PROOF THEORY

GAISI TAKEUTI

Professor of Mathematics
The University of Illinois
Urbana, Illinois, U.S.A.

1975

NORTH-HOLLAND PUBLISHING COMPANY - AMSTERDAM · OXFORD
AMERICAN ELSEVIER PUBLISHING COMPANY, INC. - NEW YORK

Library of Congress Catalog Card Number 75-23164
North-Holland ISBN S 0 7204 2200 0
0 7204 2277 9
American Elsevier ISBN 0 444 10492 5

Published by:
North-Holland Publishing Company – Amsterdam
North-Holland Publishing Company, Ltd. – Oxford

Sole distributors for the U.S.A. and Canada:
American Elsevier Publishing Company, Inc.
52 Vanderbilt Avenue
New York, N.Y. 10017

Library of Congress Cataloging in Publication Data

Takeuti, Gaisi, 1926-
Proof theory.

(Studies in logic and the foundations of mathematics ; v. 81)
1. Proof theory. I. Title. II. Series.
QA9.54.T34 1975 511'.3 75-23164
ISBN 0-444-10492-5 (American Elsevier)

PRINTED IN THE NETHERLANDS

PREFACE

This book is based on a series of lectures that I gave at the Symposium on Intuitionism and Proof Theory held at Buffalo in the summer of 1968. Lecture notes, distributed at the Buffalo symposium, were prepared with the help of Professor John Myhill and Akiko Kino. Mariko Yasugi assisted me in revising and extending the original notes. This revision was completed in the summer of 1971. At this point Jeffery Zucker read the first three chapters, made improvements, especially in Chapter 2, and my colleague Wilson Zaring provided editorial assistance with the final draft of Chapters 4–6.

To all who contributed, including our departmental secretaries, who typed versions of the material for use in my classes, I express my deep appreciation.

Gaisi Takeuti

Urbana, March 1975

CHAPTER 1

FIRST ORDER PREDICATE CALCULUS

In this chapter we shall present Gentzen's formulation of the first order predicate calculus **LK** (logischer klassischer Kalkül), which is convenient for our purposes. We shall also include a formulation of institutionistic logic, which is known as **LJ** (logischer intuitionistischer Kalkül). We then proceed to the proofs of the cut-elimination theorems for **LK** and **LJ**, and their applications.

§1. Formalization of statements

The first step in the formulation of a logic is to make the formal language and the formal expressions and statements precise.

DEFINITION 1.1. A first order (formal) language consists of the following symbols.

1) *Constants*:
   1.1) Individual constants: $k_0, k_1, \ldots, k_j, \ldots$ $(j = 0, 1, 2, \ldots)$.
   1.2) Function constants with i argument-places $(i = 1, 2, \ldots)$: $f^i_0, f^i_1, \ldots, f^i_j, \ldots$ $(j = 0, 1, 2, \ldots)$.
   1.3) Predicate constants with i argument-places $(i = 0, 1, 2, \ldots)$: $R^i_0, R^i_1, \ldots, R^i_j, \ldots$ $(j = 0, 1, 2, \ldots)$.
2) *Variables*:
   2.1) Free variables: $a_0, a_1, \ldots, a_j, \ldots$ $(j = 0, 1, 2, \ldots)$.
   2.2) Bound variables: $x_0, x_1, \ldots, x_j, \ldots$ $(j = 0, 1, 2, \ldots)$.
3) *Logical symbols*:
   $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\supset$ (implies), $\forall$ (for all) and $\exists$ (there exists). The first four are called propositional connectives and the last two are called quantifiers.

4) *Auxiliary symbols*:
   (,) and , (comma).

We say that a first order language L is given when all constants are given. In every argument, we assume that a language L is fixed, and hence we omit the phrase "of L".

There is no reason why we should restrict the cardinalities of various kinds of symbols to exactly $\aleph_0$. It is, however, a standard approach in elementary logic to start with countably many symbols, which are ordered with order type ω. Therefore, for the time being, we shall assume that the language consists of the symbols as stated above, although we may consider various other types of language later on. In any case it is essential that each set of variables is infinite and there is at least one predicate symbol. The other sets of constants can have arbitrary cardinalities, even 0.

We shall use many notational conventions. For example, the superscripts in the symbols of 1.2) and 1.3) are mostly omitted and the symbols of 1) and 2) may be used as meta-symbols as well as formal symbols. Other letters such as $g, h, \ldots$ may be used as symbols for function constants, while $a, b, c, \ldots$ may be used for free variables and $x, y, z, \ldots$ for bound variables.

Any finite sequence of symbols (from a language L) is called an *expression* (of L).

DEFINITION 1.2. *Terms* are defined inductively (recursively) as follows:

1) Every individual constant is a term.

2) Every free variable is a term.

3) If f^i is a function constant with i argument-places and $t_1, \ldots, t_i$ are terms, then $f^i(t_1, \ldots, t_i)$ is a term.

4) Terms are only those expressions obtained by 1)–3). Terms are often denoted by $t, s, t_1, \ldots$.

Since in proof theory inductive (recursive) definitions such as Definition 1.2 often appear, we shall not mention it each time. We shall normally omit the last clause which states that the objects which are being defined are only those given by the preceding clauses.

DEFINITION 1.3. If R^i is a predicate constant with i argument-places and $t_1, \ldots, t_i$ are terms, then $R^i(t_1, \ldots, t_i)$ is called an *atomic formula*. *Formulas* and their outermost logical symbols are defined inductively as follows:

1) Every atomic formula is a formula. It has no outermost logical symbol.

2) If A and B are formulas, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$ and $(A \supset B)$ are formulas. Their outermost logical symbols are $\neg$, $\wedge$, $\vee$ and $\supset$, respectively.

3) If A is a formula, a is a free variable and x is a bound variable not occurring in A, then $\forall x\, A'$ and $\exists x\, A'$ are formulas, where A' is the expression obtained from A by writing x in place of a at each occurrence of a in A. Their outermost logical symbols are $\forall$ and $\exists$, respectively.

4) Formulas are only those expressions obtained by 1)–3).

Henceforth, $A, B, C, \ldots, F, G, \ldots$ will be metavariables ranging over formulas. A formula without free variables is called a *closed formula* or a *sentence*. A formula which is defined without the use of clause 3) is called *quantifier-free*. In 3) above, A' is called the *scope* of $\forall x$ and $\exists x$, respectively.

When the language L is to be emphasized, a term or formula in the language L may be called an L-*term* or L-*formula*, respectively.

Remark. Although the distinction between free and bound variables is not essential, and is made only for technical convenience, it is extremely useful and simplifies arguments a great deal. This distinction will, therefore, be maintained unless otherwise stated.

It should also be noticed that in clause 3) of Definition 1.3, x must be a variable which does not occur in A. This eliminates expressions such as $\forall x\, (C(x) \wedge \exists x\, B(x))$. This restriction does not essentially narrow the class of formulas, since e.g. this expression $\forall x\, (C(x) \wedge \exists x\, B(x))$ can be replaced by $\forall y\, (C(y) \wedge \exists x\, B(x))$, preserving the meaning. This restriction is useful in formulating formal systems, as will be seen later.

In the following we shall omit parentheses whenever the meaning is evident from the context. In particular the outermost parentheses will always be omitted. For the logical symbols, we observe the following convention of priority: the connective $\neg$ takes precedence over each of $\wedge$ and $\vee$, and each of $\wedge$ and $\vee$ takes precedence over $\supset$. Thus $\neg A \wedge B$ is short for $(\neg A) \wedge B$, and $A \wedge B \supset C \vee D$ is short for $(A \wedge B) \supset (C \vee D)$. Parentheses are omitted also in the case of double negations: for example $\neg\neg A$ abbreviates $\neg(\neg A)$. $A \equiv B$ will stand for $(A \supset B) \wedge (B \supset A)$.

Definition 1.4. Let A be an expression, let $\tau_1, \ldots, \tau_n$ be distinct primitive symbols, and let $\sigma_1, \ldots, \sigma_n$ be any symbols. By

$$\left(A\,\frac{\tau_1,\ldots,\tau_n}{\sigma_1,\ldots,\sigma_n}\right)$$

we mean the expression obtained from A by writing $\sigma_1,\ldots,\sigma_n$ in place of $\tau_1,\ldots,\tau_n$, respectively, at each occurrence of $\tau_1,\ldots,\tau_n$ (where these symbols are replaced simultaneously). Such an operation is called the (*simultaneous*) *replacement of* $(\tau_1,\ldots,\tau_n)$ *by* $(\sigma_1,\ldots,\sigma_n)$ *in* A. It is not required that $\tau_1,\ldots,\tau_n$ actually occur in A.

PROPOSITION 1.5. (1) *If A contains none of $\tau_1,\ldots,\tau_n$, then*

$$\left(A\,\frac{\tau_1,\ldots,\tau_n}{\sigma_1,\ldots,\sigma_n}\right)$$

is A itself.

(2) *If $\sigma_1,\ldots,\sigma_n$ are distinct primitive symbols, then*

$$\left(\left(A\,\frac{\tau_1,\ldots,\tau_n}{\sigma_1,\ldots,\sigma_n}\right)\frac{\sigma_1,\ldots,\sigma_n}{\theta_1,\ldots,\theta_n}\right)$$

is identical with

$$\left(A\,\frac{\tau_1,\ldots,\tau_n}{\theta_1,\ldots,\theta_n}\right).$$

DEFINITION 1.6. (1) Let A be a formula and $t_1,\ldots,t_n$ be terms. If there is a formula B and n distinct free variables $b_1,\ldots,b_n$ such that A is

$$\left(B\,\frac{b_1,\ldots,b_n}{t_1,\ldots,t_n}\right),$$

then for each i $(1 \leqslant i \leqslant n)$ the occurrences of t_i resulting from the above replacement are said to be indicated in A, and this fact is also expressed (less accurately) by writing B as $B(b_1,\ldots,b_n)$, and A as $B(t_1,\ldots,t_n)$. A may of course contain some other occurrences of t_i; this happens if B contains t_i.

(2) We say that a term t is fully indicated in A, or every occurrence of t in A is indicated, if every occurrence of t is obtained by such a replacement (from some formula B as above, with $n = 1$ and $t = t_1$).

It should be noted that the formula B and the free variables from which A can be obtained by replacement are not unique; the indicated occurrences of some terms of A are specified relative to such a formula B and such free variables.

Proposition 1.7. *If $A(a)$ is a formula (in which a is not necessarily fully indicated) and x is a bound variable not occurring in $A(a)$, then $\forall x\, A(x)$ and $\exists x\, A(x)$ are formulas.*

Proof. By induction on the number of logical symbols in $A(a)$.

In the following, let Greek capital letters $\Gamma, \Delta, \Pi, \Lambda, \Gamma_0, \Gamma_1, \ldots$ denote finite (possibly empty) sequences of formulas separated by commas. In order to formulate the sequential calculus, we must first introduce an auxiliary symbol $\rightarrow$.

Definition 1.8. For arbitrary Γ and Δ in the above notation, $\Gamma \rightarrow \Delta$ is called a *sequent*. Γ and Δ are called the *antecedent* and *succedent*, respectively, of the sequent and each formula in Γ and Δ is called a *sequent-formula*.

Intuitively, a sequent $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n$ (where $m, n \geqslant 1$) means: if $A_1 \wedge \ldots \wedge A_m$, then $B_1 \vee \ldots \vee B_n$. For $m \geqslant 1$, $A_1, \ldots, A_m \rightarrow$ means that $A_1 \wedge \ldots \wedge A_m$ yields a contradiction. For $n \geqslant 1$, $\rightarrow B_1, \ldots, B_n$ means that $B_1 \vee \ldots \vee B_n$ holds. The empty sequent $\rightarrow$ means there is a contradiction. Sequents will be denoted by the letter S, with or without subscripts.

§2. Formal proofs and related concepts

Definition 2.1. An *inference* is an expression of the form

$$\frac{S_1}{S} \quad \text{or} \quad \frac{S_1 \quad S_2}{S},$$

where S_1, S_2 and S are sequents. S_1 and S_2 are called the *upper sequents* and S is called the *lower sequent* of the inference.

Intuitively this means that when S_1 (S_1 and S_2) is (are) asserted, we can infer S from it (from them). We restrict ourselves to inferences obtained from the following rules of inference, in which $A, B, C, D, F(a)$ denote formulas.

1) Structural rules:

1.1) *Weakening*:

$$\text{left:} \quad \frac{\Gamma \rightarrow \Delta}{D, \Gamma \rightarrow \Delta}; \qquad \text{right:} \quad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, D}.$$

D is called the weakening formula.

1.2) *Contraction*:

$$\text{left: } \frac{D, D, \Gamma \to \Delta}{D, \Gamma \to \Delta}; \qquad \text{right: } \frac{\Gamma \to \Delta, D, D}{\Gamma \to \Delta, D}.$$

1.3) *Exchange*:

$$\text{left: } \frac{\Gamma, C, D, \Pi \to \Delta}{\Gamma, D, C, \Pi \to \Delta}; \qquad \text{right: } \frac{\Gamma \to \Delta, C, D, \Lambda}{\Gamma \to \Delta, D, C, \Lambda}$$

We will refer to these three kinds of inferences as "weak inferences", while all others will be called "strong inferences".

1.4) *Cut*:

$$\frac{\Gamma \to \Delta, D \qquad D, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}.$$

D is called the *cut formula* of this inference.

2) Logical rules:

2.1) $$\neg : \text{left: } \frac{\Gamma \to \Delta, D}{\neg D, \Gamma \to \Delta}; \qquad \neg : \text{right: } \frac{D, \Gamma \to \Delta}{\Gamma \to \Delta, \neg D}.$$

D and $\neg D$ are called the *auxiliary formula* and the *principal formula*, respectively, of this inference.

2.2) $$\wedge : \text{left: } \frac{C, \Gamma \to \Delta}{C \wedge D, \Gamma \to \Delta} \quad \text{and} \quad \frac{D, \Gamma \to \Delta}{C \wedge D, \Gamma \to \Delta};$$

$$\wedge : \text{right: } \frac{\Gamma \to \Delta, C \qquad \Gamma \to \Delta, D}{\Gamma \to \Delta, C \wedge D}.$$

C and D are called the auxiliary formulas and $C \wedge D$ is called the principal formula of this inference.

2.3) $$\vee : \text{left: } \frac{C, \Gamma \to \Delta \quad D, \Gamma \to \Delta}{C \vee D, \Gamma \to \Delta};$$

$$\vee : \text{right: } \frac{\Gamma \to \Delta, C}{\Gamma \to \Delta, C \vee D} \quad \text{and} \quad \frac{\Gamma \to \Delta, D}{\Gamma \to \Delta, C \vee D}.$$

C and D are called the auxiliary formulas and $C \vee D$ the principal formula of this inference.

2.4) $\supset$: left: $\dfrac{\Gamma \to \Delta, C \quad D, \Pi \to \Lambda}{C \supset D, \Gamma, \Pi \to \Delta, \Lambda}$; $\supset$: right: $\dfrac{C, \Gamma \to \Delta, D}{\Gamma \to \Delta, C \supset D}$.

C and D are called the auxiliary formulas and $C \supset D$ the principal formula.

2.1)–2.4) are called *propositional inferences*.

2.5) $\forall$: left: $\dfrac{F(t), \Gamma \to \Delta}{\forall x\, F(x), \Gamma \to \Delta}$, $\forall$: right: $\dfrac{\Gamma \to \Delta, F(a)}{\Gamma \to \Delta, \forall x\, F(x)}$,

where t is an arbitrary term, and a does not occur in the lower sequent. $F(t)$ and $F(a)$ are called the auxiliary formulas and $\forall x\, F(x)$ the principal formula. The a in $\forall$: right is called the *eigenvariable* of this inference.

Note that in $\forall$: right all occurrences of a in $F(a)$ are indicated. In $\forall$: left, $F(t)$ and $F(x)$ are

$$\left(F(a)\,\frac{a}{t}\right) \quad \text{and} \quad \left(F(a)\,\frac{a}{x}\right),$$

respectively (for some free variable a), so not every t in $F(t)$ is necessarily indicated.

2.6) $\exists$: left: $\dfrac{F(a), \Gamma \to \Delta}{\exists x\, F(x), \Gamma \to \Delta}$, $\exists$: right: $\dfrac{\Gamma \to \Delta, F(t)}{\Gamma \to \Delta, \exists x\, F(x)}$,

where a does not occur in the lower sequent, and t is an arbitrary term.

$F(a)$ and $F(t)$ are called the auxiliary formulas and $\exists x\, F(x)$ the principal formula. The a in $\exists$: left is called the eigenvariable of this inference.

Note that in $\exists$: left a is fully indicated, while in $\exists$: right not necessarily every t is indicated. (Again, $F(t)$ is $(F(a)\frac{a}{t})$ for some a.)

2.5) and 2.6) are called *quantifier inferences*. The condition, that the eigenvariable must not occur in the lower sequent in $\forall$: right and $\exists$: left, is called the *eigenvariable condition* for these inferences.

A sequent of the form $A \to A$ is called an *initial sequent*, or *axiom*.

We now explain the notion of formal proof, i.e., proof in **LK**.

DEFINITION 2.2. A *proof* P (in **LK**), or **LK**-*proof*, is a tree of sequents satisfying the following conditions:

1) The topmost sequents of P are initial sequents.

2) Every sequent in P except the lowest one is an upper sequent of an inference whose lower sequent is also in P.

The following terminology and conventions will be used in discussing formal proofs in **LK**.

DEFINITION 2.3. From Definition 2.2, it follows that there is a unique lowest sequent in a proof P. This will be called the *end-sequent* of P. A proof with end-sequent S is called a *proof ending with* S or a *proof of* S. A sequent S is called *provable* in **LK**, or **LK**-*provable*, if there is an **LK**-proof of it. A formula A is called **LK**-*provable* (or a *theorem of* **LK**) if the sequent $\to A$ is **LK**-provable. The prefix "**LK**-" will often be omitted from "**LK**-proof" and "**LK**-provable".

A proof without the cut rule is called *cut-free*.

It will be standard notation to abbreviate part of a proof by $\ddots\vdots\cdot^{\cdot^{\cdot}}$. Thus, for example,

$$\begin{array}{c}\ddots\vdots\cdot^{\cdot^{\cdot}}\\ S\end{array}\qquad\text{and}\qquad\begin{array}{c}S_1\quad S_2\\ \ddots\vdots\cdot^{\cdot^{\cdot}}\\ S\end{array}$$

denote a proof of S, and a proof of S from S_1 and S_2, respectively. Proofs are mostly denoted by letters $P, Q, \ldots$. An expression such as $P(a)$ means that all the occurrences of a in P are indicated. (Of course such notation is useful only when replacement of a by another term is being considered.) Then $P(t)$ is the result of replacing all occurrences of a in $P(a)$ by t.

Let us consider some slightly modified rules of inference, e.g.,

$$J\quad \frac{\Gamma \to \Delta, A \qquad \Pi \to \Lambda, B}{\Gamma, \Pi \to \Delta, \Lambda, A \wedge B}.$$

This is not a rule of inference of **LK**. However, from the two upper sequents we can infer the lower sequent in **LK** using several structural inferences and an $\wedge$: right:

$$(*)\qquad \wedge:\text{right}:\quad \dfrac{\dfrac{\Gamma \to \Delta, A}{\Gamma, \Pi \to \Delta, \Lambda, A}\ \text{several weakenings and exchanges}\qquad \dfrac{\Pi \to \Lambda, B}{\Gamma, \Pi \to \Delta, \Lambda, B}\ \text{several weakenings and exchanges}}{\Gamma, \Pi \to \Delta, \Lambda, A \wedge B}$$

Conversely, from the sequents $\Gamma \to \Delta, A$ and $\Gamma \to \Delta, B$ we can infer $\Gamma \to \Delta, A \wedge B$ using several structural inferences and an instance of the inference-schema J:

$$J\quad \dfrac{\dfrac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma, \Gamma \to \Delta, \Delta, A \land B}}{\dfrac{\text{several contractions and exchanges}}{\Gamma \to \Delta, A \land B}}.$$

Thus we may regard J as an abbreviation of (*) above. In such a case we will use the notation

$$\begin{array}{c} \Gamma \to \Delta, A \qquad \Pi \to \Lambda, B \\ \hline\hline \Gamma, \Pi \to \Delta, \Lambda, A \land B \end{array}.$$

As in this example we often indicate abbreviation of several steps by double lines.

Another remark we wish to make here is that the restriction on bound variables (in the definition of formulas) prohibits an unwanted inference such as

$$\begin{array}{c} A(a), B(b) \to A(a) \land B(b) \\ \hline A(a), B(b) \to \exists x\, (A(x) \land B(b)) \\ \hline A(a), B(b) \to \exists x\, \exists x\, (A(x) \land B(x)) \\ \hline \exists x\, A(x), \exists x\, B(x) \to \exists x\, \exists x\, (A(x) \land B(x)). \end{array}$$

In our system this can never happen, since $\exists x\, \exists x\, (A(x) \land B(x))$ is not a formula.

The quantifier-free part of **LK**, that is, the subsystem of **LK** which does not involve quantifiers, is called the *propositional calculus*.

EXAMPLE 2.4. The following are **LK**-proofs.

1)

$$\begin{array}{r|c} & A \to A \\ \neg : \text{right} & \\ \hline & \to A, \neg A \\ \lor : \text{right} & \\ \hline & \to A, A \lor \neg A \\ \text{exchange} : \text{right} & \\ \hline & \to A \lor \neg A, A \\ \lor : \text{right} & \\ \hline & \to A \lor \neg A, A \lor \neg A \\ \text{contraction} : \text{right} & \\ \hline & \to A \lor \neg A. \end{array}$$

2) Suppose that a is fully indicated in $F(a)$.

$$\begin{array}{r|c} & F(a) \to F(a) \\ \exists : \text{right} & \\ \hline & F(a) \to \exists x\, F(x) \\ \neg : \text{right} & \\ \hline & \to \exists x\, F(x), \neg F(a) \\ \forall : \text{right} & \\ \hline & \to \exists x\, F(x), \forall y\, \neg F(y) \\ \neg : \text{left} & \\ \hline & \neg \forall y\, \neg F(y) \to \exists x\, F(x) \\ \supset : \text{right} & \\ \hline & \to \neg \forall y\, \neg F(y) \supset \exists x\, F(x) \end{array}$$

It should be noted that the lower sequent of $\forall$: right does not contain the eigenvariable a.

EXERCISE 2.5. Prove the following in **LK**.

1) $A \vee B \equiv \neg(\neg A \wedge \neg B)$.
2) $A \supset B \equiv \neg A \vee B$.
3) $\exists x\, F(x) \equiv \neg\forall y\, \neg F(y)$.
4) $\neg\forall y\, F(y) \equiv \exists x\, \neg F(x)$.
5) $\neg(A \wedge B) \equiv \neg A \vee \neg B$.

EXERCISE 2.6. Prove the following in **LK**.

1) $\exists x\, (A \supset B(x)) \equiv A \supset \exists x\, B(x)$.
2) $\exists x\, (A(x) \supset B) \equiv \forall x\, A(x) \supset B$, where B does not contain x.
3) $\exists x\, (A(x) \supset B(x)) \equiv \forall x\, A(x) \supset \exists x\, B(x)$.
4) $\neg A \supset B \rightarrow \neg B \supset A$.
5) $\neg A \supset \neg B \rightarrow B \supset A$.

EXERCISE 2.7. Construct a cut-free proof of $\forall x\, A(x) \supset B \rightarrow \exists x\, (A(x) \supset B)$, where $A(a)$ and B are atomic and distinct.

DEFINITION 2.8. (1) When we consider a formula, term or logical symbol together with the place that it occupies in a proof, sequent or formula respectively, we refer to it as a formula, term or logical symbol in the proof, sequent or formula, respectively.

(2) A sequence of sequents in a proof P is called a *thread* (of P) if the following conditions are satisfied:

2.1) The sequence begins with an initial sequent and ends with the end-sequent.

2.2) Every sequent in the sequence except the last is an upper sequent of an inference, and is immediately followed by the lower sequent of this inference.

(3) Let S_1, S_2 and S_3 be sequents in a proof P. We say S_1 is *above* S_2 or S_2 is *below* S_1 (in P) if there is a thread containing both S_1 and S_2 in which S_1 appears before S_2. If S_1 is above S_2 and S_2 is above S_3, we say S_2 is *between* S_1 and S_3.

(4) An inference in P is said to be *below a sequent* S (in P) if its lower sequent is below S.

(5) Let P be a proof. A part of P which itself is a proof is called a *subproof* of P. This can also be described as follows. For any sequent S in P, that part of P which consists of all sequents which are either S itself or which occur above S, is called a subproof of P (with end-sequent S).

(6) Let P_0 be a proof of the form

$$\begin{array}{l} \quad\ \ \Gamma \to \Theta \\ (*)\{ \end{array}$$

where (*) denotes the part of P_0 under $\Gamma \to \Theta$, and let Q be a proof ending with $\Gamma, D \to \Theta$. By a copy of P_0 from Q we mean a proof P of the form

$$\begin{array}{l} \quad\ \ Q\begin{cases} \\ \Gamma, D \to \Theta \end{cases} \\ (**)\{ \end{array}$$

where (**) differs from (*) only in that for each sequent in (*), say $\Pi \to \Lambda$, the corresponding sequent in (**) has the form $\Pi, D \to \Lambda$. That is to say, P is obtained from P_0 by replacing the subproof ending with $\Gamma \to \Theta$ by Q, and adding an extra formula D to the antecedent of each sequent in (*). Likewise, a copy can be defined for the case of an extra formula in the succedent. We can also extend the definition to the case where there are several of these formulas.

The precise definition can be carried out by induction on the number of inferences in (*). However this notion is intuitive, simple, and will appear often in this book.

(7) Let $S(a)$, or $\Gamma(a) \to \Delta(a)$, denote a sequent of the form $A_1(a), \ldots, A_m(a) \to B_1(a), \ldots, B_n(a)$. Then $S(t)$, or $\Gamma(t) \to \Delta(t)$, denotes the sequent $A_1(t), \ldots, A_m(t) \to B_1(t), \ldots, B_n(t)$.

We can define: t is fully indicated in $S(t)$, or $\Gamma(t) \to \Delta(t)$, by analogy with Definition 1.6.

In order to prove a basic property of provability, i.e., that provability is preserved under substitution of terms for free variables, we shall first list some lemmas, which themselves assert basic properties of proofs. We first define an important concept.

DEFINITION 2.9. A proof in **LK** is called *regular* if it satisfies the condition that firstly, all eigenvariables are distinct from one another, and secondly, if a free variable a occurs as an eigenvariable in a sequent S of the proof, then a occurs only in sequents above S.

LEMMA 2.10. (1) *Let* $\Gamma(a) \to \Delta(a)$ *be an* (**LK**-)*provable sequent in which a is fully indicated, and let* $P(a)$ *be a proof of* $\Gamma(a) \to \Delta(a)$. *Let* b *be a free variable not occurring in* $P(a)$. *Then the tree* $P(b)$, *obtained from* $P(a)$ *by replacing* a *by* b *at each occurrence of* a *in* $P(a)$, *is also a proof and its end-sequent is* $\Gamma(b) \to \Delta(b)$.

(2) *For an arbitrary* **LK**-*proof there exists a regular proof of the same end-sequent. Moreover, the required proof is obtained from the original proof simply by replacing free variables (in a suitable way).*

PROOF. (1) By induction on the number of inferences in $P(a)$. If $P(a)$ consists of simply an initial sequent $A(a) \to A(a)$, then $P(b)$ consists of the sequent $A(b) \to A(b)$, which is also an initial sequent. Let us suppose that our proposition holds for proofs containing at most n inferences and suppose that $P(a)$ contains $n + 1$ inferences. We treat the possible cases according to the last inferences in $P(a)$. Since other cases can be treated similarly, we consider only the case where the last inference, say J, is a $\forall$: right. Suppose the eigenvariable of J is a, and $P(a)$ is of the form

$$\begin{array}{r} Q(a)\left\{ \vphantom{\vdots} \right. \\ J \end{array} \dfrac{\Gamma \to \Lambda, A(a)}{\Gamma \to \Lambda, \forall x\, A(x)},$$

where $Q(a)$ is the subproof of $P(a)$ ending with $\Gamma \to \Lambda, A(a)$. It should be remembered that a does not occur in Γ, Λ or $A(x)$. By the induction hypotheses the result of replacing all a's in $Q(a)$ by b is a proof whose end-sequent is $\Gamma \to \Lambda, A(b)$. Γ and Λ contain no b's. Thus we can apply a $\forall$: right to this sequent using b as its eigenvariable:

$$Q(b)\left\{ \dfrac{\Gamma \to \Lambda, A(b)}{\Gamma \to \Lambda, \forall x\, A(x)} \right.$$

and so $P(b)$ is a proof ending with $\Gamma \to \Lambda, \forall x\, A(x)$. If a is not the eigenvariable of J, $P(a)$ is of the form

$$Q(a)\left\{ \dfrac{\Gamma(a) \to \Lambda(a), A(a, c)}{\Gamma(a) \to \Lambda(a), \forall x\, A(a, x)} \right. .$$

By the induction hypothesis the result of replacing all a's in $Q(a)$ by b is a proof and its end-sequent is $\Gamma(b) \to \Lambda(b), A(b, c)$.

Since by assumption b does not occur in $P(a)$, b is not c, and so we can apply a $\forall$: right to this sequent, with c as its eigenvariable, and obtain a proof $P(b)$ whose end-sequent is $\Gamma(b) \to \Delta(b), \forall x\, A(b, x)$.

(2) By mathematical induction on the number l of applications of $\forall$: right and $\exists$: left in a given proof P. If $l = 0$, then take P itself. Otherwise, P can be represented in the form:

$$(*)\left\{\begin{array}{c} P_1\, P_2 \ldots P_k \\ \\ S \end{array}\right. ,$$

where P_i is a subproof of P of the form

$$I_i\,\frac{\Gamma_i \to \Delta_i, F_i(b_i)}{\Gamma_i \to \Delta_i, \forall y_i\, F_i(y_i)} \quad \text{or} \quad I_i\,\frac{F_i(b_i), \Gamma_i \to \Delta_i}{\exists y_i\, F_i(y_i), \Gamma_i \to \Delta_i}$$

and I_i is a lowermost $\forall$: right or $\exists$: left in P $(i = 1, \ldots, k)$, i.e., there is no $\forall$: right or $\exists$: left in the part of P denoted by (*).

Let us deal with the case where I_i is $\forall$: right. P_i has fewer applications of $\forall$: right or $\exists$: left than P, so by the induction hypothesis there is a regular proof P_i' of $\Gamma_i \to \Delta_i, F_i(b_i)$. Note that no free variable in $\Gamma_i \to \Delta_i, F(b_i)$ (including b_i) is used as an eigenvariable in P_i'. Suppose $c_1, \ldots, c_m$ are all the eigenvariables in all the P_i''s which occur in P above $\Gamma_i \to \Delta_i, \forall y_i\, F_i(y_i)$, $i = 1, \ldots, k$. Then change $c_1, \ldots, c_m$ to $d_1, \ldots, d_m$, respectively, where $d_1, \ldots, d_m$ are the first m variables which occur neither in P nor in P_i', $i = 1, \ldots, k$. If b_i occurs in P below $\Gamma_i \to \Delta_i, \forall y_i\, F_i(y_i)$, then change it to d_{m+i}.

Let P_i'' be the proof which is obtained from P_i' by the above replacement of variables. Then $P_1'', \ldots, P_k''$ are each regular. P' is defined to be

$$(*)\left\{\begin{array}{c} P_1'' \ldots \dfrac{P_i''}{\Gamma_i \to \Delta_i, \forall y_i\, F_i(y_i)} \ldots P_n'' \\ \\ \\ S \end{array}\right.$$

where (*) is the same as in P, except for the replacement of b_i by d_{m+i}. This completes the proof.

From now on we will assume that we are dealing with regular proofs whenever convenient, and will not mention it on each occasion.

By a method similar to that in Lemma 2.10 we can prove the following.

LEMMA 2.11. *Let t be an arbitrary term. Let $\Gamma(a) \rightarrow \Delta(a)$ be a provable (in* **LK***) sequent in which a is fully indicated, and let $P(a)$ be a proof ending with $\Gamma(a) \rightarrow \Delta(a)$ in which every eigenvariable is different from a and not contained in t. Then $P(t)$ (the result of replacing all a's in $P(a)$ by t) is a proof whose end-sequent is $\Gamma(t) \rightarrow \Delta(t)$.*

LEMMA 2.12. *Let t be an arbitrary term, $\Gamma(a) \rightarrow \Delta(a)$ a provable (in* **LK***) sequent in which a is fully indicated, and $P(a)$ a proof of $\Gamma(a) \rightarrow \Delta(a)$. Let $P'(a)$ be a proof obtained from $P(a)$ by changing eigenvariables (not necessarily replacing distinct ones by distinct ones) in such a way that in $P'(a)$ every eigenvariable is different from a and not contained in t. Then $P'(t)$ is a proof of $\Gamma(t) \rightarrow \Delta(t)$.*

PROOF. By induction on the number of eigenvariables in $P(a)$ which are either a or contained in t, using Lemmas 2.10 and 2.11.

We rewrite part of Lemma 2.11 as follows.

PROPOSITION 2.13. *Let t be an arbitrary term and $S(a)$ a provable (in* **LK***) sequent in which a is fully indicated. Then $S(t)$ is also provable.*

We will point out a simple, but useful fact about the formal proofs of our system, which will be used repeatedly.

PROPOSITION 2.14. *If a sequent is provable, then it is provable with a proof in which all the initial sequents consist of atomic formulas. Furthermore, if a sequent is provable without cut, then it is provable without cut with a proof of the above sort.*

PROOF. It suffices to show that for an arbitrary formula A, $A \rightarrow A$ is provable without cut, starting with initial sequents consisting of atomic formulas. This, however, can be easily shown by induction on the complexity of A.

DEFINITION 2.15. We say that two formulas A and B are *alphabetical variants* (of one another) if for some $x_1, \ldots, x_n, y_1, \ldots, y_n$

$$\left(A\,\frac{x_1,\ldots,x_n}{z_1,\ldots,z_n}\right)$$

is

$$\left(B\,\frac{y_1,\ldots,y_n}{z_1,\ldots,z_n}\right),$$

where $z_1,\ldots,z_n$ are bound variables occurring neither in A nor in B: that is to say, if A and B are different, it is only because they have a different choice of bound variables. The fact that A and B are alphabetical variants will be expressed by $A \sim B$.

One can easily prove that the relation $A \sim B$ is an equivalence relation. Intuitively it is obvious that changing bound variables in a formula does not change its meaning. We can prove by induction on the number of logical symbols in A that if $A \sim B$, then $A \equiv B$ is provable without cut (indeed in **LJ**, which is to be defined in the next section). Thus two alphabetical variants will often be identified without mention.

§3. A formulation of intuitionistic predicate calculus

DEFINITION 3.1. We can formalize the intuitionistic predicate calculus as a subsystem of **LK**, which we call **LJ**, following Gentzen. (**J** stands for "intuitionistic".) **LJ** is obtained from **LK** by modifying it as follows (cf. Definitions 2.1 and 2.2 for **LK**):

1) A sequent in **LJ** is of the form $\Gamma \rightarrow \Delta$, where Δ consists of at most one formula.

2) Inferences in **LJ** are those obtained from those in **LK** by imposing the restriction that the succedent of each upper and lower sequent consists of at most one formula; thus there are no inferences in **LJ** corresponding to contraction : right or exchange : right.

The notions of proof, provable and other concepts for **LJ** are defined similarly to the corresponding notions for **LK**.

Every proof in **LJ** is obviously a proof in **LK**, but the converse is not true. Hence:

PROPOSITION 3.2. *If a sequent S of* **LJ** *is provable in* **LJ**, *then it is also provable in* **LK**.

Lemmas 2.10–2.12 and Propositions 2.13 and 2.14 hold, reading "**LJ**-provable" in place of "provable" or "provable (in **LK**)". We shall refer to these results (for **LJ**) as Lemma 3.3, Lemma 3.4, Lemma 3.5, Proposition 3.6 and Proposition 3.7, respectively. We omit the statements of these.)

EXAMPLE 3.8. The following are **LJ**-proofs.

1)

$$
\begin{array}{rc}
 & A \to A \\ \cline{2-2}
\wedge : \text{left} & A \wedge \neg A \to A \\ \cline{2-2}
\neg : \text{left} & \neg A,\, A \wedge \neg A \to \\ \cline{2-2}
\wedge : \text{left} & A \wedge \neg A,\, A \wedge \neg A \to \\ \cline{2-2}
\text{contraction} : \text{left} & A \wedge \neg A \to \\ \cline{2-2}
\neg : \text{right} & \to \neg(A \wedge \neg A)
\end{array}
$$

2) Suppose a is fully indicated in $F(a)$.

$$
\begin{array}{rc}
 & F(a) \to F(a) \\ \cline{2-2}
\exists : \text{right} & F(a) \to \exists x\, F(x) \\ \cline{2-2}
\neg : \text{left} & \neg\exists x\, F(x),\, F(a) \to \\ \cline{2-2}
\text{exchange} : \text{left} & F(a),\, \neg\exists x\, F(x) \to \\ \cline{2-2}
\neg : \text{left} & \neg\exists x\, F(x) \to \neg F(a) \\ \cline{2-2}
\forall : \text{right} & \neg\exists x\, F(x) \to \forall y\, \neg F(y).
\end{array}
$$

EXERCISE 3.9. Prove the following in **LJ**.

1) $\neg A \vee B \to A \supset B$.
2) $\exists x\, F(x) \to \neg\forall y\, \neg F(y)$.
3) $A \wedge B \to A$.
4) $A \to A \vee B$.
5) $\neg A \vee \neg B \to \neg(A \wedge B)$.
6) $\neg(A \vee B) \equiv \neg A \wedge \neg B$.
7) $(A \vee C) \wedge (B \vee C) \equiv (A \wedge B) \vee C$.
8) $\exists x\, \neg F(x) \to \neg\forall x\, F(x)$.
9) $\forall x\, (F(x) \wedge G(x)) \equiv \forall x\, F(x) \wedge \forall x\, G(x)$.
10) $A \supset \neg B \to B \supset \neg A$.
11) $\exists x\, (A \supset B(x)) \to A \supset \exists x\, B(x)$.
12) $\exists x\, (A(x) \supset B) \to \forall x\, A(x) \supset B$.
13) $\exists x\, (A(x) \supset B(x)) \to \forall x\, A(x) \supset \exists x\, B(x)$.

EXERCISE 3.10. Prove the following in **LJ**.

1) $\neg\neg(A \supset B), A \rightarrow \neg\neg B$.
2) $\neg\neg B \supset B, \neg\neg(A \supset B) \rightarrow A \supset B$.
3) $\neg\neg\neg A \equiv \neg A$.

EXERCISE 3.11. Define **LJ**$'$ to be the system which is obtained from **LJ** by adding to it, as initial sequents, all sequents $\neg\neg R \rightarrow R$, where R is atomic. Let A be a formula which does not contain $\vee$ or $\exists$. Then $\neg\neg A \rightarrow A$ is **LJ**$'$-provable. [*Hint:* By induction on the number of logical symbols in A; cf. Exercise 3.10.]

PROBLEM 3.12. For every formula A define A^* as follows.

1) If A is atomic, then A^* is $\neg\neg A$.

2) If A is of the forms $\neg B$, $B \wedge C$, $B \vee C$ or $B \supset C$, then A^* is $\neg B^*$, $B^* \wedge C^*$, $\neg(\neg B^* \wedge \neg C^*)$ or $B^* \supset C^*$, respectively.

3) If A is of the form $\forall x\, F(x)$ or $\exists x\, F(x)$, then A^* is $\forall x\, F^*(x)$ or $\neg\forall x\, \neg F^*(x)$, respectively.

(Thus A^* does not contain $\vee$ or $\exists$.) Prove that for any A, A is **LK**-provable if and only if A^* is **LJ**-provable. [*Hint:* Follow the prescription given below.]

1) For any A, $A \equiv A^*$ is **LK**-provable.

2) Let S be a sequent of the form $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n$. Let S' be the sequent

$$A_1^*, \ldots, A_m^*, \neg B_1^*, \ldots, \neg B_n^* \rightarrow .$$

Prove that S is **LK**-provable if and only if S' is **LK**-provable.

3) $A^* \equiv \neg\neg A^*$ in **LJ**, from Exercise 3.11.

4) Show that if S is **LK**-provable, then S' is **LJ**-provable. (Use mathematical induction on the number of inferences in a proof of S.)

What must be proved is now a special case of 4).

§4. Axiom systems

DEFINITION 4.1. The basic system is **LK**.

1) A finite or infinite set $\mathscr{A}$ of sentences is called an *axiom system*, and each of these sentences is called an *axiom* of $\mathscr{A}$. Sometimes an axiom system is called a *theory*. (Of course this definition is only significant in certain contexts.)

2) A finite (possibly empty) sequence of formulas consisting only of axioms of $\mathscr{A}$ is called an *axiom sequence* of $\mathscr{A}$.

3) If there exists an axiom sequence Γ_0 of $\mathscr{A}$ such that $\Gamma_0, \Gamma \rightarrow \Delta$ is **LK**-provable, then $\Gamma \rightarrow \Delta$ is said to be *provable from* $\mathscr{A}$ (in **LK**). We express this by $\mathscr{A}, \Gamma \rightarrow \Delta$.

4) $\mathscr{A}$ is *inconsistent* (with **LK**) if the empty sequent $\rightarrow$ is provable from $\mathscr{A}$ (in **LK**).

5) If $\mathscr{A}$ is not inconsistent (with **LK**), then it is said to be *consistent* (with **LK**).

6) If all function constants and predicate constants in a formula A occur in $\mathscr{A}$, then A is said to be *dependent on* $\mathscr{A}$.

7) A sentence A is said to be *consistent* (*inconsistent*) if the axiom system $\{A\}$ is consistent (inconsistent).

8) $\mathbf{LK}_{\mathscr{A}}$ is the system obtained from **LK** by adding $\rightarrow A$ as initial sequents for all A in $\mathscr{A}$.

9) $\mathbf{LK}_{\mathscr{A}}$ is said to be *inconsistent* if $\rightarrow$ is $\mathbf{LK}_{\mathscr{A}}$-provable, otherwise it is *consistent*.

The following propositions, which are easily proved, will be used quite often.

PROPOSITION 4.2. *Let $\mathscr{A}$ be an axiom system. Then the following are equivalent:*

(a) *$\mathscr{A}$ is inconsistent* (*with* **LK**) (*as defined above*);

(b) *for every formula A* (*of the language*), *A is provable from $\mathscr{A}$*;

(c) *for some formula A, A and $\neg A$ are both provable from $\mathscr{A}$.*

PROPOSITION 4.3. *Let $\mathscr{A}$ be an axiom system. Then a sequent $\Gamma \rightarrow \Delta$ is $\mathbf{LK}_{\mathscr{A}}$-provable if and only if $\Gamma \rightarrow \Delta$ is provable from $\mathscr{A}$* (*in* **LK**).

COROLLARY 4.4. *An axiom system $\mathscr{A}$ is consistent* (*with* **LK**) *if and only if $\mathbf{LK}_{\mathscr{A}}$ is consistent.*

These definitions and the propositions hold also for **LJ**.

§5. The cut-elimination theorem

A very important fact about **LK** is the cut-elimination theorem, also known as Gentzen's Hauptsatz:

THEOREM 5.1 (the cut-elimination theorem: Gentzen). *If a sequent is* (**LK**)-*provable, then it is* (**LK**-)*provable without a cut.*

This means that any theorem in the predicate calculus can be proved without detours, so to speak. We shall come back to this point later. The purpose of the present section is to prove this theorem. We shall follow Gentzen's original proof.

First we introduce a new rule of inference, the mix rule, and show that the mix rule and the cut rule are equivalent. Let A be a formula. An inference of the following form is called a *mix* (with respect to A):

$$\frac{\Gamma \to \Delta \qquad \Pi \to \Lambda}{\Gamma, \Pi^* \to \Delta^*, \Lambda} \quad (A)$$

where both Δ and Π contain the formula A, and Δ^* and Π^* are obtained from Δ and Π respectively by deleting all the occurrences of A in them. We call A the mix formula of this inference, and the mix formula of a mix is normally indicated in parentheses (as above).

Let us call the system which is obtained from **LK** by replacing the cut rule by the mix rule, **LK***. The following is easily proved.

LEMMA 5.2. **LK** *and* **LK*** *are equivalent, that is, a sequent S is* **LK**-*provable if and only if S is* **LK***-*provable.*

By virtue of the Lemma 5.2, it suffices to show that the mix rule is redundant in **LK***, since a proof in **LK*** without a mix is at the same time a proof in **LK** without a cut.

THEOREM 5.3 (cf. Theorem 5.1). *If a sequent is provable in* **LK***, *then it is provable in* **LK*** *without a mix.*

This theorem is an immediate consequence of the following lemma.

LEMMA 5.4. *If P is a proof of S* (*in* **LK***) *which contains* (*only*) *one mix, occurring as the last inference, then S is provable without a mix.*

The proof of Theorem 5.3 from Lemma 5.4 is simply by induction on the number of mixes occurring in a proof of S.

The rest of this section is devoted to proving Lemma 5.4. We first define two scales for measuring the complexity of a proof. The *grade* of a formula A (denoted by $g(A)$) is the number of logical symbols contained in A. The grade of a mix is the grade of the mix formula. When a proof P has a mix (only) as the last inference, we define the grade of P (denoted by $g(P)$) to be the grade of this mix.

Let P be a proof which contains a mix only as the last inference:

$$J \frac{\Gamma \to \Delta \qquad \Pi \to \Lambda}{\Gamma, \Pi^* \to \Delta^*, \Lambda} (A).$$

We refer to the left and right upper sequents as S_1 and S_2, respectively, and to the lower sequent as S. We call a thread in P a *left* (*right*) *thread* if it contains the left (right) upper sequent of the mix J. The *rank* of a thread $\mathscr{F}$ in P is defined as follows: if $\mathscr{F}$ is a left (right) thread, then the rank of $\mathscr{F}$ is the number of consecutive sequents, counting upward from the left (right) upper sequent of J, that contains the mix formula. Since the left (right) upper sequent always contains the mix formula, the rank of a thread in P is at least 1. The rank of a thread $\mathscr{F}$ in P is denoted by $\text{rank}(\mathscr{F}; P)$. We define

$$\text{rank}_l(P) = \max_{\mathscr{F}}(\text{rank}(\mathscr{F}; P)),$$

where $\mathscr{F}$ ranges over all the left threads in P, and

$$\text{rank}_r(P) = \max_{\mathscr{F}}(\text{rank}(\mathscr{F}; P)),$$

where $\mathscr{F}$ ranges over all the right threads in P. The rank of P, $\text{rank}(P)$, is defined as

$$\text{rank}(P) = \text{rank}_l(P) + \text{rank}_r(P).$$

Notice that $\text{rank}(P)$ is always $\geqslant 2$.

PROOF OF LEMMA 5.4. We prove the Lemma by double induction on the grade g and rank r of the proof P (i.e., transfinite induction on $\omega \cdot g + r$). We divide the proof into two main cases, namely $r = 2$ and $r > 2$ (regardless of g).

Case 1: $r = 2$, viz. $\text{rank}_l(P) = \text{rank}_r(P) = 1$.

We distinguish cases according to the forms of the proofs of the upper sequents of the mix.

1.1) The left upper sequent S_1 is an initial sequent. In this case we may assume P is of the form

$$J\ \frac{A \to A \qquad \Pi \to \Lambda}{A, \Pi^* \to \Lambda}\,.$$

We can then obtain the lower sequent without a mix:

$$\cfrac{\cfrac{\Pi \to \Lambda}{A, \ldots, A, \Pi^* \to \Lambda}\ \text{some exchanges}}{A, \Pi^* \to \Lambda}\ \text{some contractions}$$

1.2) The right upper sequent S_2 is an initial sequent. Similarly:

1.3) Neither S_1 nor S_2 is an initial sequent, and S_1 is the lower sequent of a structural inference J_1. Since $\mathrm{rank}_l(P) = 1$, the formula A cannot appear in the upper sequent of J_1, i.e., J_1 must be a weakening : right, whose weakening formula is A:

$$J\ \frac{J_1\ \dfrac{\Gamma \to \Delta_1}{\Gamma \to \Delta_1, A} \qquad \Pi \to \Lambda}{\Gamma, \Pi^* \to \Delta_1, \Lambda}\quad (A),$$

where Δ_1 does not contain A. We can eliminate the mix as follows:

$$\cfrac{\cfrac{\Gamma \to \Delta_1}{\Pi^*, \Gamma \to \Delta_1, \Lambda}\ \text{some weakenings}}{\Gamma, \Pi^* \to \Delta_1, \Lambda}\ \text{some exchanges}$$

1.4) None of 1.1)–1.3) holds but S_2 is the lower sequent of a structural inference. Similarly:

1.5) Both S_1 and S_2 are the lower sequents of logical inferences. In this case, since $\mathrm{rank}_l(P) = \mathrm{rank}_r(P) = 1$, the mix formula on each side must be the principal formula of the logical inference. We use induction on the grade, distinguishing several cases according to the outermost logical symbol of A. We treat here two cases and leave the others to the reader.

(i) The outermost logical symbol of A is $\wedge$. In this case S_1 and S_2 must be the lower sequents of $\wedge$: right and $\wedge$: left, respectively:

$$\cfrac{\cfrac{\Gamma \to \Delta_1, B \qquad \Gamma \to \Delta_1, C}{\Gamma \to \Delta_1, B \wedge C} \qquad \cfrac{B, \Pi_1 \to \Lambda}{B \wedge C, \Pi_1 \to \Lambda}}{\Gamma, \Pi_1 \to \Delta_1, \Lambda} \quad (B \wedge C),$$

where by assumption none of the proofs ending with $\Gamma \to \Delta_1, B$; $\Gamma \to \Delta_1, C$ or $B, \Pi_1 \to \Lambda$ contain a mix. Consider the following:

$$\frac{\Gamma \to \Delta_1, B \qquad B, \Pi_1 \to \Lambda}{\Gamma, \Pi_1^{\#} \to \Delta_1^{\#}, \Lambda} \quad (B),$$

where $\Pi_1^{\#}$ and $\Delta_1^{\#}$ are obtained from Π_1 and Δ_1 by omitting all occurrences of B. This proof contains only one mix, a mix that occurs as its last inference. Furthermore the grade of the mix formula B is less than $g(A)$ $(= g(B \wedge C))$. So by the induction hypothesis we can obtain a proof which contains no mixes and whose end-sequent is $\Gamma, \Pi_1^{\#} \to \Delta_1^{\#}, \Lambda$. From this we can obtain a proof without a mix with end-sequent $\Gamma, \Pi_1 \to \Delta_1, \Lambda$.

