\|T\| = \sup\{\langle Tx, y^*\rangle : \|x\| \le 1,\ \|y^*\| \le 1,\ x \in X,\ y^* \in Y^*\}\,.$$

12:7.5 Let X, Y be Banach spaces with duals X^* and Y^*. Show that to each $T \in B(X,Y)$ corresponds a unique $T^* \in B(X^*, Y^*)$ defined by

$$\langle Tx, y^*\rangle = \langle x, T^*y^*\rangle \quad (x \in X,\ y^* \in Y^*)$$

and that $\|T\| = \|T^*\|$.

12:7.6$\diamondsuit$ A Banach space X has a dual X^* that is also a Banach space and so has its own dual, denoted by X^{**}.

(a) Show that the mapping $\phi : X \to X^*$ defined by

$$\langle x, x^*\rangle = \langle x, \phi(x)\rangle$$

is a linear isometry of X to a closed subspace of X^{**}.

If $X^{**} = \phi(X)$, we say that X is *reflexive*. (In this case X is isomorphic, in the sense defined in Section 12.10, to its second dual X^{**}.)

(b) Prove that X is reflexive if and only if X^* is reflexive.

(c) Prove that if X is reflexive, then every continuous linear functional on X assumes a maximum on the closed unit ball of X. [Hint: Use Theorem 12.37 to obtain an element x^{**} of X^{**} such that $\|x^{**}\| = 1$ and $\langle x^*, x^{**}\rangle = \|x^*\|$. Use reflexivity to find $x \in X$ with $\langle x, x^*\rangle = \|x^*\|$.]

(d) Refer to Example 12.6. Define x^* as follows: For each element $a = \{a_1, a_2, \dots\}$ of the space c_0, we require

$$\langle a, x^*\rangle = \sum_{k=1}^{\infty} a_k/k!\ .$$

Show that $x^* \in c_0$ and that $\|x^*\| = \sum_{k=1}^{\infty} 1/k!$. Use part (c) to show that c_0 is not reflexive. (In fact it can be shown that $c_0^* = \ell_1$ and $c_0^{**} = \ell_\infty$.)

(e) Prove that $\mathcal{C}[a,b]$ is not reflexive.

12.8 The Riesz Representation Theorem

Given a concrete Banach space X, what are the continuous linear functionals on X? What precisely is the dual space X^*? What we want is a "representation" of the elements of X^* that is given at least as explicitly as we have been given the elements of X. This is an obvious and natural mathematical problem, but it has practical import: if the space X is useful in applications, then the dual space X^* is an important tool to use in working with X.

We shall find a representation for the continuous linear functionals on the Banach space $\mathcal{C}[a,b]$ and describe $\mathcal{C}[a,b]^*$. It is easy to come up with *some* if not all continuous linear functionals on $\mathcal{C}[a,b]$. The functions

$$F_1(f) = f(x_0),$$
$$F_2(f) = \sum_{i=0}^{\infty} 2^{-i} f(x_i),$$
$$F_3(f) = \int_a^b f(t)\,dt,$$
$$F_4(f) = \int_a^b f(t)g(t)\,dt,$$

where $x_0,\ x_1,\ x_2,\ \ldots$ are points of $[a,b]$ and where g is integrable on $[a,b]$, are all evidently continuous linear functionals. But what idea captures *all* continuous linear functionals on this space.

Jacques Hadamard (1865–1963) showed in 1903 that every continuous linear functional must be of the form

$$F(f) = \lim_{n\to\infty} \int_a^b k_n(t)f(t)\,dt$$

for some sequence of continuous functions k_n. Incidentally, this paper contains perhaps the first use of the term "functional" *(fonctionelle)* in our subject.

This representation is inadequate to characterize the dual space. By 1909, F. Riesz had reconsidered the problem and arrived at the solution we now present. His representation was in terms of Stieltjes integrals, a concept that had received no attention since its introduction by the Dutch mathematician T. J. Stieltjes (1856–1894) many years earlier. We shall characterize precisely the space $\mathcal{C}[a,b]^*$.

The essence of the Riesz representation theorem[4] is that it identifies each continuous linear functional on $\mathcal{C}[a,b]$ with some Lebesgue–Stieltjes signed measure μ_g. Each such signed measure determines a unique function g of bounded variation on $[a,b]$ and right continuous on (a,b) with $g(a)=0$. Conversely each such function g determines a Lebesgue–Stieltjes signed measure μ_g. Because we are dealing with continuous functions and Lebesgue–Stieltjes signed measures, we can take the integrals in the simpler Riemann–Stieltjes sense. We begin our preparation for the Riesz representation theorem by recalling the definition of the Riemann–Stieltjes integral.

Let $f \in \mathcal{C}[a,b]$ and let $g \in \mathrm{BV}[a,b]$. Let P be a partition of $[a,b]$, say

$$a = x_0 < x_1 < \cdots < x_n = b,$$

[4] Many different theorems go by this same name in the literature, testimony to the importance that Riesz (and his brother M. Riesz) played in the early decades of the development of functional analysis.

and let $t_1, t_2, \ldots, t_n$ satisfy

$$x_{i-1} \le t_i \le x_i \quad \text{for all } i = 1, \ldots, n.$$

Finally, let

$$\triangle(P) = \max_i (x_i - x_{i-1}).$$

A standard theorem asserts that

$$\int_a^b f\, dg = \lim_{\triangle(P)\to 0} \sum_{i=1}^n f(t_i)(g(x_i) - g(x_{i-1})) \tag{14}$$

exists. This means that there exists $\alpha \in \mathbb{R}$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ for which

$$\left| \sum_{i=1}^n f(t_i)(g(x_i) - g(x_{i-1})) - \alpha \right| < \varepsilon$$

for every partition P with $\triangle(P) < \delta$ and for arbitrary choice of the points $t_i \in [x_{i-1}, x_i]$.

It is clear that the integral (14) is linear in f and g. Our sole requirement on g is that $g \in \mathrm{BV}[a,b]$, but it is trivial that the value of the integral does not change if we add a constant to g.

The Riesz representation theorem gives a correspondence between the bounded linear functionals on $\mathcal{C}[a,b]$ and the space $\mathrm{BV}[a,b]$. In order to make the correspondence a bijection, we shall restrict our attention to the space $\mathrm{NBV}[a,b]$ of those functions g of bounded variation on $[a,b]$ that are right continuous on (a,b) and that satisfy $g(a) = 0$. One verifies that the variation

$$\|g\| = V(g; [a,b])$$

is a norm on $\mathrm{NBV}[a,b]$, and that

$$\left| \int_a^b f\, dg \right| \le \|f\|_\infty \, \|g\|, \tag{15}$$

where

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|.$$

We leave verification of (15) as Exercise 12:8.2.

To this point we have obtained a one to one linear correspondence between $\mathrm{NBV}[a,b]$ and a subset of the bounded linear functionals on $\mathcal{C}[a,b]$ that does not increase norms. The Riesz representation theorem states that this mapping is a linear isometry between $\mathrm{NBV}[a,b]$ and *all* of $\mathcal{C}^*[a,b]$, and every continuous linear functional on $\mathcal{C}[a,b]$ can be represented by a Riemann–Stieltjes integral.

Theorem 12.40 (Riesz) *Let F be a bounded linear functional on $\mathcal{C}[a, b]$. Then there exists $g \in \mathrm{NBV}[a, b]$ such that*

$$F(f) = \int_a^b f\,dg \quad \textit{for all } f \in \mathcal{C}[a, b].$$

Furthermore,

$$\|g\| = V(g; [a, b]) = \|F\|.$$

Proof. We shall use the Hahn–Banach theorem to obtain a function $g \in \mathrm{NBV}[a, b]$ that has all the required properties.

The functional F is linear on the space $\mathcal{C}[a, b]$. By Theorem 12.29, it can be extended to a linear functional, which we also denote by F, on all of $M[a, b]$, with preservation of $\|F\|$.

Consider now the family of step functions of the form

$$\phi_x(t) = \begin{cases} 1 & a \le t < x, \\ 0 & x \le t \le b, \end{cases}$$

for $a < x \le b$, with $\phi_a(t) = 0$ for all $t \in [a, b]$. Define g on $[a, b]$ by $g(x) = F(\phi_x)$. We show that $g \in \mathrm{BV}[a, b]$ and that $V(g; [a, b]) \le \|F\|$.

Let $a = x_0 < x_1 < \cdots < x_n = b$ be an arbitrary partition of $[a, b]$. To simplify our notation, define a function sgn by

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x = 0; \\ -1 & \text{if } x < 0, \end{cases}$$

and let $\alpha_i = \mathrm{sgn}(g(x_i) - g(x_{i-1}))$, $i = 1, \ldots, n$. Then

$$\begin{aligned} \sum_{i=1}^n |g(x_i) - g(x_{i-1})| &= \sum_{i=1}^n \alpha_i (g(x_i) - g(x_{i-1})) \\ &= \sum_{i=1}^n \alpha_i F(\phi_{x_i} - \phi_{x_{i-1}}) = F\left(\sum_{i=1}^n \alpha_i(\phi_{x_i} - \phi_{x_{i-1}})\right) \\ &\le \|F\| \left\| \sum_{i=1}^n \alpha_i (\phi_{x_i} - \phi_{x_{i-1}}) \right\| \le \|F\|, \end{aligned}$$

the last inequality following from the fact that the function

$$\sum_{i=1}^n \alpha_i (\phi_{x_i} - \phi_{x_{i-1}})$$

can take only the values 0, -1, and 1.

Thus

$$\sum_{i=1}^n |g(x_i) - g(x_{i-1})| \le \|F\|.$$

Since this is true for every partition of $[a,b]$, we see that $g \in \mathrm{BV}[a,b]$ and that $V(g;[a,b]) \leq \|F\|$. Now $g(a) = F(\phi_a) = F(0) = 0$. It follows that, by modifying g to be right continuous on (a,b) if necessary, we can take g to be in $\mathrm{NBV}[a,b]$, so

$$V(g;[a,b]) = \|g\|.$$

Thus $\|g\| \leq \|F\|$. Since we have already observed the reverse inequality in (15), we have $\|g\| = \|F\|$.

We now show that F can be represented in the desired form

$$F(f) = \int_a^b f\,dg.$$

Let $f \in \mathcal{C}[a,b]$, and let $k \in \mathbb{N}$. Since f is uniformly continuous on $[a,b]$, there exists $\delta_k > 0$ such that $\delta_k < 1/k$ and

$$|f(x) - f(y)| < \frac{1}{k} \tag{16}$$

whenever $x, y \in [a,b]$ and $|x-y| < \delta_k$.

Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$ with $|x_i - x_{i-1}| < \delta_k$ for $i = 1, \ldots, n$. Define a function f_k by

$$f_k(t) = \sum_{i=1}^n f(x_i)(\phi_{x_i}(t) - \phi_{x_{i-1}}(t)).$$

Each of the functions f_k is a step function having the value $f(x_i)$ in the interval $x_{i-1} \leq t < x_i$. By (16), $|f(t) - f_k(t)| < 1/k$ for all $t \in [a,b]$. Thus $\|f - f_k\|_\infty < 1/k$ and $f_k \to f$ [unif].

From the definition of the function g, we see that

$$\begin{aligned} F(f_k) &= \sum_{i=1}^n f(x_i)(F(\phi_{x_i}) - F(\phi_{x_{i-1}})) \\ &= \sum_{i=1}^n f(x_i)(g(x_i) - g(x_{i-1})). \end{aligned}$$

This shows that $F(f_k)$ is an approximating sum for $\int_a^b f\,dg$. Since $\delta_k < 1/k$ for each k, it follows that $\{F(f_k)\}$ converges to $\int_a^b f\,dg$.

But F is continuous on $M[a,b]$, and $\{f_k\}$ converges to f in $M[a,b]$, so $\{F(f_k)\}$ converges to $F(f)$. Thus

$$F(f) = \int_a^b f\,dg,$$

as was to be shown. ∎

Observe that the mapping Φ defined by

$$(\Phi(g))(f) = \int_a^b f\,dg$$

is a linear norm-preserving mapping of $\mathrm{NBV}[a,b]$ onto $\mathcal{C}^*[a,b]$. Thus, these two spaces can be identified as Banach spaces: Φ preserves both the linear and the norm structures of the spaces. Since we can also associate with each $g \in \mathrm{NBV}[a,b]$ a Lebesgue–Stieltjes signed measure μ_g, we can view the dual space $\mathcal{C}^*[a,b]$ of $\mathcal{C}[a,b]$ as a space of finite Borel signed measures. Note also that we found it convenient to use the language of the Riemann–Stieltjes integral in our development. For continuous functions f,

$$\int_a^b f\,dg = \int_{[a,b]} f\,d\mu_g.$$

The material in this section has been generalized to obtain the dual space for $\mathcal{C}(X)$, where X is a much more general space than an interval of the real line. Our methods were very much in the setting of the real line, and this extension, while it still involves integrals and measures, is quite different. In greater generality, the theorem is known as the Riesz–Kakutani theorem[5], and it provides a connection between the dual of the complex Banach space $\mathcal{C}(X)$, where X is a compact Hausdorff space, and the space of complex, regular, Borel measures on X.

Exercises

12:8.1 Show that each of the four functionals F_1, F_2, F_3, and F_4 defined at the beginning of this section is a continuous linear functional on $\mathcal{C}[a,b]$, and determine its norm. How may each of these be represented by a Stieltjes integral?

12:8.2 (a) Verify that the Riemann–Stieltjes integral is linear in f and g.

(b) Verify that $\|g\| = V(g;[a,b])$ is a norm on $\mathrm{NBV}[a,b]$.

(c) Verify the inequality

$$\left|\int_a^b f\,dg\right| \le \|f\|_\infty\,\|g\|.$$

12:8.3 Let $g(x) = x^2$ on $[-1,1]$, and let

$$\mathcal{F} = \{f \in \mathcal{C}[-1,1] : |f(x)| \le 5 \text{ for all } x \in [-1,1]\}.$$

Calculate $\sup_{f\in\mathcal{F}} \int_{-1}^1 f\,dg$.

[5] For example, see N. Dunford and J. T. Schwartz, *Linear Operators I*, Interscience, 1958, pp. 262–265.

12:8.4 For g to be in NBV$[a,b]$, we required that g be right continuous on (a,b). Why did we not require right continuity at $x=a$? What about right continuity at b? [Hint: Let $F(f)=f(a)$. Then $F \in \mathcal{C}^*[a,b]$. How do we represent F as a Riemann–Stieltjes integral $\int_a^b f\,dg$ if g is right continuous at $x=a$?]

Observe that for g of bounded variation g is continuous except on a countable set D_f. If one changes the values of g on D_f it will not affect the value of $\int_a^b f\,dg$, provided that arbitrarily fine partitions can avoid points of D_f. This is possible if and only if a, which must be in every partition, is not a member of D_f.

12.9 Separation of Convex Sets

The Hahn–Banach theorem assumes one more form, this time as a separation theorem. In two dimensions the ideas are transparent. It is intuitively plausible that, if C_1 and C_2 are arbitrary closed nonintersecting convex sets in the plane, then there is a line L separating C_1 from C_2; C_1 lies on one side of L, C_2 on the other. We can formulate our conjecture analytically: there exists $a,b,c \in \mathbb{R}$ such that

$$\begin{aligned} ax+by &\geq c \quad \text{for all } (x,y) \in C_1, \quad \text{and} \\ ax+by &\leq c \quad \text{for all } (x,y) \in C_2. \end{aligned}$$

Thus our notion of separation can be couched in the language of linear functionals. We shall obtain a similar separation theorem for arbitrary linear spaces.

Let X be a (real) linear space, and let $x,y \in X$. The *closed segment* joining x and y is the set

$$\{\alpha x + (1-\alpha)y : 0 \leq \alpha \leq 1\}.$$

The *open segment* joining x and y is the set

$$\{\alpha x + (1-\alpha)y : 0 < \alpha < 1\}.$$

The *interior* of a set $S \subset X$ is the set

$$\{s \in S : \text{For each } x \in X,\ \exists\, \varepsilon > 0 \text{ such that } s+tx \in S \text{ if } |t| < \varepsilon\}.$$

Thus s is in the interior of S if and only if the intersection with S of each line through s contains an open segment about s. The set S is *convex* if, whenever $x,y \in S$, the closed segment joining x and y is contained in S. A convex set is called a *convex body* if it has nonempty interior. For example, a ball in a normed linear space is a convex body. On the other hand, a proper subspace of a normed linear space is a convex set, but cannot be a convex body. The class of convex sets in a linear space is closed under various operations. (See Exercise 12:9.1.)

We have already mentioned that a ball in a normed linear space is a convex body. This is a special case of Theorem 12.41.

Theorem 12.41 *Let p be a nonnegative, positively homogeneous, subadditive functional on a linear space X. Then for every $k > 0$ the set*

$$S = \{x : p(x) \leq k\}$$

is a convex body. Its interior is the set $\{x : p(x) < k\}$.

Proof. Let $x, y \in S$, and let $\alpha \in [0,1]$. Then

$$p(\alpha x + (1-\alpha)y) \leq \alpha p(x) + (1-\alpha)p(y) \leq \alpha k + (1-\alpha)k = k,$$

so the closed segment joining x and y is in S, and S is convex.

To verify the statement about the interior of S, let $p(s) < k$, let $t > 0$, and let $x \in X$. Then

$$p(s + tx) \leq p(s) + tp(x)$$

and

$$p(s - tx) \leq p(s) + tp(-x).$$

If $p(x) = p(-x) = 0$, then $s \pm tx = s \in S$ for all t. If $p(x) \neq 0$ or $p(-x) \neq 0$, then for

$$t < \frac{k - p(s)}{\max(p(x), p(-x))}$$

we find that $p(s \pm tx) < k$, so $s \pm tx \in S$. ∎

Consider now the set $S = \{x : p(x) \leq 1\}$. Since $p(\mathbf{0}) = 0$, $\mathbf{0} \in S$. Thus p determines a convex body S with $\mathbf{0}$ as an interior point. We can turn the process around: Let S be a convex body having $\mathbf{0}$ as an interior point. Let $p = p_S$ be defined by

$$p_S(x) = \inf\left\{r > 0 : r^{-1}x \in S\right\}.$$

This functional is called the *Minkowski functional* of the convex body S. It is clear that $S = \{x : p_S(x) \leq 1\}$.

Theorem 12.42 *The Minkowski functional p is nonnegative, positively homogeneous, and subadditive.*

Proof. Since $\mathbf{0}$ is an interior point of S, it is clear that for every $x \in X$, $r^{-1}x \in S$ for r sufficiently large. Thus p is finite and nonnegative. It is also clear that $p(\mathbf{0}) = 0$.

To check for positive homogeneity of p, let $a > 0$. Then

$$\begin{aligned} p(ax) &= \inf\left\{r > 0 : ar^{-1}x \in S\right\} \\ &= \inf\left\{ar > 0 : r^{-1}x \in S\right\} \\ &= a \inf\left\{r > 0 : r^{-1}x \in S\right\} = ap(x). \end{aligned}$$

Finally, we verify that p is subadditive. Let $\varepsilon > 0$, and let x_1 and x_2 be arbitrary elements of the space. Choose r_1 and r_2 such that

$$p(x_1) < r_1 < p(x_1) + \varepsilon$$

and

$$p(x_2) < r_2 < p(x_2) + \varepsilon.$$

Then

$$x = \frac{1}{r_1 + r_2}(x_1 + x_2) = \left(\frac{r_1}{r_1 + r_2}\right)\frac{x_1}{r_1} + \left(\frac{r_2}{r_1 + r_2}\right)\frac{x_2}{r_2},$$

so x is in the segment joining x_1/r_1 and x_2/r_2. Since S is convex, $x \in S$. Thus we see from the way r_1 and r_2 were chosen that

$$p(x_1 + x_2) \leq p(x_1) + p(x_2) < r_1 + r_2 < p(x_1) + p(x_2) + 2\varepsilon.$$

Since ε is arbitrary,

$$p(x_1 + x_2) \leq p(x_1) + p(x_2),$$

so p is subadditive, completing the proof of the theorem. ■

We turn now to the question of separation of convex sets in a linear space X. Let A and B be subsets of X, and let f be a linear functional on X. If there exists $c \in \mathbb{R}$ such that $f(x) \geq c$ for all $x \in A$ and $f(x) \leq c$ for all $x \in B$, we say that f *separates* A and B. Then f separates A and B if and only if f separates the sets $\{\mathbf{0}\}$ and

$$A - B = \{z : z = x - y \text{ for some } x \in A \text{ and } y \in B\}.$$

This is also equivalent to the statement that, for every $x_0 \in X$, f separates the sets $A - \{x_0\}$ and $B - \{x_0\}$. We omit the easy verifications of these statements.

Theorem 12.43 *Let A and B be disjoint convex sets in a linear space. If A is a convex body, then there exists a nontrivial linear functional f on X that separates A and B.*

Proof. We may assume that $\mathbf{0}$ is an interior point of A; otherwise, we would simply apply our proof to the sets $A - \{x_0\}$ and $B - \{x_0\}$, where x_0 is an interior point of A.

Let $y_0 \in B$. Then $-y_0$ is an interior point of the set $A - B$, and $\mathbf{0}$ is an interior point of the set

$$A - B + y_0 = \{z : z = x - y + y_0 \text{ with } x \in A, y \in B\}.$$

Now A and B are disjoint by hypothesis, so

$$\mathbf{0} \notin A - B \text{ and } y_0 \notin A - B + y_0.$$

Let p be the Minkowski functional for the set $A - B + y_0$. Since y_0 is not in the set $A - B + y_0$, it follows that $p(y_0) \geq 1$. Define a linear functional f on $Y = \{ay_0 : a \in \mathbb{R}\}$ by $f(ay_0) = ap(y_0)$. For $a > 0$,

$$f(ay_0) = ap(y_0) = p(ay_0).$$

For $a < 0$,

$$f(ay_0) = af(y_0) < 0 \leq p(ay_0),$$

since p is nonnegative by definition. Thus $f \leq p$ on Y.

We now apply the Hahn–Banach theorem, obtaining a linear functional F defined on all of X such that $F(x) \leq p(x)$ for all $x \in X$. Since p is the Minkowski functional for the set $A - B + y_0$, we have $F(x) \leq p(x) \leq 1$ on that set. On the other hand, $F(y_0) = f(y_0) = p(y_0) \geq 1$. This means that F separates the sets $A - B + y_0$ and $\{y_0\}$. But, as we observed before stating the theorem, this implies that F separates A and B. Since $F(y_0) \geq 1$, F is nontrivial. ■

Theorem 12.43 is often called the "separation" form of the Hahn–Banach theorem. The condition that one of the sets A or B has interior points cannot be dropped from the hypothesis of Theorem 12.43.

Example 12.44 Let X be the linear space of polynomials. Let A consist of those polynomials whose highest-order coefficient is positive. Then A is convex and $\mathbf{0} \notin A$. Let f be a linear functional on X with $f \geq 0$ on A. Consider now any polynomial of the form $au^n + u^{n+1}$, $a \in \mathbb{R}$, $n \geq 0$. This polynomial is in A, and

$$af(u^n) + f(u^{n+1}) = f(au^n + u^{n+1}) \geq 0.$$

The inequality is valid for all n and a (even $a < 0$), so $f(u^n) = 0$ for all $n \geq 0$. Since each member of X is a linear combination of the elements u^n in X and f is linear, we infer that $f \equiv 0$ on X. Similarly, if $f \leq 0$ on A then $f \equiv 0$ on X. Thus there is no nontrivial linear functional separating A and $\{\mathbf{0}\}$.

Example 12.45 Let $X = \mathcal{C}[0,1]$. Let $h(t) = e^t$, and let

$$A = \{f \in X : \|f\| \leq 1\} \quad \text{and} \quad B = \{f \in X : \|f - h\| \leq 1\}.$$

Then A and B are disjoint convex bodies in X. We find a linear functional that separates A and B.

If $\|f - h\| \leq 1$, then $f(t) \geq e^t - 1$ for all $t \in [0,1]$; in particular, $f(t) \geq 1$ on $[\ln 2, 1]$. On the other hand, if $\|f\| \leq 1$, then $f(t) \leq 1$ on $[\ln 2, 1]$. To separate A from B, we seek a function $g \in \mathrm{BV}$ such that the linear functional

$$F(f) = \int_0^1 f\,dg$$

separates A and B.

We can obtain such a g easily; let $g(t) = 0$ for $0 \le t \le \ln 2$ and $g(t) = t - \ln 2$ for $\ln 2 \le t \le 1$. If $\|f\| \le 1$, then

$$F(f) \le \int_0^1 1\,dg = \int_{\ln 2}^1 dg = 1 - \ln 2.$$

For $\|f - h\| \le 1$,

$$F(f) \ge \int_0^1 (e^t - 1)\,dg = \int_{\ln 2}^1 (e^t - 1)\,dg \ge \int_{\ln 2}^1 1\,dg = 1 - \ln 2.$$

The functional F therefore separates A and B.

Exercises

12:9.1 Let X be a linear space. Verify the following statements:

(a) Any subspace of X is convex.

(b) If A and B are convex subsets of X and $a, b \in \mathbb{R}$, then the set
$$aA + bB = \{ax + by : x \in A, y \in B\}$$
is convex.

(c) If $\mathcal{A}$ is a family of convex sets in X, then $\displaystyle\bigcap_{A \in \mathcal{A}} A$ is convex.

(d) For each set $S \subset X$ there exists a smallest convex set in X containing S. This set is called the *convex hull* of S.

12:9.2 (Refer to Example 12.44.) Let x be a member of A, say
$$x = a_0 + a_1 u + \cdots + a_n u^n \ , \ a_n > 0.$$
Show that x is not an interior point of A by considering polynomials of the form $x + tu^{n+1}$.

12:9.3 (Refer to Example 12.45.) Let $h(t) = ae^t$, $a \ge 0$.

(a) Find the smallest value of a for which the functional F given in the example separates A and B.

(b) Is there a smallest value of a for which *some* linear functional separates A and B? If so, what is it? If not, find the infimum of
$$\{a > 0 : \exists \text{ a linear functional } F \text{ that separates } A \text{ and } B\}\,.$$

(c) How would the answer to (b) change if the question were asked for open balls rather than closed balls?

12.10 An Embedding Theorem

The notion of an abstract normed linear space is a large one to grasp. It is defined axiomatically, and it encompasses a seemingly inexhaustible variety of concrete examples. Often in mathematics in such a situation there is some way of realizing all instances of an abstract structure as aspects of one single thing. In this section we shall see that all normed linear spaces can be viewed as spaces of functions equipped with the sup norm. Specifically, we embed every normed linear space as a subspace of $M(A)$ for some set A.

In Section 9.6 we discussed embeddings of a metric space X into a metric space Y. The mapping that defined the embedding was required to be an isometry, thereby preserving metric properties. We did not require linearity, since no linear structure was imposed on X. Thus we can identify X and Y if these spaces are isometric.

In our present setting we are dealing with normed linear spaces. We wish to identify two such spaces X and Y if there is a linear mapping ϕ from X onto Y such that

$$\|\phi(x)\| = \|x\|$$

for all $x \in X$. Such a mapping ϕ is called an *isomorphism* or *linear isometry*, and we say that X and Y are *isomorphic*. If $Y = \phi(X)$ is contained in a normed linear space Z, we say that X is *embedded* in Z, or X is isomorphic to the subspace Y of Z.

The main theorem of this section involves embedding X into the Banach space $M(A)$ of bounded functions on an appropriate set A, with the sup norm. (See Example 12.7.)

We can now state our theorem. Observe in the proof that the expression $f_\alpha(x)$ appears repeatedly and with varying interpretations. Exercise 12:10.1 may be helpful in distinguishing these interpretations.

Theorem 12.46 *Let X be a normed linear space. Then there exists a set A such that X is isomorphic with a subspace of the Banach space $M(A)$ of bounded real-valued functions on A with norm*

$$\|f\|_\infty = \sup_{t \in A} |f(t)|.$$

Proof. We begin by choosing any dense subset of X and indexing this set as $\{x_\alpha : \alpha \in A\}$. The index set A will be the domain for the functions in the Banach space that we construct. For each $\alpha \in A$, there exists by Theorem 12.37 a linear functional f_α on X such that

$$\|f_\alpha\| = 1 \quad \text{and} \quad f_\alpha(x_\alpha) = \|x_\alpha\|.$$

For each $\alpha \in A$ and $x \in X$, we have

$$|f_\alpha(x)| \leq \|f_\alpha\| \, \|x\| = \|x\|. \tag{17}$$

To this point, we have viewed each f_α as a function of x. We now change our perspective. For each $x \in X$, $f_\alpha(x) \in \mathbb{R}$, for every $\alpha \in A$.

Thus for each $x \in X$ we can view $f_\alpha(x)$ as a function of α, which by (17) is bounded on A. Now define $\phi : X \to M(A)$ by $(\phi(x))(\alpha) = f_\alpha(x)$. Thus, for each $x \in X$, $\phi(x)$ is a bounded function on A. We therefore view ϕ as a mapping from X to $M(A)$ and show that ϕ is an isomorphism of X onto $\phi(X) \subset M(A)$.

To check the linearity of ϕ, let $x, y \in X$. From the linearity of the functionals f_α, we see that

$$\phi(x+y) = f_\alpha(x+y) = f_\alpha(x) + f_\alpha(y) = \phi(x) + \phi(y).$$

Similarly, for $x \in X$ and $a \in \mathbb{R}$, we obtain

$$\phi(ax) = f_\alpha(ax) = af_\alpha(x) = a\phi(x).$$

Thus ϕ is linear.

It remains to show that ϕ is norm preserving; that is, for every $x \in X$,

$$\|x\| = \sup_{\alpha \in A} |f_\alpha(x)| = \|\phi(x)\|_\infty.$$

From (17), we see that

$$\sup_{\alpha \in A} |f_\alpha(x)| \leq \|x\|, \tag{18}$$

so we need only establish the reverse inequality.

For each $\alpha \in \mathbb{R}$ and $x \in X$, $f_\alpha(x_\alpha) = \|x_\alpha\|$, so

$$|\,|f_\alpha(x_\alpha)| - \|x\|\,| = |\,\|x_\alpha\| - \|x\|\,| \leq \|x_\alpha - x\|.$$

Also,

$$\begin{aligned} |\,|f_\alpha(x)| - |f_\alpha(x_\alpha)|\,| &\leq |f_\alpha(x) - f_\alpha(x_\alpha)| \\ &= |f_\alpha(x - x_\alpha)| \leq \|x - x_\alpha\|, \end{aligned}$$

the last inequality following from (17). Thus

$$|\,|f_\alpha(x)| - \|x\|\,| \leq 2\|x_\alpha - x\|.$$

Finally, we recall the fact that the set $\{x_\alpha : \alpha \in A\}$ is dense in X. We can therefore choose $\alpha \in A$ such that $\|x_\alpha - x\|$ is arbitrarily small, so $f_\alpha(x)$ is arbitrarily close to $\|x\|$. It follows that

$$\|x\| \leq \sup_{\alpha \in A} |f_\alpha(x)|. \tag{19}$$

It follows from (18) and (19) that

$$\|x\| = \sup_{\alpha \in A} |f_\alpha(x)| = \|\phi(x)\|_\infty.$$

Thus ϕ is norm preserving. This completes the proof. ■

If X is a separable normed linear space, we can choose A to be a countable set. Since A is only an index set (it has no metric or measure associated with it), we can take A to be $\mathbb{N}$. Thus we have proved that every separable normed linear space is isomorphic to a subspace of the space ℓ_∞ of bounded sequences with norm

$$\|x\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}.$$

Corollary 12.47 *Every separable normed linear space is isomorphic to a subspace of the space ℓ_∞.*

Exercises

12:10.1 Consider the functions appearing in the proof of Theorem 12.46.

(a) For each $\alpha \in \mathbb{R}$, $f_\alpha(x) \in \mathbb{R}$ for all $x \in X$, so $f_\alpha : X \to \mathbb{R}$ is a bounded linear functional on X.

(b) For each $x \in X$, $f_\alpha(x) \in M(A)$, so $f_\alpha(x) \in \mathbb{R}$ for all $\alpha \in A$.

(c) For each $x \in X$, $(\phi(x))(\alpha) = f_\alpha(x)$; hence $\phi : X \to M(A)$.

Thus the expression $f_\alpha(x)$ appears in three different contexts. Clarify for yourself the differences in the three usages of the notation $f_\alpha(x)$. For example, are the functions in (a) continuous? What are their domains and ranges? The same questions are relevant for (b). What about ϕ? Is ϕ continuous? Is ϕ one to one? Is ϕ an isometry?

12.11 The Uniform Boundedness Principle

The study of linear operators in Banach spaces is dominated by four powerful and important ideas: the Hahn–Banach theorem, the uniform boundedness principle, the open mapping theorem, and the closed graph theorem. Many arguments in the subject will touch on one or more of these themes. We have already discussed at some length some of the ideas surrounding the Hahn–Banach theorem. In this section we turn to the uniform boundedness principle.

Commonly, this is attributed to Banach and to Hugo Steinhaus (1887–1972) and may appear cited as the Banach–Steinhaus theorem. The original conception appears in an argument of Lebesgue in 1908, and his ideas in turn might be traced back to the condensation of singularities method of Cantor.

We inquire as to the continuity behavior of a collection $\mathcal{F}$ of linear operators from a Banach space X to a normed linear space Y. We already know that boundedness and continuity are related for a single operator. What conditions will give equicontinuity for the family? We have already seen in Sections 9.11 and 9.12 that this notion of equicontinuity plays a vital role in some investigations. It is easy to see that equicontinuity for the family is related to a uniform boundedness of the operators in the family: if $\|T\| \leq M$ for all $T \in \mathcal{F}$, then the inequality

$$\|T(x) - T(y)\| \leq M\|x - y\|$$

holds throughout the family and the space giving equicontinuity. The uniform boundedness principle allows us to claim such a condition from an apparently much weaker pointwise boundedness condition. The proof employs a category argument, and this is why we need the domain space X to be complete.

Theorem 12.48 (Uniform boundedness) *Let X be a Banach space, let Y be a normed linear space, and let $\mathcal{F}$ be a family of bounded linear operators from X to Y. Suppose that for each $x \in X$ there exists a constant M_x such that*

$$\|Tx\| \leq M_x \quad \textit{for all } T \in \mathcal{F}.$$

Then there exists a constant M such that

$$\|T\| \leq M \quad \textit{for all } T \in \mathcal{F}.$$

Proof. For each $n \in \mathbb{N}$, let

$$A_n = \{x \in X : \|Tx\| \leq n \ \text{ for all } T \in \mathcal{F}\}.$$

Since each T is continuous, the set $\{x : \|Tx\| \leq n\}$ is closed. Since

$$A_n = \bigcap_{T \in \mathcal{F}} \{x : \|Tx\| \leq n\} \tag{20}$$

these sets are closed, too. The assumption in the statement of the theorem means that every point in the space is in one of the sets A_n. By the Baire category theorem, we conclude that there exists $n_0 \in \mathbb{N}$ and a ball $B(x_0, \delta) \subset X$ such that$\|Tx\| \leq n_0$ for all $x \in B(x_0, \delta)$ and $T \in \mathcal{F}$. Let $z \in X$ with $\|z\| < \delta$. Then $x_0 + z \in B(x_0, \delta)$. It follows that, for $T \in \mathcal{F}$,

$$\|Tz\| = \|T(x_0 + z) - T(x_0)\| \leq \|T(x_0 + z)\| + \|Tx_0\| \leq 2n_0.$$

Thus $\|Tz\| \leq 2n_0$ on $B(\mathbf{0}, \delta)$, so $\|Tx\| \leq 2n_0/\delta$ for all $x \in B(\mathbf{0}, 1)$. This means that

$$\|T\| \leq 2n_0/\delta$$

for all $T \in \mathcal{F}$ and the theorem is proved with $M = 2n_0/\delta$. ∎

We can use the uniform boundedness principle to obtain a contrast between the structure of Baire-1 functions on $\mathbb{R}$ and pointwise limits of continuous linear operators on a Banach space X. Recall that a function in the first Baire class is one that is a pointwise limit of a sequence of continuous functions. Such a function can be discontinuous almost everywhere on $\mathbb{R}$. [See Exercise 5:5.5(f).]

Theorem 12.49 *Let $\{T_n\}$ be a sequence of continuous linear operators on a Banach space X to a normed linear space Y. If $\{T_n\}$ converges pointwise to a function T on X, then T is a continuous linear operator on X.*

Proof. That T is linear is clear since

$$\begin{aligned} T(ax+by) &= \lim_{n\to\infty} T_n(ax+by) = \lim_{n\to\infty}(aT_n(x)+bT_n(y)) \\ &= aT(x)+bT(y). \end{aligned}$$

We have only to show that T is continuous on X. Let $x \in X$, with $\|x\| = 1$. Since $\{T_n x\}$ converges to Tx, $\{\|T_n x\|\}$ is bounded, say $\|T_n x\| \le M_x$. From the uniform boundedness principle, we infer the existence of a constant M such that $\|T_n\| \le M$ for all $n \in \mathbb{N}$. For every $z \in X$ with $\|z\| = 1$, we have

$$\|Tz\| = \lim_{n\to\infty} \|T_n z\| \le \limsup_{n\to\infty} \|T_n\|\,\|z\| = \limsup_{n\to\infty} \|T_n\| \le M.$$

Thus $\|T\| \le M$, so T is bounded and therefore continuous. ■

Thus continuity is preserved under pointwise limits of sequences of continuous linear operators on Banach spaces. For example, suppose that $\{g_n\}$ is a sequence of functions of bounded variation on $[a,b]$, and

$$T_n(f) = \int_a^b f\,dg_n$$

converges to $T(f)$ for all $f \in \mathcal{C}[a,b]$. By Theorem 12.49, T is a bounded linear functional on $\mathcal{C}[a,b]$. It follows from the Riesz representation theorem that there exists $g \in \mathrm{NBV}[a,b]$ such that

$$T(f) = \int_a^b f\,dg \quad \text{for all } f \in \mathcal{C}[a,b].$$

There is another important way in which the uniform boundedness principle can be used. Suppose that the family $\mathcal{F}$ of bounded linear operators in Theorem 12.48 is *not* uniformly bounded. Then each of the closed sets $\{A_n\}$ in (20) must be nowhere dense; otherwise, the conclusion of the theorem would be reached. This gives us another interpretation of the theorem, which is known as the principle of the condensation of singularities for linear operators on a Banach space. We apply this idea to a double sequence $\{T_{mn}\}$ of operators $(m, n \in \mathbb{N})$ and obtain a a first-category set for each m. The union over m of those sets is first category. We state this as a theorem.

Theorem 12.50 *Let X be a Banach space, let Y be a normed linear space, and let $\{T_{mn}\}$ be a doubly indexed sequence of bounded linear operators from X to Y such that for each m there is some $x_m \in X$ for which*

$$\limsup_{n\to\infty} \|T_{mn}(x_m)\| = \infty.$$

Then the set of points $x \in X$ for which

$$\limsup_{n\to\infty} \|T_{mn}(x)\| = \infty \quad \textit{(all } m = 1, 2, 3, \dots \textit{)} \tag{21}$$

is residual in X.

In the language of Chapter 10 we could say that, for the typical point $x \in X$, the assertion (21) holds. In Chapter 15 we shall apply the uniform boundedness principle in this form to show that there are continuous functions whose Fourier series diverge at many points. Applications such as this indicate the power of these methods.

Exercises

12:11.1 Completeness is essential in the proof of Theorem 12.48. Let X be the linear space of polynomials $p(t) = a_0 + a_1 t + \cdots a_m t^m$ of any degree m equipped with norm $\|p\| = \max_i |a_i|$. Define

$$f_n(p) = \sum_{i=0}^{n-1} a_i.$$

Show that $\{f_n\}$ is a sequence of bounded linear functionals on X, that

$$|f_n(p)| \leq (m+1)\|p\|$$

for every polynomial $p(t) = a_0 + a_1 t + \cdots a_m t^m$ and n, but that the norms $\{\|f_n\|\}$ are unbounded. How does this not contradict Theorem 12.48?

12.12 An Application to Summability

In this section we show how some functional analytic ideas can be applied to a very classical problem, the convergence of infinite sequences. The interesting aspect of this example is the shift in viewpoint: A problem that starts out with an investigation of convergent sequences finds its proper expression in the language of linear functionals on a Banach space where it can draw on such powerful tools at the uniform boundedness principle.

Our problem is that of assigning a "limit" to a divergent infinite sequence $\{x_i\}$. By a *summability method,* we mean that we are given a doubly infinite matrix

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

and we use, if possible, as the new version of the limit of the sequence $\{x_i\}$ the expression

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} a_{ni} x_i.$$

For example, a simple choice of matrix $\mathcal{A}$ would give a method of summation that merely takes averages

$$\lim_{n\to\infty} \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n},$$

and this has proved most useful in applications. It is normally named after Ernesto Cesàro (1859–1906) who studied it in 1890, but it had been employed much earlier.

It is clear that there are some restrictions to be imposed on the matrix $\mathcal{A}$ in order for this to be profitable. We need the sums $\sum_{i=1}^{\infty} a_{ni}x_i$ to be defined; in order for this to work for all bounded sequences $\{x_i\}$, we should ask for $\sum_{i=1}^{\infty} |a_{ni}|$ to converge. The method applied to the constant sequence $e_0 = (1, 1, 1, 1, \dots)$ should naturally produce 1 as limit, and this cannot happen unless

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} a_{ni} = 1.$$

The method applied to the sequence

$$e_m = (0, 0, \dots, 0, 1, 0, 0, 0, 0 \dots) \qquad (m \geq 1),$$

where the solitary 1 occurs in the m–th place, should naturally produce 0 as limit, and this cannot happen unless

$$\lim_{n\to\infty} a_{nm} = 0.$$

These considerations and a bit of hard work led Otto Toeplitz (1881–1940) to impose the following conditions, which should seem entirely natural. The first condition appears a bit strong at first glance, since we are asking for a *uniform* bound on the sums $\sum_{i=1}^{\infty} |a_{ni}|$.

Definition 12.51 A summability method defined by a matrix

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

is said to be *regular* provided that

1. $\sup_n \sum_{i=1}^{\infty} |a_{ni}| < \infty$,
2. $\lim_{n\to\infty} a_{nm} = 0$ for each $m = 1, 2, 3, \dots$, and
3. $\lim_{n\to\infty} \sum_{i=1}^{\infty} a_{ni} = 1$.

The most important features of regular summability methods is that they assign to sequences that are already convergent the limit that we would have assigned anyway. What is more remarkable is that the only summability methods that have this property for all convergent sequences are the regular ones.

Theorem 12.52 (Toeplitz) *In order for a summability method defined by a matrix $\mathcal{A}$ to assign the value $\lim x_n$ for every convergent sequence $\{x_n\}$, it is necessary and sufficient that $\mathcal{A}$ be regular.*

Proof. It is the proof of this theorem that is of primary interest to us. The theorem itself, though, will be needed for some discussions of summability of trigonometric series in Chapter 15.