(ii) The outermost logical symbol of A is $\forall$. So A is of the form $\forall x\, F(x)$ and the last part of P has the form:

$$\cfrac{\cfrac{\Gamma \to \Delta_1, F(a)}{\Gamma \to \Delta_1, \forall x\, F(x)} \qquad \cfrac{F(t), \Pi_1 \to \Lambda}{\forall x\, F(x), \Pi_1 \to \Lambda}}{\Gamma, \Pi_1 \to \Delta_1, \Lambda} \quad (A)$$

(a being fully indicated in $F(a)$). By the eigenvariable condition, a does not occur in Γ, Δ_1 or $F(x)$. Since by assumption the proof ending with $\Gamma \to \Delta_1, F(a)$ contains no mix, we can obtain a proof without a mix, ending with $\Gamma \to \Delta_1, F(t)$ (cf. Lemma 2.12). Consider now

$$\frac{\Gamma \to \Delta_1, F(t) \qquad F(t), \Pi_1 \to \Lambda}{\Gamma, \Pi_1^{\#} \to \Delta_1^{\#}, \Lambda} \quad (F(t)),$$

where $\Pi_1^{\#}$ and $\Delta_1^{\#}$ are obtained from Π_1 and Δ_1 by omitting all occurrences of $F(t)$. This has only one mix. It occurs as the last inference and the grade of the mix formula is less than $g(A)$. Thus by the induction hypothesis we can eliminate this mix and obtain a proof ending with $\Gamma, \Pi_1^{\#} \to \Delta_1^{\#}, \Lambda$, from which we can obtain a proof, without a mix, ending with $\Gamma, \Pi_1 \to \Delta_1, \Lambda$.

Case 2. $r > 2$, i.e., $\mathrm{rank}_l(P) > 1$ and/or $\mathrm{rank}_r(P) > 1$.

The induction hypothesis is that from every proof Q which contains a mix only as the last inference, and which satisfies either $g(Q) < g(P)$, or $g(Q) = g(P)$ and $\mathrm{rank}(Q) < \mathrm{rank}(P)$, we can eliminate the mix.

2.1) $\text{rank}_r(P) > 1$.

2.1.1) Γ (in S_1) contains A. Construct a proof as follows.

$$\dfrac{\dfrac{\Pi \to \Lambda}{A, \Pi^* \to \Lambda}\ \text{some exchanges and contractions}}{\Gamma, \Pi^* \to \Delta^*, \Lambda}\ \text{some weakenings and exchanges}$$

2.1.2) S_2 is the lower sequent of an inference J_2, where J_2 is not a logical inference whose principal formula is A. The last part of P looks like this:

$$\dfrac{\Gamma \to \Delta \qquad J_2\ \dfrac{\Phi \to \Psi}{\Pi \to \Lambda}}{\Gamma, \Pi^* \to \Delta^*, \Lambda}\ (A),$$

where the proofs of $\Gamma \to \Delta$ and $\Phi \to \Psi$ contain no mixes and Φ contains at least one A. Consider the following proof P':

$$\text{mix}\ \dfrac{\Gamma \to \Delta \qquad \Phi \to \Psi}{\Gamma, \Phi^* \to \Delta^*, \Psi}\ (A).$$

In P', the grade of the mix is equal to $g(P)$, $\text{rank}_l(P') = \text{rank}_l(P)$ and $\text{rank}_r(P') = \text{rank}_r(P) - 1$. Thus by the induction hypothesis, $\Gamma, \Phi^* \to \Delta^*, \Psi$ is provable without a mix. Then we construct the proof

$$J_2\ \dfrac{\dfrac{\Gamma, \Phi^* \to \Delta^*, \Psi}{\Phi^*, \Gamma \to \Delta^*, \Psi}\ \text{some exchanges}}{\Pi^*, \Gamma \to \Delta^*, \Lambda}$$

2.1.3) Γ contains no A's, and S_2 is the lower sequent of a logical inference whose principal formula is A. Although there are several cases according to the outermost logical symbol of A, we treat only two examples here and leave the rest to the reader.

(i) A is $B \supset C$. The last part of P is of the form:

$$J\ \frac{\Gamma \to \Delta \qquad J_2\ \dfrac{\Pi_1 \to \Lambda_1, B \qquad C, \Pi_2 \to \Lambda_2}{B \supset C, \Pi_1, \Pi_2 \to \Lambda_1, \Lambda_2}}{\Gamma, \Pi_1^*, \Pi_2^* \to \Delta^*, \Lambda_1, \Lambda_2}\ (B \supset C)$$

Consider the following proofs P_1 and P_2:

$$P_1\ \frac{\overset{\vdots}{\Gamma \to \Delta} \qquad \overset{\vdots}{\Pi_1 \to \Lambda_1, B}}{\Gamma, \Pi_1^* \to \Delta^*, \Lambda_1, B}\ (B \supset C) \qquad P_2\ \frac{\overset{\vdots}{\Gamma \to \Delta} \qquad \overset{\vdots}{C, \Pi_2 \to \Lambda_2}}{\Gamma, C, \Pi_2^* \to \Delta^*, \Lambda_2}\ (B \supset C)$$

assuming that $B \supset C$ is in Π_1 and Π_2. If $B \supset C$ is not in Π_i ($i = 1$ or 2), then Π_i^* is Π_i and P_i is defined as follows.

$$P_1\ \frac{\dfrac{\overset{\vdots}{\Pi_1 \to \Lambda_1, B}}{\text{weakenings and exchanges}}}{\Gamma, \Pi_1^* \to \Delta^*, \Lambda_1, B} \qquad P_2\ \frac{\dfrac{\overset{\vdots}{C, \Pi_2 \to \Lambda_2}}{\text{weakenings and exchanges}}}{\Gamma, C, \Pi_2^* \to \Delta^*, \Lambda_2}$$

Note that $g(P_1) = g(P_2) = g(P)$, $\mathrm{rank}_l(P_1) = \mathrm{rank}_l(P_2) = \mathrm{rank}_l(P)$ and $\mathrm{rank}_r(P_1) = \mathrm{rank}_r(P_2) = \mathrm{rank}_r(P) - 1$. Hence by the induction hypothesis, the end-sequents of P_1 and P_2 are provable without a mix (say by P_1' and P_2'). Consider the following proof P':

$$J\ \frac{\Gamma \to \Delta \qquad \dfrac{\overset{P_1'}{\overset{\vdots}{\Gamma, \Pi_1^* \to \Delta^*, \Lambda_1, B}} \qquad \dfrac{\dfrac{\overset{P_2'}{\overset{\vdots}{\Gamma, C, \Pi_2^* \to \Delta^*, \Lambda_2}}}{\text{some exchanges}}}{C, \Gamma, \Pi_2^* \to \Delta^*, \Lambda_2}}{B \supset C, \Gamma, \Pi_1^*, \Gamma, \Pi_2^* \to \Delta^*, \Lambda_1, \Delta^*, \Lambda_2}}{\Gamma, \Gamma, \Pi_1^*, \Gamma, \Pi_2^* \to \Delta^*, \Delta^*, \Lambda_1, \Delta^*, \Lambda_2}\ (B \supset C).$$

Then $g(P') = g(P)$, $\mathrm{rank}_l(P') = \mathrm{rank}_l(P)$, $\mathrm{rank}_r(P') = 1$, for Γ contains no occurrences of $B \supset C$ and $\mathrm{rank}(P') < \mathrm{rank}(P)$. Thus the end-sequent of P' is provable without a mix by the induction hypothesis, and hence so is the end-sequent of P.

(ii) A is $\exists x\, F(x)$. The last part of P looks like this:

$$J\ \frac{\Gamma \to \Delta \qquad \dfrac{F(a), \Pi_1 \to \Lambda}{\exists x\, F(x), \Pi_1 \to \Lambda}}{\Gamma, \Pi_1^* \to \Delta^*, \Lambda}\ (\exists x\, F(x)).$$

Let b be a free variable not occurring in P. Then the result of replacing a by b throughout the proof ending with $F(a), \Pi_1 \to \Lambda$ is a proof, without a mix, ending with $F(b), \Pi_1 \rightarrow \Lambda$, since by the eigenvariable condition, a does not occur in Π_1 or Λ (cf. Lemma 2.11).

Consider the following proof:

$$J \ \frac{\Gamma \rightarrow \Delta \qquad F(b), \Pi_1 \rightarrow \Lambda}{\Gamma, F(b), \Pi_1^* \rightarrow \Delta^*, \Lambda} \ (\exists x\, F(x)).$$

By the induction hypothesis, the end-sequent of this proof can be proved without a mix (say by P'). Now consider the proof

$$J \ \dfrac{\Gamma \rightarrow \Delta \qquad \dfrac{\dfrac{\dfrac{\begin{matrix} P' \\ \Gamma, F(b), \Pi_1^* \rightarrow \Delta^*, \Lambda \end{matrix}}{\text{some exchanges}}}{F(b), \Gamma, \Pi_1^* \rightarrow \Delta^*, \Lambda}}{\exists x\, F(x), \Gamma, \Pi_1^* \rightarrow \Delta^*, \Lambda}}{\Gamma, \Gamma, \Pi_1^* \rightarrow \Delta^*, \Delta^*, \Lambda} \ (\exists x\, F(x)),$$

where b occurs in none of $\exists x\, F(x), \Gamma, \Pi_1, \Delta, \Lambda$. This mix can then also be eliminated, by the induction hypothesis.

2.2) $\text{rank}_r(P) = 1$ (and $\text{rank}_l(P) > 1$).
This case is proved in the same way as 2.1) above.

This completes the proof of Lemma 5.4 and hence of the cut-elimination theorem.

It should be emphasized that the proof is constructive, i.e., a new proof is effectively constructed from the given proof in Lemma 5.2 and again in Lemma 5.4, and hence in Theorem 5.1.

The cut-elimination theorem also holds for **LJ**. Actually the above proof is designed so that it goes through for **LJ** without essential changes: we only have to keep in mind that there can be at most one formula in each succedent. The details are left to the reader; we simply state the theorem.

THEOREM 5.5. *The cut-elimination theorem holds for* **LJ**.

§6. Some consequences of the cut-elimination theorem

There are numerous applications of the cut-elimination theorem, some of which will be listed in this section, others as exercises. In order to facilitate discussion of this valuable, productive and important theorem, we shall first define the notion of subformula, which will be used often.

DEFINITION 6.1. By a *subformula* of a formula A we mean a formula used in building up A. The set of subformulas of a formula is inductively defined as follows, by induction on the number of logical symbols in the formula.

(1) An atomic formula has exactly one subformula, viz. the formula itself. The subformulas of $\neg A$ are the subformulas of A and $\neg A$ itself. The subformulas of $A \wedge B$ or $A \vee B$ or $A \supset B$ are the subformulas of A and of B, and the formula itself. The subformulas of $\forall x\, A(x)$ or $\exists x\, A(x)$ are the subformulas of any formula of the form $A(t)$, where t is an arbitrary term, and the formula itself.

(2) Two formulas A and B are said to be *equivalent* in **LK** if $A \equiv B$ is provable in **LK**.

(3) We shall say that in a formula A an occurrence of a logical symbol, say $\sharp$, is *in the scope* of an occurrence of a logical symbol, say $\natural$, if in the construction of A (from atomic formulas) the stage where $\sharp$ is the outermost logical symbol precedes the stage where $\natural$ is the outermost logical symbol (cf. Definition 1.3). Further, a symbol $\sharp$ is said to be in the left scope of a $\supset$ if $\supset$ occurs in the form $B \supset C$ and $\sharp$ occurs in B.

(4) A formula is called *prenex* (in prenex form) if no quantifier in it is in the scope of a propositional connective. It can easily be seen that any formula is equivalent (in **LK**) to a prenex formula, i.e., for every formula A there is a prenex formula B such that $A \equiv B$ is **LK**-provable.

One can easily see that in any rule of inference except a cut, the lower sequent is no less complicated than the upper sequent(s); more precisely, every formula occurring in an upper sequent is a subformula of some formula occurring in the lower sequent (but not necessarily conversely). Hence a proof without a cut contains only subformulas of the formulas occurring in the end-sequent (the "subformula property"). So the cut-elimination theorem tells us that if a formula is provable in **LK** (or **LJ**) at all, it is provable by use of its subformulas only. (This is what we meant by saying that a theorem in the predicate calculus could be proved without detours.)

From this observation, we can convince ourselves that the empty sequent $\rightarrow$ is not **LK**- (or **LJ**-) provable. This leads us to the consistency proof of **LK** and **LJ**.

THEOREM 6.2 (consistency). **LK** *and* **LJ** *are consistent.*

PROOF. Suppose $\rightarrow$ were provable in **LK** (or **LJ**). Then, by the cut-elimination theorem, it would be provable in **LK** (or **LJ**) without a cut. But this is impossible, by the subformula property of cut-free proofs.

An examination of the proof of this theorem (including the cut-elimination theorem) shows that the consistency of **LK** (and **LJ**) was proved by quantifier-free induction on the ordinal ω^2. We shall not, however, go into the details of the consistency problem at this stage.

For convenience, we re-state the subformula property of cut-free proofs as a theorem.

THEOREM 6.3. *In a cut-free proof in* **LK** (*or* **LJ**) *all the formulas which occur in it are subformulas of the formulas in the end-sequent.*

PROOF. By mathematical induction on the number of inferences in the cut-free proof.

In the rest of this section, we shall list some typical consequences of the cut-elimination theorem. Although some of the results are stated for **LJ** as well as **LK**, we shall give proofs only for **LK**; those for **LJ** are left to the reader.

THEOREM 6.4 (1) (Gentzen's midsequent theorem for **LK**). *Let S be a sequent which consists of prenex formulas only and is provable in* **LK**. *Then there is a cut-free proof of S which contains a sequent* (*called a midsequent*), *say S', which satisfies the following*:

1. *S' is quantifier-free.*
2. *Every inference above S' is either structural or propositional.*
3. *Every inference below S' is either structural or a quantifier inference.*

Thus a midsequent splits the proof into an upper part, which contains the propositional inferences, and a lower part, which contains the quantifier inferences.

(2) (The midsequent theorem for **LJ** without $\vee$: left.) *The above holds reading "***LJ** *without $\vee$: left" in place of "***LK***".*

PROOF (outline). Combining Proposition 2.14 and the cut-elimination theorem, we may assume that there is a cut-free proof of S, say P, in which all the initial sequents consist of atomic formulas only. Let I be a quantifier inference in P. The number of propositional inferences under I is called the order of I. The sum of the orders for all the quantifier inferences in P is called the order of P. (The term "order" is used only temporarily here.) The proof is carried out by induction on the order of P.

Case 1: The order of a proof P is 0. If there is a propositional inference, take the lowermost such, and call its lower sequent S_0. Above this sequent there is no quantifier inference. Therefore, if there is a quantifier in or above S_0, then it is introduced by weakenings. Since the proof is cut-free, the weakening formula is a subformula of one of the formulas in the end-sequent. Hence no propositional inferences apply to it. We can thus eliminate these weakenings and obtain a sequent S_0' corresponding to S_0. By adding some weakenings under S_0' (if necessary), we derive S, and S_0' serves as the midsequent.

If there is no propositional inference in P, then take the uppermost quantifier inference. Its upper sequent serves as a midsequent.

Case 2: The order of P is not 0. Then there is at least one propositional inference which is below a quantifier inference. Moreover, there is a quantifier inference I with the following property: the uppermost logical inference under I is a propositional inference. Call it I'. We can lower the order by interchanging the positions of I and I'. Here we present just one example: say I is $\forall$: right.

P:

$$(*)\ \left\{\begin{array}{l} I \\[2ex] I' \end{array}\right. \quad \dfrac{\dfrac{\Gamma \to \Theta, F(a)}{\Gamma \to \Theta, \forall x\, F(x)}}{\Delta \to \Lambda},$$

where the (*)-part of P contains only structural inferences and Λ contains $\forall x\, F(x)$ as a sequent-formula. Transform P into the following proof P':

$$\begin{array}{ll}
 & \Gamma \to \Theta, F(a) \\ \cline{2-2}
 & \text{structural inferences} \\ \cline{2-2}
 & \Gamma \to F(a), \Theta, \forall x\, F(x) \\[1ex]
I' & \overline{\Delta \to F(a), \Lambda} \\
I & \overline{\Delta \to \Lambda, \forall x\, F(x)} \\ \hline\hline
 & \Delta \to \Lambda
\end{array}$$

It is obvious that the order of P' is less than that of P.

Prior to the next theorem, Craig's interpolation theorem*, we shall first state and prove a lemma which itself can be regarded as an interpolation theorem for provable sequents and from which the original form of the interpolation theorem follows immediately. We shall present the argument for **LK** only, although everything goes through for **LJ** as well.

For technical reasons we introduce the predicate symbol $\top$, with 0 argument places, and admit $\rightarrow \top$ as an additional initial sequent. ($\top$ stands for "true".) The system which is obtained from **LK** thus extended is denoted by **LK♯**.

LEMMA 6.5. *Let* $\Gamma \rightarrow \Delta$ *be* **LK**-*provable, and let* (Γ_1, Γ_2) *and* (Δ_1, Δ_2) *be arbitrary partitions of* Γ *and* Δ, *respectively (including the cases that one or more of* $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ *are empty). We denote such a partition by* $[\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$ *and call it a partition of the sequent* $\Gamma \rightarrow \Delta$. *Then there exists a formula* C *of* **LK♯** *(called an interpolant of* $[\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$ *such that*:

(i) $\Gamma_1 \rightarrow \Delta_1, C$ *and* $C, \Gamma_2 \rightarrow \Delta_2$ *are both* **LK♯**-*provable*;

(ii) *All free variables and individual and predicate constants in* C *(apart from* $\top$*) occur both in* $\Gamma_1 \cup \Delta_1$ *and* $\Gamma_2 \cup \Delta_2$.

We will first prove the theorem (from this lemma) and then prove the lemma.

THEOREM 6.6 (Craig's interpolation theorem for **LK**). (1) *Let* A *and* B *be two formulas such that* $A \supset B$ *is* **LK**-*provable. If* A *and* B *have at least one predicate constant in common, then there exists a formula* C, *called an interpolant of* $A \supset B$, *such that* C *contains only those individual constants, predicate constants and free variables that occur in both* A *and* B, *and such that* $A \supset C$ *and* $C \supset B$ *are* **LK**-*provable. If* A *and* B *contain no predicate constant in common, then either* $A \rightarrow$ *or* $\rightarrow B$ *is* **LK**-*provable.*

(2) *As above, with* **LJ** *in place of* **LK**.

PROOF. Assume that $A \supset B$, and hence $A \rightarrow B$, is provable, and A and B have at least one predicate constant in common. Then by Lemma 6.5, taking A as Γ_1 and B as Δ_1 (with Γ_2 and Δ_1 empty), there exists a formula C satisfying (i) and (ii). So $A \rightarrow C$ and $C \rightarrow B$ are **LK♯**-provable. Let R be a predicate constant which is common to A and B and has k argument places. Let R' be $\forall y_1 \ldots \forall y_k\, R(y_1, \ldots, y_k)$, where $y_1, \ldots, y_k$ are new bound variables.

* A strong general theory on interpolation theorems is established in N. Motohasi: Interpolation theorem and characterization theorem, Ann. Japan Assoc. Philos. Sci., 4 (1972) pp. 15–80.

By replacing $\top$ by $R' \supset R'$, we can transform C into a formula C' of the original language, such that $A \to C'$ and $C' \to B$ are **LK**-provable. C' is then the desired interpolant.

If there is no predicate common to $\Gamma_1 \cup \Delta_1$ and $\Gamma_2 \cup \Delta_2$ in the partition described in Lemma 6.5, then, by that lemma, there is a C such that $\Gamma_1 \to \Delta_1, C$ and $C, \Gamma_2 \to \Delta_2$ are provable, and C consists of $\top$ and logical symbols only. Then it can easily be shown, by induction on the complexity of C, that either $\to C$ or $C \to$ is provable. Hence either $\Gamma_1 \to \Delta_1$ or $\Gamma_2 \to \Delta_2$ is provable. In particular, this applies to $A \to B$ when A is taken as Γ_1 and B as Δ_2.

This method is due to Maehara and its significance lies in the fact that an interpolant of $A \supset B$ can be constructively formed from a proof of $A \supset B$. Note also that we could state the theorem in the following form: *If neither $\neg A$ nor B is provable, then there is an interpolant of $A \supset B$.*

Proof of Lemma 6.5. The lemma is proved by induction on the number of inferences k, in a cut-free proof of $\Gamma \to \Delta$. At each stage there are several cases to consider; we deal with some examples only.

1) $k = 0$. $\Gamma \to \Delta$ has the form $D \to D$. There are four cases: (1) $[\{D; D\}, \{\ ;\ \}]$, (2) $[\{\ ;\ \}, \{D; D\}]$, (3) $[\{D;\ \}, \{\ ; D\}]$, and (4) $[\{\ ; D\}, \{D;\ \}]$. Take for C: $\neg\top$ in (1), $\top$ in (2), D in (3) and $\neg D$ in (4).

2) $k > 0$ and the last inference is $\wedge$: right:

$$\frac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B.}$$

Suppose the partition is $[\{\Gamma_1; \Delta_1, A \wedge B\}, \{\Gamma_2; \Delta_2\}]$. Consider the induced partition of the upper sequents, viz. $[\{\Gamma_1; \Delta_1, A\}, \{\Gamma_2; \Delta_2\}]$ and $[\{\Gamma_1; \Delta_1, B\}, \{\Gamma_2; \Delta_2\}]$, respectively. By the induction hypotheses applied to the subproofs of the upper sequents, there exist interpolants C_1 and C_2 so that $\Gamma_1 \to \Delta_1, A, C_1$; $C_1, \Gamma_2 \to \Delta_2$; $\Gamma_1 \to \Delta_1, B, C_2$; and $C_2, \Gamma_2 \to \Delta_2$ are all **LK#**-provable. From these sequents, $\Gamma_1 \to \Delta_1, A \wedge B, C_1 \vee C_2$ and $C_1 \vee C_2, \Gamma_2 \to \Delta_2$ can be derived. Thus $C_1 \vee C_2$ serves as the required interpolant.

3) $k > 0$ and the last inference is $\forall$: left:

$$\frac{F(s), \Gamma \to \Delta}{\forall x\, F(x), \Gamma \to \Delta}.$$

Suppose $b_1, \ldots, b_n$ are all the free variables (possibly none) which occur in s. Suppose the partition is $[\{\forall x\, F(x), \Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$. Consider the induced partition of the upper sequent and apply the induction hypothesis. So there exists an interpolant $C(b_1, \ldots, b_n)$ so that

$$F(s), \Gamma_1 \to \Delta_1, C(b_1, \ldots, b_n) \quad \text{and} \quad C(b_1, \ldots, b_n), \Gamma_2 \to \Delta_2$$

are **LK**$\sharp$-provable. Let $b_{i_1}, \ldots, b_{i_m}$ be all the variables among $b_1, \ldots, b_n$ which do not occur in $\{F(x), \Gamma_1; \Delta_1\}$. Then

$$\forall y_1 \ldots \forall y_m\, C(b_1, \ldots, y_1, \ldots, y_m, \ldots, b_n),$$

where $b_{i_1}, \ldots, b_{i_m}$ are replaced by the bound variables, serves as the required interpolant.

4) $k > 0$ and the last inference is $\forall$: right:

$$\frac{\Gamma \to \Delta, F(a)}{\Gamma \to \Delta, \forall x\, F(x)},$$

where a does not occur in the lower sequent.

Suppose the partition is $[\{\Gamma_1; \Delta_1, \forall x\, F(x)\}, \{\Gamma_2; \Delta_2\}]$. By the induction hypothesis there exists an interpolant C so that $\Gamma_1 \to \Delta_1, F(a), C$ and $C, \Gamma_2 \to \Delta_2$ are provable. Since C does not contain a, we can derive

$$\Gamma_1 \to \Delta_1, \forall x\, F(x), C,$$

and hence C serves as the interpolant.

All other cases are treated similarly.

EXERCISE 6.7. Let A and B be prenex formulas which have only $\forall$ and $\wedge$ as logical symbols. Assume furthermore that there is at least one predicate constant common to A and B. Suppose $A \supset B$ is provable.

Show that there exists a formula C such that

1) $A \supset C$ and $C \supset B$ are provable;
2) C is a prenex formula;
3) the only logical symbols in C are $\forall$ and $\wedge$;
4) the predicate constants in C are common to A and B.

[*Hint*: Apply the cut-elimination theorem and the midsequent theorem.]

DEFINITION 6.8. (1) A semi-term is an expression like a term, except that bound variables are (also) allowed in its construction. (The precise definition is left to the reader.) Let t be a term and s a semi-term. We call s a sub-semi-term of t if

(i) s contains a bound variable (that is, s is not a term),

(ii) s is not a bound variable itself,

(iii) some subterm of t is obtained from s by replacing all the bound variables in s by appropriate terms.

(2) A semi-formula is an expression like a formula, except that bound variables are (also) allowed to occur free in it (i.e., not in the scope of a quantifier).

THEOREM 6.9. *Let t be a term and S a provable sequent satisfying:*

(*) *There is no sub-semi-term of t in S.*

Then the sequent which is obtained from S by replacing all the occurrences of t in S by a free variable is also provable.

PROOF (outline). Consider a cut-free regular proof of S, say P. From the observation that if (*) holds for the lower sequent of an inference in P then it holds for the upper sequent(s), the theorem follows easily by mathematical induction on the number of inferences in P.

DEFINITION 6.10. Let $R_1,\ldots,R_m,R$ be predicate constants. Let $A(R,R_1,\ldots,R_m)$ be a sentence in which all occurrences of $R, R_1,\ldots, R_m$ are indicated. Let R' be a predicate constant with the same number of argument-places as R. Let B be $\forall x_1\ldots\forall x_k\,(R(x_1,\ldots,x_k)\equiv R'(x_1,\ldots,x_k))$, where the string of quantifiers is empty if $k=0$, and let C be $A(R,R_1,\ldots,R_m)\wedge A(R',R_1,\ldots,R_m)$. We say that $A(R,R_1,\ldots,R_m)$ defines (in **LK**) R implicitly in terms of $R_1,\ldots,R_m$ if $C\supset B$ is (**LK**-)provable and we say that $A(R,R_1,\ldots,R_m)$ defines (in **LK**) R explicitly in terms of $R_1,\ldots,R_m$ and the individual constants in $A(R,R_1,\ldots,R_m)$ if there exists a formula $F(a_1,\ldots,a_k)$ containing only the predicate constants $R_1,\ldots,R_m$ and the individual constants in $A(R,R_1,\ldots,R_m)$ such that

$$A(R,R_1,\ldots,R_m)\rightarrow\forall x_1\ldots\forall x_k\,(R(x_1,\ldots,x_k)\equiv F(x_1,\ldots,x_k))$$

is **LK**-provable.

PROPOSITION 6.11 (Beth's definability theorem for **LK**). *If a predicate constant R is defined implicitly in terms of $R_1,\ldots,R_m$ by $A(R,R_1,\ldots,R_m)$, then R can be defined explicitly in terms of $R_1,\ldots,R_m$ and the individual constants in $A(R,R_1,\ldots,R_m)$.*

PROOF (outline). Let $c_1, \ldots, c_n$ be free variables not occurring in A. Then

$$A(R, R_1, \ldots, R_m), A(R', R_1, \ldots, R_m) \to R(c_1, \ldots, c_n) \equiv R'(c_1, \ldots, c_n)$$

and hence also

$$A(R, R_1, \ldots, R_m) \wedge R(c_1, \ldots, c_k) \to A(R', R_1, \ldots, R_m) \supset R'(c_1, \ldots, c_n)$$

are provable. Now apply Craig's theorem (i.e., part (1) of Theorem 6.6) to the latter sequent.

We now present a version of Robinson's theorem (for **LK**).

PROPOSITION 6.12 (Robinson). *Assume that the language contains no function constants. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two consistent axiom systems. Suppose furthermore that, for any sentence A which is dependent on $\mathscr{A}_1$ and $\mathscr{A}_2$, it is not the case that $\mathscr{A}_1 \to A$ and $\mathscr{A}_2 \to \neg A$ (or $\mathscr{A}_1 \to \neg A$ and $\mathscr{A}_2 \to A$) are both provable. Then $\mathscr{A}_1 \cup \mathscr{A}_2$ is consistent. (See Definition* 4.1 *for the technical terms.)*

PROOF (outline). Suppose $\mathscr{A}_1 \cup \mathscr{A}_2$ is not consistent. Then there are axiom sequences Γ_1 and Γ_2 from $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively such that $\Gamma_1, \Gamma_2 \to$ is provable. Since $\mathscr{A}_1$ and $\mathscr{A}_2$ are each consistent, neither Γ_1 nor Γ_2 is empty. Apply Lemma 6.5 to the partition $[\{\Gamma_1;\}, \{\Gamma_2;\}]$.

Let **LK**$'$ and **LJ**$'$ denote the quantifier-free parts of **LK** and **LJ**, respectively, viz. the formulations (in tree form) of the classical and intuitionistic propositional calculus, respectively.

THEOREM 6.13. *There exist decision procedures for* **LK**$'$ *and* **LJ**$'$.

PROOF (outline). The following decision procedure was given by Gentzen. A sequent of **LK**$'$ (or **LJ**$'$) is said to be reduced if in the antecedent the same formula does not occur at more than three places as sequent-formulas, and likewise in the succedent. A sequent S' is called a *reduct* of a sequent S if S' is reduced and is obtained from S by deleting some occurrences of formulas. Now, given a sequent S of **LK**$'$ (or **LJ**$'$), let S' be any reduct of S. We note the following.

1) S is provable or unprovable according as S' is provable or unprovable.

2) The number of all reduced sequents which contain only subformulas of the formulas in S is finite.

Consider the finite system of sequents as in 2), say $\mathscr{S}$. Collect all initial sequents in the systems. Call this set $\mathscr{S}_0$. Then examine $\mathscr{S} - \mathscr{S}_0$ to see if there is a sequent which can be the lower sequent of an inference whose upper sequent(s) is (are) one (two) sequent(s) from $\mathscr{S}_0$. Call the set of all sequents which satisfy this condition $\mathscr{S}_1$. Now see if there is a sequent in $(\mathscr{S} - \mathscr{S}_0) - \mathscr{S}_1$ which can be the lower sequent of an inference whose upper sequent(s) is (are) one (two) of the sequent(s) in $\mathscr{S}_0 \cup \mathscr{S}_1$. Continue this process until either the sequent S' itself is determined as provable, or the process does not give any new sequent as provable. One of the two must happen. If the former is the case, then S is provable. Otherwise S is unprovable. (Note that the whole argument is finitary.)

THEOREM 6.14 (1) (Harrop). *Let Γ be a finite sequence of formulas such that in each formula of Γ every occurrence of $\vee$ and $\exists$ is either in the scope of a $\neg$ or in the left scope of a $\supset$* (cf. *Definition* 6.1, *part* 3)). *This condition will be referred to as* (*) *in this theorem. Then*

1) *$\Gamma \to A \vee B$ is* **LJ**-*provable if and only if $\Gamma \to A$ or $\Gamma \to B$ is* **LJ**-*provable,*

2) *$\Gamma \to \exists x\, F(x)$ is* **LJ**-*provable if and only if for some term s, $\Gamma \to F(s)$ is* **LJ**-*provable.*

(2) *The following sequents (which are* **LK**-*provable) are not (in general)* **LJ**-*provable.*

$$\neg(\neg A \wedge \neg B) \to A \vee B; \qquad \neg\forall x\, \neg F(x) \to \exists x\, F(x);$$

$$A \supset B \to \neg A \vee B; \qquad \neg\forall x\, F(x) \to \exists x\, \neg F(x);$$

$$\neg(A \wedge B) \to \neg A \vee \neg B.$$

PROOF. (1) part 1): The "if" part is trivial. For the "only if" part, consider a cut-free proof of $\Gamma \to A \vee B$. The proof is carried out by induction on the number of inferences below all the inferences for $\vee$ and $\exists$ in the given proof. If the last inference is $\forall$: right, there is nothing to prove. Notice that the last inference cannot be $\vee$, $\neg$, or $\exists$: left.

Case 1: The last inference is $\wedge$: left:

$$\frac{C, \Gamma \to A \vee B}{C \wedge D, \Gamma \to A \vee B}.$$

It is obvious that C satisfies the condition (*). Thus the induction hypothesis applies to the upper sequent; hence either $C, \Gamma \to A$ or $C, \Gamma \to B$ is provable. In either case, the end-sequent can be derived in **LJ**.

Case 2: The last inference is $\supset$: left:

$$\frac{\Gamma \to C \qquad D, \Gamma \to A \vee B}{C \supset D, \Gamma \to A \vee B}.$$

It is obvious that D satisfies the condition (*); thus, by the induction hypothesis applied to the right upper sequent, $D, \Gamma \to A$ or $D, \Gamma \to B$ is provable. In either case the end-sequent can be derived.

Other cases are treated likewise. The proofs of (1) part 2), and (2), are left to the reader.

§7. The predicate calculus with equality

DEFINITION 7.1. The predicate calculus with equality (denoted $\mathbf{LK_e}$) can be obtained from **LK** by specifying a predicate constant of two argument places ($=$: read equals) and adding the following sequents as additional initial sequents ($a = b$ denoting $= (a, b)$):

$$\to s = s;$$

$$s_1 = t_1, \ldots, s_n = t_n \to f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)$$

for every function constant f of n argument-places ($n = 1, 2, \ldots$);

$$s_1 = t_1, \ldots, s_n = t_n, R(s_1, \ldots, s_n) \to R(t_1, \ldots, t_n)$$

for every predicate constant R (including $=$) of n argument-places ($n = 1, 2, \ldots$); where $s, s_1, \ldots, s_n, t_1, \ldots, t_n$ are arbitrary terms.

Each such sequent may be called an equality axiom of $\mathbf{LK_e}$.

PROPOSITION 7.2. *Let $A(a_1, \ldots, a_n)$ be an arbitrary formula. Then*

$$s_1 = t_1, \ldots, s_n = t_n, A(s_1, \ldots, s_n) \to A(t_1, \ldots, t_n)$$

is provable in $\mathbf{LK_e}$, for any terms s_i, t_i ($1 \leqslant i \leqslant n$). Furthermore, $s = t \to t = s$ and $s_1 = s_2, s_2 = s_3 \to s_1 = s_3$ are also provable.

DEFINITION 7.3. Let Γ_e be the set (axiom system) consisting of the following sentences:

$$\forall x(x = x),$$

$$\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n [x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)]$$

for every function constant f with n argument-places ($n = 1, 2, \ldots$),

$$\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n [x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge R(x_1, \ldots, x_n) \supset R(y_1, \ldots, y_n)]$$

for every predicate constant R of n argument-places ($n = 1, 2, \ldots$). Each such sentence is called an equality axiom.

PROPOSITION 7.4. *A sequent $\Gamma \to \Delta$ is provable in* $\mathbf{LK}_e$ *if and only if $\Gamma, \Gamma_e \to \Delta$ is provable in* **LK**.

PROOF. Only if: It is easy to see that all initial sequents of $\mathbf{LK}_e$ are provable from Γ_e. Therefore the proposition is proved by mathematical induction on the number of inferences in a proof of the sequent $\Gamma \to \Delta$.

If: All formulas of Γ_e are $\mathbf{LK}_e$-provable.

DEFINITION 7.5. If the cut formula of a cut in $\mathbf{LK}_e$ is of the form $s = t$, then the cut is called inessential. It is called essential otherwise.

THEOREM 7.6 (the cut-elimination theorem for the predicate calculus with equality, $\mathbf{LK}_e$). *If a sequent of* $\mathbf{LK}_e$ *is* $\mathbf{LK}_e$*-provable, then it is* $\mathbf{LK}_e$*-provable without an essential cut.*

PROOF. The theorem is proved by removing essential cuts (mixes as a matter of fact), following the method used for Theorem 5.1.

If the rank is 2, S_2 is an equality axiom and the cut formula is not of the form $s = t$, then the cut formula is of the form $P(t_1, \ldots, t_n)$. If S_1 is also an equality axiom, then it has the form

$$s_1 = t_1, \ldots, s_n = t_n, P(s_1, \ldots, s_n) \to P(t_1, \ldots, t_n).$$

From this and S_2, i.e.,

$$t_1 = r_1, \ldots, t_n = r_n, P(t_1, \ldots, t_n) \to P(r_1, \ldots, r_n),$$

we obtain by a mix

$$s_1 = t_1, \ldots, s_n = t_n, t_1 = r_1, \ldots, t_n = r_n, P(s_1, \ldots, s_n) \to P(r_1, \ldots, r_n).$$

This may be replaced by

$$s_i = t_i, t_i = r_i \to s_i = r_i \quad (i = 1, 2, \ldots, n);$$

$$s_1 = r_1, \ldots, s_n = r_n, P(s_1, \ldots, s_n) \to P(r_1, \ldots, r_n);$$

and then repeated cuts of $s_i = r_i$ to produce the same end-sequent. All cuts (or mixes) introduced here are inessential.

If $P(t_1, \ldots, t_n)$ in S_2 is a weakening formula, then the mix inference is:

$$\frac{s_1 = t_1, \ldots, s_n = t_n, P(s_1, \ldots, s_n) \to P(t_1, \ldots, t_n) \qquad P(t_1, \ldots, t_n), \Pi \to \Lambda}{s_1 = t_1, \ldots, s_n = t_n\ P(s_1, \ldots, s_n), \Pi \to \Lambda.}$$

Transform this into:

$$\frac{\Pi \to \Lambda}{\overline{\text{end-sequent.}}}$$

The rest of the argument in Theorem 5.1 goes through.

PROBLEM 7.7. A sequent of the form

$$s_1 = t_1, \ldots, s_n = t_n \to s = t \quad (n = 0, 1, 2, \ldots)$$

is said to be simple if it is obtained from sequents of the following four forms by applications of exchanges, contractions, cuts, and weakening left.

1) $\to s = s$.
2) $s = t \to t = s$.
3) $s_1 = s_2, s_2 = s_3 \to s_1 = s_3$.
4) $s_1 = t_1, \ldots, s_m = t_m \to f(s_1, \ldots, s_m) = f(t_1, \ldots, t_m)$.

Prove that if $s_1 = s_1, \ldots, s_m = s_m \to s = t$ is simple, then $s = t$ is of the form $s = s$. As a special case, if $\to s = t$ is simple, then $s = t$ is of the form $s = s$.

Let $\mathbf{LK}_e'$ be the system which is obtained from $\mathbf{LK}$ by adding the following sequents as initial sequents:

a) simple sequents,
b) sequents of the form

$$s_1 = t_1, \ldots, s_m = t_m, R(s_1', \ldots, s_n') \to R(t_1', \ldots, t_n'),$$

where $s_1 = t_1, \ldots, s_m = t_m \to s_i' = t_i'$ is simple for each i $(i = 1, \ldots, n)$. First prove that the initial sequents of $\mathbf{LK}_e'$ are closed under cuts and that if

$$R(s_1, \ldots, s_n) \to R(t_1, \ldots, t_n)$$

is an initial sequent of $\mathbf{LK}_e'$ (where R is not $=$), then it is of the form $D \to D$. Finally, prove that the cut-elimination theorem (without the exception of inessential cuts) holds for $\mathbf{LK}_e'$.

PROBLEM 7.8. Show that if a sequent S without the $=$ symbol is $\mathbf{LK}_e$-provable, then it is provable in **LK** (without $=$).

PROBLEM 7.9. Prove that Theorems 6.2–6.4, 6.6, 6.9, and 6.14, Propositions 6.11 and 6.12 and Exercise 6.7 hold for $\mathbf{LK}_e$ when they are modified in the following way: References to **LK**- (or **LJ**-) provability are replaced throughout by references to $\mathbf{LK}_e$-provability, and further, when the statement demands that a formula can contain only certain constants, $=$ can be added as an exception.

The general technique of proof is to change a condition that a sequent $\Gamma \to \Delta$ be provable in **LK** to one that a sequent $\Pi, \Gamma \to \Delta$ be provable in **LK**, where Π is a set of equality axioms, and in this way to reduce the problem to **LK**.

§8. The completeness theorem

Although we do not intend to develop model theory in this book, we shall outline a proof of the completeness theorem for **LK**. The completeness theorem for the first order predicate calculus was first proved by Gödel. Here we follow Schütte's method, which has a close relationship to the cut-elimination theorem. In fact the cut-elimination theorem is a corollary of the completeness theorem as formulated below. (The importance of the proof of cut-elimination in §5 lies in its constructive nature.)

DEFINITION 8.1. (1) Let L be a language as described in §1. By a *structure* for L (an L-structure) we mean a pair $\langle D, \phi\rangle$, where D is a non-empty set and ϕ is a map from the constants of L such that

(i) if k is an individual constant, then ϕk is an element of D;

(ii) if f is a function constant of n arguments, then ϕf is a mapping from D^n into D;

(iii) if R is a predicate constant of n arguments, then ϕR is a subset of D^n.

(2) An *interpretation* of L is a structure $\langle D, \phi \rangle$ together with a mapping ϕ_0 from variables into D. We may denote an interpretation $(\langle D, \phi \rangle, \phi_0)$ simply by $\mathfrak{I}$. ϕ_0 is called an assignment from D.

(3) We say that an interpretation $\mathfrak{I} = (\langle C, \phi \rangle, \phi_0)$ *satisfies* a formula A if this follows from the following inductive definition. In fact we shall define the notion of "satisfying" for all semi-formulas (cf. Definition 6.8).

0) Firstly, we define $\phi(t)$, for every semi-term t, inductively as follows. We define $\phi(a) = \phi_0(a)$ and $\phi(x) = \phi_0(x)$ for all free variables a and bound variables x. Next, if f is a function constant and t is a semi-term for which ϕt is already defined, then $\phi(f(t))$ is defined to be $(\phi f)(\phi t)$.

1) If R is a predicate constant of n arguments and $t_1, \ldots, t_n$ are semi-terms, then $\mathfrak{I}$ satisfies $R(t_1, \ldots, t_n)$ if and only if $\langle \phi t_1, \ldots, \phi t_n \rangle \in \phi R$.

2) $\mathfrak{I}$ satisfies $\neg A$ if and only if it does not satisfy A; $\mathfrak{I}$ satisfies $A \wedge B$ if and only if it satisfies both A and B; $\mathfrak{I}$ satisfies $A \vee B$ if and only if it satisfies either A or B; $\mathfrak{I}$ satisfies $A \supset B$ if and only if either it does not satisfy A or it satisfies B.

3) $\mathfrak{I}$ satisfies $\forall x\, B$ if and only if for every ϕ_0' such that ϕ_0 and ϕ_0' agree, except possibly on x, $(\langle D, \phi \rangle, \phi_0')$ satisfies B; $\mathfrak{I}$ satisfies $\exists x\, B$ if and only if for some ϕ_0' such that ϕ_0 and ϕ_0' agree, except possibly on x, $(\langle D, \phi \rangle, \phi_0')$ satisfies B.

If $\mathfrak{I} = (\langle D, \phi \rangle, \phi_0)$ satisfies a formula A, we say that A is satisfied in $\langle D, \phi \rangle$ by ϕ_0, or simply A is satisfied by $\mathfrak{I}$.

(4) A formula is called valid in $\langle D, \phi \rangle$ if and only if for every ϕ_0, $(\langle D, \phi \rangle, \phi_0)$ satisfies that formula. It is called valid if it is valid in every structure.

(5) A sequent $\Gamma \to \Delta$ is satisfied in $\langle D, \phi \rangle$ by ϕ_0 (or $\mathfrak{I} = (\langle D, \phi \rangle, \phi_0)$ satisfies $\Gamma \to \Delta$) if either some formula in Γ is not satisfied by $\mathfrak{I}$, or some formula in Δ is satisfied by $\mathfrak{I}$. A sequent is valid if it is satisfied in every interpretation.

(6) A structure may also be denoted as

$$\langle D; \phi k_0, \phi k_1, \ldots, \phi f_0, \phi f_1, \ldots, \phi R_0, \phi R_1, \ldots \rangle.$$

A structure is called a model of an axiom system Γ if every sentence of Γ is valid in it. It is called a counter-model of Γ if there is a sentence of Γ which is not valid in it.

THEOREM 8.2 (completeness and soundness). *A formula is provable in* **LK** *if and only if it is valid.*

NOTES. (1) The "if" part of the theorem is the statement of the completeness of **LK**. In general, a system is said to be complete if and only if every valid formula is provable in the system (for a suitable definition of validity).

Soundness means: all provable sequents are valid, i.e., the "only if" part of the theorem. Soundness ensures consistency.

(2) The theorem connects proof theory with semantics, where semantics means, very roughly, the study of the interpretation of formulas in a structure (of a language), and hence of their truth or falsity.

PROOF OF THEOREM 8.2. The "only if" part is easily proved by induction on the number of inferences in a proof of the formula. We prove the "if" part in the following generalized form:

LEMMA 8.3. *Let S be a sequent. Then either there is a cut-free proof of S, or there is an interpretation which does not satisfy S (and hence S is not valid).*

PROOF. We will define, for each sequent S, a (possibly infinite) tree, called the reduction tree for S, from which we can obtain either a cut-free proof of S or an interpretation not satisfying S. (This method is due to Schütte.) This reduction tree for S contains a sequent at each node. It is constructed in stages as follows.

Stage 0: Write S at the bottom of the tree.

Stage k ($k > 0$): This is defined by cases:

Case I. Every topmost sequent has a formula common to its antecedent and succedent. Then stop.

Case II. Not Case I. Then this stage is defined according as

$$k \equiv 0, 1, 2, \ldots, 11, 12 \pmod{13}.$$

$k \equiv 0$ and $k \equiv 1$ concern the symbol $\neg$; $k \equiv 2$ and $k \equiv 3$ concern $\wedge$; $k \equiv 4$ and $k \equiv 5$ concern $\vee$; $k \equiv 6$ and $k \equiv 7$ concern $\supset$; $k \equiv 8$ and $k \equiv 9$ concern $\forall$; and $k \equiv 10$ and $k \equiv 11$ concern $\exists$.

Since the formation of reduction trees is a common technique and will be used several times in this text, we shall describe these stages of the so-called reduction process in detail. In order to make the discussion simpler, let us assume that there are no individual or function constants.

All the free variables which occur in any sequent which has been obtained at or before stage k are said to be "available at stage k". In case there is none, pick any free variable and say that it is available.

0) $k \equiv 0$. Let $\Pi \to \Lambda$ be any topmost sequent of the tree which has been defined by stage $k-1$. Let $\neg A_1, \ldots, \neg A_n$ be all the formulas in Π whose outermost logical symbol is $\neg$, and to which no reduction has been applied in previous stages. Then write down

$$\Pi \to \Lambda, A_1, \ldots, A_n$$

above $\Pi \to \Lambda$. We say that a $\neg$: left reduction has been applied to $\neg A_1, \ldots, \neg A_n$.

1) $k \equiv 1$. Let $\neg A_1, \ldots, \neg A_n$ be all the formulas in Λ whose outermost logical symbol is $\neg$ and to which no reduction has been applied so far. Then write down

$$A_1, \ldots, A_n, \Pi \to \Lambda$$

above $\Pi \to \Lambda$. We say that a $\neg$: right reduction has been applied to $\neg A_1, \ldots, \neg A_n$.

2) $k \equiv 2$. Let $A_1 \wedge B_1, \ldots, A_n \wedge B_n$ be all the formulas in Π whose outermost logical symbol is $\wedge$ and to which no reduction has been applied yet. Then write down

$$A_1, B_1, A_2, B_2, \ldots, A_n, B_n, \Pi \to \Lambda$$

above $\Pi \to \Lambda$. We say that an $\wedge$: left reduction has been applied to

$$A_1 \wedge B_1, \ldots, A_n \wedge B_n.$$

3) $k \equiv 3$. Let $A_1 \wedge B_1, A_2 \wedge B_2, \ldots, A_n \wedge B_n$ be all the formulas in Λ whose outermost logical symbol is $\wedge$ and to which no reduction has been applied yet. Then write down all sequents of the form

$$\Pi \to \Lambda, C_1, \ldots, C_n,$$

where C_i is either A_i or B_i, above $\Pi \to \Lambda$. Take all possible combinations of such: so there are 2^n such sequents above $\Pi \to \Lambda$. We say that an $\wedge$: right reduction has been applied to $A_1 \wedge B_1, \ldots, A_n \wedge B_n$.

4) $k \equiv 4$. $\vee$: left reduction. This is defined in a manner symmetric to 3).

5) $k \equiv 5$. $\vee$: right reduction. This is defined in a manner symmetric to 2).

6) $k \equiv 6$. Let $A_1 \supset B_1, \ldots, A_n \supset B_n$ be all the formulas in Π whose outermost logical symbol is $\supset$ and to which no reduction has been applied yet. Then write down the following sequents above $\Pi \to \Lambda$:

$$B_1, B_2, \ldots, B_n, \Pi \to \Lambda$$

and

$$\Pi \to \Lambda, A_i \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

We say that an $\supset$: left reduction has been applied to $A_1 \supset B_1, \ldots, A_n \supset B_n$.

7) $k \equiv 7$. Let $A_1 \supset B_1, \ldots, A_n \supset B_n$ be all the formulas in Λ whose outermost logical symbol is $\supset$ and to which no reduction has been applied yet. Then write down

$$A_1, A_2, \ldots, A_n, \Pi \to \Lambda, B_1, B_2, \ldots, B_n$$

above $\Pi \to \Lambda$. We say that an $\supset$: right reduction has been applied to

$$A_1 \supset B_1, \ldots, A_n \supset B_n.$$

8) $k \equiv 8$. Let $\forall x_1\, A_1(x_1), \ldots, \forall x_n\, A_n(x_n)$ be all the formulas in Π whose outermost logical symbol is $\forall$. Let a_i be the first variable available at this stage which has not been used for a reduction of $\forall x_i\, A_i(x)$ for $1 \leqslant i \leqslant n$. Then write down

$$A_1(a_1), \ldots, A_n(a_n), \Pi \to \Lambda$$

above $\Pi \to \Lambda$. We say that a $\forall$: left reduction has been applied to

$$\forall x_1\, A_1(x), \ldots, \forall x_n\, A_n(x_n).$$

9) $k \equiv 0$. Let $\forall x_1\, A_1(x_1), \ldots, \forall x_n\, A_n(x_n)$ be all the formulas in Λ whose outermost logical symbol is $\forall$ and to which no reduction has been applied so far. Let $a_1, \ldots, a_n$ be the first n free variables (in the list of variables) which are *not* available at this stage. Then write down

$$\Pi \to \Lambda, A_1(a_1), \ldots, A_n(a_n)$$

above $\Pi \to \Lambda$. We say that a $\forall$: right reduction has been applied to $\forall x_1\, A_1(x_1), \ldots, \forall x_n\, A_n(x_n)$. Notice that $a_1, \ldots, a_n$ are new available free variables.

10) $k \equiv 10$. $\exists$: left reduction. This is defined in a manner symmetric to 9).

11) $k \equiv 11$. $\exists$: right reduction. This is defined in a manner symmetric to 8).

12) If Π and Λ have any formula in common, write nothing above $\Pi \to \Lambda$ (so this remains a topmost sequent). If Π and Λ have no formula in common, write the same sequent $\Pi \to \Lambda$ again above it.

So the collection of those sequents which are obtained by the above reduction process, together with the partial order obtained by this process, is the reduction tree (for S). It is denoted by $T(S)$. We will construct "reduction trees" like this again.

Now a (finite or infinite) sequence $S_0, S_1, S_2, \ldots$ of sequents in $T(S)$ is called a branch if (i) $S_0 = S$; (ii) S_{i+1} stands immediately above S_i; (3) if the sequence is finite, say $S_1, \ldots, S_n$, then S_n has the form $\Pi \to \Lambda$, where Π and Λ have a formula in common.

Now, given a sequent S, let T be the reduction tree $T(S)$. If each branch of T ends with a sequent whose antecedent and succedent contain a formula in common, then it is a routine task to write a proof without a cut ending with S by suitably modifying T. Otherwise there is an infinite branch. Consider such a branch, consisting of sequents $S = S_0, S_1, \ldots, S_n, \ldots$.

Let S_i be $\Gamma_i \to \Delta_i$. Let $\bigcup \Gamma$ be the set of all formulas occurring in Γ_i for some i, and let $\bigcup \Delta$ be the set of all formulas occurring in Δ_j for some j. We shall define an interpretation in which every formula in $\bigcup \Gamma$ holds and no formula in $\bigcup \Delta$ holds. Thus S does not hold in it.

First notice that from the way the branch was chosen, $\bigcup \Gamma$ and $\bigcup \Delta$ have no atomic formula in common. Let D be the set of all the free variables. We consider the interpretation $\mathfrak{I} = (\langle D, \phi \rangle, \phi_0)$, where ϕ and ϕ_0 are defined as follows: $\phi_0(a) = a$ for all free variables a, $\phi_0(x)$ is defined arbitrarily for all bound variables x. For an n-ary predicate constant R, ϕR is any subset of D^n such that: if $R(a_1, \ldots, a_n) \in \bigcup \Gamma$, then $(a_1, \ldots, a_n) \in \phi R$, and if $R(a_1, \ldots, a_n) \in \bigcup \Delta$, then $(a_1, \ldots, a_n) \notin \phi R$.

We claim that this interpretation $\mathfrak{I}$ has the required property: it satisfies every formula in $\bigcup \Gamma$, but no formula in $\bigcup \Delta$. We prove this by induction on the number of logical symbols in the formula A. We consider here only the case where A is of the form $\forall x\, F(x)$ and assume the induction hypothesis:

Subcase 1. A is in $\bigcup \Gamma$. Let i be the least number such that A is in Γ_i. Then A is in Γ_j for all $j > i$. It is sufficient to show that all substitution instances $A(a)$, for $a \in D$, are satisfied by $\mathfrak{I}$, i.e., all these substitution instances are in $\bigcup \Gamma$. But this is evident from the way we construct the tree.

Subcase 2. A is in $\bigcup \Delta$. Consider the step at which A was used to define an upper sequent from $\Gamma_i \to \Delta_i$ (or $\Gamma_i \to \Delta_i^1, A, \Delta_i^2$). It looks like this:

$$\frac{\Gamma_{i+1} \to \Delta_{i+1}^1, F(a), \Delta_{i+1}^2}{\Gamma_i \to \Delta_i^1, A, \Delta_i^2}.$$

Then by the induction hypothesis, $F(a)$ is not satisfied by $\mathfrak{I}$, so A is not satisfied by $\mathfrak{I}$ either. This completes the proof.

PROBLEM 8.4 (Scott). (1) Consider a language, denoted by $L(R_1, \ldots, R_k)$, which contains only finitely many predicate constants $R_1, \ldots, R_k$ and no individual or function constants. If $I \subseteq \{1, \ldots, k\}$, we define an I-formula to be a formula containing only predicates with indices in I. Let $\mathrm{F} \subseteq P(\{1, \ldots, k\})$ (the power set of $\{1, \ldots, k\}$) and $\mathrm{F} \neq \emptyset$. An F-formula is defined to be a propositional combination of I-formulas for I in F, viz. a formula consisting of I-formulas, for various I's in F, joined together by $\vee$ and $\wedge$. If $\mathfrak{A} = \langle A, \bar{R}_1, \ldots, \bar{R}_k \rangle$ is a structure for our language and $I = \{i_1, \ldots, i_m\}$, let $\mathfrak{A}_I$ be the structure obtained from $\mathfrak{A}$ by restricting $\mathfrak{A}$ to the predicates with indices in I: thus $\mathfrak{A}_I$ is $\langle A, \bar{R}_{i_1}, \ldots, \bar{R}_{i_m} \rangle$. If $\mathfrak{A}$ and $\mathfrak{B}$ are two structures of $\mathrm{L}(R_1, \ldots, R_k)$, they are said to be F-isomorphic if $\mathfrak{A}_I$ and $\mathfrak{B}_I$ are isomorphic for each I in F. Prove the following interpolation theorem concerning F-isomorphic models:

Let $\mathscr{T}$ be a theory (axiom system) in $\mathrm{L}(R_1, \ldots, R_k)$ and A and B two sentences in $\mathrm{L}(R_1, \ldots, R_k)$. Suppose that whenever $\mathfrak{A}$ and $\mathfrak{B}$ are F-isomorphic models of $\mathscr{T}$ and $\mathfrak{A}$ satisfies A then $\mathfrak{B}$ satisfies B. Then there is an F-sentence C such that $\mathscr{T}, A \to C$ and $\mathscr{T}, C \to B$ are provable in **LK**. [*Hint* (Africk's method): We first introduce a new predicate constant S_i for each R_i in the language. Each S_i has the same number of arguments as R_i. If A is an expression in the language $\mathrm{L}(R_1, \ldots, R_k)$, then A^* denotes the expression obtained from A by replacing all occurrences of R_i by S_i for each $i = 1, \ldots, k$.]

(2) Corresponding to each I in F we adjoin to the language function constants f_I and g_I. f_I will represent an isomorphism between $\mathfrak{A}_I$ and $\mathfrak{B}_I$ when $\mathfrak{A}$ and $\mathfrak{B}$ are F-isomorphic, and g_I will represent the mapping inverse to f_I.

(3) Consider the language $\mathrm{L}' = \mathrm{L}(R_1, \ldots, R_k, S_1, \ldots, S_k, f_I, g_I \colon I \in \mathrm{F})$, in which the notion of "F-isomorphism between two structures" can be syntactically expressed. Let $\cong_{\mathrm{F}}$ be such a sentence.

(4) With the notion of F-isomorphism formulated syntactically, the problem now boils down to proving the following lemma.

LEMMA 8.5. *Let Φ_1, Ψ_1 be finite (possibly empty) sequences of formulas in* $\mathrm{L}(R_1, \ldots, R_k, f_I, g_I \colon I \in \mathrm{F})$ *such that no function constant contains a bound variable.*

Let Φ_2^*, Ψ_2^* be finite (possibly empty) sequences of formulas in $\mathrm{L}(S_1, \ldots, S_k, f_I, g_I \colon I \in \mathrm{F})$ *such that no function constant contains a bound variable.*

Let Φ_3 be a finite (possibly empty) sequence of subformulas of formulas in $\cong_{\mathrm{F}}$. Suppose $\Phi_1, \Phi_2^*, \Phi_3 \to \Psi_1 \Psi_2^*$ is provable. Then there exists an F*-formula Σ in the language* $\mathrm{L}(R_1, \ldots, R_k, f_I, g_I \colon I \in \mathrm{F})$ *and an* F*-formula Σ^* in the language* $\mathrm{L}(S_1, \ldots, S_k, f_I, g_I \colon I \in \mathrm{F})$ *such that $\Phi_1 \to \Psi_1, \Sigma$ and $\Sigma^*, \Phi_2^* \to \Psi_2^*$ are provable, and further:*

1) *No function constant of Σ or Σ^* contains a bound variable.*

2) *Every free variable of* Σ *or* Σ^* *must occur among the free variables of* Φ_1, Φ_2^*, Ψ_1 *and* Ψ_2^*.

3) Σ^* *may be obtained from* Σ *by replacing each predicate constant* R_i *by* S_i, *and each term* t *by* t^*; *where either* (a) t^* *is* $f_l(t)$, *or* (b) t *is* $g_l(t^*)$; *and furthermore if* (a) *holds, we require that* t^* *be an argument of a predicate* S_k *such that* $k \in I$, *and if* (b) *holds, we require that* t *be an argument of a predicate* R_k *such that* $k \in I$.

PROOF. The proof is by mathematical induction on the number of inferences in a cut-free proof of the given sequent. In most cases, the construction of Σ is routine; for $\supset$: left and those inferences which introduce quantifiers, we need the following result:

SUBLEMMA 8.6. *Let* Σ_1 *and* Σ_1^* *be* F-*formulas such that* $\Phi_1 \to \Psi_1, \Sigma_1$ *and* $\Sigma_1^*, \Phi_2^* \to \Psi_2^*$ *are* (**LK**-) *provable and having properties* 1) *and* 3) *of Lemma* 8.5. *Suppose that the only term containing the free variable* a *which occurs in* Φ_1, Φ_2^*, Ψ_1 *or* Ψ_2^* *is* a *itself. Then there exist* F-*formulas* Σ *and* Σ^* *such that* $\Phi_1 \to \Psi_1, \Sigma$ *and* $\Sigma^*, \Phi_2^* \to \Psi_2^*$ *are provable, and with properties* 1) *and* 3) *of Lemma* 8.5, *and such that* a *does not occur in either* Σ *or* Σ^*, *and all free variables of* Σ *and* Σ^* *are contained in* Σ_1 *and* Σ_1^*.

Such a Σ (resp. Σ^*) can be constructed from Σ_1 (resp. Σ_1^*) by reducing the number of occurrences of a step by step. This can be done by noting the following facts.

(i) We may assume that if a term t (t^*) occurs in Σ_1 (Σ_1^*) in the context of (b) ((a)) in part 3) of Lemma 8.5 and contains a, then t (t^*) is not of the form $g_l(f_l(t'))$ ($f_l(g_l(t'))$).

(ii) Σ_1 can be expressed in the form $\bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} A_{ij}$, where A_{ij} is an I_j-formula. For a fixed j, a term with an occurrence of a, which is not contained in some other term, occurs in A_{ij} either in the context of (a) for all i or in the context of (b) for all i. Similarly with t^*.

(iii) Take a term t which contains a and is not contained in some other term, and is the most complicated such term. If t occurs in A_{ij}, say $A_{ij}(t)$, then we can replace A_{ij} in Σ_1 by $\forall x\, A_{ij}(x)$. Likewise, we can change Σ_1^* in this way.

Continue this process until there is no further occurrence of a.

With the help of Sublemma 8.6, the problem of excessive free variables in constructing an interpolant from the induction hypothesis (in the cases of $\supset$, $\forall$ and $\exists$) is easily solved.

PROBLEM 8.7 (Feferman). Let J be a non-empty set. Each element of J is called a *sort*. A many-sorted language for the set of sorts J, say $\mathrm{L}(J)$, consists of the following.

1) Individual constants: $k_0, k_1, \ldots, k_i, \ldots$, where to each k_i is assigned one sort.

2) Predicate constants: $R_0, R_1, \ldots, R_i, \ldots$, where to each R_i is assigned a number n ($\geqslant 0$) (the number of arguments) and sorts $j_1, \ldots, j_n$. We say that $(n; j_1, \ldots, j_n)$ is assigned to R_i.

3) Function constants: $f_0, f_1, \ldots, f_i, \ldots$, where to each f_i is assigned a number n ($\geqslant 1$) (the number of arguments) and sorts $j_1, \ldots, j_n, j$. We say that $(n; j_1, \ldots, j_n, j)$ is assigned to f_i.

4) Free variables of sort j for each j in J: $a_0^j, a_1^j, \ldots, a_i^j, \ldots$.

5) Bound variables of sort j for each j in J: $x_0^j, x_1^j, \ldots, x_i^j, \ldots$.

6) Logical symbols: $\neg, \wedge, \vee, \supset, \forall, \exists$.

Terms of sort j for each j are defined as follows. Individual constants and free variables of sort j are terms of sort j; if f is a function constant with $(n; j_1, \ldots, j_n, j)$ assigned to it and $t_1, \ldots, t_n$ are terms of sort $j_1, \ldots, j_n$, respectively, then $f(t_1, \ldots, t_n)$ is a term of sort j.

If R is a predicate constant with $(n; j_1, \ldots, j_n)$ assigned to it and $t_1, \ldots, t_n$ are terms of sort $j_1, \ldots, j_n$, respectively, then $R(t_1, \ldots, t_n)$ is an atomic formula. If $F(a^j)$ is a formula and x^j does not occur in $F(a^j)$, then $\forall x^j\, F(x^j)$ and $\exists x^j\, F(x^j)$ are formulas; the other steps in building formulas of $\mathrm{L}(J)$ are as usual. The sequents of $\mathrm{L}(J)$ are defined as usual.

The rules of inference are those of **LK**, except that in the rules for $\forall$ and $\exists$, terms and free variables must be replaced by bound variables of the same sort.

Prove the following:

(1) The cut-elimination theorem holds for the system just defined.

Next, define Sort, Ex, Un, Fr, Cn and Pr as follows. $\mathrm{Sort}(A)$ is the set of j in J such that a symbol of sort j occurs in A; $\mathrm{Ex}(A)$ and $\mathrm{Un}(A)$ are the sets of sorts of bound variables which occur in some essentially existential, respectively universal quantifier in A. (An occurrence of $\exists$, say $\sharp$, is said to be essentially existential or universal according to the following definition. Count the number of $\neg$ and $\supset$ in A such that $\sharp$ is either in the scope of $\neg$, or in the left scope of $\supset$. If this number is even, then $\sharp$ is essentially existential in A, while if it is odd then $\sharp$ is essentially universal. Likewise, we define, dually, an occurrence of $\forall$ to be essentially existential or universal.) $\mathrm{Fr}(A)$ is the set of free variables in A; $\mathrm{Cn}(A)$ is the set of individual constants in A; $\mathrm{Pr}(A)$ is the set of predicate constants in A.

(2) Suppose $A \supset B$ is provable in the above system and at least one of $\mathrm{Sort}(A) \cap \mathrm{Ex}(B)$ and $\mathrm{Sort}(B) \cap \mathrm{Un}(A)$ is not empty. Then there is a formula C such that $\sigma(C) \subseteq \sigma(A) \cap \sigma(B)$, where σ stands for Fr, Cn, Pr or Sort, and such that $\mathrm{Un}(C) \subseteq \mathrm{Un}(A)$ and $\mathrm{Ex}(C) \subseteq \mathrm{Ex}(B)$. [*Hint*: Re-state the above theorem for sequents and apply (1), viz. the cut-elimination theorem.]

PROBLEM 8.8 (Feferman: an extension of a theorem of Łos and Tarski). We can define a structure for a many-sorted language (cf. Problem 8.7) as follows. Let $\mathrm{L}(J)$ be a many-sorted language. A *structure* for $\mathrm{L}(J)$ is a pair $\langle D, \phi \rangle$, where D is a set of non-empty sets $\{D_j; j \in J\}$ and ϕ is a map from the constants of $\mathrm{L}(J)$ into appropriate objects. We call D_j the *domain of the structure of sort j*. We leave the listing of the conditions on ϕ to the reader; we only have to keep in mind that an individual constant of sort j is a member of D_j. Let $\mathscr{M} = \langle D, \phi \rangle$ and $\mathscr{M}' = \langle D', \phi' \rangle$ be two structures for $\mathrm{L}(J)$. Let $J_0 \subseteq J$. We say that $\mathscr{M}'$ is an *extension*$_{J_0}$ of $\mathscr{M}$ and write $\mathscr{M} \subseteq_{J_0} \mathscr{M}'$ if

(i) for each j in J, $D_j \subseteq D'_j$,

(ii) for every j in J_0, $D'_j = D_j$,

(iii) for each individual constant k, $\phi' k = \phi k$,

(iv) for each predicate constant R with $(n; j_1, \ldots, j_n)$ assigned to it,

$$\phi R = \phi' R \cap (D_{j_1} \times \ldots \times D_{j_n}),$$

(v) for each function constant f with $(n; j_1, \ldots, j_n, j)$ assigned to it and $(d_1, \ldots, d_n) \in D_{j_1} \times \ldots \times D_{j_n}$,

$$(\phi' f)(d_1, \ldots, d_n) = (\phi f)(d_1, \ldots, d_n).$$

A formula is said to be *existential*$_{J_0}$ if $\mathrm{Un}(A) \subset J_0$.

Suppose given $J_0 \subseteq J$ and a formula of $L(J)$, say A, whose free variables are (only) $b_1, \ldots, b_n$, of sorts $j_1, \ldots, j_n$, respectively. Show that the following two statements are equivalent.

(1) For arbitrary structures $\mathscr{M}$ and $\mathscr{M}'$, where $\mathscr{M} \subseteq_{J_0} \mathscr{M}'$, and arbitrary maps ϕ_0 and ϕ'_0 from variables into the domains (of the correct sorts) of $\mathscr{M}$ and $\mathscr{M}'$ respectively which agree on $b_1, \ldots, b_n$, if $(\mathscr{M}, \phi_0)$ satisfies A then so does $(\mathscr{M}', \phi'_0)$.

(2) There is a formula B which is existential$_{J_0}$ for which $A \equiv B$ is provable, and $\mathrm{Fr}(B) \subseteq \mathrm{Fr}(A)$. [*Hint* (Feferman): Assume (2). It can be easily shown, by induction on the complexity of B, that (1) holds for B, from which follows (1) for A. In order to prove the converse, proceed as follows.

We assume (for simplicity) that the language has no individual and function constants. The major task is to write down the conditions in (1) syntactically, by considering an extended language in which we can express the relation of extension$_{J_0}$ between two structures.

Let $\mathscr{M}$ and $\mathscr{M}'$ be two structures of the form

$$\mathscr{M} = \langle \{D_j\}_{j\in J}, \{R_i\}_{i\in I}\rangle, \qquad \mathscr{M}' = \langle \{D_j'\}_{j\in J'}, \{R_i'\}_{i\in I}\rangle,$$

where J and J' are disjoint and in one-to-one correspondence. We denote corresponding elements in J and J' by j and j', respectively. Let J^+ be $J \cup J'$. $(J^+, I, \langle k_i\rangle_{i\in I})$ will determine a "type" of structures. Let L^+ be a corresponding language. It contains the original language L as a sublanguage. For each bound variable u, say the nth bound variable of sort j, let u' be the nth bound variable of sort j'. If C is an L-formula, then C' denotes the result of replacing each bound variable u in C by u'; hence $\mathrm{Fr}(C) = \mathrm{Fr}(C')$. With this notation, define Ext to be the set of sentences of the form $\forall u'\, \exists u\, (u' = u)$ for each sort of variable u in J, and $\forall u\, \exists u'\, (u = u')$ for each sort of variable in J_0. Then Ext and $\exists u_i'\, (u_i' = b_i)$ for $i = 1,\ldots, n$ yield $A' \to A$. So there is a finite subset Ext_1 of Ext and a cut-free proof of

$$(*) \qquad \mathrm{Ext}_1, \{\exists u_i'\, (u_i' = b_i)\}_{i=1}^n, A' \to A.$$

Now apply the interpolation theorem (2) of Problem 8.7. An interpolant B can be chosen so as to satisfy:

(i) $\mathrm{Fr}(B) \subseteq \mathrm{Fr}(A) = \{b_1,\ldots, b_n\}$,

(ii) $\mathrm{Rel}(B) \subseteq \mathrm{Rel}(A)$,

(iii) every bound variable in B is of sort in L,

(iv) $\mathrm{Un}(B) \subseteq J_0$.

Hence B is an existential$_{J_0}$ formula of L. Since

$$\mathrm{Ext}_1, \{\exists u_i'\, (u_i' = b_i)\}_{i=1}^n, \quad A' \to B \quad \text{and} \quad B \to A$$

are provable, we obtain that $A \equiv B$ is provable.]

Let $\mathscr{M} = \langle D, \phi_0\rangle$ and $\mathscr{M}' = \langle D', \phi_0'\rangle$ be two structures for the same language. $\mathscr{M}'$ is said to be an extension of $\mathscr{M}$ if $D \subseteq D'$, $\phi_0 k = \phi_0' k$ for each individual constant k, and $\phi_0 f$ is $\phi_0' f$ restricted to D for each function or predicate constant f.

Corollary 8.9 (Łos-Tarski). *The following are equivalent: let A be a formula of an ordinary (i.e., single-sorted) language* L.

(i) *For any structure* $\mathcal{M}$ *(for* L*) and extension* $\mathcal{M}'$, *and any assignments* ϕ, ϕ' *from the domains of* $\mathcal{M}$, $\mathcal{M}'$, *respectively, which agree on the free variables of* A, *if* $(\mathcal{M}, \phi)$ *satisfies* A, *then so does* $(\mathcal{M}', \phi')$.

(ii) *There exists an (essentially) existential formula* B *such that* $A \equiv B$ *is provable and the free variables of* B *are among those of* A.

PROOF. From the above problem, where J is a single sort and J_0 is the empty set.

PROBLEM 8.10. Let $\mathcal{A}$ be an axiom system in a language L, $\forall x\, \exists y\, A(x, y)$ a sentence of L provable from $\mathcal{A}$, and f a function symbol not in L. Then any L-formula which is provable from $\mathcal{A} \cup \{\forall x\, A(x, f(x))\}$ is also provable from $\mathcal{A}$ in L. (That is to say, the introduction of f in this way does not essentially extend the system.) [*Hint* (Maehara's method): This is a corollary of the following lemma.]

LEMMA 8.11. *Let* $\forall x\, \exists y\, A(x, y)$ *be a sentence of* L, f *a function symbol not in* L, *and* Γ *and* Θ *finite sequences of* L-*formulas. If* $\forall x\, A(x, f(x)), \Gamma \to \Theta$ *is* (**LK**-) *provable, then* $\forall x\, \exists y\, A(x, y), \Gamma \to \Theta$ *is provable in* L.

PROOF. Let P be a cut-free regular proof of $\forall x\, A(x, f(x)), \Gamma \to \Theta$. Let $t_1, \ldots, t_n$ be all the terms in P (i.e. proper terms, not semi-terms) whose outermost function symbol is f. These are arranged in an order such that t_i is not a subterm of t_j for $i < j$. Suppose t_i is $f(s_i)$ for $i = 1, \ldots, n$. P is transformed in three steps.

Step (1): Let $a_1, \ldots, a_n$ be distinct free variables not occurring in P. Transform P by replacing t_1 by a_1, then t_2 by a_2, and so on. The resulting figure P' has the same end-sequent as P, but is not, in general, a proof (as we will see below) and must be further transformed.

Step (2): Since P is cut-free and f does not occur in Γ or Θ, it can be seen that the only occurrences of f in P are in the context. $\forall x\, A(x, f(x))$, and further, all these $\forall x\, A(x, f(x))$ occur in antecedents of sequents in P', and the corresponding occurrences of $\forall x\, A(x, f(x))$ in P are introduced (in P) only by weakening : left or by some inferences of the form

$$I \quad \frac{A(s_i, f(s_i)), \Pi \to \Lambda}{\forall x\, A(x, f(x)), \Pi \to \Lambda}$$

(for some of the i, $1 \leqslant i \leqslant n$). Suppose the upper sequent of I is transformed into

$$A(s_i', a_i), \Pi' \to \Lambda'$$

in P'. (So I is not transformed by step (1) into a correct inference in P'.) Now replace all occurrences of $\forall x\, A(x, f(x))$ in P' by

$$A(s_1', a_1), \ldots, A(s_n', a_n)$$

(where s_i' is formed by replacing all t_j in s_i by a_j). Then the lower sequent of (the transform of) I can be derived from the upper sequent by several weakenings.

The result (after applying some contractions etc.) is a figure P'' with end-sequent

$$A(s_1', a_1), \ldots, A(s_n', a_n), \Gamma \to \Theta.$$

However it may still not be a proof, as we now show, and must be transformed further.

Step (3): Consider a $\exists$: left in P:

$$J \quad \frac{B(b), \Delta \to \Psi}{\exists z\, B(z), \Delta \to \Psi}$$

and suppose this is transformed in P'' (by steps (1) and (2)) to

$$J' \quad \frac{B'(b), \Delta' \to \Psi'}{\exists z\, B'(z), \Delta' \to \Psi'}.$$

Now it may happen that for some i, the eigenvariable b occurs in s_i (and also s_i'), and further, the formula $A(s_i', a_i)$ occurs in Δ' or Ψ'; so that the eigenvariable condition is no longer satisfied in J'.

So we transform all J' in P'' (arising from $\exists$: left inferences J in P) as follows:

$$\supset : \text{left} \quad \frac{\exists z\, B'(z) \to \exists z\, B'(z) \qquad B'(b), \Delta' \to \Psi'}{\exists z\, B'(z) \supset B'(b), \exists z\, B'(z), \Delta' \to \Psi'}$$

and carry the extra formula $\exists z\, B'(z) \supset B'(b)$ down to the end-sequent.

For the same reason, for every $\forall$: right in P

$$J \quad \frac{\Delta \to \Psi, B(b)}{\Delta \to \Psi, \forall z\, B(z)}$$

we replace its transform in P''

$$J' \; \dfrac{\Delta' \to \Psi', B'(b)}{\Delta' \to \Psi', \forall z\, B'(z)}$$

by

$$\supset: \text{left} \quad \dfrac{\begin{matrix} \vdots \\ \to \forall z\, B'(z), \exists z\, \neg B'(z) \end{matrix} \qquad \dfrac{\Delta' \to \Psi', B'(b)}{\neg B'(b), \Delta' \to \Psi'}}{\exists z\, \neg B'(z) \supset \neg B'(b), \Delta' \to \Psi', \forall z\, B'(z)}$$

(and carry the extra formula down to the end).

The result (after some obvious adjustments with structural inferences) is a proof, without $\exists$: left or $\forall$: right, whose end-sequent has the form

$$(S_1) \qquad \exists z\, B'(z) \supset B'(b), \ldots, A(s_i', a_i), \ldots, \Gamma \to \Theta.$$

Now apply $\exists$: left and $\forall$: left inferences in a suitable order (see below) (and contractions, etc.) to derive

$$(S_2) \qquad F, \ldots, \forall x\, \exists y\, A(x, y), \Gamma \to \Theta,$$

where F is the formula obtained from $\exists u\, (\exists z\, B'(z) \supset B'(u))$ by universal quantification over all its free variables.

Finally, applying cuts with sequents $\to F$, we obtain a proof, as desired, of

$$\forall x\, \exists y\, A(x, y), \Gamma \to \Theta.$$

We must still check that it is indeed possible to find a suitable order for applying the quantifier inferences in proceeding from (S_1) to (S_2) above, so that they all satisfy the eigenvariable condition. To this end, we use the following (temporary) notation. For terms s and t and a formula B, $s \subset t$ means that s is a (proper) subterm of t, $s \subseteq t$ means that s is a subterm of t or t itself, and $s \subset B$ means that s is contained in B.

Now note that the following condition (C) is satisfied for any of the auxiliary formulas $B(b)$ of P with eigenvariable b, considered above, and $1 \leqslant i \leqslant n$:

(C) If $b \subset t_i$, then $t_i \not\subset B(b)$.

(For suppose $b \subset t_i$ and also t_i, which we write as $f(s_i(b))$, occurs in $B(b)$. Then in the lower sequent of the inference J with auxiliary formula $B(b)$, f would occur in the principal formula $\exists z\, B(z)$ (or $\forall z\, B(z)$) in the context of the semi-term $f(s_i(z))$, and so, since P is cut-free, f would also occur (in a similar context) in all sequents of P below this, and hence in Γ or Θ.)

Now let $J_1, \ldots, J_m$ be all the $\exists$: left and $\forall$: right inferences in P, with eigenvariables $b_1, \ldots, b_m$ and auxiliary formulas $B_1(b_1), \ldots, B_m(b_m)$, respectively. Consider the partial order on $a_1, \ldots, a_n, b_1, \ldots, b_m$, generated by the relation $\prec$, which is defined by the following conditions:

(1a) If $t_j \subset t_i$, then $a_i \prec a_j$.

(1b) If $b_j \subset t_i$, then $a_i \prec b_j$.

(2a) If $t_j \subset B_i(b_i)$, then $b_i \prec a_j$.

(2b) If $b_j \subset B_i(b_i)$ $(j \neq i)$, then $b_i \prec b_j$.

We will prove below that this does indeed generate a partial order, i.e., no circularities are formed. Assume this for the moment. Then, starting with sequent (S_1), we apply, in any $\prec$-increasing order, the quantifier inferences

$$\frac{\dfrac{A(s_i', a_i), \ldots}{\exists : \text{left and } \forall : \text{left}}}{\forall x\, \exists y\, A(x, y), \ldots}$$

and

$$\frac{\dfrac{\exists z\, B_j(z, a_i, \ldots, b_k, \ldots) \supset B_j(b_j, a_i, \ldots, b_k, \ldots), \ldots}{\exists : \text{left and } \forall : \text{left}}}{\forall x \ldots \forall y \ldots \exists u\, (\exists z\, B_j(z, x, \ldots, y, \ldots) \supset B_j(u, x, \ldots, y, \ldots)), \ldots}$$

so as to obtain (S_2). We can see that the eigenvariable condition is satisfied throughout, from the way in which $\prec$ was defined (and since $a_j \subset s_i' \Rightarrow t_j \subset t_i$, $b_j \subset s_i' \Rightarrow b_j \subset t_i$, $a_j \subset B_i'(b_i) \Rightarrow t_j \subset B_i(b_i)$, and $b_j \subset B_i'(b_i) \Rightarrow b_j \subset B_i(b_i)$).

Finally we must show that the relation $\prec$ does generate a partial order. This follows from the following two sublemmas.

Sublemma 8.12 (in the notation of Lemma 8.11). (a) *For any $\prec$-increasing sequence $b_i \prec \ldots \prec b_j$, J_i lies above J_j in P. (So $i \neq j$.)*

(b) *For any $\prec$-increasing sequence $a_i \prec \ldots \prec a_j$, we have $t_i \nsubseteq t_j$. (So, in particular, $i \neq j$.)*

Proof of (a). The proof is by induction on the length of this sequence.

(i) If the length is 2, i.e., $b_i \prec b_j$, this follows from the definition of $\prec$ (part 2b) and the eigenvariable condition in P.

(ii) For the case $b_i \prec a_k \prec b_j$: we have $t_k \subset B_i(b_i)$ (by 2a) and $b_j \subset t_k$ (by 1b). Hence $b_j \subset B_i(b_i)$. Also $i \neq j$, by condition (C). So again $b_i \prec b_j$ (by 2b) and J_i is above J_j.

(iii) For the case $b_i \prec a_k \prec \ldots \prec a_l \prec b_j$ (with only a's between b_i and b_j): notice that $t_l \subset t_k$ (from 1a). The argument is now similar to that in (ii).

(iv) For the remaining case, $b_i \prec \ldots \prec b_k \prec \ldots \prec b_j$, use the induction hypothesis.

PROOF OF (b). The proof is by induction on the length of this sequence.

(i) If the length is 2, i.e. $a_i \prec a_j$, this follows from the definition (part 1a).

(ii) For the case $a_i \prec b_k \prec a_j$: we have $b_k \subset t_i$ and $t_j \subset B_k(b_k)$. So $t_i \subseteq t_j$ would imply $t_i \subset B_k(b_k)$, contradicting (C).

(iii) For the case $a_i \prec b_k \prec \ldots \prec b_l \prec a_j$ (with anything between b_k and b_l) we have $b_k \subset t_i$, $t_j \subset B_l(b_l)$ and J_k is above J_l (by Sublemma 8.12(a)). So $t_i \subseteq t_j$ would imply $b_k \subset B_l(b_l)$, contradicting the eigenvariable condition in P.

For the remaining two cases:

(iv) $a_i \prec a_k \prec \ldots \prec a_j$,

(v) $a_i \prec \ldots \prec a_k \prec a_j$,

use (1a) and the induction hypothesis.

This completes the proof of the sublemmas, and hence of Lemma 8.11.

PROBLEM 8.13. Prove the following, sharpened version of the interpolation theorem for **LK** (Maehara-Takeuti).

Let A and B be formulas with a predicate constant in common, let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two finite sequences of free variables of the same length such that all the variables in $\boldsymbol{a}$ are distinct from one another (while some of the variables in $\boldsymbol{b}$ may be the same), and let $A\binom{\boldsymbol{a}}{\boldsymbol{b}}$ and $B\binom{\boldsymbol{a}}{\boldsymbol{b}}$ be the formulas obtained from A and B by replacing each variable in $\boldsymbol{a}$ by the corresponding variable in $\boldsymbol{b}$. Suppose $A\binom{\boldsymbol{a}}{\boldsymbol{b}} \supset B\binom{\boldsymbol{a}}{\boldsymbol{b}}$ is **LK**-provable. Then there is a formula C such that the individual constants, predicate constants and free variables of C (apart from those in $\boldsymbol{a}$) occur in both A and B and such that $A\binom{\boldsymbol{a}}{\boldsymbol{b}} \supset C\binom{\boldsymbol{a}}{\boldsymbol{b}}$ and $C \supset B$ are both provable. [*Hint*: State and prove the theorem for sequents. The technique of the proof of Theorem 6.6 works.]

The following proposition is not strictly proof-theoretical in nature; however, it is useful for the next topic (in the proof of Proposition 8.16). We first give some definitions.

Let R be a set and suppose a set W_p is assigned to every $p \in R$. If $R_1 \subseteq R$ and $f \in \prod_{p \in R_1} W_p$, then f is called a *partial function* (*over* R) with domain $\mathrm{Dom}(f) = R_1$. If $\mathrm{Dom}(f) = R$, then f is called a *total function* (*over* R). If f

and g are partial functions and $\mathrm{Dom}(f) = D_0 \subseteq \mathrm{Dom}(g)$ and $f(x) = g(x)$ for every $x \in D_0$, then we call g an *extension* of f and write $f \prec g$ and $f = g \restriction D_0$.

PROPOSITION 8.14 (a generalized König's lemma). *Let R be any set. Suppose a finite set W_p is assigned to every $p \in R$. Let* P *be a property of partial functions f over R (defined as above) satisfying the following conditions*:

1) $\mathrm{P}(f)$ *holds if and only if there exists a finite subset N of R satisfying* $\mathrm{P}(f \restriction N)$,

2) $\mathrm{P}(f)$ *holds for every total function f.*

Then there exists a finite subset N_0 of R such that $\mathrm{P}(f)$ *holds for every f with* $N_0 \subseteq \mathrm{Dom}(f)$.

Note that R can have arbitrarily large cardinality. The case that R is the set of natural numbers is the original König's lemma.

PROOF. Let $X = \prod_{p \in R_1} W_p$, and give each W_p the discrete topology, and X the product topology. Since each W_p is compact, so is X (Tychonoff's theorem). For each g such that $\mathrm{Dom}(g)$ is finite, let

$$N_g = \{f \mid f \text{ is total and } g \prec f\}.$$

Let

$$C = \{N_g \mid \mathrm{Dom}(g) \text{ is finite, and } \mathrm{P}(g)\}.$$

C is an open cover of X. Therefore C has a finite subcover, say

$$N_{g_1}, \ldots, N_{g_k}.$$

Let $N_0 = \mathrm{Dom}(g_1) \cup \ldots \cup \mathrm{Dom}(g_k)$. We will show that N_0 satisfies the condition of the theorem. If $N_0 \subseteq \mathrm{Dom}(g)$, then let $g \prec f$, f total. Then $\mathrm{P}(f)$ and $f \in N_{g_1} \cup \ldots \cup N_{g_k}$. Say $f \in N_{g_i}$. So $g_i \prec f$, $\mathrm{P}(g_i)$ and $g_i \prec g$. Therefore $\mathrm{P}(g)$. This completes the proof.

What happens if we wish to apply to **LJ** the technique which has been used in proving completeness for **LK**? This leads us naturally to the study of Kripke models of **LJ**, relative to which one can prove the completeness of **LJ**. In order to simplify the discussion, we assume again that our language does not contain individual or function constants. Again, there should be no essential difficulty in extending the argument to the case where individual and function constants are included.

For technical reasons, we will deal with a system which is an equivalent modification of **LJ**. This system, invented by Maehara, will be called **LJ′**. **LJ′** is defined by restricting **LK** (rather than **LJ**) as follows: The inferences $\neg$: right, $\supset$: right and $\forall$: right are allowed only when the principal formulas are the only formulas in the succedents of the lower sequents. (These are called the "critical inferences" of **LJ′**.) Thus, for instance, $\neg$: right will take a form:

$$\frac{D, \Gamma \rightarrow}{\Gamma \rightarrow \neg D}.$$

As is obvious from the definition, the sequents of **LJ′** are those of **LK** (so the restriction on the sequents of **LJ**, that there can be at most one formula in the succedent of a sequent, is lifted here). It should be noted that all the other inferences are exactly those of **LK**. In particular, in $\vee$: right, the inference

$$\frac{\Gamma \rightarrow \Delta, A}{\Gamma \rightarrow \Delta, A \vee B}$$

is allowed even if Δ is not empty.

By interpreting a sequent of **LJ′**, say $\Gamma \rightarrow B_1, \ldots, B_n$, as $\Gamma \rightarrow B_1 \vee \ldots \vee B_n$, it is a routine matter to prove that **LJ′** and **LJ** are equivalent. Also, the cut-elimination theorem holds for **LJ′**. (Combine the proofs of cut-elimination for **LK** and **LJ**.)

The question now arises: Given a sequent of **LJ′**, say $\Gamma \rightarrow \Delta$, is there a cut-free proof of $\Gamma \rightarrow \Delta$ in **LJ′**?

Starting with a given $\Gamma \rightarrow \Delta$, we can carry out the reduction process which was defined for the classical case (cf. Lemma 8.3), except that we omit the stages 1) ($\neg$: right reduction), 7) ($\supset$: right reduction) and 9) ($\forall$: right reduction); in other words, all the reductions are as for the classical case, except those which concern the critical inferences of **LJ′**, which are simply omitted. We return to consider this point later.

As an example of the case where the reduction process does not terminate, consider a sequent of the form

$$\forall x\, \exists y\, A(x, y) \rightarrow$$

where A is a predicate constant.

The tree obtained by the above reduction process is (again) called the reduction tree for $\Gamma \rightarrow \Delta$.

In preparation for Kripke's semantics for intuitionistic systems and the completeness theorem for **LJ**, we will generalize the above reduction process to the case where Γ and/or Δ are infinite; i.e., we define reduction trees for infinite sequents $\Gamma \to \Delta$.

DEFINITION 8.15. Let Γ and Δ be well-ordered sequences of formulas, which may be infinite. We say that $\Gamma \to \Delta$ is provable (cut-free provable) (in **LJ**′) if there are finite subsequences of Γ and Δ, say $\tilde{\Gamma}$ and $\tilde{\Delta}$, respectively, such that $\tilde{\Gamma} \to \tilde{\Delta}$ is provable (cut-free provable).

(It is clear that $\Gamma \to \Delta$ is provable (in **LJ**′) if and only if it is provable without cut, even when Γ and/or Δ are infinite, by the cut-elimination theorem of §5, adapted to **LJ**′.)

The reduction process which has just been described can be generalized immediately to the case of infinite sequents. We shall only point out a few modifications in the stages. Note: for the reduction process, we assume that the language is augmented by uncountably many new free and bound variables (in a well-ordered sequence).

8) $k \equiv 8$. Let $\forall x_1\, A_1(x_1), \ldots, \forall x_\alpha\, A_\alpha(x_\alpha), \ldots$ be all the formulas in Π whose outermost logical symbol is $\forall$. Let $a_1, \ldots, a_\beta, \ldots$ be all the free variables available at this stage. Then write down

$$A_1(a_1), \ldots, A_1(a_\beta), \ldots, A_\alpha(a_1), \ldots, A_\alpha(a_\beta), \ldots, \Pi \to \Lambda$$

above $\Pi \to \Lambda$.

10) $k \equiv 10$. Let $\exists x_1\, A_1(x_1), \ldots, \exists x_\alpha\, A_\alpha(x_\alpha), \ldots$ be all the formulas in Π whose outermost logical symbol is $\exists$. Introduce new free variables $b_1, b_2, \ldots, b_\alpha, \ldots$. Then write down

$$A_1(b_1), \ldots, A_\alpha(b_\alpha), \ldots, \Pi \to \Lambda$$

above $\Pi \to \Lambda$.

PROPOSITION 8.16. (a) *If a sequent $\Gamma \to \Delta$ is provable (in* **LJ**′*), then every sequent of the reduction tree for $\Gamma \to \Delta$ is provable.*

(b) *If a sequent $\Gamma \to \Delta$ is unprovable, then there is a branch (in the tree for $\Gamma \to \Delta$) in which every sequent is unprovable.*

PROOF. (a) is obvious. In order to prove (b), we shall first prove the following: Let $\Pi \to \Lambda$ be a sequent in the tree and let $\Pi_\lambda \to \Lambda_\lambda$, $\lambda = 1, 2, \ldots, \alpha, \ldots$

be all its upper sequents, given by a reduction. If each is provable, then $\Pi \to \Lambda$ is provable. In other words, if for each λ, $\lambda = 1, 2, \ldots, \alpha, \ldots$, there are finite subsets of Π_λ and Λ_λ, say Π'_λ and Λ'_λ respectively, such that $\Pi'_\lambda \to \Lambda'_\lambda$ is provable, then there are finite subsets of Π and Λ, say Π' and Λ' respectively, such that $\Pi' \to \Lambda'$ is provable. We shall only deal with a few cases.

1) A $\exists$: left reduction has been applied to $\Pi \to \Lambda$. Then its upper sequent is of the form

$$A_1(b_1), \ldots, A_\alpha(b_\alpha), \ldots, \Pi \to \Lambda,$$

where $\exists x_\alpha\, A_\alpha(x_\alpha)$ is in Π for each α, and $b_1, \ldots, b_\alpha, \ldots$ are newly introduced free variables. By the hypothesis, there are finite subsets of $A_1(b_1), \ldots, A_\alpha(b_\alpha), \ldots$ (say $B_1(c_1), \ldots, B_n(c_n)$), of Π (say Π'), and of Λ (say Λ'), such that

$$B_1(c_1), \ldots, B_n(c_n), \Pi' \to \Lambda'$$

is provable. By repeated $\exists$: left and some weak inferences, we obtain $\Pi \to \Lambda'$, which is a subsequent of $\Pi \to \Lambda$. Notice that since $B_1(c_1), \ldots, B_n(c_n), \Pi' \to \Lambda'$ is provable (with a finite proof), we may regard $c_1, \ldots, c_n$ as free variables of our original language.

2) An $\wedge$: right reduction has been applied to $\Pi \to \Lambda$. Then its upper sequents are of the form

$$\Pi \to \Lambda, C_1, \ldots, C_\alpha, \ldots,$$

where $A_1 \wedge B_1, \ldots, A_\alpha \wedge B_\alpha, \ldots$ are all the formulas of Λ whose outermost logical symbol is $\wedge$ and each C_α is A_α or B_α. We shall distinguish these cases by denoting C_α by $C_{0,\alpha}$ if C_α is A_α and by $C_{1,\alpha}$ if C_α is B_α. Then the upper sequents are the sequents

$$\Pi \to \Lambda, C_{i_1,1}, \ldots, C_{i_\alpha,\alpha}, \ldots,$$

where $i_\alpha = 0$ or 1, for all possible combinations of values of $i_1, \ldots, i_\alpha, \ldots$. Let f denote any sequence $(i_1, \ldots, i_\alpha, \ldots)$. By assumption, there is a finite subsequent of each sequent, say $\Pi^f \to \Lambda^f, C_1^f, \ldots, C_{n_f}^f$, which is provable, where $C_1^f, \ldots, C_{n_f}^f$ is a finite subset of $C_{i_1,1}, \ldots, C_{i_\alpha,\alpha}, \ldots$.

In order now to exploit the generalized König's lemma (Proposition 8.14), we let R be a set with the order type of the sequence $C_1, C_2, \ldots, C_\alpha, \ldots$ (say $R = \{1, 2, \ldots, \alpha, \ldots\}$). Define $W_\alpha = 2$ $(= \{0, 1\})$. For any subset $R_1 \subseteq R$ and any $f \in \prod_{\alpha \in R_1} W_\alpha$, we say that a finite sequence of formulas

$$(C_{f(\alpha_1),\alpha_1}, \ldots, C_{f(\alpha_n),\alpha_n})$$

(with $\alpha_1, \ldots, \alpha_n \in R_1$) is selected for f if there are finite subsets of Π and Λ, say Π' and Λ', respectively, such that

$$\Pi' \to \Lambda', C_{f(\alpha_1),\alpha_1}, \ldots, C_{f(\alpha_n),\alpha_n}$$

is provable. From the observation above, there is such a selected subset for any total function f. Now, for any $R_1 \subseteq R$ and any $f \in \prod_{\alpha \in R_1} W_\alpha$, we define

$$P(f) \Leftrightarrow_{df} \exists k\, \exists \alpha_1 \ldots \exists \alpha_k\, (\alpha_1, \ldots, \alpha_k \text{ are in the domain of } f \text{ and } (C_{f(\alpha_1),\alpha_1}, \ldots, C_{f(\alpha_k),\alpha_k}) \text{ is selected}), \text{ where } k \text{ ranges over the natural numbers.}$$

Then conditions 1 and 2 in the hypothesis of the generalized König's lemma are satisfied; hence by this lemma, there exists a finite subset of R, say $N_0 = \{\gamma_1, \ldots, \gamma_l\}$, such that if Dom$(f)$ contains N_0, then P(f) holds.

Let

$$F = \{f \mid \mathrm{Dom}(f) = N_0\} = \prod_{j=1}^{l} W_{\gamma_j}.$$

F is a finite set and, for every f in F, P(f) holds, i.e., there is a subset of $\gamma_1, \ldots, \gamma_l$, say $\alpha_1, \ldots, \alpha_k$, such that $(C_{f(\alpha_1),\alpha_1}, \ldots, C_{f(\alpha_k),\alpha_k})$ is selected for f; i.e., there exist finite subsets of Π and Λ, say Π' and Λ' respectively, such that

$$\Pi' \to \Lambda', C_{f(\alpha_1),\alpha_1}, \ldots, C_{f(\alpha_k),\alpha_k}$$

is provable. Therefore, for every possible combination of values of $(i_1, \ldots, i_k)$ $(= i)$, there are finite subsets of Π and Λ, say Π^i and Λ^i respectively, such that

$$\Pi^i \to \Lambda^i, C_{i_1,\alpha_1}, \ldots, C_{i_k,\alpha_k}$$

is provable. Hence by weakenings and repeated $\wedge$: right, we obtain

$$\tilde{\Pi}' \to \tilde{\Lambda}', A_{\alpha_1} \wedge B_{\alpha_1}, \ldots, A_{\alpha_k} \wedge B_{\alpha_k},$$

where $\tilde{\Pi}'$ consists of all the Π''s for f in F, and likewise with $\tilde{\Lambda}'$.

Now, from the argument just completed, if the given sequent $\Gamma \to \Delta$ is not provable, then there is one branch in which every sequent is unprovable.

Having finished these preparations, we now define Kripke (intuitionistic) structures (for a language L).

DEFINITION 8.17. (1) A *partially ordered structure* $P = \langle O, \leqslant \rangle$ consists of a set O together with a binary relation $\leqslant$ satisfying the following:

a) $p \leqslant p$,

b) $p \leqslant q$ and $q \leqslant p$ imply $p = q$,

c) $p \leqslant q$ and $q \leqslant r$ imply $p \leqslant r$,

where p, q and r range over elements of O.

(2) A Kripke structure for a language L is an ordered triple $\langle P, U, \phi \rangle$ such that:

1) $P = \langle O, \leqslant \rangle$ is a partially ordered structure.

2) U is a map which assigns to every member of O, say p, a non-empty set, say U_p, such that, if $p \leqslant q$, then $U_p \subseteq U_q$ (where $\subseteq$ means set inclusion).

3) ϕ is a binary function $\phi(R, p)$, where R ranges over predicate constants in the language L and p ranges over members of O. Further:

3.1) Suppose the number of argument places of R is 0. Then $\phi(R, p) = \mathsf{T}$ or F, and if $\phi(R, p) = \mathsf{T}$ and $p \leqslant q$, then $\phi(R, q) = \mathsf{T}$.

3.2) Suppose R is an n-ary predicate ($n \geqslant 1$). Then $\phi(R, p)$ is a subset of

$$U_p^n = \underbrace{U_p \times \ldots \times U_p}_{n \text{ times}},$$

and $p \leqslant q$ implies $\phi(R, p) \subseteq \phi(R, q)$.

We define $U = \bigcup_{p \in O} U_p$. Then U can be thought of as the universe of the model or structure, and the elements of O as stages (see below).

Suppose that there is an assignment of objects of U to all the free variables; i.e., to each free variable a_i an object of U, say c_i, is assigned. Let $F(a_1, \ldots, a_n)$ be a formula with free variables $a_1, \ldots, a_n$ (at most). The *interpretation* of $F(a_1, \ldots, a_n)$ at (the stage) p (under this assignment) is defined as follows by induction on the number of logical symbols in $F(a_1, \ldots, a_n)$, and this interpretation is expressed as $\phi(F(c_1, \ldots, c_n), p)$. The value of such an interpretation is T or F.

a) $\phi(R(c_1, \ldots, c_n), p) = \mathsf{T}$ if and only if $\langle c_1, \ldots, c_n \rangle \in \phi(R, p)$ (for $n > 0$).

b) $\phi(A \wedge B, p) = \mathsf{T}$ if and only if $\phi(A, p) = \mathsf{T}$ and $\phi(B, p) = \mathsf{T}$.

c) $\phi(A \vee B, p) = \mathsf{T}$ if and only if $\phi(A, p) = \mathsf{T}$ or $\phi(B, p) = \mathsf{T}$.

d) $\phi(A \supset B, p) = \mathsf{T}$ if and only if for all q such that $p \leqslant q$, either $\phi(A, q) = \mathsf{F}$ or $\phi(B, q) = \mathsf{T}$.

e) $\phi(\neg A, p) = \mathsf{T}$ if and only if for all q such that $p \leqslant q$, $\phi(A, q) = \mathsf{F}$.

f) $\phi(\exists x\, A(c_1, \ldots, c_n, x), p) = \mathsf{T}$ if and only if there is a c in U_p such that $\phi(A(c_1, \ldots, c_n, c), p) = \mathsf{T}$.

g) $\phi(\forall x\, F(c_1, \ldots, c_n, x), p) = \mathsf{T}$ if and only if for all q such that $p \leqslant q$, and for all c in U_q, $\phi(F(c_1, \ldots, c_n, c), p) = \mathsf{T}$.

We can generalize the definition of interpretation which has just been given to the case of sequents (finite or infinite). Let $\Gamma \rightarrow \Delta$ be a sequent. Then $\phi(\Gamma \rightarrow \Delta, p)$ is defined to be T if and only if, for all q such that $p \leqslant q$, either $\phi(A, q) = \mathsf{F}$ for some A in Γ or $\phi(B, q) = \mathsf{T}$ for some B in Δ.

A sequent $\Gamma \rightarrow \Delta$ is said to be valid in a Kripke structure $\langle P, U, \phi \rangle$ (with $P = \langle O, \leqslant \rangle$) if $\phi(\Gamma \rightarrow \Delta, p) = \mathsf{T}$ for all p in O.

PROPOSITION 8.18. *Suppose $\Gamma \rightarrow \Delta$ is provable in* **LJ**′, *and $\langle P, U, \phi \rangle$ is a Kripke structure. Then $\Gamma \rightarrow \Delta$ is valid in $\langle P, U, \phi \rangle$.*

PROOF. This is only a routine matter: by mathematical induction on the number of inferences in a proof of $\Gamma \rightarrow \Delta$ (or a subsequent of it).