The highlight of the proof is the reinterpretation of the statement into the language of linear functionals. The Banach space that is clearly present in the statement of the theorem is the space c of convergent sequences with the sup norm $\|x\|_\infty = \sup |x_i|$. For the matrix $\mathcal{A} = (a_{ij})$ and $x \in c$, write

$$T_{mn}(x) = \sum_{i=1}^{m} a_{ni} x_i \, , \quad T_n(x) = \sum_{i=1}^{\infty} a_{ni} x_i$$

and observe that each $T_{mn}, T_n : c \to \mathbb{R}$ is a linear functional on c (assuming that the series converge) and with norms

$$\|T_{mn}\| = \sum_{i=1}^{m} |a_{ni}| \, , \quad \|T_n\| = \sum_{i=1}^{\infty} |a_{ni}|.$$

For example, the inequality

$$\left| \sum_{i=1}^{m} a_{ni} x_i \right| \leq (\sup |x_i|) \sum_{i=1}^{m} |a_{ni}|$$

shows that

$$\|T_{mn}\| \leq \sum_{i=1}^{m} |a_{ni}|,$$

and the choice of $x \in c$ with $x_i = \pm 1$ so that $a_{ni} x_i = |a_{ni}|$ shows that the value of the norm is correct.

Thus the summability method consists of taking for the limit of the sequence $\{x_i\}$ the expression

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \lim_{m\to\infty} T_{mn}(x) = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{m} a_{ni} x_i.$$

Thus we are now in a setting with some powerful tools: sequences of bounded linear functionals on a Banach space.

Suppose that the method $\mathcal{A}$ assigns the ordinary limit to all sequences $x \in c$. Conditions (2) and (3) of Definition 12.51 must hold as we already noted in the discussion before the definition. We wish to establish condition (1) of Definition 12.51. The limit

$$\lim_{m\to\infty} T_{mn}(x) = \lim_{m\to\infty} \sum_{i=1}^{m} a_{ni} x_i$$

must exist for all $x \in c$, and so, by Theorem 12.49, T_n must be a continuous linear functional for each n, and

$$\|T_n\| = \lim_{m\to\infty} \|T_{mn}\| = \sum_{i=1}^{\infty} |a_{ni}|.$$

Once again the limit

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{i=1}^{\infty} a_{ni}x_i$$

must exist for all $x \in c$, and so by the uniform boundedness principle (Theorem 12.48) the norms $\|T_n\|$ are uniformly bounded. Hence we have

$$\sum_{i=1}^{\infty} |a_{ni}| \le \sup_n \|T_n\| < \infty,$$

which is exactly condition (1) of Definition 12.51.

Conversely, suppose that $\mathcal{A}$ is regular. Let $T(x) = \lim_{i\to\infty} x_i$ for each $x \in c$. This is a continuous linear functional on c assigning to each convergent sequence its limit. Clearly, $\|T\| = 1$. We wish to show that $\mathcal{A}$ assigns this same value to every $x \in c$; that is, $T_n(x) \to T(x)$. Consider the special elements of c that we have already indicated as $e_0, e_1, e_2, \ldots$ in the preamble to our definition. Note that every element of the space c can be approximated by a finite linear combination of these (Exercise 12:12.5). Conditions (1), (2), and (3) of Definition 12.51 show easily that $T_n(e_k) \to T(e_k)$ for each of these special sequences. In fact, from this and condition (1) we can show that $T_n(x) \to T(x)$ for every $x \in c$.

To this end, let $\varepsilon > 0$ and $M = \sup_n \|T_n\|$. Choose a finite linear combination

$$x_0 = \sum_{k=0}^{K} \lambda_k e_k$$

so that

$$\|x_0 - x\|_\infty < \frac{\varepsilon}{3M+3}.$$

Since $T_n(e_k) \to T(e_k)$ for each $1 \le k \le K$, there is an integer N so that

$$|T_n(x_0) - T(x_0)| < \varepsilon/3$$

for all $n \ge N$. Thus

$$\begin{aligned}|T_n(x) - T(x)| &\le |T_n(x) - T_n(x_0)| + |T_n(x_0) - T(x_0)| + |T(x_0) - T(x)| \\ &\le M\|x - x_0\|_\infty + \tfrac{1}{3}\varepsilon + \|T\|\|x - x_0\|_\infty < \varepsilon\end{aligned}$$

for all $n \ge N$. We have shown that $T_n(x) \to T(x)$ for every $x \in c$., Hence, this summability method assigns the correct value $\lim x_n$ for every convergent sequence $\{x_n\}$. ■

Exercises

12:12.1 Show that condition (1) of Definition 12.51 is a consequence of condition (3) if all the a_{ni} are nonnegative.

12:12.2 What summability matrix would yield the Cesàro method of summation that takes averages
$$\lim_{n\to\infty}\frac{x_1+x_2+x_3+\cdots+x_n}{n}.$$
Is this matrix regular?

12:12.3 Let $\mathcal{A}$ be a regular matrix, each of whose entries is nonnegative. Set $\sigma_n=\sum_{i=1}^{\infty}a_{ni}x_i$. Show that
$$\liminf_{n\to\infty}x_n\le\liminf_{n\to\infty}\sigma_n\le\limsup_{n\to\infty}\sigma_n\le\limsup_{n\to\infty}x_n.$$
(Is this condition that all the entries are nonnegative necessary?)

12:12.4 Let p_1, p_2, p_3, ... be a sequence of real numbers and q_1, q_2, q_3, ... a sequence of positive numbers. Write $P_n=p_1+p_2+\cdots+p_n$ and $Q=q_1+q_2+\cdots+q_n$. Show that if $p_n/q_n\to x$ then $P_n/Q_n\to x$. What does this reduce to if all $q_i=1$? [Hint: Let $x_i=p_i/q_i$, and check that $P_n/Q_n=\sum_{i=1}^{n}q_iQ_n^{-1}x_i$.]

12:12.5 Show that every element of the space c can be approximated by a finite linear combination from the set $\{e_0,e_1,e_2,\dots\}$. [Hint: If $x\in c$, then $x_k\to\lambda$. Compute
$$\left\|\lambda e_0+\sum_{k=1}^{n}(x_k-\lambda)e_k-x\right\|.$$

12:12.6 There is a general principle at work in the proof of Theorem 12.52. Let T, T_n be continuous linear functionals on a Banach space X. Show that $T_n(x)\to T(x)$ for every $x\in X$ (we say $T_n\to T$ weakly) if and only if $\sup_n\|T_n\|<\infty$ and $T_n(x)\to T(x)$ for every x belonging to some set whose linear span is dense in X.

12.13 The Open Mapping Theorem

A mapping $T:X\to Y$ is said to be *open* if the image of every open set in X is open in Y. Note the comparison with continuity: T is continuous if the preimage $T^{-1}(G)$ of every open set $G\subset Y$ is open. A bijection that is continuous and open is evidently a homeomorphism.

In the setting of Banach spaces, bounded linear transformations that are surjections are always open. This fact is known as the open mapping theorem and is due to Banach. Like the uniform boundedness principle of the previous section and the closed graph theorem of the next, it is one of the preeminent tools of the theory.

Theorem 12.53 (Open mapping) *A bounded linear operator T mapping a Banach space X onto a Banach space Y is an open mapping; that is, it maps open sets in X onto open sets in Y. Thus, if T is one to one, then T^{-1} is continuous.*

Proof. To prepare for the proof, we recall some notation. If A and B are subsets of X and $\alpha \in \mathbb{R}$, then αA denotes the set $\{\alpha a : a \in A\}$ and $A - B$ denotes the set $\{a - b : a \in A, b \in B\}$.

We first show that the closure of the set $T(B(\mathbf{0}, 1))$ contains a ball centered at $\mathbf{0}$ in Y. Let $B = B(\mathbf{0}, 1/2)$ in X. For $x \in X$, the sequence $\{x/n\}$ converges to $\mathbf{0}$, so there exists $n \in \mathbb{N}$ such that $x \in nB$. Thus

$$Y = \bigcup_{n=1}^{\infty} nT(B).$$

Since Y is complete, the Baire category theorem implies that there exists $n \in \mathbb{N}$ such that the closure of the set $nT(B)$ contains a ball in Y. It follows readily (see Exercise 12:13.3) that the closure of the set $T(B)$ also contains a ball $B(y_0, s)$ and that y_0 can be chosen in $T(B)$.

Choose $x_0 \in B$ such that $y_0 = Tx_0$. For each $y \in B(y_0, s)$, there exists $x \in X$ such that $Tx = y$. Now

$$T(x_0 - x) = y_0 - y \in B(\mathbf{0}, s).$$

By allowing y to vary over $\overline{T(B)}$, we see that

$$\left\{ y_0 - y : y \in \overline{T(B)} \right\} \supset B(\mathbf{0}, s).$$

But $T(B)$ is dense in $\overline{T(B)}$, and for each $y \in T(B)$, there exists $x \in B$ such that $Tx = y$. For such an x, we have

$$x_0 - x \in B(\mathbf{0}, 1).$$

It follows that $T(B(\mathbf{0}, 1))$ is dense in $B(\mathbf{0}, s)$, so

$$\overline{T(B(\mathbf{0}, 1))} \supset B(\mathbf{0}, s). \tag{22}$$

To complete the proof, let

$$B_n = B\left(\mathbf{0}, 2^{-n}\right)$$

in the space X, and let

$$B_n^* = B\left(\mathbf{0}, s2^{-n}\right)$$

in the space Y for each $n = 0, 1, 2, \ldots$. From (22) and the linearity of T, we infer that

$$\overline{T(B_n)} \supset B_n^*.$$

We now show that $T(B_0) \supset B_1^*$. Let $y \in B_1^*$. Since $y \in \overline{T(B_1)}$ and T is continuous, there exists $x_1 \in B_1$ such that

$$\|y - Tx_1\| < \frac{s}{4}.$$

Proceeding inductively, we obtain a sequence $\{x_k\}$ of points in X such that, for each k, $x_k \in B_k$ and

$$\left\| y - \sum_{k=1}^{n} T(x_k) \right\| < \frac{s}{2^{n+1}} \tag{23}$$

for all $n \in \mathbb{N}$.

The sequence $\{\sum_{k=1}^{n} x_k\}$ is a Cauchy sequence in X, since $\|x_k\| < 2^{-k}$ for each k. Because X is complete by hypothesis, the sequence converges to a point $x = \sum_{k=1}^{\infty} x_k$ in X. Now T is continuous, so

$$T(x) = \lim_{n\to\infty} T\left(\sum_{k=1}^{n} x_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} T(x_k) = \sum_{k=1}^{\infty} T(x_k) = y,$$

the last equality following from (23). Thus $y \in T(B_0)$. This shows that $T(B_0) \supset B_1^*$.

We have shown that the image under T of the unit ball $B(\mathbf{0}, 1)$ in X contains a ball $B(\mathbf{0}, \frac{1}{2}s)$ centered at $\mathbf{0}$ in Y. The linearity of T implies that T maps every open ball $B(x, r)$ onto a set containing a ball with center $T(x)$. It follows that T maps open sets onto open sets, as was to be proved. ■

As an application illustrating how this theorem can be used, we obtain a theorem showing how to check that two norms on a Banach space are *equivalent*. This is meant in the same sense as the equivalence of metrics that we discussed in Section 9.4. For example, we know that the Euclidean plane $\mathbb{R}^2$ is complete under the norm

$$\|x\|_1 = |x_1| + |x_2|,$$

as well as under the norm

$$\|x\|_\infty = \max\{|x_1|, |x_2|\},$$

and that these norms are equivalent. We obtain this from the fact that there exist constants c_1 and c_2 such that, for each $x \in \mathbb{R}^2$,

$$\|x\|_1 \leq c_1 \|x\|_\infty \quad \text{and} \quad \|x\|_\infty \leq c_2 \|x\|_1.$$

The open mapping theorem shows that it is enough to establish just one since either of the inequalities above implies the other.

Theorem 12.54 *Let X be a linear space that is complete with respect to each of two norms $\|\cdot\|_a$ and $\|\cdot\|_b$. If there exists a constant c_1 such that*

$$\|x\|_a \leq c_1 \|x\|_b$$

for all $x \in X$, then there exists a constant c_2 such that

$$\|x\|_b \leq c_2 \|x\|_a$$

for all $x \in X$.

Proof. Apply the open mapping theorem to the identity map $Tx = x$. ■

We conclude this section with an application of the open mapping theorem to obtain a result about perturbations in differential equations.

Example 12.55 Consider the differential equation

$$x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t). \tag{24}$$

Here a_1, a_2, and y are members of $\mathcal{C}[a, b]$. An initial-value problem for (24) calls for finding a twice continuously differentiable function x on $[a, b]$ satisfying (24) and satisfying the initial conditions

$$x(a) = x'(a) = 0.$$

A standard theorem in differential equations asserts that this initial value problem has a unique solution. We wish to study the dependency of the function x on y, and vice versa.

Let $X = \mathcal{C}^2$, the space of twice continuously differentiable functions. Then X becomes a Banach space under the norm

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\}, \tag{25}$$

where $\|\cdot\|_\infty$ denotes the usual supremum norm in $\mathcal{C}[a, b]$. (See Exercise 12:13.1.) Let $Y = C[a, b]$. Let $T : X \to Y$ be defined by $Tx = y$, where

$$y(t) = x''(t) + a_1(t)x'(t) + a_2(t)x(t), \quad (a \leq t \leq b). \tag{26}$$

The standard theorem in differential equations that we mentioned before asserts that T is one to one on X and maps X onto Y. We show that T is a continuous operator.

Let

$$A = 1 + \|a_1\|_\infty + \|a_2\|_\infty.$$

Then

$$\|Tx\|_\infty = \|y\|_\infty \leq \|x''\|_\infty + \|a_1\|_\infty \|x'\|_\infty + \|a_2\|_\infty \|x\|_\infty \leq A\|x\|,$$

so T is continuous. By the open mapping theorem, T^{-1} is also continuous. We can interpret this as saying that small perturbations of the function y will result in small perturbations of the solution $x \in \mathcal{C}^2$. This means that such a perturbed solution x_1 will be "$\mathcal{C}^2$-close" to x; that is, x_1, x_1', and x_1'' will be (uniformly) close to x, x', and x'', respectively.

Exercises

12:13.1 Show that $\mathcal{C}^2$ is a Banach space under the norm indicated in (25).

12:13.2 Let X be the linear space of continuous functions on [0,1]. Let $\|\cdot\|_\infty$ be the norm of $\mathcal{C}[0,1]$, and let $\|\cdot\|_1$ be the L_1 norm.

(a) Show that there exists a constant c_1 such that $\|f\|_1 \leq c_1\|f\|_\infty$ for all $f \in X$.

(b) Show that there is no constant c_2 such that $\|f\|_\infty \leq c_2\|f\|_1$ for all $f \in X$.

(c) Explain why the results in (a) and (b) do not violate Theorem 12.54.

12:13.3 (Refer to the proof of Theorem 12.53.) Verify that $\overline{T(B)}$ contains a ball $B(y_0, s)$ with $y_0 \in T(B)$.

12.14 The Closed Graph Theorem

Let f be a real-valued function defined on an interval $[a, b]$. In elementary courses, the function is often studied in terms of its graph

$$G_f = \{(x, y) : x \in [a, b],\ y = f(x)\},$$

which is a subset of the space $\mathbb{R}^2$. If f is continuous, then the graph G_f is closed. To prove this, just suppose that $(x_n, y_n) \in G_f$ converges to (x, y) then

$$f(x) = \lim f(x_n) = \lim y_n = y,$$

and so $(x, y) \in G_f$, hence this set is closed. Does the converse hold? If f has a closed graph, must f be continuous? If that were so, it would be a useful observation, since checking for the closure of a set may be easier than checking continuity everywhere. It is not hard to come up with an example of a discontinuous function with a closed graph, but it is if you are looking for bounded functions. In fact, the following theorem is easy to prove using our metric space methods of Chapter 9.

Theorem 12.56 *Let $f : X \to Y$, where X and Y are metric spaces and Y is compact. If f has a closed graph*

$$G_f = \{(x, f(x)) : x \in X\}$$

(as a subset of $X \times Y$), then f is continuous.

Can we use these ideas in the study of linear operators on Banach spaces? If a linear operator has a closed graph, must it be continuous? The answer is yes, and this is a most useful fact, but it is true for different reasons than the theorem above might suggest. We have no hope of using compactness arguments here since, as we have seen in Section 12.2, compact sets in infinite-dimensional spaces are rather small. The closed graph theorem for linear operators is due, as is much of the material of this chapter, to Banach. It can be obtained almost immediately from the open mapping theorem. Observe that to check for a closed graph for an operator T we merely show that the assertion of the following definition holds.

Definition 12.57 Let X and Y be Banach spaces, and let T be a linear operator defined on a subspace X_1 of X with $T(X_1) \subset Y$. Then T is called *closed* if

$$x_n \in X_1\,, \ \lim_{n\to\infty} x_n = x\,, \ \text{ and } \ \lim_{n\to\infty} Tx_n = y$$

implies that $x \in X_1$ and $y = Tx$.

It is clear that a bounded linear operator defined on a closed subspace of a Banach space is a closed operator. While the converse is not true in general, as the operator D in Example 12.59 will illustrate, it is true under the assumptions that the domain X_1 of the operator T is closed. This result, called the *closed graph theorem* for reasons that will be apparent, follows easily from the *open mapping theorem*. We can now state and prove the closed graph theorem.

Theorem 12.58 (Closed graph theorem) *Let T be a closed linear operator from X to Y, where X and Y are Banach spaces. Then T is continuous.*

Proof. Denote the norm in X by $\|\cdot\|$. Define a norm $\|\cdot\|_a$ by

$$\|x\|_a = \|x\| + \|Tx\|.$$

If $\{x_n\}$ is a Cauchy sequence relative to $\|\cdot\|_a$, then $\{x_n\}$ is also a Cauchy sequence relative to $\|\cdot\|$, and $\{Tx_n\}$ is a Cauchy sequence in Y. Since X and Y are complete, both $\{x_n\}$ and $\{Tx_n\}$ converge, say

$$\|x_n - x\| \to 0 \ \text{ and } \ \|Tx_n - y\| \to 0.$$

Since T is closed, $y = Tx$. Therefore, $\|x_n - x\|_a \to 0$, and X is complete relative to $\|\cdot\|_a$. By Theorem 12.54, there exists a constant c such that $\|x\|_a \le c\|x\|$, so $\|Tx\| \le c\|x\|$. Thus T is continuous, as required. ■

Example 12.59 We observed in Section 12.1 that not all important linear operators on a normed linear space need be continuous. Generally, differential operators tend not to be continuous, but they are usually closed and this compensates. This does not violate the closed graph theorem since the domain of the operator will not be complete and so not a Banach space.

Let $\mathcal{C}_1$ be the subspace of $\mathcal{C} = \mathcal{C}[0,1]$ consisting of those functions on $[0,1]$ that are continuously differentiable. Define D on $\mathcal{C}_1$ by $Df = f'$. Then D is linear and maps $\mathcal{C}_1$ onto $\mathcal{C}$. But D is not bounded on $\mathcal{C}_1$. To see this, let $f_n(t) = t^n$, so

$$(Df_n)(t) = f_n'(t) = nt^{n-1}.$$

Now

$$\|f_n\| = \max_{t\in[0,1]} |f_n(t)| = 1,$$

but

$$\|Df_n\| = \max_{t\in[0,1]} |nt^{n-1}| = n,$$

so D is unbounded on the unit ball in $\mathcal{C}_1$. Observe that neither the sequence $\{f_n\}$ nor the sequence $\{Df_n\} = f_n'$ converges.

Suppose now that $\{f_n\}$ is a sequence in $\mathcal{C}_1$ that converges to f in $\mathcal{C}$, and $\{Df_n\}$ converges to g in $\mathcal{C}$. Since convergence in these spaces is uniform convergence, we can calculate, for $0 \le t \le 1$, that

$$\begin{aligned}\int_0^t g(u)\,du &= \int_0^t \lim_{n\to\infty} f_n'(u)\,du = \lim_{n\to\infty}\int_0^t f_n'(u)\,du \\ &= \lim_{n\to\infty}(f_n(t) - f_n(0)) = f(t) - f(0).\end{aligned}$$

Thus

$$f(t) = f(0) + \int_0^t g(u)\,du$$

for all $t \in [0,1]$. It follows that $f \in \mathcal{C}_1$ and

$$Df = g = \lim_{n\to\infty} f_n'.$$

We see, then, that D is closed. Thus, for $X = Y = \mathcal{C}[0,1]$ and $X_1 = \mathcal{C}_1$, the differential operator $Df = f'$ is closed, but not continuous.

Exercises

12:14.1 Give an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ with a closed graph. (Why does this not contradict the closed graph theorem? After all, $\mathbb{R}$ is a Banach space.)

12:14.2 Prove the metric space theorem, Theorem 12.56.

12:14.3 Formulate and prove a version of the closed graph theorem that applies to a closed linear operator defined only on a subspace of the Banach space X.

12.15 Additional Problems for Chapter 12

12:15.1 (Hamel bases) Let X be a real linear space. A linearly independent set $H = \{x_\alpha\}$ of vectors in X is called a *Hamel basis* for X if every $x \in X$ has a representation of the form

$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n, \quad x_i \in H, \ \alpha_i \in \mathbb{R}.$$

(a) Prove that every real linear space has a Hamel basis. [Hint: Use Zorn's lemma.]

(b) Show that every $x \in X$ has a unique representation as a finite linear combination of elements of H.

(c) Prove that every Hamel basis for X has the same cardinality (called the *algebraic dimension* of X).

(d) Show that $H = \{1, x, x^2, \dots\}$ is a Hamel basis for the linear space of real polynomials.

Let H_1 be a Hamel basis for $\mathcal{C}[a,b]$ such that $H_1 \supset H$. Let $h_1 \in H_1 \setminus H$. Let $Th_1 = 1$ and $Th = 0$ for all $h \in H$, $h \neq h_1$. Extend T to H_1 by linearity.

(e) Show that T is *not* continuous on $\mathcal{C}[a,b]$ (with respect to the usual norm in $\mathcal{C}[a,b]$). [Hint: If T were continuous, the set

$$\{f \in \mathcal{C}[a,b] : Tf = 0\}$$

would be a closed subspace of $\mathcal{C}[a,b]$ containing the polynomials $\mathcal{P}$. But $\mathcal{P}$ is dense in $\mathcal{C}[a,b]$.]

This problem shows that there exist discontinuous linear functionals on $\mathcal{C}[a,b]$.

12:15.2 *(Geometric interpretation of the norm of a linear functional.)* Let $X = \mathbb{R}^2$, with the usual Euclidean norm

$$\|x\| = \sqrt{x_1^2 + x_2^2}.$$

For $x = (x_1, x_2)$, let $f(x) = x_1 + x_2$, and let $A = \big\{x \in \mathbb{R}^2 : f(x) = 1\big\}$.

(a) Calculate $\|f\|$.

(b) Calculate $d = \inf\{\|x\| : x \in A\}$.

(c) Compare the results in (a) and (b) and make and prove a conjecture about $\|f\|$ and d when $f(x) = a_1x_1 + a_2x_2$. Interpret $\|f\|$ geometrically.

Let X be a normed linear space, and let f be a nontrivial continuous linear functional on X. Let $A = \{x \in X : f(x) = 1\}$, and let

$$d = \inf\{\|x\| : x \in A\}.$$

(d) Verify that if $x \in A$ then $\|x\| \geq (\|f\|)^{-1}$ so $d \geq (\|f\|)^{-1}$. [Hint: $|f(x)| \leq \|f\|\,\|x\|$ for all $x \in X$.]

(e) Verify that $d \leq (\|f\|)^{-1}$. [Hint: Show that for every $\varepsilon > 0$ there exists $x_0 \in A$ such that

$$(\|f\| - \varepsilon)\|x_0\| < 1.$$

From (d) with (e), we see that the norm of a continuous linear functional is the reciprocal of the distance between the set $\{x : f(x) = 1\}$ and the origin. The set

$$A = \{x : f(x) = \text{constant}\}$$

is called a *hyperplane*. For $X = \mathbb{R}^2$, A is a line; for $X = \mathbb{R}^3$, A is a genuine plane.

12:15.3 Let X be a linear space, and let f be a linear functional on X. The *kernel* K of f is the set $\{x : f(x) = 0\}$.

(a) Show that K is a linear space.

(b) For $\alpha \in \mathbb{R}$, let $K_\alpha = \{x : f(x) = \alpha\}$. Let $x_1 \in K_\alpha$. Show that $K_\alpha = K + x_1 = \{x + x_1 : x \in K\}$.

(c) Let X be a normed linear space. Show that if K is closed in X then K_α is closed for every $\alpha \in \mathbb{R}$.

(d) Show that f is continuous if and only if K is closed. [Hint: If f is discontinuous at $\mathbf{0}$, there exists a sequence $\{x_n\}$ such that $x_n \to \mathbf{0}$, $f(x_n) \to \alpha \neq 0$, and for each n, $f(x_n) \neq 0$. Let $y_n = x_n/f(x_n)$. Then $y_n \to \mathbf{0}$ and $f(y_n) \to 1$, so K_α is not closed.]

12:15.4 Let D denote the differentiation operator $f \to f'$ defined from X into $\mathcal{C}[a,b]$ where

$$X = \{f \in \mathcal{C}[a,b] : f(a) = 0 \text{ and } f' \text{ is continuous on } [a,b]\}.$$

(a) Show that X is a normed linear space, but is not a Banach space.

(b) Is D a bounded linear operator on X?

(c) Is D a surjection of X to $\mathcal{C}[a,b]$?

(d) Is D a closed operator?

(e) Does D map open sets onto open sets?

(f) Is D a contraction map?

Chapter 13

THE L_P SPACES

In this chapter we collect some interesting facts about the scale of L_p function spaces. As a special case, this includes the scale of sequence spaces ℓ_p.

The three spaces L_1, L_2, and L_∞ and the part of the scale L_p for $1 < p < \infty$ all exhibit different behavior in different settings, and herein lies their importance. In many parts of modern analysis (Fourier analysis, operator theory, differential equations, etc.), these spaces play a key role. The space L_2 of square integrable functions on an interval was early on recognized as important in the study of Fourier series; indeed, these spaces characterize Hilbert spaces, as we shall see in Chapter 14. Many of the linear operators that arise in applications are bounded on the scale L_p ($1 < p < \infty$), but not at the extremes L_1 or L_∞. The whole scale provides, too, a fundamental class of examples of Banach spaces for functional analysts seeking to gain an understanding of the geometry of normed linear spaces and Banach spaces.

The full scale of L_p spaces was first studied by F. Riesz in 1910. The basic ideas go back, however, to Minkowski, who considered the metrics

$$\rho_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$$

on $\mathbb{R}^n$ (see Section 9.1). The ℓ_p spaces can be considered infinite-dimensional analogs of these metric spaces and the L_p spaces as further generalizations.

13.1 The Basic Inequalities

Let $(X, \mathcal{M}, \mu)$ be a measure space and f a real or complex (possibly infinite valued) measurable function defined on the space X. For any $0 < p < \infty$ the function $|f|^p$ is measurable, too (since it is the composition of a continuous function with f), and so we can define what is called the "p–norm" of f by writing

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}.$$

(This might be infinite.) This does have the usual properties that $\|cf\|_p = |c|\|f\|_p$ (unless $c = 0$ and $\|f\|_p = \infty$) and $\|f\|_p = 0$ if and only if $f = 0$ a.e. But, in fact, this is not a norm in the usual sense if $0 < p < 1$ because the triangle inequality fails. Nonetheless, it does not trouble us to use the term norm for this expression. For $1 \leq p < \infty$, the triangle inequality does hold, and so it is a norm in that part of the scale. By $L_p(X, \mathcal{M}, \mu)$ we denote the collection of all real (complex) valued measurable functions f on X for which $\|f\|_p < \infty$. It is this scale of function spaces that is our concern throughout the chapter.

For most of this chapter we present the arguments needed if our functions are real valued; the resulting linear spaces are then over the real numbers. All statements apply as well if the functions are taken as complex valued and the spaces are linear spaces over the complex field. In our studies in Sections 13.5, 13.9, and 13.11, however, we specifically give the arguments for the complex case as it differs in some details from the real.

The cases

$$0 < p < 1, \; p = 1, \; p = 2, \text{ and } 1 < p < \infty$$

are different in character, importance, and technique. We have separated the case $0 < p < 1$ from the discussion; this class is found in Section 13.7. The situation for $p = 1$ we mostly know already. The part of the scale $1 < p < \infty$ receives most of our attention throughout. The special case $p = 2$ (Hilbert space) has its own problems and is discussed in Section 13.5 and again in Chapter 14.

The essential tools in discussing the L_p spaces for $1 < p < \infty$ are the inequalities of Otto Hölder (1860–1937) and Hermann Minkowski (1864–1909). These are so frequently used and so fundamental to our discussion that they deserve a central place, and we obtain them immediately. One crucial concept is already apparent in Hölder's inequality: to any index $1 < p < \infty$ there is a *conjugate index* $1 < q < \infty$ for which

$$\frac{1}{p} + \frac{1}{q} = 1$$

[or, alternatively, $p + q = pq$ or, again, $q = p(p-1)^{-1}$]. The role of the conjugate index appears clearly in the two inequalities and finds later a deeper meaning in the study of the dual spaces. Some special cases to notice: $p = 2$ and $q = 2$ are conjugate indices, and as a limiting case we shall later consider also $p = 1$ and $q = \infty$. (The reader may wish to review related material from Section 9.1 on these inequalities.)

Theorem 13.1 (Hölder's inequality) *Let $(X, \mathcal{M}, \mu)$ be a measure space, let p, q be conjugate indices, and let $f \in L_p(\mu)$, $g \in L_q(\mu)$. Then the product fg is integrable and*

$$\int_X |fg|\, d\mu \le \|f\|_p \|g\|_q.$$

The inequality is strict except precisely in the case where there exist constants c_1, c_2 $(c_1 c_2 \neq 0)$ for which $c_1|f|^p = c_2|g|^q$ μ–a.e.

Proof. Recall that we have already established the elementary inequality

$$\alpha\beta \le p^{-1}\alpha^p + q^{-1}\beta^q \tag{1}$$

and that equality can hold in (1) if and only if $\alpha^p = \beta^q$ (see Section 9.1). Assume, first, that $\|f\|_p = \|g\|_q = 1$. Then the inequality (1) gives us

$$|f(x)|\,|g(x)| \le p^{-1}|f(x)|^p + q^{-1}|g(x)|^q, \tag{2}$$

and an integration of (2) over the space yields

$$\begin{aligned} \int_X |fg|\, d\mu &\le \int_X |f|\,|g|\, d\mu \\ &\le \quad p^{-1}\int_X |f|^p\, d\mu + q^{-1}\int_X |g|^q\, d\mu = p^{-1} + q^{-1} = 1. \end{aligned} \tag{3}$$

The general case follows from this. If $\|f\|_p\|g\|_q = 0$, there is nothing to prove. Otherwise, in (2) replace f by $f/\|f\|_p$ and replace g by $g/\|g\|_p$. Then the inequality

$$\int_X |fg|\, d\mu \le \|f\|_p \|g\|_q$$

follows. From this, too, we see that if $\|f\|_p$ and $\|g\|_q$ are finite then fg is integrable.

Finally, it remains to sort out when equality can occur. We can have equality in (3) if and only if we have equality for μ-almost every x in (2). We know that this occurs if and only if $|f(x)|^p = |g(x)|^q$ for μ-almost every x. This, remember, is under our additional assumption that $\|f\|_p = \|g\|_q = 1$. In the general case, we replace f by $f/\|f\|_p$ and g by $g/\|g\|_p$ as before and so obtain equality in the statement of the theorem if and only if

$$|f(x)|^p/\|f\|_p = |g(x)|^q/\|g\|_q$$

for μ-almost every x. It is now an easy matter to see that the inequality is strict except precisely in the case where $c_1|f|^p = c_2|g|^q$ μ–a.e. for some constants c_1, c_2 (e.g., $c_1 = \|g\|_q$ and $c_2 = \|f\|_p$). ∎

Theorem 13.2 (Minkowski inequality) *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $1 < p < \infty$. For all functions f, $g \in L_p(\mu)$,*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

The inequality is strict except precisely in the case where there are nonnegative constants c_1, c_2 ($c_1c_2 \neq 0$) so that $c_1 f = c_2 g$ μ–almost everywhere.

Proof. Under the assumptions of the theorem, both f and g are finite μ–a.e., and so we can ignore the possibility of infinite values, which would interfere with many statements.

An easy inequality is

$$\begin{aligned} |f(x) + g(x)|^p &= |f(x) + g(x)|\,|f(x) + g(x)|^{p-1} \\ &\leq |f(x)|\,|f(x) + g(x)|^{p-1} + |g(x)|\,|f(x) + g(x)|^{p-1}, \end{aligned} \tag{4}$$

which holds at every point x where $f(x)$ and $g(x)$ are finite. We integrate the inequality (4) to obtain

$$\|f + g\|_p^p = \int_X |f|\,|f + g|^{p-1}\,d\mu + \int_X |g|\,|f + g|^{p-1}\,d\mu. \tag{5}$$

An application of Hölder's inequality to each of the terms on the right-hand side of (5) will complete the proof.

One application yields

$$\begin{aligned} &\int_X |f|\,|f + g|^{p-1}\,d\mu \\ &\leq \ \|f\|_p \left(\int_X |f + g|^{(p-1)p/(p-1)}\,d\mu\right)^{(p-1)/p} \\ &= \ \|f\|_p\,\|f + g\|_p^{p-1}, \end{aligned} \tag{6}$$

and, in the same way, we would have

$$\int_X |g|\,|f + g|^{p-1}\,d\mu \leq \|g\|_p\,\|f + g\|_p^{p-1}. \tag{7}$$

Using (6) and (7) in (5), we obtain

$$\begin{aligned} \|f + g\|_p^p &\leq \|f\|_p\,\|f + g\|_p^{p-1} + \|g\|_p\,\|f + g\|_p^{p-1} \\ &= (\|f\|_p + \|g\|_p)\,\|f + g\|_p^{p-1}, \end{aligned}$$

and the Minkowski inequality follows immediately.

It remains to sort out when equality could occur. This happens if there is equality in (4) almost everywhere as well as equality in the two applications of Hölder's inequality when we obtained (6) and (7). Equality in (4) occurs at points where $f(x)$ and $g(x)$ are the same sign. Equality in (6) and (7) (by Theorem 13.1) will hold if

$$c_1|f(x)|^p = |f(x) + g(x)|^q = c_2|g(x)|^q$$

holds a.e., and from this the assertion of the theorem follows. ■

Exercises

13:1.1$\diamondsuit$ If $f \in L_p(X, \mathcal{M}, \mu)$ then, for all $t > 0$,

$$\mu\left(\{x \in X : |f(t)| > t\}\right) \le \left(\frac{\|f\|_p}{t}\right)^p.$$

[Hint: Use Fubini's theorem.]

13:1.2 Show that for all $0 < p < \infty$ the collections L_p of measurable functions defined on a measure space $(X, \mathcal{M}, \mu)$ such that

$$\int_X |f|^p \, d\mu < \infty$$

are linear spaces. [Hint: Use the inequality $(a+b)^p \le 2^p(a^p + b^p)$.]

13:1.3 Prove the Minkowski inequality for the case $p = 1$: Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f, $g \in L_1(\mu)$. Then

$$\|f + g\|_1 \le \|f\|_1 + \|g\|_1.$$

The inequality is strict except precisely in the case where there is a nonnegative measurable function h so that $fh = g$ μ–a.e. on the set where neither f nor g vanishes. [Hint: The inequality is trivial; it is the conditions for equality that need to be looked into here.]

13:1.4$\diamondsuit$ [A Minkowski inequality for integrals] For $1 \le p < \infty$ and for any nonnegative measurable function on an interval $[a, b]$,

$$\left(\int_a^b \left(\int_a^b f(x,y)\, dx\right)^p dy\right)^{1/p} \le \int_a^b \left(\int_a^b f(x,y)^p \, dy\right)^{1/p} dx.$$

[Hint: Use Fubini's theorem and Hölder's inequality.]

13.2 The ℓ_p and L_p Spaces $(1 \le p < \infty)$

We proceed now to study the ℓ_p and L_p spaces for that part of the scale $(1 \le p < \infty)$. Later we will add on to the high end of the scale by introducing (in Section 13.3) the spaces ℓ_∞ and L_∞ and to the low end of the scale by studying (in Section 13.7) the spaces ℓ_p and L_p for $(0 < p < 1)$.

Definition 13.3 Let $(X, \mathcal{M}, \mu)$ be a given measure space. We denote by $L_p(X, \mathcal{M}, \mu)$ or merely $L_p(\mu)$ the collection of those measurable real (or complex) functions defined on X such that

$$\int_X |f|^p \, d\mu < \infty;$$

that is, those functions having a finite p–norm.

As usual for function spaces associated with measure theory, we identify functions that are equal almost everywhere with respect to the underlying measure. Then, since $\|f\|_p = 0$ if and only if f vanishes almost everywhere, we can consider that $\|f\|_p = 0$ only for the zero function. It is easy to check that each of these spaces is a real (complex) linear space and that $f \to \|f\|_p$ is a norm. The Minkowski inequality supplies the only difficult parts of the proofs.

The ℓ_p spaces $(1 \le p < \infty)$ are particular cases of the general L_p spaces, but deserve attention on their own merit.

Definition 13.4 By ℓ_p, we denote the collection of all sequences $x = (x_1, x_2, x_3 \dots)$ of real (complex) numbers such that

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} < \infty.$$

Since this is precisely the space L_p taken over the measure space on $\mathbb{N}$ with counting measure μ, all our theory applies to these sequence spaces too. These spaces are infinite-dimensional analogs of the spaces introduced in Section 9.1.

Our main theorem here is that the L_p–spaces are Banach spaces. For this, we need to construct a completeness proof. We recall the elements of a standard completeness proof (cf. Section 9.6). We construct an "object" from an arbitrary Cauchy sequence that will be the desired function, we show that object is a member of the space, and, finally, we show that the sequence converges to the object in the space.

Theorem 13.5 *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then the spaces $L_p(\mu)$ for $1 \le p < \infty$ are Banach spaces furnished with the norm $\|f\|_p$.*

Proof. We prove that each Cauchy sequence in the space converges to an element of the space. Let $\{f_n\}$ be Cauchy in $L_p(\mu)$. We can pass to a subsequence so that

$$\|f_{n_{i+1}} - f_{n_i}\|_p < 2^{-i}$$

for $i = 1, 2, 3, \dots$. Write

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|$$

and

$$g = \lim_{k\to\infty} g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|.$$

Note that the function g is defined everywhere, but may be infinite. By using the Minkowski inequality, we see that

$$\|g_k\|_p \le \sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p \le \sum_{i=1}^{k} 2^{-i} < 1$$

and hence Fatou's lemma supplies us with the inequality

$$\int_X |g|^p \, d\mu \le \liminf_{k\to\infty} \int_X |g_k|^p \, d\mu \le 1.$$

In particular, $g(x) < \infty$ for μ–almost every $x \in X$ and, consequently, the limit

$$f(x) = \lim_{i\to\infty} f_{n_i}(x) = f_{n_1}(x) + \sum_{i=1}^{k} \left(f_{n_{i+1}}(x) - f_{n_i}(x)\right)$$

provides a finite value for μ–a.e. point x (since the series converges absolutely). We can define $f(x) = 0$ at all other points, and this gives a finite-valued measurable function defined everywhere on the space.

This is our candidate for the limit of the Cauchy sequence $\{f_n\}$. Let $\varepsilon > 0$, and choose N so large that $\|f_n - f_m\|_p < \varepsilon$ for all m, $n \ge N$. Fix $m \ge N$, and apply Fatou's lemma to the sequence $\{f_{n_i}\}$, this time obtaining

$$\int_X |f - f_m|^p \, d\mu \le \liminf_{i\to\infty} \int_X |f_{n_i} - f_m|^p \, d\mu \le \varepsilon^p.$$

This gives the p–norm estimate $\|f - f_m\|_p \le \varepsilon$ for all $m \ge N$. By Minkowski's inequality,

$$\|f\|_p \le \|f - f_m\|_p + \|f_m\|_p < \infty,$$

and so f is a member of the space L_p and evidently $\|f - f_m\|_p \to 0$ as $m \to \infty$, as required. ■

Exercises

13:2.1◇ Let X denote any set. The $\ell_p(X)$ spaces $(1 \le p < \infty)$ are defined as the set of all functions $x : X \to \mathbb{R}$ (or $\mathbb{C}$ if preferred) such that

$$\|x\|_p = \left(\sum_{\alpha\in X} |x_\alpha|^p\right)^{1/p} < \infty.$$

Show that this is precisely the space L_p taken for an appropriate measure space.

13:2.2 Let $\{f_n\}$ be convergent to a function f in $L_p(\mu)$. Show that there is a subsequence $\{f_{n_k}\}$ that is almost everywhere convergent to f. [Hint: This is essentially contained in the proof of Theorem 13.5. Also, this is related to an earlier result from Section 4.2 on subsequences of sequences converging in measure.]

13.3 The Spaces ℓ_∞ and L_∞

Let us move to the high end of the scale. This can be motivated in several ways. For one thing, we notice that the duality between conjugate pairs of indices p, q with $p^{-1} + q^{-1} = 1$ collapses for $p = 1$, unless we allow $q = \infty$. A space corresponding to L_∞ seems to be needed just for symmetry. On the other hand the p–norm itself can be extended to the end of the scale by taking limits:

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$

We proceed directly. Let $(X, \mathcal{M}, \mu)$ be a given measure space. For any measurable function (real or complex), we write

$$\|f\|_\infty = \operatorname{ess\,sup} |f(x)| = \inf \{t > 0 : \mu(\{x : |f(x)| > t\}) = 0\} \qquad (8)$$

and refer to this as the *essential supremum* or ∞–norm of the function f. The functions for which this is finite are called *essentially bounded functions.* This is perhaps easier to understand if one notes that an ordinary supremum of a bounded function f could be obtained as

$$\sup |f(x)| = \inf \{t > 0 : \{x : |f(x)| > t\} = \emptyset\}.$$

(In both of these, one uses the convention that $\inf \emptyset = \infty$.)

By $L_\infty(X, \mathcal{M}, \mu)$ or merely $L_\infty(\mu)$, we denote those measurable real (or complex) functions defined on X such that $\|f\|_\infty < \infty$; that is, those functions having a finite ∞–norm. Again, as usual for function spaces associated with measure theory, we identify functions that are equal almost everywhere with respect to the underlying measure. Then, since $\|f\|_\infty = 0$ if and only if f vanishes almost everywhere, we can consider that $\|f\|_\infty = 0$ only for the zero function. We shall check that $L_\infty(\mu)$ is a real (complex) linear space and that $\|f\|_\infty$ is a norm; like the other L_p spaces, this too is a Banach space.

In the special case where $X = \mathbb{N}$ and μ is taken as the counting measure, the space L_∞ reduces to the sequence space ℓ_∞ of bounded sequences with the supremum norm.

Note that the spaces $L_p(\mu)$ for $p < \infty$ depend very much on the underlying measure and would be sensitive to any changes in μ. The space $L_\infty(\mu)$ depends only on the class of μ–measure zero sets and not on any values of the measure itself.

The essential supremum norm can be used for continuous functions. In that case, in almost all settings the usual sup norm and the norm $\|\cdot\|_\infty$ would be identical. Certainly, in the case of Lebesgue measure on the line this is so: thus the collection of bounded continuous functions on $\mathbb{R}$ is a closed subspace of the space $L_\infty(\mathbb{R}, \mathcal{L}, \lambda)$.