Now, in order to finish the completeness proof for **LJ**′, we shall start with an unprovable sequent $\Gamma \rightarrow \Delta$ and construct a counter-model in the sense of Kripke. This will be constructed from the reduction tree for $\Gamma \rightarrow \Delta$. Let us call this tree T. (Remember, in the construction of T, the $\neg$: right, $\supset$: right and $\forall$: right reductions were omitted.) This situation, i.e., with just this tree present, is called stage 0. By Proposition 8.16, there is a branch of T, say B_0, containing (only) unprovable sequents. If B_0 is finite, let $\Gamma_0 \rightarrow \Delta_0$ be its uppermost sequent. If B_0 is infinite, let Γ_0 and Δ_0 be the union of all formulas in the antecedents and succedents respectively of the sequents in B_0 (each arranged in a well-ordered sequence), and consider the (possibly infinite) sequent $\Gamma_0 \rightarrow \Delta_0$. Single out all the formulas in Δ_0 whose outermost symbols are $\neg$, $\supset$ or $\forall$. (If there is no such formula, then stop.) Let the symbol p range over all such formulas. We call each such p an immediate successor of 0 (and 0 an immediate predecessor of p.)

Case 1. p is a formula of the form $\neg A$. Then consider the sequent $A, \Gamma_0 \rightarrow$.

Case 2. p is $B \supset C$. Then consider the sequent $B, \Gamma_0 \rightarrow C$.

Case 3. p is $\forall x\, F(x)$. Let a be a free variable which does not belong to U_0. (This can always be done by introducing a new symbol if necessary.) Then consider the sequent $\Gamma_0 \rightarrow F(a)$.

It is easily shown that (in each case) this new sequent is not provable, since otherwise $\Gamma_0 \rightarrow \Delta_0$ would be provable. Let us call this new sequent $\tilde{\Gamma}_p \rightarrow \tilde{\Delta}_p$, and let T_p be the reduction tree for $\tilde{\Gamma}_p \rightarrow \tilde{\Delta}_p$.

As before, let B_p be a branch of T_p containing unprovable sequents, and let $\Gamma_p \to \Delta_p$ be either the topmost sequent of B_p, or (if B_p is infinite) the "union" of all sequents in B_p, as before. Then follow exactly the same process as the preceding paragraph. Namely, let q range over all formulas in Δ_p whose outermost logical symbol is $\neg$, $\supset$ or $\forall$. (If there is no such formula, then stop.) Again, for all such q and p (with q in Δ_p as above), we call q an immediate successor of p and p an immediate predecessor of q. Then define as before the tree T_q and branch B_q.

Continue this procedure ω times. Let O be the set of all these p's, and let $\leqslant$ be the transitive reflexive relation on O generated by the immediate predecessor relation defined above. O is partially ordered by $\leqslant$. Now define U_p to be the set of all free variables occurring in B_p, for all $p \in O$, and define $U = \bigcup_{p \in O} U_p$. Notice the following.

1) If $p \leqslant q$, then $U_p \subseteq U_q$.

2) If q is an immediate successor of p, then all formulas in Γ_p occur in the antecedents of all sequents in T_q (and hence in B_q).

We now define the function ϕ as follows. For any n-ary predicate symbol R $(n > 0)$, and any $p \in O$,

$$\phi(R, p) = \{\langle a_1, \ldots, a_n \rangle \mid a_1, \ldots, a_n \in U_p \text{ and } R(a_1, \ldots, a_n) \text{ occurs in } \Gamma_p\}$$

(and for $n = 0$, $\phi(R, p) = \mathsf{T}$ if and only if R occurs in Γ_p).

So we have defined a Kripke structure $\langle P, U, \phi \rangle$. We shall consider the interpretation of formulas in this structure relative to the (natural) assignment of each free variable to itself.

PROPOSITION 8.19 (with the above notation). *Let A be a formula in B_p. If A occurs in the antecedent of a sequent in B_p, then $\phi(A, p) = \mathsf{T}$; if it occurs in the succedent, then $\phi(A, p) = \mathsf{F}$.*

PROOF. By induction on the number of logical symbols in A. First it should be noticed that if a formula occurs in the antecedent of a sequent in B_p, then it does not occur in the succedent of any sequent in B_p. The same holds with "antecedent" and "succedent" interchanged. Also, once a formula appears on one side of a sequent, it will appear on the same side of all higher sequents of B_p, and hence of the sequent $\Gamma_p \to \Delta_p$.

1) A is an atomic formula $R(a_1, \ldots, a_n)$. If A occurs in an antecedent, hence in Γ_p, then by definition $\langle a_1, \ldots, a_n \rangle \in \phi(R, p)$, which implies, again by definition, that $\phi(A, p) = \mathsf{T}$. If A occurs in a succedent, then $\langle a_1, \ldots, a_n \rangle \notin \phi(R, p)$, so $\phi(A, p) = \mathsf{F}$.

2) A is $\neg B$. Suppose A occurs in the antecedent. Then A occurs in Γ_p. This implies that, given any q such that $p \leqslant q$, A occurs in the antecedent of all the sequents in B_q; hence B occurs in the succedent of a sequent in B_q; therefore, by the induction hypothesis, $\phi(B, q) = \mathsf{F}$. So $\phi(B, q) = \mathsf{F}$ for any q such that $p \leqslant q$. This means that $\phi(B, p) = \mathsf{T}$.

Suppose next that A occurs in the succedent of a sequent in B_p. Then there exists a next stage, say q. It starts with $B, \Gamma_p \rightarrow$. By the induction hypothesis, $\phi(B, q) = \mathsf{T}$. That is to say, there is a q such that $p \leqslant q$ and $\phi(B, q) = \mathsf{T}$. Therefore by definition $\phi(A, p) = \mathsf{F}$.

3) A is $B \wedge C$ or $B \vee C$. Those cases are easy; so they are left to the reader.

4) A is $\forall x\, F(x)$. Suppose A occurs in the antecedent of a sequent in B_p and suppose $p \leqslant q$. Then A occurs in the antecedent of a sequent in B_q. Let a be an element of U_q. Then $F(a)$ occurs in the antecedent of a sequent in B_q. Hence, by the induction hypothesis, $\phi(F(a), q) = \mathsf{T}$. So for any q such that $p \leqslant q$ and any a in U_q, $\phi(F(a), q) = \mathsf{T}$, which means that $\phi(A, p) = \mathsf{T}$.

Suppose next that A occurs in the succedent of a sequent in B_p. So the next stage, say q, starts with $\Gamma_p \rightarrow F(a)$, where a is a (new) variable in U_q. By the induction hypothesis, $\phi(F(a), q) = \mathsf{T}$. So there exists a q such that $p \leqslant q$, and a member a of U_q, such that $\phi(F(a), q) = \mathsf{F}$. This means that

$$\phi(\forall x\, F(x), p) = \mathsf{F}.$$

5) A is of the form $\exists x\, F(x)$. This case is left as an exercise.

6) A is of the form $B \supset C$. Suppose that A occurs in the antecedent of a sequent in B_p. Then either C occurs in Γ_p or B occurs in Δ_p. Let $p \leqslant q$. Then either C occurs in the antecedent or B occurs in the succedent of a sequent in B_q. So for any q, with $p \leqslant q$, either $\phi(C, q) = \mathsf{T}$ or $\phi(B, q) = \mathsf{F}$. So $\phi(B \supset C, p) = \mathsf{T}$.

Suppose next that A occurs in the succedent of a sequent in B_p. Then the next stage, say q, starts with $B, \Gamma_p \rightarrow C$. Hence there is a q such that $p \leqslant q$, $\phi(B, q) = \mathsf{T}$ and $\phi(C, q) = \mathsf{F}$; so $\phi(B \supset C, p) = \mathsf{F}$.

So now we can conclude that if $\Gamma \rightarrow \Delta$ is unprovable, then we can construct a Kripke structure $\langle P, U, \phi \rangle$ such that (under a suitable assignment to free variables) every formula in Γ assumes the value T and every formula in Δ assumes the value F; in other words, there is a Kripke counter-model for $\Gamma \rightarrow \Delta$. This ends the completeness proof. Thus we have obtained:

THEOREM 8.20 (completeness of the intuitionistic predicate calculus: a generalized version; cf. Theorem 8.2). *Let $\Gamma \rightarrow \Delta$ be a sequent (finite or infinite).*

If $\Gamma \to \Delta$ is valid in all Kripke structures, then $\Gamma \to \Delta$ is provable. In particular, **LJ** *is complete.*

(Recall that the soundness of **LJ** was established by Proposition 8.18). Notice that the method which has been prescribed here for completeness of **LJ** works even when the language is not countable, while the method for **LK** works only for a countable language. Although we could in fact use a method for **LK** similar to this one for **LJ**′, we do not attempt to do so, since Henkin's simple method is sufficient for that purpose.

EXERCISE 8.21. Construct a Kripke counter-model for each of the following sequents.

1) $\to P \vee \neg P$, where P is a predicate symbol.

2) $\forall x\,(P(x) \vee Q) \to \forall x\, P(x) \vee Q$, where P and Q are predicate symbols of the indicated numbers of argument.

3) $\to \exists x\,(\exists y\, P(y) \supset P(x))$, where P is a unary predicate.

[*Hint* for 1): At stage 0:

$$\frac{\to P \vee \neg P,\ P, \neg P}{\to P \vee \neg P}\,.$$

Let p be $\neg P$. Then at stage p:

$$\frac{P \to}{\to \neg P}\,.$$

So define $O = \{0, p\}, 0 \leqslant p, U_0 = U_p = \{a\}, \phi(P, 0) = \mathsf{F}$. Then $\phi(P \vee \neg P, 0) = \mathsf{F}$ can be easily proved.]

CHAPTER 2

PEANO ARITHMETIC

In this chapter we shall formulate first-order Peano arithmetic, prove Gödel's incompleteness theorem, develop a constructive theory of ordinals up to the first ε-number ε_0, and then present a consistency proof of the system, due to Gentzen.

§9. A formulation of Peano arithmetic

DEFINITION 9.1. The language of the system, which will be called Ln, contains finitely many constants, as follows. (See also Definition 1.1.)

Individual constant: 0;

Function constants: $'$, $+$, $\cdot$;

Predicate constant: $=$;

where $'$ is unary while the other constants are binary.

The intended interpretation of the above constants should be obvious. We shall use expressions like $s = t$, $s + t$, $s \cdot t$ and s' rather than formal expressions like $+(s, t)$.

A numeral is an expression of the form $0'^{\cdots}{}'$, i.e., zero followed by n primes for some n, which is used as a formal expression for the natural number n, and is denoted by $\bar{n}$. Further, if s is a closed term of Ln denoting a number m (in the intended interpretation), then $\bar{s}$ denotes the numeral $\bar{m}$ (e.g., if s is $\bar{2} + \bar{3}$, then $\bar{s}$ denotes $\bar{5}$).

DEFINITION 9.2. The first axiom system of Peano arithmetic which we consider, CA, consists of the following sentences.

A1 $\forall x\, \forall y\, (x' = y' \supset x = y)$;

A2 $\forall x\, (\neg x' = 0)$;

A3 $\forall x\, \forall y\, \forall z\, (x = y \supset (x = z \supset y = z))$;

A4 $\forall x\, \forall y\, (x = y \supset x' = y')$;
A5 $\forall x\, (x + 0 = x)$;
A6 $\forall x\, \forall y\, (x + y' = (x + y)')$;
A7 $\forall x\, (x \cdot 0 = 0)$;
A8 $\forall x\, \forall y\, (x \cdot y' = x \cdot y + x)$.

The second axiom system of Peano arithmetic which we consider, VJ, consists of all sentences of the form

$$\forall z_1 \ldots \forall z_n\, \forall x\, (F(0, \mathbf{z}) \wedge \forall y\, (F(y, \mathbf{z}) \supset F(y', \mathbf{z})) \supset F(x, \mathbf{z})),$$

where $\mathbf{z}$ is an abbreviation for the sequence of variables $z_1, \ldots, z_n$; and all the free variables of $F(x, \mathbf{z})$ are among $x, \mathbf{z}$.

The basic logical system of Peano arithmetic is **LK**. Then CA $\cup$ VJ is an axiom system with equality, regarding $=$ as the distinguished predicate constant in §7. Furthermore, $\forall x\, \forall y\, (x = y \supset (F(x) \equiv F(y)))$ is provable for every formula of Ln (cf. Proposition 7.2).

As an example of the strength of CA $\cup$ VJ, we mention that the theory of primitive recursive functions can be developed in this system. Although this point will not be discussed further here, such knowledge is assumed.

DEFINITION 9.3. The system **PA** (Peano arithmetic) is obtained from **LK** (in the language Ln) by adding extra initial sequents (called the *mathematical initial sequents*) and a new rule of inference called "*ind*", stated below.

1) Mathematical initial sequents:

$$s' = t' \rightarrow s = t;$$

$$s' = 0 \rightarrow \quad ;$$

$$s = t, s = r \rightarrow t = r;$$

$$s = t \rightarrow s' = t';$$

$$\rightarrow s + 0 = s;$$

$$\rightarrow s + t' = (s + t)';$$

$$\rightarrow s \cdot 0 = 0;$$

$$\rightarrow s \cdot t' = s \cdot t + s,$$

where s, t, r are arbitrary terms of Ln.

2) Ind:

$$\frac{F(a),\, \Gamma \to \Delta,\, F(a')}{F(0),\, \Gamma \to \Delta,\, F(s)}$$

where a is not in $F(0)$, Γ or Δ; s is an arbitrary term (which may contain a); and $F(a)$ is an arbitrary formula of Ln.

$F(a)$ is called the *induction formula*, and a is called the *eigenvariable* of this inference. Further, we call $F(a)$ and $F(a')$ the *left* and *right auxiliary formula*, respectively, and $F(0)$ and $F(s)$ the *left* and *right principal formula*, respectively, of this inference.

The initial sequents of the form $D \to D$ are called *logical* initial sequents (in contrast to the mathematical initial sequents defined above).

To summarize, then: there are two kinds of initial sequents of **PA**: logical and mathematical; and three kinds of inference rules: structural, logical and ind (cf. Definition 2.1).

Finally, a *weak inference* is a structural inference other than cut.

We shall adapt the concepts concerning proofs which were defined in Chapter I with some modifications; the new inference "ind" must be taken into account in every definition. In particular, the successor of $F(a)$ (respectively, $F(a')$) in ind is $F(0)$ (respectively, $F(s)$). Otherwise all definitions in Chapter 1 are relevant here.

As an easy corollary of the definitions we have

PROPOSITION 9.4. *A sequent is provable from* CA $\cup$ VJ (*in* **LK**) *if and only if it is provable in* **PA**. *Hence the axiom system* CA $\cup$ VJ *is consistent if and only if* $\to$ *is not provable in* **PA**.

Thus we can restrict our attention to the system **PA**. In the rest of this chapter, "provability" means provability in **PA**. It was Gentzen's great development to formulate first-order arithmetic in the form of **PA**.

Similarly to Lemma 2.11, we can prove the following proposition, which we shall use without mention.

PROPOSITION 9.5. *Let P be a proof in* **PA** *of a sequent $S(a)$, where all the occurrences of a in $S(a)$ are indicated. Let s be an arbitrary term. Then we may construct a* **PA**-*proof P' of $S(s)$ such that P' is regular* (*cf. Lemma* 2.9, *part* (2)) *and P' differs from P only in that some free variables are replaced by some other free variables and some occurrences of a are replaced by s.*

The following lemma will be used later.

LEMMA 9.6. (1) *For an arbitrary closed term* s, *there exists a unique numeral* $\bar{n}$ *such that* $s = \bar{n}$ *is provable without an essential cut and without ind. (See Definition 7.5 for "essential cut".)*

(2) *Let* s *and* t *be closed terms. Then either* $\rightarrow s = t$ *or* $s = t \rightarrow$ *is provable without an essential cut or ind.*

(3) *Let* s *and* t *be closed terms such that* $s = t$ *is provable without an essential cut or ind and let* $q(a)$ *and* $r(a)$ *be two terms with some occurrences of* a *(possibly none). Then* $q(s) = r(s) \rightarrow q(t) = r(t)$ *is provable without an essential cut or ind.*

(4) *Let* s *and* t *be as in (3). For an arbitrary formula* $F(a)$: $s = t, F(s) \rightarrow F(t)$ *is provable without an essential cut or ind.*

PROOF. (1) By induction on the complexity of s.

We defined some notions concerning formal proofs in §2. In order to carry out the consistency proof for **PA**, however, we need some more of these. We shall list them all here.

DEFINITION 9.7. When we consider a formula or a logical symbol together with the place that it occupies in a proof, in a sequent or in a formula, we refer to it (respectively) as a formula or a logical symbol in the proof, in the sequent or in the formula. A formula in a sequent is also called a *sequent-formula*.

(1) Successor. If a formula E is contained in the upper sequent of an inference using one of the rules of inference in §1 or "ind", then the *successor* of E is defined as follows:
   (1.1) If E is a cut formula, then E has no successor.
   (1.2) If E is an auxiliary formula of any inference other than a cut or exchange, then the principal formula is the successor of E. (For the case of ind, see above.)
   (1.3) If E is the formula denoted by C (respectively, D) in the upper sequent of an exchange (in Definition 2.1), then the formula C (respectively, D) in the lower sequent is the successor of E.
   (1.4) If E is the kth formula of Γ, Π, Δ, or Λ in the upper sequent (in Definition 2.1), then the kth formula of Γ, Π, Δ or Λ, respectively, in the lower sequent is the successor of E.
(2) Thread. The notion of a sequence of sequents in a proof, called a *thread*, has been defined in Definition 2.8.

(3) The notions of a sequent being above or below another, and of a sequent being between two others, were defined in Definition 2.8; so was the notion of an inference being below a sequent.

(4) A sequent formula is called an *initial formula* or an *end-formula* if it occurs, respectively, in an initial sequent or an end-sequent.

(5) Bundle. A sequence of formulas in a proof with the following properties is called a *bundle*:

(5.1) The sequence begins with an initial formula or a weakening formula.

(5.2) The sequence ends with an end-formula or a cut-formula.

(5.3) Every formula in the sequence except the last is immediately followed by its successor.

(6) Ancestor and descendant. Let A and B be formulas. A is called an *ancestor* of B and B is called a *descendent* of A if there is a bundle containing both A and B in which A appears above B.

(7) Predecessor. Let A and B be formulas. If A is the successor of B, then B is called a *predecessor* of A.

Some principal formulas, e.g., $\wedge$: right, has two predecessors. In such cases we call a predecessor the *first* or the *second* predecessor of A, according as it is in the left or right upper sequent.

(8) The concepts of explicit and implicit.

(8.1) A bundle is called *explicit* if it ends with an end formula.

(8.2) It is called *implicit* if it ends with a cut-formula.

A formula in a proof is called explicit or implicit according as the bundles containing the formula are explicit or implicit.

A sequent in a proof is called implicit or explicit according as this sequent contains an implicit formula or not.

A logical inference in a proof is called explicit or implicit according as the principal formula of this inference is explicit or implicit.

(9) End-piece. The *end-piece* of a proof is defined as follows:

(9.1) The end-sequent of the proof is contained in the end-piece.

(9.2) The upper sequent of an inference other than an implicit logical inference is contained in the end-piece if and only if the lower sequent is contained in it.

(9.3) The upper sequent of an implicit logical inference is not contained in the end-piece.

We can rephrase this definition as follows: A sequent in a proof is in the end-piece of the proof if and only if there is no implicit logical inference below this sequent.

(10) An inference of a proof is said to be *in the end-piece* of the proof if the lower sequent of the inference is in the end-piece.

(11) Boundary. Let J be an inference in a proof. We say J *belongs to the boundary* (or J is a *boundary inference*) if the lower sequent of J is in the end-piece and the upper sequent is not. It should be noted that if J belongs to the boundary, then it is an implicit logical inference.

(12) Suitable cut. A cut in the end-piece is called *suitable* if each cut formula of this cut has an ancestor which is the principal formula of a boundary inference.

(13) Essential and inessential cuts. A cut is called *inessential* if the cut formula contains no logical symbol; otherwise it is called essential.

In **PA**, the cut formulas of inessential cuts are of the form $s = t$.

(14) A proof P is *regular* if: (i) the eigenvariables of any two distinct inferences ($\forall$: right, $\exists$: left or induction) in P are distinct from each other; and (ii) if a free variable a occurs as an eigenvariable of a sequent S of P, then a only occurs in sequents above S.

PROPOSITION 9.8. *For an arbitrary proof of* **PA**, *there exists a regular proof of the same end-sequent, which can be obtained from the original proof by simply replacing free variables.*

PROOF. The proof is as for Lemma 2.10, part (2).

§10. The incompleteness theorem

In this section we shall prove the incompleteness of **PA**. This is a celebrated result of Gödel. We shall actually consider any axiomatizable system which contain **PA** as a subsystem.

DEFINITION 10.1. An axiom system $\mathscr{A}$ (cf. §4) is said to be *axiomatizable* if there is a finite set of schemata such that $\mathscr{A}$ consists of all the instances of these schemata. A formal system **S** is called axiomatizable if there is an axiomatizable axiom system $\mathscr{A}$ such that **S** is equivalent to $\mathbf{LK}_{\mathscr{A}}$ (cf. §4). (Two systems are called equivalent if they have exactly the same theorems.)

A system **S** is called an extension of **PA** if every theorem of **PA** is provable in **S**. Throughout this section we deal with axiomatizable systems which are extensions of **PA**. They are denoted by **S**. Such an **S** is arbitrary but fixed; so is the language of **S**, say L (which will always extend Ln).

DEFINITION 10.2. The class of primitive recursive functions is the smallest class of functions generated by the following schemata. (These can be thought of as the clauses of an inductive definition, or as the defining equations of the function being defined.)

(i) $f(x) = x'$, where $'$ is the successor function.

(ii) $f(x_1, \ldots, x_n) = k$, where $n \geqslant 1$ and k is a natural number.

(iii) $f(x_1, \ldots, x_n) = x_i$, where $1 \leqslant i \leqslant n$.

(iv) $f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n))$, where $g, h_1, \ldots, h_m$ are primitive recursive functions.

(v) $f(0) = k$, $f(x') = g(x, f(x))$, where k is a natural number and g is a primitive recursive function.

(vi) $f(0, x_2, \ldots, x_n) = g(x_2, \ldots, x_n)$, $f(x', x_2, \ldots, x_n) = h(x, f(x, x_2, \ldots, x_n), x_2, \ldots, x_n)$, where g and h are primitive recursive functions.

This formulation is due to Kleene.

An n-ary relation R (of natural numbers) is said to be primitive recursive if there is a primitive recursive function f which assumes values 0 and 1 only such that $R(a_1, \ldots, a_n)$ is true if and only if $f(a_1, \ldots, a_n) = 0$.

EXERCISE 10.3. We define $+$ and $\cdot$ as follows:

$$a + 0 = a, \qquad a \cdot 0 = 0,$$

$$a + b' = (a + b)', \qquad a \cdot b' = a \cdot b + a.$$

Prove the following from the above equations in **PA**.

(1) $a + b = b + a$.

(2) $a \cdot b = b \cdot a$.

(3) $a \cdot (b + c) = a \cdot b + a \cdot c$.

EXERCISE 10.4. Prove that $=$ and $<$ are primitive recursive relations of natural numbers.

Here we shall state a basic metamathematical lemma without proof, which we shall use later.

LEMMA 10.5. *The consistency of* S *(i.e.,* S*-unprovability of* $\rightarrow$*) is equivalent to the* S*-unprovability of* $0 = 1$. *In other words,* $0 = 1$ *is* S*-provable if and only if every formula of* L *is* S*-provable.* (Cf. Proposition 4.2.)

PROPOSITION 10.6 (Gödel). (1) *The graphs of all the primitive recursive functions can be expressed in* Ln, *so that (the translations of) their defining equations are provable in* **PA**.

Thus the theory of primitive recursive functions can be translated into our formal system of arithmetic. We may therefore assume that **PA** *(or any of its extensions) actually contains the function symbols for primitive recursive functions and their defining equations, as well as predicate symbols for the primitive recursive relations.*

We must distinguish between informal objects and their formal expressions (although this will lead to notational complications). For example, the formal expression (function symbol) for a primitive recursive function f will be denoted by $\bar{f}$; if R is a predicate (of natural numbers) which can be expressed in the formal language, then its formal expression will be denoted by $\bar{R}$. Also, as stated earlier, for any closed term t, $\bar{t}$ is the numeral of the number denoted by t. Although in later sections we may omit such a rigorous distinction between formal and informal expressions, it is essential in this section.

(2) *Let R be a primitive recursive relation of n arguments. It can be represented in* **PA** *by a formula $\bar{R}(a_1, \ldots, a_n)$, namely $\bar{f}(a_1, \ldots, a_n) = \bar{0}$, where f is the characteristic function of R. Then, for any n-*tuple *of numbers $(m_1, \ldots, m_n)$, if $R(m_1, \ldots, m_n)$ is true, then $\bar{R}(\bar{m}_1, \ldots, \bar{m}_n)$ is* **PA**-*provable.*

PROOF. The proof of (1) is by induction on the inductive definition of the primitive recursive functions (i.e., by induction on their construction).

The proof of (2) is carried out as follows. We prove that for any primitive recursive function f (of n arguments) and any numbers $m_1, \ldots, m_n, p$, if $f(n_1, \ldots, m_n) = p$, then $\bar{f}(\bar{m}_1, \ldots, \bar{m}_n) = \bar{p}$ is **PA**-provable. The proof is by induction on the construction of f (according to its defining equations). Then, finally, if f is a primitive recursive function which is the characteristic function of R, we have, for all $m_1, \ldots, m_n$, if $R(m_1, \ldots, m_n)$ is true, then $\bar{f}(\bar{m}_1, \ldots, \bar{m}_n) = \bar{0}$ is **PA**-provable.

Since the rest of the argument depends heavily on this proposition, we shall use it without quoting it each time.

Note that the converse proposition (i.e., for primitive recursive R, if $\bar{R}(\bar{m}_1, \ldots, \bar{m}_n)$ is **PA**-provable, then $R(m_1, \ldots, m_n)$ is true) follows from the consistency of **PA**.

DEFINITION 10.7 (Gödel numbering). We shall define a one-to-one map from the formal expressions of the language L, such as symbols, terms, formulas, sequents and proofs, to the natural numbers. (The following is only one example of a suitable map.) For an expression X, we use $\ulcorner X \urcorner$ to denote the corresponding number, which we call the Gödel number of X.

(1) First assign different odd numbers to the symbols of Ln. (We include $\rightarrow$ and $-$ among the symbols of the language here.)

(2) Let X be a formal expression $X_0 X_1 \ldots X_n$, where each X_i, $0 \leqslant i \leqslant n$, is a symbol of L. Then X is defined to be $2^{\ulcorner X_0 \urcorner} 3^{\ulcorner X_1 \urcorner} \ldots p_n^{\ulcorner X_n \urcorner}$, where p_n is the nth prime number.

(3) If P is a proof of the form

$$\frac{Q}{S} \quad \text{or} \quad \frac{Q_1 \quad Q_2}{S}$$

then $\ulcorner P \urcorner$ is $2^{\ulcorner Q \urcorner} 3^{\ulcorner - \urcorner} 5^{\ulcorner S \urcorner}$ or $2^{\ulcorner Q_1 \urcorner} 3^{\ulcorner Q_2 \urcorner} 5^{\ulcorner - \urcorner} 7^{\ulcorner S \urcorner}$, respectively.

If an operation or relation defined on a class of formal objects (e.g., formulas, proofs, etc.) is thought of in terms of the corresponding number-theoretic operation or relation on their Gödel numbers, we say that the operation or relation has been *arithmetized*. More precisely, suppose ψ is an operation defined on n-tuples of formal objects of a certain class, and f is a number-theoretic function such that for all formal objects $X_1, \ldots, X_n, X$ (of the class considered), if ψ applied to $X_1, \ldots, X_n$ produces X, then $f(\ulcorner X_1 \urcorner, \ldots, \ulcorner X_n \urcorner) = \ulcorner X \urcorner$. Then f is called the *arithmetization* of ψ. Similarly with relations.

LEMMA 10.8. (1) *The operation of substitution can be arithmetized primitive recursively, i.e., there is a primitive recursive function* sb *of two arguments such that if $X(a_0)$ is an expression of* L *(where all occurrences of a_0 in X are indicated), and Y is another expression, then* $\mathrm{sb}(\ulcorner X(a_0) \urcorner, \ulcorner Y \urcorner) = \ulcorner X(Y) \urcorner$, *where $X(Y)$ is the result of substituting Y for a_0 in X.*

(2) *There is a primitive recursive function ν such that* $\nu(m) = \ulcorner$the mth numeral$\urcorner$. *In terms of our notation,* $\nu(m) = \ulcorner \bar{m} \urcorner$.

(3) *The notion that P is a proof (of the system* S*) of a formula A (or a sequent S) is arithmetized primitive recursively; i.e., there is a primitive recursive relation* $\mathrm{Prov}(p, a)$ *such that* $\mathrm{Prov}(p, a)$ *is true if and only if there is a proof P and a formula A (or a sequent S) such that* $p = \ulcorner P \urcorner$, $a = \ulcorner A \urcorner$ *(or* $a = \ulcorner S \urcorner$*) and P is a proof of A (or S).*

(4) Prov *may be written as* $\mathrm{Prov}_{\mathrm{S}}$ *to emphasize the system* S.

(5) *As was mentioned before, the formal expression for* Prov *will be denoted by* $\overline{\mathrm{Prov}}$.

We shall not prove this lemma. It is important to note that the axiomatizability of S is used in (3); (3) is crucial in the subsequent argument. We also use the following fact about Gödel numbering: we can go effectively from formal objects to their Gödel numbers, and back again (i.e., decide effectively whether a given number is a Gödel number, and if so, of what formal object).

$\exists x\, \overline{\mathrm{Prov}}(x, \overline{\ulcorner A \urcorner})$ is often abbreviated to $\overline{\mathrm{Pr}}(\overline{\ulcorner A \urcorner})$ or $\vdash \overline{\ulcorner A \urcorner}$.

PROPOSITION 10.9. (1) *If A is S-provable, then $\vdash \overline{\ulcorner A \urcorner}$ is S-provable.*

(2) *If $A \leftrightarrow B$ is S-provable, then $\overline{\mathrm{Pr}}(\overline{\ulcorner A \urcorner}) \leftrightarrow \overline{\mathrm{Pr}}(\overline{\ulcorner B \urcorner})$, i.e., $\vdash \overline{\ulcorner A \urcorner} \leftrightarrow \vdash \overline{\ulcorner B \urcorner}$, is S-provable.*

(3) $\vdash \overline{\ulcorner A \urcorner} \rightarrow (\vdash \overline{\ulcorner \vdash \overline{\ulcorner A \urcorner} \urcorner})$ *is S-provable.*

PROOF. (1) Suppose A is provable with a proof P. Then by (3) of Lemma 10.5, $\mathrm{Prov}(\ulcorner P \urcorner, \ulcorner A \urcorner)$ is true, which implies, by (2) of Proposition 10.6, that $\exists x\, \overline{\mathrm{Prov}}(x, \overline{\ulcorner A \urcorner})$, i.e., $\vdash \overline{\ulcorner A \urcorner}$, is S-provable.

(2) Suppose $A \equiv B$ is provable with a proof P and A is provable with a proof Q. There is a prescription for constructing a proof of B from P and Q, uniform in P and Q, which can be arithmetized by a primitive recursive function f. Thus $\mathrm{Prov}(q, \ulcorner A \urcorner) \rightarrow \mathrm{Prov}(f(p, q), \ulcorner B \urcorner)$ is true, from which it follows by (2) of Proposition 10.6 that $\vdash \overline{\ulcorner A \urcorner} \rightarrow \vdash \overline{\ulcorner B \urcorner}$ is provable. The same argument works for $\vdash \overline{\ulcorner B \urcorner} \rightarrow \vdash \overline{\ulcorner A \urcorner}$.

(3) If P is a proof of A, then we can construct a proof Q of $\vdash \overline{\ulcorner A \urcorner}$ by (1). This process is uniform in P; in other words, there is a uniform prescription for obtaining Q from P. Thus

$$\mathrm{Prov}(p, \ulcorner A \urcorner) \rightarrow \mathrm{Prov}(f(p), \ulcorner \overline{\mathrm{Pr}}(\overline{\ulcorner A \urcorner}) \urcorner)$$

is true for some primitive recursive function f, from which it follows that $\vdash \overline{\ulcorner A \urcorner} \rightarrow \vdash \overline{\ulcorner \vdash \overline{\ulcorner A \urcorner} \urcorner}$ is provable.

We shall now consider the notion of truth definition and Tarski's theorem concerning it.

DEFINITION 10.10. A formula of L (the language of S) with one free variable, say $T(a_0)$, is called a *truth definition* for S if for every sentence A of L,

$$T(\overline{\ulcorner A \urcorner}) \equiv A$$

is S-provable.

THEOREM 10.11 (Tarski). *If* **S** *is consistent, then it has no truth definition.*

PROOF. Suppose otherwise. Then there is a formula $T(a_0)$ of L such that for every sentence A of L, $T(\overline{\ulcorner A \urcorner}) \equiv A$ is provable in **S**. Consider the formula $F(a_0)$, with sole free variable a_0, defined as: $\neg T(\overline{\mathrm{sb}}(a_0, \bar{\nu}(a_0)))$. Put $p = \ulcorner F(a_0) \urcorner$, and let A_T be the sentence $F(\bar{p})$. Then by definition:

$$A_T \equiv \neg T(\overline{\mathrm{sb}}(\bar{p}, \bar{\nu}(\bar{p}))). \tag{1}$$

Also, since $\ulcorner A_T \urcorner = \mathrm{sb}(p, \nu(p))$, we can prove in **S** the equivalences:

$$\begin{aligned} A_T &\equiv T(\overline{\ulcorner A_T \urcorner}) \qquad \text{(by assumed property of } T\text{)} \\ &\equiv T(\overline{\mathrm{sb}}(\bar{p}, \bar{\nu}(\bar{p}))). \end{aligned} \tag{2}$$

(1) and (2) together contradict the consistency of **S**.

An interesting consequence of Theorem 10.11 is the following. First note that in the proof of Theorem 10.11, we need *not* assume that **S** is axiomatizable (cf. Def. 10.1). So we may take as the axioms of **S** the set of all sentences of Ln which are *true* in the intended interpretation (or standard model) $\mathfrak{M}$ of **PA** (using the ordinary semantic or model-theoretic definition of truth in $\mathfrak{M}$). We then obtain that there is no formula $T(a_0)$ of Ln such that for any sentence A of Ln:

$$A \text{ is true} \Leftrightarrow T(\overline{\ulcorner A \urcorner}) \text{ is true}$$

(i.e., true in $\mathfrak{M}$). This corollary of Theorem 10.11 can be stated in the form: "The notion of arithmetical truth is not arithmetical" (i.e., cannot be expressed by a formula of Ln). This is often taken as the statement of Tarski's theorem.

DEFINITION 10.12. **S** is called *incomplete* if for some sentence A, neither A nor $\neg A$ is provable in **S**.

Next we introduce "Gödel's trick" for use in Theorem 10.16.

DEFINITION 10.13. Consider a formula $F(\alpha)$ with a metavariable α (i.e., a new predicate variable, not in L, which we only use temporarily for notational convenience), where α is regarded as an atomic formula in $F(\alpha)$ and $F(\alpha)$ is closed. $F(\vdash \overline{\mathrm{sb}(a_0,\ \bar{\nu}(a_0))})$ is a formula with a_0 as its sole free variable. Define $p = \ulcorner F(\vdash \overline{\mathrm{sb}(a_0,\ \bar{\nu}(a_0))})\urcorner$ and A_F as $F(\vdash \mathrm{sb}(\bar{p},\ \bar{\nu}(\bar{p})))$. Note that A_F is a sentence of L.

LEMMA 10.14. *$A_F \equiv F(\vdash \overline{\ulcorner A_F \urcorner})$ is provable in* S.

PROOF. Since $\ulcorner A_F \urcorner = \mathrm{sb}(p,\ \nu(p))$ by definition,

$$\overline{\ulcorner A_F \urcorner} = \mathrm{sb}(\bar{p},\ \bar{\nu}(\bar{p})) \text{ is provable in } \mathbf{S}.$$

Hence $A_F \equiv F(\vdash \overline{\ulcorner A_F \urcorner})$ is provable in S.

From now on, we shall use the abbreviation $\vdash A$ for $\vdash \overline{\ulcorner A \urcorner}$.

DEFINITION 10.15. S is called *ω-consistent* if the following condition is satisfied. For every formula $A(a_0)$, if $\neg A(\bar{n})$ is provable in S for every $n = 0, 1, 2, \ldots$, then $\exists x\, A(x)$ is not provable in S. Note that ω-consistency of S implies consistency of S.

THEOREM 10.16 (Gödel's first incompleteness theorem). *If* S *is ω-consistent, then* S *is incomplete.*

PROOF. There exists a sentence A_G of L such that $A_G \equiv \neg \vdash A_G$ is provable in S. (Any such sentence will be called a Gödel sentence for S.) This follows from Lemma 10.14, by taking $F(\alpha)$ in Definition 10.13 to be $\neg\alpha$. Then $A_G \equiv \neg \vdash A_G$ is provable in S. First we shall show that A_G is not provable in S, assuming only the consistency of S (but without assuming the ω-consistency of S). Suppose that A_G were provable in S. Then by (1) of Proposition 10.9, $\vdash A_G$ is provable in S; thus by the definition of Gödel sentence, $\neg A_G$ is provable in S, contradicting the consistency of S.

Next we shall show that $\neg A_G$ is not provable in S, assuming the ω-consistency of S. Since we have proved that A_G is not provable in S, for each $n = 0, 1, 2, \ldots$ $\neg\mathrm{Prov}(\bar{n}, \overline{\ulcorner A_G \urcorner})$ is provable in S. By the ω-consistency of S, $\exists x\, \overline{\mathrm{Prov}}(x, \overline{\ulcorner A_G \urcorner})$ is not provable in S. Since $\neg A_G \equiv \vdash A_G$ is provable in S, $\neg A_G$ is not provable in S.

REMARK. In fact, A_G, although unprovable, is (intuitively) true, since it asserts its own unprovability.

DEFINITION 10.17. $\overline{\text{Consis}_{\mathbf{S}}}$ is the sentence $\neg \vdash 0 = 1$. (So $\overline{\text{Consis}_{\mathbf{S}}}$ asserts the consistency of $\mathbf{S}$.)

THEOREM 10.18 (Gödel's second incompleteness theorem). *If* $\mathbf{S}$ *is consistent, then* $\overline{\text{Consis}_{\mathbf{S}}}$ *is not provable in* $\mathbf{S}$.

PROOF. Let A_G be a Gödel sentence. In the proof of Theorem 10.16, we proved that A_G is not provable, assuming only consistency of $\mathbf{S}$. Now we shall prove a stronger theorem: that $A_G \equiv \overline{\text{Consis}_{\mathbf{S}}}$ is provable in $\mathbf{S}$.

(1) To show $A_G \to \overline{\text{Consis}_{\mathbf{S}}}$ is provable in $\mathbf{S}$. By Lemma 10.5, $\overline{\neg\text{Consis}_{\mathbf{S}}} \equiv \forall \ulcorner A \urcorner (\vdash A)$ is provable (where $\forall \ulcorner A \urcorner$ means: for all Gödel numbers of formulas A). Therefore, $A_G \to \neg \vdash A_G \to \overline{\neg\forall \ulcorner A \urcorner} (\vdash A) \to \overline{\text{Consis}_{\mathbf{S}}}$.

(2) To show $\overline{\text{Consis}_{\mathbf{S}}} \to A_G$ is provable in $\mathbf{S}$. Again by Lemma 10.5, $\overline{\text{Consis}_{\mathbf{S}}}, \vdash A_G \to \neg \vdash \neg A_G \to \neg \vdash \vdash A_G$, since $\neg A_G \equiv \vdash A_G$ (cf. (3) of Lemma 10.8). But $\vdash A_G \to \vdash \vdash A_G$, by Proposition 10.9. So $\overline{\text{Consis}_{\mathbf{S}}}, \vdash A_G \to \neg \vdash \vdash A_G \wedge \vdash \vdash A_G$, and so $\overline{\text{Consis}_{\mathbf{S}}} \to \neg \vdash A_G \to A_G$.

EXERCISE 10.19. Define the system **QA** as the quantifier-free part of **PA**. Show that the following are provable in **QA** for free variables a, b, c.

(1) $a = a$,

(2) $a = b \to b = a$,

(3) $a + b = b + a$,

(4) $a \cdot b = b \cdot a$,

(5) $a \cdot (b + c) = a \cdot b + a \cdot c$.

EXERCISE 10.20. In Gödel's trick (cf. Definition 10.13) we may replace $\text{sb}(a_0, \nu(a_0))$ by $e(\text{sb}(a_0, \nu(a_0)))$ for some primitive recursive function e which satisfies that if A is a formula then $e(\ulcorner A \urcorner)$ is Gödel number of a formula obtained from A by adding some more stages of the definition of formula; for example, $e(\ulcorner A \urcorner) = \ulcorner \neg A \urcorner$. Show that if $e(\ulcorner A \urcorner) = \ulcorner \neg A \urcorner$, $p = \ulcorner F(\vdash \bar{e}(\overline{\text{sb}}(a_0, \bar{\nu}(a_0)))) \urcorner$ and B_F is $F(\vdash \bar{e}(\overline{\text{sb}}(\bar{p}, \bar{\nu}(\bar{p}))))$, then $\ulcorner B_F \urcorner = \text{sb}(p, \nu(p))$, i.e., B_F is $F(\vdash \neg B_F)$.

PROBLEM 10.21 (Löb). Show that for any sentence A, if $(\vdash A) \to A$ is **PA**-provable, then A is itself provable. [*Hint*: By Gödel's trick there is a sentence B such that $B \equiv (\vdash B \supset A)$. For such B, if B is provable then $\vdash B$ is provable (cf. Proposition 10.9) and $(\vdash B) \to A$ is provable; thus A is provable. This procedure is uniform in the proofs of B; hence by formalizing the entire process we obtain $(\vdash B) \to (\vdash A)$. This and the assumption $(\vdash A) \to A$ imply $(\vdash B) \to A$. But, by the definition of B, the last sequent implies B itself, and hence $\vdash B$ (Proposition 10.9). So, since $\vdash B$ and $(\vdash B) \to A$ are both provable, so is A.]

PROBLEM 10.22 (Rosser). Let e be a primitive recursive function satisfying $e(\ulcorner A \urcorner) = \ulcorner \neg A \urcorner$ as in Exercise 10.20. Let $F(a_0)$ be

$$\forall x_1\, (\overline{\mathrm{Prov}}(x_1, \overline{\mathrm{sb}}(a_0, \bar{\nu}(a_0))) \supset \exists x_2\, (x_2 \leqslant x_1 \wedge \overline{\mathrm{Prov}}(x_2, \bar{e}(\overline{\mathrm{sb}}(x_1, \bar{\nu}(x_0)))))).$$

Define $p = \ulcorner F(a_0) \urcorner$ and A_R as $F(\bar{p})$. Prove that if $\mathbf{S}$ is consistent, then neither A_R nor $\neg A_R$ is provable in $\mathbf{S}$.

REMARK. This strengthens Gödel's first incompleteness theorem. Namely, the hypothesis of the ω-consistency in Theorem 10.16 is weakened to the consistency.

§11. A discussion of ordinals from a finitist standpoint

When one is concerned with consistency proofs, their philosophical interpretation is always a paramount problem. There is no doubt that Hilbert's "finitist standpoint" which considers only a finite number of symbols concretely given and arguments concretely given about finite sequences of these symbols (called expressions) is an ideal standpoint in proving consistency. From this standpoint, one defines expressions in the following way (as we have, in fact, done already).

(0) Firstly, we give a finite set of symbols, called an alphabet.

(1) Next, we give a finite set of finite sequences of these symbols, called initial expressions.

(2) Next, we give a finite set of concrete operations, for constructing or generating expressions from expressions already obtained.

(3) Finally, we restrict ourselves to considering only expressions obtained by starting with step (1) and iterating step (2).

As a special case of the above, let us suppose that we are given symbols $a_1, \ldots, a_n$ by (1), and concrete operations $f_1, \ldots, f_j$, to obtain new expressions from expressions we already have, and let $\mathscr{D}$ be the collection of all expressions thus obtained. Then the definition of $\mathscr{D}$ is as follows:

(0) The alphabet consists of $\{a_1, \ldots, a_n\}$.

(1) $a_1, \ldots, a_n$ (considered as sequences of length 1) are in $\mathscr{D}$.

(2) If $x_1, \ldots, x_{k_i}$ are in $\mathscr{D}$, then $f_i(x_1, \ldots, x_{k_i})$ is in $\mathscr{D}$ $(i = 1, \ldots, j)$.

(3) $\mathscr{D}$ consists of only these objects (expressions) obtained by (1) and (2).

This is called a *recursive* or *inductive definition* of the class $\mathscr{D}$. Corresponding to this inductive definition, we have a principle of "*proof by induction*" on (the elements of) $\mathscr{D}$, namely, let A be any property (of expressions), and suppose we can do the following.

(1) Prove that $A(a_1), \ldots, A(a_n)$ hold;

(2) Assuming $A(x_1), \ldots, A(x_k)$ hold for $x_1, \ldots, x_k$ in $\mathscr{D}$, infer that

$$A(f_1(x_1, \ldots, x_{k_1})), \ldots, A(f_j(x_1, \ldots, x_{k_j}))$$

hold.

Then we conclude that $A(x)$ holds for all x in $\mathscr{D}$. This follows since for any x in $\mathscr{D}$ that is concretely given, one can show that $A(x)$ holds by following the steps in constructing this x, by applying (1) and (2) above step by step. According to this viewpoint, we can regard "induction" simply as a general statement of a concrete method of proof applicable for any given expression x, and not as an axiom that is accepted a priori.

Though nobody denies that the above way of thinking is contained in Hilbert's standpoint, there are many opinions about where to set the boundary of this standpoint: for example, assuming that transfinite induction up to each of $\omega, \omega \cdot 2, \omega \cdot 3, \ldots$ is accepted, whether transfinite induction up to ω^2 should also be accepted; or, assuming that transfinite induction up to each of $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ is accepted, whether transfinite induction up to the first ε-number (denoted by ε_0) should be. If we consider each concretely given expression (in this case an ordinal less than ε_0), then it must be less than some ω_n, and so should be accepted—or should it? Here ω_n denotes the ordinal

$$\left.\omega^{\cdot^{\cdot^{\omega}}}\right\} n.$$

When one thinks about this in a very skeptical way, how far can one accept induction? One might even perhaps doubt whether induction up to ω itself is already beyond Hilbert's standpoint.

However, if we interpret Hilbert's finitist standpoint in an extremely pure and restricted way so as to forbid both transfinite induction and all abstract notions such as Gödel's primitive recursive functionals of finite types, then by Gödel's incompleteness theorem, it is clear that *the consistency of* **PA** *cannot be proved if one adheres to this standpoint,* since (presumably) such strictly finitist methods can be formalized in **PA** (in fact, in "primitive recursive arithmetic": see below).

Therefore in a consistency proof it is always very interesting to see what is used that goes beyond Hilbert's finitist standpoint, and on what basis it can be justified.

At present, the methods used mainly for consistency proofs are firstly those using transfinite induction (initiated by Gentzen), and, secondly, those using higher type functionals (initiated by Gödel).

We explain the first method, that of Gentzen. First, in order to make sure of our standpoint, let us consider an inductive definition of natural numbers that adheres most closely to the above scheme:

N 1 1 is a natural number.

N 2 If a is a natural number, then $a1$ is a natural number.

N 3 Only those objects obtained by N 1 and N 2 are natural numbers.

Although we normally consider a definition like this to be obvious, it seems that this is because much knowledge is often unconsciously presupposed. In order to clarify our unconsciously-arrived-at standpoint, let us ask ourselves questions that a person E who has no understanding of N 1–N 3 might ask.

First, E might say he did not understand N 2 and N 3. For E it is impossible to understand N 2 using the notion of natural number when one does not understand "natural numbers" (a "vicious circle"). Moreover, E cannot understand in N 3 what "those objects obtained by N 1 and N 2" means. There are many possible answers to these doubts. The most practical one from the didactic point of view will be as follows: 1 is a natural number by N 1. Now that we know 1 is a natural number, 11 is a natural number by N 2; now that we know 11 is a natural number, 111 is a natural number by N 2. Everything obtained in this way by starting with N 1 and iterating the operation N 2 is a natural number. N 3 says on the other hand, that only those things obtained in this way are natural numbers. Of course E might ask more questions about the above explanation: "What do you mean by 'iterating the operation N 2'?", "What do you mean by 'everything obtained in this way'?" etc., and this kind of discussion can be continued endlessly. I hope that E will finally get the idea. The important fact is that the general concept of a (potentially) infinite process of creating new things by iterating a concrete operation a finite number of times is presupposed in order to understand the definition N 1–N 3 of natural numbers, and that the purpose of the definition N 1–N 3 is to specify the process of defining natural numbers by such a procedure.

When we analyze precisely the discussion repeated endlessly with E, we will realize that we must accept or presuppose to some extend the notion of finite sequence (or finite iteration of an operation) as our basic notion. Here an important remark should be made: this does not mean that we must accept large amounts of knowledge about sequences and finiteness separately; only that which seems absolutely necessary to understand the single notion of finite sequence.

In order to clarify our standpoint further, let us consider the inductive definition of the finite (non-empty) sequences of natural numbers:

S 1 If n is a natural number, then n itself is a finite sequence of natural numbers.

S 2 If m is a natural number and s is a finite sequence of natural numbers, then $s * m$ is a finite sequence of natural numbers.

S 3 Only those objects obtained by S 1 and S 2 are finite sequences of natural numbers.

It should be realized that this kind of definition is regarded as basic and clear, no matter what standpoint one assumes.

We shall present some more examples of such inductively defined classes of concrete objects, and properties of them.

For instance, the notion of length of a finite sequence of natural numbers is defined inductively as follows:

L 1 If s is a sequence of natural numbers consisting of a natural number n only, then the length of s is 1.

L 2 If s is a sequence of natural numbers of the form $s_0 * n$, and the length of s_0 is l, then the length of s is $l + 1$.

We can certainly take an alternative definition: given a sequence of natural numbers, say s, examine s and count the number of $*$'s in it. If the number of $*$'s is l, then the length of s is $l + 1$. (Each of these definitions presents an operation which applies to the concretely given figures in a general form.)

These finitist inferences often present striking similarities to the arguments in the following formalism, which we call primitive recursive arithmetic.

(1) The basic logical system is the propositional calculus.

(2) The defining equations of primitive recursive functions are assumed as axioms.

(3) No quantifiers are introduced.

(4) Mathematical induction (for quantifier-free formulas) is admitted:

$$\frac{A(a), \Gamma \to \Delta, A(a')}{A(0), \Gamma \to \Delta, A(t)},$$

where a does not occur in $A(0)$, Γ or Δ, and t is an arbitrary term.

From the above discussion, it seems quite reasonable to characterize Hilbert's finitist standpoint as that which can be formalized in *primitive recursive arithmetic.* This standpoint shall be called the "purely finitist standpoint". It is therefore of paramount importance to clarify where a consistency proof exceeds this formalism, i.e., the purely finitist standpoint. (Thus, in the following, we shall not bother with arguments which can be carried out within the above formalism.) In order to pursue this point, we shall first present the recursive definition of ordinal numbers up to ε_0 (the first ε-number); temporarily, by "ordinal" we mean: ordinal less than ε_0.

O 1 0 is an ordinal.

O 2 Let μ and $\mu_1, \mu_2, \ldots, \mu_n$ be ordinals. Then $\mu_1 + \mu_2 + \ldots + \mu_n$ and ω^μ are ordinals.

O 3 Only those objects obtained by O 1 and O 2 are ordinals.

ω^0 will be denoted by 1. Regarding 1 as the natural number 1, $1 + 1$ as 2, etc., we may assume that the natural numbers are included in the ordinals. (We may also include 0 among the natural numbers if we wish.)

We can now define the relations $=$ and $<$ on ordinals so that they match the notions of equality and the natural ordering of ordinals which we know from set theory, and develop the theory of ordinals for these relations within the purely finitist standpoint. We can actually inductively define $=$, $<$, $+$, and $\cdot$ simultaneously so that they satisfy the following.

(1) $<$ is a linear ordering and 0 is its least element.

(2) $\omega^\mu < \omega^\nu$ if and only if $\mu < \nu$.

(3) Let μ be an ordinal containing an occurrence of the symbol 0 but not 0 itself, and let μ' be the ordinal obtained from μ by eliminating this occurrence of 0 as well as excessive occurrences of $+$. Then $\mu = \mu'$.

As a consequence of (3) it can be easily shown that

(4) Every ordinal which is not 0 can be expressed in the form

$$\omega^{\mu_1} + \omega^{\mu_2} + \ldots + \omega^{\mu_n},$$

where each of $\mu_1, \mu_2, \ldots, \mu_n$ which is not 0 has the same property. (Each term ω^{μ_i} is called a monomial of this ordinal.)

(5) Let μ and ν be of the forms

$$\omega^{\mu_1} + \omega^{\mu_2} + \ldots + \omega^{\mu_k} \quad \text{and} \quad \omega^{\nu_1} + \omega^{\nu_2} + \ldots + \omega^{\nu_l},$$

respectively. Then $\mu + \nu$ is defined as

$$\omega^{\mu_1} + \omega^{\mu_2} + \ldots + \omega^{\mu_k} + \omega^{\nu_1} + \omega^{\nu_2} + \ldots + \omega^{\nu_l}.$$

(6) Let μ be an ordinal which is written in the form of (4) and contains two consecutive terms ω^{μ_j} and $\omega^{\mu_{j+1}}$ with $\mu_j < \mu_{j+1}$, i.e., μ is of the form

$$\ldots + \omega^{\mu_j} + \omega^{\mu_{j+1}} + \ldots,$$

and let μ' be an ordinal obtained from μ be deleting "$\omega^{\mu_j} +$", so that μ' is of the form

$$\ldots \omega^{\mu_{j+1}} + \ldots .$$

Then $\mu = \mu'$.

As a consequence of (6) we can show that

(7) For every ordinal μ (which is not 0) there is an ordinal of the form

$$\omega^{\mu_1} + \omega^{\mu_2} + \ldots + \omega^{\mu_n},$$

where $\mu_1 \geqslant \ldots \geqslant \mu_n$ such that $\mu = \omega^{\mu_1} + \ldots + \omega^{\mu_n}$, where $\mu \geqslant \nu$ means: $\nu < \mu$ or $\nu = \mu$. The latter is called the normal form of μ. (This normal form of μ is unique, since the same holds for every ordinal which is used in constructing μ: see O 2.)

(8) Let μ have the normal form

$$\omega^{\mu_1} + \omega^{\mu_2} + \ldots + \omega^{\mu_n}$$

and ν be > 0. Then $\mu \cdot \omega^{\nu} = \omega^{\mu_1 + \nu}$.

(9) Let μ and ν be as in (5). Then

$$\mu \cdot \nu = \mu \cdot \omega^{\nu_1} + \mu \cdot \omega^{\nu_2} + \ldots + \mu \cdot \omega^{\nu_l}.$$

(10) $(\omega^{\mu})^n$ is defined as $\omega^{\mu} \cdot \omega^{\mu} \ldots \omega^{\mu}$ (n times) for any natural number n. Then $(\omega^{\mu})^n = \omega^{\mu \cdot n}$.

As a consequence of our definitions, it can easily be shown that for an arbitrary ordinal μ an ordinal of the form ω_n which satisfies $\mu < \omega_n$ can be constructed.

It is obvious that for any given natural number n the length of a strictly decreasing sequence of ordinals which starts with n is at most $n + 1$; in other words, there can be no strictly decreasing sequence of ordinals which starts with n and has length $n + 2$. This fact tells us that the notion of arbitrary, strictly decreasing sequences of ordinals which start with n is a clear notion.

At this point it is not very meaningful to object to this on the grounds that if we write the statement that a strictly decreasing sequence terminates, in terms of expressions in the Kleene hierarchy, it turns out to belong to the Π^1_1-class. The important fact is not to which class of the hierarchy it belongs but how evident it is. We shall come to this point later.

In the following section, a consistency proof (for **PA**) will be given in the following way. In order to emphasize the concrete or "figurative" aspect of the arguments, we say "proof-figure" for formal proof.

1) We present a uniform method such that, if a proof-figure P is concretely given, then the method enables us to concretely construct another proof-figure P'; furthermore, the end-sequent of P' is the same as that of P if the end-sequent of P does not contain quantifiers. The process of constructing P' from P is called the "reduction" (of P) and may be denoted by r. Thus $P' = r(P)$.

2) There is a uniform method by which every proof-figure is assigned an ordinal $< \varepsilon_0$. The ordinal assigned to P (the ordinal of P) may be denoted by $o(P)$.

3) o and r satisfy: whenever a proof-figure P contains an application of ind or cut, then $o(P) > \omega$ and $o(r(P)) < o(P)$, and if P does not contain any such application, then $o(P) < \omega$.

Suppose we have concretely shown that any strictly decreasing sequence of natural numbers is finite, and that whenever a concrete method of constructing decreasing sequences of ordinals $< \varepsilon_0$ is given it can be recognized that any decreasing sequence constructed this way is finite (or such a sequence terminates). (By "decreasing sequence" we will always mean strictly decreasing sequence.) We can then conclude, in the light of 1)–3) above, that, for any given proof-figure P whose end-sequent does not contain quantifiers, there is a concrete method of transforming it into a proof-figure with the same end-sequent and containing no applications of the rules cut and ind. It can be easily seen, on the other hand, that no proof-figure without applications of a cut or ind can be a proof of the empty sequent. Thus we can claim that the consistency of the system has been proved.

The crucial point in the process described above is to demonstrate:

(*) Whenever a concrete method of constructing decreasing sequences of ordinals is given, any such decreasing sequence must be finite.

We are going to present a version of such a demonstration, which the author believes represents the most illuminating approach to the consistency proof.

Suppose $a_0 > a_1 > \ldots$ is a decreasing sequence concretely given.

(I) Assume $a_0 < \omega$, or a_0 is a natural number.

Consider a decreasing sequence which starts with a concretely given natural number. As soon as one writes down its first term n, one can recognize that its length must be at most $n+1$. Hence we can assume that a_0 is not a natural number.

In order to deal with all ordinals $< \varepsilon_0$, we shall define the concept of α-sequence and α-eliminator for all $\alpha < \varepsilon_0$. We start, however, with a simple example rather than the general definition.

(II) Suppose each a_i in $a_0 > a_1 > \ldots$ is written in the canonical form; a_i has the form

$$\omega^{\mu^i_1} + \omega^{\mu^i_2} + \ldots + \omega^{\mu^i_{n_i}} + k_i,$$

where $\mu^i_j > 0$ and k_i is a natural number. (This includes the case where $+\, k_i$ does not actually appear.) A sequence in which k_i does not appear for any a_i will be called a 1-sequence. We call $\omega^{\mu^i_1} + \omega^{\mu^i_2} + \ldots + \omega^{\mu^i_{n_i}}$ in a_i the 1-major part of a_i. We shall give a concrete method (M_1) which enables us to do the following: given a descending sequence $a_0 > a_1 > \ldots$, where each a_i is written in its canonical form, the method M_1 concretely produces a (decreasing) 1-sequence $b_0 > b_1 > \ldots$ so as to satisfy the condition

(C_1) b_0 is the 1-major part of a_0, and we can concretely show that if $b_0 > b_1 > \ldots$ is a finite sequence, then so is $a_0 > a_1 > \ldots$.

This method M_1 (a 1-eliminator) is defined as follows. Put $a_i = a'_i + k_i$, where a'_i is the 1-major part of a_i. Then $a_0 > a_1 > a_2 > \ldots$ can be expressed as $a'_0 + k_0 > a'_1 + k_1 > a'_2 + k_2 > \ldots$.

Put $b_0 = a'_0$. Suppose $b_0 > b_1 > \ldots > b_m$ has been constructed in such a manner that b_m is a'_j for some j. Then either $a'_j = a'_{j+1} = \ldots = a'_{j+p}$ for some p and a_{j+p} is the last term in the sequence, or $a'_j = a'_{j+1} = \ldots = a'_{j+p} > a'_{j+p+1}$. This is so, since $a'_j = a'_{j+1} = \ldots = a'_{j+p} = \ldots$ implies $k_j > k_{j+1} > \ldots > k_{j+p} > \ldots$, but such a sequence (of natural numbers) must stop (cf. (I)). Therefore, as stated above, either the whole sequence stops, or

$a'_{j+p} > a'_{j+p+1}$ for some p. If the former is the case, then stop. If the latter holds, then put $b_{m+1} = a'_{j+p+1}$.

From the definition, it is obvious that $b_0 > b_1 > \ldots > b_m > \ldots$. Suppose this sequence is finite, say $b_0 > b_1 > \ldots > b_m$. Then according to the prescribed construction of b_{m+1} the original sequence is finite. Thus the sequence $b_0 > b_1 > \ldots$ satisfies (C_1), and we have completed the definition of M_1.

(III) Suppose we are given a decreasing sequence $a_0 > a_1 > \ldots$, in which $a_0 < \omega^2$. Then by a 1-eliminator M_1 applied to this sequence, we can construct a 1-sequence $b_0 > b_1 > \ldots$, where $b_0 \leqslant a_0$. Then $b_0 > b_1 > \ldots$ can be written in the form $\omega \cdot k_0 > \omega \cdot k_1 > \ldots$, which implies $k_0 > k_1 > \ldots$. Then by (I), $k_0 > k_1 > \ldots$ must be finite, which successively implies that $b_0 > b_1 > \ldots$ and $a_0 > a_1 > \ldots$ are finite.

(IV) We now define "n-sequences" as follows. Let $a_0 > a_1 > \ldots$ be a descending sequence which is written in the form $a'_0 + c_0 > a'_1 + c_1 > \ldots$, where if $a_i = a'_i + c_i$ then each monomial in a'_i is $\geqslant \omega^n$ and each monomial in c_i is $< \omega^n$. (a'_i is called the n-major part of a_i.) Such a sequence is called an n-sequence if every c_i is empty.

Now assume (as an induction hypothesis) that any descending sequence $d_0 > d_1 > \ldots$, with $d_0 < \omega^n$, is finite. We shall define a concrete method M_n (an n-eliminator) such that, given a decreasing sequence $a_0 > a_1 > \ldots$, M_n concretely produces an n-sequence, say $b_0 > b_1 > \ldots$, which satisfies:

(C_n) b_0 is the n-major part of a_0, and if $b_0 > b_1 > \ldots$ is finite then we can concretely show that $a_0 > a_1 > \ldots$ is also finite.

The prescription for M_n is as follows. Write each a_i as $a'_i + c_i$, where a'_i is the n-major part of a_i. The definition now proceeds very much like that for 1-sequences in (II). Namely, put $b_0 = a'_0$. Suppose $b_0 > b_1 > \ldots > b_m$ has been constructed and b_m is a'_j. If $a'_j = a'_{j+1} = \ldots = a'_{j+p}$ and a'_{j+p} is the last term in the given sequence, then stop. Otherwise $a'_j = a'_{j+1} = \ldots = a'_{j+p} > a'_{j+p+1}$ for some p, since $a'_j = a'_{j+1} = \ldots = a'_{j+p}$ implies that $c_j > c_{j+1} > \ldots > c_{j+p}$, which, by the induction hypothesis, is finite; hence for some p, $c_{j+p+1} \geqslant c_{j+p}$, which implies $a'_{j+p} > a'_{j+p+1}$. Then define $b_m = a'_{j+p+1}$. Then the sequence $b_0 > b_1 > \ldots$ satisfies (C_n), and so we have successfully defined M_n.

(V) By means of the n-eliminator M_n, we shall prove that a decreasing sequence $a_0 > a_1 > \ldots$, where $a_0 < \omega^{n+1}$, must be finite. By applying M_n to $a_0 > a_1 > \ldots$, we can construct concretely an n-sequence, say $b_0 > b_1 > \ldots$, where $b_0 \leqslant a_0$. Moreover, b_i can be written as $\omega^n \cdot k_i$, where k_i is a natural number. So, $\omega^n \cdot k_0 > \omega^n \cdot k_1 > \ldots$, and this implies $k_0 > k_1 > \ldots$, which is a finite sequence by (I), hence $b_0 > b_1 > \ldots$ is finite, which in turn implies that $a_0 > a_1 > \ldots$ is finite.

(VI) From (III) and (V) we conclude: given (concretely) any natural number n, we can concretely demonstrate that any decreasing sequence $a_0 > a_1 > \ldots$ with $a_0 < \omega^n$ is finite.

(VII) Any decreasing sequence $a_0 > a_1 > \ldots$ is finite if $a_0 < \omega^\omega$, for this means that $a_0 < \omega^n$ for some n, and hence (VI) applies.

(VIII) Now the general theory of α-sequences and (α, n)-eliminators will be developed, where α ranges over all ordinals $< \varepsilon_0$ and n ranges over natural numbers > 0. A descending sequence $d_0 > d_1 > \ldots$ is called an α-sequence if in each d_i all the monomials are $\geqslant \omega^\alpha$. If $a = a' + c$ where each monomial in a' is $\geqslant \omega^\alpha$ and each monomial in c is $< \omega^\alpha$, then we say that a' is the α-major part of a. An α-eliminator has the property that given any concrete descending sequence, say $a_0 > a_1 > \ldots$, it concretely produces an α-sequence $b_0 > b_1 > \ldots$ such that

(i) b_0 is the α-major part of a_0,

(ii) if $b_0 > b_1 > \ldots$ is a finite sequence then we can concretely demonstrate that $a_0 > a_1 > \ldots$ is finite.

(Clearly $a_0 \geqslant b_0$.)

We delay the definition of α-eliminators. Assuming that an α-eliminator has been defined for every α, we can show that any decreasing sequence is finite. For consider $a_0 > a_1 > \ldots$. There exists an α such that $a_0 < \omega^{\alpha+1}$. An α-eliminator concretely gives an α-sequence $b_0 > b_1 > \ldots$ satisfying (i) and (ii) above. Since $b_0 \leqslant a_0$, each b_i can be written in the form $\omega^\alpha \cdot k_i$; thus $\omega^\alpha \cdot k_0 > \omega^\alpha \cdot k_1 > \ldots$, which implies $k_0 > k_1 > \ldots$. By (I) this means that $k_0 > k_1 > \ldots$ is finite, hence so is $b_0 > b_1 > \ldots$; so $a_0 > a_1 > \ldots$ is finite. This proves our objective (*). Therefore, what must be done is to define (construct) α-eliminators for all $\alpha < \varepsilon_0$.

(IX) We rename an α-eliminator to be an $(\alpha, 1)$-eliminator. Suppose that (α, n)-eliminators have been defined. A $(\beta, n+1)$-eliminator is a *concrete* method for constructing an $(\alpha \cdot \omega^\beta, n)$-eliminator from any given (α, n)-eliminator. We must go through the following procedure.

(X) Suppose $\{\mu_m\}_{m<\omega}$ is an increasing sequence of ordinals whose limit is μ (where there is a concrete method for obtaining μ_m for each m), and suppose g_m is a μ_m-eliminator. Then the g defined as follows is a μ-eliminator. Suppose $a_0 > a_1 > \ldots$ is a concretely given sequence. If a_0 is written as $a_0' + c_0$, where a_0' is the μ-major part of a_0, then there exists an m for which $c_0 < \omega^{\mu_m}$; so we may assume that each a_i is written as $a_i' + c_i$, where a_i' is the μ_m-major part of a_i. Then g_m can be applied to the sequence $a_0 > a_1 > \ldots$ and hence it concretely produces a μ_m-sequence

$$b_{10} > b_{11} > b_{12} > \ldots \tag{1}$$

satisfying (i) and (ii) above (with μ_m in place of α), with $b_{10} = a_0'$, so that in fact b_{10} is the μ-major part of a_0. Write $b_0 = b_{10}$.

Now consider the sequence $b_{11} > b_{12} > \ldots$. Suppose $b_{11} \geqslant \omega^{\mu}$. Then repeat the above procedure: i.e., for the sequence (1), write $b_{10} = b_{10}' + c_{10}$, where b_{10}' is the μ-major part of b_{10}. Then there exists an m_1 such that $c_{10} < \omega^{\mu_{m_1}}$. So apply g_{m_1} to the sequence $b_{11} > b_{12} > b_{13} > \ldots$, to obtain a μ_{m_1}-sequence

$$b_{21} > b_{22} > b_{23} > \ldots$$

satisfying (i) and (ii) (with μ_{m_1} in place of α), with b_{21} the μ-major part of b_{10}. Put $b_1 = b_{21}$. Suppose $b_{22} \geqslant \omega^{\mu}$. Then repeat this procedure with the sequence $b_{22} > b_{23} > \ldots$ to obtain a sequence

$$b_{32} > b_{33} > b_{34} > \ldots,$$

and put $b_2 = b_{32}$. Continuing in this way, we obtain a μ-sequence

$$b_0 > b_1 > b_2 > \ldots.$$

If this sequence is finite with last term (say) $b_l = b_{l+1,l}$, then it follows that in the sequence

$$b_{l+1,l} > b_{l+1,l+1} > b_{l+1,l+2} > \ldots \tag{2}$$

we must have $b_{l+1,l+1} < \omega^{\mu}$. So $b_{l+1,l+1} < \omega^{\mu_{m'}}$ for some m'. Apply $g_{m'}$ to the sequence (2); we then obtain a finite $\mu_{m'}$-sequence with only the term 0; hence the sequence (2) is finite (by definition of $\mu_{m'}$-eliminator); hence the sequence $b_{l,l-1} > b_{l,l} > \ldots$ is finite; and so on (backwards), until we deduce that the original sequence $a_0 > a_1 > \ldots$ is finite.

(XI) Suppose $\{\mu_m\}_{m<\omega}$ is a sequence of ordinals whose limit is μ and suppose, for each m, a $(\mu_m, n + 1)$-eliminator is concretely given. Then we can define a $(\mu, n + 1)$-eliminator g as follows. The definition is by induction on n. For $n = 0$ (so $n + 1 = 1$), (X) applies. Assume (XI) for n; so there is an operation k_n such that for any sequence $\{\gamma_m\}_{m<\omega}$ with limit γ and any (γ_m, n)-eliminator g_m', k_n applied to g_m' concretely produces a (γ, n)-eliminator. Now for $n + 1$, suppose a sequence $\{\beta_m\}_{m<\omega}$ with limit β and an (α, n)-eliminator p are given. Since g_m is a $(\beta_m, n + 1)$-eliminator, it produces concretely an $(\alpha \cdot \omega^{\beta_m}, n)$-eliminator from p, which we denote by $g_m(p)$. So, by taking $\alpha \cdot \omega^{\beta_m}$ for γ_m, $g_m(p)$ for g'_m and $\alpha \cdot \omega^\beta$ for γ, we can apply the induction hypothesis; thus k_n applied to $\{g'_m\}$ defines an $(\alpha \cdot \omega^\beta, n)$-eliminator q. This procedure for defining q from p is concrete, and so serves as a $(\beta, n + 1)$-eliminator.

(XII) Suppose g is a $(\mu, n + 1)$-eliminator. Then we will construct a $(\mu \cdot \omega, n + 1)$-eliminator. In virtue of (XI) it suffices to show that we can concretely construct (from g) a $(\mu \cdot m, n + 1)$-eliminator for every $m < \omega$. Suppose an (α, n)-eliminator, say f, is given. Note that

$$\alpha \cdot \omega^{\mu \cdot m} = \alpha \cdot \underbrace{\omega^\mu \cdot \omega^\mu \ldots \omega^\mu}_{m}.$$

Since g is a $(\mu, n + 1)$-eliminator, g concretely constructs an $(\alpha \cdot \omega^\mu, n)$-eliminator from f, which we denote by $g(f)$. Now apply g to this, to obtain an $(\alpha \cdot \omega^\mu \cdot \omega^\mu, n)$-eliminator $g(g(f))$. Repeating this procedure m times, we obtain the $(\alpha \cdot \omega^{\mu \cdot m}, n)$-eliminator $g(g(\ldots g(f)\ldots))$.

(XIII) We can now construct a $(1, m + 1)$-eliminator for every $m \geqslant 0$. The construction is by induction on m. We may take M_1 as a $(1, 1)$-eliminator. For $m > 0$, suppose f is an (α, m)-eliminator. Then, by (XII) (with $n + 1 = m$), we can construct an $(\alpha \cdot \omega, m)$-eliminator concretely from f. Hence we have a $(1, m + 1)$-eliminator.

(XIV) *Conclusion: An (α, n)-eliminator can be constructed for every α of the form ω_m*, i.e.,

$$\left.\omega^{\cdot^{\cdot^{\omega}}}\right\} m.$$

The construction is by induction on m. If $m = 0$, then we define α to be $1 = \omega^0$. Then an (α, n)-eliminator has been defined in (XIII) for every n. Suppose f is a $(1, n)$-eliminator, and g is an $(\alpha, n + 1)$-eliminator, which we assume to have been defined. Then g operates on f and produces the required $(1 \cdot \omega^\alpha, n) = (\omega^\alpha, n)$-eliminator. This completes the proof.

Our standpoint, which has been discussed above, is like Hilbert's in the sense that both standpoints involve "Gedankenexperimente" only on clearly defined operations applied to some concretely given figures and on some clearly defined inferences concerning these operations. An α-eliminator is a concrete operation which operates on concretely given figures. A $(\beta, 2)$-eliminator is a concrete method which enables one to exercise a Gedanken-experiment in constructing an $\alpha \cdot \omega^\beta$-eliminator from any concretely given α-eliminator. So if an ordinal, say ω_k is given, then we have a method for concretely constructing an ω_k-eliminator.

We believe that the most illuminating way to view the consistency proof of **PA**, to be described in §12, is in terms of the notion of eliminators, as described above. (In fact, it is not difficult to generalize this notion, so as to include, say, the concept of (α, ω)-eliminator, and so on; however, this is unnecessary for the consistency proof for **PA**.)

The ideas we have presented are normally formulated in terms of the notion of accessibility. It may be helpful to reformulate our ideas in terms of this notion, which (we believe) is a rough but convenient way of expressing the idea of eliminators.

We say that an ordinal μ is accessible if it has been demonstrated that every strictly decreasing sequence starting with μ is finite. More precisely, we consider the notion of accessibility only when we have actually seen, or demonstrated constructively, that a given ordinal is accessible. Therefore we never consider a general notion of accessibility, and hence we do not define the negation of accessibility as such. If we mention "the negation of accessibility", it means that we are concretely given an infinite, strictly decreasing sequence.

First, we assume we have arithmetized the construction of the ordinals (less than ε_0) given by clauses O 1–O 3. In other words, we assume a Gödel numbering of these (expressions for) ordinals, with certain nice properties: namely, the induced number-theoretic relations and functions corresponding to the ordinal relations and functions $=$, $<$, $+$, $\cdot$, and exponentiation by ω (which we will often continue to denote by the same symbols) are primitive recursive; also we can primitive recursively represent any (Gödel number of an) ordinal in its normal form, and hence decide primitive recursively whether it represents a limit or successor ordinal, etc. The ordering of the natural numbers corresponding to $<$ (on the ordinals) will be called a "standard well-ordering of type ε_0", or just "standard ordering of ε_0".

Our method for proving the accessibility of ordinals will be as follows. (We work with our standard well-ordering of type ε_0.)

(1) When it is known that $\mu_1 < \mu_2 < \mu_3 \ldots \to \nu$ (i.e., ν is the limit of the increasing sequence $\{\mu_i\}$) and that every μ_i is accessible, then ν is also accessible.

(2) A method is given by which, from the accessibility of a subsystem, one can deduce the accessibility of a larger system.

(3) By repeating (1) and (2), we show that every initial segment of our ordering is accessible, and hence so is the whole ordering.

The fact that every decreasing sequence which starts with a natural number is finite can be proved as in (I) above.

Let us proceed to the next stage: decreasing sequences of ordinals less than $\omega + \omega$. Here we can again see that every decreasing sequence terminates. This is done as follows. Consider the first term μ_0 of such a sequence. We can effectively decide whether it is of the form n or of the form $\omega + n$, where n is a natural number. If it is of the form n, then it suffices to repeat the above argument for natural numbers. If it is of the form $\omega + n$, consider the first $n + 2$ terms of the sequence

$$\mu_{n+1} < \ldots < \mu_2 < \mu_1 < \mu_0.$$

It is easily seen that μ_{n+1} cannot be of the form $\omega + m$ for any natural number m and hence must be a natural number, so we now repeat the proof for natural numbers. This method can be extended to the cases of decreasing sequences of ordinals less than $\omega \cdot n$, less than ω^2, less than ω^ω, etc.

A more mathematical presentation of this idea now follows.

LEMMA 11.1. *If μ and ν are accessible, then so is $\mu + \nu$.*

PROOF. We just generalize the proof that $\omega + \omega$ is accessible and make use of the following fact which is easily seen: given ordinals μ, ξ, ν such that $\mu \leqslant \xi < \nu$, we can effectively find a ν_0 such that $\nu_0 < \nu$ and $\xi = \mu + \nu_0$.

LEMMA 11.2. *If μ is accessible, then so is $\mu \cdot \omega$.*

PROOF. We use the following fact, which is easy to show: if $\nu < \mu \cdot \omega$, then we can find an n such that $\nu < \mu \cdot n$.

With these lemmas, let us prove that all ordinals less than ε_0 are accessible. First we introduce the technical term: "n-accessible", for every n, by induction on n.

DEFINITION 11.3. μ is said to be 1-*accessible* if μ is accessible. μ is said to be $(n + 1)$-*accessible* if for every ν which is n-accessible, $\nu \cdot \omega^\mu$ is n-accessible.

It should be emphasized that "ν being n-accessible" is a clear notion only when it has been concretely demonstrated that ν is n-accessible.

LEMMA 11.4. *If μ is n-accessible and $\nu < \mu$, then ν is n-accessible.*

LEMMA 11.5. *Suppose $\{\mu_m\}$ is an increasing sequence of ordinals with limit μ. If each μ_m is n-accessible, then so is μ.*

LEMMA 11.6. *If ν is $(n + 1)$-accessible, then so is $\nu \cdot \omega$.*

PROOF. We must show that for any n-accessible μ, $\mu \cdot \omega^{\nu \cdot \omega}$ is n-accessible. For this purpose it suffices to show that $\mu \cdot \omega^{\nu \cdot m}$ is n-accessible for each m (cf. Lemma 11.5). This is, however, obvious, since

$$\mu \cdot \omega^{\nu \cdot m} = \mu \cdot (\omega^{\nu})^m = \mu \cdot \omega^{\nu} \ldots \omega^{\nu}$$

and ν is $(n + 1)$-accessible.

PROPOSITION 11.7. 1 *is $(n + 1)$-accessible.*

PROOF. Suppose μ is n-accessible. Then by Lemma 11.6, $\mu \cdot \omega = \mu \cdot \omega^1$ is n-accessible, which means by definition that 1 is $(n + 1)$-accessible.

DEFINITION 11.8. $\omega_0 = 1$; $\omega_{n+1} = \omega^{\omega_n}$.

PROPOSITION 11.9. *ω_k is $(n - k)$-accessible for an arbitrary $n > k$.*

PROOF. By induction on k. If $k = 0$, then $\omega_k = 1$ and hence is n-accessible for all n (cf. Proposition 11.7). Suppose ω_k is $(n - k)$-accessible. Since 1 is $[n - (k + 1)]$-accessible, $1 \cdot \omega^{\omega^k}$ is $[n - (k + 1)]$-accessible by Definition 11.3, i.e., ω_{k+1} is $[n - (k + 1)]$-accessible.

As a special case of Proposition 11.9 we have:

PROPOSITION 11.10. *ω_k is accessible for every k.*

Given any decreasing sequence of ordinals (less than ε_0), there is an ω_k such that all ordinals in the sequence are less than ω_k. Therefore the sequence must be finite by Proposition 11.10. Thus we can conclude:

PROPOSITION 11.11. *ε_0 is accessible.*

An important point to note is this. Our proof of the accessibility of ε_0 (by the method of eliminators, (I)–(XIV), or by the method of Proposition 11.11) depends essentially on the fact that we are using a standard well-ordering of type ε_0, for which the successive steps in the argument are evident. Of course this is not so for an arbitrary well-ordering of type ε_0, nor for the general notion of well-ordering or ordinal.

Comparison of our standpoint with some other standpoints may help one to understand our standpoint better. First, consider set theory. Our standpoint does not assume the absolute world as set theory does, which we can think of as being based on the notion of an "infinite mind". It is obvious that, on the contrary, it tries to avoid the absolute world of an "infinite mind" as much as possible. It is true that in the study of number theory, which does not involve the notion of sets, the absolute world of numbers 0, 1, 2,... is not such a complicated notion; to an infinite mind it would be quite clear and transparent. Nevertheless, our minds being finite, it is, after all, an imaginary world to us, no matter how clear and transparent it may appear. Therefore we need reassurance of such a world in one way or another.

Next, consider intuitionism. Although our standpoint and that of intuitionism have much in common, the difference may be expressed as follows.

Our standpoint avoids abstract notions as much as possible, except those which are eventually reduced to concrete operations or Gedankenexperimente on concretely given sequences. Of course we also have to deal with operations on operations, etc. However, such operations, too, can be thought of as Gedankenexperimente on (concrete) operations.

By contrast, intuitionism emphatically deals with abstract notions. This is seen by the fact that its basic notion of "construction" (or "proof") is absolutely abstract, and this abstract nature also seems necessary for its impredicative concept of "implication". It is not the aim of intuitionism to reduce these abstract notions to concrete notions as we do.

We believe that our standpoint is a natural extension of Hilbert's finitist standpoint, similar to that introduced by Gentzen, and so we call it the Hilbert-Gentzen finitist standpoint.

Now a Gentzen-style consistency proof is carried out as follows:

(1) Construct a suitable standard ordering, in the strictly finitist standpoint.

(2) Convince oneself, in the Hilbert-Gentzen standpoint, that it is indeed a well-ordering.

(3) Otherwise use only strictly finitist means in the consistency proof.

We now present a consistency proof of this kind for **PA**.

§12. A consistency proof of PA

We assume from now on that **PA** is formalized in a language which includes a constant $\bar{f}$ for every primitive recursive function f. We call this language L.

As initial sequents of **PA** we will also take from now on the defining equations for all primitive recursive functions, as well as all sequents $\rightarrow s = t$, where s, t are closed terms of L denoting the same number, and all sequents $s = t \rightarrow$, where s, t are closed terms of L denoting different numbers.

We shall follow Gentzen's second version of his consistency proof for first order arithmetic. This involves a "reduction method". Since this method will recur often, we shall abstract the concept here. (We assume that the ordinals less than ε_0 are represented as notations in a fixed standard well-ordering, as described in §11.)

First, suppose that ordinals less than ε_0 are effectively assigned to proofs. Now let R be a property of proofs such that:

(*) For any proof P satisfying R, we can find (effectively from P) a proof P' satisfying R such that P' has a smaller ordinal than P.

We can then infer from (*), and the accessibility of ε_0:

(**) No proof satisfies R.

The procedure of finding (or constructing) P' from P in (*) is called: a *reduction of P to P'* (for the property R).

The property R of proofs that we will be interested in, is the property of having $\rightarrow$ as an end-sequent.

By giving a uniform reduction procedure for this property (Lemma 12.8), we will have shown (by (**)) that no proof of **PA** ends with $\rightarrow$; in other words:

THEOREM 12.1. *The system* **PA** *is consistent.*

Of course the importance of this theorem exists in its proof, which, apart from the assumption of the accessibility of ε_0, is strictly finitist. (Nobody suspects the consistency of Peano arithmetic!)

Theorem 12.1 follows from Lemma 12.8 (as just stated). First, we need:

DEFINITION 12.2. A proof in **PA** is *simple* if no free variables occur in it, and it contains only mathematical initial sequents, weak inferences and inessential cuts.

(Recall that a weak inference is a structural inference other than a cut. Cf. §9 for other definitions.)

LEMMA 12.3. *There is no simple proof of* $\rightarrow$.

PROOF. Let P be any simple proof. All the formulas in P are of the form $s = t$ with s and t closed. Note that with the natural interpretation of the constants, it can be determined (finitistically) whether $s = t$ is true or false (since this only involves the evaluation of certain primitive recursive functions). A sequent in P is then given the value T if at least one formula in the antecedent is false, or at least one formula in the succedent is true, and it is given the value F otherwise. It is easy to see that all mathematical initial sequents take the value T, and weak inferences and inessential cuts preserve the value T downward for sequents. So all sequents of P have the value T. But $\rightarrow$ has the value F.

DEFINITION 12.4. (1) The *grade of a formula*, is (as defined in §5) the number of logical symbols it contains. The *grade of a cut* is the grade of the cut formula; the *grade of an ind inference* is the grade of the induction formula.

(2) The *height of a sequent* S in a proof P (denoted by $h(S; P)$ or, for short, $h(S)$) is the maximum of the grades of the cuts and ind's which occur in P below S.

PROPOSITION 12.5. (1) *The height of the end-sequent of a proof is* 0.

(2) *If* S_1 *is above* S_2 *in a proof, then* $h(S_1) \geqslant h(S_2)$; *if* S_1 *and* S_2 *are the upper sequents of an inference, then* $h(S_1) = h(S_2)$.

Before defining the assignment of ordinals to proofs, we introduce the following notation. For any ordinal α and natural number n, $\omega_n(\alpha)$ is defined by induction on n; $\omega_0(\alpha) = \alpha$, $\omega_{n+1}(\alpha) = \omega^{\omega_n(\alpha)}$. So

$$\omega_n(\alpha) = \underbrace{\omega^{\cdot^{\cdot^{\omega^{\alpha}}}}}_{n} .$$

DEFINITION 12.6. Assignment of ordinals (less than ε_0) to the proofs of **PA**. First we assign ordinals to the sequents in a proof. The ordinal assigned to a sequent S in a proof P is denoted by $o(S; P)$ or $o(S)$. Now suppose a proof P is given. We shall define $o(S) = o(S; P)$, for all sequents S in P.

We shall henceforth assume that the ordinals are expressed in normal form (cf. §11). If μ and ν are ordinals of the form $\omega^{\mu_1} + \omega^{\mu_2} + \ldots + \omega^{\mu_n}$ and $\omega^{\nu_1} + \omega^{\nu_2} + \ldots + \omega^{\nu_n}$ respectively (so that $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_m$ and $\nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_n$), then $\mu \,\#\, \nu$ denotes the ordinal $\omega^{\lambda_1} + \omega^{\lambda_2} + \ldots + \omega^{\lambda_{m+n}}$, where $\{\lambda_1, \lambda_2, \ldots, \lambda_{m+n}\} = \{\mu_1, \mu_2, \ldots, \nu_1, \nu_2, \ldots\}$ and $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{m+n}$. $\mu \,\#\, \nu$ is called the natural sum of μ and ν.

(1) An initial sequent (in P) is assigned the ordinal 1.

(2) If S is the lower sequent of a weak inference, then $o(S)$ is the same as the ordinal of its upper sequent.

(3) If S is the lower sequent of $\wedge$: left, $\vee$: right, $\supset$: left or an inference involving a quantifier, and the upper sequent has the ordinal μ, then $o(S) = \mu + 1$.

(4) If S is the lower sequent of $\wedge$: right, $\vee$: left, or $\supset$: left and the upper sequents have ordinals μ and ν, then $o(S) = \mu \,\#\, \nu$.

(5) If S is the lower sequent of a cut and its upper sequents have the ordinals μ and ν, then $o(S)$ is $\omega_{k-l}(\mu \,\#\, \nu)$, i.e.,

$$\left.\omega^{\cdot^{\cdot^{\omega^{\mu\#\nu}}}}\right\} k - l,$$

where k and l are the heights of the upper sequents and of S, respectively.

(6) If S is the lower sequent of an ind and its upper sequent has the ordinal μ, then $o(S)$ is $\omega_{k-l+1}(\mu_1 + 1)$, i.e.,

$$\left.\omega^{\cdot^{\cdot^{\omega^{\mu_1+1}}}}\right\} (k - l) + 1,$$

where μ has the normal form $\omega^{\mu_1} + \omega^{\mu_2} + \ldots + \omega^{\mu_n}$ (so that $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$), and k and l are the heights of the upper sequent and of S, respectively.

(7) The ordinal of a proof P, $o(P)$, is the ordinal of its end-sequent.

We use the notation

$$P: \quad \Gamma \xrightarrow{\mu} \Delta$$

to denote a proof P of $\Gamma \rightarrow \Delta$ such that $o(\Gamma \rightarrow \Delta; P) = o(P) = \mu$.

LEMMA 12.7. *Suppose P is a proof containing a sequent S_1, there is no ind below S_1, P_1 is the subproof of P ending with S_1, P_1' is any other proof of S_1, and P' is the proof formed from P by replacing P_1 by P_1':*

$$P:\ P_1\left\{\ S_1\right. \qquad P':\ P_1'\left\{\ S_1\right.$$

Suppose also that $o(S_1; P') < o(S_1; P)$. *Then* $o(P') < o(P)$.

PROOF. Consider a thread of P passing through S_1. We show that for any sequent S of this thread at or below S_1: if S' is the sequent "corresponding to" S in P', then

$$(*) \qquad o(S'; P') < o(S; P).$$

This is true for $S = S_1$ by assumption, and this property (*) is preserved downwards by all the inference rules, as can be checked. (We use the fact that the natural sum is strictly monotonic in each argument, i.e., $\alpha < \beta \Rightarrow \alpha \# \gamma < \beta \# \gamma$, etc.) Finally, letting S be the end-sequent of P, we obtain the desired conclusion.

This lemma is used repeatedly in the consistency proof.

Now let R be the property of proofs of ending with the sequent $\rightarrow$; i.e., for any proof P, $\mathrm{R}(P)$ holds if and only if P is a proof of $\rightarrow$.

Notice first that *if P is a proof of $\rightarrow$, then every logical inference of P is implicit*! (cf. Definition 9.7) (since otherwise a bundle containing the principal formula of this inference would end with an end-formula).

Hence the definition of end-piece for such proofs can be simply stated as follows.

The end-piece of a proof of $\rightarrow$ consists of all those sequents that are encountered as we ascend each thread from the end-sequent and stop as soon as we arrive at a logical inference. (Then the upper sequent of this inference no longer belongs to the end-piece, but the lower sequent, and all sequents below it, do.) This inference belongs to the boundary.

LEMMA 12.8. *If P is a proof of $\rightarrow$, then there is another proof P' of $\rightarrow$ such that* $o(P') < o(P)$.

PROOF. Let P be a proof of $\rightarrow$. We can assume, by Proposition 9.8, that P is regular. We describe a "reduction" of P to obtain the desired P'. The reduction consists of a number of steps, described below. Each step is performed, perhaps finitely often (as will be clear), and at each step, we assume that the previous steps have been performed (as often as possible).

At each step, the ordinal of the resulting proof does not increase, and at least at one step, the ordinal decreases.

Step 1. Suppose the end-piece of P contains a free variable, say a, which is not used as an eigenvariable. Then replace a by the constant 0. This results in a proof of $\rightarrow$ (using the analogue of Lemma 2.10 for **PA**), with the same ordinal.

Step 1 is performed repeatedly until there is no free variable in the end-piece which is not used as an eigenvariable.

Step 2. Suppose the end-piece of P contains an ind. Then take a lowermost one, say I. Suppose I is of the following form:

$$\begin{array}{ll} P_0(a) & \left\{ \begin{array}{c} \vdots \\ F(a), \Gamma \xrightarrow{\mu} \Delta, F(a') \end{array} \right. \\ I & \quad \overline{\quad F(0), \Gamma \rightarrow \Delta, F(s) \quad} \\ & \qquad\qquad \vdots \\ & \qquad\qquad \rightarrow \end{array},$$

where $P_0(a)$ is the subproof ending with $F(a), \Gamma \rightarrow \Delta, F(a')$, and let l and k be the heights of the upper sequent (call it S) and the lower sequent (call it S_0) of I, respectively. Then

$$o(S_0) = \omega_{l-k}(\mu_1 + 1),$$

where $\mu = o(S) = \omega^{\mu_1} + \omega^{\mu_2} + \ldots + \omega^{\mu_n}$ and $\mu_n \leqslant \ldots \leqslant \mu_2 \leqslant \mu_1$. Since no free variable occurs below I, s is a closed term and hence there is a number m such that $\rightarrow s = \bar{m}$ is **PA**-provable without an essential cut or ind (cf. Lemma 9.6); hence there is a proof Q of $F(\bar{m}) \rightarrow F(s)$ without an essential cut or ind (cf. Lemma 9.6). Let $P_0(\bar{n})$ be the proof which is obtained from P_0 by replacing a by $\bar{n}$ throughout. Consider the following proof P'.

$$\begin{array}{ll}
 & \dfrac{\dfrac{\dfrac{\begin{array}{c} P_0(\bar{0}) \\ \vdots \\ F(0), \Gamma \rightarrow \Delta, F(0') \end{array} \quad \begin{array}{c} P_0(\bar{1}) \\ \vdots \\ F(0'), \Gamma \rightarrow \Delta, F(0'') \end{array}}{F(0), \Gamma \rightarrow \Delta, F(0'')} \quad \begin{array}{c} P_0(\bar{2}) \\ \vdots \\ F(0''), \Gamma \rightarrow \Delta, F(0''') \end{array}}{F(0), \Gamma \rightarrow \Delta, F(0''')}}{} \\
 & \qquad\qquad \vdots \\
 & \dfrac{F(0), \Gamma \rightarrow \Delta, F(\bar{m}) \qquad \begin{array}{c} Q \\ \vdots \\ F(\bar{m}) \xrightarrow{q} F(s) \end{array}}{F(0), \Gamma \rightarrow \Delta, F(s)} \\
 & \qquad\qquad \vdots \\
 & \qquad\qquad \rightarrow
\end{array}$$

S_1, S_2, S_3, S_m, S_0

where $S_1, S_2, \ldots, S_0$ denote the sequents shown on their right, $S_1, \ldots, S_m$ all have height l, since the formulas $F(\bar{n})$, $n = 0, \ldots, m$, all have the same grade. Therefore,

$$o(F(\bar{n}), \Gamma \to \Delta, F(\bar{n}'); P') = \mu \quad \text{for} \quad n = 0, 1, \ldots, m.$$

Since Q has no essential cut or ind, $o(F(\bar{m}) \to F(s); P') = q$ (say) $< \omega$, $o(S_2) = \mu \# \mu$; $o(S_3) = \mu \# \mu \# \mu$; $, \ldots,$ and in general, writing $\mu * n = \mu \# \mu \# \ldots \# \mu$ (n times), $o(S_n) = \mu * n$ for $n = 1, 2, \ldots, m$. Thus

$$o(S_0) = \omega_{l-k}(\mu * m + q)$$

and $\mu * m + q < \omega^{\mu_1+1}$, since $q < \omega$. Therefore

$$o(S_0; P') = \omega_{l-k}(\mu * m + q) < \omega_{l-k}(\mu_1 + 1) = o(S_0; P).$$

Thus $o(S_0; P') < o(S_0; P)$, and hence by Lemma 12.7, $o(P') < o(P)$.

Thus, if P has an ind in the end-piece, we are done: we have reduced P to a proof P' of $\to$ with $o(P') < o(P)$. Otherwise, we assume from now on that P has no ind in its end-piece, and go to Step 3.

Step 3. Suppose the end-piece of P contains a logical initial sequent $D \to D$. Since the end-sequent is empty, both D's (or more strictly, descendants of both D's) must disappear by cuts. Suppose that (a descendant of) the D in the antecedant is a cut formula first (viz. in the following figure a descendent of the D in the succedent of $D \to D$ occurs in Ξ).

$$\begin{array}{c} \qquad\qquad\qquad D \to D \\ \vdots \qquad\qquad \vdots \\ S \qquad \dfrac{\Gamma \to \Delta, D \qquad D, \Pi \to \Xi}{\Gamma, \Pi \to \Delta, \Xi} . \\ \vdots \\ \to \end{array}$$

P is reduced to the following P':

$$\begin{array}{c} \vdots \\ S' \qquad \dfrac{\dfrac{\Gamma \to \Delta, D}{\text{weakenings and exchanges}}}{\Gamma, \Pi \to \Delta, \Xi} . \\ \vdots \\ \to \end{array}$$

Then $o(S'; P') < o(S; P)$. (This is to be expected, since the ordinal of a proof is a measure of its complexity, and the subproof of S' in P' is clearly simpler than the subproof of S in P. However, the proof is not trivial, since the height of $\Gamma \to \Delta, D$ and of sequents above it may drop if the grade of D is greater than $h(S; P)$. The proof uses the inequality $\omega^\alpha \# \omega^\beta < \omega^{\alpha \# \beta}$ for $\alpha, \beta \neq 0$.)

Hence, by Lemma 12.2, $o(P') < o(P)$.

The other case is proved likewise.

So, if the end-piece of P contained a logical initial sequent, we have found a P' as desired. Otherwise, we assume from now on that the end-piece of P contains no logical initial sequents, and go on to Step 4.

Step 4. Suppose there is a weakening in the end-piece of P. Then we shall define a "weakening elimination". It is actually convenient to define this weakening elimination for proofs P which satisfy the conclusion of steps 1–3 (i.e., their end-piece contains no free variables other than eigenvariables, no ind, and no logical initial sequent), but with the end-sequent possibly non-empty. Let P be such a proof. We define another such proof P^* which satisfies the further conditions that its end-piece contains no weakenings, its end-sequent is obtained from that of P by eliminating some (possibly none) of its formulas, and $o(P^*) \leqslant o(P)$. In particular, if P is a proof of $\to$, then so is P^*.

P^* is obtained by eliminating all the weakenings in the end-piece of P. The definition of P^* is by induction on the number of inferences in the end-piece of P.

(1) If the end-piece of P does not contain any weakening, then P^* is P.

Suppose the end-piece of P contains a weakening. We define P^* according to the last inference I of P.

We use the following notation below. Γ^*, Δ^*, etc. will denote throughout sequences of formulas formed from Γ, Δ, etc. (respectively) by deleting some formulas (possibly none).

(2) I is a weakening : left.

$$I \quad \frac{\Gamma \to \Delta}{D, \Gamma \to \Delta}$$

Let P' be the subproof of P ending with the upper sequent of I. By the induction hypothesis, P'^* is defined. Take P'^* as P^*.

If I is a weakening : right, then P^* is defined similarly.

(3) I is a cut. Suppose P is of the form

$$\frac{P_1 \left\{ \Gamma \to \Delta, D \right. \qquad P_2 \left\{ D, \Pi \to \Lambda \right.}{\Gamma, \Pi \to \Delta, \Lambda}.$$

By the induction hypothesis, P_1^* and P_2^* have been defined.

(3.1) The end-sequent of P_1^* is $\Gamma^* \to \Delta^*$. Then P^* is P_1^*.

(3.2) If (3.1) is not the case but the end-sequent of P_2^* is $\Pi^* \to \Lambda^*$, then P^* is P_2^*.

(3.3) If the end-sequent of P_1^* is $\Gamma^* \to \Delta^*, D$ and that of P_2^* is $D, \Pi^* \to \Lambda^*$, then P^* is

$$\dfrac{P_1^* \left\{ \begin{array}{c} \vdots \\ \Gamma^* \to \Delta^*, D \end{array} \right. \qquad P_2^* \left\{ \begin{array}{c} \vdots \\ D, \Pi^* \to \Lambda^* \end{array} \right.}{\Gamma^*, \Pi^* \to \Delta^*, \Lambda^*}.$$

(4) I is a contraction : left.

$$\dfrac{P_0 \left\{ \begin{array}{c} \vdots \\ D, D, \Gamma \to \Delta \end{array} \right.}{D, \Gamma \to \Delta}.$$

By the induction hypothesis, P_0^* is defined.

(4.1) The end-sequent of P_0^* is $D, \Gamma^* \to \Delta^*$ or $\Gamma^* \to \Delta^*$. Then P^* is P_0^*.

(4.2) The end-sequent of P^* is $D, D, \Gamma^* \to \Delta^*$. P^* is defined to be

$$\dfrac{P_0^* \left\{ \begin{array}{c} \vdots \\ D, D, \Gamma^* \to \Delta^* \end{array} \right.}{D, \Gamma^* \to \Delta^*}.$$

If I is a contraction : right; similarly.

(5) I is an exchange : left. Suppose P is of the form:

$$\dfrac{P_0 \left\{ \begin{array}{c} \vdots \\ \Gamma_1, C, D, \Gamma_2 \to \Delta \end{array} \right.}{\Gamma, D, C, \Gamma_2 \to \Delta}.$$

By the induction hypothesis, P_0^* is defined.

(5.1) The end-sequent of P_0^* is $\Gamma_1^*, \Gamma_2^* \to \Delta^*$ or $\Gamma_1^*, C, \Gamma_2^* \to \Delta^*$ or $\Gamma_1^*, D, \Gamma_2^* \to \Delta$. Define P^* to be P_0^*.

(5.2) The end-sequent of P_0^* is $\Gamma_1^*, C, D, \Gamma_2^* \to \Delta^*$. P^* is defined as

$$\dfrac{\begin{array}{c} \vdots \\ \Gamma_1^*, C, D, \Gamma_2^* \to \Delta^* \end{array}}{\Gamma_1^*, D, C, \Gamma_2^* \to \Delta^*}.$$

Similarly if I is an exchange : right.

This completes the definition of P^*. It is easily seen that $o(P^*) \leqslant o(P)$. So (returning to the case where P is a proof of $\to$) we assume from now on that the end-piece of P has no weakening (by replacing P by P^*).

Step 5. We can now assume that P is not its own end-piece, since otherwise it would be simple (Definition 12.2), as is easily seen, and hence by Lemma 12.3, could not end with $\rightarrow$.