Theorem 13.6 *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then the space $L_\infty(\mu)$ is a Banach space furnished with the norm $\|f\|_\infty$.*

Proof. It is easy to see that a linear combination of essentially bounded functions remains essentially bounded, and so the space is linear. It is almost immediate that $\|f\|_\infty$ is a norm on this space. The triangle inequality, that

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

(which can also be considered as the extension of Minkowski's inequality to the case $p=\infty$), follows from the set inclusion

$$\{x : |f(x)+g(x)| > \|f\|_\infty + \|g\|_\infty\}$$
$$\subset \{x : |f(x)| > \|f\|_\infty\} \cup \{x : |g(x)| > \|g\|_\infty\}\,.$$

Exercise 13:3.2 shows that each of the sets on the right side of the inclusion has μ–measure zero and so, too, must the set on the left. This gives the triangle inequality.

The completeness part of the proof is rather simpler than the completeness proof for the L_p spaces with $1 \leq p < \infty$. Let $\{f_n\}$ be Cauchy in $L_\infty(\mu)$. Define A_i to be the set of points x in X for which $|f_i(x)| > \|f_i\|_\infty$, and define $B_{j,k}$ to be the set of points x in X for which $|f_j(x) - f_k(x)| > \|f_k\|_\infty$. All these sets have measure zero by definition. Let E be the totality of all these points, that is, the union of these sets taken over all integers i, j, k. Then E has measure zero, and the sequence $\{f_n(x)\}$ converges for every $x \in X \setminus E$, and indeed it converges uniformly to some bounded function f defined on $X \setminus E$. We can extend f to all of X in any arbitrary fashion [or simply set $f(x) = 0$ for $x \in E$], and it is easy to see that $f \in L_\infty(\mu)$ and that $\|f - f_n\|_\infty \to 0$ as $n \to \infty$. ■

We have already indicated that Minkowski's inequality extends to the case $p = \infty$, that

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

We now extend Hölder's inequality. Note that we interpret $p = 1$ and $q = \infty$ as conjugate indices by considering that we still have the conjugate relation

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 13.7 (Hölder's inequality) *Let $(X, \mathcal{M}, \mu)$ be a measure space, consider the conjugate indices 1, ∞, and let $f \in L_1(\mu)$, $g \in L_\infty(\mu)$. Then the product fg is integrable and*

$$\int_X |fg|\, d\mu \leq \|f\|_1 \|g\|_\infty.$$

The inequality is strict except precisely in the case where $|g| = \|g\|_\infty$ μ–a.e.

Proof. The elementary inequality

$$|f(x)g(x)| \leq \|g\|_\infty |f(x)|$$

holds almost everywhere. Just integrate this to obtain the theorem. The final statement of the theorem is easily checked, too. ■

Exercises

13:3.1 Show that a sequence f_n converges to a function f in the space $L_\infty(X, \mathcal{M}, \mu)$ if and only if there is a set $E \in \mathcal{M}$ with $\mu(E) = 0$ so that $f_n \to f$ uniformly on $X \setminus E$.

13:3.2 Show that the infimum in equation (8) is attained, in fact that

$$\mu\left(\{x : |f(x)| > \|f\|_\infty\}\right) = 0.$$

13:3.3 Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Show that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty \text{ for all } f \in L_\infty.$$

13.4 Separability

Let us now look at the question of separability of the ℓ_p and L_p spaces. Recall that to show that a metric space is separable we must demonstrate the existence of a countable dense subset of the space. For the ℓ_p spaces $(1 \le p < \infty)$, this presents no challenge. The space ℓ_∞ is *not* separable, however. Let S denote the family of all sequences of 0's and 1's. If $x, y \in S$ are distinct then $\|x - y\|_\infty \ge 1$. Since S is an uncountable subset of ℓ_∞ and every pair of points in S is at least a unit distance apart, there can be no countable dense subset of ℓ_∞.

Generally, we would expect similar assertions for the L_p spaces. Normally, L_∞ is not separable and, normally, L_p $(1 \le p < \infty)$ can be seen to be separable. For example, $L_1([0,1], \mathcal{L}, \lambda)$ is separable: the family of rational linear combinations of the characteristic functions of those sets that are finite unions of intervals with rational endpoints provides a countable dense subset. More generally, if the underlying space is $\mathbb{R}^n$ and μ is a Borel measure, it is not too much trouble to show that all the spaces L_p $(1 \le p < \infty)$ are separable. Here we shall address this problem more abstractly. What properties of the underlying measure space allow the function space L_1 to be separable?

Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall (Example 9.12) that we have defined a metric on equivalence classes of $\mathcal{M}$:

$$\rho(A, B) = \mu(A \triangle B).$$

The resulting metric space may or may not be separable. Our next result shows that separability of $L_1(X, \mathcal{M}, \mu)$ and separability of $\mathcal{M}$ coincide.

Theorem 13.8 *Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Then the Banach space $L_1(X, \mathcal{M}, \mu)$ is separable if and only if the space $\mathcal{M}$ with metric $\rho(A, B) = \mu(A \triangle B)$ is separable.*

Proof. Suppose that $\mathcal{M}$ is separable. Let $\{A_n\}$ be a countable dense subset of $\mathcal{M}$. We may assume that $\{A_n\}$ is an algebra $\mathcal{A}$, since the algebra generated by $\{A_n\}$ is also countable. Let $\mathcal{S}$ denote the family of simple functions of the form

$$f(x) = \sum_{k=1}^{n} c_k f_k(x), \tag{9}$$

where $f_k = \chi_{A_k}$ and $c_k \in \mathbb{Q}$. The family $\mathcal{S}$ is countable, since both $\mathbb{Q}$ and $\mathcal{A}$ are countable. We show that $\mathcal{S}$ is dense in L_1.

It follows from the definition of the integral that the collection of all simple functions is dense in L_1 (Exercise 13:4.3). Since each simple function can be approximated in the L_1 norm by a simple function taking only rational values, we need show only that such functions can be approximated in the L_1 norm by functions in $\mathcal{S}$.

To verify that this is possible, let

$$g = \sum_{k=1}^{n} c_k \chi_{E_k}$$

be a simple function with $c_k \in \mathbb{Q}$ for all $k = 1, \ldots, n$. Let

$$c = \max \{|c_k| : k = 1, \ldots, n\},$$

and let $\varepsilon > 0$. We may assume that $c \neq 0$. By hypothesis, there exist sets $A_1, A_2, \ldots, A_n$ from $\mathcal{A}$ such that

$$\mu(E_k \triangle A_k) < \frac{\varepsilon}{2nc} \quad (k = 1, \ldots, n). \tag{10}$$

For each $k = 1, \ldots, n$, let

$$B_k = A_k \setminus \bigcup_{j=1}^{k-1} A_j.$$

The sets B_k are members of $\mathcal{A}$ and are pairwise disjoint. Furthermore, for each $k \leq n$, $A_k \subset B_k$ and

$$\bigcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} A_k.$$

It follows from (10) that

$$\mu\left(X \setminus \bigcup_{k=1}^{n} B_k\right) < \frac{\varepsilon}{2c}.$$

Let

$$f(x) = \begin{cases} c_k, & \text{if } x \in B_k; \\ 0, & \text{if } x \in X \setminus \bigcup_{j=1}^{n} B_j. \end{cases}$$

Thus $f \in \mathcal{S}$ and

$$\mu(\{x : f(x) \neq g(x)\}) < \frac{\varepsilon}{2c}.$$

Now

$$\int_X |f - g| \, d\mu \leq 2c\left(\frac{\varepsilon}{2c}\right) = \varepsilon,$$

so $\|f - g\| \leq \varepsilon$. This completes the proof that L_1 is separable.

To prove the converse, we need only note that if L_1 is separable then so too is the subset of characteristic functions of measurable sets. But this space is isometric to the space $\mathcal{M}$ (Example 9.27). Thus $\mathcal{M}$ is separable. ■

Exercises

13:4.1 Show that $L_\infty([0,1], \mathcal{L}, \lambda)$ is not separable.

13:4.2 Give the details to show that $L_1([0,1], \mathcal{L}, \lambda)$ is separable. [Hint: Check that the family of those sets that are equivalent to finite unions of intervals with rational endpoints provides a countable dense subset of the space $\mathcal{M}$ of Theorem 13.8.]

13:4.3 Show that the set of simple functions $f = \sum_{i=1}^n a_i \chi_{E_i}$, where each $\mu(E_i) < \infty$, is dense in any space $L_p(\mu)$ $(1 \leq p < \infty)$.

13:4.4 Prove a version of Theorem 13.8 for $1 < p < \infty$.

13:4.5 When are the spaces $\ell_p(X)$ separable? (See Exercise 13:2.1.)

13.5 The Spaces ℓ_2 and L_2

The spaces ℓ_2 and L_2 play a very special role in many investigations and are worth looking at much more closely. This rests entirely on the fact that the conjugate index to $p = 2$ is $q = 2$: the spaces are self-dual in a sense that will be made precise in Section 13.6. We examine now the structure of these spaces and see how they are distinguished from the rest of the scale of function spaces. This furnishes a brief introduction to the world of Hilbert spaces that the reader can continue in Chapter 14.

All our functions in this section will be taken to be complex valued, although the real case is included and is interesting on its own. Now $|\cdot|$ denotes complex modulus rather than absolute value, and $\overline{c}$ is the complex conjugate of the complex number c.

Definition 13.9 Let $(X, \mathcal{M}, \mu)$ be a measure space. We denote by $L_2(X, \mathcal{M}, \mu)$ the set of all complex-valued functions f defined on X for which

$$\int_X |f|^2 \, d\mu < \infty.$$

As with L_1, we are technically dealing with equivalence classes of functions, thus each $f \in L_2$ consists of a family of functions each pair of which agree a.e. In the special case $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$ and μ is counting measure, L_2 becomes the space ℓ_2 of sequences $\{x_k\}$ of complex numbers such that $\sum |x_k|^2 < \infty$.

Definition 13.10 By ℓ_2, we mean the space of all sequences of complex numbers $x = (x_1, x_2, x_3, \dots)$ for which the sum

$$\sum_{k=1}^{\infty} |x_k|^2$$

is finite.

We first check directly that L_2 is a linear space. It is clear that if $a \in \mathbb{C}$ and $f \in L_2$ then $af \in L_2$. Suppose now that $f, g \in L_2$. Since $|fg| \le \frac{1}{2}(|f|^2 + |g|^2)$, we see that $fg \in \mathrm{L}_1$. That $f + g \in L_2$ now follows from integrating the inequality

$$|f+g|^2 \le |f|^2 + 2|fg| + |g|^2.$$

We now define a scalar function called an *inner product* or *scalar product* $(f, g) : L_2 \times L_2 \to \mathbb{C}$ by

$$(f, g) = \int_X f\overline{g}\, d\mu.$$

It is clear that (f, g) satisfies the following conditions:

13.11 *(Properties of the inner product)*

1. $(f, f) \ge 0$ *for all* $f \in L_2$*, with equality if and only if* $f = 0$*.*
2. $(f, g) = \overline{(g, f)}$ *for all* $f, g \in L_2$*.*
3. $(af, g) = a(f, g)$ *for all* $f, g \in L_2$ *and* $a \in \mathbb{C}$*.*
4. $(f, g_1 + g_2) = (f, g_1) + (f, g_2)$ *for all* $f, g_1, g_2 \in L_2$*.*

The inner product can be used to define notions of orthogonality and norm. Two elements $f, g \in L_2$ are said to be *orthogonal* if $(f, g) = 0$. We define the norm on L_2 by

$$\|f\|_2 = \sqrt{(f, f)}. \tag{11}$$

Note that this is the same 2–norm with which we started the chapter. We shall show that this definition does, in fact, provide a norm. For the moment, $\|f\|_2$ is only an alternative notation for $\sqrt{(f, f)}$.

Lemma 13.12 (Cauchy–Schwarz inequality) *For all elements* $f, g \in L_2$*, the following inequality holds:*

$$|(f, g)| \le \|f\|_2 \|g\|_2. \tag{12}$$

Proof. This is a special case of Hölder's inequality (Theorem 13.1) with $p = q = 2$ as conjugate indices. Since it is very special, it has a much easier proof that requires only some algebraic properties of the inner product. We present that here, although it can be skipped if the reader has not skipped over Theorem 13.1.

Let us assume first that (f, g) [and so also (g, f)] is real. Consider the polynomial

$$\begin{aligned} p(\alpha) &= (\alpha f + g, \alpha f + g) = \alpha^2 (f, f) + 2\alpha (f, g) + (g, g) \\ &= \|f\|_2^2 \alpha^2 + 2(f, g)\alpha + \|g\|_2^2. \end{aligned}$$

The definition of p together with condition (1) of (13.11) implies that $p(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}$. It follows from the quadratic formula that

$$(f, g)^2 - \|f\|_2{}^2 \|g\|_2^2 \leq 0. \tag{13}$$

Otherwise, the quadratic p would have two distinct roots and would therefore not be nonnegative. The inequality (12) follows from (13).

Now let f, g be any elements of the space. There is a real θ so that $(f, g) = e^{i\theta} |(f, g)|$. Let $f_1 = e^{-i\theta} f$ and observe that $(f_1, g) = |(f, g)|$. Since (f_1, g) is real, we can obtain from the argument of the first paragraph that $(f_1, g) \leq \|f_1\|_2 \, \|g\|_2$. Since $\|f_1\|_2 = \|f\|_2$, the inequality (12) follows. ■

Theorem 13.13 L_2 *is a normed linear space with the norm given by*

$$\|f\|_2 = \sqrt{(f, f)}.$$

Proof. It is obvious that conditions (1) and (2) of the norm definition (Definition 12.1) are satisfied. It remains only to check the triangle inequality. Note that, for all $f, g \in L_2$,

$$\begin{aligned} \|f + g\|_2^2 &= (f + g, f + g) = (f, f) + (f, g) + (g, f) + (g, g) \\ &\leq (f, f) + 2|(f, g)| + (g, g) \\ &\leq \|f\|_2^2 + 2\|f\|_2 \|g\|_2 + \|g\|_2^2 = (\|f\|_2 + \|g\|_2)^2, \end{aligned}$$

the last inequality following from the Cauchy–Schwarz inequality. Thus

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$$

establishing the triangle inequality, and the proof is complete. ■

The norm has a feature that is available only in this kind of setting and distinguishes inner product norms from all others.

Theorem 13.14 (Parallelogram law) *The identity*

$$\|f + g\|_2^2 + \|f - g\|_2^2 = 2(\|f\|_2^2 + \|g\|_2^2) \tag{14}$$

holds for all pairs f, g.

Proof. To verify this parallelogram law (the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of its sides), note that

$$\|f+g\|_2^2 + \|f-g\|_2^2 = (f+g, f+g) + (f-g, f-g)$$
$$= (f,f) + (f,g) + (g,f) + (g,g) + (f,f) - (f,g) - (g,f) + (g,g)$$
$$= 2(\|f\|_2^2 + \|g\|_2^2).$$

as required. ■

The reader may have observed that in the proof of the Cauchy–Schwarz inequality and in the verification that L_2 is a normed linear space, we made no explicit mention of the integral $\int_X |f|^2\,d\mu$ itself. All our results derived from the fact that $\|f\|_2$ is defined in terms of the inner product and that the inner product satisfies conditions 13.11. We have, in fact, shown that if a linear space X has an inner product defined on it, and one defines

$$\|x\| = \sqrt{(x,x)},$$

then the Schwarz inequality

$$|(x,y)| \le \|x\|_2\,\|y\|_2$$

is valid for all $x, y \in X$, and $\|x\|$ is a norm on X.

A set X furnished with an inner product is called an *inner product space*. Some authors use the term *Euclidean space* because such spaces are direct extensions of $\mathbb{R}^n$. For example, if $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$, and μ is counting measure, L_2 becomes the space ℓ_2 of sequences $\{x_k\}$ of real or complex numbers such that $\sum |x_k|^2 < \infty$. The norm

$$\|x\|_2 = \sqrt{\sum_{k=1}^{\infty} |x_k|^2}$$

on ℓ_2 is just the infinite analog of the usual norm in $\mathbb{R}^n$ or $\mathbb{C}^n$.

An inner product space X that is complete (as a metric space) is called a Hilbert space. A more extended account of the theory of Hilbert space is given in Chapter 14. Our next result shows that L_2 is a Hilbert space.

Theorem 13.15 *The space $L_2(X, \mathcal{M}, \mu)$ is complete.*

Proof. This can be proved by the methods of Theorem 13.5, where we showed the completeness of all the real L_p spaces. Note that the case $p = 2$ is only a little simpler, since it can use the more elementary Cauchy–Schwarz inequality in place of the Minkowski inequality in the proof, but still needs to employ many basic measure-theoretic tools. ■

Let us now characterize the continuous linear functionals on the Hilbert space L_2. In fact, this characterization follows just from the Hilbert space structure itself and requires no special features of the underlying measure space. For the space L_2, this theorem can be interpreted as providing the characterization

$$\Gamma(f) = \int_X f\overline{g}\, d\mu \quad (f \in L_2(\mu))$$

for some unique $g \in L_2(\mu)$ and, for the space ℓ_2, as providing the characterization

$$\Gamma(x) = \sum_{i=1}^{\infty} x_i \overline{y_i} \quad (x \in \ell_2)$$

for some unique $y \in \ell_2$.

Theorem 13.16 *Let $\Gamma : L_2(X, \mathcal{M}, \mu) \to \mathbb{C}$ be a continuous linear functional. Then there is a unique $g \in L_2(X, \mathcal{M}, \mu)$ so that*

$$\Gamma(f) = (f, g) \quad (f \in L_2(X, \mathcal{M}, \mu)).$$

Proof. The proof is omitted since it can be constructed from other material. The argument for a general Hilbert space is given in Theorem 14.14 and that can be repeated here with only notational changes. Also, the argument for general L_p spaces $1 < p < \infty$ is given in Theorem 13.18 and that, too, could be repeated here with $p = 2$, where it becomes only slightly simpler. ■

Example 13.17 This example provides an L_2 version of Example 12.27. Let K be an L_2 function on $[a, b] \times [a, b]$. For $f \in L_2[a, b]$, consider the integral

$$\int_a^b K(x, y) f(y)\, dy.$$

By Fubini's theorem (Theorem 6.6) the function $K(x, \cdot) \in L_2[a, b]$ for almost every $x \in [a, b]$. We can thus define an operator A on $L_2[a, b]$ by $Af = g$, where, for $f \in L_2[a, b]$,

$$g(x) = \int_a^b K(x, y) f(y)\, dy$$

The function g is defined only a.e.

From the Cauchy–Schwarz inequality, we see that, for almost every x,

$$|g(x)|^2 \leq \int_a^b |K(x, y)|^2\, dy \int_a^b |f(y)|^2\, dy,$$

and hence

$$\int_a^b |g(x)|^2\, dx \leq \left(\int_a^b \int_a^b |K(x, y)|^2\, dx dy \right) \left(\int_a^b |f(y)|^2\, dy \right).$$

It follows that $g \in L_2[a,b]$ so A maps $L_2[a,b]$ into itself. It follows, too, that

$$\|A\| \leq \int_a^b \int_a^b |K(x,y)|^2 \, dx \, dy.$$

Consider now the Fredholm equation

$$\phi(x) = f(x) - \int_a^b K(x,y) f(y) \, dy. \tag{15}$$

Given $\phi \in L_2[a,b]$, we seek $f \in L_2[a,b]$ such that equation (15) holds for almost every x. By Theorem 12.26, we know that for every such ϕ there will be a unique solution to (15) provided that $\|A\| < 1$, and this will be the case in particular when

$$\int_a^b \int_a^b |K(x,y)|^2 \, dx \, dy < 1.$$

The solution can then take the form $f = \phi + A\phi + A^2\phi + \cdots$. The difference between this example and previous examples involving Fredholm equations is that here we are dealing with L_2 functions, rather than with continuous functions.

Incidentally, one can show that the operator A is a compact operator even when $\|A\| \geq 1$. The corresponding assertion in Example 12.27 was easy to verify because of Ascoli's theorem. In the present setting of L_2, Ascoli's theorem is not available, and verification of compactness of A is more delicate. We omit a proof.

Exercises

13:5.1 Write out the Schwarz inequality and the triangle inequality explicitly for L_2 using integrals instead of norms and inner products.

13:5.2 Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Prove that $L_2(X, \mathcal{M}, \mu)$ is separable if and only if the space $\mathcal{M}$ of Example 9.12, is separable.

13:5.3 Show that the parallelogram law fails for L_1 so that L_1 is not a Hilbert space. [Hint: Use $X = [0,1]$, $f = \chi_{[0,1/2]}$, and $g = \chi_{[1/2,1]}$, and calculate $\|f+g\|_1 = \|f-g\|_1 = 1$ and $\|f\|_1 = \|g\|_1 = \frac{1}{2}$.]

13:5.4 Prove the Pythagorean theorem: If f_1, f_2, $\ldots$, f_n are orthogonal in L_2 [i.e., if $(f_i, f_j) = 0$ for all $i \neq j$], then

$$\|\sum_{i=1}^n f_i\|_2^2 = \sum_{i=1}^n \|f_i\|_2^2.$$

13:5.5 If $f_1,\ f_2,\ \dots,\ f_n$ are orthonormal in L_2 [i.e., if $(f_i, f_j) = 0$ for all $i \neq j$ and $\|f_i\|_2 = 1$], then
$$\|\sum_{i=1}^{n} c_i f_i\|_2^2 = \sum_{i=1}^{n} |c_i|^2.$$

13.6 Continuous Linear Functionals

Our main theorem in this section provides us with a representation for the continuous linear functionals on $L_p(\mu)$ for $1 \leq p < \infty$. This theorem is often called the Riesz representation theorem, too, but that would conflict with our usage of this for the theorem in Section 12.8 that established the form for the continuous linear functionals on the space $\mathcal{C}[a, b]$.

Note that the case $p = \infty$ is not resolved by our theorem and, indeed, the linear functionals on $L_\infty(\mu)$ are of a different nature. There is a natural duality implied between the spaces $L_p(\mu)$ and $L_q(\mu)$ for conjugate indices p and q, but only for $1 \leq p < \infty$. Continuous linear functionals on $L_1(\mu)$ are represented by functions $L_\infty(\mu)$, but we will have no representation for *all* the continuous linear functionals on $L_\infty(\mu)$ by functions in $L_1(\mu)$. (This is related to Exercise 12:7.6.) Another thing to note is the special case of the conjugate indices $p = 2$ and $q = 2$.

The proof of the theorem offers us an interesting application of the Radon–Nikodym theorem. (See Theorem 5.29 and also Exercise 5:12.17 for the complex version.)

Theorem 13.18 *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let p, q be conjugate indices with $1 \leq p < \infty$. Then to each continuous linear functional Γ on $L_p(\mu)$ there corresponds a unique $g \in L_q(\mu)$ so that*
$$\Gamma(f) = \int_X f g \, d\mu \tag{16}$$
for all $f \in L_p(\mu)$ and, moreover, the norm of the functional $\|\Gamma\|$ is exactly $\|g\|_q$.

Proof. For the proof, we shall assume that $\mu(X) < \infty$. The σ-finite case is sketched in the exercises; for $p > 1$, one does not need to assume that μ is σ-finite in fact, although the proof requires some different details.

We obtain immediately a candidate g for the representation (16) by constructing a (real or complex) signed measure ν on the space that is absolutely continuous with respect to μ. The Radon-Nikodym derivative of ν with respect to μ will be our function g.

For each $E \in \mathcal{M}$, write $\nu(E) = \Gamma(\chi_E)$. Since we are assuming that μ is finite, the function χ_E is in L_p and so ν is well defined. In fact, ν is a signed measure on $\mathcal{M}$. The linearity of Γ shows that ν is finitely additive: if A, B are disjoint members of $\mathcal{M}$, then apply Γ to
$$\chi_{A \cup B} = \chi_A + \chi_B$$

to obtain

$$\nu(A \cup B) = \nu(A) + \nu(B).$$

The continuity of Γ allows us to extend this to countable additivity. For if $A_1,\ A_2,\ A_3\ \ldots$ are disjoint members of $\mathcal{M}$ and $E = \bigcup_1^\infty A_i,\ E_k = \bigcup_1^k A_i$, then

$$\|\chi_E - \chi_{E_k}\|_p^p = \int_X |\chi_E - \chi_{E_k}|^p \, d\mu = \mu(E \setminus E_k),$$

which tends to zero as $k \to \infty$, since $E \setminus E_k = \bigcup_{k+1}^\infty A_i$ and μ is finite. It follows that $\chi_{E_k} \to \chi_E$ in the space L_p and so, from the continuity of Γ, $\Gamma(\chi_{E_k}) \to \Gamma(\chi_E)$ and hence also $\sum_1^k \nu(A_i) = \nu(E_k) \to \nu(E)$. Finally, we see that ν is absolutely continuous with respect to μ: for if $\mu(A) = 0$, then χ_A is equivalent to the zero function in L_p and, consequently, $\nu(A) = \Gamma(\chi_A) = 0$.

The Radon-Nikodym theorem (Theorem 5.29) provides a function $g \in L_1$ for which

$$\nu(E) = \Gamma(\chi_E) = \int_X \chi_E \, g \, d\mu \qquad (E \in \mathcal{M}). \tag{17}$$

We do not yet know that $g \in L_q$ nor that the representation in (16) holds for all functions $f \in L_p$, nor do we have the norm identity $\|\Gamma\| = \|g\|_q$.

We claim first that the representation in (16) holds for all functions $f \in L_\infty$ (and our goal of all functions in $f \in L_p$ will come later). First, from (17), we see that the representation holds for all functions f that are characteristic functions of sets in $\mathcal{M}$. By the linearity of Γ and the integral, it follows that the representation holds for all functions f that are simple measurable functions. But every function $f \in L_\infty$ is a uniform limit of a sequence f_n of simple measurable functions. If such a sequence $f_n \to f$ uniformly, then $f_n \to f$ in L_p, and so $\Gamma(f_n) \to \Gamma(f)$ and also $\int_X f_n g \, d\mu \to \int_X f g \, d\mu$. This shows that the representation in (16) holds for all functions $f \in L_\infty$.

At this stage we wish to show that $g \in L_q$, indeed that

$$\|g\|_q \leq \|\Gamma\|.$$

The two cases $p = 1, q = \infty$ and $1 < p,\ q < \infty$ are handled differently. Choose a measurable function h so that everywhere

$$|h| = 1 \text{ and } \ hg = |g|$$

(for real functions this is trivial, while for complex functions see Theorem 5.45).

If $p = 1,\ q = \infty$, then we show that $\|g\|_\infty \leq \|\Gamma\|$ by showing that $|g(x)| \leq \|\Gamma\|$ almost everywhere. Let E_n be the set of points x, where $|g(x)| \geq \|\Gamma\| + n^{-1}$. Then

$$\|\Gamma\| \, \|\chi_{E_n}\|_1 \geq |\Gamma(h\chi_{E_n})| = \left| \int_X \chi_{E_n} hg \, d\mu \right| \geq (\|\Gamma\| + n^{-1})\mu(E_n),$$

which can happen only if $\mu(E_n) = 0$. It follows that $|g(x)| \leq \|\Gamma\|$ almost everywhere, since the set of points where this is not true is the union of the sequence of zero measure sets $\{E_n\}$.

In the case $1 < p$, $q < \infty$, we use Hölder's inequality (Theorem 13.1). Let E_n be the set of points x where $|g(x)| \leq n$, and write

$$f(x) = \chi_{E_n}(x)|g(x)|^{q-1}h(x).$$

Then f is bounded, $|f|^p = |g|^q$ on E_n, and hence

$$\int_X |g|^q \, d\mu = \int_X fg \, d\mu = \Gamma(f) \leq \|\Gamma\| \left\{ \int_{E_n} |g|^q \, d\mu \right\}^{1/p}.$$

This gives

$$\int_{E_n} |g|^q \, d\mu \leq \|\Gamma\|^q$$

for each integer n. Letting $n \to \infty$ in this inequality, we obtain

$$\int_X |g|^q \, d\mu \leq \|\Gamma\|^q,$$

and so we have $\|g\|_q \leq \|\Gamma\|$, as required.

Now consider the two functionals

$$\Gamma(f) \ \text{ and } \ \Gamma_1(f) = \int_X fg \, d\mu$$

defined and continuous on L_p. The first of these was given to us and the second, Γ_1, is defined and continuous on L_p because of Hölder's inequality and the fact that $g \in L_q$. The two functionals agree on the dense subspace L_∞ and consequently must agree on the whole of the space L_p. This proves that the representation in (16) holds for all functions $f \in L_p$.

To complete the proof, note that we already have $\|g\|_q \leq \|\Gamma\|$. The opposite direction comes from Hölder's inequality, since

$$|\Gamma(f)| = |\Gamma_1(f)| = \left| \int_X fg \, d\mu \right| \leq \|f\|_p \, \|g\|_q$$

so that $\|g\|_q \geq \|\Gamma\|$. This completes the proof for the case of a finite measure; the exercises show how to push the proof to take care of a σ-finite measure space. ■

How can we represent the functionals on L_∞, the missing case? We do not have to worry in general about the spaces L_p for $0 < p < 1$ at the extreme low end of our scale because, as it turns out, there are often *no* continuous linear functionals at all! (See Section 13.7.) The exact form for the continuous linear functionals on L_∞ would require us to develop an integration theory for finitely additive measures. Rather than do this, we merely state the theorem and leave a fuller exploration to the reader.[1]

Theorem 13.19 *Let $(X, \mathcal{M}, \mu)$ be a measure space. To each continuous linear functional Γ on $L_\infty(\mu)$ there corresponds a bounded, finitely additive set function ϕ defined on $\mathcal{M}$ and satisfying the condition $|\phi|(E) = 0$ whenever $\mu(E) = 0$, so that*

$$\Gamma(f) = \int_X f\, d\phi$$

for all $f \in L_\infty(\mu)$.

Exercises

13:6.1 Let $L_\infty[0,1]$ be the space of functions on the interval $[0,1]$ that are essentially bounded with respect to sets of Lebesgue measure zero. Show that the mapping $\Gamma : f \to \int_0^1 f(x)g(x)\, dx$ is a continuous linear functional on L_∞ for every $g \in L_1[0,1]$.

13:6.2 Show that there is a continuous linear functional on $L_\infty[0,1]$ that vanishes on the (closed) subspace of continuous functions.

13:6.3 Show that there is a continuous linear functional on $L_\infty[0,1]$ that is not of the form $\Gamma : f \to \int_0^1 f(x)g(x)\, dx$ for any $g \in L_1[0,1]$.

13:6.4 Interpret Theorem 13.18 as asserting that the spaces $(L_p)^*$ and L_q are isometrically isomorphic.

13:6.5 Let X be uncountable, let $\mathcal{M}$ denote the family of all sets that are at most countable or whose complements are at most countable, and let μ denote the counting measure on X. Characterize $L_\infty(X, \mathcal{M}, \mu)$. Describe the continuous linear functionals on $L_1(X, \mathcal{M}, \mu)$. Show that they cannot all be represented as integrals using functions in the space $L_\infty(X, \mathcal{M}, \mu)$. Does this contradict Theorem 13.18?

13:6.6 In Theorem 13.19, suppose that $\mu(X) = \infty$, but the measure space is σ-finite. Show that there is an everywhere positive function $w \in L_1$. Define a measure on $\mathcal{M}$ by $\tilde{\mu}(E) = \int_E w\, d\mu$. Show that $\tilde{\Gamma}(f) = \Gamma(w^{1/p} f)$ defines a continuous linear functional on the space $L_p(\tilde{\mu})$ with $\|\tilde{\Gamma}\| = \|\Gamma\|$.

13:6.7 Continuing the argument of Exercise 13:6.6, determine a function $\tilde{g} \in L_q(\tilde{\mu})$ so that

$$\tilde{\Gamma}(f) = \int_X f \tilde{g}\, d\tilde{\mu}$$

and define $g = w^{1/q}\tilde{g}$ (in the case $q = \infty$, just take $g = \tilde{g}$). Show that g satisfies the conclusion of Theorem 13.19.

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[1] For example, see E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag (1965).

13.7 The L_p Spaces $(0 < p < 1)$

Thus far we have not looked at the other end of the scale of L_p spaces, the lower end $(0 < p < 1)$. In our study we have used frequently and necessarily that the p–norm is a genuine norm; for $p = 1$ and $p = \infty$ this is trivial, for $1 < p < \infty$ this is Minkowksi's inequality, and for $0 < p < 1$ this is simply false.

Thus we cannot hope to have a similar theory for the low end of this scale and the spaces will not be Banach spaces. Even so, there is a linear structure and a metric structure, just not given by a norm. To find the right metric we need an elementary inequality.

Lemma 13.20 *For $0 < p < 1$ and for all nonnegative real a, b, the inequality*

$$(a + b)^p \leq a^p + b^p \tag{18}$$

holds, with equality occurring only if one of a, b is zero.

Proof. To prove the inequality (18), consider the function

$$\phi(t) = (1 + t)^p - 1 - t^p \qquad (t \geq 0).$$

One computes $\phi(0) = 0$ and checks that $\phi'(t)$ is negative for $t > 0$. It follows that $\phi(t) < 0$ for $t > 0$. Hence, if a and b are not zero, we replace t by a/b and obtain

$$\left(1 + \frac{a}{b}\right)^p - 1 - \left(\frac{a}{b}\right)^p < 0,$$

and the lemma follows easily. ■

Let $(X, \mathcal{M}, \mu)$ be a given measure space. It follows from the inequality (18) that the L_p spaces $(0 < p < 1)$ are linear spaces and that

$$\rho_p(f, g) = \int_X |f - g|^p \, d\mu$$

is a metric on L_p. It can be shown that the space is a complete metric space.

We must beware of applying Banach space ideas to these spaces. The metric is not defined by a norm and so, while we may seem to be in a familiar setting (a complete linear metric space), we do not have certain tools at hand. For example the Hahn–Banach theorem supplies an abundance of continuous linear functionals in any Banach space. The following theorem, due to M. M. Day, then is quite strange and illustrates a remarkable difference between Banach spaces and general linear metric spaces.

Theorem 13.21 (Day) *Using Lebesgue measure, the spaces $L_p[0, 1]$ for $0 < p < 1$ admit no continuous linear functionals apart from the zero functional.*

Proof. In order to obtain a contradiction, let us suppose that there is a continuous linear functional Γ on $L_p[0,1]$ that is not identically zero. There must accordingly be at least one function $f \in L_p[0,1]$ for which $\Gamma(f) = 1$. The mapping $x \to f\chi_{[0,x]}$ is a continuous function from $[0,1]$ to $L_p[0,1]$. Hence the composition

$$\phi(x) = \Gamma(f\chi_{[0,x]})$$

is a continuous real-valued map on $[0,1]$ for which $\phi(0) = 0$ and $\phi(1) = 1$. By the intermediate-value property, we can choose a point $x_0 \in (0,1)$ for which $\phi(x_0) = 1/2$. Consider the two functions $g_1 = f\chi_{[0,x_0]}$ and $g_2 = f\chi_{[x_0,1]}$. Since

$$\Gamma(g_1) + \Gamma(g_2) = \Gamma(g_1 + g_2) = \Gamma(f) = 1$$

and $\Gamma(g_1) = 1/2$, it follows that $\Gamma(g_2) = 1/2$.

But

$$\int_0^1 (|g_1(x)|^p + |g_2(x)|^p)\, dx = \int_0^1 |f(x)|^p\, dx$$

so one of the values $\|g_1\|_p^p$ and $\|g_2\|_p^p$ must be no greater than $\|f\|_p^p/2$. In particular, then,

$$\|2g_i\|_p^p \le 2^{p-1}\|f\|_p^p$$

either for $i = 1$ or for $i = 2$. Thus there is a function f_1 (taken as either $2g_1$ or $2g_2$, whichever is appropriate) for which

$$\Gamma(f_1) = 1 \text{ and } \|f_1\|_p^p \le 2^{p-1}\|f\|_p^p. \tag{19}$$

A repetition of this argument, applied now to f_1 rather than f, would yield a function f_2 for which

$$\Gamma(f_2) = 1 \text{ and } \|f_1\|_p^p \le 2^{p-1}\|f_2\|_p^p \le 2^{2(p-1)}\|f\|_p^p. \tag{20}$$

By induction, we arrive at a sequence of functions $\{f_n\}$ in the space $L_p[0,1]$, and for each n we have

$$\Gamma(f_n) = 1 \text{ and } \|f_n\|_p^p \le 2^{n(p-1)}\|f\|_p^p. \tag{21}$$

But this last assertion is impossible, since then $f_n \to 0$ in $L_p[0,1]$, and yet $\Gamma(f_n) = 1$, which cannot happen for a continuous functional. From this contradiction the theorem follows. ■

Exercises

13:7.1 Show that the functions in the $L_p[0,1]$ spaces $(0 < p < 1)$ are not necessarily integrable.

13:7.2 Show that the L_p spaces $(0 < p < 1)$ are complete.

13:7.3 Show that the "metric"

$$\rho(f,g) = \left(\int_X |f-g|^p \, d\mu\right)^{1/p}$$

does not satisfy the triangle inequality on the L_p spaces for the values $0 < p < 1$. Indeed, take two disjoint measurable sets A and B of positive measure, take f and g as their characteristic functions, and show that

$$\rho(f,g) \geq \rho(f,0) + \rho(0,g),$$

an inequality opposite to what one might expect.

13:7.4 In contrast to Theorem 13.21, construct a continuous linear functional on the spaces ℓ_p $(0 < p < 1)$. [Hint: If $y \in \ell_\infty$, then the mapping $x \to \sum x_i y_i$ is continuous.]

13:7.5 The limit of the p–norm is interesting at the lower end, too. Show that if $\|f\|_p$ is finite for some positive value of p and $\mu(X) = 1$ then $\|f\|_q$ is finite for all $0 < q < p$, and

$$\lim_{q \to 0+} \|f\|_q = \exp\left\{\int_X \log|f| \, d\mu\right\}.$$

13:7.6 In these spaces are continuous linear functionals the same as bounded linear functionals? [Hint: yes and no. If you interpret *bounded* using the metric then no. There is another way that bounded is usually interpreted though.]

13.8 Relations

We have been studying a scale of spaces without mentioning an obvious question. What happens as the index of the scale changes? Do we pick up some new functions or do we lose some? Both can happen. An example from the elementary calculus (the improper integrals) illustrates this. Consider the existence of the two integrals

$$\int_0^1 \left(\frac{1}{\sqrt{t}}\right)^p dt \quad \text{and} \quad \int_1^\infty \left(\frac{1}{\sqrt{t}}\right)^p dt.$$

The index $p = 2$ is critical for both integrals, but in a different way. For $p < 2$, the first integral exists, while for $p > 2$, the second integral exists. Once the nature of this, admittedly trivial, distinction is grasped, there are really no other surprises, and the first two theorems are nearly immediate. In general, we do not expect an inclusion $L_q(\mu) \subset L_p(\mu)$ for any values of p and q; we do obtain some relations of this kind in special cases.

Theorem 13.22 *If $\mu(X) < +\infty$ and $0 < p < q \leq \infty$, then $L_q(\mu) \subset L_p(\mu)$ and*

$$\|f\|_p \leq \|f\|_q \, (\mu(X))^{1/p - 1/q}$$

for any $f \in L_p(\mu)$.

Proof. Hölder's inequality (Theorem 13.1) gives most of this. If q is finite, then the two indices q/p and $q/(q-p)$ are conjugate, since

$$(q/p)^{-1} + (q/(q-p))^{-1} = p/q + (q-p)/q = 1,$$

and hence

$$\|f\|_p^p = \int_X 1 \cdot |f|^p \, d\mu \leq \left(\int_X |f|^{p(q/p)} \, d\mu \right)^{p/q} \left(\int_X d\mu \right)^{(q-p)/q}$$

On taking the $1/p$ power of both sides, we obtain the inequality of the theorem. In particular, since $\mu(X)$ is finite, the norm $\|f\|_p$ is finite if $\|f\|_q$ is finite, and so

$$L_q(\mu) \subset L_p(\mu).$$

Since

$$\|f\|_p^p = \int_X 1 \cdot |f|^p \, d\mu \leq \|f\|_\infty^p \left(\int_X d\mu \right) = \|f\|_\infty^p \, \mu(X),$$

the case $q = \infty$ is immediate. ■

Theorem 13.23 *If $0 < p < q < r \leq \infty$, then*

$$L_q(\mu) \subset L_p(\mu) + L_r(\mu)$$

[i.e., each $f \in L_q(\mu)$ can be decomposed into a sum of two functions, $f = f_1 + f_2$ where $f_1 \in L_p(\mu)$ and $f_2 \in L_r(\mu)$].

Proof. Let $f \in L_q(\mu)$, and split the space X into two parts: $A = \{x \in X : |f(x)| > 1\}$ and $B = \{x \in X : |f(x)| \leq 1\}$. Set $f_1 = f\chi_A$ and $f_2 = f\chi_B$. Then $f = f_1 + f_2$ and $f_1 \in L_p(\mu)$ and $f_2 \in L_r(\mu)$. To see this, we merely note that

$$\int_X |f_1|^p \, d\mu = \int_A |f|^p \, d\mu \leq \int_A |f|^q \, d\mu \leq \|f\|_q < \infty,$$

and $f_2 \in L_\infty(\mu)$ since $|f_2| \leq 1$, while, if $r < \infty$, we can use

$$\int_X |f_2|^r \, d\mu = \int_B |f|^r \, d\mu \leq \int_B |f|^q \, d\mu \leq \|f\|_q < \infty.$$

■

Theorem 13.24 *If $0 < p < q < r \leq \infty$, then*

$$L_q(\mu) \supset L_p(\mu) \cap L_r(\mu)$$

and

$$\|f\|_q \leq (\|f\|_p)^\kappa \, (\|f\|_r)^{1-\kappa},$$

where

$$\frac{1}{q} = \frac{\kappa}{p} + \frac{1-\kappa}{r}.$$

Proof. Suppose that f is in both $L_p(\mu)$ and $L_r(\mu)$. We apply Hölder's inequality (Theorem 13.1). If r is finite, then the two indices $p/(\kappa q)$ and $r/((1-\kappa)q)$ are conjugate since

$$\left(\frac{p}{\kappa q}\right)^{-1} + \left(\frac{r}{(1-\kappa)q}\right)^{-1} = q\left(\frac{\kappa}{p} + \frac{1-\kappa}{r}\right) = 1,$$

and hence

$$\|f\|_q^q = \int_X |f|^{\kappa q} \cdot |f|^{(1-\kappa)q}\, d\mu \leq \left(\int_X |f|^p\, d\mu\right)^{\kappa q/p} \left(\int_X |f|^r d\mu\right)^{(1-\kappa)q/r}$$

On taking the $1/q$ power of both sides, we obtain the inequality of the theorem and also that $f \in L_q(\mu)$.

The case $r = \infty$ is immediate, since

$$\|f\|_q^q = \int_X |f|^{q-p} \cdot |f|^p\, d\mu \leq \|f\|_\infty^{q-p} \left(\int_X |f|^p d\mu\right),$$

and the norm inequality follows directly. ■

For the ℓ_p spaces (to which Theorem 13.22 cannot apply), we have the following simple theorem. Note that the inclusion is proper since all the spaces in this scale are distinct.

Theorem 13.25 *For any $0 < p < q \leq \infty$, the inclusion $\ell_p \subset \ell_q$ holds and*

$$\|x\|_q \leq \|x\|_p$$

for each $x \in \ell_p$.