Under these assumptions, we shall prove that the end-piece of P contains a suitable cut (cf. Definition 9.7). We actually prove a stronger result, which is used again later (for Problem 12.11):

SUBLEMMA 12.9. *Suppose that a proof in* **PA**, *say P, satisfies the following.*

(1) *P is not its own end-piece.*

(2) *The end-piece of P does not contain any logical inference, ind or weakening.*

(3) *If an initial sequent belongs to the end-piece of P, then it does not contain any logical symbol.*

Then there exists a suitable cut in the end-piece of P.

(Notice that we do not assume here that the end-sequent is $\rightarrow$.)

PROOF. This is proved by induction on the number of essential cuts in the end-piece of P. The end-piece of P contains an essential cut, since P is not its own end-piece. Take a lowermost such cut, say I. If I is a suitable cut, then the sublemma is proved. Otherwise, let P be of the form

$$I\quad \frac{P_1\left\{\Gamma \rightarrow \Delta, D \right. \qquad P_2\left\{D, \Pi \rightarrow \Lambda\right.}{\Gamma, \Pi \rightarrow \Delta, \Lambda}\,.$$

Since I is not a suitable cut, one of two cut formulas of I is not a descendent of the principal formula of a boundary inference. Suppose that D in $\Gamma \rightarrow \Delta, D$ is not a descendent of the principal formula of a boundary inference. Now we prove:

(i) P_1 contains a boundary inference of P.

Suppose otherwise. Then D in $\Gamma \rightarrow \Delta, D$ is a descendent of D in an initial sequent in the end-piece of P, by (2). This contradicts the assumption that I is an essential cut, by (3).

(ii) If an inference J in P_1 is a boundary inference of P, then J is a boundary inference of P_1.

This is easily seen by the fact that I is a lowermost essential cut of P and D is not a descendent of the principal formula of a boundary inference.

(iii) P_1 is not its own end-piece and the end-piece of P_1 is the intersection of P_1 and the end-piece of P.

This follows immediately from (i), (ii) and (1).

Now from the induction hypothesis, the end-piece of P_1 has a suitable cut. This cut is a suitable cut in the end-piece of P.

Returning to our proof P of $\rightarrow$ which satisfies the conclusion of steps 1–4, we have, as an immediate consequence of Sublemma 12.9, that the end-piece of P contains a suitable cut. We now define an *essential reduction* of P.

Take a lowermost suitable cut in the end-piece of P, say I.

Case 1. The cut formula of I is of the form $A \wedge B$. Suppose P is of the form

$$
\begin{array}{c}
I_1 \; \dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma' \rightarrow \Theta', A & \Gamma' \rightarrow \Theta', B\end{array}}{\Gamma' \rightarrow \Theta', A \wedge B} \qquad I_2 \; \dfrac{\begin{array}{c}\vdots \\ A, \Pi' \rightarrow \Lambda'\end{array}}{A \wedge B, \Pi' \rightarrow \Lambda'} \\
\begin{array}{cc}\vdots \qquad\qquad & \qquad\qquad \vdots\end{array} \\
I \; \dfrac{\Gamma \overset{\mu}{\rightarrow} \Theta, A \wedge B \qquad\qquad A \wedge B, \Pi \overset{\nu}{\rightarrow} \Lambda}{\Gamma, \Pi \rightarrow \Theta, \Lambda} \quad (l) \\
\vdots \\
\Delta \overset{\lambda}{\rightarrow} \Xi \quad (k) \\
\vdots \\
\rightarrow
\end{array}
$$

where $\Delta \rightarrow \Xi$ denotes the uppermost sequent below I whose height is less than that of the upper sequents of I. Let l be the height of each upper sequent of I, and k that of $\Delta \rightarrow \Xi$. Then $k < l$. Notice that $\Delta \rightarrow \Xi$ may be the lower sequent of I, or the end-sequent. The existence of such a sequent follows from Proposition 12.5.

$\Delta \rightarrow \Xi$ must be the lower sequent of a cut J (since there is no ind below I). Let $\mu = o(\Gamma \rightarrow \Theta, A \wedge B)$, $\nu = o(A \wedge B, \Pi \rightarrow \Lambda)$, $\lambda = o(\Delta \rightarrow \Xi)$ as shown. Consider the following proofs:

$$
P_1: \quad
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\vdots \\ \Gamma' \rightarrow \Theta', A\end{array}}{\Gamma' \rightarrow A, \Theta'}}{\Gamma' \rightarrow A, \Theta', A \wedge B} \;\text{(weakening : right)} \\
\begin{array}{cc}\vdots \qquad\qquad & \qquad\qquad \vdots\end{array} \\
J_1 \; \dfrac{\Gamma \overset{\mu_1}{\rightarrow} A, \Theta, \; A \wedge B \qquad A \wedge B, \Pi \overset{\nu_1}{\rightarrow} \Lambda}{\Gamma, \Pi \rightarrow A, \Theta, \Lambda} \quad (l) \\
\vdots \\
\dfrac{\Delta \overset{\lambda_1}{\rightarrow} A, \Xi}{\Delta \rightarrow \Xi, A} \quad (m)
\end{array}
$$

P_2:

$$\begin{array}{c} \vdots \\ \dfrac{\dfrac{A, \Pi' \to \Lambda'}{\Pi', A \to \Lambda'}}{A \wedge B, \Pi', A \to \Lambda'} \ \text{(weakening : left)} \end{array}$$

$$J_2 \quad \dfrac{\begin{array}{c}\vdots \\ \Gamma \xrightarrow{\mu_2} \Theta, A \wedge B\end{array} \qquad \begin{array}{c}\vdots \\ A \wedge B, \Pi, A \xrightarrow{\nu_2} \Lambda \quad (l)\end{array}}{\Gamma, \Pi, A \to \Theta, \Lambda}$$

$$\begin{array}{c} \vdots \\ \dfrac{\Delta, A \xrightarrow{\lambda_2} \Xi}{A, \Delta \to \Xi} \ (m) \end{array}$$

(where l and m are the heights of the sequents shown, not in P_1 and P_2, but in P', defined below, which contains these as subproofs.)

Define P' to be the proof:

$$I' \quad \begin{array}{c} \dfrac{\begin{array}{c} P_1 \\ \vdots \\ \Delta \xrightarrow{\lambda_1} \Xi, A \quad (m) \end{array} \qquad \begin{array}{c} P_2 \\ \vdots \\ A, \Delta \xrightarrow{\lambda_2} \Xi \quad (m) \end{array}}{\dfrac{\Delta, \Delta \xrightarrow{\lambda_0} \Xi, \Xi \quad (k)}{\Delta \to \Xi}} \ \text{(cut for } A\text{)} \\ \vdots \\ \to \end{array}$$

So m is the height of the upper sequents of I' (the cut for A). Note that the height of the lower sequent of I' is k.

It is obvious that $m = k$ if $k >$ grade of A and $m =$ grade of A otherwise. In either case $k \leqslant m < l$.

$$h(\Gamma \to A, \Theta, A \wedge B; P') = h(A \wedge B, \Pi \to \Lambda; P') = l,$$

since all cut formulas below I in P occur in P' below J_1, all cut formulas below J_1 in P' except A occur in P under I, and grade of $A <$ grade of $A \wedge B \leqslant l$. Similarly,

$$h(\Gamma \to \Theta, A \wedge B; P') = h(A \wedge B, \Pi, A \to \Lambda; P') = l.$$

Let

$$\mu_1 = o(\Gamma \to A, \Theta, A \wedge B; P'), \quad \nu_1 = o(A \wedge B, \Pi \to \Lambda; P'), \quad \lambda_1 = o(\Delta \to A, \Xi; P'),$$

$$\mu_2 = o(\Gamma \to \Theta, A \wedge B; P'), \qquad \nu_2 = o(A \wedge B, \Pi, A \to \Lambda; P'),$$

$$\lambda_2 = o(\Delta, A \to \Xi; P'), \qquad \lambda_0 = o(\Delta, \Delta \to \Xi, \Xi; P').$$

Then $\mu_1 < \mu$, $\nu_1 = \nu$, $\mu_2 = \mu$ and $\nu_2 < \nu$.

Now let

$$J' \quad \frac{S_1' \quad S_2'}{S'} \quad \begin{matrix}(k_1)\\(k_2)\end{matrix}$$

be an arbitrary inference between J_1 and $\Delta \to A, \Xi$ and let

$$J \quad \frac{S_1 \quad S_2}{S}$$

be the corresponding inference between I and $\Delta \to \Xi$. Let

$$\alpha_1' = o(S_1'; P'), \qquad \alpha_2' = o(S_2'; P'), \qquad \alpha' = o(S'; P'),$$

$$\alpha_1 = o(S_1; P), \qquad \alpha_2 = (S_2, P), \qquad \alpha = o(S; P),$$

$$k_1 = h(S_1', P') = h(S_2', P'), \qquad k_2 = h(S', P').$$

Then $\alpha = \alpha_1 \# \alpha_2$ if S is not $\Delta \to A, \Xi$, and $\alpha = \omega_{l-k}(\alpha_1 \# \alpha_2)$ if S' is $\Delta \to A, \Xi$. On the other hand $\alpha' = \omega_{k_1-k_2}(\alpha_1 \# \alpha_2)$.

Starting with $\mu_1 < \mu$ and $\nu_1 = \nu$, it is easily seen by induction on the number of inferences between J_1 and S that

$$\alpha' < \omega_{l-k_2}(\alpha), \tag{1}$$

if S is not $\Delta \to A, \Xi$. Let $\lambda = \omega_{l-k}(\kappa)$. Then (1) implies that $\lambda_1 < \omega_{l-m}(\kappa)$. Similarly, $\lambda_2 < \omega_{l-m}(\kappa)$. Hence

$$\omega_{m-k}(\lambda_1 + \lambda_2) < \omega_{l-k}(\kappa),$$

since $l - k = (l - m) + (m - k)$. Therefore $\lambda_0 < \lambda$. Finally, from $\lambda_0 < \lambda$ it follows that $o(P') < o(P)$.

Case 2. The cut formula of I is of the form $\forall x\, F(x)$. So P has the form:

$$\begin{array}{c}
I_1 \; \dfrac{\Gamma' \to \Theta', F(a)}{\Gamma' \to \Theta', \forall x\, F(x)} \qquad I_2 \; \dfrac{F(s), \Pi' \to \Lambda'}{\forall x\, F(x), \Pi' \to \Lambda'} \\
\vdots \qquad\qquad\qquad\qquad \vdots \\
I \; \dfrac{\Gamma \to \Theta, \forall x\, F(x) \qquad\qquad \forall x\, F(x), \Pi \to \Lambda}{\Gamma, \Pi \to \Theta, \Lambda} \\
\vdots \\
\Delta \to \Xi \\
\vdots \\
\to
\end{array}$$

The definition of $\Delta \to \Xi$ is the same as in case 1. The proof P' is then defined in terms of the following two subproofs P_1 and P_2:

P_1:

$$
\dfrac{\dfrac{\vdots}{\dfrac{\dfrac{\dfrac{\Gamma' \to \Theta', F(s)}{\Gamma' \to F(s), \Theta'}}{\Gamma' \to F(s), \Theta', \forall x\, F(x)}}{\vdots}}
\qquad
\dfrac{\vdots}{\Gamma \to F(s), \Theta, \forall x\, F(x) \qquad \forall x\, F(x), \Pi \to \Lambda}}
{\dfrac{\Gamma, \Pi \to F(s), \Theta, \Lambda}{\dfrac{\vdots}{\dfrac{\Delta \to F(s), \Xi}{\Delta \to \Xi, F(s)}}}}
$$

P_2:

$$
\dfrac{\Gamma \to \Theta, \forall x\, F(x) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{F(s), \Pi' \to \Lambda'}}{\Pi', F(s) \to \Lambda'}}{\forall x\, F(x), \Pi', F(s) \to \Lambda'}}{\vdots}\ \ \forall x\, F(x), \Pi, F(s) \to \Lambda}
{\dfrac{\Gamma, \Pi, F(s) \to \Theta, \Lambda}{\dfrac{\vdots}{\dfrac{\Delta, F(s) \to \Xi}{F(s), \Delta \to \Xi}}}}\ .
$$

P' is defined to be

$$
\dfrac{\dfrac{\dfrac{P_1}{\vdots}\ \Delta \to \Xi, F(s) \qquad \dfrac{P_2}{\vdots}\ F(s), \Delta \to \Xi}{\Delta, \Delta \to \Xi, \Xi}}{\dfrac{\Delta \to \Xi}{\dfrac{\vdots}{\to}}}\ .
$$

Note that $o(\Gamma' \to \Theta', F(s); P') = o(\Gamma' \to \Theta', F(a); P)$. The argument on ordinals goes through as in case 1.

For the other cases, the proof is similar.

This completes the proof of Lemma 12.8 and hence the consistency proof for **PA** (Theorem 12.1).

REMARK 12.10. We wish to point out the following. One often says that the consistency of **PA** is proved by transfinite induction on the ordinals of proofs, as if we were using a general principle of transfinite induction in order to prove the consistency of mathematical induction.

This is misleading, however. The point is that the consistency proof uses the notion of accessibility of ε_0, as explained in §11, and otherwise strictly finitist methods. To re-state the matter from a more formal viewpoint:

The principle of *transfinite induction* on some (definable) well-ordering $\prec$ of the natural numbers can be expressed (in first-order formal systems) by the schema

$$TI(\prec, F(x)): \qquad \forall x\, [\forall y\, (y \prec x \supset F(y)) \supset F(x)] \rightarrow \forall x\, F(x)$$

for arbitrary formulas $F(x)$ of the system considered.

Now Gentzen's consistency proof of **PA** can be formalized in the system of primitive recursive arithmetic, together with the axiom $TI(\prec, F(x))$, where $\prec$ is the standard well-ordering of type ε_0 and $F(x)$ is a certain *quantifier-free* formula.

PROBLEM 12.11. We can extend the reduction procedure of Lemma 12.8 to the following situation.

A sequent S (of the language of **PA**) is said to satisfy the property P if:

(1) All sequent-formulas of S are closed;

(2) Each sequent-formula in the succedent of S is either quantifier-free or of the form $\exists y_1, \ldots, \exists y_m\, R(y_1, \ldots, y_m)$, where $R(y_1, \ldots, y_m)$ is quantifier-free;

(3) Each sequent formula in the antecedent of S is either quantifier-free or of the form $\forall y_1, \ldots, \forall y_m\, R(y_1, \ldots, y_m)$, where $R(y_1, \ldots, y_m)$ is quantifier-free.

Show that if a sequent satisfying P is provable in **PA**, then it is provable without an essential cut or ind. [*Hint*: We may assume that there is no free variable which is not used as an eigenvariable in the end-piece of a proof of such a sequent.]

If the end-piece has an explicit logical inference, take the lowermost explicit logical inference I. Without loss of generality, we assume that the proof is of the following form:

$$\begin{array}{c} \vdots \\ I \quad \dfrac{\Gamma \rightarrow \Delta, \exists y_2 \ldots \exists y_m\, R(t, y_2, \ldots, y_m)}{\Gamma \rightarrow \Delta, \exists y_1 \ldots \exists y_m\, R(y_1, y_2, \ldots, y_m)} \\ \vdots \\ \Gamma_0 \rightarrow \Delta_0, \exists y_1 \ldots \exists y_m\, R(y_1, \ldots, y_m), \Delta_1 \end{array}$$

where $\Gamma_0 \rightarrow \Delta_0, \exists y_1 \ldots \exists y_m\, R(y_1, \ldots, y_m), \Delta_1$ is the end-sequent of the proof. We can eliminate I by replacing the proof by a proof whose end-sequent is either of the form

$$\Gamma_0 \to \Delta_0, \exists y_2 \ldots \exists y_m\, R(t, y_2, \ldots, y_m), \Delta_1$$

or of the form

$$\Gamma_0 \to \Delta_0, \exists y_1 \ldots \exists y_m\, R(y_1, \ldots, y_m), \Delta_1, \exists y_2 \ldots \exists y_m\, R(t, y_2, \ldots, y_m).$$

PROBLEM 12.12. Intuitionistic arithmetic can be formalized as the subsystem of **PA** defined by the condition that in the succedent of every sequent there can be at most one sequent-formula which contains quantifiers. This system may be called **HA** (for Heyting arithmetic). The reduction method for **PA** works for **HA** with a slight modification: roughly, in an essential reduction, if the cut formula of the suitable cut under consideration contains a quantifier then the weakening : right will not be introduced.

Define the reduction for **HA** precisely, thus proving the consistency of **HA** directly (not as a subsystem of **PA**).

PROBLEM 12.13. Let (*) be the property of formulas defined in Theorem 6.14, i.e., a formula satisfies (*) if every $\vee$ and $\exists$ in it is either in the scope of a $\neg$ or in the left scope of a $\supset$. Show that, if each formula in Γ satisfies (*) and all formulas in Γ, A, B and $\exists x\, F(x)$ are closed, then in **HA** (cf. Problem 12.12):

(1) $\Gamma \to A \vee B$ if and only if $\Gamma \to A$ or $\Gamma \to B$,

(2) $\Gamma \to \exists x\, F(x)$ if and only if for some closed term s, $\Gamma \to F(s)$.

[*Hint* (B. Scarpellini): By transfinite induction on the ordinal of a proof P of $\Gamma \to A \vee B$ (for 1) or $\Gamma \to \exists x\, F(x)$ (for 2), respectively, following the reduction method for the consistency of **PA**. First deal with explicit logical inferences in the end-piece of P.]

REMARK 12.14. As an application of Gentzen's reduction method, one can easily prove the following.

The consistency of arithmetic in which the induction formulas are restricted to those which have at most k quantifiers can be proved by transfinite induction on ω_{k+1}.

The outline of the proof is as follows. Suppose there is a proof of $\to$ in this system. We shall carry out a reduction of such a proof.

(1) We assume that the induction formulas are in prenex normal form.

(2) A formula A in a proof (in this system) will be temporarily called free if either it has no ancestor which is an induction formula, or it has an induction

formula as an ancestor but a logical symbol is introduced in an ancestor of A between any such induction formula and A itself. A cut is called free if both cut formulas are free. Notice that if a formula is not free, then it is in prenex form with at most k quantifiers. Now we can prove the following partial cut-elimination theorem:

If a sequent is provable in our system, then it is provable without free cuts.

(We simply adapt the cut-elimination proof for **LK**.)

Thus we obtain a proof of $\rightarrow$ in which there are no free cuts, and so all the cut formulas, as well as induction formulas, are in prenex form with at most k quantifiers. We assume $k \geqslant 1$.

(3) Further we can assume, for convenience, that the inference rules are modified in such a way that all formulas in the proof are in prenex form, with at most k quantifiers.

This system is called $\mathbf{PA}_k$.

We must now modify some notions slightly. The grade of a formula A is now defined to be: the number of quantifiers in A, minus 1; the grade of a cut or induction inference is the grade of the cut formula or the induction formula, respectively. The height of a sequent in a proof is defined as before, using the new definition of grade. The ordinals are assigned as before, except that the initial sequents are assigned the ordinal 0 and the propositional inferences as well as quantifier-free cuts are treated in the same manner as the weak inferences, i.e., the ordinals do not change. (In case there are two upper sequents, take the maximum of the two ordinals.) It can easily be seen that the ordinal of a proof (of the kind we are considering) is less than $\omega_k(l)$ for some natural number l.

A boundary inference is defined to be an inference which introduces a quantifier and is a boundary inference in the previous sense. A suitable cut is a cut whose cut formula contains quantifiers and which is suitable in the previous sense. In eliminating initial sequents from the end-piece, one eliminates only those which have quantifiers. The existence of a suitable cut (under certain conditions) can be proved just as before.

(4) In an essential reduction, if the suitable cut is of grade > 0, then we can proceed as before (Step 5 in the proof of Lemma 12.8). If its grade is 0, then the cut formula is either of the form $\forall x\, F(x)$ or $\exists x\, F(x)$, where F is quantifier-free. Let us take the first case as an example. Let $F(s)$ be the auxiliary formula of a boundary inference which is an ancestor of the cut formula $\forall x\, F(x)$. s is a closed term, and so either $\rightarrow F(s)$ or $F(s) \rightarrow$ is a mathematical initial

sequent (with ordinal 0). Suppose $\to F(s)$ is a mathematical initial sequent. Consider the proof:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \to \Theta, \forall x\, F(x)\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{\to F(s) \qquad \begin{array}{c}\vdots\\ F(s), \Pi' \to \Lambda'\end{array}}{\Pi' \to \Lambda'}}{\forall x\, F(x), \Pi' \to \Lambda'}\\ \vdots\\ \forall x\, F(x), \Pi \to \Lambda\end{array}}{\begin{array}{c}\Gamma, \Pi \to \Theta, \Lambda\\ \vdots\\ \to\end{array}}$$

(taking $\Gamma, \Pi \to \Theta, \Lambda$ as the sequent $\Delta \to \Xi$ shown in Lemma 12.8, Step 5). It is easy to see that the ordinal decreases again.

REMARK 12.15. Here we define an extended notion of primitive recursiveness. Let $<\cdot$ be a primitive recursive well-ordering of natural numbers. The class of $<\cdot$-primitive recursive functions is defined as the class of functions f generated by the following schemata:

(i) $f(a) = a + 1$,

(ii) $f(a_1, \ldots, a_n) = 0$,

(iii) $f(a_1, \ldots, a_n) = a_i \ (1 \leqslant i \leqslant n)$,

(iv) $f(a_1, \ldots, a_n) = g(h_1(a_1, \ldots, a_n), \ldots, h_m(a_1, \ldots, a_n))$, where g and $h_i (1 \leqslant i \leqslant m)$ are $<\cdot$-primitive recursive.

(v) $f(0, a_2, \ldots, a_n) = g(a_2, \ldots, a_n)$,
$f(a + 1, a_2, \ldots, a_n) = h(a, f(a, a_2, \ldots, a_n), a_2, \ldots, a_n)$,
where g and h are $<\cdot$-primitive recursive.

(vi) (Definition by $<\cdot$-recursion.)

$$f(a_1, \ldots, a_n) = \begin{cases} h(f(\tau(a_1, \ldots, a_n), a_2, \ldots, a_n), a_1, \ldots, a_n) & \\ \qquad\qquad\qquad \text{if} \quad \tau(a_1, \ldots, a_n) <\cdot\, a_1, & \\ g(a_1, \ldots, a_n) \quad \text{otherwise}, & \end{cases}$$

where g, h and τ are $<\cdot$-primitive recursive.

The idea of (vi) is that $f(a, a_2, \ldots, a_n)$ is defined either outright or in terms of $f(b, a_2, \ldots, a_n)$ for certain $b <\cdot\, a$.

The consistency proof for $\mathbf{PA}_k$ which has just been presented has the following application.

COROLLARY 12.16. *Suppose R is a primitive recursive predicate and there is a proof of* $\rightarrow \exists x\, R(a, x)$ *in* $\mathbf{PA}_k$, *with ordinal* $< \omega_k(l)$ *for some numbers k and l* (as defined just above Definition 12.6). *Then the number-theoretic function f defined by*

$$f(m) = \text{the least } n \text{ such that } R(m, n)$$

is $<\cdot$-primitive recursive, where $<\cdot$ is the initial segment of the standard ordering of ε_0, of order type $\omega_k(l)$.

PROOF. We divide the proof into steps.

(i) Let $P(a)$ be a proof in $\mathbf{PA}_k$ of $\rightarrow \exists x\, R(a, x)$ (where all occurrences of a are indicated). Then for all m, $P(\bar{m})$ is a proof in $\mathbf{PA}_k$ of $\rightarrow \exists x\, R(\bar{m}, x)$ with the same ordinal, and with Gödel number primitive recursive in m. Also note that $\rightarrow \exists x\, R(\bar{m}, x)$ satisfies property P of Problem 12.11.

(ii) We (temporarily) call a proof *reducible* if it is a proof in $\mathbf{PA}_k$, with ordinal $< \omega_k(l)$, containing an essential cut or ind, and with end-sequent satisfying P. If P is reducible, then by applying repeatedly the reduction procedure of Lemma 12.8 (modified for $\mathbf{PA}_k$ as in Remark 12.14), we obtain a proof in $\mathbf{PA}_k$ of the same sequent, without an essential cut or ind. Let r be the function such that if p is a Gödel number of a reducible proof, then $r(p)$ is the Gödel number of the proof obtained by applying this reduction procedure (once), otherwise $r(p) = p$. Clearly r is primitive recursive.

Let O be the function such that if p is a Gödel number of a proof in $\mathbf{PA}_k$ with ordinal $< \omega_k(l)$, then $O(p)$ is the Gödel number of its ordinal (and, say $O(p) = 0$ otherwise). Clearly O is primitive recursive. Note also that for all p, $O(r(p)) <\cdot O(p) \Leftrightarrow p$ is the Gödel number of a reducible proof.

(iii) Now given a proof P of $\rightarrow \exists x\, R(\bar{m}, x)$ without an essential cut or ind, we can effectively find from P a number n such that $R(m, n)$ holds (and in fact the least such n). This is done in the following way.

First, we may assume that no free variables appear in P. Hence if $\Gamma \rightarrow \Delta$ is a sequent in P, every formula in Γ is a closed atomic formula and every formula in Δ is either $\exists x\, R(\bar{m}, x)$ or a closed atomic formula.

Now consider the following property Q of sequents: Every atomic formula in the antecedent is true and every atomic formula in the succedent is false.

Notice that the end-sequent of P satisfies Q; and if the lower sequent of a cut in P satisfies Q, then so does one upper sequent (since the cut formula is closed and atomic). Now start to construct a thread of sequents in P satisfying Q, working from the bottom upwards: the end-sequent is in the thread, and if the lower sequent of an inference is in the thread, take an upper sequent which satisfies Q. Since no initial sequent of P satisfies Q, this procedure must stop before we reach an initial sequent. The only way for this to happen is in the following case:

$$\frac{\Gamma \to \Delta, R(\bar{m}, \bar{k})}{\Gamma \to \Delta, \exists x\, R(\bar{m}, x)}$$

where $R(\bar{m}, \bar{k})$ is true. Finally, take the least $n \leqslant k$ for which $R(m, n)$ holds. Clearly there is a primitive recursive function h such that if P is a proof of $\to \exists x\, R(\bar{m}, x)$ without an essential cut or ind, then $h(\ulcorner P \urcorner)$ is the number n found as above.

(iv) Now we can define a $<\cdot$-primitive recursive function g such that if P is a proof of $\to \exists x\, R(\bar{m}, x)$ in $\mathbf{PA}_k$, with ordinal $< \omega_k(l)$, then $g(\ulcorner P \urcorner) =$ the least n such that $R(m, n)$ holds:

$$g(p) = \begin{cases} g(r(p)) & \text{if } \; 0(r(p)) <\cdot\, 0(p), \\ h(p) & \text{otherwise.} \end{cases}$$

Then it is easily seen that g is $<\cdot$-primitive recursive function.

(v) Finally, let $P(a)$ be a proof of $\to \exists x\, R(a, x)$ in $\mathbf{PA}_k$, with ordinal $< \omega_k(l)$ as stated. Then we define f by:

$$f(m) = g(\ulcorner P(\bar{m}) \urcorner).$$

As a special case of Corollary 12.16 we have: if $\to \exists x\, R(x, a)$ is provable within the system whose induction formulas have at most one quantifier, then f (defined as above) is primitive recursive (by a theorem of R. Peter that ω^l-primitive recursiveness implies primitive recursiveness for any finite l).

§13. Provable well-orderings

In this section, in order to distinguish between the natural ordering of natural numbers and the order relation on numbers given by the standard ordering of type ε_0, we denote the latter by $\prec$ in this section.

A partial function is a number-theoretic function that may not be defined at all arguments.

DEFINITION 13.1. (1) The class of *partial recursive functions* is the class of partial functions generated by the schemata (i)–(vi) for primitive recursive functions (cf. Definition 10.2), and also the schema:

(vii) $f(x_1, \ldots, x_n) \simeq \mu y[g(x_1, \ldots, x_n, y) = 0]$, where g is partial recursive; the right-hand side means the least y such that $\forall z<y\ (g(x_1, \ldots, x_n, z)$ is defined and $\neq 0)$ and $g(x_1, \ldots, x_n, y) = 0$, if such a y exists, and undefined otherwise; and $\simeq$ means that the left-hand side is defined if and only if the right-hand side is, in which case they are equal.

(2) A *general recursive* or *total recursive* or *recursive* function is a partial recursive function which is *total*, i.e., defined at all arguments.

(3) A relation on natural numbers, say R, is called *recursive* if there is a recursive function f which assumes values 0 and 1 only such that $R(x_1, \ldots, x_n)$ holds if and only if $f(x_1, \ldots, x_n) = 0$.

(4) A Σ_1^0-formula of the language L is a formula of the form

$$\exists y\ (\bar{f}(x_1, \ldots, x_n, y) = 0),$$

$\bar{f}$ a primitive recursive function symbol. A Π_1^0-formula is similarly of the form $\forall y\ (\bar{f}(x_1, \ldots, x_n, y) = 0$, $\bar{f}$ primitive recursive.

It can be shown that any recursive relation R can be represented in **PA** by a Σ_1^0-formula, i.e., there is a Σ_1^0-formula $\bar{R}(x_1, \ldots, x_n)$ of the language L such that, for all $m_1, \ldots, m_n$:

$$R(m_1, \ldots, m_n) \text{ holds} \leftrightarrow \bar{R}(\bar{m}_1, \ldots, \bar{m}_n) \text{ is } \mathbf{PA}\text{-provable}.$$

Also, any recursive relation can be represented in **PA** by a Π_1^0-formula.

DEFINITION 13.2. Let ε be a new predicate constant. $\mathrm{L}(\varepsilon)$ is the language extending L (cf. §12), formed by admitting $\varepsilon(t)$ as an atomic formula for all terms t.

PA(ε) is the system **PA** in the language $\mathrm{L}(\varepsilon)$; more precisely, we extend **PA** by admitting as mathematical initial sequents $s = t, \varepsilon(s) \rightarrow \varepsilon(t)$ for all terms s, t and applying the rule ind to all formulas of $\mathrm{L}(\varepsilon)$.

DEFINITION 13.3. Let $<\cdot$ be a recursive (infinite) linear ordering of the natural numbers which is actually a well-ordering. (Without loss of generality we may assume that the domain of $<\cdot$ is the set of all natural numbers and the least element with respect to $<\cdot$ is 0.) We use the same symbol $<\cdot$ in order to denote the Σ_1^0-formula in **PA** which represents the ordering $<\cdot$.

Consider the sequent

$$TI(<\cdot): \qquad \forall x\ (\forall y<\cdot x\ (\varepsilon(y) \supset \varepsilon(x))) \rightarrow \varepsilon(a)$$

(cf. the formula $TI(\prec, F(x))$ of Remark 12.10). If $TI(<\cdot)$ is provable in $\mathbf{PA}(\varepsilon)$, then we say that $<\cdot$ is a *provable well-ordering* of $\mathbf{PA}$.

The following theorem is proved by analyzing Gentzen's proof of the unprovability of the well-ordering of $\prec$ (where $\prec$ was defined at the beginning of this section).

THEOREM 13.4 (Gentzen). *If $<\cdot$ is a provable well-ordering of* $\mathbf{PA}$, *then there exists a recursive function which is a $<\cdot$ — $\prec$ order-preserving map into an initial segment of α. That is to say, there is a recursive function f such that $a <\cdot b$ if and only if $f(a) \prec f(b)$, and there is an ordinal $\mu(< \varepsilon_0)$ such that for every a, $f(a) \prec \bar{\mu}$ (where $\bar{\mu}$ is the Gödel number of μ).*

This section is devoted to Gentzen's proof, and the arithmetization of it, which proves Theorem 13.4.

From now on, let $<\cdot$ be a fixed provable well-ordering of $\mathbf{PA}$.

13.1) First we define $\mathbf{TJ}$-proofs, where $\mathbf{TJ}$ stands for "transfinite induction". $\mathbf{TJ}$-proofs are defined as $\mathbf{PA}(\varepsilon)$-proofs with some modifications:

(1) The initial sequents of a $\mathbf{TJ}$-proof are those of $\mathbf{PA}(\varepsilon)$, and the following sequents, called $\mathbf{TJ}$-initial sequents:

$$\forall x\ (x <\cdot t \supset \varepsilon(x)) \rightarrow \varepsilon(t)$$

for arbitrary terms t.

(2) The end-sequent of a $\mathbf{TJ}$-proof must be of the form

$$\rightarrow \varepsilon(\bar{m}_1), \ldots, \varepsilon(\bar{m}_n),$$

where $\bar{m}_1, \ldots, \bar{m}_n$ are numerals.

Let $|m|_{<\cdot}$ be the ordinal denoted by m with respect to $<\cdot$, i.e., the order type of the initial segment of $<\cdot$ determined by m. Then the minimum of $|m_1|_{<\cdot}, \ldots, |m_n|_{<\cdot}$ is called the end-number of the $\mathbf{TJ}$-proof.

13.2) Since $<\cdot$ is a provable well-ordering of $\mathbf{PA}$, the sequent $TI(<\cdot)$ (Definition 13.3) is $\mathbf{PA}(\varepsilon)$-provable, and hence we can obtain in the system formed from $\mathbf{PA}(\varepsilon)$ by adjoining $\mathbf{TJ}$-initial sequents, a proof $P(a)$ of $\rightarrow \varepsilon(a)$ (for a free variable a). Note that for each number m, $P(\bar{m})$ is a $\mathbf{TJ}$-proof of $\rightarrow \varepsilon(\bar{m})$.

13.3) A $\mathbf{TJ}$-proof is called non-critical if one of the reduction steps for $\mathbf{PA}$ (in the proof of Lemma 12.8) which lower the ordinal (i.e., step 2, 3 or 5) applies to it. Otherwise it is called critical.

13.4) We shall assign ordinals (less than ε_0) to **TJ**-proofs and define a reduction for **TJ**-proofs following the reduction method for **PA** given in the proof of Lemma 12.8: if a **TJ**-proof is critical, then more manipulation is required. The reduction is defined in such a manner that a **TJ**-proof P with end-number > 0 is reduced to another with the same end-number if P is not critical and with an arbitrary end-number which is smaller than the original one if P is critical. At the same time the ordinal decreases.

13.5) If we can define an ordinal assignment and a reduction method with the properties stated in 13.4), we can prove:

LEMMA 13.5 (Fundamental Lemma). *For any* **TJ**-*proof, its end-number is not greater than its ordinal.*

PROOF. By transfinite induction on the ordinal of the proof. Let P be a **TJ**-proof with ordinal μ and end-number σ. We assume as the induction hypothesis that the lemma is true for any **TJ**-proof whose ordinal is less than μ and show that $\sigma \leqslant \mu$. If P is non-critical then P is reduced to a **TJ**-proof P' with the same end-number σ and an ordinal $\nu < \mu$. By the induction hypothesis $\sigma \leqslant \nu$, and hence $\sigma \leqslant \mu$. Now suppose P is critical. If σ were greater than μ, we could reduce P to a **TJ**-proof whose end-number is μ and whose ordinal is less than μ, contradicting the induction hypothesis.

Now let us proceed to the reduction method for **TJ**-proofs.

13.6) The ordinals are assigned to the sequents of the **TJ**-proofs as in §12; the ordinal of a **TJ**-initial sequent is 7, i.e., $\omega^0 + \ldots + \omega^0$ (7 times). The lower sequent of a term-replacement inference is assigned the same ordinal as the upper sequent. For convenience, the formula in the succedent of a **TJ**-initial sequent will be considered as a principal formula.

13.7) We can follow the reduction steps given for the consistency proof of **PA** up to Step 4 (in the proof of Lemma 12.8), i.e., until we reach a **TJ**-proof P with the following properties p 1–p 4.

p 1. The end-piece of P contains no free variable.

p 2. The end-piece of P contains no induction.

p 3. The end-piece of P contains no logical initial sequent.

p 4. If the end-piece of P contains a weakening I, then any inference below I is a weakening.

REMARK. Since the end-piece of a **TJ**-proof is not empty, the end-sequent S' of the proof obtained from P by eliminating weakenings in the end-piece (in Step 4) may be different from the end-sequent of P. In this case we add weakenings below S' so that the end-sequent becomes the same as the end-sequent of P.

13.8) We can easily show the following. Let P be a **TJ**-proof satisfying p 1–p 4. Then P contains at least one logical inference (which must be implicit) or **TJ**-initial sequent. Therefore the end-piece of P contains a principal formula at the boundary or in a **TJ**-initial sequent.

13.9) Let P be a **TJ**-proof satisfying p 1–p 4. By 13.8), the end-piece of P contains a principal formula either at the boundary or in a **TJ**-initial sequent. We call a formula A in the end-piece of P a principal descendant or a principal **TJ**-descendant, according as A is a descendant of a principal formula at the boundary or a descendant of the principal formula of a **TJ**-initial sequent in the end-piece of P.

Note that a principal **TJ**-descendant in the end-piece of P always occurs in the succedent cf a sequent, and has the form $\varepsilon(t)$.

13.10) Let P be a **TJ**-proof satisfying p 1–p 4, and S a sequent in the end-piece of P. If S contains a formula B with a logical symbol, then there exists a formula A in S or in a sequent above S such that A is a principal descendant or a principal **TJ**-descendant.

PROOF. Suppose S contains a formula with a logical symbol. Then S is above the uppermost weakening in the end-piece. The property of sequents, of containing a logical symbol, is preserved upwards, to one of the upper sequents of each inference in the end-piece (but not necessarily beyond a boundary inference), or a **TJ**-initial sequent, when we follow upward the string to which S belongs. Notice that B may not be A, since B may be a descendant of a formula which is "passive" at a boundary inference.

13.11) Let P be a **TJ**-proof satisfying p 1–p 4 and not containing a suitable cut. Then its end-sequent contains a principle **TJ**-descendant.

PROOF. It suffices to prove that the end-sequent of P contains a principal descendant or a principal **TJ**-descendant, since the end-sequent contains no logical symbol. Suppose not. Since the end-piece contains a principal descendant or a principal **TJ**-descendant by 13.8), let us consider the following property (P) of cuts in the end-piece of P: A cut in the end-piece of P is said to have the property (P) if (at least) one of its upper sequents contains such a formula and its lower sequent contains no such formula. Since the end-piece contains such a formula, but the end-sequent does not (by assumption), there must be such a cut. Let I be an uppermost cut with the property (P) in the end-piece of P:

$$I\ \frac{\Gamma \to \Delta, D \qquad D, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}$$

Let S_1 and S_2 be the left and right upper sequents of I, respectively. By our assumption one of the cut formulas is a principal descendant or a principal **TJ**-descendant. First suppose D in S_1 has this property. If D contains a logical symbol, then it is a principal descendant. Then also, S_2 contains a formula with a logical symbol (namely D). Therefore, by 13.10), there is a formula A in S_2 or above it such that A is a principal descendant or a principal **TJ**-descendant. If there is no such formula in S_2, there must be a cut having the property (P) above I, contradicting our choice of I. If such a formula A is in S_2, A must be D itself, which contradicts our assumption that P does not contain a suitable cut. Thus D must be of the form $\varepsilon(t)$. Now suppose S_2 contains a logical symbol. Then there exists a principal descendant or a principal **TJ**-descendant either in S_2 or above it. If it is in S_2, it cannot be D (since D is $\varepsilon(t)$ and is in the left side of a sequent, it cannot be a principal **TJ**-descendant), and so it must also appear in the lower sequent of I, contradicting our assumption that I has the property (P). This means that such a formula is in a sequent above S but not in S itself, contradicting our assumption that I is an uppermost cut with the property (P). Thus S_2 cannot contain a formula with a logical symbol. Since I is an uppermost cut with the property (P), no logical inference at the boundary or **TJ**-initial sequent in the end-piece is above S_2. Therefore the proof down to S_2 is included in the end-piece and no logical initial sequents or **TJ**-initial sequents occur there and it is impossible that S_2 contains $\varepsilon(t)$, and so D cannot be $\varepsilon(t)$. Hence we have shown that D in S_1 cannot be a principal descendant or principal **TJ**-descendant. Next, suppose that the cut formula in S_2 is a principal descendant or principal **TJ**-descendant. As was seen above, D cannot be a principal **TJ**-descendant: D must contain a logical symbol. Hence there is a principal descendant or a principal **TJ**-descendant either in S_1 or in a sequent above S_1. If such a formula is not in S_1, there must be a cut having the property (P) above S_1, which contradicts our assumption about I. Therefore D in S_1 must have that property, since the lower sequent of I cannot contain such a formula. This again contradicts our assumption that P does not contain a suitable cut.

13.12) Now let P be a critical **TJ**-proof to which the reduction of Lemma 12.8 has been applied as far as possible (i.e., up to Step 4). Then P satisfies p 1–p 4 and does not contain a suitable cut (since it is critical). We define the notion of critical reduction. By 13.11), the end-sequent of P contains a principal **TJ**-descendant, $\varepsilon(\bar{m}_i)$, say, the descendant of a principal formula $\varepsilon(r)$ (where the closed term r denotes the number m_i). Let m be any number such that $|m|_{<\cdot}$ is less than the end-number of P. Then $\bar{m} <\cdot r$ is a true Σ_1^0-sentence of **PA**, and hence the sequent $\rightarrow \bar{m} <\cdot r$ can be derived from a mathematical initial sequent of **PA** (say $\rightarrow F$) by one application of $\exists$: right. So we replace the **TJ**-initial sequent

$$\forall x\,(x <\cdot r \supset \varepsilon(x)) \rightarrow \varepsilon(r)$$

in P by an ordinary proof in $\mathbf{PA}(\varepsilon)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\rightarrow F}{\rightarrow \bar{m} <\cdot r} \qquad \varepsilon(\bar{m}) \rightarrow \varepsilon(\bar{m})}{\bar{m} <\cdot r \supset \varepsilon(\bar{m}) \rightarrow \varepsilon(\bar{m})}}{\forall x\,(x <\cdot r \supset \varepsilon(x)) \rightarrow \varepsilon(\bar{m})}}{\forall x\,(x <\cdot r \supset \varepsilon(x)) \rightarrow \varepsilon(\bar{m}), \varepsilon(r).}$$

The ordinal of this proof is 6 and is less than that of a **TJ**-initial sequent (which is 7). By this replacement and some obvious changes, P is transformed into a **TJ**-proof P' whose end-sequent is

$$\rightarrow \varepsilon(\bar{m}), \varepsilon(\bar{m}_1), \ldots, \varepsilon(\bar{m}_n),$$

where $\rightarrow \varepsilon(\bar{m}_1), \ldots, \varepsilon(\bar{m}_n)$ is the end-sequent of P, and such that the ordinal of P' is less than that of P and the end-number of P' is $|m|_{<\cdot}$. We shall refer to P' as the proof obtained from P by an application of a critical reduction at m.

Now suppose P is any **TJ**-proof (not necessarily critical), and $|m|_{<\cdot}$ is less than the end-number of P. We shall define what is meant by the proof obtained from P by an application of a critical reduction at m.

If P is critical, the definition is as above. Otherwise, apply a sequence of reductions (as in the proof of Lemma 12.8). At each reduction, the ordinal of the proof *decreases*, so this process must terminate after a finite number of steps with a *critical* proof satisfying p 1–p 4. Now take the proof obtained from *this* proof as above.

13.13) Adjoining the reduction in 13.12) to the previous reductions, and applying the fundamental lemma in 13.5), we obtain the original form of Gentzen's theorem:

THEOREM 13.6. *The order type of* $<\cdot$ *is less than* ε_0.

13.14) Let $P(a)$ be a proof of $\rightarrow \varepsilon(a)$, obtained as described in 13.2. Let us define for each number k a **TJ**-proof P_k by induction on k, where the end-number of P_k is $|k|_{<\cdot}$.

(1) The case where $\forall n<k\ (n <\cdot k)$. We define P_k to be the proof $P(\bar{k})$ obtained from $P(a)$ by replacing a by the numeral $\bar{k}$ throughout $P(a)$.

(2) The case where $\exists n<k\ (k <\cdot n)$. Let

$$(**) \qquad n_0 <\cdot \ldots <\cdot n_{j-1}(=k) <\cdot n_{j+1} <\cdot \ldots <\cdot n_k$$

be the re-ordering of the numbers $\leqslant k$ with respect to $<\cdot$. Then we define P_k to be the proof obtained from $P_{n_{j+1}}$ by applying a critical reduction at k (cf. 13.12)). It is obvious that this definition is recursive.

13.15) We now define a map f, which will turn out to be an order-preserving recursive map as required for Theorem 13.4, by making use of the P_k. Define $f(k)$ by induction on k:

$$f(0) = \omega^{o(P_0)},$$

and for $k > 0$, $f(k) = f(n_{j-1}) + \omega^{o(P_k)}$ where $o(P)$ is (the Gödel number of) the ordinal of P, $+$ is (the primitive recursive function representing) addition of ordinals, ω^a is (the primitive recursive function representing) exponentiation by ω, and n_{j-1} is as in (**) (such a number always existing if $k > 0$).

13.16) Let $m_0 <\cdot m_1 <\cdot \ldots <\cdot m_i$ be the re-ordering of the numbers $< i+1$ with respect to $<\cdot$. Then

$$f(m_{j+1}) = f(m_j) + \omega^{o(P_{m_{j+1}})},$$

where $0 \leqslant j < i$. This is proved by mathematical induction on i. For $i = 0$, this is trivial. Assume it for i. For the case of $i+1$, it is sufficient to show (with $m_0, \ldots, m_i$ as above):

$$f(i+1) = f(m_j) + \omega^{o(P_{i+1})} \tag{1}$$

and

$$f(m_{j+1}) = f(i+1) + \omega^{o(P_{m_{j+1}})}, \tag{2}$$

where $m_j <\cdot i+1 <\cdot m_{j+1}$. Here (1) holds by definition of f, and (2) follows from (1) and $f(m_{j+1}) = f(m_j) + \omega^{o(P_{m_{j+1}})}$ (by induction hypothesis) and $o(P_{i+1}) \prec o(P_{m_{j+1}})$ (by definition of P_{i+1}). The second point of Theorem 13.4 is also easily seen if one puts $\mu = \omega^{o(P(a))+1}$. This completes the proof of Theorem 13.4.

To end this section, another result of Gentzen will be stated. The proof is straightforward.

THEOREM 13.7. *Let $\prec_n$ be the standard well-ordering of ε_0, restricted to ω_n. Then $\prec_n$ is a provable well-ordering of* **PA**.

Kleene's T-predicate (for unary, i.e., one-argument functions) is a primitive recursive relation T such that for an arbitrary partial recursive function f (of one argument) there exists a number e for which

$$f(x) \simeq U(\mu y\, T(e, x, y))$$

for all x. (U is a fixed primitive recursive function). Such an e is called a Gödel number of f. The definition can be extended to functions of many arguments.

If e is the Gödel number of a unary partial recursive function, then clearly

$$f \text{ is (total) recursive if and only if } \forall x\, \exists y\, T(e, x, y).$$

Further, f is called *provably recursive* (in **PA**) if it has a Gödel number e such that $\forall x\, \exists y\, T(\bar{e}, x, y)$ is **PA**-provable. Having discussed the Gödel numbering of recursive functions, we can now state a problem which should, in its correct context, actually have been placed in §12. The idea is due to Schütte.

PROBLEM 13.8. Let **PA*** be the system obtained by modifying **PA** as follows. The language is the same as that of **PA**; the initial sequents are those of **PA**; the rules of inference are those of **PA** except cut, $\forall$: right and ind; the constructive ω-rule, which is described below, is added as a new rule of inference:

$$\frac{P_1 \ldots P_i \ldots}{\Gamma \to \Delta,\, \forall x\, A(x)} \quad (i < \omega)$$

where P_i is a proof ending with $\Gamma \to \Delta,\, A(\bar{i})$, and there is a recursive function f such that $f(i) = \ulcorner P_i \urcorner$. Let e be a Gödel number of f. Then the proof ending with $\Gamma \to \Delta,\, \forall x\, A(x)$ is assigned the number

$$5^e \cdot 7^{\ulcorner \Gamma \to \Delta, \forall x A(x) \urcorner}.$$

Show that if a sequent S is **PA**-provable and contains no free variable, then S is provable in **PA***. [*Hint*: We adapt the method of the consistency proof of **PA** as follows. Let P be a (regular) proof in **PA**, with ordinal α

(according to the assignment of Definition 12.4). Then assign $\omega^\alpha + m$ to P, where m is the number of free variables in the end-piece of P. The reduction process for the consistency proof goes through almost unchanged, except that if P contains an explicit logical inference and the lowermost such is a $\forall$: right, then replace it by the ω-rule, which is applied at the end of the proof.]

PROBLEM 13.9. Let f be a provably recursive function in **PA**. Then there exists an ordinal μ (less than ε_0) such that f is $\prec^\mu$-primitive recursive, where $\prec^\mu$ is the standard ordering of ε_0 restricted to μ. [*Hint*: Let e be a Gödel number of f such that $\forall x\, \exists y\, T(\bar{e}, x, y)$ is **PA**-provable. Then there is a proof, say $P(a)$, of $\exists y\, T(\bar{e}, a, y)$, with free variable a. Let μ be the ordinal assigned to $P(a)$, and let P_m denote $P(\bar{m})$ for each natural number m. By the method of Problem 13.8, P_m can be transformed into a cut-free proof in **PA*** of the same end-sequent. It can be easily shown that the resulting proof does not contain the ω-rule, since $P(\bar{m})$ does not contain any explicit $\forall$: right. The transformation is actually $\prec^\mu$-primitive recursive. Thus there is a $\prec^\mu$-primitive recursive function τ such that $\tau(\ulcorner P_m \urcorner)$ is (the Gödel number of) a cut-free proof of $\exists y\, T(\bar{e}, \bar{m}, y)$. By examining this proof, we can find (primitive recursively in its Gödel number) a number n satisfying $T(e, m, n)$. Then n is a $\prec^\mu$-primitive recursive function of m and $f(m) = U(n)$. Thus f is $\prec^\mu$-primitive recursive.]

§14. An additional topic

Here we assume again that all the primitive recursive functions are included in the language of **PA** and their defining equations are included as initial sequents.

PROPOSITION 14.1. *Let Φ_n be the set of sentences of* **PA** *which have at most n logical symbols. Then there exists a truth definition for Φ_n in* **PA**, *i.e., a formula $T_n(a)$ of* **PA** *such that for every sentence A of Φ_n*

$$T_n(\overline{\ulcorner A \urcorner}) \equiv A$$

is **PA**-*provable.*

PROOF. T_n is defined by induction on n. We shall present only the induction step, in passing from T_n to T_{n+1}.