Proof. All the ℓ_p spaces evidently consist of bounded sequences (indeed, for $0 < p < \infty$ all consist of sequences converging to zero), and so all spaces are contained in ℓ_∞. It is also easy to check that $\|x\|_\infty \leq \|x\|_p$ for any sequence x. Now apply Theorem 13.24 with $0 < p < q < r = \infty$ and $\kappa = p/q$, using the measure space $\mathbb{N}$ with the counting measure; then

$$\|x\|_q \leq (\|x\|_p)^{(p/q)} (\|x\|_\infty)^{1-p/q} \leq (\|x\|_p)^{(p/q)} (\|x\|_p)^{1-p/q} = \|x\|_p,$$

as required. ■

Exercises

13:8.1 Show that all the spaces in the scale ℓ_p are distinct.

13:8.2 Let $1 \leq p < q \leq \infty$, and suppose that $\mu(X) < \infty$. Show that the identity mapping from L_p into L_q is continuous.

13:8.3 If X contains a disjoint sequence of measurable sets $\{E_i\}$ with $0 < \mu(E_i) < 2^{-i}$ and $0 < p < q \leq \infty$, then show that there is a function in L_q that is not in L_p.

13:8.4 If X contains a disjoint sequence of measurable sets $\{E_i\}$ with $1 \leq \mu(E_i) < \infty$ and $0 < p < q \leq \infty$ then show that there is a function in L_p that is not in L_q.

13:8.5 Let $0 < p < q \leq \infty$. Show that there is a function in L_p that is not in L_q if and only if X contains sets of arbitrarily small positive measure.

13:8.6 Let $0 < p < q \leq \infty$. Show that there is a function in L_q that is not in L_p if and only if X contains sets of arbitrarily large finite positive measure.

13:8.7 Let $1 \leq p < q \leq \infty$. Show that $L_p \cap L_q$ is a Banach space when furnished with the norm $\|f\| = \|f\|_p + \|f\|_q$.

13:8.8 Let $1 \leq p < \kappa < q \leq \infty$. Show that the identity mapping from $L_p \cap L_q$ into L_κ is continuous (when $L_p \cap L_q$ is furnished with the norm $\|f\| = \|f\|_p + \|f\|_q$).

13.9 The Banach Algebra $L_1(\mathbb{R})$

In this section we investigate the structure of the space $L_1(\mathbb{R})$ more closely. For this purpose we take the functions now as complex valued. $L_1(\mathbb{R})$ is a complex linear space furnished with a norm that makes it a Banach space. It is also a Banach algebra when an appropriate multiplication operation is defined. In Section 12.4 we saw that the operators on a Banach space form such a structure. Let us recall the definition here of a Banach algebra.

Definition 13.26 A Banach algebra is a Banach space A on which is defined a multiplication operation that satisfies the following conditions:

1. The multiplication operation is associative; that is,
$$x(yz) = (xy)z.$$
2. The multiplication operation is distributive; that is,
$$x(y+z) = xy + xz\,, \quad (x+y)z = xz + yz.$$
3. Scalar multiplication operation associates with the multiplication operation; that is,
$$(\lambda x)y = \lambda(xy) = x(\lambda y).$$
4. The norm satisfies $\|xy\| \leq \|x\|\,\|y\|$.

The appropriate multiplication operation in $L_1(\mathbb{R})$ is defined by *convolution*:
$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

Lemma 13.27 *The convolution*

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy \tag{22}$$

is defined for all f, $g \in L_1(\mathbb{R})$, the function $(f \star g)$ is an element of $L_1(\mathbb{R})$ and $\|f \star g\|_1 \leq \|f\|_1\,\|g\|_1$.

Proof. We give a proof without worrying about the measurability problem that arises. In the exercises we allow the reader to take on this worry.

Assume first that f, g as given are nonnegative. Then, since they are measurable, the function

$$F(x,y) = f(x-y)g(y)$$

is a measurable function with respect to two-dimensional Lebesgue measure in $\mathbb{R}^2$. (Is it?)

Thus we can apply Tonelli's theorem (Theorem 6.7) to obtain

$$\begin{aligned}\int\int_{\mathbb{R}^2} F &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-y)g(y)\,dy\right)\,dx \\ &= \int_{\mathbb{R}} g(y)\left(\int_{\mathbb{R}} f(x-y)\,dx\right)\,dy = \left(\int_{\mathbb{R}} g(y)\,dy\right)\left(\int_{\mathbb{R}} f(x)\,dx\right).\end{aligned}$$

We have also used the translation invariance of the integral in one of these computations. It follows from this that the function

$$x \to \int_{\mathbb{R}} f(x-y)g(y)\,dy$$

is almost everywhere finite and integrable and that

$$\int_{\mathbb{R}} (f \star g)(x)\,dx = \left(\int_{\mathbb{R}} g(y)\,dy\right)\left(\int_{\mathbb{R}} f(x)\,dx\right)$$

for nonnegative functions. Thus $\|f \star g\|_1 = \|f\|_1\,\|g\|_1$ for such functions. Then we use

$$\int_{\mathbb{R}} |f \star g|\,dx \leq \int_{\mathbb{R}} |f| \star |g|\,dx = \|f\|_1\,\|g\|_1$$

for the general case. ■

Theorem 13.28 *$L_1(\mathbb{R})$ is a Banach algebra when multiplication is defined by convolution.*

Proof. Lemma 13.27 supplies part of this. The only part that is not completely direct is to show that the multiplication is associative, that is

$f \star (g \star h) = (f \star g) \star h$. This is an interesting exercise in the use of the Fubini–Tonelli theorems:

$$(f \star g) \star h(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(y)g(x - z - y)h(z)\, dy\, dz$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y)g(x - y - z)h(z)\, dz\, dy = f \star (g \star h)(x).$$

■

A major point of investigation for Banach spaces is to determine the nature of the continuous linear functionals, that is the continuous mappings from the space into $\mathbb{C}$ that preserve the linear structure. For Banach algebras, this program requires us to focus on mappings that preserve also the multiplicative structure.

Definition 13.29 A mapping $\phi : B \to \mathbb{C}$ from a Banach space into the complex field is a *complex homomorphism* if ϕ is a linear functional preserving multiplication, that is, for which

$$\phi(\lambda_1 x + \lambda_2 y) = \lambda_1 \phi(x) + \lambda_2 \phi(y)\,, \quad \phi(xy) = \phi(x)\phi(y)$$

for all complex numbers λ_1, λ_2 and all x, $y \in B$.

We are interested only in continuous homomorphisms. It is rather curious that all complex homomorphisms on a Banach algebra are continuous in any case.

Theorem 13.30 *If ϕ is a complex homomorphism on a Banach algebra B, then ϕ is continuous.*

Proof. We show that the norm of ϕ as a linear functional is at most 1 so that $|\phi(x)| \leq \|x\|$ for all $x \in B$, and continuity is evident.

Suppose not. Then there exists an element $x_0 \in B$ for which $|\phi(x_0)| > \|x_0\|$. Let $x = (1/\phi(x_0))x_0$. Then

$$\|x\| = |1/\phi(x_0)|\|x_0\| < 1$$

and

$$\phi(x) = \phi([1/\phi(x_0)]x_0) = 1.$$

The series $\sum_{i=1}^{\infty} x^n$ must converge in the Banach space to an element z. Note that $x + x\left(\sum_{i=1}^{\infty} x^n\right) = y$ so that $y = x + xy$. Consequently,

$$\phi(y) = \phi(x + xy) = \phi(x) + \phi(x)\phi(y),$$

and so $\phi(y) = 1 + \phi(y)$, which is impossible. ■

We now turn to a most natural and most important problem: to determine all the complex homomorphisms on the Banach algebra $L_1(\mathbb{R})$. Since we already know the nature of all continuous linear functionals on $L_1(\mathbb{R})$ it is enough to examine these to see which are also complex homomorphisms. The special form of the answer is given in equation (23), called a Fourier transform.

Theorem 13.31 *To every nonzero complex homomorphism on the Banach algebra $L_1(\mathbb{R})$ there is a unique real number t so that*

$$\phi(f) = \int_{-\infty}^{\infty} f(x)e^{-ixt}\,dx. \tag{23}$$

Proof. We know from Theorem 13.18 that there is a function $h \in L_\infty(\mathbb{R})$ for which

$$\phi(f) = \int_{-\infty}^{\infty} f(x)h(x)\,dx, \tag{24}$$

so we have merely to show that $h(x) = e^{-ixt}$ for some t. We obtain this directly from Exercise 13:9.13 by showing that h is a nonzero, bounded, complex-valued continuous function on $\mathbb{R}$ that everywhere satisfies the functional equation

$$h(x+y) = h(x)h(y).$$

Since ϕ is a homomorphism on $L_1(\mathbb{R})$, we know that

$$\phi(f \star g) = \phi(f)\phi(g).$$

We apply (24) to both sides of this equation to get a relation involving h. First,

$$\begin{aligned}
\phi(f \star g) &= \int_{\mathbb{R}} f \star g(x)h(x)\,dx \\
&= \int_{\mathbb{R}} h(x)\,dx \int_{\mathbb{R}} f(x-y)g(y)\,dy \\
&= \int_{\mathbb{R}} g(y)\,dy \int_{\mathbb{R}} f_y(x)h(x)\,dx \\
&= \int_{\mathbb{R}} g(y)\phi(f_y)\,dy,
\end{aligned}$$

where we have used the notation $f_y(x) = f(x-y)$ and have applied (24) twice. Also,

$$\phi(f)\phi(g) = \phi(f)\int_{\mathbb{R}} g(y)h(y)\,dy,$$

so, putting these together, we have

$$\int_{\mathbb{R}} g(y)\phi(f)h(y)\,dy = \int_{\mathbb{R}} g(y)\phi(f_y)\,dy$$

for every $g \in L_1(\mathbb{R})$. This can happen only if

$$\phi(f)h(y) = \phi(f_y) \tag{25}$$

for almost every real y. There is no harm in redefining h on a set of measure zero so that (25) holds everywhere, and so we now know precisely what h is in terms of ϕ.

The functions $y \to f_y$ and ϕ are both continuous (see Exercise 13:9.11), and so h is continuous. We know already that h cannot be identically zero, and we know that h is bounded. We now wish to show that it satisfies the functional equation $h(x+y) = h(x)h(y)$. Using (25), we have

$$\phi(f)h(x+y) = \phi(f_{x+y}) = \phi((f_x)_y)$$

and, using (25) twice more,

$$\phi((f_x)_y) = \phi(f_x)h(y) = \phi(f)h(x)h(y),$$

so $h(x+y) = h(x)h(y)$ as required. It follows from Exercise 13:9.13 that $h(x) = e^{-ixt}$ for some t, and the proof is complete. ■

A final note: in the study of the Fourier transform many formulas simplify if Lebesgue measure on $\mathbb{R}$ is rescaled by a factor of $1/\sqrt{2\pi}$, and so the reader will often see the Fourier transform in a slightly different form than this theorem provides with that factor in front of the integral sign.

Exercises

13:9.1 Let f, $g \in L_1(\mathbb{R})$. Show that $f \star g = g \star f$ at every point where one of the two functions is defined.

13:9.2 Let $f \in L_1(\mathbb{R})$ and $g \in L_p(\mathbb{R})$, $1 < p < \infty$. Show that the convolution $f \star g$ is defined, that the function $(f \star g)$ is an element of $L_p(\mathbb{R})$, and $\|f \star g\|_p \leq \|f\|_1 \|g\|_p$. [Hint: Use ideas from the proof of Lemma 13.27 along with Hölder's inequality.]

13:9.3 Let f, $g \in L_1(\mathbb{R}^n)$. Define what should be meant by the convolution $f \star g$ and extend Lemma 13.27 to this setting.

13:9.4 Show that the algebra $L_1(\mathbb{R}^n)$ has no unit element [i.e., there is no function $u \in L_1(\mathbb{R}^n)$ so that $f \star u = u \star f = f$ for all $f \in L_1(\mathbb{R}^n)$]. [Hint: Take a function $f = \chi_{[0,1]}$ and show that

$$f(x) = \int_{-\infty}^{\infty} u(t) f(x - nt)\, dt,$$

but that $f(x - nt) \to 0$ everywhere as $n \to \infty$, and the dominated convergence theorem applies.]

13:9.5 Are any of the L_p spaces Banach algebras if multiplication is defined pointwise; that is, $(fg)(x) = f(x)g(x)$? [Hint: Are the spaces closed under such an operation?]

13:9.6 In the proof of Lemma 13.27, why would it not have been enough merely to say that $f(x - y)$ and $g(y)$ are integrable and hence so is the product.

13:9.7 Handle the measurability problem in Lemma 13.27. Let $F_1(x, y) = f(x)$. Show that F_1 is a measurable function in $\mathbb{R}^2$. Consider the transformation $T : (\xi, \eta) \to (x, y) = (\xi - \eta, \xi + \eta)$. Show that the composition

$$F(\xi, \eta) = F_1 \circ T(\xi, \eta) = F_1(\xi - \eta, \xi + \eta) = f(\xi - \eta)$$

is measurable.

13:9.8 Avoid the measurability problem in Lemma 13.27. Argue that f and g can be replaced by Borel functions f_0 and g_0 that are almost everywhere equal and so the integrals do not change. Is the function $F_0(x, y) = f_0(x - y)g_0(y)$ a Borel function in $\mathbb{R}^2$?

13:9.9 Let $1 \leq p \leq \infty$. Prove that the convolution

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)\, dy$$

is defined for all $f \in L_1(\mathbb{R})$ and $g \in L_p(\mathbb{R})$, that the function $(f \star g)$ is an element of $L_p(\mathbb{R})$, and that $\|f \star g\|_p \leq \|f\|_1 \|g\|_p$.

13:9.10 Let $1 \leq p, q \leq \infty$ (not necessarily conjugate) such that

$$r^{-1} = p^{-1} + q^{-1} - 1 \geq 0.$$

Prove that the convolution

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)\, dy$$

is defined for all $f \in L_p(\mathbb{R})$ and $g \in L_q(\mathbb{R})$, that the function $(f \star g)$ is an element of $L_r(\mathbb{R})$, and that $\|f \star g\|_r \leq \|f\|_p \|g\|_q$.

13:9.11 Let $f \in L_1(\mathbb{R})$ and $y \in \mathbb{R}$. Define the translate f_y by $f_y(x) = f(x - y)$. Show that the mapping $y \to f_y$ is a continuous map of $\mathbb{R}$ into $L_1(\mathbb{R})$. [Hint: First approximate f by a continuous function g that vanishes outside some interval.]

13:9.12 Show that

$$(f \star g)_y = f_y \star g = f \star g_y,$$

where the notation is as in Exercise 13:9.11.

13:9.13 Let h be a nonzero, bounded, complex-valued continuous function on $\mathbb{R}$ that everywhere satisfies the functional equation

$$h(x + y) = h(x)h(y).$$

Show that $h = e^{-itx}$ for some t. [Hint: Show first that $h(0) = 1$. Choose $\delta > 0$ so that $\int_0^\delta h(x)\, dx = c \neq 0$, and show that $ch(x) = \int_x^{x+\delta} h(y)\, dy$. Conclude that h is differentiable. Obtain that $h'(x) = h'(0)h(x)$ and hence that $h(x) = e^{h'(0)x}$. You will need to remember that h is bounded.]

13.10 Weak Sequential Convergence

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let p, q be conjugate indices with $1 \le p < \infty$. A sequence of functions $\{f_n\}$ *converges in the sense of the norm* in $L_p(\mu)$ to a function f if $\|f_n - f\|_p \to 0$; that is, if

$$\int_X |f_n - f|^p \, d\mu \to 0.$$

Often we must work with a weaker version of convergence in these spaces.

Definition 13.32 A sequence of functions $\{f_n\}$ *converges weakly* in $L_p(\mu)$ to a function f if

$$\int_X f_n g \, d\mu \to \int_X f g \, d\mu$$

for all $g \in L_q(\mu)$.

By using Theorem 13.19, we see that this is the requirement that $\Gamma(f_n) \to \Gamma(f)$ for all continuous linear functionals Γ on $L_p(\mu)$.

One of the most useful applications of this notion of weak convergence is in compactness arguments. A sequence may be bounded in $L_p(\mu)$ and yet have no convergent subsequence (with convergence interpreted in the norm sense). This would seem to imply that we are unable to use any kind of compactness arguments when dealing with bounded sets in $L_p(\mu)$. But if we can be satisfied with weak convergence, a convergent subsequence can be found.

Theorem 13.33 (Weak sequential compactness)
Suppose that $(X, \mathcal{M}, \mu)$ is a σ-finite measure space, let $1 < p < \infty$, and suppose that $L_q(X, \mathcal{M}, \mu)$ is separable, where q is the conjugate index to p. Suppose that $\{f_n\}$ is a sequence of functions with

$$\|f_n\|_p \le M$$

for some M. Then there is a function $f \in L_p(\mu)$ with

$$\|f\|_p \le M$$

and a subsequence $\{f_{n_k}\}$ that converges weakly in $L_p(\mu)$ to f.

Proof. Fix an element $g_1 \in L_q$. We show how to determine a subsequence so that

$$\lim_{k\to\infty} \int_X f_{n_k} g_1 \, d\mu \tag{26}$$

exists. By Hölder's inequality, we know that

$$\left| \int_X f_n g \, d\mu \right| \le M \|g\|_q < \infty,$$

and so this sequence of real (complex) numbers is bounded. Thus a subsequence for which the limit (26) exists can be found merely from the Bolzano–Weierstrass theorem.

Fix elements $g_1, g_2, \dots, g_m \in L_q$. We can determine a subsequence so that

$$\lim_{k\to\infty} \int_X f_{n_k} g_i \, d\mu \tag{27}$$

exists for each $i = 1, 2, 3, \dots m$. We just apply the same argument for each i and pass to subsequences of subsequences.

Finally, and much more generally, let $g_1, g_2, \dots$ be an infinite sequence of elements of L_q that forms a dense subset. Once again we can determine a subsequence so that

$$\lim_{k\to\infty} \int_X f_{n_k} g_i \, d\mu \tag{28}$$

exists for each $i = 1, 2, 3, \dots$. We cannot quite use a "subsequence of a subsequence" argument indefinitely, but we can use a Cantor diagonalization argument to get a single subsequence $\{f_{n_k}\}$ that works for each g_i (Exercise 13:10.6).

Define a functional Γ on L_q by first writing

$$\Gamma(g_i) = \lim_{k\to\infty} \int_X f_{n_k} g_i \, d\mu \tag{29}$$

for each g_i in our dense set and then extending to all of L_q by continuity. By Hölder's inequality applied to (29), we have

$$|\Gamma(g_i) - \Gamma(g_j)| \le M\|g_i - g_j\|,$$

and so Γ is uniformly continuous on the dense subset, allowing therefore a unique extension to a continuous functional. We claim that Γ is linear. It is certainly linear on the dense subset formed from the $\{g_i\}$ because of its definition as an integral in (29). This linearity is preserved in the limit, too, when extended to all of L_q. Note as well that $\|\Gamma\| \le M$.

We apply Theorem 13.18 to obtain an element $f \in L_p$ so that

$$\Gamma(g) = \int_X fg \, d\mu$$

for all $g \in L_q$. It is easy to see now that f is precisely the element of L_p that we want, that $f_{n_k} \to f$ weakly, and that $\|f\|_p \le M$. ■

Exercises

13:10.1 Give an example of a sequence in ℓ_p for $(1 < p < \infty)$ that converges weakly but not in norm. Check that your example does not converge weakly in ℓ_1.

13:10.2 As a project, find and present a proof of the following theorem.

> **Theorem (Shur)** *A weakly convergent sequence in ℓ_1 is necessarily also norm convergent.*

13:10.3 Let $1 < p < \infty$, and suppose that $\{f_n\}$ is a bounded sequence in $L_p[0,1]$. Show that if $f_n \to f$ almost everywhere then $f_n \to f$ weakly in $L_p[0,1]$.

13:10.4 Let $1 < p < \infty$, and suppose that $\{f_n\}$ is a sequence in $L_p[0,1]$. Show that $f_n \to f$ weakly in $L_p[0,1]$ if and only if $\{f_n\}$ is bounded and $\int_E f_n(x)\,dx$ converges to $\int_E f(x)\,dx$ for every measurable subset of $[0,1]$.

13:10.5 Let $1 < p < \infty$, and suppose that $\{f_n\}$ is a sequence in $L_p[0,1]$. Show that $f_n \to f$ weakly in $L_p[0,1]$ if and only if $\{f_n\}$ is bounded and $f_n \to f$ in measure. (What if $p = 1$?)

13:10.6 In the proof of Theorem 13.33 we left out some details involving "subsequences of subsequences." How might these be provided?

13.11 Closed Subspaces of the L_p Spaces

In this section we prove a property of closed subspaces of the L_p spaces as an interesting application of the closed graph theorem from Section 12.14. The theorem is due to A. Grothendieck.

Theorem 13.34 *Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let W be a closed subspace of $L_p(\mu)$ consisting of essentially bounded functions [i.e., $W \subset L_\infty(\mu)$]. Then W is finite dimensional.*

Proof. For our application of the closed graph theorem, the reader need recall only that, if the map $\Gamma : W \to L_\infty$ defined by $\Gamma(f) = f$ (i.e., the identity injection) has a closed graph, then it is continuous. To see that the graph is closed, consider a sequence $\{f_n\}$ in W so that $f_n \to f$ in W and $\Gamma(f_n) = f_n \to g$ in L_∞: then $f = g$ a.e. and this shows that the graph of Γ is closed. Hence, by a basic property of continuous operators,

$$\|f\|_\infty \le M\|f\|_p \quad (\text{for all } f \in W). \tag{30}$$

Here we are considering W as a Banach space itself using the L_p–norm; since W is a closed subset of L_p, this is justified. This is the only use made of the hypothesis that W is closed.

We need to sharpen our inequality (30) to obtain

$$\|f\|_\infty \le M_1\|f\|_2 \quad (\text{for all } f \in W), \tag{31}$$

thus allowing us to use some special features of L_2. If $1 \le p \le 2$, then (31) is immediate with $M_1 = M$, since then

$$\|f\|_p \le \|f\|_2.$$

If $2 < p < \infty$, then, since $\mu(X)$ is finite and f is essentially bounded, we can obtain (31) with an appropriate M_1 by integrating the inequality

$$|f|^p \leq (\|f\|_\infty)^{p-2}\, |f|^2$$

and using (30).

Now that we have placed W inside L_2, we can use special features of the latter space to show that W must be finite dimensional. Let $\{f_1, f_2, \dots, f_n\}$ be a linearly independent set in W; without loss of generality, we can assume that these are orthonormal in L_2 (by applying the elementary Gram–Schmidt process for example). Thus

$$\int_X f_i f_j \, d\mu$$

is 0 or 1 depending on $i \neq j$ or $i = j$. Our goal is to show that n cannot be too big, in fact, that $n \leq M_1^2 \mu(X)$.

For each choice of rational numbers $c = (c_1, c_2, \dots, c_n)$ with $\sum_{i=1}^n |c_i|^2 \leq 1$, define a function

$$F_c = \sum_{i=1}^n c_i f_i.$$

Note that $F_c \in W$ and that

$$\|F_c\|_2 = \|\sum_{i=1}^n c_i f_i\|_2 = \sum_{i=1}^n |c_i|^2 \leq 1$$

follows from the Pythagorean theorem (Exercise 13:5.4). Consequently, by (31),

$$\|F_c\|_\infty \leq M_1 \|F_c\|_2 \leq M_1.$$

Thus there is a set of measure zero E_c such that

$$\left| \sum_{i=1}^n c_i f_i(x) \right| \leq M_1$$

for all $x \in X \setminus E_c$. Let E denote the intersection of the countable family of all sets E_c taken over rational numbers $c = (c_1, c_2, \dots, c_n)$ with $\sum_{i=1}^n |c_i|^2 \leq 1$. Then we have

$$\left| \sum_{i=1}^n c_i f_i(x) \right| \leq M_1$$

for all $x \in X \setminus E$ and any choice of rational numbers $(c_1, c_2, \dots, c_n)$ with $\sum_{i=1}^n |c_i|^2 \leq 1$. By continuity, this same inequality holds for all real numbers $c = (c_1, c_2, \dots, c_n)$ with $\sum_{i=1}^n |c_i|^2 \leq 1$. But, at any x for which this is true, we must have

$$\sum_{i=1}^n |f_i(x)|^2 \leq M_1^2.$$

This inequality holds almost everywhere on X, and so an integration [remember that $\int_X |f_i(x)|^2\, d\mu = 1$] gives us

$$n \leq M_1^2 \mu(X)$$

as required to complete the proof. ■

We should not leave this theorem without constructing a closed infinite-dimensional proper subspace of an L_p space that lies in some later L_q space (but not in L_∞ because of the theorem). This also proves interesting for us because it exploits some of the basic tools in the subject (Hölder's inequality, Cauchy sequences) and previews some ideas from trigonometric series that are to reappear in a fuller light in Chapter 15.

Theorem 13.35 *Let L_1 be the space of Lebesgue integrable functions on the interval $[-\pi, \pi]$. Then there is an infinite-dimensional closed subspace of L_1 that forms a closed subspace of L_4.*

Proof. The computations are simpler if we use the measure

$$\mu = (2\pi)^{-1}\lambda,$$

that is, Lebesgue measure divided by 2π. This makes the family $\{e^{ijt}\}$ orthonormal in the L_2–norm, and combinations can be used to form our subspace. Let E be the set of integers 2^k for $k = 1, 2, 3, \dots$. The only significant feature for us is that E is infinite and that no integer can be written as a sum of members of E in more than one way. Define W_1 to be the vector space of all functions of the form

$$f(e^{it}) = \sum_{j=1}^{\infty} c_j e^{ijt},$$

where $c_j = 0$ if $j \notin E$. Let W be the closure in $L_1[-\pi, \pi]$ of W_1. It is obvious that W is closed and infinite dimensional; we must show that every function in W is also from L_4 and that W is also closed in L_4.

Let

$$f(e^{it}) = \sum_{j=1}^{\infty} c_j e^{ijt},$$

where $c_j = 0$ if $j \notin E$ be any member of W_1. Then squaring, we have

$$f^2(e^{it}) = \sum_j c_j^2 e^{2ijt} + \sum_{j \neq k} c_j c_k e^{i(j+k)t}.$$

Under our assumptions on E, we see that a nonzero coefficient for the term $e^{i(j+k)t}$ can occur only once as $c_j c_k$ for $j,\ k \in E$. Thus, using the

Pythagorean theorem (Exercise 13:5.5),

$$\int_{[-\pi,\pi]} |f^2|^2\,d\mu = \sum_j |c_j|^4 + 2\sum_{j\neq k} |c_j|^2|c_k|^2$$
$$\leq \quad 2\left(\sum_j |c_j|^2\right)^2 = 2\left(\int_{[-\pi,\pi]} |f^2|\,d\mu\right)^2 .$$

Consequently, $\|f\|_4^4 \leq 2\|f\|_2^4$ or $\|f\|_4 \leq 2^{1/4}\|f\|_2$.

To improve this estimate, we need to relate it to the L_1-norm. Use Hölder's inequality and the conjugate indices $p=3$, $q=3/2$ to obtain

$$\int_{[-\pi,\pi]} |f|^2\,d\mu = \int_{[-\pi,\pi]} |f|^{4/3}\cdot|f|^{2/3}\,d\mu$$
$$\leq \quad \left(\int_{[-\pi,\pi]} \left(|f|^{4/3}\right)^3\,d\mu\right)^{1/3}\left(\int_{[-\pi,\pi]} \left(|f|^{2/3}\right)^{3/2}\,d\mu\right)^{2/3},$$

and so
$$\|f\|_2^2 \leq \|f\|_4^{4/3}\|f\|_1^{2/3}.$$

If we combine this with the inequality

$$\|f\|_4 \leq 2^{1/4}\|f\|_2$$

obtained above, we have (after some arithmetic) that

$$\|f\|_4 \leq 2^{3/4}\|f\|_1.$$

This shows that every L_1-Cauchy sequence in the set W_1 is also an L_4-Cauchy sequence. It follows that the L_1-closure W must be a subset of L_4. Since it is closed by definition in the L_1-norm and in general we have $\|f\|_1 \leq \|f\|_4$, we see that W is closed in L_4. ■

Exercises

13:11.1 In the proof of Theorem 13.34, go through the computations necessary to establish (31): for $2 < p < \infty$ show that

$$\|f\|_\infty \leq M^{p/2}\|f\|_2 \quad \text{(for all } f \in W).$$

13.12 Additional Problems for Chapter 13

13:12.1 Show that if $f \in \mathbb{L}_p(X,\mathcal{M},\mu)$ then

$$\lim_{t\to\infty} t^p\mu\left(\{x\in X : |f(x)| > t\}\right) = 0.$$

13:12.2 Let $1 < p < \infty$. A necessary and sufficient condition that a function F on $[0,1]$ be an indefinite integral of a function $f \in L_p[0,1]$ is that

$$\sup \sum_{i=1}^{n} \frac{F(x_i) - F(x_{i-1})}{(x_i - x_{i-1})^{p-1}} < \infty$$

where the supremum is taken over all partitions

$$0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$$

of the interval $[0,1]$. What are the appropriate characterizations for the cases $p = 1$ and $p = \infty$?

13:12.3 Let $1 < p < \infty$. A sequence $\{\xi_n\}$ converges weakly to ξ in $\ell_p(X)$ if and only if $\sup_n \|\xi_n\| < \infty$ and $\xi_n(x) \to \xi(x)$ for every $x \in X$. (See Exercise 13:2.1 for the terminology.)

13:12.4 Show that the sequence of functions

$$f_n(x) = n\chi_{(0,1/n)}$$

converges a.e. and in measure on $[0,1]$, but not weakly in $L_p[0,1]$.

13:12.5 Let $f \in L_p(X, \mathcal{M}, \mu)$, and define the function

$$F_f(t) = \mu\left(\{x \in X : |f(x)| > t\}\right).$$

(a) Show that F_f is a nondecreasing function on $(0,\infty)$ and continuous on the right.

(b) If $|f| \le |g|$ almost everywhere, then show that $F_f \le F_g$ everywhere.

(c) Show that

$$\int_X |f|^p \, d\mu = -\int_0^\infty t^p \, dF_f(t) = p\int_0^\infty t^{p-1} F_f(t)\, dt.$$

13:12.6 Let $1 \le p, q \le \infty$ be conjugate indices, and suppose that $f \in L_p(\mathbb{R})$ and $g \in L_q(\mathbb{R})$. Show that $f \star g(x)$ exists everywhere, that $f \star g$ is bounded and continuous on $\mathbb{R}$, and that

$$\|f \star g\|_\infty \le \|f\|_p \, \|g\|_q.$$

[Hint: Consider $\|(f \star g)_y - f \star g\|_\infty$.]

13:12.7 Prove Steinhaus's theorem:

> **Theorem (Steinhaus)** *Let $E \subset \mathbb{R}$ be a measurable set of positive measure. Then the set*
>
> $$E - E = \{x - y : x, y \in E\}$$
>
> *contains an interval $(-\delta, \delta)$.*

[Hint: Show that $\phi = \chi_E \star \chi_E$ is continuous and that $\phi(0) > 0$. Then $\phi(x) > 0$ on some interval $(-\delta,\delta)$, and so for each x in that interval there is a t with $\chi_E(t)\chi_E(t-x) > 0$.]

Chapter 14

HILBERT SPACES

The spaces that we study in this chapter have been named in honor of David Hilbert (1861–1943), by folklore the last of the great mathematicians. To be sure there are today and will be tomorrow great algebraists, great analysts, great topologists, and others, but Hilbert is considered by many to be the end of the line of the great universal mathematicians that contains such important figures as Gauss, Euler, and Riemann. It seems to have been F. Riesz who first named Hilbert spaces this way *(un espace Hilbertien)* in his study of the sequence space ℓ_2 of square summable sequences. Hilbert himself did not use the term space in his studies or use any explicit geometric language, but the methods that he developed in investigations of infinite systems of linear equations, infinite quadratic forms, and integral equations we would certainly consider as Hilbert space methods. The theory came into its own as a recognizable subject by the 1920s when John von Neumann (1903–1957) showed that these spaces were fundamental to an understanding of quantum mechanics.

The basic underlying idea goes back as far as Gauss and Legendre in the familiar method of least squares. It was not, however, until this century that this was placed in the context of infinite-dimensional spaces and the theory developed in the direction we see now.

Hilbert spaces are very special Banach spaces. Indeed they are extraordinary in many ways. The geometry is more transparent, the proofs easier and more beautiful, and the results farther reaching. The study of Banach spaces is much messier and less organized. The fundamental reason is that Hilbert spaces are self-adjoint; that is, the space of continuous linear functionals on a Hilbert space is the space itself. This relation is described by an inner product that, along with the linear structure, carries the full structure of the space.

Our chapter is only an introduction. Many long treatises have been written on this subject, and the reader here will see only the fundamentals and a few highlights. The basic theory of orthogonal series is covered. A few ideas related to weak sequential convergence appear, and a version of the spectral theorem for compact operators is given. This is more than enough to give the flavor of Hilbert space and provides all the basic tools of the subject.

14.1 Inner Products

We shall insist that our Hilbert spaces use complex scalars, although real Hilbert spaces are of use. The reason for this is similar to the situation in elementary algebra: to study real $n \times n$ matrices requires exploring their eigenvalues, and eigenvalues even of real matrices are frequently complex numbers. It is better at the outset of such a study to investigate complex $n \times n$ matrices.

Let X be first of all a complex linear space. As before, we use $\mathbf{0}$ to denote the origin. Rather than place a norm on X directly, we shall assume that X is equipped with an inner product; from the inner product we will derive a norm so that X is then furnished with a norm structure, too.

Definition 14.1 A scalar function $(f, g) : X \times X \to \mathbb{C}$ (or $\mathbb{R}$ if X is a real linear space) is said to be an *inner product* or *scalar product* if it satisfies the following conditions:

1. $(f, f) \geq 0$ for all $f \in X$, with equality if and only if $f = \mathbf{0}$.
2. $(f, g) = \overline{(g, f)}$ for all $f, g \in X$.
3. $(af, g) = a(f, g)$ for all $f, g \in X$ and $a \in \mathbb{C}$ (or $\mathbb{R}$).
4. $(f_1 + f_2, g) = (f_1, g) + (f_2, g)$ for all $f_1, f_2, g \in X$.

We say that X is an *inner product space* if X is a linear space equipped with an inner product. We will see, in due course, that

$$\|f\| = \sqrt{(f, f)}$$

is a norm on X. Thus an inner product space is also a normed linear space and inherits all the terminology of Chapter 12. In particular, it can be a Banach space too: if so, we shall say that X is a Hilbert space. Specifically, then, a *Hilbert space* is an inner product space that is complete as a metric space when furnished with the norm $\|f\| = \sqrt{(f, f)}$. It will be clear from the context whether a real or a complex Hilbert space is intended; assume the latter in most cases.

For the rest of this section we discuss only the most rudimentary properties. Notice first the linearity of an inner product in the first variable:

$$(\lambda_1 f_1 + \lambda_2 f_2, g) = \lambda_1(f_1, g) + \lambda_2(f_2, g).$$

There is a kind of linearity in the second variable, but complex conjugation enters in:

$$(f, \lambda_1 g_1 + \lambda_2 g_2) = \overline{\lambda_1}(f, g_1) + \overline{\lambda_2}(f, g_2).$$

If the space is real, then, the form is linear in each variable.

We define a norm on X by

$$\|f\| = \sqrt{(f,f)}. \tag{1}$$

We shall show that this definition does, in fact, provide a norm. For the moment, $\|f\|$ is only an alternative notation for $\sqrt{(f,f)}$.

Lemma 14.2 (Cauchy–Schwarz inequality) *For f, g in an inner product space X, the following inequality holds:*

$$|(f,g)| \le \|f\|\,\|g\| \tag{2}$$

with equality if and only if f and g are linearly dependent in X.

Proof. We can repeat the proof of Lemma 13.12 here since that proof did not use any special features of the space L_2 that are not also true in a general inner product space.

First, suppose that X is a real inner product space. Consider the polynomial

$$\begin{aligned} p(\alpha) &= (\alpha f + g, \alpha f + g) = \alpha^2(f,f) + 2\alpha(f,g) + (g,g) \\ &= \|f\|^2\alpha^2 + 2(f,g)\alpha + \|g\|^2. \end{aligned}$$

The definition of p, together with the fact that $(\alpha f + g, \alpha f + g)$ must be nonnegative, implies that $p(\alpha) \ge 0$ for all $\alpha \in \mathbb{R}$. It follows from the quadratic formula that

$$(f,g)^2 - \|f\|^2\|g\|^2 \le 0. \tag{3}$$

Otherwise, the quadratic p would have two distinct roots and would therefore not be nonnegative. The inequality (2) follows from (3).

Now suppose that the space is a complex inner product space and fix f, g. There is a real θ so that $(f,g) = e^{i\theta}|(f,g)|$. Let $f_1 = e^{-i\theta}f$, and observe that $(f_1, g) = |(f,g)|$. Since (f_1, g) is real, we can obtain from the argument of the first paragraph that $(f_1, g) \le \|f_1\|\,\|g\|$. Since $\|f_1\| = \|f\|$, the inequality (2) follows. ■

We verify that $\|f\|$ as given in (1) is actually a norm on X. It is obvious that $\|f\| = 0$ if and only if $f = 0$ and $\|af\| = |a|\|f\|$ for all $a \in \mathbb{C}$. It remains to check the triangle inequality.

Lemma 14.3 *For all f, g in an inner product space X,*

$$\|f+g\| \le \|f\| + \|g\|.$$

Proof. For all $f, g \in X$,

$$\begin{aligned} \|f+g\|^2 &= (f+g, f+g) \\ &= (f,f) + (g,f) + (f,g) + (g,g) \\ &\le (f,f) + 2|(f,g)| + (g,g) \\ &\le \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 = (\|f\| + \|g\|)^2, \end{aligned}$$

the last inequality following from the Cauchy–Schwarz inequality. Thus

$$\|f+g\| \leq \|f\| + \|g\|.$$

This establishes the triangle inequality. ■

The next theorem is usually called the parallelogram law because of its geometric interpretation: the sum of the squares of the diagonals of a parallelogram is the sum of the squares of the sides. This characterizes the norm in a Hilbert space. If a normed linear space has a norm that satisfies the parallelogram law, then there is an inner product on the space that expresses this norm (see Exercises 14:1.4 and 14:1.8).

Theorem 14.4 (Parallelogram law) *In any inner product space the identity*

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2) \tag{4}$$

holds for all pairs f, g.

Proof. A direct computation yields

$$\begin{aligned} \|f+g\|^2 \quad &+ \quad \|f-g\|^2 = (f+g, f+g) + (f-g, f-g) \\ &= \quad 2(\|f\|^2 + \|g\|^2) \end{aligned}$$

as required. ■

Orthogonality in an inner product space or a Hilbert space is defined as in $\mathbb{R}^n$. Two elements f, g are *orthogonal* if $(f, g) = 0$. By this definition, the zero element is orthogonal to every element, but is the only such element. A family of elements is said to be *orthonormal* if every pair of members is orthogonal and all elements have unit length. The reader should recall that this is precisely how orthogonality and orthonormality in Euclidean spaces are defined. Thus there should be no surprise that a Pythagorean theorem is available.

Theorem 14.5 (Pythagorean theorem) *If f_1, f_2, ..., f_n are pairwise orthogonal elements of an inner product space, then*

$$\left\|\sum_{i=1}^{n} f_i\right\|^2 = \sum_{i=1}^{n} \|f_i\|^2 .$$

Proof. This is an entirely elementary computation starting with

$$\left\|\sum_{i=1}^{n} f_i\right\|^2 = (\sum_{i=1}^{n} f_i, \sum_{j=1}^{n} f_j)$$

and continuing in the obvious manner, making use of the fact that $(f_i, f_j) = 0$ for $i \neq j$. ■

Here is another variant of the Pythagorean theorem.

Corollary 14.6 *If f_1, f_2, $\dots f_n$ are orthonormal elements of an inner product space, then*

$$\left\| \sum_{i=1}^{n} c_i f_i \right\|^2 = \sum_{i=1}^{n} |c_i|^2.$$

We conclude this section with some standard examples of Hilbert spaces. In fact, these are the only examples of Hilbert spaces: any other Hilbert spaces that might be differently given will turn out to be identical to one of these.

Example 14.7 (Euclidean space) The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are examples of real and complex Hilbert spaces. The inner product is the familiar $(x, y) = \sum_{i=1}^{n} x_i y_i$ in the real case and $(x, y) = \sum_{i=1}^{n} x_i \overline{y_i}$ in the complex case. In both cases, the norm is

$$\|x\| = \sqrt{(x, x)} = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

Since these spaces are complete, they are Hilbert spaces.

Example 14.8 (The space ℓ_2) The space ℓ_2 has been defined as the collection of all sequences $\{x_k\}$ of real or complex numbers such that $\sum |x_k|^2 < \infty$. The inner product is the infinite-dimensional analog of that for the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ and just substitutes an infinite sum for a finite one: $(x, y) = \sum_{i=1}^{\infty} x_i y_i$ in the real case and $(x, y) = \sum_{i=1}^{\infty} x_i \overline{y_i}$ in the complex case. In both cases, the norm is

$$\|x\| = \sqrt{(x, x)} = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}.$$

Since ℓ_2 is complete, it is a Hilbert space. (See also Section 13.5.)

Example 14.9 (The space $\ell_2(I)$) Let I be any nonempty set. We can generalize both of the preceding examples by defining $\ell_2(I)$ to be the collection of all real or complex functions x on I such that $\sum_{i \in I} |x_i|^2 < \infty$. The inner product is defined to be

$$(x, y) = \sum_{i \in I} x_i y_i$$

in the real case and

$$(x, y) = \sum_{i \in I} x_i \overline{y_i}$$

in the complex case. In both cases, the norm is

$$\|x\| = \sqrt{(x, x)} = \sqrt{\sum_{i \in I} |x_i|^2}.$$

The space $\ell_2(I)$ can be shown to be complete and so is a Hilbert space. Note that already this includes the preceding examples for different index sets either by using $I = \{1, 2, 3, \ldots, n\}$ or $I = \mathbb{N}$.

To interpret this example, the reader must be informed as to the meaning of infinite, unordered sums of the form $A = \sum_{i \in I} a_i$. The meaning is taken that for every $\varepsilon > 0$ there is a finite set $F \subset I$ so that

$$|A - \sum_{i \in F'} a_i| < \varepsilon$$

for every finite set F' with $F \subset F' \subset I$.

Example 14.10 (The space $L_2(X, \mathcal{M}, \mu)$) The special space studied in Section 13.5 is a Hilbert space. Recall that the inner product is taken as

$$(f, g) = \int_X f\overline{g}\, d\mu$$

and the norm as

$$\|f\| = \sqrt{(f, f)} = \sqrt{\int_X |f|^2\, d\mu}.$$

In fact, this space will also turn out to be identical with ℓ_2 or $\ell_2(I)$ for some appropriate index set I.

Exercises

14:1.1 If $f_n \to f$ and $g_n \to g$, show that $(f_n, g_n) \to (f, g)$.

14:1.2 Show that the mapping $f \to (f, g)$ for a fixed $g \in H$ is a continuous linear functional on H and that its norm (as a linear functional) is precisely $\|g\|$.

14:1.3 Show that the parallelogram law fails for L_1, and thus there is no choice of inner product giving that norm. [Hint: For $X = [0, 1]$, $f = \chi_{[0,1/2]}$, and $g = \chi_{[1/2,1]}$, calculate $\|f + g\|_1 = \|f - g\|_1 = 1$ and $\|f\|_1 = \|g\|_1 = \frac{1}{2}$.]

14:1.4$\diamondsuit$ Show that if the parallelogram law (4) is valid for a real normed linear space X then

$$(f, g) = \frac{1}{4}(\|f + g\|^2 - \|f - g\|^2)$$

is a real inner product on X that gives rise to the norm via (1).

14:1.5 Show that the spaces L_p and ℓ_p are inner product spaces if and only if $p = 2$.

14:1.6$\diamondsuit$ Give a converse to Theorem 14.5 in a *real* inner product space: if

$$\|f_1 + f_2\|^2 = \|f_1\|^2 + \|f_2\|^2,$$

then f_1 and f_2 are orthogonal.

14:1.7◇ (Polarization identity) For any f, g in a complex inner product space,

$$4(f, g) = \|f + g\|^2 - \|f - g\|^2 + i\|f + ig\|^2 - i\|f - ig\|^2.$$

14:1.8◇ Show that if the parallelogram law (4) is valid for a complex normed linear space X then there is an inner product on X that gives rise to the norm (cf. Exercises 14:1.4 and 14:1.7).

14:1.9◇ Let $\{f_n\}$ be an orthogonal sequence in a Hilbert space H. Show that the series $\sum_{i=1}^{\infty} f_i$ converges in H if and only if $\sum_{i=1}^{\infty} \|f_i\|^2 < \infty$.

14:1.10 For any set S in an inner product space P, write

$$S^{\perp} = \{f \in H : (f, g) = 0 \text{ for all } g \in S\}$$

so that $S^{\perp}$ is the set of all elements of the space orthogonal to each element of S. Possibly $S^{\perp}$ may consist of just the zero vector, but often it is much more. Let A and B be nonempty subsets of an inner product space. Prove the following:

(a) $A^{\perp}$ is a closed subspace.

(b) If $A \subset B$, then $B^{\perp} \subset A^{\perp}$.

(c) If $A \subset B$, then $(A^{\perp})^{\perp} \subset (B^{\perp})^{\perp}$.

(d) $(A^{\perp})^{\perp}$ is the smallest closed subspace containing A.

(e) $A^{\perp\perp\perp} = A^{\perp\perp}$. [Here $A^{\perp\perp\perp}$ means $((A^{\perp})^{\perp})^{\perp}$.]

(f) $A^{\perp} \cap A$ is either empty or $\{\mathbf{0}\}$.

(g) $\{\mathbf{0}\}^{\perp}$ is the entire space P and $P^{\perp} = \{\mathbf{0}\}$.

(h) If A is dense in P, then $A^{\perp} = \{\mathbf{0}\}$.

14:1.11 Let E be a linear subspace of a Hilbert space H.

(a) Show that $E^{\perp\perp} = \overline{E}$, the closure of E.

(b) Show that if E is closed then $E^{\perp\perp} = E$.

(c) Show that $E^{\perp} = \{\mathbf{0}\}$ if and only if E is dense in H.

(d) Show that if E is closed and $E^{\perp} = \{\mathbf{0}\}$ then $E = H$.

14:1.12 In quantum mechanics the inner product (f, g) would be written instead as $< g|f >$. Change the axioms so that they work for this notation.

14.2 Convex Sets

A set E in a vector space is said to be *convex* if the line segment joining any pair of points in E is itself in E. In algebraic terms, this merely requires that whenever f, $g \in E$ and $0 < t < 1$ the point $tf + (1 - t)g$ is in E. In particular, subspaces are always convex.

We now prove an interesting and useful, but elementary property of convex sets in a Hilbert space. This arises from a best-approximation problem: given a subset E of a normed linear space and an element x of the space, is there a nearest point in E to x? Recall we have used the notation $\operatorname{dist}(x, E)$ in a general metric space to represent the distance from a point x to a set E. Here in a normed space

$$\operatorname{dist}(x, E) = \inf\{\|x - y\| : y \in E\}.$$

A closest approximation to x from E would be an element $x_a \in E$ with

$$\|x_a - x\| = \operatorname{dist}(x, E).$$

Even in one or two dimensions, it is easy to see that closest approximations need not exist and, if they do, they need not be unique. It is natural to ask for E to be closed, but even then a nearest point in E may not be found. In a Hilbert space, we require only that E be closed and convex in order for a completely satisfactory solution of the problem to be found; in a Banach space, this is not generally true. What is most remarkable about the following theorem is that the proof requires very little more than the parallelogram law; this also shows why we might not expect such a statement in an arbitrary Banach space.

Theorem 14.11 *Let C be a closed, nonempty convex set in a Hilbert space H, and let $f \in H$. Then there exists a unique point g in the set C such that*

$$\operatorname{dist}(f, C) = \|f - g\|.$$

Proof. We can assume that $f = \mathbf{0}$ (by replacing C by $C - f$). Let

$$c = \operatorname{dist}(\mathbf{0}, C) = \inf\{\|g\| : g \in C\}.$$

For any g_1, $g_2 \in C$, the parallelogram law yields

$$\tfrac{1}{4}\|g_1 - g_2\|^2 = \tfrac{1}{2}\|g_1\|^2 + \tfrac{1}{2}\|g_2\|^2 - \|\tfrac{1}{2}(g_1 + g_2)\|^2,$$

and from this inequality and the fact that $\frac{1}{2}(g_1 + g_2)$ must also be in the convex set C, we obtain

$$\|g_1 - g_2\|^2 \le 2\|g_1\|^2 + 2\|g_2\|^2 - 4c^2. \tag{5}$$

Uniqueness in the statement of the theorem is immediate from (5), for if $\|g_1\| = \|g_2\| = c$, then $\|g_1 - g_2\| = 0$.

There is a sequence $g_n \in C$ with $\|g_n\| \to c$. From (5) we have

$$\|g_n - g_m\|^2 \le 2\|g_n\|^2 + 2\|g_m\|^2 - 4c^2, \tag{6}$$

which shows that $\{g_n\}$ is a Cauchy sequence. Since H is a Hilbert space, g_n converges to some element f_a and, since C is closed, $f_a \in C$. But

$$|\|g_n\| - \|f_a\|| \le \|g_n - f_a\| \to 0,$$

so $\|f_a\| = c$, as required for our closest approximation. ■

Corollary 14.12 *Every closed, convex set in a Hilbert space has a unique element of smallest norm.*

Let us think for a moment about what this theorem says if E is a closed subspace of a Hilbert space H and we want (and we frequently do) a closest approximation to an element f by some member from E. Since subspaces are convex, the theorem applies to show that there is a unique element $f_a \in E$ closest to f, that is with $\mathrm{dist}(f,E) = \|f - f_a\|$. Consider the geometry here: in a finite-dimensional space, we would expect the nearest element to be the orthogonal projection onto the subspace. Is there a similar statement in any Hilbert space, too? For any $g \in E$, $g \neq \mathbf{0}$ we can check the inner product $(f - f_a, g)$. Let $f_p = f - f_a$ and λ be a scalar. Since E is a subspace,

$$\|f_p - \lambda g\|^2 = \|f - (f_a + \lambda g)\|^2 \geq \mathrm{dist}(f,E)^2 = \|f - f_a\|^2.$$

We can write this as inner products and obtain

$$-\overline{\lambda}(f_p, g) - \lambda(g, f_p) + |\lambda|^2(g,g) \geq 0.$$

Inserting $\lambda = (f_p, g)/(g,g)$, we can conclude that $|(f_p,g)|^2 \leq 0$ and hence that $(f_p, g) = (f - f_a, g) = 0$.

This shows that we have obtained a decomposition $f = f_a + f_p$, where f_a is the nearest element in the subspace E, and f_p is orthogonal to every element in E. In short, exactly the same geometric picture can be used here in a general Hilbert space as we are accustomed to in finite-dimensional spaces. Nearest points and orthogonal projections are intimately related. We can express this as a theorem. The uniqueness part of the statement is left as an exercise.

Theorem 14.13 *Let H be a Hilbert space and E a closed subspace. Then every element f of H can be written uniquely in the form*

$$f = f_a + f_p,$$

where $f_a \in E$ and f_p is orthogonal to each element of E. Moreover, $\|f\|^2 = \|f_a\|^2 + \|f_p\|^2$.

Exercises

14:2.1 In the statement of Theorem 14.11, show that we can take H as an inner product space (not necessarily complete), provided that we insist that C is complete and not merely closed.

14:2.2 Give examples in $\mathbb{R}^2$ showing that in Theorem 14.11 there may be no nearest point, or, if there is, it is not unique in the case where C is not convex or is not closed.

14:2.3 Show that the representation $f = f_a + f_p$ in Theorem 14.13 is unique.

14:2.4 Let E be a bounded, convex set in a Hilbert space H, and suppose that $\|f_n\| \to \mathrm{dist}(\mathbf{0}, E)$. Show that $\{f_n\}$ is convergent, but not necessarily to an element of E.

14.3 Continuous Linear Functionals

Our main theorem showing that the inner product supplies all the continuous linear functionals on a Hilbert space is due to Maurice Fréchet and F. Riesz.

Theorem 14.14 (Fréchet–Riesz) *Suppose that Γ is a continuous linear functional on a Hilbert space H. Then there exists a unique element $g \in H$ so that $\Gamma(f) = (f, g)$ for all $f \in H$.*

Proof. If Γ is the zero functional, then the choice of g is easy, the zero function. Otherwise, let E be the set of all $f \in H$ for which $\Gamma(f) = 0$. This set E forms a subspace (since Γ is linear), and it is closed (since Γ is continuous). There must be an element g_1 that is not in the closed subspace E (since Γ is not identically zero) and hence, by Theorem 14.13, there is an element g_2 that is not in E and is orthogonal to every element of E. We can assume that $\|g_2\| = 1$. Let $\lambda = \overline{\Gamma(g_2)}$ and $g = \lambda g_2$.

It remains only to verify that this element g satisfies $\Gamma(f) = (f, g)$ for all $f \in H$. Let $f \in H$ be arbitrary, and write $h = \Gamma(f)g_2 - \Gamma(g_2)f$. We can check that

$$\Gamma(h) = \Gamma(f)\Gamma(g_2) - \Gamma(g_2)\Gamma(f) = 0,$$

which means that h must belong to E and is thus orthogonal to g_2. Consequently, since $\|g_2\| = 1$,

$$0 = (h, g_2) = (\Gamma(f)g_2 - \Gamma(g_2)f, g_2)$$

$$= \Gamma(f)(g_2, g_2) - \Gamma(g_2)(f, g_2) = \Gamma(f)(g_2, g_2) - \overline{\lambda}(f, g_2)$$

$$= \Gamma(f) - (f, \lambda g_2) = \Gamma(f) - (f, g),$$

and so $\Gamma(f) = (f, g)$, as required.

The uniqueness part of the proof is easy enough to be left as an entertainment (Exercise 14:3.1). ■

This theorem shows us that every continuous linear functional is of a very special, easily understood type. In the language of Banach spaces, we can say more. If the reader has mastered the language of the dual spaces (Section 12.3), then the following more precise formulation can be understood. The dual of a Hilbert space H (or any Banach space in fact) is written H^* and defined as the linear space of all continuous linear functionals on H, furnished with the usual operator norm. By the theory we that know so far, H^* will be a Banach space; if we find the correct inner product then, it is also a Hilbert space.

Theorem 14.15 *Let H be a Hilbert space. Then the conjugate space H^* is also a Hilbert space.*

Proof. We can associate with every element $\Gamma \in H^*$ a unique element $g \in H$ so that $\Gamma(f) = (f, g)$ for every $f \in H$. This defines a mapping $\Phi : H \to H^*$ by

$$\Phi(g)(f) = (f, g).$$

The mapping Φ is evidently, by Theorem 14.14, both one-one and onto. One might have hoped at this stage to see what to do, but this is hampered a bit by the fact that this mapping is not quite linear: a quick check reveals that

$$\Phi(c_1 f_1 + c_2 f_2) = \overline{c_1}\Phi(f_1) + \overline{c_2}\Phi(f_2). \tag{7}$$

With that in mind, the inner product on H is easily lifted over to H^*. Define for any pair Γ_1, $\Gamma_2 \in H^*$,

$$(\Gamma_1, \Gamma_2) = (\Phi^{-1}\Gamma_2, \Phi^{-1}\Gamma_1)$$

and the rest of the proof is just computational. We need to show that this is indeed an inner product on H^* and that this inner product is associated with the norm that we usually use on the dual space. The details are left to the reader. ■

Finally, we can see that H and H^* are identical as structures, justifying the loose statement that a Hilbert space is its own dual. The mapping that connects the two spaces preserves all elements of the structure except that it is not linear, but rather *conjugate-linear,* meaning that it satisfies the relation (7). We remember that an isometry is norm preserving; this is all that remains to be checked (Exercise 14:3.2) to justify the following theorem.

Theorem 14.16 *Let Φ be the mapping from H to H^* defined by*

$$\Phi(g)(f) = (f, g)$$

for all $f \in H$. Then Φ is a conjugate-linear isometry of H onto H^*.

Exercises

14:3.1◇ Let Γ be a continuous linear functional on a Hilbert space H. Show that the representation $\Gamma(f) = (f, g)$ $(f \in H)$ in Theorem 14.14 is unique.

14:3.2◇ Let Φ be the mapping from a Hilbert space H to its dual H^* defined by $\Phi(g)(f) = (f, g)$ for all $f \in H$. Show that $\|\Phi(g)\| = \|g\|$.

14:3.3◇ (Compare with Exercise 14:1.9.) Let $\{f_n\}$ be an orthogonal sequence in a Hilbert space H. Show that the series $\sum_{i=1}^{\infty} f_i$ converges in H if and only if the series of numbers $\sum_{i=1}^{\infty}(f_i, g)$ converges for every $g \in H$. [Hint: Define

$$\Gamma_n(g) = \sum_{i=1}^{n}(f_i, g)$$

and apply the uniform boundedness principle (Section 12.11) to the sequence $\{\Gamma_n\}$.]

14:3.4 In the language of Exercise 12:7.6, show that every Hilbert space is reflexive.

14.4 Orthogonal Series

In finite-dimensional vector spaces, one quickly learns the utility of the notion of a basis for the space. In the special spaces $\mathbb{R}^n$ or $\mathbb{C}^n$, all computations and geometric notions become even simpler if that basis is an orthonormal system; that is, the vectors are mutually orthogonal and have unit length. Most of these ideas can be lifted to Hilbert space. Here, however, the concepts cannot be purely algebraic because infinite sums must frequently be used. Even so, much of our work is merely algebraic and very familiar.

We need to recall what is meant by a linearly independent set in a linear space. A set is *linearly independent* if no finite linear combination of the elements (other than a zero combination) can produce the zero element. An *orthogonal system* in an inner product space consists of pairwise orthogonal elements. An orthogonal system containing no zero elements is linearly independent. An *orthonormal system* consists of pairwise orthogonal elements each of unit length. An orthonormal system is always linearly independent. The well known Gram–Schmidt process of elementary algebra allows one to take any linearly independent sequence $\{f_1, f_2, f_3, \dots\}$ in an inner product space and produce from it an orthonormal system $\{g_1, g_2, g_3, \dots\}$ so that the linear span of the set $\{f_1, f_2, f_3, \dots, f_n\}$ is precisely the linear span of the set $\{g_1, g_2, g_3, \dots, g_n\}$ for each n.

By Zorn's lemma, there is always a maximal orthonormal system in any Hilbert space H. This is usually said to be an *orthonormal basis* for the space. We will check that this maximal system is countable and so forms a sequence if and only if H is separable. Since nearly all important examples of Hilbert spaces arising in applications are separable, there will always be a maximal orthonormal sequence available in most studies. Indeed, it turns out, as the main theorem in this section shows, that all such spaces are identical with ℓ_2 even if the definition obscures this.

Theorem 14.17 *A maximal orthonormal system in a Hilbert space H is countable if and only if H is separable.*

Proof. Certainly, if S is a maximal orthonormal system in H and S is not countable, then H cannot be separable. For if f, g are distinct elements of

S, then

$$\|f - g\|^2 = \|f\|^2 + \|-g\|^2 = 2$$

so that $\|f - g\| = \sqrt{2}$. There is no hope then of a countable set approximating each member of S.

On the other hand, if S is countable, then the linear span of S is dense in H and so, too, is the set of all finite rational combinations. This latter is countable and so is a countable dense set as needed to show that H is separable. To check that the linear span of S is dense in H, we use the maximality of S. If W is the span of H and it is not dense, then $\overline{W}$ is a closed, proper linear subspace of H. Take an element f of H with $\|f\| = 1$ that is orthogonal to all of W (using Theorem 14.13). Then $S \cup \{f\}$ is now a larger orthonormal set than S, contradicting the maximality of S. ■

Example 14.18 A natural orthonormal sequence that is maximal in ℓ_2 is given by the sequence $\{e_1, e_2, \dots\}$, where

$$e_j = (0, 0, 0, \dots, 0, 1, 0, \dots)$$

and the solitary 1 occurs in the jth position in the sequence.

It is an easy enough exercise to check that this sequence is both orthonormal and maximal. This sequence is very useful in studying properties of the space ℓ_2 .

Example 14.19 (Trigonometric functions) The sequence of functions

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{int} \right\} \qquad (n = 0, \pm 1, \pm 2, \dots)$$

is a maximal, orthonormal sequence in $\mathbf{L}_2[0, 2\pi]$. It is an easy exercise in integration to check that this sequence is orthonormal, but it is by no means obvious that it is maximal (see Section 15.11). It is fundamental to the study of Fourier series that this be so. Indeed, this example is the inspiration for many of the ideas that now follow, and we shall label some of them with Fourier's name. The full development of these ideas comes only in Chapter 15.

For the real Hilbert space $\mathbf{L}_2[0, 2\pi]$ one would use the real and imaginary parts of the sequence $\left\{1/\sqrt{2\pi}\, e^{int}\right\}$ and discover that the functions

$$\frac{1}{\sqrt{2\pi}} \sin nt, \quad \frac{1}{\sqrt{2\pi}} \cos nt \quad (n = 0, 1, 2, \dots)$$

comprise a maximal orthonormal sequence.

Example 14.20 (Laguerre functions) The functions

$$\phi_n(x) = \frac{1}{n!} e^{-x/2} L_n(x),$$

where

$$L_n(x) = e^x \frac{d^n}{dx^n} \left(x^n e^{-x} \right),$$

are called the Laguerre polynomials. The sequence of functions ϕ_0, ϕ_1, ϕ_2, ... forms an orthonormal basis for $L_2[0,\infty)$. This basis plays a role in many studies in applied mathematics.

Example 14.21 (Legendre functions) The functions

$$\phi_n(x) = \sqrt{\frac{2n+1}{2}} P_n(x),$$

where

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n,$$

are called the Legendre polynomials. The sequence of functions ϕ_0, ϕ_1, ϕ_2, ... forms an orthonormal basis for $L_2[-1,1]$.

If f is an element of a Hilbert space that is in the span of a collection $\{f_1, f_2, f_3, \dots, f_n\}$ forming an orthonormal system, then the coefficients are particularly easy to deduce (as students of elementary linear algebra will quickly recall). Let

$$f = \sum_{j=1}^{n} \lambda_j f_j,$$

and then take the inner product of both sides with each f_i. The linearity of the inner product allows us to handle the sum immediately to obtain $\lambda_j = (f, f_j)$ for each j. These numbers are called the *Fourier coefficients* of f with respect to the orthonormal system. The term is taken from the study of trigonometric series, where the theory is formally identical. The same argument applies to infinite sums as well by taking limits. Thus, if $\{f_1, f_2, f_3, \dots\}$ forms an orthonormal system and

$$f = \sum_{j=1}^{\infty} \lambda_j f_j,$$

then, in fact,

$$f = \sum_{j=1}^{\infty} (f, f_j) f_j,$$

which is often referred to as a *Fourier series* or perhaps a generalized Fourier series.

This addresses the situation in which an element f is to be expressed exactly as a (possibly infinite) linear combination of elements of some orthonormal system. Now we consider how best to approximate f by such a finite combination. This leads to better insight into the nature of these Fourier series.

14.22 (Best approximation) *Suppose that*

$$\{f_1, f_2, f_3, \dots, f_n\}$$

is an orthonormal system in a Hilbert space H, and let $f \in H$. Then the minimum value of the expression

$$\left\| f - \sum_{j=1}^{n} \lambda_j f_j \right\| \tag{8}$$

is obtained by setting $\lambda_i = (f, f_i)$. Moreover, that minimum value can be obtained from

$$\left\| f - \sum_{j=1}^{n} (f, f_j) f_j \right\|^2 = \|f\|^2 - \sum_{j=1}^{n} |(f, f_j)|^2 . \tag{9}$$

Proof. Take any linear combination of the $\{f_j\}$, and compute

$$\begin{aligned}\left\| f - \sum_{j=1}^{n} \alpha_j f_j \right\|^2 &= (f - \sum_{j=1}^{n} \alpha_j f_j, f - \sum_{j=1}^{n} \alpha_j f_j) \\ &= \|f\|^2 - (\sum_{j=1}^{n} \alpha_j f_j, f) - (f, \sum_{j=1}^{n} \alpha_j f_j) + \sum_{j=1}^{n} |\alpha_j|^2 \\ &= \|f\|^2 - \sum_{j=1}^{n} \alpha_j |(f_j, f)|^2 + \sum_{j=1}^{n} \alpha_j |(f_j, f) - \alpha_j|^2 .\end{aligned}$$

From this computation, the minimum of the expression (8) is easily deduced: it occurs precisely when each $\alpha_j = (f_j, f)$ and the value of that minimum is as stated in (9). ■

As a corollary, we obtain immediately an inequality for the sum of the squares of the Fourier coefficients that was obtained originally for the trigonometric system by Freidrich Bessel (1784–1846).

14.23 (Bessel's inequality) *Let*

$$\{f_1, f_2, f_3, \ldots, f_n\}$$

be an orthonormal system in a Hilbert space H, and let $f \in H$. Then

$$\sum_{j=1}^{n} |(f, f_j)|^2 \le \|f\|^2$$

Proof. This follows immediately from the identity (8) since the left-hand side is nonnegative. ■

From this theorem we can also derive Parseval's identity, which can be viewed as an infinite form of the Pythagorean theorem, as well as a condition under which equality holds in Bessel's inequality.

14.24 (Parseval's identity) *Let $\{f_1, f_2, f_3, \dots,\}$ be an orthonormal system in a Hilbert space H, and let $f \in H$. Then*

$$\sum_{j=1}^{\infty} |(f, f_j)|^2 = \|f\|^2$$

if and only if

$$\lim_{n\to\infty} \left\| f - \sum_{j=1}^{\infty} (f, f_j) f_j \right\| = 0;$$

that is, if the series $\sum_{j=1}^{\infty}(f, f_j)f_j$ converges to f in the Hilbert space.

Proof. This is an immediate consequence of (8). ■

This convergence condition gives away our program: we wish to know precisely when we can write $f = \sum_{j=1}^{\infty}(f, f_j)f_j$ for an orthonormal system and any element f of the Hilbert space. Parseval's identity is part of a larger answer. We make one major simplifying assumption: our Hilbert space is assumed to be separable. The reason for this is that if we hope for such a representation then evidently the space must have a countable dense set: the set of finite rational combinations of elements of the system $\{f_1, f_2, f_3, \dots,\}$. This assumption can be avoided by working with orthonormal systems $\{f_i\}_{i\in I}$ over some larger index set I and then undertaking to interpret infinite sums of the form $\sum_{j\in I}(f, f_j)f_j$. This extension does not present any really fundamental problems, but obscures some of the presentation; thus we prefer the simpler setting.

Theorem 14.25 *Let $\{f_1, f_2, f_3, \dots\}$ be an orthonormal system in a separable Hilbert space H. Then the following assertions are equivalent:*

(Maximality) *The orthonormal system $\{f_1, f_2, f_3, \dots\}$ is maximal.*

(Denseness) *The set of finite linear combinations from the collection $\{f_1, f_2, f_3, \dots\}$ is dense in H.*

(Parseval's identity) *For every $f \in H$,*

$$\sum_{j=1}^{\infty} |(f, f_j)|^2 = \|f\|^2.$$

(Convergence of the Fourier series) *The series*

$$\sum_{j=1}^{n} (f, f_j) f_j$$

converges to f in the Hilbert space; that is,

$$\lim_{n\to\infty} \left\| f - \sum_{j=1}^{n} (f, f_j) f_j \right\| = 0.$$

(Parseval's identity: polar version) *For every f, $g \in H$,*

$$\sum_{i,j=1}^{\infty} (f, f_j)\overline{(g, f_i)} = (f, g).$$

Proof. Let us note first that the two versions of Parseval's identity are equivalent. One direction is trivial: just substitute $f = g$ in the polar version and we obtain the ordinary version. In the other direction, we compute the norms of $f + g$, $f - g$, $f + ig$, and $f - ig$ using Parseval's identity and then use the relation

$$4(f, g) = \|f + g\|^2 - \|f - g\|^2 + i\|f + ig\|^2 - i\|f - ig\|^2$$

from Exercise 14:1.7 to complete the proof.

We already know from assertion 14.24 that Parseval's identity is equivalent to the convergence of the Fourier series.

The fact that the maximality of a system is equivalent to the denseness statement is addressed now. Let W be the linear span of the maximal orthonormal sequence $\{f_1, f_2, f_3, \dots\}$. If this is not dense, then $\overline{W}$ is a closed, proper linear subspace of H. Take an element f of H with $\|f\| = 1$ that is orthogonal to all of W (using Theorem 14.13); then f can be added to the sequence $\{f_1, f_2, f_3, \dots\}$ to form a larger orthonormal sequence, contradicting the assumed maximality. Conversely, if $\{f_1, f_2, f_3, \dots\}$ is not maximal, then there is a nonzero element f orthogonal to each of these, and f cannot be in the closure of the linear span of the $\{f_1, f_2, f_3, \dots\}$. This is because the set $\{g \in H : (f, g) = 0\}$ is closed and contains all members of the sequence.

Finally, the convergence of the Fourier series is equivalent to the denseness. If $f \in H$ and $\varepsilon > 0$ and

$$\|f - \sum_{i=1}^{N} \alpha_i f_i\| < \varepsilon,$$

then also

$$\|f - \sum_{i=1}^{N} (f, f_i) f_i\| < \varepsilon$$

by our best approximation result (assertion 14.22). Thus, if the denseness condition, holds the Fourier series converges. Conversely, if the convergence of the Fourier series holds, then every element in the space can indeed be approximated by a finite linear combination from the sequence $\{f_1, f_2, f_3, \dots\}$, that is, by the sum $\sum_{i=1}^{m} (f, f_i) f_i$ for m sufficiently large.
■

Exercises

14:4.1 Let $F = \{f_1, f_2, f_3, \dots, f_n\}$ be an orthonormal system in a Hilbert space H. Show that the set of all finite linear combinations from F is a closed subspace.

14:4.2 Let E be a closed subspace of a Hilbert space H, and let $f \in H$. Show that the set of all linear combinations $\lambda f + g$ for $g \in E$ and $\lambda \in \mathbb{C}$ is a closed subspace.

14:4.3 Show that a subset S of a Hilbert space is dense if and only if the only element of H orthogonal to every element of S is the zero element.

14:4.4 Define inductively the Gram–Schmidt orthonormalization process: given any linearly independent sequence $\{f_1, f_2, f_3, \dots\}$ in an inner product space, produce from it an orthonormal system $\{g_1, g_2, g_3, \dots\}$ so that the linear span of the set $\{f_1, f_2, f_3, \dots, f_n\}$ is precisely the linear span of the set $\{g_1, g_2, \dots, g_n\}$ for each n. [Hint: Start with $g_1 = \|f_1\|^{-1} f_1$ and use Theorem 14.13 to obtain a nonzero element in the span of $\{f_1, f_2\}$ that is orthogonal to g_1.]

14:4.5 Develop the theory of this section for nonseparable Hilbert spaces. [Hint: You will need to know how to interpret a sum $\sum_{i \in I} x_i$ for arbitrary index sets I. See Example 14.9.]

14:4.6 Show that if H is an infinite-dimensional separable Hilbert space then H is isometrically isomorphic to ℓ_2.

14.5 Weak Sequential Convergence

For many applications in Hilbert space, the norm convergence asks too much. For example, bounded sequences $\{f_n\}$ need not have convergent subsequences $\{f_{n_k}\}$ if by convergence we mean that

$$\|f_{n_k} - f\| \to 0$$

as $k \to \infty$. But if we require much less, then such a statement is true and, more importantly, extremely useful.

Definition 14.26 A sequence $\{f_n\}$ in a Hilbert space H is said to be *weakly convergent* to f if

$$(f_n, g) \to (f, g)$$

for every element $g \in H$.

We have seen weak convergence before in a special setting in Section 13.10. It is easy to see that a norm convergent sequence is also weakly convergent, but the converse does not hold. Note that the definition could be rephrased (because of Theorem 14.14) as the requirement that $\Gamma(f_n) \to \Gamma(f)$ for every continuous linear functional on H. This is the usual definition of weak convergence supplied in normed linear spaces where there is no inner product.

We shall obtain some immediate and easy results for the notion of weak convergence. Some of this holds in general normed linear spaces, too, but with different proofs. The first result can be called a weak sequential compactness result: from a bounded sequence a convergent subsequence is found, much as in the original Bolzano–Weierstrass theorem.

Theorem 14.27 *A bounded sequence $\{f_n\}$ in a Hilbert space H has a weakly convergent subsequence.*

Proof. Let us first assume that the space is separable. (We use an argument similar to that for Theorem 13.33; if the reader has studied that proof, it would be best to read no further, but try constructing one independently.)

Fix an element $g_1 \in H$. We show how to determine a subsequence so that

$$\lim_{k\to\infty} (f_{n_k}, g_1) \tag{10}$$

exists. We know that

$$|(f_n, g)| \leq M\|g\| < \infty,$$

where $M = \sup_n \|f_n\|$, and so this sequence of complex numbers is bounded. Thus a subsequence for which the limit (10) exists can be found merely from the Bolzano–Weierstrass theorem.

Fix element g_1, g_2, $\dots$, $g_m \in H$. We can determine a subsequence so that

$$\lim_{k\to\infty} (f_{n_k}, g_i) \tag{11}$$

exists for each $i = 1, 2, 3, \dots, m$. We just apply the same argument for each i and pass to subsequences of subsequences.

Finally, and much more generally, let g_1, g_2, $\dots$ be an infinite sequence of elements of H that forms a dense subset. Once again we can determine a subsequence so that

$$\lim_{k\to\infty} (f_{n_k}, g_i) \tag{12}$$

exists for each $i = 1, 2, 3, \dots$. We cannot quite use a "subsequence of a subsequence" argument indefinitely, but we can use a Cantor diagonalization argument to get a single subsequence $\{f_{n_k}\}$ that works for each g_i.

Define a functional Γ on H by first writing

$$\Gamma(g_i) = \lim_{k\to\infty} (f_{n_k}, g_i) \tag{13}$$

for each g_i in our dense set and then extending to all of H by continuity. By the Cauchy–Schwarz inequality applied to (13), we have

$$|\Gamma(g_i) - \Gamma(g_j)| \leq M\|g_i - g_j\|,$$

and so Γ is uniformly continuous on the dense subset, allowing therefore a unique extension to a continuous functional. We claim that Γ is linear. It is certainly linear on the dense subset formed from the $\{g_i\}$ because of its definition using the inner product in (13). This linearity is preserved in the limit, too, when extended to all of H. Note, as well, that $\|\Gamma\| \leq M$.

We apply Theorem 14.14 to obtain an element $f \in H$ so that

$$\Gamma(g) = (g, f)$$

for all $g \in H$. It is easy to see now that f is precisely the element of H that we want, that $f_{n_k} \to f$ weakly, and that $\|f\| \leq M$.

This completes the proof in a separable Hilbert space. In a general non-separable Hilbert space, a maximal orthonormal system $\{g_\alpha\}_{\alpha \in A}$ forming a basis for the space can be found. For each f_n there are only countably many $\{g_\alpha\}$ for which $(f, g_\alpha) \neq 0$ (this follows, for example, from Bessel's inequality). If we collect these indices together, we obtain a countable set $A_1 \subset A$ and a closed subspace $H_1 \subset H$ for which $\{g_\alpha\}_{\alpha \in A_1}$ forms a basis. H_1 is a separable Hilbert space, and if we apply the first part of our argument to obtain a subsequence $\{f_{n_k}\}$ converging weakly in H_1, that same subsequence will evidently converge weakly in H as required. ■

Norm convergence in a Hilbert space of a sequence f_n to an element f requires that $\|f_n - f\| \to 0$. This implies, in particular, that $\|f_n\| \to \|f\|$ and that f is in the closure of the set of elements of the sequence $\{f_n\}$. For weak convergence, we get somewhat less.

Theorem 14.28 *Suppose that the sequence $\{f_n\}$ converges weakly to an element f in a Hilbert space H. Then the following assertions are true:*

1. $\sup \|f_n\| < \infty$.
2. *f is in the closure of the subspace spanned by the sequence $\{f_n\}$.*
3. $\|f\| \leq \liminf_{n\to\infty} \|f_n\|$.

Proof. Define a sequence of bounded linear functionals on H by writing

$$\Gamma_n(g) = (g, f_n - f) \quad (g \in H).$$

Note that $\Gamma_n(g) = (g, f_n - f) \to 0$ as $n \to \infty$ so that, in particular,

$$\sup_n |\Gamma_n(g)| < \infty.$$

Applying the uniform boundedness principle (Section 12.11), we obtain

$$\sup_{g \in H} \sup_n |\Gamma_n(g)| < \infty$$

and hence that $\sup_n \|f_n - f\| < \infty$. It follows that $\sup_n \|f_n\| < \infty$, as required for (1).

To prove (2), suppose that f is not in the closure of the subspace spanned by the sequence $\{f_n\}$. Then, by Theorem 14.13, there must be an element $g \in H$ orthogonal to each member of $\{f_n\}$, but not orthogonal to f. But $(f_n, g) \to (f, g)$, and yet $(f_n, g) = 0$ for all n and $(f, g) \neq 0$. This is a contradiction, and so (2) follows.

For (3), we can find a subsequence $\{f_{n_k}\}$ so that

$$\lim_{k\to\infty} \|f_{n_k}\| = \liminf_{n\to\infty} \|f_n\| = M.$$

We know that

$$|(f, g)| = \lim_{k\to\infty} |(f_{n_k}, g)| \leq \lim_{k\to\infty} \|f_{n_k}\|\,\|g\| \leq M\|g\|.$$

But the inequality $|(f, g)| \leq M\|g\|$ holding for all $g \in H$ implies that $\|f\| \leq M$ too, as required. ■

When does weak convergence suffice to ensure that, in fact, the convergence is taking place in the norm sense? One extra condition on the norms is sufficient.

Theorem 14.29 *Suppose that the sequence $\{f_n\}$ converges weakly to an element f in a Hilbert space H. If, in addition,*

$$\|f\| = \lim_{n\to\infty} \|f_n\|,$$

then $\{f_n\}$ converges to f in H.

Proof. We assume that $(f_n, g) \to (f, g)$, and hence it is also true that $(g, f_n) \to (g, f)$. Then from the identity

$$\|f - f_n\|^2 = (f - f_n, f - f_n) = \|f\|^2 + \|f_n\|^2 - (f, f_n) - (f_n, f)$$

along with the extra assumption that $\|f\| = \lim_{n\to\infty} \|f_n\|$, we get

$$\lim_{n\to\infty} \|f - f_n\|^2 = \|f\|^2 + \|f\|^2 - (f, f) - (f, f) = 0,$$

as required. ■

Finally, let us conclude with one more result of this type. A series can certainly converge weakly without converging in norm; but if it is an orthogonal series, then the two notions are equivalent. The proof has appeared in Exercises 14:1.9 and 14:3.3.

Theorem 14.30 *Suppose that $\{f_n\}$ is a sequence of pairwise orthogonal elements of a Hilbert space H. Then the following are equivalent:*

1. *$\sum_{i=1}^{\infty} f_i$ is convergent in H.*
2. *$\sum_{i=1}^{\infty} \|f_i\|^2 < \infty$.*
3. *$\sum_{i=1}^{\infty} (f_i, g)$ converges for every element $g \in H$.*

Exercises

14:5.1 In the proof of Theorem 14.27, replace the "subsequence of a subsequence" argument by a formal Cantor diagonalization argument.

14:5.2 Find a sequence converging weakly to zero in ℓ_2 but not convergent.

14:5.3 Suppose that the sequence $\{f_n\}$ converges weakly to an element f in a Hilbert space H. Show that there is a subsequence $\{f_{n_k}\}$ such that the means

$$\sigma_k = \frac{f_{n_1} + f_{n_2} + \cdots f_{n_k}}{k}$$

converge to f in H.

14:5.4 (The unit sphere is weakly dense in the unit ball.) For every f in a Hilbert space H for which $\|f\| \leq 1$, there exists a sequence $\{f_n\}$ for which $\|f_n\| = 1$ that converges weakly to f in H.

14.6 Compact Operators

We begin by recalling some information about linear operators on normed linear spaces for readers who have skipped Chapter 12 or who may be in need of a review. In the next few sections we shall find a way of seeing precisely how certain operators can be realized.

A mapping from a Hilbert space into itself is called a *linear operator* if it preserves the linear structure. Thus $T : H \to H$ must satisfy

$$T(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 T(f_1) + \lambda_2 T(f_2).$$

Naturally, we wish it to preserve the rest of the structure. The most obvious condition to impose is continuity: if $f_n \to f$, then $T(f_n) \to T(f)$. This condition is precisely equivalent to a boundedness property of the mapping: T is continuous if and only if T maps bounded sets in H into bounded sets. This allows us to define a norm on the operator that it inherits from the space:

$$\|T\| = \sup\left\{\frac{\|T(f)\|}{\|f\|} : f \neq \mathbf{0}\right\}.$$

We should remember how to prove that boundedness is equivalent to continuity. If $\|T\| < \infty$, then

$$\|T(f_n - f)\| \leq \|T\|\|f_n - f\|$$

so that clearly if $f_n \to f$ in H it follows that $T(f_n) \to T(f)$ in H. Hence boundedness implies continuity. If T is continuous and yet not bounded, we can obtain a contradiction. There must be a sequence $\{f_n\}$ with $\|T(f_n\| > n\|f_n\|$. Set $g_n = (n^{-1}\|f_n\|)f_n$; then $g_n \to \mathbf{0}$ and yet $\|T(g_n\| > 1$, which is not possible if T is continuous.

The study of general continuous linear operators on a Hilbert space is of fundamental importance, but a bit too ambitious for us to tackle. Instead, we will ask more of the operators and seek to determine the structure of what are called compact operators. An operator is called *compact* if not only does

$$T(f_n) \to T(f)$$

for every sequence for which $f_n \to f$ in norm, but in fact for every sequence for which $f_n \to f$ weakly. (Note the distinction between the two modes of convergence: we require that

$$\|T(f_n) - T(f)\| \to 0$$

whenever $f_n \to f$ weakly.) There are more weakly convergent sequences than convergent sequences, and so this demands very much more of the operator. Every compact operator is continuous, but not all continuous operators are compact.

A continuous operator maps convergent sequences to convergent sequences, while a compact operator maps weakly convergent sequences to convergent sequences. It is not immediately clear that a continuous operator maps weakly convergent sequences to weakly convergent sequences. We prove this now.

The proof will also give us an excuse to introduce an important idea in the study of operators on a Hilbert space, the notion of the *adjoint operator.* For this proof, it enters only as a notational convenience. A continuous functional Γ_g can be defined on H by

$$\Gamma_g(f) = (T(f), g) \quad (f \in H)$$

for any fixed $g \in H$. It is easy to check that Γ_g is continuous and linear. Thus $\Gamma_g(f) = (f, g_1)$ for some $g_1 \in H$. So for every $g \in H$ there is an element g_1 for which $(T(f), g) = (f, g_1)$. This mapping is written as $g_1 = T^*(g)$, so T^* is a mapping from H to itself for which

$$(T(f), g) = (f, T^*(g)) \tag{14}$$

holds for all f, $g \in H$. We will explore this in more detail later, but for now we might mention that T^* is a linear operator, it is continuous and its operator norm is the same as the operator norm for T itself; that is, $\|T\| = \|T^*\|$.

Theorem 14.31 *A continuous linear operator on a Hilbert space maps weakly convergent sequences to weakly convergent sequences.*

Proof. Suppose that $f_n \to f$ weakly. We must show that

$$T(f_n) \to T(f)$$

weakly if T is a continuous linear operator on a Hilbert space H. The adjoint notation just introduced allows us to prove our theorem. We observe, using two applications of (14) and the weak convergence of $f_n \to f$, that, for all $g \in H$,

$$\lim_{n\to\infty}(T(f_n), g) = \lim_{n\to\infty}(f_n, T^*(g)) = (f, T^*(g)) = (T(f), g),$$

and so $T(f_n) \to T(f)$ weakly as required. ■

The next result shows that compact operators map bounded sets into relatively compact ones (i.e., sets whose closures are compact). If $\{f_n\}$ is a bounded sequence and T is a continuous linear operator, then $\{T(f_n)\}$ is also a bounded sequence and so has a *weakly* convergent subsequence $\{T(f_{n_k})\}$, but it may not have a convergent subsequence in the norm sense. To demand that of T is precisely to ask that T be compact, as this theorem now shows.

Theorem 14.32 *Let T be a continuous linear operator on a Hilbert space H. Then T is a compact operator if and only if for every bounded sequence $\{f_n\}$ there is a subsequence $\{T(f_{n_k})\}$ that is convergent in the norm sense.*

Proof. Suppose that $\{f_n\}$ is bounded. Then, by Theorem 14.27, there is a subsequence and an element $f \in H$ so that $f_{n_k} \to f$ weakly. By definition, since T is a compact operator $T(f_{n_k}) \to T(f)$, with convergence in the norm sense.

Conversely, suppose that T has the stated property and yet, contrary to the theorem, there is a bounded sequence $\{f_n\}$ for which there is no subsequence $\{T(f_{n_k})\}$ that is convergent in the norm sense. Using Theorem 14.27 and passing to a subsequence if necessary, we may assume that in fact there is a an element $f \in H$ so that $f_n \to f$ weakly. By our assumptions, there must be $\varepsilon > 0$ and $n_1 < n_2 < n_3 < \dots$ so that

$$\|T(f_{n_k}) - T(f)\| \geq \varepsilon, \tag{15}$$

even though we do know (Theorem 14.31) that $T(f_{n_k}) \to T(f)$ weakly.

By our assumptions on T, we know that there is a further subsequence $T(f_{n_{k_m}})$ that is norm convergent to an element $g \in H$. But $T(f_{n_{k_m}})$ converges weakly to $T(f)$ so that this is possible only if $T(f) = g$. Thus we have a subsequence $T(f_{n_{k_m}})$ that is norm convergent to $T(f)$ in contradiction to (15). This contradiction proves the assertion. ■

Exercises

14:6.1 Show that every linear operator on a finite-dimensional Hilbert space is compact.

14:6.2 If f, g are nonzero elements of a Hilbert space, show that there is a compact operator T for which $T(f) = g$.

14:6.3 Show that the identity map on ℓ_2 is a continuous linear operator, but is not a compact operator.

14:6.4 Show that the identity map on any Hilbert space H is compact if and only if H is finite dimensional.