A sequence number, say x, is a number which can be decomposed into the form $2^{x_0} \cdot 3^{x_1} \cdot \ldots \cdot p_{n-1}^{x_{n-1}}$, where $x_i = 0$ or 1 for each i, $0 \leqslant i \leqslant n$. Let $\mathrm{seq}(x, n)$ be a (primitive recursive) predicate which expresses that x is a sequence number of the above form. We call n the length of x. The ith exponent of x, x_i, will be denoted $x(i)$. Let $\mathrm{st}(\ulcorner A \urcorner)$ express "A is a sentence", and let $\mathrm{ls}(\ulcorner A \urcorner)$ be the number of logical symbols in A. Then T_{n+1} is defined as follows.

$$
\begin{aligned}
& T_{n+1}(\ulcorner A \urcorner) \leftrightarrow \\
& \leftrightarrow \mathrm{st}(\ulcorner A \urcorner) \wedge \mathrm{ls}(\ulcorner A \urcorner) \leqslant n + 1 \\
& \quad \wedge \exists x\, [\mathrm{seq}(x, \ulcorner A \urcorner) \wedge \forall i\, (0 \leqslant i \leqslant \ulcorner A \urcorner \supset \\
& \qquad (\forall \ulcorner B \urcorner [i = \ulcorner \neg B \urcorner \supset (x(i) = 1 \equiv x(\ulcorner B \urcorner) = 0)] \\
& \qquad \wedge \forall \ulcorner B \urcorner \forall \ulcorner C \urcorner [i = \ulcorner B \wedge C \urcorner \\
& \qquad\qquad \supset (x(i) = 1 \equiv x(\ulcorner B \urcorner) = 1 \wedge x(\ulcorner C \urcorner) = 1)] \\
& \qquad \wedge \forall \ulcorner \forall y\, B(y) \urcorner [i = \ulcorner \forall y\, B(y) \urcorner \supset (x(i) = 1 \equiv \forall y\, T_n(\ulcorner B(\bar{y}) \urcorner))] \\
& \qquad \wedge \forall \ulcorner \exists y\, B(y) \urcorner [i = \ulcorner \exists y\, B(y) \urcorner \supset (x(i) = 1 \equiv \exists y\, T_n(\ulcorner B(\bar{y}) \urcorner))])) \\
& \qquad \wedge x(\ulcorner A \urcorner) = 1].
\end{aligned}
$$

It is easily seen that

$$T_{n+1}(\overline{A(\bar{b}_1, \ldots, \bar{b}_n)}) \equiv A(b_1, \ldots, b_n)$$

is **PA**-provable for every A in Φ_n, where all the free variables of A are among $b_1, \ldots, b_n$.

Let $S : A_1, \ldots, A_m \rightarrow B_1, \ldots, B_l$ be a sequent such that all of $A_1, \ldots, A_m$, $B_1, \ldots, B_l$ are in Φ_n. Then $T_n(\ulcorner S \urcorner)$ is defined to be

$$\exists i\, (1 \leqslant i \leqslant m \wedge \neg T_n(\ulcorner A_i \urcorner)) \vee \exists i\, (1 \leqslant i \leqslant l \wedge T_n(\ulcorner B_i \urcorner))).$$

Here of course m and l are primitive recursive functions of $\ulcorner S \urcorner$ and A_i and B_i are determined primitive recursively from $\ulcorner S \urcorner$ and i.

PROPOSITION 14.2. **PA** *cannot be formulated with finitely many axioms; in other words, mathematical induction cannot be expressed by finitely many formulas.*

PROOF. First note that

$$\mathbf{PA} \vdash (\vdash \ulcorner S \urcorner \rightarrow \underset{CF}{\vdash} \ulcorner S \urcorner), \tag{1}$$

by formalizing the cut-elimination theorem for **LK** in **PA**.

Next, suppose P is a cut-free proof of a sequent S, and all the formulas in S are in F_n. Then every formula in P is in F_n. Further, if P is in the language of **PA**, then we can prove in **PA** that every numerical instance of S is true; in other words:

$$\mathbf{PA} \vdash \underset{CF}{\vdash} \ulcorner S(b_1, \ldots, b_m) \urcorner \rightarrow \forall x_1 \ldots x_m \, T_n(\ulcorner S(\bar{x}_1, \ldots, \bar{x}_m) \urcorner), \tag{2}$$

where all the free variables of S are among $b_1, \ldots, b_m$. The proof of (2) is by induction on the number of sequents in P.

Now let Γ_0 be any finite set (or rather sequence) of axioms of CA $\cup$ VJ (Definition 9.5) and let n be the maximum number of logical symbols in any formula of Γ_0. Letting S be $\Gamma_0 \rightarrow \overline{0} = \overline{1}$, we obtain from (2):

$$\mathbf{PA} \vdash \underset{CF}{\vdash} \ulcorner \Gamma_0 \rightarrow \overline{0} = \overline{1} \urcorner \rightarrow T_n(\ulcorner \Gamma_0 \rightarrow \overline{0} = \overline{1} \urcorner). \tag{3}$$

Further (of course):

$$\mathbf{PA} \vdash \neg T_n(\ulcorner \Gamma_0 \rightarrow \overline{0} = \overline{1} \urcorner)$$

and hence, from (1) and (3):

$$\mathbf{PA} \vdash \neg \vdash \ulcorner \Gamma_0 \rightarrow \overline{0} = \overline{1} \urcorner.$$

This sentence, $\neg \vdash \ulcorner \Gamma_0 \rightarrow \overline{0} = \overline{1} \urcorner$, can be taken as expressing the consistency of Γ_0, which, as we see, is provable in **PA**. Hence, by Gödel's second incompleteness theorem (Theorem 10.18), Γ_0 cannot be proof-theoretically equivalent to **PA**.

EXERCISE 14.3. Show that ZF (Zermelo-Fraenkel set theory) cannot be formulated with finitely many axioms; in other words, the axiom of replacement cannot be expressed by finitely many formulas.

CHAPTER 3

SECOND ORDER SYSTEMS AND SIMPLE TYPE THEORY

§15. Second order predicate calculus

DEFINITION 15.1. A language for second order predicate calculus (a second order language) is defined by extending a language for first order predicate calculus (Definition 1.1) by adding the following.

5) *Second order variables*:

5.1) Free variables with i argument-places ($i = 0, 1, 2, \ldots$):

$$\alpha_0^i, \alpha_1^i, \ldots, \alpha_j^i, \ldots \qquad (j = 0, 1, 2, \ldots).$$

5.2) Bound variables with i argument places ($i = 0, 1, 2, \ldots$):

$$\varphi_0^i, \varphi_1^i, \ldots, \varphi_j^i, \ldots \qquad (j = 0, 1, 2, \ldots).$$

We shall call the variables in 2) of Definition 1.1 ($a_0, a_1, \ldots$ and $x_0, x_1, \ldots$) the *first order variables* in order to distinguish them from the second order variables.

Terms are defined as in Definition 1.2.

As in the preceding sections, we use α and φ both as formal and meta-variables; $\alpha, \beta, \gamma, \ldots$ may be used for second order free variables (with or without subscripts) and φ, ψ, χ may be used for second order bound variables. The superscripts i in α_j^i and φ_j^i are mostly omitted.

DEFINITION 15.2. The formulas for a second order language are defined as in Definition 1.3 with the following alteration.

If R^i is a predicate constant or a second order free variable with i argument-places and $t_1, \ldots, t_i$ are terms, then $R^i(t_1, \ldots, t_i)$ is an atomic formula.

In 3) of Definition 1.3 "a is a free variable" and "x is a bound variable" should read "a is a first order free variable" and "x is a first order bound variable", respectively.

We also add the clause:

3') If A is a formula, α a second order free variable and φ a second order bound variable not occurring in A, which has the same number of argument-places as α, then $\forall\varphi\, A'$ and $\exists\varphi\, A'$ are formulas, where A' is the expression obtained from A by writing φ in place of α at each occurrence of α in A. The outermost logical symbols of $\forall\varphi\, A'$ and $\exists\varphi\, A'$ are $\forall$ and $\exists$, respectively.

The quantifier-free formulas and closed formulas (i.e., sentences) are defined as before.

The replacement of symbols, and the notions of indicated and fully indicated occurrences of certain symbols, are defined as in Definitions 1.4 and 1.6, respectively. Thus from $F(\alpha)$ we obtained $F(R)$ by replacing the indicated occurrences of α by R. Also the notion of alphabetical variant is defined as in Definition 2.15 (where we assume, of course, that bound variables are replaced by other bound variables of the same order and, for second order variables, the same number of argument places).

A sequent is an expression of the form $\Gamma \to \Delta$, where Γ and Δ are finite sequences of formulas of our language.

In the following, we shall assume we have a fixed second order language, which we call L_2.

We shall first define a second order system which does not contain any "comprehension axiom", and is simply **LK** with second order variables. Since this system is basic to second order systems, we shall call it the basic calculus for second order systems and abbreviate it **BC**.

DEFINITION 15.3. The formulas of **BC** are those of L_2 and the sequents of **BC** are those of L_2. The rules of inference of **BC** are defined as those for **LK**: only the following should be added to those in Definition 2.1.

2.5') Second order $\forall$:

$$\text{left:}\quad \frac{F(R), \Gamma \to \Delta}{\forall\varphi\, F(\varphi), \Gamma \to \Delta},$$

where R is an arbitrary second order free variable or predicate constant and φ has the same number of argument-places as R.

$$\text{right:}\quad \frac{\Gamma \to \Delta, F(\alpha)}{\Gamma \to \Delta, \forall\varphi\, F(\varphi)},$$

where α is a second order free variable which is fully indicated in $F(\alpha)$ and does not occur in the lower sequent, and φ is a second order bound variable of the same number of argument-places as α (and does not occur in $F(\alpha)$, of course). Here α is called the *eigenvariable* of the inference.

2.6′) Second order $\exists$:

$$\text{left:}\quad \frac{F(\alpha), \Gamma \rightarrow \Delta}{\exists\varphi\, F(\varphi), \Gamma \rightarrow \Delta},$$

where α is a second order free variable which is fully indicated in $F(\alpha)$ and does not occur in the lower sequent, and φ is a second order bound variable of the same number of argument-places as α. Then α is called the eigenvariable of the inference.

$$\text{right:}\quad \frac{\Gamma \rightarrow \Delta, F(R)}{\Gamma \rightarrow \Delta, \exists\varphi\, F(\varphi)},$$

where R is an arbitrary second order free variable or predicate constant and φ has the same number of argument-places as R.

The auxiliary and principal formulas of these inferences are defined as for the other cases.

In contrast to 2.5′) and 2.6′), 2.5) and 2.6) will be called "first order $\forall$" and "first order $\exists$", respectively.

DEFINITION 15.4. The proofs of **BC** and the related notions and terminologies are defined as in §2 (cf. Definitions 2.2, 2.3 and 2.8); thus, we can define "a proof ending with S, or of S". "S is provable", "a thread of sequents", the concept of one sequent being "below" or "above" another, etc. The consistency of the system is defined exactly as before (Definition 4.1).

Similarly to Lemma 2.10 we can prove the following.

PROPOSITION 15.5. (1) *Let $P(R)$ be a* **BC**-*proof of a sequent $S(R)$, where R is an arbitrary second order free variable or predicate constant. Let R' be an arbitrary second order free variable or a predicate constant which does not occur in $P(R)$. Assume that R and R' have the same number of argument-places. Then $P(R')$ is a proof of $S(R')$.*

(2) *A proof is called regular if it satisfies the condition that, firstly, all second order eigenvariables are distinct from one another, and, secondly, if a second order α occurs as an eigenvariable in a sequent S of the proof, then α occurs only in sequents above S. If a sequent S is* **BC**-*provable then S is provable with a regular proof.*

From now on we assume that we deal with regular proofs whenever necessary.

DEFINITION 15.6. The concept of "axiom system" is defined as in Definition 4.1; an axiom system $\mathscr{A}$ (of L_2) is a set of sentences (of L_2). Clauses 2)–7) in Definition 4.1 can be adapted to the second order case. Proposition 4.4 is re-stated here.

PROPOSITION 15.7. *Let $\mathscr{A}$ be an axiom system and let $\mathbf{BC}_{\mathscr{A}}$ be the system obtained from* **BC** *by adding $\rightarrow A$ as initial sequents for all A in $\mathscr{A}$. Then a sequent $\Gamma \rightarrow \Delta$ is $\mathbf{BC}_{\mathscr{A}}$-provable if and only if for some $A_1, \ldots, A_m$ of $\mathscr{A}$, $A_1, \ldots, A_m, \Gamma \rightarrow \Delta$ is* **BC**-*provable.*

When dealing with second order systems it is convenient to work with semi-terms and semi-formulas.

DEFINITION 15.8. (1) Semi-terms are defined as follows. Individual constants and first order variables (free or bound) are semi-terms; if $t_1, \ldots, t_n$ are semi-terms and f is a function constant with n argument-places, then $f(t_1, \ldots, t_n)$ is a semi-term.

(2) Semi-formulas and the free occurrences of bound variables are defined as follows. Let R be a predicate constant or a second order variable (free or bound) with i argument-places, and let $t_1, \ldots, t_i$ be semi-terms. Then $R(t_1, \ldots, t_i)$ is an atomic semi-formula; the bound variables in $t_1, \ldots, t_i$ occur free in $R(t_1, \ldots, t_i)$, and if R is a bound variable, then R occurs free in $R(t_1, \ldots, t_i)$. If B and C are semi-formulas, then so is $B \wedge C$, and the free occurrences of bound variables in $B \wedge C$ are those of B and C. For other propositional connectives, the definition is analogous. If $F(x)$ is a semi-formula in which x is fully indicated, then $\forall x\, F(x)$ is a semi-formula; the free occurrences of bound variables in $\forall x\, F(x)$ are those in $F(x)$ except x. If $F(\varphi)$ is a semi-formula in which φ is fully indicated, then $\forall \varphi\, F(\varphi)$ is a semi-formula and the free occurrences in $\forall \varphi\, F(\varphi)$ are those in $F(\varphi)$ except φ. For $\exists$ the definition is analogous.

It is obvious that terms are semi-terms without bound variables, and formulas are semi-formulas without free occurrences of bound variables.

Now we shall define two important notions of abstracts and substitution.

DEFINITION 15.9. Let $A(b_1, \ldots, b_m)$ be a formula where some occurrences of $b_1, \ldots, b_m$ are indicated. (Some of $b_1, \ldots, b_m$ may not occur in the formula at all.) Let $y_1, \ldots, y_m$ be bound first order variables which do not occur in $A(b_1, \ldots, b_m)$. Then the meta-expression $\{y_1, \ldots, y_m\}A(y_1, \ldots, y_m)$ is called an *abstract* of $A(b_1, \ldots, b_m)$.

We should emphasize that this is a meta-expression, i.e., not a formal expression of L_2, and will be used only as an auxiliary aid.

An abstract of the form $\{y_1, \ldots, y_m\}A(y_1, \ldots, y_m)$ is said to have m *argument-places*. Abstracts are mostly denoted by $V, U, \ldots$. An abstract of the form $\{y_1, \ldots, y_m\}\alpha(y_1, \ldots, y_m)$ is often identified with α. If V denotes the abstract $\{y_1, \ldots, y_m\}A(y_1, \ldots, y_m)$ and $t_1, \ldots, t_m$ are semi-terms, then $V(t_1, \ldots, t_m)$ stands for $A(t_1, \ldots, t_m)$.

Definition 15.10. Substitution of an abstract for a second order free variable in a semi-formula is defined as follows. Let $F(\alpha)$ be a semi-formula where some of the occurrences of α are indicated, and let V be an abstract with the same number of argument-places as α. (In the following we shall not mention the last condition, as the substitution is defined only for α and V which have the same number of argument-places.) We define *substitution* of V for α in $F(\alpha)$, denoting the result by $F\,\binom{\alpha}{V}$ or $F(V)$. In order to simplify the notation, we assume that α and V have one argument-place. One can easily generalize the definition to the case of more than one argument-place. So let V be of the form $\{y\}A(y)$. $F\,\binom{\alpha}{V}$ is defined by induction on the logical complexity of $F(\alpha)$.

1) (i) $F(\alpha)$ is $\alpha(s)$ and this α is indicated in $F(\alpha)$. Then $F\,\binom{\alpha}{V}$ is $A(s)$. (ii) $F(\alpha)$ is $\alpha(s)$ and this α is not indicated, or $F(\alpha)$ is $\beta(s)$ for some β other than α. Then $F\,\binom{\alpha}{V}$ is $F(\alpha)$ itself.

In the subsequent cases we first replace all the bound variables in F which occur in V by bound variables which do not occur in V in a manner such that each variable is replaced by another of the same order, distinct variables are replaced by distinct ones and a second order variable of i argument-places is replaced by another of i argument-places. Thus we may assume that F does not contain bound variables which occur in V.

2) $F(\alpha)$ is one of $\neg B(\alpha)$, $B(\alpha) \wedge C(\alpha)$, $B(\alpha) \vee C(\alpha)$, and $B(\alpha) \supset C(\alpha)$. Then $F\,\binom{\alpha}{V}$ is, respectively, $\neg B\,\binom{\alpha}{V}$, $B\,\binom{\alpha}{V} \wedge C\,\binom{\alpha}{V}$, $B\,\binom{\alpha}{V} \vee C\,\binom{\alpha}{V}$ and $B\,\binom{\alpha}{V} \supset C\,\binom{\alpha}{V}$.

3) $F(\alpha)$ has one of the forms $\forall x\, G(x)(\alpha)$, $\exists x\, G(x)(\alpha)$, $\forall \varphi\, G(\varphi)(\alpha)$ and $\exists \varphi\, G(\varphi)(\alpha)$. Then $F\,\binom{\alpha}{V}$ is, respectively, $\forall x\, (G(x)\,\binom{\alpha}{V})$, $\exists x\, (G(x)\,\binom{\alpha}{V})$, $\forall \varphi\, (G(\varphi)\,\binom{\alpha}{V})$ and $\exists \varphi\, (G(\varphi)\,\binom{\alpha}{V})$.

It is obvious that $F\,\binom{\alpha}{V}$ is a semi-formula. It is also obvious that if $F(\alpha)$ is a formula then so is $F\,\binom{\alpha}{V}$.

The ambiguity in 2) and 3), viz. the choice of new bound variables, can be eliminated by requiring that these are the first variables in the list of first and second order bound variables which satisfy the conditions. This is not an essential restriction, by virtue of the following.

PROPOSITION 15.11. *Let A and B be two formulas which are alphabetical variants of each other. Then $A \equiv B$ is* **BC**-*provable.*

Thus we shall henceforth deal with any of the alphabetical variants of a given formula.

EXAMPLE 15.12. (1) Let $F(\alpha)$ be $\forall x\, \forall y\, (x = y \supset (\alpha(x) \equiv \alpha(y)))$, where both occurrences of α are indicated, and let V be $\{u\}\, \exists x\, (x + u = 5)$, where it is assumed that 5 is an individual constant, $+$ is a function constant and $=$ is a predicate constant in the language. Since x in $F(\alpha)$ occurs in V, first change it to, say, z: $\forall z\, \forall y\, (z = y \supset (\alpha(z) \equiv \alpha(y)))$. Let us call this formula $F'(\alpha)$. We shall carry out the substitution of V for α in $F'(\alpha)$ step by step.

$$\alpha(z) \begin{pmatrix} \alpha \\ V \end{pmatrix}: \quad \exists x\, (x + z = 5)$$

$$\alpha(y) \begin{pmatrix} \alpha \\ V \end{pmatrix}: \quad \exists x\, (x + y = 5)$$

$$(\alpha(z) \equiv \alpha(y)) \begin{pmatrix} \alpha \\ V \end{pmatrix}: \quad \exists x\, (x + z = 5) \equiv \exists x\, (x + y = 5)$$

$$F' \begin{pmatrix} \alpha \\ V \end{pmatrix}, \quad \text{i.e.,} \quad \forall z\, \forall y\, (z = y \supset (\alpha(z) \equiv \alpha(y))) \begin{pmatrix} \alpha \\ V \end{pmatrix}:$$

$$\forall z\, \forall y\, (z = y \supset (\exists x\, (x + z = 5) \equiv \exists y\, (y + z = 5))).$$

This is a familiar formula, in fact an equality axiom. If we did not first replace x by z, the result would be

$$\forall x\, \forall y\, (x = y \supset (\exists x\, (x + x = 5) \equiv \exists x\, (x + y = 5))),$$

which is not even a formula.

This can be generalized to an arbitrary abstract $\{u\}B(u)$ (assuming there is no clash of bound variables), thus obtaining $\forall x\, \forall y\, (x = y \supset (B(x) \equiv B(y)))$, which is an equality axiom. That is to say, the simple schema

$$\forall x\, \forall y\, (x = y \supset (\alpha(x) \equiv \alpha(y)))$$

and substitution produce all the equality axioms.

(2) Let $F(\alpha)$ be $\alpha(0) \wedge \forall x\, (\alpha(x) \supset \alpha(x')) \supset \forall x\, \alpha(x)$, where all occurrences of α are indicated, and let V be $\{u\}B(u)$. Let us assume that x does not occur in V. Then $F\,(^{\alpha}_{V})$ is $B(0) \wedge \forall x\, (B(x) \supset B(x')) \supset \forall x\, B(x)$, which is an induction axiom in arithmetic (in an appropriate language).

(3) Let $F(\alpha)$ be

$$\forall x\, \forall y\, \forall z\, (\alpha(x, y) \wedge \alpha(x, z) \supset y = z) \supset$$
$$\supset \exists v\, \forall y\, (y \in v \equiv \exists x\, (x \in u \wedge \alpha(x, y))),$$

with all occurrences of α indicated, and let V be $\{x^1, y^1\}B(x^1, y^1)$, in the language of set theory. Then $F\,(^{\alpha}_{V})$ is

$$\forall x\, \forall y\, \forall z\, (B(x, y) \wedge B(x, z) \supset y = z) \supset$$
$$\supset \exists v\, \forall y\, (y \in v \equiv \exists x\, (x \in u \wedge B(x, y)),$$

which is an axiom of replacement in ZF set theory. Note that $B(x, y)$ may contain variables other than x and y, including u, but not v (since this is bound in $F(\alpha)$).

We shall return to those examples later.

The following is easily proved by induction on the number of logical symbols in $F(\alpha)$.

PROPOSITION 15.13. *For an arbitrary formula $F(\alpha)$ and arbitrary abstracts U and V, the sequent*

$$\forall x\, (U(x) \equiv V(x)),\, F(U) \to F(V),$$

(where it is assumed that the bound variables are properly taken care of) is **BC**-*provable.*

DEFINITION 15.14. (1) Let $A(b_1, \ldots, b_m, c_1, \ldots, c_n, \beta_1, \ldots, \beta_k)$ be a formula, all of whose free variables are among $b_1, \ldots, b_m, c_1, \ldots, c_n, \beta_1, \ldots, \beta_k$ (though not necessarily all of these occur in A), and where all occurrences of these free variables are indicated. Then a sentence of the form

$$(*) \qquad \forall z_1 \ldots \forall z_n\, \forall \psi_1 \ldots \forall \psi_k\, \exists \varphi\, \forall y_1 \ldots \forall y_m\, (\varphi(y_1, \ldots, y_m)$$
$$\equiv A(y_1, \ldots, y_m, z_1, \ldots, z_n, \psi_1, \ldots, \psi_k))$$

is called a comprehension axiom.

Let V be the abstract

$$\{y_1 \ldots y_m\}A(y_1, \ldots, y_m, c_1, \ldots, c_n, \beta_1, \ldots, \beta_k).$$

Then the above comprehension axiom may be written as

$$(**) \quad \forall z_1, \ldots, \forall z_n \forall \psi_1 \ldots \forall \psi_k \exists \varphi \forall y_1 \ldots \forall y_m (\varphi(y_1, \ldots, y_m) \equiv U(y_1, \ldots, y_m)),$$

where U is obtained from V by replacing c's and β's by z's and ψ's, respectively.

(2) Let K be an arbitrary set of formulas. (This use of K is only temporary.) A formula which belongs to K is called a K-formula, and if a formula $A(b_1, \ldots, b_m)$ is a K-formula, then the abstract $\{y_1 \ldots y_m\}A(y_1, \ldots, y_m)$ is called a K-abstract. If the formula A in a comprehension axiom (cf. (*)) is a K-formula, then (*) is called a K-comprehension axiom.

(3) A set of formulas K is said to be closed under substitution if for every K-formula or K-abstract $A(\alpha)$ and for every K-abstract V, $A(V)$ again belongs to K.

DEFINITION 15.15. Let K be a set of formulas.

1) The K-system is obtained from **BC** by adding to it all K-comprehension axioms as initial sequents (viz. sequents of the form $\rightarrow A$, where A is a K-comprehension axiom).

2) **KC** is the system obtained from **BC** by adding the following inferences, for arbitrary formulas $F(\alpha)$, and K-abstracts V (where $F(V)$ and $F(\varphi)$ are obtained by replacing the indicated α by, respectively, V and φ):

$$\text{Second order } \forall : \text{left:} \quad \frac{F(V), \Gamma \rightarrow \Delta}{\forall \varphi\, F(\varphi), \Gamma \rightarrow \Delta};$$

$$\text{Second order } \exists : \text{right:} \quad \frac{\Gamma \rightarrow \Delta, F(V)}{\Gamma \rightarrow \Delta, \exists \varphi\, F(\varphi)}.$$

The auxiliary and principal formulas of these inferences are defined as usual.

Since a system **KC** has interest only if K is closed under substitution, we shall henceforth assume that K is closed under substitution.

PROPOSITION 15.16. *For an arbitrary set K of formulas (closed under substitution), the K-system is equivalent to* **KC**.

PROOF. It can easily be shown that the K-comprehension axioms are provable in **KC**, while the lower sequents of second order $\forall$: left and second order $\exists$: right are provable in the K-system from their upper sequents.

Due to the above proposition we shall henceforth only deal with K-comprehension axioms in the form of the system **KC**.

DEFINITION 15.17. If K is the set of all second order formulas, then **KC** is called the second order predicate calculus with full comprehension, and is denoted by $\mathbf{G^1LC}$.

PROPOSITION 15.18. *If the cut-elimination theorem holds for* $\mathbf{G^1LC}$, *then* $\mathbf{G^1LC}$ *is consistent.*

PROOF. The proof is immediate, as for Theorem 6.2.

In fact the cut-elimination theorem does hold for $\mathbf{G^1LC}$, as we will see later (§20). The reason why we put Proposition 15.18 in this form is that the proof of cut-elimination for $\mathbf{G^1LC}$ is non-constructive, and hence, on the basis of our finitist standpoint, we cannot claim the consistency of $\mathbf{G^1LC}$ from that proof.

§16. Some systems of second order predicate calculus

In this section we shall deal with some inessential extensions of the first order predicate calculus.

DEFINITION 16.1. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two formal systems which contain **LK**. $\mathbf{S}_2$ is called an inessential extension of $\mathbf{S}_1$ if $\mathbf{S}_1$ is a subsystem of $\mathbf{S}_2$ and for any sequent S of the language of $\mathbf{S}_1$, if S is $\mathbf{S}_2$-provable, then S is $\mathbf{S}_1$-provable.

PROPOSITION 16.2. *The cut-elimination theorem holds for* **BC**.

The proof is exactly as for **LK**, so we shall not repeat the argument.

As consequences of this proposition, consistency, the subformula property, the midsequent property, etc., all hold for **BC**. As another consequence we can claim:

COROLLARY 16.3. **BC** *is an inessential extension of* **LK**.

PROOF. If any inference for a second order quantifier is used in a cut-free proof of **BC**, then that quantifier will occur in all sequents below that inference (as is easily shown by induction on the number of inferences in such a proof).

DEFINITION 16.4. (1) A *first order formula* is one which contains no second order quantifiers (although it may contain second order variables). Such a formula is also called arithmetical if the language is that of second order arithmetic (i.e., **PA**, with second order variables).

A *first order abstract* is one obtained from a first order formula.

(2) K_1 is the set of all first order formulas.

(3) The predicative comprehension axioms are those in which the U in (**) of Definition 15.14 is a first order abstract; in other words, the K_1-comprehension axioms.

THEOREM 16.5 (cut-elimination theorem for the system with predicative comprehension axioms). *If a sequent S is provable in the system* $\mathbf{K}_1\mathbf{C}$ (*cf. Definition* 15.15), *then it is provable in* $\mathbf{K}_1\mathbf{C}$ *without cut.*

PROOF. The proof for **LK** almost goes through. Here we use triple induction instead of double induction (cf. Proof of Lemma 5.4). Let A be a formula of a second order language. Define a function c by: $c(A) =_{\mathrm{df}}$ the number of second order quantifiers in A. It is easily seen that $c(F(\alpha)) = c(F(V))$ if V is first order. Let $c =_{\mathrm{df}} c(P) =_{\mathrm{df}} c(D)$, where D is the mix formula of P (assuming P has a mix at most as the last inference). Then Lemma 5.4 is proved now by transfinite induction on $\omega^2 \cdot c + \omega \cdot g(P) + \mathrm{rank}(P)$. We may follow the proof in §5 but there are some additional cases here. After 1.5) (i) there, add the cases that D is $\forall\varphi\, F(\varphi)$ and $\exists\varphi\, F(\varphi)$. P has the form

$$\dfrac{\dfrac{\Gamma \to \Delta_0, F(\alpha)}{\Gamma \to \Delta_0, \forall\varphi\, F(\varphi)} \qquad \dfrac{F(V), \Pi_0 \to \Lambda}{\forall\varphi\, F(\varphi), \Pi_0 \to \Lambda}}{\Gamma, \Pi_0 \to \Delta_0, \Lambda}\ (\forall\varphi\, F(\varphi)),$$

where V is a first order abstract. From the above remark, $c(F(V)) = c(F(\alpha)) = c(\forall\varphi\, F(\varphi)) - 1$. As α does not occur in Γ, Δ_0 or $F(\varphi)$; $\Gamma \to \Delta, F(V)$ is provable without a mix (cf. 1.5) of Proof of Lemma 5.4). Define P' as

$$\frac{\Gamma \to \Delta_0, F(V) \qquad F(V), \Pi_0 \to \Lambda}{\Gamma, \Pi_0^{\#} \to \Delta_0^{\#}, \Lambda} \quad (F(V)).$$

Since $c(P') = c(F(V)) < c(\forall\varphi\, F(\varphi)) = c(P)$, the induction hypothesis applies to P'. Thus we can obtain a proof without a mix of $\Gamma, \Pi_0^{\#} \to \Delta_0^{\#}, \Lambda$, and hence a proof without a mix of $\Gamma, \Pi_0 \to \Delta_0, \Lambda$.

Finally, after 2.1.3 (ii) of the Proof of Lemma 5.4, add the cases where D is $\forall\varphi\, F(\varphi)$ and $\exists\varphi\, F(\varphi)$.

COROLLARY 16.6. $\mathbf{K_1C}$ *is an inessential extension of* $\mathbf{LK}$. *Hence, in particular,* $\mathbf{K_1C}$ *is consistent.*

PROOF. The proof is as for $\mathbf{BC}$ (cf. Corollary 16.3).

PROPOSITION 16.7. *Let* $\mathbf{LK}^+$ *be the system which is like* $\mathbf{LK}$ *except that the language includes free second order variables.*

Let $\Gamma \to \Theta$ *be a sequent consisting of first order formulas only and let* $F_i(\beta_i)$ *be a first order formula which has a free second order variable* β_i, $i = 1, 2, \ldots, m$. *Then*

$$\forall\varphi_1\, F_1(\varphi_1), \ldots, \forall\varphi_m\, F_m(\varphi_m), \Gamma \to \Theta \tag{1}$$

is $\mathbf{K_1C}$*-provable if and only if the following is satisfied:*

(*) *For each* $i = 1, 2, \ldots, m$ *there exist first order abstracts* $V_{i,1}, \ldots, V_{i,l_i}$ $(l_i \geqslant 1)$ *such that*

$$\{\forall \mathbf{z}_{1,j}\, F_1(V'_{1,j})\}_{j=1,\ldots,l_1}, \ldots, \{\forall \mathbf{z}_{n,j}\, F_n(V'_{n,j})\}_{j=1,\ldots,l_n}, \Gamma \to \Theta \tag{2}$$

is $\mathbf{LK}^+$*-provable;*

Here $\{A_j\}_{j=1,\ldots,m}$ *denotes a sequence of formulas* $A_1, \ldots, A_m$, $\forall \mathbf{z}_{i,j}$ *denotes a (possibly empty) sequence of universally quantified first order variables* $\forall z_1 \forall z_2 \ldots \forall z_k$, *where* k *(depending on* i *and* j*) is the number of free first order variables in* $V_{i,j}$, *and* V' *is obtained from* V *by changing the free first order variables in* V *which do not occur in* (1) *to* $z_1, \ldots, z_k$.

PROOF. If: Suppose (*) holds. First we shall prove that for every formula $F(\alpha)$; $\forall\varphi\, F(\varphi) \to \forall \mathbf{z}\, F(V')$ is $\mathbf{K_1C}$-provable, if V is first order.

$$\cfrac{\cfrac{F(V) \to F(V)}{\forall\varphi\, F(\varphi) \to F(V)}}{\cfrac{\text{(repeated } \forall : \text{right)}}{\forall\varphi\, F(\varphi) \to \forall \mathbf{z}\, F(V).}}$$

Thus we have $\forall\varphi_i\, F_i(\varphi_i) \rightarrow \forall \mathbf{z}_{i,j}\, F_i(V'_{i,j})$ for $j = 1, 2, \ldots, l_i$ and $i = 1, 2, \ldots, n$. From these and (2), by repeated cuts and contractions, we can construct a $\mathbf{K}_1\mathbf{C}$-proof of (1).

Only if: Suppose (1) is $\mathbf{K}_1\mathbf{C}$-provable. Then there exists a cut-free proof of (1) in $\mathbf{K}_1\mathbf{C}$. Therefore it is sufficient to prove the following proposition.

PROPOSITION 16.8. *Suppose P is a cut-free proof in $\mathbf{K}_1\mathbf{C}$ of a sequent of the form* (1) *above. Then for the end-sequent of P,* (*) *holds.*

Notice that since P is cut-free, all sequents in P have the form (1). The proposition may now be proved by mathematical induction on the number of inferences in P.

PROOF. (1) If P consists of an initial sequent $D \rightarrow D$, then D has no second order quantifier. Therefore $P \rightarrow D$ itself has the form (2) above.

(2) The induction steps are proved according to the last inference I in P. Notice that the only possible inference in P concerning a second order quantifier is second order $\forall$: left.

2.1) I is second order $\forall$: left. P is of the form

$$\frac{F(V), \Pi \rightarrow \Lambda}{\forall\varphi\, F(\varphi), \Pi \rightarrow \Lambda},$$

where V and $F(\varphi)$ are first order. By the induction hypothesis, when $F(V), \Pi \rightarrow \Lambda$ is taken for the sequent in (1), there are appropriate abstracts for which a sequent like (2) is provable in $\mathbf{LK}^+$. Denote such a sequent by $F(V), \Pi^* \rightarrow \Lambda$. Now add V to the set of abstracts obtained by the induction hypothesis. If V has no first order free variable which does not occur in $\forall\varphi\, F(\varphi), \Pi \rightarrow \Lambda$, then take $F(V), \Pi^* \rightarrow \Lambda$ itself for the sequent (2). If V has free variables $b_1, \ldots, b_k$ which do not occur in the above sequent, then replace them by new bound variables $z_1, \ldots, z_k$ and call the result V'. The required sequent is then $\forall z_1, \ldots, \forall z_k\, F(V'), \Pi^* \rightarrow \Lambda$.

2.2) I is not a second order $\forall$: left. Such a case is proved trivially from the induction hypothesis.

Then replace free second order variables by $0 = 0$.

Notice that it is the inference contraction : left that results in more than one first order abstract $V_{i,1}, V_{i,2}, \ldots$ being associated with the same formula $F_i(\beta_i)$ in (2).

PROPOSITION 16.9. *If a formula* $\exists\varphi\, F(\varphi)$ *is provable in* $\mathbf{K_1C}$, *where* $F(\varphi)$ *does not have second order quantifiers, then there exist first order abstracts* $V_1, \ldots, V_n$ *such that* $\exists \mathbf{z}_1\, F(V_1') \vee \ldots \vee \exists \mathbf{z}_n\, F(V_n')$ *is provable in* $\mathbf{LK}^+$.

This is the dual of Proposition 16.7 and is proved similarly.

PROBLEM 16.10. We define $\mathbf{PA}'$, the predicative (second order) extension of Peano arithmetic, as follows. Let VJ′ and Eq′ be respectively the sentences:

VJ′ $\quad \forall\varphi\, (\varphi(0) \wedge \forall x\, (\varphi(x) \supset \varphi(x')) \supset \forall x\, \varphi(x))$;

Eq′ $\quad \forall\varphi\, \forall x\, \forall y\, (x = y \wedge \varphi(x) \supset \varphi(y))$.

VJ′ is the second order formulation of the principle of mathematical induction and Eq′ is the second order formulation of one of the equality axioms.

$\mathbf{PA}'$ is then obtained from $\mathbf{K_1C}$ (in the language of $\mathbf{PA}$ augmented by second order variables) by adding to it the axioms of CA $\cup$ VJ′ $\cup$ Eq′ as initial sequents. (CA was defined in definition 9.2.)

Show that $\mathbf{PA}'$ is an inessential extension of $\mathbf{PA}$. [*Hint*: Let A be a formula of the language of $\mathbf{PA}'$. Then A is $\mathbf{PA}'$-provable if and only if CA $\cup$ VJ′ $\cup$ Eq′ $\to A$ is $\mathbf{K_1C}$-provable. Noting that VJ′ and Eq′ each have one second order $\forall$ in front, apply Proposition 16.7.]

PROBLEM 16.11. Consider ZF (Zermelo-Fraenkel set theory). The language consists of $\in$ (a binary predicate symbol), first order variables and logical symbols ($a = b$ is an abbreviation of $\forall x\, (a \in x \equiv b \in x)$). The axioms of extensionality, pairs, sum, power, regularity and infinity can be stated as single sentences. However, the axiom of replacement is actually an axiom schema, which is formulated as

$$\forall x\, \forall y\, \forall z\, (B(x, y) \wedge B(x, z) \supset y = z) \supset$$
$$\supset \exists v\, \forall y\, (y \in v \equiv \exists x\, (x \in u \wedge B(x, y))$$

(cf. Example 15.12, (3)). The basic logical system is $\mathbf{LK}$.

On the other hand BG (Bernays-Gödel set theory) is formulated in a second order language. The language is that of ZF augmented by second order variables. The axioms are those of ZF plus an axiom of equality

$$\forall\varphi\, \forall x\, \forall y\, (x = y \supset \varphi(x) \equiv \varphi(y))$$

except that the axiom of replacement is now formulated in a single sentence:

$$\forall\varphi\ (\forall x\ \forall y\ \forall z\ (\varphi(x, y) \wedge \varphi(x, z) \supset y = z) \supset$$

$$\supset \exists v\ \forall y\ (y \in v \equiv \exists x\ (x \in u \wedge \varphi(x, y)))).$$

The basic logical system is $\mathbf{K_1C}$.

Show that BG is an inessential extension of ZF. [*Hint*: As for the previous problem.]

DEFINITION 16.12. Let us assume that the only logical symbols are $\neg$, $\wedge$ and $\forall$. (For convenience, we let **BC** denote also the basic calculus restricted to these logical symbols.)

1) A formula is said to be *positive* if every occurrence of a second order quantifier is in the scope of an even number of $\neg$'s. A sequent

$$F_1, \ldots, F_m \rightarrow H_1, \ldots, H_n$$

is called positive if $\neg(F_1 \wedge \ldots \wedge F_m) \vee (H_1 \vee \ldots \vee H_n)$ is positive (where $\vee$ is defined in terms of $\neg$ and $\wedge$). Thus, for example, $\neg(\neg\forall\varphi\ F(\varphi) \wedge \forall\psi\ H(\psi))$ is not positive since $\forall\psi$ is in the scope of one $\neg$, while $\neg(B \wedge \neg\forall\varphi\ F(\varphi))$, where B and F do not contain second order quantifiers, is positive.

2) The Π^1-predicate calculus, or $\mathbf{\Pi^1PC}$, is the system obtained from **BC** by restricting it as follows. (For the sake of simplicity, we assume there are no function or predicate constants.)

(1) The initial sequents consist of first order formulas only.

(2) There is no second order $\forall$: left.

(3) There is no cut rule.

It is obvious that any sequent provable in $\mathbf{\Pi^1PC}$ is positive.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ denote finite sequences of free variables such that all variables of $\boldsymbol{a}$ are distinct while in $\boldsymbol{b}$ there may be repetitions, the length of $\boldsymbol{a}$ and $\boldsymbol{b}$ are the same, the ith variable of $\boldsymbol{b}$ is first or second order according as the ith variable of $\boldsymbol{a}$ is first or second order, and if the ith variable of $\boldsymbol{a}$ has j argument-places then so does the ith variable of $\boldsymbol{b}$. Let

$$F\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix} \quad \text{be} \quad \left(F\,\frac{\boldsymbol{a}}{\boldsymbol{b}}\right)$$

(i.e., the replacement of $\boldsymbol{a}$ by $\boldsymbol{b}$, cf. Definition 1.4).

(Notice that omitting predicate constants from the language is not an essential restriction, since the free variables which are not used as eigenvariables can be regarded as such.)

PROPOSITION 16.13 (Maehara-Takeuti). *If a formula* $F\binom{\boldsymbol{a}}{\boldsymbol{b}} \supset G\binom{\boldsymbol{a}}{\boldsymbol{b}}$ *is provable in* $\mathbf{\Pi^1 PC}$, *then there exists a first order formula C satisfying the following.*

1) *Every variable in C other than those in* $\boldsymbol{a}$ *occurs both in F and G.*
2) $F\binom{\boldsymbol{a}}{\boldsymbol{b}} \supset C\binom{\boldsymbol{a}}{\boldsymbol{b}}$ *and* $C \supset G$ *are provable in* $\mathbf{\Pi^1 PC}$.

(Notice that the provability of $F \supset G$ is not assumed.)

PROOF. The proof is almost the same as that of Theorem 6.6 (Craig's interpolation theorem for **LK**). State the proposition for arbitrary partitions of provable sequents, introduce $\rightarrow \top$ as an auxiliary initial sequent (cf. Lemma 6.5) and prove the statement in this system. The conditions (1)–(3) in the definition of $\mathbf{\Pi^1 PC}$ are indeed crucial.

PROBLEM 16.14 (Chang). Let $\boldsymbol{x}$ be a sequence $X_1, \ldots, X_n$ of bound variables and $\mathsf{Q}\boldsymbol{x}$ be $\mathsf{Q}_1 X_1 \ldots \mathsf{Q}_n X_n$ where Q_i is $\forall$ or $\exists$ and Q_i is always $\forall$ if X_i is a second order variable. If $\mathsf{Q}\boldsymbol{x}\,(F(\boldsymbol{x}) \supset G(\boldsymbol{x}))$ is provable in $\mathbf{\Pi^1 PC}$, then there exists a first order formula $C(\boldsymbol{a})$ such that

$$\mathsf{Q}\boldsymbol{x}\,((F(\boldsymbol{x}) \supset C(\boldsymbol{x})) \wedge (C(\boldsymbol{x}) \supset G(\boldsymbol{x})))$$

is provable in $\mathbf{\Pi^1 PC}$. [*Hint*: Maehara-Takeuti method. This is a trivial consequence of Proposition 16.16, which is a consequence of Proposition 16.13.

DEFINITION 16.15. Let $G(\boldsymbol{a})$ be a formula whose free variables are all in $\boldsymbol{a}$.

For convenience we temporarily introduce, on the meta-level, the third order variable $\mathscr{A}$, and new atomic formula $\mathscr{A}(\boldsymbol{a})$, and extend the notion of formula accordingly. (However this variable, and any formula containing it, are not part of our formal system). Let F be a formula containing $\mathscr{A}$.

$$F\begin{pmatrix} \mathscr{A} \\ \lambda \boldsymbol{x}\, G(\boldsymbol{x}) \end{pmatrix}$$

is the formula obtained from F by substituting $G(\boldsymbol{b})$ for $\mathscr{A}(\boldsymbol{b})$ in F. If S is a sequent $F_1, \ldots, F_m \rightarrow H_1, \ldots, H_n$, then

$$S\begin{pmatrix} \mathscr{A} \\ \lambda \boldsymbol{x}\, G(\boldsymbol{x}) \end{pmatrix}$$

is

$$F_1\begin{pmatrix}\mathscr{A}\\ \lambda\boldsymbol{x}\, G(\boldsymbol{x})\end{pmatrix},\ldots,F_m\begin{pmatrix}\mathscr{A}\\ \lambda\boldsymbol{x}\, G(\boldsymbol{x})\end{pmatrix}\to H_1\begin{pmatrix}\mathscr{A}\\ \lambda\boldsymbol{x}\, G(\boldsymbol{x})\end{pmatrix},\ldots,H_n\begin{pmatrix}\mathscr{A}\\ \lambda\boldsymbol{x}\, G(\boldsymbol{x})\end{pmatrix}.$$

An occurrence of $\mathscr{A}$ in a formula F is defined as positive or negative, inductively as follows. (We assume, as stated above, that we have only $\neg$, $\wedge$ and $\forall$ as logical symbols.)

1) The occurrence of $\mathscr{A}$ in $\mathscr{A}(\boldsymbol{b})$ is positive.

2) If the occurrence of $\mathscr{A}$ in F is positive (negative), then that occurrence of $\mathscr{A}$ in $F \wedge G$ or $G \wedge F$ is also positive (negative). An occurrence of $\mathscr{A}$ in $\neg F$ is positive or negative according as that occurrence of $\mathscr{A}$ in F is negative or positive.

3) An occurrence of $\mathscr{A}$ in $\forall x\, F(x)$ or $\forall \varphi\, F(\varphi)$ is positive or negative according as that occurrence of $\mathscr{A}$ in $F(a)$ or $F(\alpha)$ is positive or negative.

An occurrence of $\mathscr{A}$ in a sequent $F_1,\ldots, F_m \to G_1,\ldots, G_n$ is positive or negative according as that occurrence of $\mathscr{A}$ in $\neg(F_1 \wedge \ldots \wedge F_m) \vee G_1 \vee \ldots \vee G_n$ is positive or negative (where $\vee$ is defined in terms of $\neg$ and $\wedge$).

PROPOSITION 16.16. *Let $G(\boldsymbol{a})$ be a formula all of whose free variables are in $\boldsymbol{a}$, and let S be a sequent in which all the occurrences of $\mathscr{A}$ are positive. If*

$$S\begin{pmatrix}\mathscr{A}\\ \lambda\boldsymbol{x}\, G(\boldsymbol{x})\end{pmatrix}$$

is provable in $\mathbf{\Pi^1PC}$, *then there exists a first order formula $C(\boldsymbol{a})$ satisfying the following conditions:*

1) $S\begin{pmatrix}\mathscr{A}\\ \lambda\boldsymbol{x}\, C(\boldsymbol{x})\end{pmatrix}$ is provable in $\mathbf{\Pi^1PC}$;

2) *all the free variables of $C(\boldsymbol{a})$ are in $\boldsymbol{a}$;*

3) $\forall \boldsymbol{x}\, (C(\boldsymbol{x}) \supset G(\boldsymbol{x}))$ *is provable in* $\mathbf{\Pi^1PC}$.

PROOF. The proof is by induction on the number of inferences in a proof of

$$S\begin{pmatrix}\mathscr{A}\\ \lambda\boldsymbol{x}\, G(\boldsymbol{x})\end{pmatrix}.$$

Use Proposition 16.13 for the case second order $\forall$: right.

DEFINITION 16.17. The *satisfaction relation* for second order formulas in a given structure $\mathscr{D} = \langle D, \phi\rangle$ (cf. §8) is defined as follows. Let ϕ_0 be a map from variables (first and second order) such that its values for first order variables are elements of D, while its values for second order variables of i argument-places are subsets of $D \times \ldots \times D$ $(= D^i)$.

1) $(\mathscr{D}, \phi_0)$ satisfies $\alpha(t_1, \ldots, t_i)$ $(\varphi(t_1, \ldots, t_i))$ if and only if $(\phi_0 t_1, \ldots, \phi_0 t_i)$ belongs to $\phi_0\alpha$ $(\phi_0\varphi)$.

2) $(\mathscr{D}, \phi_0)$ satisfies $\forall\varphi\, F(\varphi)$ $(\exists\varphi\, F(\varphi))$ if and only if for every ϕ_0' (there exists a ϕ_0') which agrees with ϕ_0 except at φ, (such that) $(\mathscr{D}, \phi_0')$ satisfies $F(\varphi)$.

For other cases the definition is the same as in §8.

A formula is valid if for every structure $\mathscr{D} = \langle D, \phi\rangle$ and map ϕ_0 as above, it is satisfied by $(\mathscr{D}, \phi_0)$.

PROPOSITION 16.18. **Π^1PC** *is complete for positive formulas (or sequents); i.e., every valid positive formula is provable in* **Π^1PC**. *This implies that the cut-rule is admissible in* **Π^1PC**, *i.e., if $\Gamma \to \Delta, D$ and $D, \Gamma \to \Delta$ are provable in* **Π^1PC**, *then so is $\Gamma \to \Delta$.*

PROOF. This can be proved by following the proof of Theorem 8.2 (the completeness of **LK**); namely, construct the reduction tree of a given positive sequent, and if there is an infinite branch, define a structure in which the sequent is false. For the induction steps in the construction of the tree, the only new case is the step in which formulas whose outermost logical symbol is $\forall\varphi$ are under consideration. These formulas occur only in the succedent of a sequent since the sequent is positive.

Thus (for this step) suppose the sequent $\Pi \to \Lambda$ is under consideration, and let $\forall\varphi_1\, F(\varphi_1), \ldots, \forall\varphi_n\, F(\varphi_n)$ be all the formulas in Λ whose outermost logical symbol is $\forall$ (second order). Write the sequent $\Pi \to \Lambda, F_1(\alpha_1), \ldots, F_n(\alpha_n)$ above this, where $\alpha_1, \ldots, \alpha_n$ are the first n (second order) free variables not used yet. Note that if $F(\alpha)$ is false in a structure, then so is $\forall\varphi\, F(\varphi)$.

PROBLEM 16.19. Let L be the language consisting of 0, ′, =, and <, and let Γ_0 be the first order Peano axioms without mathematical induction for this language. We also assume that every axiom in Γ_0 is in prenex normal form and no $\exists$ occur in Γ_0. Let $N(a)$ be an abbreviation of the formula:

$$\forall\varphi\,(\varphi(0) \wedge \forall x\,(\varphi(x) \supset \varphi(x'))$$
$$\wedge\, \forall x\, \forall y\, (x = y \wedge \varphi(x) \supset \varphi(y)) \supset \varphi(a)).$$

If $\Gamma_0 \to F(\{x\}N(x))$ is provable in **Π^1PC**, then there exists a formula $A(a)$ of the form $0 = 0$ or $a = \bar{n}_1 \vee a = \bar{n}_2 \vee \ldots \vee a = \bar{n}_k$ for some numerals $\bar{n}_1, \ldots, \bar{n}_k$, such that $\Gamma_0 \to F(\{x\}A(x))$ is provable in **Π^1PC**.

PROOF. By Problem 16.14, there exists a first order formula $A(a)$ of L such that

1) a is the only free variable in $A(a)$,
2) $\Gamma_0, A(a) \to N(a)$ is provable in $\mathbf{\Pi^1 PC}$,
3) $\Gamma_0 \to F(\{x\}A(x))$ is provable in $\mathbf{\Pi^1 PC}$.

By a well-known decision method, we can assume that $A(a)$ has one of the following forms:

a) $0 = 0$,
b) $0 = 1$,
c) $a = \bar{n}_1 \vee a = \bar{n}_2 \vee \ldots \vee a = \bar{n}_k$,
d) $\bar{n} < a \vee a = \bar{n}_1 \vee \ldots \vee a = \bar{n}_k$.

Case 1. $A(a)$ is $0 = 1$.