14:6.5 Let T be a linear operator on a Hilbert space H such that

$$(T(f), g) = (f, T(g))$$

for all f, $g \in H$. Then T is continuous. [Hint: If $f_n \to \mathbf{0}$, then $T(f_n) \to \mathbf{0}$ weakly. Use the closed graph theorem: if $f_n \to f$, $T(f_n) \to g$, and $h \in H$, then $(T(f_n - f), h) \to 0$, so $(T(f), h) = \lim(T(f_n), h) = (g, h)$.]

14:6.6 Show that the adjoint notion introduced in Theorem 14.31 has the following properties. If T_1, T_2 are continuous linear operators on a Hilbert space, then

(a) $\|T_1^* T_1\| = \|T_1\|^2$,

(b) $(\alpha_1 T_1 + \alpha_2 T_2)^* = \overline{\alpha_1} T_1^* + \overline{\alpha_2} T_2^*$,

(c) $(T_1 T_2)^* = T_2^* T_1^*$, and

(d) $(T_1^*)^* = T_1$.

14:6.7$\diamondsuit$ If T_1 and T_2 are self-adjoint operators on a Hilbert space, then the product $T_1 T_2$ is also self-adjoint if and only if the operators commute.

14:6.8 Show that if T is a compact operator on a Hilbert space then the adjoint T^* is also compact.

14:6.9 Show that the limit (in the sense of the operator norm) of a convergent sequence of compact operators is compact. [Hint: If $T_n \to T$ and $\|f_n\| \leq 1$, construct a convergent subsequence $\{T(f_{n_k})\}$ by finding first a convergent subsequence $\{T_1(f_{n_k})\}$ and continue using subsequences of subsequences followed by a diagonal argument.]

14:6.10$\diamondsuit$ Show that if T_1, T_2 are compact linear operators on a Hilbert space then so too is their product and any linear combination.

14:6.11 Show that if T_1, T_2 are continuous linear operators on a Hilbert space and one of them is compact then $T_1 T_2$ and $T_2 T_1$ are compact.

14:6.12 Let f_1, f_2, f_3, ... be an orthonormal basis for a Hilbert space H, let α_1, α_2, α_3, ... be a sequence of scalars converging to zero. Show that the map

$$T(f) = \sum_{i=1}^{\infty} \alpha_i (f, f_i) f_i$$

defines a compact linear operator on H.

14.7 Projections

The simplest operators on a Hilbert space are the projections. It is easy to see what a projection is doing, and projections manipulate quite naturally. Often in analysis one finds that complicated objects can be expressed in terms of simpler ones; this suggests that more complicated linear operators might be analyzed in some way that expresses them as combinations of the simpler projections. This is one of our goals. To start on that goal let us first examine projections rather closely.

Let E be a closed subspace of a Hilbert space H. Theorem 14.13 shows that every element f in the space has a decomposition

$$f = f_a + f_p,$$

where $f_a \in E$ and f_p is orthogonal to each element of E. We call f_a the *orthogonal projection* of f onto E, and the mapping $f \to f_a$ is denoted P_E and is called the *projection operator* onto the subspace E. We use the notation

$$E^\perp = \{f \in H : (f, g) = 0 \text{ for all } g \in E\}$$

to denote the orthogonal complement of E. This, too, is a closed subspace of H and the decomposition $f = f_a + f_p$ shows that every vector in H can be written uniquely as the sum of two vectors, one from E and one from $E^\perp$. Our theorem summarizes all the elementary information we can extract from these notions.

Theorem 14.33 *Let E be a closed subspace of a Hilbert space H, and let $P = P_E$ denote the projection onto E. The operator P has the following properties:*

1. *P is a linear operator.*
2. *$P(f) = f$ for all $f \in E$, and $P(g) = 0$ for all $g \in E^\perp$.*
3. *P is self-adjoint in the sense that $(P(f), g) = (f, P(g))$ for all f, $g \in H$.*
4. *$P^2 = P$.*
5. *$(P(f), f) = \|P(f)\|^2 \leq \|f\|^2$.*
6. *$E =$ range of $P = \{f \in H : P(f) = f\}$.*
7. *$E^\perp =$ null space of $P = \{f \in H : P(f) = \mathbf{0}\}$.*

Proof. Each of these assertions follows almost immediately from definitions and obvious considerations. We prove only the assertion that P is self-adjoint in the sense that $(P(f), g) = (f, P(g))$ for all f, $g \in H$. Recall from the proof of Theorem 14.31 that this means that P is its own adjoint; that is, $P^* = P$.

Let

$$f = f_a + f_p \quad \text{and} \quad g = g_a + g_p,$$

where f_a, $g_a \in E$ and f_p, $g_p \in E^\perp$. Then $P(f) = f_a$, $P(g) = g_a$,

$$(P(f), g) = (f_a, g_a + g_p) = (f_a, g_a) + (f_a, g_p) = (f_a, g_a),$$

and

$$(f, P(g)) = (f_a + f_p, g_a) = (f_a, g_a) + (f_p, g_a) = (f_a, g_a),$$

from which the identity $(P(f), g) = (f, P(g))$ follows. ∎

Projections can also be characterized by these elementary properties. We know from the theorem that a projection P must be self-adjoint and that $P^2 = P$. Conversely, any such operator is a projection onto some closed subspace. Readers with more algebraic than geometric insight will appreciate that projections are simply the idempotents in the algebra of operators.

Theorem 14.34 *Let P be a self-adjoint linear operator on Hilbert space H for which $P^2 = P$. Then P is a projection.*

Proof. Let us show first that such an operator P is bounded, indeed that $\|P(f)\| \leq \|f\|$ for all $f \in H$ so that $\|P\| \leq 1$. This follows from the inequality

$$\|P(f)\|^2 = (P(f), P(f)) = (f, P^2(f)) = (f, P(f)) \leq \|f\| \, \|P(f)\|$$

in which we have used the Cauchy–Schwarz inequality and the hypotheses of the theorem.

Let $E = \{P(f) : f \in H\}$ be the range of the operator P. Since P is linear, E is a subspace. We claim that E is also a closed subspace. To see this, suppose that $\{g_n\}$ is a sequence of points in E with $g_n \to g$. Then there are $\{f_n\}$ with $P(f_n) = g_n$, and so also

$$P(g_n) = P^2(f_n) = P(f_n) = g_n.$$

Since P is continuous, we can take limits in this identity and obtain $P(g) = g$ so that $g \in E$, as required to show that E is closed.

Let $f \in E$ and $g \in E^\perp$. We show that

$$P(f) = f \text{ and } \ P(g) = \mathbf{0}.$$

It will follow that $P = P_E$, the projection onto the subspace E, and the proof is complete. Since $f \in E$, there is a $P(f_1) = f$, and so also $P(f) = P^2(f_1) = P(f_1) = f$, as required. Since $g \in E^\perp$, we know that $(g, g_1) = 0$ for all $g_1 \in E$, so $(g, P(g_2)) = 0$ for all $g_2 \in H$. Thus $(g, P(g_2)) = (P(g), g_2) = 0$ for all $g_2 \in H$. But this can only happen if $P(g) = \mathbf{0}$, again as required. ∎

Exercises

14:7.1 Prove each of the parts of Theorem 14.33. If this is too tedious, at least prove that P is linear and check the inequality

$$(P(f), f) = \|P(f)\|^2 \leq \|f\|^2.$$

14:7.2 Let E_1 and E_2 be closed subspaces of a Hilbert space. Show that $E_1 \perp E_2$ if and only if $P_{E_1}P_{E_2} = P_{E_2}P_{E_1} = \mathbf{0}$.

14:7.3 Let E_1 and E_2 be closed subspaces of a Hilbert space. Show that the sum $P_{E_1} + P_{E_2}$ is again a projection if and only if $E_1 \perp E_2$. (This is harder than Exercise 14:7.2.)

14:7.4 Let E_1 and E_2 be closed subspaces of a Hilbert space. Show that the product $P_{E_1}P_{E_2}$ is again a projection if and only if $P_{E_1}P_{E_2} = P_{E_2}P_{E_1}$. If $P_{E_1}P_{E_2}$ is a projection, what is its range?

14:7.5 Let E_1 and E_2 be closed subspaces of a Hilbert space H. Show that the following three assertions are equivalent:

(a) $E_1 \subset E_2$.

(b) $P_{E_2}P_{E_1} = P_{E_1}P_{E_2} = P_{E_1}$.

(c) $(P_{E_1}(f), f) \leq (P_{E_2}(f), f)$ for all $f \in H$.

14:7.6 When is a projection operator compact? [Hint: Use Theorem 12.16.]

14.8 Eigenvectors and Eigenvalues

One of the most successful and applicable of the results one learns in elementary linear algebra is that of characterizing the $n \times n$ matrices in terms of their eigenvalues. The reader should recall (fondly) that every symmetric real or complex Hermetian $n \times n$ matrix M has a full set of eigenvalues that allows a representation as a sum of multiples of projection matrices

$$M = \sum_j \lambda_j P_{E_{\lambda_j}}, \tag{16}$$

where $P_{E_{\lambda_j}}$ is the projection operator taking $\mathbb{C}^n$ onto the eigenspace E_{λ_j} corresponding to the eigenvalue λ_j. Put another way, there is an orthonormal basis for the space consisting solely of eigenvectors, and this basis permits a "diagonalization" of the matrix M.

This theory has been generalized to higher dimensions. One considers, naturally enough, an $n \times n$ matrix $M = (\alpha_{ij})$ to be a linear operator on the Euclidean space $\mathbb{C}^n$ and notes that M has the important and familiar property $\alpha_{ij} = \overline{\alpha_{ji}}$ precisely when the operator is self-adjoint. Eigenvalues and eigenvectors are defined for linear operators on a Hilbert space in much the same way as in matrix theory, and we find that eigenvalues and eigenvectors do exist for compact, self-adjoint operators. In this special case, a theory emerges that is very close to the finite-dimensional situation. For noncompact operators, a different theory is required, one that we do not develop here.

The set of eigenvalues of an operator forms part of what is known as the spectrum of the operator. The theory is then called spectral theory, and representations similar to or analogous to (16) are called spectral representations. We pursue these ideas only within the setting of eigenvalues, eigenvectors and eigenspaces, which terms we now define.

Definition 14.35 Let T be a linear operator on a Hilbert space H. If there exist $\lambda \in \mathbb{C}$ and a nonzero $f \in H$ for which

$$T(f) = \lambda f,$$

then λ is said to be an *eigenvalue* for T, and f to be a corresponding *eigenvector*. If λ is an eigenvalue for T then

$$E_\lambda = \{f \in H : T(f) = \lambda f\}$$

is called the *eigenspace* corresponding to λ.

It is easy to see that E_λ is a nonzero subspace of H whenever λ is an eigenvalue. If T is also continuous, then it is easy to see that the eigenspace E_λ must be closed. If, moreover, T is compact and $\lambda \neq 0$, then the eigenspace E_λ must be finite-dimensional (Exercise 14:8.1).

We are interested in operators that are both compact and self-adjoint in the sense of the next definition, since without these assumptions there may be no nonzero eigenvalues, and hope for a kind of spectral decomposition must follow some other plan. (Exercise 14:8.2 exhibits a compact operator and Exercise 14:8.3 exhibits a self-adjoint operator neither of which has any nonzero eigenvalues.)

Definition 14.36 A linear operator T on a Hilbert space H is said to be *self-adjoint* if

$$(T(f), g) = (f, T(g))$$

for all f, $g \in H$.

In elementary linear algebra, this idea corresponds to symmetric matrices (in the real case) or Hermetian matrices (in the complex case). For Fredholm operators with L_2 kernels, as in Example 13.17, this corresponds to the equality $K(x, y) = K(y, x)$ a.e. for the kernel function. We have seen the notion of the adjoint concept arise in Theorem 14.31, but here we do not need to use anything beyond the simple property expressed in the definition. The mere fact that a linear operator is self-adjoint is enough to ensure that it is continuous; we shall not insist on this result, however (which is proved in Exercise 14:6.5 using the closed graph theorem of Chapter 12), and so we add an unnecessary hypothesis to the theorem.

Theorem 14.37 *Let T be a continuous linear operator on a Hilbert space H, and suppose that T is self-adjoint. Then the following are true:*

1. *$(T(f), f)$ is real for all $f \in H$.*
2. *$\|T\| = \sup\{(T(f), f) : \|f\| = 1\}$.*
3. *All eigenvalues of T are real numbers contained in the interval $[-\|T\|, \|T\|]$.*
4. *Eigenspaces H_{λ_1},H_{λ_2} corresponding to distinct eigenvalues λ_1, λ_2 are orthogonal.*
5. *If P_λ denotes the projection onto the eigenspace H_λ corresponding to an eigenvalue λ, then*

$$\lambda P_\lambda = TP_\lambda = P_\lambda T.$$

Proof. To prove the first assertion, we merely use the self-adjoint assumption to obtain

$$(T(f), f) = (f, T(f)) = \overline{(T(f), f)},$$

so that $(T(f), f)$ must be real.

For the second assertion, let $M = \sup\{(T(f), f) : \|f\| = 1\}$. It is clear that $M \leq \|T\|$ since, if $\|f\| = 1$, then

$$|(T(f), f)| \leq \|T(f)\| \, \|f\| \leq \|T\| \|f\|^2 = \|T\|.$$

The other direction takes more computations. Note first that, for any f,

$$|(T(f), f)| \leq M\|f\|^2. \tag{17}$$

Let $\|f\| = \|g\| = 1$, and suppose that $(T(f), f)$ is real. We first compute

$$\begin{aligned}(T(f), g) = \tfrac{1}{4}[&(T(f+g), f+g) - (T(f-g), f-g) \\ + \quad &i(T(f+ig), f+ig) - i(T(f-ig), f-ig)].\end{aligned}$$

Noting that all terms here are real, we have

$$(T(f), g) = \tfrac{1}{4}\left[(T(f+g), f+g) - (T(f-g), f-g)\right].$$

From (17) and the parallelogram law, we obtain then

$$(T(f), g) \leq \tfrac{1}{4}M\left(\|f+g\|^2 + \|f-g\|^2\right) = \tfrac{1}{4}M\left(2\|f\|^2 + 2\|g\|^2\right) = M,$$

which, with $f = g$, is precisely what is needed. This proves the second assertion of the theorem.

For assertion (3), let λ be an eigenvalue for T and f a corresponding eigenvector. Then $g = f/\|f\|$ is also an eigenvector, and

$$\lambda = \lambda(g, g) = (\lambda g, g) = (T(g), g),$$

which we know, by assertion (1), is real.

For assertion (4), suppose that λ_1 and λ_2 are eigenvalues with corresponding eigenspaces H_{λ_1},H_{λ_2}, and suppose that $f_1 \in H_{\lambda_1}$ and $f_2 \in H_{\lambda_2}$. Then

$$\lambda_1(f,g) = (\lambda_1 f, g) = (T(f), g) = (f, T(g)) = (f, \lambda_2 g) = \lambda_2(f,g),$$

since λ_2 must be real. Since $\lambda_1 \neq \lambda_2$, it follows that $(f,g) = 0$, which is the required orthogonality condition.

For the final assertion, we note first that, because the eigenspaces are closed, the projection operator is well defined. Now let f, g be arbitrary members of H. Then

$$(\lambda P_\lambda(f), g) = (TP_\lambda(f), g) = (P_\lambda T(f), g) \tag{18}$$

because $\lambda P_\lambda(f) = T(P_\lambda(f))$ and because

$$\begin{aligned}(\lambda P_\lambda(f), g) &= (f, \lambda P_\lambda(g)) = (f, T(P_\lambda(y))) \\ &= (T(f), P_\lambda(y)) = (P_\lambda(T(f)), y).\end{aligned}$$

But if (18) holds for all f, g, then

$$\lambda P_\lambda = TP_\lambda = P_\lambda T,$$

as required. ■

Before carrying on, let us suppose we are in a situation where a continuous linear operator $T : H \to H$ permits an orthonormal basis for H consisting of a sequence $\{f_1, f_2, f_3, \dots\}$ of eigenvectors corresponding to eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \dots\}$ (not necessarily distinct). Then any $f \in H$ can be written $f = \sum_{i=1}^{\infty}(f, f_i)f_i$, and so

$$T(f) = T\left(\sum_{i=1}^{\infty}(f, f_i)f_i\right) = \sum_{i=1}^{\infty}(f, f_i)T(f_i) = \sum_{i=1}^{\infty}\lambda_i(f, f_i)f_i$$

merely using linearity, the eigenvalue relation, and continuity. This seems to suggest that

$$T = \sum_{i=1}^{\infty}\lambda_i P_{\lambda_i},$$

where P_{λ_i} is the projection onto the one-dimensional subspace of H spanned by f_i. There are still some problems with claiming this. First, how can we be assured of enough eigenvectors to form a basis for the space? Second, how can we interpret the expression of T as an infinite sum of operators?

The first problem is addressed by showing that an operator that is both compact and self-adjoint does have an abundance of eigenvalues; one nonzero eigenvalue is enough for a start. The next section shows how to interpret infinite sums of operators and how to obtain the suggested representation in general.

Theorem 14.38 *Let T be a nonzero, continuous linear operator on a Hilbert space H, and suppose that T is both compact and self-adjoint. Then T has a nonzero eigenvalue.*

Proof. The eigenvalues will be real if there are any. We look for the largest (in absolute value). Remember that the eigenvalues must occur in the interval $[-\|T\|, \|T\|]$. We shall find an eigenvalue at one end or other of this interval.

Let

$$\lambda_1 = \sup\{(T(f), f) : \|f\| = 1\}$$

and

$$\lambda_2 = \inf\{(T(f), f) : \|f\| = 1\}.$$

By Theorem 14.37, we know that either $\|T\| = \lambda_1$ or else $\|T\| = -\lambda_2$. One of these two values is an eigenvalue of T depending on which of these two assertions is true. The cases are similar. Let us handle just the case $\|T\| = \lambda_1$ and show that this is an eigenvalue. There must be a sequence $\{f_n\}$ with $\|f_n\| = 1$ and

$$(T(f_n), f_n) \to \lambda_1 = \|T\|.$$

By passing to a subsequence if necessary, we can assume that the norm limit $\lim_{n\to\infty} T(f_n)$ exists in H; this uses the fact that T is compact.

We claim that

$$\|T(f_n) - \lambda_1 f_n\| \to 0,$$

which is sometimes taken as the definition of an *approximate eigenvector.* This follows from the identity

$$\begin{aligned}\|T(f_n) - \lambda_1 f_n\|^2 &= \|T(f_n)\|^2 + \lambda_1^2 - 2\lambda_1(T(f_n), f_n)\\ &\le \|T\|^2 + \lambda_1^2 - 2\lambda_1(T(f_n), f_n)\end{aligned}$$

and the fact that

$$(T(f_n), f_n) \to \lambda_1 = \|T\|.$$

In this particular case, we see that λ_1 is in fact an eigenvector. Write $g = \lim_{n\to\infty} T(f_n)$ and check that

$$f_n = {\lambda_1}^{-1}\left(T(f_n) - (T(f_n) - \lambda_1 f_n)\right) \to {\lambda_1}^{-1} g.$$

Thus $T(f_n) \to g$ and $f_n \to {\lambda_1}^{-1} g$ so that

$$g = {\lambda_1}^{-1} T(g) \quad \text{or} \quad T(g) = \lambda_1 g.$$

This is exactly what we wanted to prove, and so the proof is complete. ∎

Exercises

14:8.1◇ Show that if T is a compact operator on a Hilbert space and $\lambda \neq 0$ is an eigenvalue of T then the eigenspace E_λ must be finite-dimensional. [Hint: Use Theorem 12.16 or, more simply, assume that there is an infinite orthonormal sequence in E_λ.]

14:8.2 Define the operator $T : \ell_2 \to \ell_2$ so that if $x = (x_1, x_2, x_3, \dots)$ then $T(x) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$. Show that T is a compact linear operator and that T has no nonzero eigenvalues. Does this contradict Theorem 14.38?

14:8.3 Define the operator $T : L_2[0,1] \to L_2[0,1]$ so that if $g = T(f)$ then $g(x) = xf(x)$ a.e.. Show that T is a continuous and self-adjoint linear operator and that T has no eigenvalues. Does this contradict Theorem 14.38?

14:8.4 Let T be a self-adjoint operator on $\mathbb{C}^n$, and suppose that T has exactly n distinct eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$. Use the material of this section to prove that

$$T = \sum_{i=1}^{n} \lambda_i P_{\lambda_i},$$

where P_{λ_i} is the projection onto the eigenspace associated with the eigenvalue λ_i.

14:8.5 Let T be a compact operator on a Hilbert space, and let $\varepsilon > 0$. Show that there are only finitely many eigenvalues λ of T with $\varepsilon < |\lambda| \leq \|T\|$. [Hint: If there is a distinct sequence $\varepsilon < |\lambda_n| \leq \|T\|$ with eigenvectors $\{f_n\}$, $\|f_n\| = 1$, then, by passing to subsequences, one can assume that $\lambda_n \to \lambda \neq 0$ and $T(f_n) \to g$. Show that $\|f_n - f_m\| \to 0$, which cannot happen for an orthonormal sequence.]

14.9 Spectral Decomposition

We are now in a position to obtain the promised spectral decomposition for compact self-adjoint operators on a Hilbert space. This reveals that every such operator has a transparent form if viewed in the correct light. Because operators of this kind occur in many applications, this representation offers an important and useful tool in their study. It is also important to study operators that are not compact or not self-adjoint. In that case, however, one finds that eigenvalues and eigenvectors do not provide the means for such a representation and, indeed, that no representation as an infinite sum is available. One needs more general spectral ideas and much heavier machinery, which we do not develop. More advanced texts such as the classic of Dunford and Schwartz[1] should be consulted.

[1] N. Dunford and J. T. Schwartz, *Linear Operators*, Wiley (1971).

Theorem 14.39 *Let T be a continuous, nonzero linear operator on a Hilbert space H, and suppose that T is both compact and self-adjoint. Then the set of nonzero eigenvalues of T can be arranged into a finite or infinite sequence of elements $\{\lambda_n\}$ with*

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots |\lambda_n| \geq \cdots,$$

and the operator T can be expressed as

$$T = \sum_j \lambda_j P_{\lambda_j},$$

where P_{λ_j} is the projection operator taking H onto the eigenspace H_{λ_j} corresponding to the eigenvalue λ_j.

Proof. If there are infinitely many eigenvalues, then the convergence of the series of projections is interpreted in the strongest sense, that is, in the sense of the operator norm

$$\lim_{n\to\infty} \left\| T - \sum_{j=1}^{n} \lambda_j P_{\lambda_j} \right\| = 0.$$

We define the sequence of eigenvalues and eigenspaces inductively. Start with $T_1 = T$, choose the largest eigenvalue λ_1 of T_1 (largest in absolute value) and let $P_1 = P_{\lambda_1}$ be the projection onto the eigenspace H_{λ_1} associated with λ_1. Set $T_2 = T_1 - T_1 P_1$, repeat the process by choosing the largest eigenvalue λ_2 of T_2 (again largest in absolute value), and let $P_2 = P_{\lambda_2}$ be the projection onto the eigenspace H_{λ_2} associated with λ_2. Set $T_3 = T_2 - T_2 P_2$ and continue the process inductively. In this way we arrive at a sequence of distinct eigenvalues $\{\lambda_n\}$, operators $\{T_n\}$, and projections $\{P_n\}$ onto the eigenspaces H_{λ_n} such that

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots |\lambda_n| \to 0, \tag{19}$$

$$\|T_n\| = |\lambda_n|, \tag{20}$$

and

$$T_{n+1} = T_n - \lambda_n P_n = T - \sum_{i=1}^{n} \lambda_i P_i. \tag{21}$$

If $T_{n+1} = 0$ at some stage, then the process stops, and (21) expresses T as a finite combination, as required. If the process continues indefinitely, then (19), (20), and (21) show that

$$\|T - \sum_{i=1}^{n} \lambda_i P_i\| \to 0$$

as $n \to \infty$ and expresses T as an infinite sum, exactly as required.

This plan seems simple enough, but will require a great deal of checking to see if it goes through as described. We can apply Theorem 14.38 at the first stage to select an eigenvalue λ_1 of T_1 with $|\lambda_1| = \|T_1\|$, because $T_1 = T$ is compact and self-adjoint. To continue the process will require us to check that each T_2, T_3, ... is *also* compact and self-adjoint. At each stage, we select an eigenvalue λ_n of T_n, but we do not know that λ_n is also an eigenvalue of the *original* operator T, as we hope. We do not know that the eigenvalues are *distinct* as claimed. Finally, we do not know that $|\lambda_n| \to 0$.

This gives us quite a few details to check, but the structure of the proof is now clear. First, let us look more closely at the initial stage of the construction, where we apply Theorem 14.38 to select an eigenvalue λ_1 of T. We know that we can select this so that $\|T_1\| = |\lambda_1|$. If P_1 is the projection onto the corresponding eigenspace and $T_2 = T_1 - \lambda_1 P_1$, then, using Theorem 14.37, we have

$$T_2 = T_1 - \lambda_1 P_1 = T_1(I - \lambda_1 P_1) = (I - \lambda_1 P_1)T_1 \tag{22}$$

and

$$\lambda_1 P_1 = T_1 P_1 = P_1 T_1. \tag{23}$$

The identity (22) along with Exercises 14:6.7 and 14:6.10 show that T_2 must be compact and self-adjoint. This applies inductively to show that each operator T_n in the sequence is compact and self-adjoint, and thus, unless $T_n = \mathbf{0}$ (in which case the process stops), Theorem 14.38 can be used at each stage to select an eigenvalue λ_n of T_n so that $\|T_n\| = |\lambda_n|$.

The next step is to show that the sequence of eigenvalues is distinct. We show that λ_1 cannot be an eigenvalue of T_2; it follows inductively that at each stage we have chosen a value λ_n differing from all λ_k $(k < n)$. Suppose that $T_2(f) = \lambda_1 f$; we show that $f = \mathbf{0}$ so that λ_1 cannot be an eigenvalue. By (22) and (23),

$$T_2(f) = T_1(f) - \lambda_1 P_1(f) = \lambda_1 f, \tag{24}$$

and so

$$P_1(T_1(f)) - \lambda_1 P_1(f) = P_1(\lambda_1 f),$$

which, using (23) once again, shows that

$$\lambda_1 P_1(f) - \lambda_1 P_1(f) = \lambda_1 P_1(f) = \mathbf{0}.$$

Together with (24), this shows that $T_1(f) = \lambda_1 f$. But this means that f is in the eigenspace for T_1 and the eigenvalue λ_1, so $f = P_1(f) = \mathbf{0}$, as required.

Our next task is to show that, although we choose each λ_n as an eigenvalue for T_n, it is nonetheless true that each λ_n is an eigenvalue for the original operator T and that the eigenspaces are identical. To do this, it is enough to show that any eigenvalue $\lambda \neq 0$ for T_2 is also an eigenvalue for T and that the eigenspaces are identical. Suppose that $T_2(f) = \lambda f$ and $f \neq \mathbf{0}$. Then

$$T_1(I - P_1)(f) = \lambda f$$

and hence, also,

$$(I - P_1)T_1(I - P_1)(f) = (I - P_1)(\lambda f).$$

But the left side of this is

$$(I - P_1)T_1(I - P_1)(f) = T_1(I - P_1)(I - P_1)(f) = T_1(I - P_1)(f) = \lambda f,$$

and so we have

$$(I - P_1)(\lambda f) = \lambda f$$

or $f = (I - P_1)(f)$, from which we deduce that

$$T(f) = T(I - P_1)(f) = T_2(f) = \lambda f,$$

which is exactly what we need. We see that λ is an eigenvalue of T as well as of T_2 and that f is also an eigenvector for T as well as for T_2.

We still do not know that all λ-eigenvectors for T are also λ-eigenvectors for T_2, and this we prove now. Since λ is an eigenvalue for T_2, it cannot be equal to λ_1. Thus, if f is a λ-eigenvector for T, it is orthogonal to the eigenspace for λ_1 and hence $P_1(f) = \mathbf{0}$. Thus

$$T_2(f) = (T_1 - \lambda_1 P_1)(f) = T_1(f) = T(f) = \lambda f,$$

and so f is a λ-eigenvector for T_2, as we wished to show.

We can now turn to the proof of (19). It is certainly true, by the way in which we have constructed the sequence, that $\{|\lambda_n|\}$ forms a nonincreasing sequence. But we do not know yet that $\lambda_n \to 0$. If $T_n = \mathbf{0}$ at some stage, there is nothing to prove. Let us suppose that, contrary to what we wish to prove, there is an $\varepsilon > 0$ so that $\inf_n |\lambda_n| \geq \varepsilon$. For each n, choose an eigenvector f_n of T associated with λ_n and T and with $\|f_n\| = 1$. Since the sequence $\{f_n\}$ is bounded and the operator T is compact, there is a subsequence with $\{T(f_{n_k})\}$ convergent in norm. But, using the Pythagorean theorem (Corollary 14.6), we find that

$$\|T(f_{n_k}) - T(f_{n_j})\|^2 = \|\lambda_{n_k} f_{n_k} - \lambda_{n_j} f_{n_j}\|^2 = |\lambda_{n_k}|^2 + |\lambda_{n_j}|^2 \geq 2\varepsilon^2$$

for all j, k, which is impossible if $\{T(f_{n_k})\}$ converges. From this contradiction, it follows that $|\lambda_n|$ decreases to zero and so (19) is proved.

Having checked all the problematical details raised in the third paragraph of our proof, we see that the representation is shown to be valid. There remains one problem, because the statement of our theorem claimed rather more than this; the alert reader will spot this before attempting Exercise 14:9.1. ■

Exercises

14:9.1◇ We did not check that the process in the proof of Theorem 14.39 picks up *all* the nonzero eigenvalues of T. Use the representation of T to show that there are no more eigenvalues other than the $\{\lambda_n\}$ listed or a zero eigenvalue.

14:9.2 An operator T on a Hilbert space H is said to be *finite-dimensional* if the range of T is a finite-dimensional subspace of H. Show that a compact self-adjoint operator is finite-dimensional if and only if it has a finite number of eigenvalues.

14:9.3 Show that an operator that is both compact and self-adjoint on an infinite-dimensional Hilbert space cannot be invertible.

14:9.4 Show that an operator that is both compact and self-adjoint on an infinite-dimensional Hilbert space cannot map H onto itself.

14:9.5 Show that a self-adjoint operator on a Hilbert space is compact if and only if there is a sequence of finite-dimensional, self-adjoint operators $\{T_n\}$ with $\|T_n - T\| \to 0$.

14.10 Additional Problems for Chapter 14

14:10.1 Let f_1, f_2, f_3, ... be any orthonormal sequence in a Hilbert space H. Show that there is a unique linear operator T on H (called the *shift operator*) such that $T(f_n) = f_{n+1}$.

(a) Show that T is continuous and compute its norm.

(b) Describe the null space and range of T.

(c) Characterize the adjoint T^*.

(d) What are the null space and range of T^*?

(e) Show that $T^*T = I$, but that neither T nor T^* is invertible.

14:10.2 To clarify Exercise 14:10.1(e) show that in a finite-dimensional Hilbert space a left inverse of an operator is also a right inverse.

14:10.3 A self-adjoint operator T on a Hilbert space H is said to be *positive* if $(T(f), f) \geq 0$ for all $f \in H$. Show that every eigenvalue of a positive operator is nonnegative.

14:10.4 Show that if T is a positive, self-adjoint operator on a Hilbert space H then

$$|(T(f), g)| \leq (T(f), f)\,(T(g), g)$$

for all f, $g \in H$.

14:10.5 Show that If T is a continuous, linear operator on a Hilbert space H then TT^* and T^*T are self-adjoint and positive.

14:10.6 Show that every continuous, linear operator T on a Hilbert space H can be expressed as a linear combination of self-adjoint transformations. [Hint: For a start, $\frac{1}{2}(T+T^*)$ is self-adjoint.]

14:10.7 An ordering for self-adjoint operators T_1, T_2 on a Hilbert space H can be defined by writing $T_1 \preceq T_2$ if

$$(T_1(f), f) \leq (T_2(f), f)$$

for all $f \in H$. Show that this is a partial order on the collection of self-adjoint operators on H.

14:10.8 Let T be a continuous linear operator on a Hilbert space. Show that the following conditions are equivalent:

(a) $TT^* = T^*T = I$.

(b) T^{-1} exists and $(f, g) = (T(f), T(g))$ for all f, $g \in H$.

(c) T^{-1} exists and $\|f\| = \|T(f)\|$ for all $f \in H$.

(Operators satisfying these conditions are said to be *unitary*. The class of such operators forms a group.)

14:10.9 Let T be a continuous linear operator on a Hilbert space. A number λ is said to be an *approximate eigenvalue* for T if for any $\varepsilon > 0$ there is a vector f with $\|f\| = 1$ for which $\|T(f) - \lambda f\| < \varepsilon$. Prove the following:

(a) Every eigenvalue is an approximate eigenvalue, but not conversely.

(b) If λ is an approximate eigenvalue, then

$$|\lambda| \leq \sup\{|(T(f), f)| : \|f\| \leq 1\} \leq \|T\|.$$

(c) A necessary and sufficient condition that T have an approximate eigenvalue λ with $|\lambda| = \|T\|$ is that

$$\sup\{|(T(f), f) : \|f\| \leq 1\} = \|T\|.$$

(d) If T is also self-adjoint, then every approximate eigenvalue is real.

(e) If T is also self-adjoint, then one of the values $\|T\|$ or $-\|T\|$ is an approximate eigenvalue.

(f) If T is an isometry and λ is an approximate eigenvalue of T, then $|\lambda| = 1$.

(g) If T is normal and λ is an approximate eigenvalue of T, then $\overline{\lambda}$ is an approximate eigenvalue of T^*. (An operator T is said to be *normal* if $TT^* = T^*T$.)

[Hint: See the proof of Theorem 14.38 for ideas.]

14:10.10 Theorem 14.39 can be made the basis for an "operator calculus" as first observed by F. Riesz in greater generality. Suppose that T is a continuous linear operator on a Hilbert space H with the representation

$$T = \sum_{j}^{\infty} \lambda_j P_{\lambda_j},$$

where P_{λ_j} is the projection operator taking H onto the eigenspace H_{λ_j} corresponding to the eigenvalue λ_j of T.

(a) Show that $T^2 = \sum_{j=1}^{\infty} (\lambda_j)^2 P_{\lambda_j}$.

(b) If T is positive, compact, and self-adjoint, then it has a square root,

$$T^{1/2} = \sum_{j=1}^{\infty} \sqrt{\lambda_j} P_{\lambda_j}.$$

(Use Exercise 14:10.3.)

(c) Show that $T^n = \sum_{j=1}^{\infty} (\lambda_j)^n P_{\lambda_j}$ for every positive integer n.

(d) Assume that T is invertible (it cannot be compact then unless H is finite-dimensional). Show that $T^{-n} = \sum_{j=1}^{\infty} (\lambda_j)^{-n} P_{\lambda_j}$ for every positive integer n.

(e) Show that $e^T = \sum_{j=1}^{\infty} e^{\lambda_j} P_{\lambda_j}$ where, by definition,

$$e^T = \sum_{n=0}^{\infty} \frac{1}{n!} T^n.$$

(f) How might these ideas generalize?

Chapter 15

FOURIER SERIES

This chapter presents a short introduction to the theory of trigonometric series and Fourier series. The choice of topics is mostly directed by a wish to illustrate various applications of the analytic tools developed so far in this text: measure, integral, convergence, derivatives, metric space, Baire category, the L_p–spaces, and Banach spaces. The reader may have (we hope will have) encountered some of the ideas of Fourier analysis in more elementary courses where the more sophisticated and powerful tools we now have were not available. If so, the impression should be that the theory becomes clearer and more lucid, the methods more delicate and exact, and the results start to form a more meaningful picture.

The origins of the subject go back to the middle of the eighteenth century. Certain problems in mathematical physics seemed to require that an arbitrary function f with a fixed period (taken here as 2π) be represented in the form of a trigonometric series

$$f(t) = \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty}(a_j \cos jt + b_j \sin jt), \tag{1}$$

and such mathematicians as Daniel Bernoulli, d'Alembert, Lagrange, and Euler had debated whether such a thing should be possible. Bernoulli maintained that this would always be possible, while Euler and d'Alembert argued against it.

If we remember that Newton and Leibnitz were alive in the early 1700s it is remarkable to realize that such a discussion could take place as early as the middle of the century, and we can surely forgive them for their misconceptions as to the nature of "arbitrary" functions.

Joseph Fourier (1768–1830), as much a physicist and an egyptologist as a mathematician, saw the utility of these representations. Although he did nothing to verify his position other than to perform some specific calculations, in 1807 he accepted that the representation in (1) would be available for every function f and gave the formulas

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jt\,dt \quad \text{and} \quad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jt\,dt$$

for the coefficients. (These were exactly the formulas Euler had advanced in 1777 should the series representation be possible.)

Fourier's presentations were received with no less scepticism on the part of the professional mathematicians of the day. Nonetheless, the many methods he gave to mathematical physics and the vision that he had has let his name survive on this representation: the numbers a_j, b_j are called the Fourier coefficients, and the series itself is called a Fourier series.

Fourier series are of very great importance in physics, applied mathematics, and engineering. For analysts they are, perhaps, even more important. Many of the great mathematicians of the nineteenth century attacked problems in the subject. More than any other line of research, this program started by Fourier has led to a clarification of the concepts of function and convergence, major advances in the study of the integral (first by Riemann and then by Lebesgue), and ultimately to the creation of many fields of mathematical research. Even Cantor's set theory was developed by him in order to study the sets of uniqueness of trigonometric series.

We cover most of what may be considered a standard short introduction to the subject, including applications of the Dirichlet and Fejér kernels, some of the basics of pointwise convergence, and an account of Fourier series in the Hilbert space $L_2[-\pi,\pi]$. In addition, we have included some topics from the general theory of trigonometric series since they give a different flavor and have their own charm. Any account of this subject pasted onto the end of a beginning graduate text in real analysis will be inadequate to convey the wide range of ideas, techniques, and applications of harmonic analysis: even a casual trip to a good mathematics library will lead the reader to a wealth of deeper reading. Above all, do not pass over Zygmund's monumental *Trigonometric Series*[1] or Bari's *A Treatise on Trigonometric Series*.[2]

15.1 Notation and Terminology

We shall express our Fourier series in the language of complex exponentials rather than as sums of sines and cosines. This requires only a small effort of will in order to become accustomed to the notation and pays back considerably in ease of computation and manipulation. In addition, this language is used in more modern theories and helps to frame a natural connection with certain problems in complex analysis.

[1] A. Zygmund, *Trigonometric Series*, Cambridge University Press (1959).

[2] N. Bari, *A Treatise on Trigonometric Series*, Pergamon Press (1964).

Thus the expression

$$\sum_{|j|\le n} c_j e^{ijt}$$

is said to be a *trigonometric polynomial,* and the expression

$$\sum_{j=-\infty}^{\infty} c_j e^{ijt}$$

is called a *trigonometric series.* Here c_j are real or complex constants. The degree of a trigonometric polynomial is the highest exponent entering in the sum. We say that $\sum_{|j|\le n} c_j e^{ijt}$ has *degree* n provided that c_n and c_{-n} are not both zero.

The domain of definition of all functions is taken to be the real interval $T = (-\pi, \pi]$. In fact, we think of T as being the real line modulo the equivalence relation $x \sim y$ if $x - y$ is a multiple of 2π. Thus T can be taken as *any* interval of length 2π with this understanding, and the endpoints are identified with each other. In this way, T is actually a compact set and has the structure of an additive group under addition.

The more usual interpretation of T is to consider it as the circle group or the one-dimensional torus group: the set of complex numbers with unit modulus under the group operation of multiplication and given the usual metric as a subset of $\mathbb{C}$. The mapping $t \to e^{it}$ is a continuous isomorphism that identifies points in T with points in the circle group. This more algebraic viewpoint is needed when one wishes to undertake a generalization of Fourier analysis to different settings; it is not much needed here, other than perhaps to explain why our interval is labeled T (for torus).

Given a trigonometric polynomial

$$P(t) = \sum_{|j|\le n} c_j e^{ijt} \qquad (t \in T),$$

there is a way to determine the coefficients of the polynomial from the values of P. Indeed, since this is a finite sum of continuous (complex) functions, we have

$$c_j = \frac{1}{2\pi} \int_T P(t) e^{-ijt}\, dt \qquad (|j| \le n).$$

We would obtain precisely the same formulas for the coefficients of a trigonometric series $f(t) = \sum_{j=-\infty}^{\infty} c_j e^{ijt}$, provided that some meaning is attached to the sum of the series and the integration may be performed by integrating each term in the sum (as would be the case with uniform convergence or dominated convergence, for example).

This suggests a way of associating a trigonometric series with any integrable function without any regard to the question (at least for now) of whether the series in any way sums back to the function. We use $L_1(T)$

to denote the space of complex-valued functions defined and integrable on T; since we wish to allow T to represent any interval of length 2π, we can consider $L_1(T)$ to be the space of complex-valued, 2π-periodic functions defined on $\mathbb{R}$ and integrable on each finite interval.

In general, $L_p(T)$ $(1 \le p < \infty)$ represents the usual spaces of p@,th power integrable functions on T, again interpreted as complex-valued, 2π-periodic functions defined on $\mathbb{R}$. For norm we shall use

$$\|f\|_p = \left(\frac{1}{2\pi} \int_T |f(t)|^p \, dt \right)^{1/p},$$

which is the usual norm adjusted by a constant factor that simplifies many formulas. The space $L_\infty(T)$ has its usual essential supremum norm $\|f\|_\infty$ (not needing any such adjustment) and is, as the reader may recall from Section 13.3, the limiting value of $\|f\|_p$ as $p \to \infty$.

Definition 15.1 A *Fourier series* is a trigonometric series $\sum_j c_j e^{ijt}$ for which there is some function $f \in L_1(T)$ so that

$$c_j = c_j(f) = \frac{1}{2\pi} \int_T f(t) e^{-ijt} \, dt$$

for all j. The constants $c_j = c_j(f)$ are called the *Fourier coefficients* of f, and the relation between f and the associated series is denoted as

$$f \sim \sum_j c_j e^{ijt}.$$

The distinction between a Fourier series and a trigonometric series is easy but must be grasped. A trigonometric series is merely a series $\sum_j c_j e^{ijt}$ considered formally with no claims to convergence. A Fourier series is a trigonometric series again considered formally with no claims to convergence but associated with some function $f \in L_1(T)$ in the sense that the coefficients have been determined from f. We rather hope for a closer connection between a function and its Fourier series: in some way the series is intended to "represent" the function. But investigating this representation problem will take some time and effort.

We have now embarked on a program that is part of the subject of harmonic analysis. The first part is solved. Given a function f, we know how to resolve it into its "components" in each of the "directions" e^{ijt}. The second part of the program, the more difficult part, is the "synthesis" problem: given the components, how can we reconstruct f from the components? We hope that somehow the Fourier series can be summed to recover f.

Summing a Fourier series or a general trigonometric series will always follow this convention: we form the symmetric partial sums

$$s_n(t) = \sum_{|j| \le n} c_j e^{ijt}$$

and investigate the limit of the sequence s_n, interpreted in several senses. This sequence $\{s_n(t)\}$ is called the *sequence of partial sums* of the trigonometric series. If $\sum_j c_j e^{ijt}$ is the Fourier series of a function f, then it is useful to indicate this by the notation

$$s_n(f,t) = \sum_{|j|\le n} c_j e^{ijt}.$$

This sequence $\{s_n(f,t)\}$ is called the *sequence of partial sums* of the Fourier series. Much of our concern in what follows is how to obtain f from the sequence $s_n(f)$.

Exercises

15:1.1 Obtain the orthogonality relations; that is, determine

$$\frac{1}{2\pi}\int_T e^{ikt}e^{-ijt}\,dt$$

for $k = j$ and for $k \neq j$. Do this too for the real versions:

$$\frac{1}{2\pi}\int_T \sin(kt)\sin(jt)\,dt, \quad \frac{1}{2\pi}\int_T \cos(kt)\sin(jt)\,dt,$$

and

$$\frac{1}{2\pi}\int_T \cos(kt)\cos(jt)\,dt.$$

15:1.2 Show that the integrals $\int_T f(t)e^{-ijt}\,dt$ exist for any $f \in L_1(T)$. [Equivalently, show that the integrals

$$\int_0^{2\pi} f(t)\cos jt\,dt \quad\text{and}\quad \int_0^{2\pi} f(t)\sin jt\,dt$$

exist.]

15:1.3 Given a trigonometric polynomial $P(t) = \sum_{|j|\le n} c_j e^{ijt}$, show that

$$c_j = \frac{1}{2\pi}\int_T P(t)e^{-ijt}\,dt$$

for each $|j| \le n$.

15:1.4 Given that the limit $f(t) = \lim_{n\to\infty} s_n(t)$ holds uniformly where

$$s_n(t) = \sum_{|j|\le n} c_j e^{ijt},$$

show that

$$c_j = \frac{1}{2\pi}\int_T f(t)e^{-ijt}\,dt$$

for each $|j| \le n$.

15:1.5 Given that the limit $f = \lim_{n\to\infty} s_n$ holds in the sense of the $L_1(T)$ norm [where again $s_n(t) = \sum_{|j|\le n} c_j e^{ijt}$] for a function $f \in L_1(T)$, show that

$$c_j = \frac{1}{2\pi}\int_T f(t)e^{-ijt}\,dt$$

for each $|j| \le n$.

15:1.6 Given that the limit $f = \lim_{n\to\infty} s_n$ holds in the sense of the $L_p(T)$ norm for a function $f \in L_p(T)$ and some $1 < p < \infty$, show that

$$c_j = \frac{1}{2\pi}\int_T f(t)e^{-ijt}\,dt$$

for each $|j| \le n$.

15:1.7 Suppose that the limit $f = \lim_{n\to\infty} s_n$ holds in the sense that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} u(t)s_n(t)\,dt = \int_{-\pi}^{\pi} u(t)f(t)\,dt$$

for every infinitely differentiable, 2π-periodic function u. Show that

$$c_j = \frac{1}{2\pi}\int_T f(t)e^{-ijt}\,dt$$

for each $|j| \le n$.

15:1.8 The reader who wishes occasionally to see Fourier series in the familiar real form from elementary applications can check the details of the following. If f is real-valued, integrable, and 2π-periodic then $c_0(f)$ is real, $c_{-j}(f)$ is the complex conjugate of $c_j(f)$, and

$$\begin{aligned} s_n(f,t) = \sum_{|j|\le n} c_j e^{ijt} &= c_0 + \sum_{j=1}^{n}\left(c_j e^{ijt} + c_{-j}e^{-ijt}\right) \\ &= c_0 + \sum_{j=1}^{n}(c_j + c_{-j})\cos jt + i\,(c_j - c_{-j})\sin jt \\ &= \tfrac{1}{2}a_0 + \sum_{j=1}^{n} a_j\cos jt + b_j\sin jt, \end{aligned}$$

where $a_j = (c_j + c_{-j})$ and $b_j = i\,(c_j - c_{-j})$. In this case,

$$a_j = \frac{1}{\pi}\int_T f(t)\cos jt\,dt \quad b_j = \frac{1}{\pi}\int_T f(t)\sin jt\,dt.$$

15:1.9$\diamondsuit$ If the function $f \in L_1(T)$ is real and even [i.e., if $f(t) = f(-t)$], then show that the Fourier series assumes the form

$$f \sim \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty} a_j\cos jt$$

where

$$a_j = \frac{2}{\pi} \int_0^{\pi} f(t) \cos jt \, dt.$$

If the function $f \in L_1(T)$ is real and odd [i.e., if $f(t) = -f(-t)$] what is the appropriate form?

15:1.10 What are the exponential functions $\{e^{ijt}\}$ that play such a key role in this study? Show that they are precisely the continuous group characters of T. [A function $\chi : T \to \mathbb{C}$ such that $\chi(s+t) = \chi(s)\chi(t)$ is called a group character. We want only continuous, 2π-periodic functions: show that $\chi(0) = 1$, $\chi(-t) = \chi(t)^{-1}$, $|\chi(t)| = 1$,

$$\int_0^h \chi(s+t) \, dt = \chi(s) \int_0^h \chi(t) \, dt = \int_s^{s+h} \chi(t) \, dt,$$

and $\chi'(t) = (-i\chi'(0)) \, \chi(t)$. Compare this with Exercise 13:9.13.]

15.2 Dirichlet's Kernel

In any study of trigonometric series, some attention must be given to the partial sums of the series. In the case of the partial sums of the Fourier series of a function

$$s_n(f, x) = \sum_{|j| \le n} c_j e^{ijx},$$

the c_j are determined by an integral, and naturally one can replace each c_j by that integral and obtain

$$\begin{aligned} s_n(f, x) &= \sum_{|j| \le n} \left(\frac{1}{2\pi} \int_T f(t) e^{-ijt} \, dt \right) e^{ijx} \\ &= \frac{1}{\pi} \int_T f(t) \left(\frac{1}{2} \sum_{|j| \le n} e^{ij(x-t)} \right) dt = \frac{1}{\pi} \int_T f(t) D_n(x-t) \, dt, \end{aligned}$$

where we are writing

$$D_n(t) = \frac{1}{2} \sum_{|j| \le n} e^{ijt} \, dt.$$

Since these are just finite sums, these manipulations are not deep, and the resulting expression,

$$s_n(f, x) = \frac{1}{\pi} \int_T f(t) D_n(x-t) \, dt, \tag{2}$$

is a trivial rewriting of $s_n(f, x)$. It suggests, though, that any study of the convergence properties of Fourier series must address properties of the functions $D_n(t)$, and this is so.

The function $D_n(t)$ is called the *Dirichlet kernel of order* n after Peter Gustav Lejeune-Dirichlet (1805–1859), who was the first to obtain any rigorous results on the convergence behavior of Fourier series. (His 1829 theorem asserts that a function with at most finitely many simple discontinuities and only a finite number of maxima and minima has a Fourier series that converges everywhere, to the function at the points of continuity and to the average between the left and right limits at a discontinuity.)

We collect in a theorem all the properties of these kernels that are needed for our subsequent study.

Theorem 15.2 (Properties of the Dirichlet kernel) *The function*

$$D_n(t) = \frac{1}{2} \sum_{|j| \le n} e^{ijt}$$

is called the Dirichlet kernel of order n*, and the numbers*

$$L_n = \frac{1}{\pi} \int_T |D_n(t)|\, dt = 2\|D_n\|_1$$

are called the Lebesgue constants. The following properties hold for these concepts:

1. *Each* $D_n(t)$ *is a real-valued, continuous,* 2π*-periodic function and (for* $n > 0$*) assumes both positive and negative values.*
2. *Each* $D_n(t)$ *is an even function.*
3. *For each* n*,*
$$\frac{1}{\pi} \int_T D_n(t)\, dt = \frac{2}{\pi} \int_0^\pi D_n(t)\, dt = 1.$$
4. *For each* n*,*
$$D_n(t) = \frac{\sin\left(n + \frac{1}{2}\right) t}{2 \sin \frac{1}{2} t}.$$
5. *For each* n*,* $D_n(0) = n + \frac{1}{2}$*.*
6. *For each* n *and all* t*,* $|D_n(t)| \le n + \frac{1}{2}$*.*
7. *For each* n *and* $0 < |t| < \pi$*,*
$$|D_n(t)| \le \frac{\pi}{2|t|}.$$
8. $L_n \to \infty$ *as* $n \to \infty$*.*

Proof. Items (1), (2), (3), (5), and (6) are almost immediate from the definition of the D_n. Item (4) requires only some elementary manipulations. Since the sum defining D_n is a geometric series, we have

$$D_n(t) = \frac{1}{2}e^{-int}\left(\frac{e^{i(2n+1)t}-1}{e^{it}-1}\right),$$

and some mildly tedious applications of the standard formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

will produce (4).

Use the simple inequality

$$\frac{2\theta}{\pi} \leq \sin\theta$$

for $0 < \theta < \pi/2$ (draw a picture) to obtain that

$$\frac{1}{2\sin\frac{1}{2}t} \leq \frac{\pi}{2|t|}$$

for all $0 < t < \pi$. From this (7) follows.

Finally, we wish to show that (8) holds. In fact, one can prove that L_n asymptotically approaches

$$\frac{4\ln n}{\pi^2}$$

as $n \to \infty$. We require only to know that $L_n \to \infty$. We use the elementary inequality

$$|\sin\theta| \leq |\theta|$$

to obtain

$$L_n = \frac{1}{\pi}\int_T |D_n(t)|\,dt \geq \frac{2}{\pi}\int_0^\pi \frac{\sin\left(n+\frac{1}{2}\right)t}{|t/2|}\,dt,$$

and a change of variables shows that this is

$$\frac{4}{\pi}\int_0^{(n+1/2)\pi}\frac{|\sin\tau|}{\tau}\,d\tau \geq \frac{4}{\pi}\sum_{j=1}^{n}\frac{1}{j\pi}\int_{(j-1)\pi}^{j\pi}|\sin\tau|\,d\tau = \frac{4}{\pi^2}\sum_{j=1}^{n}\frac{1}{j}.$$

As this series diverges, the numbers L_n grow without bound. Hence we have proved (8). ∎

Some of the features of the Dirichlet kernel can be seen in Figure 15.1. The symmetry is certainly apparent (D_n is even) and that the graph oscillates above and below the horizontal axis is evident. The value of the function is small except close to 0 where the function is large, and as n increases this feature becomes more pronounced. The total area remains fixed always at π because of the cancellations: if the area is taken without

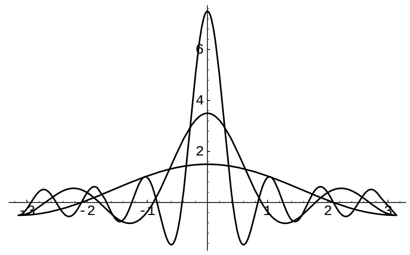

Figure 15.1: Dirichelet kernel $D_n(t)$ for $n = 1$, 3, and 7.

cancellations (i.e., the area under $|D_n|$ is computed), then this gets large with increasing n. This last fact plays a role in Section 15.9, where we show that the Fourier series of a continuous function need not converge. Item (7) is interesting for us only in the fact that we cannot improve it. In contrast, we shall see that the Fejér kernel of the next section has a better upper estimate, which can be exploited.

Exercises

15:2.1 Check the representation

$$s_n(f,x) = \frac{1}{\pi}\int_T f(x+t)D_n(t)\,dt$$

$$= \frac{1}{\pi}\int_0^\pi \left(f(x+t)+f(x-t)\right)D_n(t)\,dt.$$

15:2.2◊ Check the representation

$$s_n(f,x_0) - s = \frac{1}{\pi}\int_0^\pi \left(f(x_0+t)+f(x_0-t)-2s\right)D_n(t)\,dt$$

for any real number s.

15.3 Fejér's Kernel

A study of the convergence properties of the Fourier series will evidently require handling the Dirichlet kernel. In the preceding section we collected some of the properties of that kernel in anticipation of solving convergence problems.

We would hope to use these ideas to determine that the Fourier series of a reasonable function converges pointwise to that function. Let us confess immediately, though, to the difficulties of this task. Pointwise convergence of Fourier series is a subtle and occasionally elusive pursuit. This leaves us at the beginning of our study with a major nuisance: we do not know how to recover a function from its Fourier series. Indeed, the Fourier series of an integrable function may diverge (everywhere!), and there would seem to be no hope of "summing" the series to obtain the function.

A simple idea comes to the rescue. Average the sums. If the sequence $s_n(f,x)$ will not recover $f(x)$, consider instead the averages

$$\sigma_n(f,x) = \frac{s_0(f,x) + s_1(f,x) + s_2(f,x) + \cdots + s_n(f,x)}{n+1}.$$

The idea of forming averages for divergent series goes back over two centuries, but received its first formal study by Ernesto Cesàro (1859–1906) in 1890. A young Hungarian mathematician Leopold Fejér (1880–1959) first applied it in 1900 to the study of Fourier series and obtained the results we now study. The averages $\sigma_n(f,x)$ are called the *Cesàro means* of the Fourier series, and this method of summing a series that may possibly be divergent is called *Cesàro (C,1) summation.* It is developed a bit further in the exercises. There are many summability methods, of which the Cesàro method is but one.

We obtain a simple formula for the averages $\sigma_n(f,x)$, just as we did for the partial sums themselves. Using the Dirichlet kernel itself we have

$$\begin{aligned}\sigma_n(f,x) &= \frac{s_0(f,x) + s_1(f,x) + s_2(f,x) + \cdots + s_n(f,x)}{n+1} \\ &= \frac{1}{\pi}\int_T f(t)K_n(x-t)\,dt,\end{aligned}$$

where we are writing

$$K_n(t) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(t).$$

This representation can, with some minor computations, be written in the form

$$\sigma_n(f,x) = \frac{1}{\pi}\int_T \frac{1}{2}\left(f(x+t) + f(x-t)\right)K_n(t)\,dt$$

or in the equivalent form

$$\sigma_n(f,x) = \frac{1}{\pi}\int_0^{\pi} \left(f(x+t) + f(x-t)\right)K_n(t)\,dt. \tag{3}$$

The function $K_n(t)$ is called the *Fejér kernel of order* n. We collect in a theorem all the properties of these kernels that are needed for our study in a way that parallels Theorem 15.2 cataloging the properties of the Dirichlet kernel. The reason why this method of summing a Fourier series has better properties than ordinary partial sums can be seen by comparing these two theorems. The reason is easy to spot: the Fejér kernel is nonnegative.

Theorem 15.3 (Properties of the Fejér kernel) *The function*

$$K_n(t) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(t)$$

is called the Fejér kernel of order n and enjoys the following six properties:

1. *Each $K_n(t)$ is a real-valued, nonnegative, continuous function.*
2. *Each $K_n(t)$ is an even function.*
3. *For each n,*
$$\frac{1}{\pi}\int_T K_n(t)\,dt = \frac{2}{\pi}\int_0^{\pi} K_n(t)\,dt = 1.$$
4. *For each n,*
$$K_n(t) = \frac{1}{2(n+1)}\left(\frac{\sin\left(\frac{1}{2}(n+1)t\right)}{\sin\frac{1}{2}t}\right)^2.$$
5. *For each n, $K_n(0) = \frac{1}{2}(n+1)$.*
6. *For each n and $0 < |t| < \pi$,*
$$0 \le K_n(t) \le \frac{\pi}{(n+1)t^2}.$$

Proof. Items (1), (2), (3), and (5) are almost immediate from the definition of the K_n. That $K_n(t) \ge 0$ follows from (4). Item (4) requires elementary manipulations once again, summing a geometric series and using trigonometric identities. The details are not interesting and nowadays can be checked on a computer in any case. Again, use the simple inequality $2\pi^{-1}\theta \le \sin\theta$ for $0 < \theta < \pi/2$ on the expression in the denominator of (4) to obtain (6). ■

Some of the features of the Fejér kernel can be seen in Figure 15.2 and should be compared and contrasted with the picture for the Dirichlet kernel. Again the symmetry is certainly apparent (K_n is even), but the graph here does not oscillate above and below the horizontal axis, but remains always on or above. As before, the value of the function is small except close to 0 where the function is large, and as n increases, this feature becomes more pronounced. The total area under the graph remains fixed always at π, but this is not because of any cancellations. This last fact is the reason why the Cesàro means of the Fourier series of a continuous function can converge even though the series itself diverges. From these properties of the Fejér kernel we can, in the next section, obtain a number of convergence facts for the Cesàro means of a Fourier series.

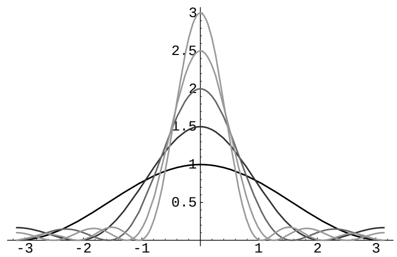

Figure 15.2: Fejér kernel $K_n(t)$ for $n = 1, 2, 3, 4$, and 5.

Exercises

15:3.1 A series $\sum_{j=1}^{\infty} c_j$ of real or complex numbers can often be summed by taking averages. Let $s_n = \sum_{j=1}^{n} c_j$ denote the usual partial sums of the series, and let $\sigma_n = (1/n)\sum_{j=1}^{n} s_j$ be the Cesàro means. The series is said to be (C,1)-summable to a value s if $\lim_{n\to\infty} \sigma_n = s$. If the series is convergent to s in the usual sense (i.e., if $\lim_{n\to\infty} s_n = s$), show that the series is also (C,1)-summable to the same value s. (Is the converse true?) [Hint: This exercise can also be done within the context of summability methods (Section 12.12).]

15:3.2 The series $\sum_{j=0}^{\infty} z^j$ diverges for all z on the unit circle $|z| = 1$. Determine the (C,1)–sum.

15:3.3 If a series of positive terms is (C,1)-summable to s $(0 \le s \le \infty)$ then, in fact, $\lim_{n\to\infty} s_n = s$.

15:3.4 [Hardy's Tauberian theorem] If a series $\sum_{j=1}^{\infty} c_j$ is (C,1)-summable to s and $\{jc_j\}$ is bounded then in fact $\lim_{n\to\infty} s_n = s$. (A theorem that asserts that, in the presence of some additional hypothesis, a sequence that is summable by some method must be convergent is called a Tauberian theorem after Alfred Tauber, who proved a very simple theorem of this type.)

15:3.5 Check the representation in (3) using appropriate properties of K_n from the theorem.

15:3.6$\diamondsuit$ Show that

$$\sigma_n(f,x) = \sum_{|j|\le n} \left(1 - \frac{|j|}{n+1}\right) c_j e^{ijx},$$

where $c_j = c_j(f)$.

15.4 Convergence of the Cesàro Means

We begin with the basic theorem first proved by Fejér and then give some variants that can be obtained by essentially the same methods.

Theorem 15.4 (Fejér) *Let $f \in L_1(T)$, and let $\sigma_n(f,x)$ denote the Cesàro means of the Fourier series of f. If the limits $f(x_0+0)$ and $f(x_0-0)$ both exist at a point x_0, then*

$$\lim_{n\to\infty} \sigma_n(f,x_0) = \tfrac{1}{2}\left(f(x_0+0)+f(x_0-0)\right).$$

If, moreover, f is continuous at each point of an interval $[a,b]$, then $\sigma_n(f,x) \to f(x)$ uniformly for $x \in [a,b]$.

Proof. Recall that $f(x_0+0)$ and $f(x_0-0)$ are our notations for the right- and left-hand limits of f at x_0. We may assume that

$$f(x_0) = \tfrac{1}{2}\left(f(x_0+0)+f(x_0-0)\right).$$

This one change in the value of f does nothing to the Fourier series, and so we are allowed this. If f is continuous, then this step can be skipped.

Let $\varepsilon > 0$, and choose $\delta > 0$ so that

$$|f(x_0+t)+f(x_0-t)-2f(x_0)| < \varepsilon$$

for every $0 \le t \le \delta$. We note that

$$\frac{2}{\pi}\int_0^\pi f(x_0)K_n(t)\,dt = f(x_0)$$

[by using property (3) of Theorem 15.3] and so from our representation of $\sigma_n(f,x)$ in (3) we have

$$\begin{aligned}|\sigma_n(f,x_0)-f(x_0)| &\le \frac{1}{\pi}\int_0^\pi |f(x_0+t)+f(x_0-t)-2f(x_0)|\,K_n(t)\,dt \\ &\le I_1+I_2,\end{aligned}$$

where I_1 is the integral taken over $[0,\delta]$ and I_2 is the integral taken over $[\delta,\pi]$. Since K_n is nonnegative, we did not need to keep it inside the absolute value in the integral. (It is here where we first see how this feature, lacking in the Dirichlet kernel D_n, can be used.)

The part I_1 will be small (for all n) because the expression in the absolute values is small for t in the interval $[0,\delta]$. The part I_2 will be small (for large n) because of the bound on the size of K_n for t away from zero in Theorem 15.3. Here are the details: for I_1 we have, using Theorem 15.3(3),

$$I_1 \le \frac{\varepsilon}{\pi}\int_0^\delta K_n(t)\,dt \le \varepsilon.$$

For I_2, let
$$\kappa_n = \sup\{K_n(t) : \delta \le t \le \pi\},$$
and note that Theorem 15.3(6) supplies us with the fact that $\kappa_n \to 0$ as $n \to \infty$. Now we have
$$I_2 \le \frac{\kappa_n \varepsilon}{\pi} \int_\delta^\pi \left(|f(x_0+t)| + |f(x_0-t)| + 2|f(x_0)|\right) dt$$
so that for large n we can make I_2 as small as we please. It follows, since ε is arbitrary, that
$$\lim_{n\to\infty} \sigma_n(f, x_0) = f(x_0),$$
as required.

If, moreover, f is continuous at each point of an interval $[a,b]$, then these arguments apply uniformly throughout so that $\sigma_n(f,x) \to f(x)$ uniformly for $x \in [a,b]$. ■

A more modern version of this same theorem is proved in somewhat the same way. Here we note that points of continuity can be replaced by the weaker notion of a Lebesgue point (see Section 7.8) and still convergence can be proved.

Theorem 15.5 (Fejér–Lebesgue) *Let $f \in L_1(T)$, and let $\sigma_n(f,x)$ be the Cesàro means of the Fourier series of f. Then*
$$\lim_{n\to\infty} \sigma_n(f,x) = f(x)$$
at every Lebesgue point of f. (Since almost every point is a Lebesgue point, this occurs almost everywhere.)

Proof. The proof is similar in its strategy to that given for Theorem 15.4, but the arguments are more delicate because a weaker assumption is made. The details will be better understood if the reader attempts a proof first along the lines of Theorem 15.4 and discovers where the difficulties arise.

Let x_0 be a Lebesgue point and write
$$F(t) = \int_0^t |f(x_0+\tau) + f(x_0-\tau) - 2f(x_0)|\, d\tau.$$
The function F is absolutely continuous and
$$F'(t) = |f(x_0+\tau) + f(x_0-\tau) - 2f(x_0)|$$
for a.e. value of t. The integral
$$\int_T |F'(t)|\, dt = M < \infty$$
and, since x_0 is a Lebesgue point for f, we know that $F(t)/t \to 0$ as $t \to 0+$.

As before, the representation of $\sigma_n(f,x)$ in (3) allows us to write

$$|\sigma_n(f,x_0) - f(x_0)| \leq \frac{1}{\pi}\int_0^\pi F'(t)K_n(t)\,dt \tag{4}$$

and we show that this is small for large n by splitting the integral over $[0,\pi]$ into integrals over three subintervals

$$[0,n^{-1}],\ [n^{-1},n^{-1/4}],\ \text{and}\ [n^{-1/4},\pi].$$

In our earlier proof the intervals chosen were independent of n, but a more delicate version of this argument is now needed.

The integral

$$\int_{n^{-1/4}}^{\pi} F'(t)K_n(t)\,dt$$

is small for large n because, using property (6) of Theorem 15.3, it is smaller than

$$\frac{\pi}{n+1}\int_{n^{-1/4}}^{\pi} F'(t)t^{-2}\,dt \leq \frac{M\pi}{(n+1)(n^{-1/4})^2},$$

and certainly this tends to zero as $n \to \infty$.

The integral

$$\int_0^{n^{-1}} F'(t)K_n(t)\,dt$$

is small for large n because, using using property (5) of Theorem 15.3, it is smaller than

$$\frac{n+1}{2}\int_0^{n^{-1}} F'(t)\,dt = \frac{n+1}{2}F(n^{-1}),$$

and this tends to zero as $n \to \infty$ since, as noted, $F(t)/t \to 0$ as $t \to 0+$.

Finally, the integral

$$\int_{n^{-1}}^{n^{-1/4}} F'(t)K_n(t)\,dt \tag{5}$$

can be seen to be small for large n after some computations. First, using property (6) of Theorem 15.3 and an integration by parts, we see it is smaller than

$$\begin{aligned}
&\frac{\pi}{n+1}\int_{n^{-1}}^{n^{-1/4}} F'(t)t^{-2}\,dt \\
&\quad = \frac{\pi}{n+1}\left(\frac{F(n^{-1/4})}{(n^{-1/4})^2} - \frac{F(n^{-1})}{(n^{-1})^2}\right) + \frac{2\pi}{n+1}\int_{n^{-1}}^{n^{-1/4}} \frac{F(t)}{t}t^{-2}\,dt.
\end{aligned}$$

Both of the terms

$$\frac{F(n^{-1/4})}{(n+1)(n^{-1/4})^2}, \quad \frac{F(n^{-1})}{(n+1)(n^{-1})^2}$$

tend to zero as $n \to \infty$ because $F(t)/t \to 0$ as $t \to 0+$. The term involving the integral can be handled by noting that

$$\int_{n^{-1}}^{n^{-1/4}} \frac{F(t)}{t} t^{-2}\, dt$$

$$\leq \left(\int_{n^{-1}}^{n^{-1/4}} t^{-2}\, dt \right) \sup \left\{ F(t)/t : t \in \left[n^{-1}, n^{-1/4} \right] \right\}$$

and the integral

$$\int_{n^{-1}}^{n^{-1/4}} t^{-2}\, dt \leq \int_{n^{-1}}^{\infty} t^{-2}\, dt = n.$$

Again,

$$\sup\{F(t)/t : t \in [n^{-1}, n^{-1/4}]\}$$

is small for large n because $F(t)/t \to 0$ as $t \to 0+$. Putting these together, we find that (5) tends to zero for $n \to \infty$ as required.

These three integrals have now been handled. We conclude that the expression in (4) also tends to zero for $n \to \infty$ and the proof is complete. ■

The same methods show that the convergence can be taken as uniform if the function is continuous on all of T and 2π-periodic.

Theorem 15.6 (Fejér) *Let f be continuous and 2π-periodic. Then*

$$\lim_{n\to\infty} \sigma_n(f, x) = f(x)$$

uniformly.

Exercises

15:4.1♢ Let $f \in L_1(T)$. Prove that for $\lim_{n\to\infty} \sigma_n(f, x_0) = s$ it is necessary and sufficient that for some $\delta > 0$ it is true that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^{\delta} \left(f(x_0 + t) + f(x_0 - t) - 2s\right) \frac{\sin^2(\frac{1}{2}nt)}{t^2}\, dt = 0.$$

(Compare this with Corollary 15.14.)

15.5 The Fourier Coefficients

We are now in a technical position to establish some facts concerning the Fourier coefficients. While the Fourier series of an arbitrary function $f \in L_1(T)$ need not converge, there is still something that can be said about the series: the terms go to zero. This was first proved by Riemann for some integrable functions and then extended by Lebesgue to all integrable functions. The proof is quite elementary once we know that the trigonometric functions are dense in $C(T)$. Even so, it is a most useful result about the Fourier coefficients and should be remembered.

Theorem 15.7 (Riemann–Lebesgue) *Let $f \in L_1(T)$, and let*

$$c_j = c_j(f)$$

denote the Fourier coefficients of f. Then

$$\lim_{|j|\to\infty} c_j = 0.$$

Proof. Let $\varepsilon > 0$. There is a trigonometric polynomial $P \in L_1(T)$ so that $\|f - P\| < \varepsilon$. If N is the degree of the polynomial P, then certainly

$$\frac{1}{2\pi}\int_T P(t)e^{-ijt}\,dt = 0$$

for all $|j| > N$. Consequently,

$$|c_j| = \frac{1}{2\pi}\left|\int_T f(t)e^{-ijt}\,dt\right| = \frac{1}{2\pi}\left|\int_T (f(t) - P(t))\,e^{-ijt}\,dt\right|$$
$$\leq \frac{1}{2\pi}\int_T |f(t) - P(t)|\,dt = \|f - P\| < \varepsilon$$

for all $|j| > N$, and this proves the theorem. ∎

In the exercises we shall ask the reader to carry through on some computations needed in applications of the Riemann–Lebesgue theorem. In particular, we need to obtain zero limits for expressions such as

$$\int_a^b f(t)\sin(n + \tfrac{1}{2})t\,dt$$

as occur in using the Dirichlet kernel.

Having obtained the Riemann–Lebesgue theorem, we ask now whether a converse is available. Let $\sum_j c_j e^{ijt}$ be a given trigonometric series. In order that this be the Fourier series of some function, then certainly, because of the Riemann–Lebesgue theorem, a necessary condition is that the coefficients tend to zero. This is not sufficient: there must exist many such sequences that are not the Fourier coefficients of a function in $L_1(T)$. An interesting proof of this can be based on the open mapping principle of Section 12.13, and we present this in Theorem 15.9.

First, we dispense with a uniqueness problem in this regard. Can two functions have the same Fourier series? If the two functions agree almost everywhere, then certainly the Fourier series are identical. The next theorem asserts that only in this case can this happen.

Theorem 15.8 *Let f, $g \in L_1(T)$, and let*

$$f \sim \sum_j c_j e^{ijt} \quad \textit{and} \quad g \sim \sum_j d_j e^{ijt}$$

be the two Fourier series. If, for all j, $c_j = d_j$, then $f = g$ almost everywhere [i.e., $f = g$ in the space $L_1(T)$].

Proof. To prove the theorem, it is enough to subtract f and g and obtain a function h all of whose Fourier coefficients are zero. Let $\sigma_n(h, x)$ be the Cesàro means for the Fourier series of h. Then, by Theorem 15.12, $\sigma_n(h, x)$ converges to h in $L_1(T)$. But since all the coefficients vanish, so too does $\sigma_n(h, x)$, and consequently h is the zero element of $L_1(T)$, as required. ■

To place our next theorem in the setting of Banach spaces, consider the mapping $f \to \hat{f}$, where $f \in L_1(T)$ and $\hat{f}$ is the function defined on the integers $\mathbb{Z}$ by

$$\hat{f}(j) = c_j(f),$$

so $\hat{f}$ is the just the sequence of Fourier coefficients of f. The space $c_0(\mathbb{Z})$ of all complex sequences $c = \{c_j\}_{-\infty}^{\infty}$ with $c_j \to 0$ as $|j| \to \infty$ is a Banach space with its usual supremum norm

$$\|c\|_\infty = \sup_j |c_j|.$$

The open mapping theorem applied to an appropriate mapping on these spaces shows that there must exist sequences in $c_0(\mathbb{Z})$ that are not the Fourier coefficients of any integrable function.

Theorem 15.9 *The mapping $f \to \hat{f}$ from $L_1(T)$ into $c_0(\mathbb{Z})$ is a continuous, one-one linear mapping that is not onto.*

Proof. If Γ denotes the mapping taking $f \to \hat{f}$, it is trivial to verify that the mapping is linear since it is defined by an integration. We verify first that $\|\Gamma\| = 1$. The constant function $f_0(t) = 1$ provides an example of a function with $\|f_0\| = 1$ and $\|\hat{f}\|_\infty = 1$, since $\hat{f}(0) = 1$ and $\hat{f}(j) = 0$ if $j \neq 1$. This shows that $\|\Gamma\| \geq 1$. On the other hand, for all $j \in \mathbb{Z}$,

$$|\hat{f}(j)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt}\, dt \right| \leq \|f\|_1$$

so that $\|\Gamma\| \leq 1$. That Γ is one-one is precisely the content of Theorem 15.8, just proved.

Finally, we show that Γ is not onto by invoking the open mapping theorem (Theorem 12.53). If, contrary to what we claim, Γ is onto, then the inverse exists and is continuous. The sequence D_n of Dirichlet kernels can be used to see that this is impossible. Each $D_n \in L_1(T)$. For each n the sequence $\hat{D}_n$ of Fourier coefficients is in the unit ball of $c_0(\mathbb{Z})$: an obvious computation shows that each of the Fourier coefficients of D_n is either $1/2$ or 0. The inverse Γ^{-1}, if it did exist, would have to map that unit ball into a bounded set, which it cannot do because $\|D_n\|_1 \to \infty$ (Theorem 15.2). ■

Exercises

15:5.1 Let $f \in L_1(T)$. Show that, for any interval $[a, b]$,

$$\lim_{|j|\to\infty} \int_a^b f(t)e^{-ijt}\,dt = 0.$$

[Hint: If $[a, b] \subset [-\pi, \pi]$, apply the Riemann–Lebesgue theorem to the function $f\chi_{[a,b]}$.]

15:5.2 Let $f \in L_1(T)$. Show that

$$\lim_{n\to\infty} \int_a^b f(t)\sin nt\,dt = \lim_{n\to\infty} \int_a^b f(t)\cos nt\,dt = 0.$$

15:5.3◇ Let $f \in L_1[a, b]$. Show that

$$\lim_{n\to\infty} \int_a^b f(t)\sin(n + \tfrac{1}{2})t\,dt = 0.$$

[Hint: Try some trigonometric identities.]

15:5.4◇ Let $0 < \delta < \pi$ and $f \in L_1[\delta, \pi]$. Show that

$$\lim_{n\to\infty} \int_\delta^\pi f(t)D_n(t)\,dt = 0.$$

[Hint: The function $\csc(t/2)$ is bounded on this interval, and so $f(t)\csc(t/2)$ is integrable there.]

15.6 Weierstrass Approximation Theorem

Fejér's theorem allows us to conclude that the trigonometric polynomials are dense in most of the spaces with which we are concerned. This is a good excuse for us to pause to harvest some results. Also, it is useful to draw a parallel between the denseness of the trigonometric polynomials and the famous Weierstrass approximation theorem asserting that continuous functions on a compact interval can be uniformly approximated by ordinary polynomials. The reader will have seen other proofs of this, for example in Section 9.13. The proof we present here shows a rather nice connection between approximations using trigonometric polynomials and approximations using ordinary polynomials.

Theorem 15.10 *Let f be a continuous, 2π-periodic, complex-valued function, and let $\varepsilon > 0$. Then there is a trigonometric polynomial $g(x)$ so that*

$$|f(x) - g(x)| < \varepsilon$$

for all x.

Proof. If f is a continuous, 2π-periodic complex-valued function, then, by Theorem 15.6, for large enough n the Cesàro means $\sigma_n(f)$ are uniformly close to f. Thus not only can we approximate f by a trigonometric polynomial, we can even do it explicitly (although we have not determined the degree). ■

To obtain the Weierstrass theorem from trigonometric polynomial approximation takes only a few ideas, interesting in themselves.

Theorem 15.11 (Weierstrass approximation) *Let f be a continuous function on an interval $[a, b]$, and let $\varepsilon > 0$. Then there is a polynomial*

$$g(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

so that

$$|f(x) - g(x)| < \varepsilon$$

for all $x \in [a, b]$.

Proof. There is nothing special about the interval $[a, b]$ for the purposes of the theorem, since an affine transformation can take $[a, b]$ into any interval, and polynomials transform into polynomials. There is something special about $[0, 1]$ for our proof, so we take it instead.

Let f be a continuous, real or complex function on $[0, 1]$, let $\varepsilon > 0$, and write $F(t) = f(|\cos t|)$. Then F is a continuous, 2π-periodic function and can be approximated by a trigonometric polynomial within ε. Since F is even [i.e., $F(t) = F(-t)$] we can figure out what form that trigonometric polynomial may take: we can find a_0, a_1, a_2, $\ldots a_n$ so that

$$\left| F(t) - \sum_0^n a_j \cos jt \right| < \varepsilon \tag{6}$$

for all t. Each $\cos jt$ can be written using elementary trigonometric identities as $T_j(\cos t)$ for some jth order (ordinary) polynomial T_j, and so, by setting $x = \cos t$ for any $x \in [0, 1]$, we have

$$\left| f(x) - \sum_0^n a_j T_j(x) \right| < \varepsilon,$$

which is exactly the polynomial approximation that we need. ■

The polynomials T_j that appear in the proof are well known as the Tchebychev polynomials and are easily generated (see Exercise 15:6.2).

As another application of these ideas let us note that the Cesàro means can also be used as approximations in other spaces.

Theorem 15.12 *Let $f \in L_p(T)$ $(1 \le p < \infty)$. Then*

$$\lim_{n\to\infty} \|\sigma_n(f) - f\|_p = 0.$$

Proof. Let

$$F(t) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+t) - f(x)|^p \, dx\right)^{1/p}.$$

We know that $F(t) \to 0$ as $t \to 0$, and so, by Theorem 15.4, it follows that $\sigma_n(F, 0) \to 0$.

With this fact we can prove that the sequence $\sigma_n(f)$ converges to f in $L_p(T)$. We use the usual representation of the Cesàro means to get

$$|\sigma_n(f,x) - f(x)| \le \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+t) - f(x)|\, K_n(t)\, dt$$

and the version of Minkowski's inequality for integrals obtained in Exercise 13:1.4 to get

$$\begin{aligned}
\|\sigma_n(f) - f\|_p &\le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+t) - f(x)|\, K_n(t)\, dt\right)^p \text{@!@!@!@!@!@!@!}dx\right)^{1/p} \\
&\le \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+t) - f(x)|^p \, dx\right)^{1/p} \text{@!@!@!@!@!@!@!}K_n(t)\, dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} F(t) K_n(t)\, dt = \sigma_n(F, 0) \to 0.
\end{aligned}$$

Since $\sigma_n(F, 0) \to 0$ as $n \to \infty$, we have our desired result. ■

Exercises

15:6.1 Check that the approximating polynomial in (6) can be written in the form as stated (cf. Exercise 15:1.9).

15:6.2 Define the Tchebychev polynomials by requiring T_j to be a polynomial so that

$$\cos jt = T_j(\cos t)$$

identically. Show that $T_0(x) = 1$, $T_1(x) = x$, and

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

Generate the first few of these polynomials.

15.7 Pointwise Convergence: Jordan's Test

The most natural question and, it might seem, the most important that we should now ask is for situations in which $s_n(f)$ converges pointwise or uniformly to f. Indeed, much of the nineteenth-century discussion of Fourier series centered on this convergence problem. The problem turned out to be difficult, subtle, and interesting. But its importance was overstated. Indeed, it is important for a Fourier series to "converge" back to the function, but pointwise convergence is not important for applications—maybe even it is unimportant. We know of many ways of interpreting convergence of functions (e.g., convergence in mean, convergence in measure, L_p-convergence) that might be better suited to the problem. One of the main difficulties with pointwise convergence we have seen many times: a representation of a function as a pointwise convergent series does not necessarily allow further operations on the series, such as differentiation and integration.

Even so, for historical reasons and for its intrinsic interest, we shall look at the situation regarding pointwise convergence of Fourier series in this section and the next two sections. The ideas prove to be challenging. They may not be essential to an exclusively practical development of the subject, but they lead us in important directions.

First, we obtain a formal requirement for convergence of a Fourier series of an integrable function f. We know (Exercise 15:2.2) that

$$s_n(f, x_0) - s = \frac{1}{\pi} \int_0^{\pi} \left(f(x_0+t) + f(x_0-t) - 2s\right) D_n(t)\, dt. \qquad (7)$$

Split the integral into $\int_0^{\delta}$ and $\int_{\delta}^{\pi}$ and consider the latter part: Exercise 15:5.4 shows that

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{\delta}^{\pi} \left(f(x_0+t) + f(x_0-t) - 2s\right) D_n(t)\, dt = 0,$$

and consequently we have just obtained a formal, but interesting, necessary and sufficient condition for the convergence of the series.

Theorem 15.13 *Let $f \in L_1(T)$. In order that*

$$\lim_{n\to\infty} s_n(f, x_0) = s$$

it is necessary and sufficient that for some $\delta > 0$ it is true that

$$\lim_{n\to\infty} \int_0^{\delta} \left(f(x_0+t) + f(x_0-t) - 2s\right) D_n(t)\, dt = 0.$$

This theorem can assume another form, which may be more suggestive. (Compare with Exercise 15:4.1.)

Corollary 15.14 *Let $f \in L_1(T)$. In order that $\lim_{n\to\infty} s_n(f, x_0) = s$, it is necessary and sufficient that for some $\delta > 0$ it is true that*

$$\lim_{n\to\infty} \int_0^\delta (f(x_0+t) + f(x_0-t) - 2s)\, \frac{\sin(n+\frac{1}{2})t}{t}\, dt = 0.$$

Proof. Note first that the function

$$h(t) = \frac{1}{t} - \frac{1}{2}\csc\left(\frac{t}{2}\right)$$

is bounded on $(0, \delta)$. Fix x_0, s and write

$$F(t) = f(x_0+t) + f(x_0-t) - 2s.$$

We recall that the Dirichlet kernel assumes the form

$$D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} = \frac{1}{2}\sin\left((n+\tfrac{1}{2})t\right)\csc\tfrac{1}{2}t$$

and so we have

$$\int_0^\delta F(t)D_n(t)\, dt = \int_0^\delta F(t)t^{-1}\sin(n+\tfrac{1}{2})t\, dt$$

$$+ \int_0^\delta F(t)h(t)\sin(n+\tfrac{1}{2})t\, dt.$$

The function $F(t)h(t)$ is integrable on $[0, \delta]$ because h is bounded. From Exercise 15:5.3, we see that the second integral on the right must converge to zero as $n \to \infty$. This shows that the criterion of Theorem 15.13 is equivalent to that stated here. ■

The criterion we now present is due to Jordan, but all the pointwise theory owes a debt first of all to Dirichlet, who was the first to find methods that rigorously establish conditions under which a Fourier series converges to its function. Jordan's version includes Dirichlet's.

Theorem 15.15 (Jordan) *Suppose that $f \in L_1(T)$ is of bounded variation on some neighborhood of a point x_0. Then*

$$s_n(f, x_0) \to \tfrac{1}{2}\left(f(x_0+0) + f(x_0-0)\right).$$

Proof. First, since the function f has bounded variation in some interval $(x_0-\delta, x_0+\delta)$, both its real and imaginary parts have bounded variation there, too. Thus we can reduce the argument to the situation in which f is real valued. In that case both the right- and left-hand limits $f(x_0+0)$ and $f(x_0-0)$ exist. Define

$$F(t) = f(x_0+t) + f(x_0-t) - f(x_0+0) - f(x_0-0).$$

By Corollary 15.14, our theorem is proved if we can show that

$$\lim_{n\to\infty}\int_0^\delta F(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt = 0.$$

But F is also of bounded variation on $(0,\delta)$ and so can be split into the sum $F = G + H$, where G, H are nonnegative, nondecreasing functions with $\lim_{t\to 0+} G(t) = \lim_{t\to 0+} H(t) = 0$. Thus we can complete the proof by showing that

$$\lim_{n\to\infty}\int_0^\delta G(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt = \lim_{n\to\infty}\int_0^\delta H(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt = 0.$$

The argument for G will do. Let $\varepsilon > 0$. Since $\lim_{t\to 0+} G(t) = 0$, there is a $0 < \delta_1 < \delta$ so that $G(\delta_1) < \varepsilon$. Then

$$\int_0^\delta G(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt = \int_0^{\delta_1} G(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt + \int_{\delta_1}^{\delta} G(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt.$$

We show that there is a number M (independent of ε) so that

$$\left|\int_0^{\delta_1} G(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt\right| < M\varepsilon, \tag{8}$$

and we know (from Exercise 15:5.3) that

$$\lim_{n\to\infty}\int_{\delta_1}^\delta G(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt = 0,$$

since G is integrable and the rest of the integrand is bounded in $[\delta_1, \delta]$. Thus the proof is obtained from (8) since ε is arbitrary.

This argument needed here is at the level of advanced calculus. The second mean-value theorem for integrals shows that there must be some point $0 < \delta_2 < \delta_1$ so that

$$\int_0^{\delta_1} G(t)\frac{\sin(n+\frac{1}{2})t}{t}\,dt = G(\delta_1 - 0)\int_{\delta_2}^{\delta_1}\frac{\sin(n+\frac{1}{2})t}{t}\,dt,$$

where $G(\delta_1 - 0) < \varepsilon$. The integral on the right side of this inequality is the same as

$$\int_{(n+1/2)\delta_2}^{(n+1/2)\delta_1}\frac{\sin\tau}{\tau}\,d\tau$$

with an appropriate change of variables. The well-known existence of the improper integral $\int_0^\infty (\sin\tau)/\tau\,d\tau$ guarantees the existence of a number M

for which $\left|\int_a^b (\sin\tau)/\tau\, d\tau\right| \le M$ for all a, b. This gives us (8), and the proof is complete. ■

We have mentioned (e.g., Sections 1.18 and 5.5) that mathematicians of the early nineteenth century often integrated series of functions term by term without justification. In Chapter 5 we provided a number of conditions under which term-by-term integration is permissible. In our setting of Fourier series (using Lebesgue integration), term by term integration is actually always justified, even when the series is not known to converge anywhere. We use Jordan's theorem (Theorem 15.15) to prove this. The real version is given here because it is the one most frequently used in applications and the most recognizable.

Theorem 15.16 *Let $f \in L_1(T)$ be a real function with a Fourier series*

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]. \tag{9}$$

Then, for any interval $[\alpha, \beta]$,

$$\int_\alpha^\beta f(t)\, dt = \left[\frac{a_0 t}{2}\right]_\alpha^\beta + \sum_{n=1}^{\infty} \left[\frac{a_n \sin nt - b_n \cos nt}{n}\right]_\alpha^\beta,$$

that is, the integral can be obtained by integrating (9) term by term, and this series converges.

Proof. As usual, we consider that f is extended periodically with period 2π to $\mathbb{R}$. Let

$$F(x) = \int_0^x [f(t) - \frac{a_0}{2}]\, dt. \tag{10}$$

Then F is absolutely continuous (and therefore continuous and of bounded variation) on every bounded interval. It follows directly from the periodicity of f that F is also periodic with period 2π. By Theorem 15.15, the Fourier series of F converges to F everywhere on $\mathbb{R}$:

$$F(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos nx + B_n \sin nx\right). \tag{11}$$

We first show that, for every $n \ge 1$, $A_n = -b_n/n$ and $B_n = a_n/n$, as would be true if we were allowed term-by-term integration of the Fourier series for $f - \frac{1}{2}a_0$. Integrating by parts (Exercise 7:9.2) and noting that $F' = f$ a.e., we obtain

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^{2\pi} F(x)\cos nx\, dx \\ &= \frac{1}{\pi}\frac{F(x)\sin nx}{n}\Big|_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi} F'(x)\sin nx\, dx \\ &= -\frac{1}{n\pi}\int_0^{2\pi} f(x)\sin nx\, dx = -\frac{b_n}{n}. \end{aligned}$$

We find, similarly, that $B_n = a_n/n$. Thus we can write (11) as

$$F(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \frac{1}{n}\left[a_n \sin nx - b_n \cos nx\right]. \tag{12}$$

Now, from (10) we see that $F(0) = 0$ and, from (11), that

$$F(0) = \frac{1}{2}A_0 - \sum_{n=1}^{\infty} \frac{b_n}{n}.$$

Thus

$$\frac{1}{2}A_0 = \sum_{n=1}^{\infty} \frac{b_n}{n}. \tag{13}$$

Substituting (13) into (12), we obtain

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} \frac{1}{n}\left[a_n \sin nx + b_n(1 - \cos nx)\right] \\ &= \sum_{n=1}^{\infty} \left[a_n \int_0^x \cos nt\, dt + b_n \int_0^x \sin nt\, dt\right]. \end{aligned} \tag{14}$$

Comparing (14) with (10), we find that

$$\int_0^x \left(f(t) - \frac{a_0}{2}\right) dt = \sum_{n=1}^{\infty} \left[a_n \int_0^x \cos nt\, dt + b_n \int_0^x \sin nt\, dt\right].$$

From this, with $x = \beta$ and $x = \alpha$, the theorem now follows. ■

Exercises

15:7.1 Deduce from Jordan's theorem the original theorem of Dirichlet: *A function with at most finitely many simple discontinuities and only a finite number of maxima and minima has a Fourier series that converges everywhere, to the function at the points of continuity and to the average between the left and right limits at a discontinuity.*

15:7.2 The identity (13), $\frac{1}{2}A_0 = \sum_1^\infty b_n/n$ (as well as the entire proof of Theorem 15.16) did not require that the Fourier series of f converge. We can use this fact to provide an example of an everywhere convergent trigonometric series that is not the Fourier series of a function in $L_1(T)$. (See the paragraph following Definition 15.1.)

(a) Show that the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\ln n}$$

converges everywhere.

Suppose that there exists $f \in L_1(T)$ such that

$$f \sim \sum_{n=2}^{\infty} \frac{\sin nx}{\ln n}. \tag{15}$$

(b) Use Theorem 15.16 and (13) to show that, in the notation of that theorem,

$$F(x) = \sum_{n=2}^{\infty} \frac{1}{n \ln n}(1 - \cos nx).$$

(c) Show that the series in (b) diverges at $x = \pi$.

(d) Conclude that the series in (a) is not the Fourier series of f, contradicting (15).

15:7.3 State and prove a uniform version of Jordan's theorem, that is, conditions under which the Fourier series of f converges to f uniformly.

15.8 Pointwise Convergence: Dini's Test

We can continue with the theme of the pointwise convergence of Fourier series almost without end. The literature is filled with special cases of convergence theorems, some very deep and difficult. We pause to prove just one more. This test, due to Dini, is of a different character from that of Jordan; in fact, the two tests are incomparable in the sense that either one may give a convergence result when the other fails (see Exercises 15:8.1 and 15:8.2).

Theorem 15.17 (Dini) *Let $f \in L_1(T)$, and suppose that, at a point $x_0 \in T$,*

$$\int_0^{\pi} \left| \frac{f(x_0+t) + f(x_0-t)}{2} - s \right| \frac{dt}{t} < \infty$$

for some number s. Then $s_n(f, x_0) \to s$.

Proof. Fix x_0, s and write

$$F(t) = \frac{1}{2}\left(f(x_0+t) + f(x_0-t) - 2s\right).$$

The function $F(t)t^{-1}$ is integrable by hypothesis, and so, from Exercise 15:5.3, we see that the integral

$$\int_0^{\delta} F(t)t^{-1} \sin(n + \tfrac{1}{2})t \, dt$$

tends to zero as $n \to \infty$. By Corollary 15.14, it follows that

$$s_n(f, x_0) \to s,$$

as required. ■

From Dini's theorem we can deduce some of the earliest of the convergence theorems in the study of Fourier series.

Corollary 15.18 *Let $f \in L_1(T)$, and suppose that f has a finite derivative at a point $x_0 \in T$. Then $s_n(f, x_0) \to f(x_0)$.*

Corollary 15.19 *Let $f \in L_1(T)$, and suppose that f has finite one-sided derivatives at a point $x_0 \in T$ in the sense that*

$$\lim_{h \to 0+} \frac{f(x_0 + h) - f(x_0 + 0)}{h}$$

and

$$\lim_{h \to 0+} \frac{f(x_0 - h) - f(x_0 - 0)}{-h}$$

both exist. Then $s_n(f, x_0) \to \frac{1}{2}(f(x_0 + 0) + f(x_0 - 0))$.

Generally, if the Fourier series of a function f converges at a point x_0 we expect (or rather hope) it will converge to the value $f(x_0)$. If x_0 is a point of continuity then this might occur. If x_0 is a simple discontinuity (either removable or a jump discontinuity) we normally expect convergence to the average at the jump; that is, $\frac{1}{2}(f(x_0+0)+f(x_0-0))$. This behavior is already apparent in Dini's theorem and its corollaries: in order for the integral there to be finite, the value s must be taken as $f(x_0)$ in the case of a point of continuity and $\frac{1}{2}(f(x_0+0)+f(x_0-0))$ at a simple discontinuity. If we relax our concerns to merely a.e. convergence, then we do not quite need continuity to get convergence to the function values. The L_1-modulus of continuity, defined as

$$\omega_1(f, x_0) = \frac{1}{2\pi} \int_T |f(x_0 + t) - f(x_0)| \, dt$$

can be used to obtain a simple but useful criterion.

Theorem 15.20 (Marcinkiewicz) *Let $f \in L_1(T)$, and suppose*

$$\int_0^\pi \omega_1(f, t) \frac{dt}{t} < \infty.$$

Then $s_n(f, x) \to f(x)$ for almost every $x \in T$.

Proof. Let

$$F(x) = \int_0^\pi |f(x + t) - f(x)| \, t^{-1} \, dt$$

and use Tonelli's theorem to evaluate the iterated integral

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) \, dx &= \int_0^\pi \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x + t) - f(x)| \, dx \right) t^{-1} \, dt \\ &= \int_0^\pi \omega_1(f, t) \frac{dt}{t}, \end{aligned}$$

which integral is finite, by hypothesis. It follows that the integrand $F(x)$ must be finite for almost every $x \in T$. But at every point x with $F(x) < \infty$ we have, in particular, that

$$\int_0^\pi \left| \frac{f(x+t)+f(x-t)}{2} - f(x) \right| \frac{dt}{t} < \infty,$$

and so, by Dini's theorem, $s_n(f,x) \to f(x)$ for almost every $x \in T$, as required. ■

Exercises

15:8.1 Let $f(-t) = f(t) = |\log(t/2\pi)|^{-1}$ for $0 < t \leq \pi$ and $f(0) = 0$. Check that Jordan's test applies at $t = 0$, but that Dini's does not.

15:8.2 Let $f(-t) = f(t) = \sqrt{t} \sin t^{-1}$ for $0 < t \leq \pi$ and $f(0) = 0$. Check that Dini's test applies at $t = 0$, but that Jordan's test does not.

15.9 Pointwise Divergence

The progression of thought in the nineteenth century was toward the conclusion that the Fourier series of any continuous function should converge pointwise back to the function. All results seemed to lead in that direction as sharper and sharper criteria were developed to establish convergence. Thus it must have been a shock when du Bois-Reymond in 1876 produced an example of a continuous function whose Fourier series diverged at a point.

In this section we look at this negative result for continuous functions. We show, as did du Bois-Reymond, that there is a continuous function with a Fourier series diverging somewhere. By an interesting application of a category argument (using the Banach–Steinhaus theorem), we get a lot more from this. We get a continuous function with a Fourier series diverging at every point of a dense $\mathcal{G}_\delta$ subset of T. Indeed, we get even more: the *typical* function in $C(T)$ has this property. (Here we are using "typical" in the sense of Section 10.1.)

Theorem 15.21 *Let $C(T)$ denote the Banach space of continuous, 2π-periodic functions equipped with the supremum norm. Then the typical function in $C(T)$ has the following property: the Fourier series of f diverges at the points of a dense $\mathcal{G}_\delta$ set in T.*

Proof. We construct a dense $\mathcal{G}_\delta$ set $E \subset C(T)$ so that, for each $f \in E$,

$$S(f,x) = \sup_n |s_n(f,x)| = +\infty$$

at the points of a dense $\mathcal{G}_\delta$ set in T.

Let us begin by working at just one point, taken for convenience as $t = 0$. Define the continuous linear functionals Γ_n on $C(T)$ by

$$\Gamma_n(f) = s_n(f, 0).$$

We need to compute the norms of the functionals Γ_n. Since

$$\Gamma_n(f) = s_n(f, 0) = \frac{1}{\pi} \int_T f(t) D_n(t)\, dt,$$

it is easy to see that

$$|\Gamma_n(f)| \leq \frac{1}{\pi} \int_T |f(t)||D_n(t)|\, dt \leq \|f\|_\infty \left(\frac{1}{\pi} \int_T |D_n(t)|\, dt \right).$$

Thus $\|\Gamma_n\| \leq L_n$, where L_n are the Lebesgue constants defined in Theorem 15.2. We wish to show that

$$\|\Gamma_n\| = L_n.$$

Fix n. Define a function g by $g(t) = 1$ at points t for which $D_n(t) \geq 0$ and as $g(t) = -1$ at points t for which $D_n(t) < 0$. While D_n is continuous, g is not. Even so, we can approximate g uniformly on T by a continuous function and, in fact, construct a sequence of continuous functions f_m with $|f_m| \leq 1$ so that $f_m \to g$ uniformly. Thus

$$\Gamma_n(f_m) = \frac{1}{\pi} \int_T f_m(t) D_n(t)\, dt \to \frac{1}{\pi} \int_T g(t) D_n(t)\, dt$$

$$= \frac{1}{\pi} \int_T |D_n(t)|\, dt = L_n.$$

This shows that $\|\Gamma_n\| \geq L_n$ and so, from what we have already proved, we have the identity $\|\Gamma_n\| = L_n$. Recall now that in Theorem 15.2 we showed that the L_n are unbounded.

By the uniform boundedness principle (Theorem 12.48), it follows that there is at least one function f in $C(T)$ for which $\Gamma_n(f)$ is unbounded. Thus we have obtained a function f whose Fourier series diverges at 0; in fact, $\sup_n |s_n(f, 0)| = \infty$.

Let $\{t_1, t_2, t_3, \dots\}$ be any sequence of points in T (which we shall choose to be dense) and define the functionals

$$\Gamma_{mn}(f) = s_n(f, t_m).$$

As before, since there was nothing special about the point 0, there is some function $f_m \in C(T)$ for which

$$\limsup_{n \to \infty} |\Gamma_{mn}(f_m)| = \infty.$$

Then it follows directly from Theorem 12.50 that the set of members $f \in C(T)$ for which

$$\limsup_{n\to\infty} |\Gamma_{mn}(f)| = \infty \quad (\text{all } m = 1, 2, 3, \dots) \tag{16}$$

is a dense $\mathcal{G}_\delta$ in $C(T)$. Any f for which (16) holds satisfies

$$S(f, t_m) = \infty$$

for every t_m. But the function $S(f,t)$, being the supremum of a sequence of continuous functions, is lower semicontinuous. (See Exercise 5:5.2.) Hence the set

$$\{t \in T : S(f,t) = \infty\}$$

is a $\mathcal{G}_\delta$ subset of T that is also dense (since it contains each of the points t_m). This proves the theorem. ■

Theorem 15.21 shows that for most continuous functions the Fourier series diverges on a large set; but a dense $\mathcal{G}_\delta$, while large in one sense, might also have measure zero. So this theorem does not answer the question as to the almost everywhere convergence of the Fourier series of a continuous function. This problem from 1915 was known as Lusin's problem and the answer remained elusive for a very long time. In 1965, it was solved by L. Carleson: he showed that the Fourier series of every function in $L_2(T)$ does converge almost everywhere. Since then, R. A. Hunt has extended this to all functions in $L_p(T)$ for $p > 1$. The case $p = 1$ must be excluded. Kolmogorov in 1926 gave a function in $L_1(T)$ whose Fourier series diverges *everywhere.*

Exercises

15:9.1 Assume that there is a function in $L_1(T)$ whose Fourier series diverges almost everywhere. Show that the typical function in $L_1(T)$ has a Fourier series diverging almost everywhere.

15.10 Characterizations

How can we recognize that a trigonometric series is a Fourier series and, if so, the Fourier series of what function? We know from Theorem 15.8 that a series can correspond to at most one function (up to equivalences), but we do not know how to recognize a Fourier series at sight. The next two theorems provide some solutions to the problem of characterizing Fourier series of functions in certain classes.

Theorem 15.22 *Let $\sum_j c_j e^{ijt}$ be a trigonometric series. A necessary and sufficient condition that it be the Fourier series of a continuous, 2π-periodic function f is that the sequence $\sigma_n(x)$ of Cesàro means converge uniformly to f.*

Proof. One direction is clear by Theorem 15.4.

Suppose that the sequence $\sigma_n(x)$ of Cesàro means of the given series converge uniformly to a function f. Certainly, f is continuous and 2π-periodic. We wish to show that each $c_j = c_j(f)$. As in Exercise 15:3.6, an elementary computation shows that

$$\sigma_n(x) = \sum_{|j|\leq n} \left(1 - \frac{|j|}{n+1}\right) c_j e^{ijx},$$

and this allows us to compute the Fourier coefficients of f. Fix j and note that for all $n > |j|$ this identity and the orthogonality relations show that

$$\frac{1}{2\pi}\int_T \sigma_n(t)e^{-ijt}\,dt = \left(1 - \frac{|j|}{n+1}\right) c_j.$$

Let $n \to \infty$ in this statement. The right-hand side converges to c_j, and the left-hand side converges to

$$\frac{1}{2\pi}\int_T f(t)e^{-ijt}\,dt = c_j(f),$$

since σ_n converges uniformly to f. Thus each coefficient $c_j = c_j(f)$, as required. ■

We characterize, also, those trigonometric series that are Fourier series of a function $f \in L_p(T)$ $(1 < p < \infty)$. This offers an interesting application of some of our weak compactness ideas developed in Section 13.10. The case $p = 1$ requires different treatment, and the stated result does not extend to this end of the scale.

Theorem 15.23 (Fejér) *Let $\sum_j c_j e^{ijt}$ be a trigonometric series. A necessary and sufficient condition that it be the Fourier series of a function $f \in L_p(T)$ $(1 < p < \infty)$ is that the sequence $\sigma_n(x)$ of Cesàro means is bounded in $L_p(T)$; that is, $\|\sigma_n\|_p \leq M$ for some real M and all n. In that case, it is also true that $\|f\|_p \leq M$.*

Proof. Let us suppose first that $\sigma_n(x) = \sigma_n(f, x)$ is the sequence of Cesàro means of a function in $f \in L_p(T)$. We use the usual representation of the Cesàro means to get

$$|\sigma_n(f,x)| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+t)|\, K_n(t)\,dt$$

and Minkowski's inequality for integrals (Exercise 13:1.4) to get

$$\begin{aligned}\|\sigma_n(f)\|_p &\leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x+t)|\,K_n(t)\,dt\right)^p dx\right)^{1/p} \\ &\leq \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+t)|^p\,dx\right)^{1/p} K_n(t)\,dt = \|f\|_p.\end{aligned}$$

This gives the inequality $\|\sigma_n\|_p \leq \|f\|_p$, so that the sequence $\sigma_n(x)$ of Cesàro means is bounded in L_p.

In the other direction, suppose that the sequence σ_n is bounded in $L_p(T)$. By weak compactness (Theorem 13.33), there must be a subsequence σ_{n_k} and an element $f \in L_p(T)$ so that σ_{n_k} converges to f weakly. In particular, since each function e^{-ijt} is a member of $L_\infty(T)$, each is also in the dual space $L_q(T)$. It follows that

$$\lim_{k\to\infty} \frac{1}{2\pi}\int_T \sigma_{n_k}(t)e^{-ijt}\,dt = \frac{1}{2\pi}\int_T f(t)e^{-ijt}\,dt = c_j(f).$$

We can compute this integral, using the elementary identity (Exercise 15:3.6)

$$\sigma_{n_k}(x) = \sum_{|s|\leq n_k}\left(1 - \frac{|s|}{n_k+1}\right)c_s e^{isx}$$

and the orthogonality relations to obtain

$$\lim_{k\to\infty} \frac{1}{2\pi}\int_T \sigma_{n_k}(t)e^{-ijt}\,dt = \lim_{k\to\infty}\left(1 - \frac{|j|}{n_k+1}\right)c_j = c_j,$$

so $c_j = c_j(f)$. Thus the series is indeed the Fourier series of f. We know from Theorem 15.12 that $\|\sigma_n - f\|_p \to 0$, so $\|\sigma_n\|_p \to \|f\|_p$, and it follows that $\|f\|_p \leq M$ since each $\|\sigma_n\|_p \leq M$. ■

We state our final theorem of this section without proof just to indicate to the interested reader how the situation develops at the $p = 1$ end of the scale.

Theorem 15.24 *Let $\sum_j c_j e^{ijt}$ be a trigonometric series. A necessary and sufficient condition that it be the Fourier series of a function $f \in L_1(T)$ is that the sequence $\sigma_n(x)$ of Cesàro means be Cauchy in L_1@, @,; that is, $\|\sigma_n - \sigma_m\|_1 \to 0$ as n, $m \to \infty$.*

15.11 Fourier Series in Hilbert Space

We have come far enough in our study of Fourier series to see a number of complexities and technical difficulties. In a simpler world, we might have hoped that all functions would have Fourier series that converge back to the function. We would have hoped to be able to recognize immediately when a trigonometric series is a Fourier series. The geometry describing the relation between a function and its Fourier series would be transparent and familiar.

In the Hilbert space $L_2(T)$, all these things are true and more. It is in this setting that the most satisfying, simple, and complete theory is available. In fact, in this setting, the belief of many nineteenth-century mathematicians that an "arbitrary" function could be represented as the sum of its Fourier series is realized.

Since we have already established the groundwork for this in Chapter 14 we can obtain almost all our results as applications of what we now know. The reader who has preferred to skip over Chapter 14 can still prove all these statements without extracting more than a few arguments from that chapter. The exercises sketch this out.

In $L_2(T)$, we use the inner product

$$(f, g) = \frac{1}{2\pi} \int_T f(t)\overline{g(t)}\, dt,$$

which is just the usual inner product adjusted with a constant. The norm then is, as elsewhere throughout this chapter,

$$\|f\|_2 = \sqrt{(f, f)} = \left(\frac{1}{2\pi} \int_T |f(t)|^2\, dt\right)^{1/2}.$$

We know that this is a Hilbert space, and in this setting we see that the trigonometric system plays a special and recognizable role. Once we have established that this system is maximal (in the formal sense required in Hilbert space), all our results follow from the simple general theory of Chapter 14 and no further proofs are needed.

15.25 (Maximality of the trigonometric system) *The functions*

$$t \to e^{ijt} \quad (j = 0, \pm 1, \pm 2, \dots)$$

form a maximal orthonormal system in $L_2(T)$.

Proof. Let e_j denote the functions $t \to e^{ijt}$ for $j \in \mathbb{Z}$; then the Fourier series is just the series $\sum_j (f, e_j)e_j$ in this notation, and it is clear from elementary computations that the system is orthogonal and, indeed, orthonormal. The fact that the trigonometric polynomials are dense in the separable Hilbert space $L_2(T)$ allows us to conclude directly from Theorem 14.25 that this is a maximal orthonormal system, as required. ■

We generally expect that for a function f the trigonometric polynomial $s_n(f)$ is somehow an approximation to f and that as $n \to \infty$ these approximations get closer to f. That vague geometric picture is not correct in all settings: in the Hilbert space setting, it is precisely correct. Not only is the polynomial $s_n(f)$ an approximation to f, it is among all approximations of this type the best.

15.26 (Best approximation in $L_2(T)$) *For any $f \in L_2(T)$ and any integer n, the best approximation to f by a trigonometric polynomial of degree n is $s_n(f, x)$@, @,; that is,*

$$\left\| f - \sum_{|j| \leq n} c_j(f) e^{ijt} \right\|_2 \leq \left\| f - \sum_{|j| \leq n} \lambda_j e^{ijt} \right\|_2$$

for any complex numbers $\lambda_1, \lambda_2, \dots, \lambda_n$.

The first main result is that the Fourier series of any function in $L_2(T)$ converges back to the function, provided that the convergence is interpreted in the $L_2(T)$ sense itself.

15.27 (Convergence of the Fourier series) *For any function f in the space $L_2(T)$,*

$$\lim_{n\to\infty} \|f - s_n(f)\|_2 = 0.$$

We also obtain Parseval's identity, which, if it is examined closely, is just a form of the Pythagorean theorem. It is named after Marc Antoine Parseval-Deschènes who stated it in 1799, although a full proof would not appear for a great many years.

15.28 (Parseval's identity) *For any $f \in L_2(T)$,*

$$\|f\|_2^2 = \frac{1}{2\pi}\int_T |f(t)|^2\,dt = \sum_{j=-\infty}^{\infty} |c_j(f)|^2.$$

There is also a more general version of Parseval's identity, which goes under the same name or is sometimes referred to as the *polarized* version.

15.29 (Parseval's identity) *For any $f,\ g \in L_2(T)$,*

$$(f,g) = \frac{1}{2\pi}\int_T f(t)\overline{g(t)}\,dt = \sum_{j=-\infty}^{\infty} c_j(f)\overline{c_j(g)}.$$

Finally, to complete the picture, we would hope that to any given trigonometric series of a recognizable type there is a unique function whose Fourier series it is. Up to this point we do not have, however, many theorems of this type. Perhaps the only moderately satisfying one is that for uniformly convergent trigonometric series there is a unique continuous function for which it is the Fourier series. In the Hilbert space setting, the geometry is clear.

15.30 (Riesz–Fischer theorem) *Suppose that*

$$\sum_{j=-\infty}^{\infty} |c_j|^2 < \infty.$$

Then there is a unique $f \in L_2(T)$ so that $c_j(f) = c_j$.

Exercises

15:11.1 (For readers who have skipped Chapter 14.) Give direct proofs for the material of this section that do not depend on any general development.

(a) Establish the maximality of the trigonometric system: if g is an element of $L_2(T)$ and g is orthogonal to every function e^{ijt} for $j \in \mathbb{Z}$, then $g = 0$. [Hint: Use the fact that $\sigma_n(g) = 0$ and $\sigma_n \to g$.]

(b) Establish the best-approximation result. [Hint: Theorem 14.22 gives the pattern.]

(c) Establish the L_2-convergence of the Fourier series. [Hint: Use the fact that $\|f - s_n(f)\|_2 \le \|f - \sigma_n(f)\|_2$, and that the latter tends to zero.]

(d) Establish Parseval's identity. [Hint: Compute $(s_n(f), s_n(g))$ and show that it converges to (f, g).]

(e) Prove the Riesz–Fischer theorem. [Hint: Show that the sequence $s_n = \sum_{|j|<n} c_j e^{ijt}$ is Cauchy in $L_2(T)$ if $\sum_j |c_j|^2$ converges.]

15.12 Riemann's Theorems

We turn now to some problems in the general study of trigonometric series. In Riemann's investigation of series of the form

$$f(t) = \sum_j c_j e^{ijt} \tag{17}$$

where the coefficients are bounded, he was led to a study of the function

$$R(t) = \frac{c_0 t^2}{2} - \sum_{j \neq 0} \frac{c_j}{j^2} e^{ijt}$$

obtained by formally integrating the series (17) twice. If we know that the coefficients are bounded, then this series converges uniformly to a continuous, 2π-periodic function R called the *Riemann function* for the series. The properties of the limits

$$\lim_{h \to 0+} \frac{R(t+h) + R(t-h) - 2R(t)}{h} \tag{18}$$

and

$$\lim_{h \to 0+} \frac{R(t+h) + R(t-h) - 2R(t)}{h^2} \tag{19}$$

constitute what is commonly called the Riemann theory of trigonometric series.

The mere existence of the limit in (18) forces that limit to be zero. A function with this property at a point t is said to be *smooth* at t. Riemann's second theorem (Theorem 15.33) says that the Riemann function is uniformly smooth. The limit in (19) is often called the *second-order symmetric derivative* and we use the terminology $\mathrm{SD}_2\, R(t)$ for this derivative. The second symmetric derivative is a generalization of the ordinary second derivative.

Theorem 15.31 *If $f''(t)$ exists, then so too does the second symmetric derivative, and both have the same value.*

Proof. By an application of the mean-value theorem of the calculus, the fraction

$$\frac{f(t+h)+f(t-h)-2f(t)}{h^2}$$

must have the value

$$\frac{f'(t+k)-f'(t-k)}{2k}$$

for some $0<k<h$. Letting $h\to 0$, $k\to 0$, we obtain

$$\frac{f'(t+k)-f'(t-k)}{2k}\to f''(t),$$

which justifies our assertion. ■

Riemann's first theorem (Theorem 15.32) asserts that the second symmetric derivative of the Riemann function for a trigonometric series recovers the sum of the series at any point at which the series converges. This should make sense: we have integrated the series twice to obtain the Riemann function, so presumably two derivatives should return us back to where we started, but the second derivative must be interpreted in a generalized sense.

Theorem 15.32 (Riemann's first theorem)
Let R denote the Riemann function for the trigonometric series (17), where $c_j\to 0$ as $|j|\to\infty$. If the series converges to a finite sum s at a point t, then $SD_2\, R(t)=s$.

Proof. Let $A_j(t)=c_je^{ijt}$, and compute the second symmetric difference quotient at t using the increment $2h$:

$$\frac{A_j(t+2h)+A_j(t-2h)-2A_j(t)}{(2h)^2}=\frac{1}{h^2}\left(\frac{e^{2ijh}+e^{-2ijh}-2}{4}\right)A_j(t).$$

The expression inside parentheses can be written as $\frac{1}{2}(\cos(2jh)-1)$ or $\sin^2(jh)$. These simple computations lead us to an expression for the second symmetric difference quotient at t of the Riemann function,

$$\frac{R(t+2h)+R(t-2h)-2R(t)}{4h^2}$$
$$=A_0+\sum_{|j|=1}^{\infty}A_j(t)\left(\frac{\sin jh}{jh}\right)^2, \tag{20}$$

which will prove most useful. Write

$$s_n = A_0 + \sum_{|j|=1}^{n} A_j(t)$$

so that, by hypothesis, $s_n \to s$.

The proof is obtained by showing that, for any sequence $h_k \searrow 0$,

$$\lim_{k\to\infty} \frac{R(t+2h_k) + R(t-2h_k) - 2R(t)}{4h_k^2} = s,$$

and the theorem is proved. Let

$$u(\theta) = \left(\frac{\sin\theta}{\theta}\right)^2$$

with $u(0) = 1$. Then we compute, using summation by parts, that

$$\sum_{j=0}^{N} A_j u(jh_k) = \sum_{j=0}^{N-1} s_j \left(u(jh_k) - u((j+1)h_k)\right) + s_N u(Nh_k).$$

We write

$$a_{kj} = u(jh_k) - u((j+1)h_k)$$

and we let $N \to \infty$ in this last identity; so, using (20), we have finally

$$\frac{R(t+2h_k) + R(t-2h_k) - 2R(t)}{4h_k^2} = \sum_{j=0}^{\infty} s_j a_{kj}. \tag{21}$$

At this stage our perspective changes, and we begin to view the problem as one of a summability method. The sequence s_n converges to s, and $\sum_{j=0}^{\infty} s_j a_{kj}$ is a transformation of that sequence using a matrix (a_{kj}). If the matrix is regular in the sense of Definition 12.51, then Theorem 12.52 shows that

$$\lim_{k\to\infty} \sum_{j=0}^{\infty} s_j a_{kj} = s$$

and we are done. Thus we must verify each of the three conditions of that definition; that is,

1. $\sup_k \sum_{j=0}^{\infty} |a_{kj}| < \infty$,
2. $\lim_{k\to\infty} a_{kj} = 0$ for each $j = 0, 1, 2, 3, \ldots$, and
3. $\lim_{k\to\infty} \sum_{j=0}^{\infty} a_{kj} = 1$.

We recall that

$$a_{kj} = u(jh_k) - u((j+1)h_k) = \left(\frac{\sin(jh_k)}{jh_k}\right)^2 - \left(\frac{\sin((j+1)h_k)}{(j+1)h_k}\right)^2.$$

From this we see that condition (2) must hold. Certainly, this expression tends to zero for large k since we know that $h_k \to 0$ and that $\sin\theta/\theta \to 1$ as $\theta \to 0$.

We next compute the sum

$$\sum_{j=0}^{N} a_{kj} = \sum_{j=0}^{N} u(jh_k) - u((j+1)h_k) = u(0) - u((N+1)h_k).$$

From this we conclude that $\sum_{j=0}^{\infty} a_{kj} = 1$ for all k, so condition (3) holds, too.

Finally, to verify condition (1), we write

$$\sum_{j=0}^{\infty} |a_{kj}| = \sum_{j=0}^{\infty} |u(jh_k) - u((j+1)h_k)|$$
$$= \sum_{j=0}^{\infty} \left| \int_{jh_k}^{(j+1)h_k} u'(\theta)\,d\theta \right| \le \int_0^{\infty} |u'(\theta)|\,d\theta.$$

This last integral is finite. There is no problem near 0 and, for large θ, the integrand is smaller than $4\theta^{-2}$. This checks condition (1), and the proof is complete. ■

Theorem 15.33 (Riemann's second theorem)
Let R denote the Riemann function for the trigonometric series in (17), where $c_j \to 0$ as $|j| \to \infty$. Then

$$\lim_{h\to 0+} \frac{R(t+h) + R(t-h) - 2R(t)}{h} = 0$$

uniformly in t.

Proof. This statement may be viewed as a regularity claim for a summability method as in the proof of Theorem 15.32. We omit the proof since it introduces no new ideas. See Exercise 15:12.3. ■

Exercises

15:12.1 Show that the second symmetric derivative arises when one integrates a first symmetric derivative. If

$$\lim_{s\to 0} \frac{f(t_0+s) - f(t_0-s)}{2s} = c,$$

if f is integrable, and if F is an indefinite integral of f, then

$$\mathrm{SD}_2\, F(t_0) = c.$$

15:12.2 A sequence $\{s_n\}$ is said to be summable (R) to a limit s if

$$\lim_{h\to 0}\sum_{j=1}^{\infty} s_j\left(\frac{\sin jh}{jh}\right)^2 = s.$$

Show that if $\{s_n\}$ converges to s then $\{s_n\}$ is summable (R) to s. [Hint: This is a matter of interpreting the proof of Theorem 15.32.]

15:12.3 A sequence $\{s_n\}$ is said to be summable (R') to a limit s if

$$\lim_{h\to 0}\frac{2}{\pi}\sum_{j=1}^{\infty} s_j\left(\frac{\sin^2 jh}{j^2 h}\right) = s.$$

Show that if $\{s_n\}$ converges to s then $\{s_n\}$ is summable (R) to s. [Hint: Study the proof of Theorem 15.32. You may use the identity

$$\frac{\pi}{2} = \frac{\theta}{2} + \sum_{j=1}^{\infty}\frac{\sin^2 j\theta}{j^2\theta}$$

for $0<\theta<\pi/2$.]

15.13 Cantor's Uniqueness Theorem

One of the most interesting and immediate applications of Riemann's theory of trigonometric series is to the problem of uniqueness. The problem was posed by Eduard Heine (1821–1881) and presented to Cantor in 1870 as an important, unsolved problem.

> Can a function have two different representations as the sum of convergent trigonometric series?

Curiously, this problem would not have been considered unsolved a few decades earlier. Term-by-term integration of series was performed routinely without justification, and such a method solves this problem easily. Heine had shown that for *uniform* convergence there was uniqueness, but recognized that for pointwise convergence the problem was more delicate. It is also instructive to note that this problem might not be considered important in our century: pointwise representations of functions are not considered to have much merit. Even so, Cantor's solution of the uniqueness problem stands as a landmark in the history of mathematics, not so much for the depth of the methods themselves, but for the fact that it marked the beginning of a research program for Cantor that culminated in the development of modern set theory and the theory of transfinite ordinals.

By subtracting two such series, we see that the problem as posed reduces to asking whether there can exist a trigonometric series

$$f(t) = \sum_j c_j e^{ijt} \tag{22}$$

converging everywhere to 0, but whose coefficients are not all zero. Cantor's solution has four simple steps:

1. If the trigonometric series (22) converges everywhere, then the coefficients $c_j \to 0$ as $|j| \to \infty$.

2. Take then the Riemann function R for the series; by Riemann's first theorem (Theorem 15.32), $\mathrm{SD}_2\, R(t) = 0$ everywhere.

3. Prove that any continuous function F with $\mathrm{SD}_2\, F(t) = 0$ everywhere must be linear.

4. Show that the coefficients vanish from the relation

$$R(t) = \alpha t + \beta = \frac{c_0 t^2}{2} - \sum_{j \neq 0} \frac{c_j}{j^2}\, e^{ijt}.$$

The first part Cantor proved himself. (Next we shall present a version of this theorem with a later improvement added by Lebesgue.) The second part is Riemann's and would have been well known at the time. The last part is easy, since the series converges uniformly. It was the third step that blocked Cantor's progress. In a letter, mailed to Cantor from Zurich, Hermann Schwarz (1843–1921) supplied the proof that any continuous function F with $\mathrm{SD}_2\, F(t) = 0$ everywhere must be linear.

We begin with the first step in the program, now known as the Cantor–Lebesgue theorem. Cantor's original version needed convergence on an interval, and the argument required eight pages of very delicate arguments. We are in a position now to prove better results more easily because of the tools of measure theory. Notice that it is Egoroff's theorem that does much of the work here.

Theorem 15.34 (Cantor–Lebesgue) *Suppose that a trigonometric series $f(t) = \sum_j c_j e^{ijt}$ converges everywhere on a set of positive Lebesgue measure. Then the coefficients $c_j \to 0$ as $|j| \to \infty$.*

Proof. Let

$$A_j(x) = c_j e^{ijt} + c_{-j} e^{-ijt}.$$

What we show is the more general statement that if $A_j(x) \to 0$ on a set E of positive Lebesgue measure then $c_j,\ c_{-j} \to 0$ as $j \to \infty$. This is the version most likely described as the Cantor–Lebesgue theorem. By Egoroff's theorem, there is a set $F \subset [-\pi, \pi]$ of positive measure on which $A_j(x) \to 0$ uniformly.

Write

$$\rho_j = \sqrt{|c_j|^2 + |c_{-j}|^2},$$

and assume for the moment that ρ_j, and hence also c_j and c_{-j}, are bounded. One easily shows that

$$|A_j(x)|^2 = |c_j|^2 + |c_{-j}|^2 + c_j \overline{c_{-j}} e^{2ijt} + + \overline{c_j} c_{-j} e^{-2ijt},$$

and multiplication by χ_F and integration yield

$$\begin{aligned}
&\int_{-\pi}^{\pi} \chi_F(x)|A_j(x)|^2\,dx \\
&\quad= \rho_j^2 \int_{-\pi}^{\pi} \chi_F(x)\,dx + (c_j\overline{c_{-j}})\left(\int_{-\pi}^{\pi} \chi_F(x)e^{2ijt}\,dx\right) \\
&\qquad + \overline{c_j}c_{-j}\left(\int_{-\pi}^{\pi} \chi_F(x)e^{-2ijt}\,dx\right).
\end{aligned}$$

The second and third expressions on the right tend to 0 as $j \to \infty$ by the Riemann–Lebesgue theorem (Theorem 15.7) (remembering that we are assuming c_j, c_{-j} bounded). Thus

$$\int_{-\pi}^{\pi} \chi_F(x)|A_j(x)|^2\,dx = \rho_j^2|F| + \alpha_j$$

where $\alpha_j \to 0$ as $j \to \infty$. But $|F| > 0$ and

$$\chi_F(x)|A_j(x)|^2$$

converges to zero uniformly. It follows that ρ_j tends to zero as required.

In our proof we have assumed that the sequence ρ_j is bounded. Exercise 15:13.1 shows how to convert the unbounded case to this, and so the proof is complete. ■

We now pass to the third step in Cantor's program, the theorem mailed to him by Schwarz. The proof uses recognizably nineteenth-century methods.

Theorem 15.35 (Schwarz) *Let F be continuous on the interval $[a,b]$ and suppose that*

$$SD_2\,F(x) = 0$$

at every point of (a,b). Then F is linear in this interval.

Proof. Let $\varepsilon > 0$, and define the functions

$$G(x) = F(x) - F(a) - \frac{F(b)-F(a)}{b-a}(x-a) + \varepsilon(x-a)(x-b) \tag{23}$$

and

$$H(x) = F(x) - F(a) - \frac{F(b)-F(a)}{b-a}(x-a) - \varepsilon(x-a)(x-b).$$

We prove that $G(x) \leq 0$ everywhere in $[a,b]$. If not, then, because $G(a) = G(b) = 0$, there is a point $c \in (a,b)$ at which a positive maximum is attained. At such a point the derivative $\mathrm{SD}_2\,G(c)$ cannot be positive, and yet this would contradict the fact that

$$\mathrm{SD}_2\,F(c) \geq 0.$$

An identical proof establishes that $H(x) \geq 0$ everywhere in $[a, b]$.

This gives

$$\left| F(x) - F(a) - \frac{F(b) - F(a)}{b - a}(x - a) \right| \leq \varepsilon (b - a)^2$$

for all $\varepsilon > 0$. From this, the linearity of F follows. ■

Exercises

15:13.1 Show that if there is a counterexample to Theorem 15.34 with

$$A_j(x) = c_j e^{ijt} + c_{-j} e^{-ijt},$$

$A_j(x) \to 0$ on a set E of positive Lebesgue measure, and

$$\rho_j = \sqrt{|c_j|^2 + |c_{-j}|^2}$$

not tending to zero then there is a counterexample for which ρ_j is bounded. [Hint: If some sequence j_k has $\rho_{j_k} \geq \varepsilon > 0$, then define

$$B_j(x) = (\rho_j)^{-1} A_j(x)$$

for $j = j_k$ and zero if $j \neq j_k$.]

15:13.2 If the relation

$$\alpha x + \beta = \frac{a_0 x^2}{4} - \sum_{n=1}^{\infty} \frac{(a_n \cos nx + b_n \sin nx)}{n^2}$$

holds at every point and the coefficients of the series are bounded show that all the coefficients vanish.

15:13.3 One of Cantor's first steps after proving the uniqueness theorem was to show that a single point may be omitted. Show that a trigonometric series $f(t) = \sum_j c_j e^{ijt}$ converging everywhere to 0 except possibly at a single point must have all its coefficients zero. [Hint: Generalize the Schwarz theorem by allowing a single exceptional point at which the function is smooth and use Theorem 15.33.]

15.14 Additional Problems for Chapter 15

15:14.1 Obtain a proof of Parseval's theorem by obtaining first the inequality

$$\|\sigma_n(f)\|_2^2 = \sum_{|j| \leq n} \left(1 - \frac{|j|}{n+1}\right)^2 |c_j|^2 \leq \sum_{|j| \leq n} |c_j|^2$$

and using Bessel's inequality and an appropriate convergence theorem.

15:14.2 (Denjoy–Lusin) If $\sum_{j=1}^{\infty} |a_j \cos jt + b_j \sin jt| < \infty$ for all t in a measurable set of positive measure, then the series converges uniformly and absolutely everywhere; in fact, $\sum_{j=1}^{\infty} (|a_j| + |b_j|) < \infty$. [Hint: Rewrite $a_j \cos jt + b_j \sin jt$ as $r_j \cos(nt - \theta_j)$, where $a_j = r_j \cos \theta_j$ and $b_j = r_j \sin \theta_j$. Find a set E of positive measure on which $\sum_j |r_j \cos(nt - \theta_j)|$ converges uniformly using Egoroff's theorem. Show that

$$\liminf_{j\to\infty} \int_E |\cos(nt - \theta_j)|\, dt > 0.$$

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