Since the occurrences of N are positive in $F(\{x\}N(x))$, the provability of $\Gamma_0 \to F(\{x\}(0 = 1))$ implies the provability of $\Gamma_0 \to F(\{x\}(0 = 0))$.

Case 2. $A(a)$ is $\bar{n} < a \vee a = \bar{n}_1 \vee \ldots \vee a = \bar{n}_k$.

In this case $\Gamma_0, \bar{n} < a \to N(a)$ is provable by 2). Hence the following sequent is provable:

$$\Gamma_0, \bar{n} < a, \alpha(0), \forall x\, (\alpha(x) \supset \alpha(x')),$$
$$\forall x\, \forall y\, (x = y \wedge \alpha(x) \supset \alpha(y)) \to \alpha(a).$$

Now introduce a new individual constant ω and substitute $a < \omega$ for $\alpha(a)$. Then we have a proof of the sequent:

$$\Gamma_0, 0 < \omega, \forall x\, (x < \omega \supset x' < \omega) \to .$$

The usual interpretation of $<$ on $0, 0', 0'', \ldots, \omega, \omega', \omega'', \ldots$ shows that this is a contradiction.

PROBLEM 16.20 (Kreisel). Consider the system $\mathbf{PA}'$ of predicative second order arithmetic defined in Problem 16.10. In order to facilitate the use of some results in recursive function theory, we use the following notation: $\forall f\, A(f)$ (resp. $\exists f\, A(f)$) is an abbreviation for $\forall \varphi\, (\varphi$ is a function $\supset A^*(\varphi))$ (resp. $\exists \varphi\, (\varphi$ is a function and $A^*(\varphi)))$, where "φ is a function" is expressed by

$$\forall x\, \exists y\, \forall z\, (\varphi(x, z) \equiv y = z)$$

and $A(f)$ and $A^*(\varphi)$ are related in a manner such that subformulas of $A(f)$ of the form $B(f(t))$ are (systematically) replaced in A^* by $\exists y\, (\varphi(t, y) \wedge B(y))$.

Next we define a Π_1^1 (resp. Σ_1^1) formula as one of the form $\forall\varphi\, A(\varphi)$ (resp. $\exists\varphi\, A(\varphi)$) where A is arithmetical. Any Π_1^1 formula is equivalent (in **PA**′) to one in "Π_1^1-normal form", i.e., $\forall f\, \exists y\, R(\bar{f}y)$, for some R which is primitive recursive (or more strictly, primitive recursive relative to the free second order variables in the formula), where $\bar{f}y$ is (the Gödel number of) the sequence $\langle f(0), \ldots, f(y-1)\rangle$. Similarly, any Σ_1^1 formula can be transformed to Σ_1^1-normal form: $\exists f\, \forall y\, R(\bar{f}y)$, for suitable primitive recursive R. Finally, a Π_1^1 predicate or relation (of k number variables say) is one which can be expressed by a Π_1^1 formula with k free first order variables (and no free second order variables). Similarly for Σ_1^1 predicates or relations.

Now let $<\cdot$ be a Σ_1^1-ordering, i.e., $<\cdot$ is a Σ_1^1 binary relation which is a linear ordering of natural numbers. Let $We(<\cdot)$ express that $<\cdot$ is a well-ordering: $\forall f\, \exists x\, \neg(f(x+1) <\cdot f(x))$. Suppose $We(<\cdot)$ is provable in **PA**′, i.e., $<\cdot$ is a provable well-ordering of **PA**′. Show that the ordinal of $<\cdot$ (the order type of $<\cdot$) is less than ε_0. [*Hint*: First express $a <\cdot b$ in Σ_1^1-normal form: $\exists f\, \forall y\, R(\bar{f}y, ab)$. Now follow the steps listed below.

1) Let $<_n$ be an enumeration of primitive recursive, binary relations (for $n = 0, 1, 2, \ldots$), and let $W(x)$ denote $We(<_x)$. Then $W(x)$ is a (provably) complete Π_1^1-form, viz. there is a primitive recursive function $S(r, x)$ such that for every Π_1^1-predicate $A(x)$, there exists a number r_0 such that $\forall x\, (A(x) \equiv W(S(\bar{r}_0, x)))$ is **PA**′-provable.

2) For any Σ_1^1 predicate $B(x)$ there is a number n such that $B(\bar{n}) \equiv \neg W(\bar{n})$ is **PA**′-provable. (We formalize an argument in recursion theory showing that a Σ_1^1 predicate cannot be Π_1^1-complete).

3) Let $W_1(x)$ be the formula expressing that there is an embedding of $<_x$ into $<\cdot$, i.e. an order-preserving function from the domain of $<_x$ to the domain of $<\cdot$ (which implies that $<_x$ is a well-ordering, with ordinal less than or equal to that of $<\cdot$). Then $W_1(x)$ is Σ_1^1, and so there is a formula $W_1^*(x)$ in Σ_1^1-normal form such that $\forall x\, (W_1(x) \equiv W_1^*(x))$ is **PA**′-provable.

4) Since $We(<\cdot)$ is provable, $\forall x\, (W_1^*(x) \supset W(x))$ is provable.

5) By 2), there is an n such that $W_1^*(\bar{n}) \equiv \neg W(\bar{n})$ is provable. Hence by 4), $W(\bar{n})$ and $\neg W_1^*(\bar{n})$ are provable.

6) $W(\bar{n})$ being provable in **PA**′ means that the primitive recursive relation $<_n$ is a provable well-ordering of **PA**′, hence of **PA**. By Gentzen's result in the previous chapter, this implies that the ordinal of $<_n$ is less than ε_0.

7) $W(\bar{n})$ and $\neg W_1^*(\bar{n})$ (see 5)) means that $<_n$ is a primitive recursive well-ordering which is not embeddable in $<\cdot$, and hence the ordinal of $<\cdot$ is less than that of $<_n$. This and 6) yield that the order type of $<\cdot$ is less than ε_0.]

§17. The theory of relativization

DEFINITION 17.1. A *system of relativization* consists of a pair of formulas $R^0(a)$ and $R^1(\alpha)$, where $R^0(a)$ and $R^1(\alpha)$ each have the one free variable a and α, respectively. One or both of $R^0(a)$ and $R^1(\alpha)$ may be missing. A system of relativization is often denoted by r.

For an arbitrary (second order) semi-formula A, A^r (the relativization of A to r) is defined inductively as follows. (Here it is assumed that r consists of two formulas. If one or both of R^0 and R^1 is missing, then the definition should be adjusted accordingly.)

1) If A has no logical symbol, then A^r is A.

2) $(A \wedge B)^r$, $(A \vee B)^r$, $(A \supset B)^r$, $(\neg A)^r$ are, respectively,

$$A^r \wedge B^r, A^r \vee B^r, A^r \supset B^r, \neg A^r.$$

3) $(\forall x\, F(x))^r$ and $(\exists x\, F(x))^r$ are $\forall y\, (R^0(y) \supset F^r(y))$ and $\exists y\, (R^0(y) \wedge F^r(y))$, respectively, where $F^r(y)$ is $(F(y))^r$ and y is a variable which does not occur in $R^0(a)$ or $F^r(a)$.

4) $(\forall \varphi\, F(\varphi))^r$ and $(\exists \varphi\, F(\varphi))^r$ are $\forall \psi\, (R^1(\psi) \supset F^r(\psi))$ and $\exists \psi\, (R^1(\psi) \wedge F^r(\psi))$, respectively, where $F^r(\psi)$ is $(F(\psi))^r$ and ψ is a variable which does not occur in $R^1(\alpha)$ or $F^r(\alpha)$.

5) $(\{y_1, \ldots, y_n\}A(y_1, \ldots, y_n))^r$ is $\{y_1, \ldots, y_u\}(A(y_1, \ldots, y_n))^r$.

LEMMA 17.2. (1) *If A has no quantifiers, than A^r is A.*

(2) *A free variable occurs in A if and only if it occurs in A^r.*

(3) *A bound variable occurs free in A if and only if it occurs free in A^r.*

(4) *A is a formula if and only if A^r is a formula.*

(5) *Let $A(t)$ denote $A(a)\,\binom{a}{t}$; then $(A(t))^r$ is the same as $A^r(t)$, i.e., $(A(a))^r\,\binom{a}{t}$ ("the same", that is, up to bound occurrences of bound variables).*

(6) *Let $A(V)$ denote $A(\alpha)\,\binom{\alpha}{V}$, then $(A(V))^r$ is the same as $A^r(V^r)$, i.e., $(A(\alpha))^r\,\binom{\alpha}{V^r}$ (again, up to bound occurrences of bound variables).*

PROOF. By mathematical induction on the number of logical symbols in A. (1)–(5) are left to the reader.

(6): Let V be $\{y\}C(y)$. (For the sake of simplicity we assume that V has only one argument place.) If $A(\alpha)$ is $\alpha(t)$, then $(A(V))^r$ is $(C(t))^r$ and $(A(\alpha))^r\,\binom{\alpha}{V^r}$ is $(\alpha(t))^r\,\binom{\alpha}{V^r}$, i.e., $C^r(t)$. These are the same by (5).

Suppose $A(\alpha)$ is $\forall x\, F(x, \alpha)$. $(\forall x\, F(x, V))^r$ is $\forall y\, (R^0(y) \supset (F(y, V))^r)$. By the induction hypothesis, this is the same as $\forall y\, (R^0(y) \supset F^r(y, \alpha)\,\binom{\alpha}{V^r})$, i.e., $(\forall x\, F(x, \alpha))^r\,\binom{\alpha}{V^r}$.

Suppose $A(\alpha)$ is $\forall\varphi\, F(\varphi, \alpha)$. $(\forall\varphi\, F(\varphi, V))^r$ is $\forall\psi\, (R^1(\psi) \supset (F(\psi, V))^r)$. By the induction hypothesis, this is the same as $\forall\psi\, (R^1(\psi) \supset F^r(\psi, \alpha)\, \binom{\alpha}{Vr})$, i.e., $(\forall\varphi\, F(\varphi, \alpha))^r\, \binom{\alpha}{Vr}$.

The other cases are left to the reader.

Definition 17.3. (1) If Γ is a sequence of formulas $A_1, \ldots, A_m$, then Γ^r denotes $A_1^r, \ldots, A_m^r$. For the sake of simplicity, we write both R^0 and R^1 as r. It will be obvious which is meant, since $r(a)$ denotes $R^0(a)$ and $r(\alpha)$ denotes $R^1(\alpha)$.

(2) Suppose a system of relativization r is given. Then Φ is the set of the following formulas.

1) $r(c)$ for every individual constant c (in the language).
2) $\forall y_1 \ldots \forall y_m\, (r(y_1) \wedge \ldots \wedge r(y_m)) \supset r(f(y_1, \ldots, y_m)))$ for every function constant f.
3) $\exists x\, r(x)$.
4) $\exists\varphi\, r(\varphi)$.

Example 17.4. The language includes $=$ as a distinguished binary predicate constant. Suppose r consists of R^1 only, where $R^1(\alpha)$ is defined to be

$$\forall x\, \forall y\, (x = y \wedge \alpha(x) \supset \alpha(y)).$$

Let $\mathscr{B}$ be the following axiom system:

$$\forall x\, (x = x),$$

$$\forall x\, \forall y\, (x = y \supset y = x),$$

$$\forall x\, \forall y\, \forall z\, (x = y \wedge y = z \supset x = z),$$

$$\forall x_1 \ldots \forall x_m\, \forall y_1 \ldots \forall y_m\, (x_1 = y_1 \wedge \ldots \wedge x_m = y_m \supset f(x_1, \ldots, x_m) = f(y, \ldots, y_m)) \text{ for every } f,$$

$$\forall x_1 \ldots \forall x_m \forall y_1 \ldots \forall y_m\, (x_1 = y_1 \wedge \ldots \wedge x_m = y_m \wedge P(x_1, \ldots, x_m) \supset P(y_1, \ldots, y_m)) \text{ for every } P.$$

To apply the theory below to this example, we want to check that $\mathscr{B} \to A$ is provable in the systems considered for every A in Φ. Since R^0 is missing, we only have to consider 4). It is a routine matter to see that this condition is satisfied. (Further, r and $\mathscr{B}$ also satisfy condition 5) in Lemma 17.5.)

LEMMA 17.5. *Let* S *be* **KC**, *where* K is *an arbitrary set of formulas which is closed under substitution* (*cf. Definitions* 15.14 *and* 15.15). *Suppose the system of relativization* r *satisfies the condition that for an arbitrary* K*-abstract* V, V^r *is also a* K*-abstract.* (*This is satisfied if, e.g.,* K *consists of all formulas in the language.*) *Let* $\mathscr{B}$ *be an axiom system such that* $\mathscr{B} \to A$ *is* S*-provable for every* A *in* Φ *and*

5) $r(b), \ldots, r(\beta), \ldots, \mathscr{B} \to r(V^r)$ *is* S*-provable for every* K*-abstract* V, *where* $b, \ldots$ *and* $\beta, \ldots$ *are all the free variables which occur in* V (*and hence in* V^r: *cf. part* (2) *of Lemma* 17.2).

Then for an arbitrary sequent $\Gamma \to \Theta$ *which is* S*-provable,*

$$r(a), \ldots, r(\alpha), \ldots, \mathscr{B}, \Gamma^r \to \Theta^r$$

is S*-provable, where* $a, \ldots, \alpha, \ldots$ *are all the free variables which occur in* Γ, Θ.

We first prove the following sublemma.

SUBLEMMA 17.6. *If* s *is a term, then* $r(b), \ldots, \mathscr{B} \to r(s)$ *is* S*-provable, where* $b, \ldots$ *are all the* (*free first order*) *variables in* s.

This is proved by mathematical induction on the number of function constants in s.

PROOF OF LEMMA 17.5. This is proved by mathematical induction on the number of inferences in a proof ending with $\Gamma \to \Theta$.

1) The proof consists of an initial sequent $D \to D$. Then $D^r \to D^r$ is also an initial sequent. Therefore $\mathscr{B}, D^r \to D^r$ is obviously S-provable, and hence so is $r(a), \ldots, r(\alpha), \ldots, \mathscr{B}, D^r \to D^r$.

2) The last inference is a first order $\forall$: left:

$$\frac{F(s), \Gamma' \to \Theta}{\forall x\, F(x), \Gamma' \to \Theta}.$$

By the induction hypothesis,

$$r(a), \ldots, r(\alpha), \ldots, \mathscr{B}, F^r(s), \Gamma'^r \to \Theta^r$$

is S-provable (cf. part (5) of Lemma 17.2). Also $r(b), \ldots, \mathscr{B} \to r(s)$ by Sublemma 17.6 (where the variables $b, \ldots$ are included among $a, \ldots$). Therefore

$$r(a), \ldots, r(\alpha), \ldots, \mathscr{B}, \forall x\, (r(x) \supset F^r(x)), \Gamma'^r \to \Theta^r.$$

3) The last inference is a first order $\forall$: right:

$$\frac{\Gamma \rightarrow \Theta', F(d)}{\Gamma \rightarrow \Theta', \forall x\, F(x)}.$$

By the induction hypothesis,

$$r(a), \ldots, r(\alpha), \ldots, \mathscr{B}, \Gamma^r \rightarrow \Theta'^r, F^r(d),$$

where d does not occur in the antecedent. So

$$r(a), \ldots, r(\alpha), \ldots, \mathscr{B}, \Gamma^r \rightarrow \Theta'^r, \forall x\, (r(x) \supset F^r(x)).$$

4) The last inference is a second order $\forall$: left:

$$\frac{F(V), \Gamma' \rightarrow \Theta}{\forall \varphi\, F(\varphi), \Gamma' \rightarrow \Theta},$$

where V is a K-abstract. By the induction hypothesis,

$$r(a), \ldots, r(\alpha), \ldots, \mathscr{B}, F^r(V^r), \Gamma'^r \rightarrow \Theta^r$$

(cf. part (6) of Lemma 17.2). Hence

$$r(a), \ldots, r(\alpha), \ldots, \mathscr{B} \rightarrow r(V^r)$$

by condition 5) (since $a, \ldots, \alpha, \ldots$ include all the free variables in V). So

$$r(a), \ldots, r(\alpha), \ldots, \mathscr{B}, r(V^r) \supset F^r(V^r), \Gamma'^r \rightarrow \Theta^r.$$

Also by the condition on K, V^r is a K-abstract, so that from the last sequent

$$r(a), \ldots, r(\alpha), \ldots, b, \forall \varphi\, (r(\varphi) \supset F^r(\varphi)), \Gamma'^r \rightarrow \Theta^r$$

is **S**-provable.

Other cases are treated similarly.

Definition 17.7. (1) An *axiom system* (in this section) is defined as a set of formulas which do not contain any free first order variables.

(2) Let $\mathscr{A}$ be an arbitrary axiom system and let r be a system of relativization. $\mathscr{A}^r$ is the set of formulas A^r for all A in $\mathscr{A}$.

Theorem 17.8. *Let $\mathscr{A}$ and $\mathscr{B}$ be axiom systems. Suppose that the formal system* **S** *and the axiom system $\mathscr{B}$ satisfy the conditions of Lemma* 17.5, *and, further*:

for every formula A of $\mathscr{A} \cup \mathscr{B}$, $\mathscr{B} \to A^r$ is S*-provable; and for every second order free variable α contained in $\mathscr{A} \cup \mathscr{B}$, $\mathscr{B} \to r(\alpha)$ is* S*-provable. Then:*

(1) *for any formula B, if $\mathscr{A} \cup \mathscr{B} \to B$ is* S*-provable, then so is $\mathscr{B} \to B^r$;*

(2) *if $\mathscr{B}$ is consistent (with* S*), then so is $\mathscr{A} \cup \mathscr{B}$.*

NOTE. We can express this result (part (1)) by saying that $\mathscr{A} \cup \mathscr{B}$ is interpretable in $\mathscr{B}$ (relative to S), or more accurately that r provides an interpretation (or "inner model") of $\mathscr{A} \cup \mathscr{B}$ in $\mathscr{B}$ (relative to S).

PROOF. (1) Suppose $\mathscr{A} \cup \mathscr{B} \to B$ (in S). Then there are finite sequences Γ and Δ from $\mathscr{A}$ and $\mathscr{B}$, respectively, such that $\Gamma, \Delta \to B$ (in S). So by Lemma 17.5,

$$r(\alpha), \ldots, \Gamma^r, \Delta^r \to B^r \text{ (in S)}.$$

(Recall that Γ and Δ contain no free first order variables.) But by our assumption, $\mathscr{B} \to A^r$ and $\mathscr{B} \to r(\alpha)$ are provable for every formula A and second order variable α in $\Gamma \cup \Delta$. Hence $\mathscr{B} \to B^r$ (in S).

Part (2) is proved from (1), by taking, for B, say $C \wedge \neg C$ (since $(C \wedge \neg C)^r$ is $C^r \wedge \neg C^r$).

DEFINITION 17.9. For the following theorem, let L′ denote the second order language with constants 0, ′ and $=$, and let $\mathscr{A}_0$ denote the following axioms (for arithmetic) in this language:

$$\forall x\, (\neg x' = 0),$$

$$\forall x\, \forall y\, (x = y \supset x' = y'),$$

$$\forall x\, \forall y\, (x' = y' \supset x = y),$$

$$\forall x\, (x = x),$$

$$\forall x\, \forall y\, (x = y \supset y = x),$$

$$\forall x\, \forall y\, \forall z\, (x = y \wedge y = z \supset x = z).$$

THEOREM 17.10 (relative consistency of classical analysis). *Consider classical analysis, formalized as* $\mathbf{G^1LC}$ *together with the axiom system $\mathscr{A}_0 \cup \{\mathrm{Eq}', \mathrm{VJ}'\}$ in the language* L′. *(*Eq′ *and* VJ′ *were defined in Problem* 16.10.*) Then:*

(1) *it is interpretable in $\mathscr{A}_0$ (relative to* $\mathbf{G^1LC}$*);*

(2) *it is consistent, assuming cut-elimination for* $\mathbf{G^1LC}$.

PROOF. The interpretation is carried out in two steps.

(i) $\mathscr{A}_0 \cup \{\mathrm{Eq}'\}$ is interpreted in $\mathscr{A}_0$ (relative to $\mathbf{G^1LC}$) by Theorem 19, with r defined by: $R^1(\alpha)$ is $\forall x\, \forall y\, (x = y \wedge \alpha(x) \supset \alpha(y))$ (and no R^0). In fact, taking V as $\{u\}(u = 0)$, we can prove in $\mathbf{G^1LC}$ $\mathscr{A}_0 \to r(V)$, and hence $\mathscr{A}_0 \to \exists\varphi\, r(\varphi)$. Further, condition 5) of Lemma 17.5 is easily proved by induction on the logical complexity of V; and Eq'^r is provable in $\mathbf{G^1LC}$.

(ii) Next, $\mathscr{A}_0 \cup \{\mathrm{Eq}' \cup \mathrm{VJ}'\}$ is interpreted in $\mathscr{A}_0 \cup \{\mathrm{Eq}'\}$, again by Theorem 17.8, this time with r defined by:

$$R^0(a) \text{ is } \quad \forall\varphi\, (\varphi(0) \wedge \forall y\, (\varphi(y) \supset \varphi(y')) \supset \varphi(a)) \quad \text{(and no } R^1\text{)}.$$

Now $r(0)$, and hence $\exists x\, r(x)$, are easily proved in $\mathbf{G^1LC}$. Further, for any formula A of $\mathscr{A}_0 \cup \{\mathrm{Eq}'\}$, the sequent $\mathscr{A}_0, \mathrm{Eq}' \to A^r$ is provable in $\mathbf{G^1LC}$; and so is $\mathscr{A}_0, \mathrm{Eq}' \to \mathrm{VJ}'^r$.

Thus part (1) is proved. In fact the two steps could be combined, so as to give an interpretation of $\mathscr{A}_0 \cup \{\mathrm{Eq}', \mathrm{VJ}'\}$ directly in $\mathscr{A}_0$, by means of a single system of relativization r. The reader is invited to define such an r.

To prove part (2), we first show that if $\mathscr{A}_0$ is consistent (with $\mathbf{G^1LC}$), then so is $\mathscr{A}_0 \cup \{\mathrm{Eq}', \mathrm{VJ}'\}$. The method is parallel to that for part (1), using Theorem 17.8 part (2) twice.

The argument is completed by showing that $\mathscr{A}_0$ is consistent (with $\mathbf{G^1LC}$), assuming cut-elimination for $\mathbf{G^1LC}$. But this is clear, since a proof of $\mathscr{A}_0 \to$ in $\mathbf{G^1LC}$ without a cut would in fact be a proof in $\mathbf{LK}$, which is impossible.

§18. Truth definition for first order arithmetic

DEFINITION 18.1. (1) Although we have mentioned second order arithmetic from time to time we shall now formulate it more systematically. The language of the systems of second order arithmetic is that of **PA** (cf. §9) augmented by second order variables. The basic logical system is **BC** (cf. Definition 15.3), and the axioms (i.e., the mathematical initial sequents) are those of **PA** and the generalized equality axioms:

$$s = t, A(s) \to A(t)$$

for arbitrary terms s and t and arbitrary formulas A. The various systems of second order arithmetic are classified according to the forms of the induction and comprehension axioms. They are both introduced as rules of inference rather than axioms, and if both are allowed for all formulas and abstracts, then the system will represent classical analysis. In order to simplify the arguments, we assume only the logical symbols $\neg$, $\wedge$, $\forall$, although other symbols may be used occasionally. We recall that the rule of induction or "ind" has the form

$$\frac{F(a), \Gamma \to \Theta, F(a')}{F(0), \Gamma \to \Theta, F(s)},$$

where a does not occur in Γ, Θ or $F(0)$, and s is an arbitrary term. $F(a)$ is called the induction formula, and a is called the eigenvariable, of this inference. The comprehension axiom, or $\forall$: left rule, has the form

$$\frac{F(V), \Gamma \to \Theta}{\forall\varphi\, F(\varphi), \Gamma \to \Theta},$$

where V and φ have the same number of argument places.

We normally deal with systems where the induction formulas belong to a certain class of formulas which is closed under substitution (cf. Definition 15.14, part (3)) and the abstracts for $\forall$: left also belong to a certain class, closed under substitution. If the induction formula or the abstract for $\forall$: left is restricted to a set K, then the corresponding ind or $\forall$: left is called a K-ind or a K-comprehension axiom, respectively.

(2) Formulas of second order arithmetic which do not contain second order quantifiers are called arithmetical. Also, abstracts are called arithmetical if they are formed from arithmetical formulas.

(3) Let Π^1_i be the class of formulas of the form $\forall\varphi_1\, \exists\varphi_2 \ldots \varphi_i F(\varphi_1, \varphi_2, \ldots, \varphi_i)$, where $\forall\varphi_1\, \exists\varphi_2 \ldots \varphi_i$ denotes a string of i alternating quantifiers with second order bound variables starting with $\forall$, and F is arithmetical. The closure of Π^1_i under substitution will be called Π^1_i (in wider sense). Σ^1_i and Σ^1_i (in wider sense) are defined likewise (with $\exists\varphi_1\, \forall\varphi_2 \ldots \varphi_i$ instead of $\forall\varphi_1\, \exists\varphi_2 \ldots \varphi_i$). (For $i = 1$, this is essentially the same as the definition in Problem 16.20, which used function instead of predicate quantification, since predicates or sets can be represented by their characteristic functions.)

The following are straightforward consequences of the definition.

PROPOSITION 18.2. (1) *The Π^1_i-comprehension axiom and the Π^1_i (in wider sense)-comprehension axiom, are equivalent (in* **BC**). *Similarly with the Σ^1_i-comprehension axiom.*

(2) *The Π^1_i-comprehension axiom, and Σ^1_i-comprehension axiom, are equivalent (in* **BC**).

This proposition enables us to identify the Π_i^1-, the Π_i^1 (in wider sense)-, the Π_i^1-, and the Σ_i^1 (in wider sense)-comprehension axioms. Therefore we shall call them all the Π_i^1-comprehension axiom.

DEFINITION 18 3. We assume a standard Gödel numbering of **PA**, and, if X is a formal object of **PA**, then $\ulcorner X \urcorner$ denotes the Gödel number of X (cf. §9 and §10). We shall list the primitive recursive functions and predicates we need.

ls(a): the number of logical symbols in a.

fl(a): a is a formula of **PA**.

st(a): a is a sentence (i.e., closed formula) of **PA**.

tm(a): a is a term.

ct(a): a is a closed term.

sub$(\ulcorner A \urcorner, \ulcorner a_i \urcorner, \ulcorner t \urcorner)$: the result of substituting t for a_i in A. This may be denoted by $\ulcorner A(t) \urcorner$.

$v(\ulcorner t \urcorner)$: the value of t (if t is closed).

$\nu(a)$: Gödel number of the ath numeral.

Abbreviated notions:

$\forall \ulcorner A \urcorner (\ldots \ulcorner A \urcorner \ldots)$: $\forall x\,(\mathrm{fl}(x) \supset \ldots x \ldots)$,

$\forall \ulcorner A \wedge B \urcorner (\ldots \ulcorner A \wedge B \urcorner \ldots)$: $\forall x\,(\mathrm{fl}(x) \wedge$ "the outermost logical symbol of x is $\wedge$" $\supset \ldots x \ldots)$,

$\forall \ulcorner t \urcorner (\ldots \ulcorner t \urcorner \ldots)$: $\forall x\,(\mathrm{tm}(x) \supset \ldots x \ldots)$.

Also we will write, for terms $t(a_i)$ and formulas $A(a_i)$:

$\ulcorner t(\nu(b)) \urcorner$ for sub$(\ulcorner t(a_i) \urcorner, \ulcorner a_i \urcorner, \nu(b))$,

$\ulcorner A(\nu(b)) \urcorner$ for sub$(\ulcorner A(a_i) \urcorner, \ulcorner a_i \urcorner, \nu(b))$.

In this section, $\mathbf{S}^1$ denotes second order arithmetic with the arithmetical comprehension axiom and ind applied to Π_1^1 (in wider sense) formulas.

PA′ is the system of second order arithmetic with the arithmetical ind and the arithmetical comprehension axiom. (This is clearly equivalent to the system of predicative arithmetic, also denoted by **PA′**, defined in Problem 16.10.)

In order to avoid too many parentheses and brackets, we use dots for punctuation in a well-known manner; $A \mathrel{.\supset.} B \equiv C$, for instance, means $A \supset (B \equiv C)$.

This section is devoted to defining the truth definition for **PA**. The argument is carried out within $\mathbf{S}^1$.

DEFINITION 18.4. $F(\alpha, n)$ stands for the following formula:

$$\forall \ulcorner t_1 \urcorner\, \forall \ulcorner t_2 \urcorner\, [\mathrm{ct}(\ulcorner t_1 \urcorner) \wedge \mathrm{ct}(\ulcorner t_2 \urcorner) \supset$$
$$\supset (\alpha(\ulcorner t_1 = t_2 \urcorner) \equiv v(\ulcorner t_1 \urcorner) = v(\ulcorner t_2 \urcorner))]$$
$$\wedge \forall \ulcorner A \urcorner\, \forall \ulcorner B \urcorner\, [\mathrm{st}(\ulcorner A \wedge B \urcorner) \wedge \mathrm{ls}(\ulcorner A \wedge B \urcorner) \leqslant n \supset$$
$$\supset (\alpha(\ulcorner A \wedge B \urcorner) \equiv \alpha(\ulcorner A \urcorner) \wedge \alpha(\ulcorner B \urcorner))]$$
$$\wedge \forall \ulcorner \neg A \urcorner\, [\mathrm{st}(\ulcorner \neg A \urcorner) \wedge \mathrm{ls}(\ulcorner \neg A \urcorner) \leqslant n \supset$$
$$\supset (\alpha(\ulcorner \neg A \urcorner) \equiv \neg\alpha(\ulcorner A \urcorner))]$$
$$\wedge \forall \ulcorner \forall x_i\, A(x_i) \urcorner\, [\mathrm{st}(\ulcorner \forall x_i\, A(x_i) \urcorner) \wedge \mathrm{ls}(\ulcorner \forall x_i\, A(x_i) \urcorner) \leqslant n \supset$$
$$\supset (\alpha(\ulcorner \forall x_i\, A(x_i) \urcorner) \equiv \forall x\, \alpha(\ulcorner A(\nu(x)) \urcorner))].$$

$F(\alpha, n)$ means: α is a truth definition for sentences of complexity $\leqslant n$. In fact, a predicate T_n, satisfying $F(\{y\}T_n(y), \bar{n})$, was defined in §14 for each n (separately). However, we can now go further, and give a truth definition for all sentences, namely:

$$T(a) \text{ abbreviates } \mathrm{st}(a) \wedge \exists\varphi\, (F(\varphi, \mathrm{ls}(a)) \wedge \varphi(a)).$$

(Note: This is not a "truth definition" according to Definition 10.10. We are generalizing the notion of truth definition. For the meaning of this, see Theorem 18.13.)

LEMMA 18.5. (1) *In* **PA**$'$:

$$F(\alpha, n), F(\beta, n), \mathrm{st}(\ulcorner A \urcorner), \mathrm{ls}(\ulcorner A \urcorner) \leqslant n \to \alpha(\ulcorner A \urcorner) \equiv \beta(\ulcorner A \urcorner).$$

(This states that any α for which $F(\alpha, n)$ holds, is unique, at least with regard to the sentences whose complexities are $\leqslant n$.)

(2) $F(\alpha, n), m \leqslant n \to F(\alpha, m)$ *in* **PA**$'$.

(3) $F(\alpha, n), E(\beta, \alpha, n) \to F(\beta, n+1)$ *in* **PA**$'$, *where* $E(\alpha, \beta, n)$ *is an abbreviation of the following*:

$$\forall x\, (\beta(x) \equiv [\mathrm{st}(x) \wedge \mathrm{ls}(x) \leqslant n \wedge \alpha(x)]$$
$$\vee [\mathrm{st}(x) \wedge \mathrm{ls}(x) = n + 1$$
$$\wedge \{\exists \ulcorner A \urcorner\, (x = \ulcorner \neg A \urcorner \wedge \neg\alpha(\ulcorner A \urcorner))$$

$$\vee \; \exists \ulcorner A \urcorner \; \exists \ulcorner B \urcorner \; (x = \ulcorner A \wedge B \urcorner \wedge \alpha(\ulcorner A \urcorner) \wedge \alpha(\ulcorner B \urcorner))$$
$$\vee \; \exists \ulcorner \forall x_i \, A(x_i) \urcorner \; (x = \ulcorner \forall x_i \, A(x_i) \urcorner \wedge \forall y \, \alpha(\ulcorner A(\nu(y)) \urcorner))\}]).$$

(*This means that* β *is an extension of* α *to the sentences of complexity* $n + 1$.)

(4) $\forall n \, \forall \varphi \, \exists \psi \, E(\psi, \varphi, n)$ *in* **PA**$'$. (*The existence of an extension.*)

(5) $G(\alpha) \to F(\alpha, 0)$ *in* **PA**$'$, *where* $G(\alpha)$ *is an abbreviation of*

$$\forall x \; (\alpha(x) \equiv \exists \ulcorner t_1 \urcorner \; \exists \ulcorner t_2 \urcorner \; [\mathrm{ct}(\ulcorner t_1 \urcorner) \wedge \mathrm{ct}(\ulcorner t_2 \urcorner)$$
$$\wedge \; (x = \ulcorner t_1 = t_2 \urcorner \wedge v(\ulcorner t_1 \urcorner) = v(\ulcorner t_2 \urcorner))]).$$

(*Truth definition for* $n = 0$.)

PROPOSITION 18.6. $\forall n \, \exists \varphi \, F(\varphi, n)$ *in* $\mathbf{S}^1$.

PROOF. By an application of ind, with induction formula $\exists \varphi \, F(\varphi, a)$, which is Σ^1_1, or Π^1_1 (in wider sense). Use (4), (2) and (3) of Lemma 18.5.

PROPOSITION 18.7. $T(a) \equiv \mathrm{st}(a) \wedge \forall \varphi \, (F(\varphi, \mathrm{ls}(a)) \supset \varphi(a))$ *in* $\mathbf{S}^1$.

PROOF. Use (1) of Lemma 18.5, and Proposition 18.6.

The following propositions (18.8–18.10) assert that T commutes with logical symbols.

PROPOSITION 18.8. $\mathrm{st}(\ulcorner A \urcorner) \to T(\ulcorner \neg A \urcorner) \equiv \neg T(\ulcorner A \urcorner)$ *in* $\mathbf{S}^1$.

PROPOSITION 18.9. $\mathrm{st}(\ulcorner B \urcorner) \wedge \mathrm{st}(\ulcorner C \urcorner) \to T(\ulcorner B \wedge C \urcorner) \equiv T(\ulcorner B \urcorner) \wedge T(\ulcorner C \urcorner)$ *in* $\mathbf{S}^1$.

PROPOSITION 18.10. $\mathrm{st}(\ulcorner \forall x_i \, B(x_i) \urcorner) \to T(\ulcorner \forall x_i \, B(x_i) \urcorner) \equiv \forall x \, T(\ulcorner B(\nu(x)) \urcorner)$ *in* $\mathbf{S}^1$.

These propositions follow from Proposition 18.6 and (1) and (2) of Lemma 18.5. It should be noted that if we assume Proposition 18.6, then the argument goes through in **PA**$'$.

LEMMA 18.11. (1) $\mathrm{ct}(\ulcorner t_1 \urcorner), \; \mathrm{ct}(\ulcorner t_2 \urcorner) \to T(\ulcorner t_1 = t_2 \urcorner) \equiv v(\ulcorner t_1 \urcorner) = v(\ulcorner t_2 \urcorner)$ *in* **PA**$'$.

(2) $v(\ulcorner \nu(b) \urcorner) = b$ *in* **PA**.

(3) *Let $t(a_1, \ldots, a_k)$ be a term with at most $a_1, \ldots, a_k$ as free variables. Then*

$$v(\ulcorner t(\nu(b_1), \ldots, \nu(b_k)) \urcorner) = t(b_1, \ldots, b_k)$$

in **PA**, *where $b_1, \ldots, b_k$ are arbitrary free variables.*

PROPOSITION 18.12. *Let $A(a_1, \ldots, a_k)$ be a formula of* **PA** *with at most $a_1, \ldots, a_k$ as free variables. Then*

$$T(\ulcorner A(\nu(b_1), \ldots, \nu(b_k)) \urcorner) \equiv A(b_1, \ldots, b_k) \text{ in } \mathbf{S}^1.$$

PROOF. By mathematical induction on the complexity of A. Use (3) of Lemma 18.11, and Propositions 18.8–18.10.

THEOREM 18.13 (property of truth definition). *Let A be a sentence of* **PA**. *Then $T(\ulcorner A \urcorner) \equiv A$ is provable in* $\mathbf{S}^1$.

PROOF. This is a special case of Proposition 18.12.

Since we have established this property of T, we can prove the consistency of **PA** in $\mathbf{S}^1$.

DEFINITION 18.14. We recall (cf. §10) that $\mathrm{Prov}_{\mathbf{PA}}(p, a)$ is the proof-predicate for **PA**: p is a proof of a in **PA**. (The subscript **PA** may be omitted.) Also, $\exists p \, \mathrm{Prov}(p, a)$ may be abbreviated to $\mathrm{Pr}(a)$.

LEMMA 18.15.

(1) $\mathrm{ct}(\ulcorner t_1 \urcorner), \mathrm{ct}(\ulcorner t_2 \urcorner), \mathrm{ct}(\ulcorner t(0) \urcorner), v(\ulcorner t_1 \urcorner) = v(\ulcorner t_2 \urcorner) \rightarrow v(\ulcorner t(t_1) \urcorner) = v(\ulcorner t(t_2) \urcorner)$ *in* **PA**.

(2) $\mathrm{ct}(\ulcorner t_1 \urcorner), \mathrm{ct}(\ulcorner t_2 \urcorner), \mathrm{st}(\ulcorner A(0) \urcorner), v(\ulcorner t_1 \urcorner) = v(\ulcorner t_2 \urcorner) \rightarrow$
$\rightarrow T(\ulcorner A(t_1) \urcorner) \equiv T(\ulcorner A(t_2) \urcorner)$ *in* $\mathbf{S}^1$.

PROOF OF (2). By induction on n applied to the following formula:

$$\begin{aligned}\forall \ulcorner A(a_i) \urcorner \, [\mathrm{st}(\ulcorner A(0) \urcorner) \wedge \mathrm{ls}(\ulcorner A(0) \urcorner) \leqslant n \,.\supset \\ \supset. \forall \ulcorner t_1 \urcorner \, \forall \ulcorner t_2 \urcorner \, (\mathrm{ct}(\ulcorner t_1 \urcorner) \wedge \mathrm{ct}(\ulcorner t_2 \urcorner) \wedge v(\ulcorner t_1 \urcorner) = v(\ulcorner t_2 \urcorner) \supset \\ \supset T(\ulcorner A(t_1) \urcorner) \equiv T(\ulcorner A(t_2) \urcorner))].\end{aligned}$$

This is Π_1^1 (in wider sense). Use (1) and Propositions 18.8–18.10.

THEOREM 18.16. $\mathrm{st}(a), \mathrm{Pr}(a) \rightarrow T(a)$ *in* $\mathbf{S}^1$.

PROOF. By induction on n applied to the following formula:

$$\forall y\ (i(y) \leqslant n \supset T(u(y))),$$

where, if y is the Gödel number of a proof of $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_p$, then $i(y)$ is the number of inferences of this proof and $u(y)$ denotes the "universal closure" (i.e., a sentence formed by repeated universal quantification) of $(A_1 \wedge \ldots \wedge A_m) \supset (B_1 \vee \ldots \vee B_p)$. Both i and u are primitive recursive functions. The above formula is Π_1^1 (in wider sense). Use Propositions 18.8–18.10 and Lemma 18.15.

THEOREM 18.17. $\mathrm{Consis}(\mathbf{PA})$ *in* $\mathbf{S}^1$.

PROOF. By Theorems 18.13 and 18.16 applied to the formula $0 = 0'$.

PROBLEM 18.18. Let ZF′ be the second order system which is defined as ZF with the basic logical system $\mathbf{K}_1\mathbf{C}$ (cf. Definitions 15.15 and 16.4) and (finite) induction applied to Π_1^1-formulas. Give a truth definition for ZF in ZF′, thus proving the consistency of ZF in ZF′. [*Hint*: Follow the arguments in this section. It is important to notice that $T(\ulcorner A \urcorner)$, where A is an axiom of replacement, is a formula of ZF′ which is again an axiom of replacement.]

§19. The interpretation of a system of second order arithmetic

DEFINITION 19.1. (1) Let $\mathbf{S}^2$ be second order arithmetic with the arithmetical comprehension axiom and full induction, and let $\mathbf{S}^3$ be second order arithmetic with the Π_1^1 (in wider sense)-comprehension axiom and Π_1^1 (in wider sense)-induction. Notice that $\mathbf{S}^2$ and $\mathbf{S}^3$ are extensions of $\mathbf{S}^1$.

(2) We shall assume a standard Gödel numbering of the second order language (of arithmetic). In particular, $\ulcorner \alpha_1 \urcorner, \ulcorner \alpha_2 \urcorner, \ldots, \ulcorner \varphi_1 \urcorner, \ulcorner \varphi_2 \urcorner, \ldots$ denote Gödel numbers of second order variables. We include abstracts among the formal objects. So we need Gödel numbers for them: $\ulcorner \{ \urcorner, \ulcorner \} \urcorner, \ulcorner \{x\}A(x) \urcorner$. Actually we only use arithmetical abstracts here.)

Notice however that although abstracts are included among the formal objects for convenience, they do not actually occur in the formulas of $\mathbf{S}^2$ (as explained in §15).

(3) We take over from §18 all the notations for primitive recursive functions and predicates pertaining to first order arithmetic (some of them, like ls(a), now adapted in an obvious way to the second order language).

Additional primitive recursive functions and predicates needed are:

fl2(a): a is a first or second order formula (of $\mathbf{S}^2$).

st2(a): a is a sentence, i.e., fl2(a) and a is closed.

ab(a): a is an arithmetical abstract.

cab(a): ab(a) and a is closed.

sub($\ulcorner A \urcorner$, $\ulcorner \alpha \urcorner$, V): the result of substituting V for α in A if A is a formula, α a second order variable and V an arithmetical abstract. This may be denoted by $\ulcorner A(^{\alpha}_{V}) \urcorner$ or $\ulcorner A(V) \urcorner$.

q2(a): The number of second order quantifiers in a, if fl2(a).

We also use the other abbreviations in §18, and T is defined as in Definition 18.4.

DEFINITION 19.2. $F'(\alpha, n)$ stands for:

$$\forall \ulcorner A \urcorner\ [\mathrm{st2}(\ulcorner A \urcorner) \wedge \mathrm{q2}(\ulcorner A \urcorner) = 0 \,.\!\supset\!.\, \alpha(\ulcorner A \urcorner) \equiv T(\ulcorner A \urcorner)]$$

$$\wedge\ \forall \ulcorner A \urcorner\ [\mathrm{st2}(\ulcorner A \urcorner) \wedge 0 < \mathrm{q2}(\ulcorner \neg A \urcorner) \leqslant n \,.\!\supset\!.\, \alpha(\ulcorner \neg A \urcorner) \equiv \neg\alpha(\ulcorner A \urcorner)]$$

$$\wedge\ \forall \ulcorner A \wedge B \urcorner\ [\mathrm{st2}(\ulcorner A \wedge B \urcorner) \wedge 0 < \mathrm{q2}(\ulcorner A \wedge B \urcorner) \leqslant n \,.\!\supset$$

$$\supset\!.\, \alpha(\ulcorner A \wedge B \urcorner) \equiv \alpha(\ulcorner A \urcorner) \wedge \alpha(\ulcorner B \urcorner)]$$

$$\wedge\ \forall \ulcorner \forall x_i\, A(x_i) \urcorner\ [\mathrm{st2}(\ulcorner \forall x_i\, A(x_i) \urcorner) \wedge 0 < \mathrm{q2}(\ulcorner \forall x_i\, A(x_i) \urcorner) \leqslant n \,.\!\supset$$

$$\supset\!.\, \alpha(\ulcorner \forall x_i\, A(x_i) \urcorner) \equiv \forall x\, \alpha(\ulcorner A(\nu(x)) \urcorner)]$$

$$\wedge\ \forall \ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner\ [\mathrm{st2}(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) \wedge 0 < \mathrm{q2}(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) \leqslant n \,.\!\supset$$

$$\supset\!.\, \alpha(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) \equiv \forall \ulcorner V \urcorner\ (\mathrm{cab}(\ulcorner V \urcorner) \supset \alpha(\ulcorner A(V) \urcorner))].$$

$F'(\alpha, n)$ means that α is an interpretation for sentences of $\mathbf{S}^2$ of (second order quantifier) complexity $\leqslant n$. We now give an interpretation for all sentences of $\mathbf{S}^2$.

$$\mathrm{I}(a)\colon\ \exists\psi\ (F'(\psi, \mathrm{q2}(a) \wedge \psi(a)).$$

The point is that we can give a kind of truth definition for $\mathbf{S}^2$ by interpreting the second order variables as ranging over arithmetical predicates or sets (i.e., sets and relations of natural numbers defined by the closed arithmetical abstracts). This is often expressed by saying that the arithmetical sets form a model of $\mathbf{S}^2$. Further (as Lemma 19.14 essentially says) this can be formalized and proved in $\mathbf{S}^3$.

LEMMA 19.3. (1) $F'(\alpha, n), \mathrm{st}(\ulcorner A \urcorner) \rightarrow \alpha(\ulcorner A \urcorner) \equiv T(\ulcorner A \urcorner)$ *in* $\mathbf{PA}'$. (α *coincides with the truth definition* T *for arithmetical formulas.*)

(2) $\mathrm{st2}(\ulcorner A \urcorner), F'(\alpha, n), F'(\beta, n), \mathrm{q2}(\ulcorner A \urcorner) \leqslant n \rightarrow \alpha(\ulcorner A \urcorner) \equiv \beta(\ulcorner A \urcorner)$ *in* $\mathbf{S}^1$.

(3) $F'(\alpha, n), m \leqslant n \rightarrow F'(\alpha, m)$ *in* $\mathbf{PA}'$.

PROOF OF (2). By double induction on $(\mathrm{q2}(\ulcorner A \urcorner), \mathrm{ls}(\ulcorner A \urcorner))$ applied to the above sequent, which is Π^1_1 (in wider sense).

DEFINITION 19.4. $E(\alpha, \beta, n, l)$ is defined to be:

$$\forall \ulcorner A \urcorner\ [\mathrm{st2}(\ulcorner A \urcorner) \wedge \mathrm{q2}(\ulcorner A \urcorner) \leqslant n \,.\supset.\, \beta(\ulcorner A \urcorner) \equiv \alpha(\ulcorner A \urcorner)]$$

$$\wedge \forall \ulcorner \neg A \urcorner\ [\mathrm{st2}(\ulcorner \neg A \urcorner) \wedge \mathrm{q2}(\ulcorner \neg A \urcorner) = n' \wedge \mathrm{ls}(\ulcorner \neg A \urcorner) \leqslant l \,.\supset$$

$$\supset.\, \beta(\ulcorner \neg A \urcorner) \equiv \neg \beta(\ulcorner A \urcorner)]$$

$$\wedge \forall \ulcorner A \wedge B \urcorner\ [\mathrm{st2}(\ulcorner A \wedge B \urcorner) \wedge \mathrm{q2}(\ulcorner A \wedge B \urcorner) = n' \wedge \mathrm{ls}(\ulcorner A \wedge B \urcorner) \leqslant l \,.\supset$$

$$\supset.\, \beta(\ulcorner A \wedge B \urcorner) \equiv \beta(\ulcorner A \urcorner) \wedge \beta(\ulcorner B \urcorner)]$$

$$\wedge \forall \ulcorner \forall x_i\, A(x_i) \urcorner [\mathrm{st2}(\ulcorner \forall x_i\, A(x_i) \urcorner) \wedge \mathrm{q2}(\ulcorner \forall x_i\, A(x_i) \urcorner) = n' \wedge \mathrm{ls}(\ulcorner \forall x_i\, A(x_i) \urcorner) \leqslant l \,.\supset$$

$$\supset.\, \beta(\ulcorner \forall x_i\, A(x_i) \urcorner) \equiv \forall x\, \alpha(\ulcorner A(\nu(x)) \urcorner)]$$

$$\wedge \forall \ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner\ [\mathrm{st2}(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) \wedge \mathrm{q2}(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) = n'$$

$$\wedge\ \mathrm{ls}(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) \leqslant l \,.\supset$$

$$\supset.\, \beta(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) \equiv \forall \ulcorner V \urcorner (\mathrm{cab}(\ulcorner V \urcorner) \supset \beta(\ulcorner A(V) \urcorner)].$$

LEMMA 19.5.

(1) $E(\alpha, \beta, n, l), \mathrm{st2}(\ulcorner A \urcorner), \mathrm{q2}(\ulcorner A \urcorner) \leqslant n \rightarrow \beta(\ulcorner A \urcorner) \equiv \alpha(\ulcorner A \urcorner)$ *in* $\mathbf{PA}'$.

(2) $E(\alpha, \beta, n, l), E(\alpha, \gamma, n, k), \mathrm{st2}(\ulcorner A \urcorner), \mathrm{q2}(\ulcorner A \urcorner) \leqslant n'$,
$\mathrm{ls}(\ulcorner A \urcorner) \leqslant l, k \rightarrow \beta(\ulcorner A \urcorner) \equiv \gamma(\ulcorner A \urcorner)$ *in* $\mathbf{PA}'$.

(*Uniqueness of extension.*)

(3) $E(\alpha, \alpha, n, 0)$ *in* $\mathbf{PA}'$.

(4) $E(\alpha, \beta, n, l) \rightarrow E(\alpha, \{x\}C(x), n, l')$ *in* $\mathbf{PA}'$, *where* $C(x)$ *stands for*:

$$[(\mathrm{st2}(x) \wedge \mathrm{q2}(x) \leqslant n \;.\vee.\; (\mathrm{q2}(x) = n' \wedge \mathrm{ls}(x) < l')) \wedge \beta(x)]$$

$$.\vee.\; [\mathrm{st2}(x) \wedge \mathrm{q2}(x) = n' \wedge \mathrm{ls}(x) = l'$$

$$\wedge\; (\exists \ulcorner A \urcorner\; (x = \ulcorner \neg A \urcorner \wedge \neg\beta(\ulcorner A \urcorner))$$

$$\vee\; \exists \ulcorner A \wedge B \urcorner\; (x = \ulcorner A \wedge B \urcorner \wedge \beta(\ulcorner A \urcorner) \wedge \beta(\ulcorner B \urcorner))$$

$$\vee\; \exists \ulcorner \forall x_i\, A(x_i) \urcorner\; (x = \ulcorner \forall x_i\, A(x_i) \urcorner \wedge \forall y\, \beta(\ulcorner A(\nu(y)) \urcorner))$$

$$\vee\; \exists \ulcorner \forall \varphi_i A(\varphi_i) \urcorner\; (x = \ulcorner \forall \varphi_i A(\varphi_i) \urcorner \wedge \forall \ulcorner V \urcorner\; (\mathrm{cab}(\ulcorner V \urcorner) \supset \beta(\ulcorner A(V) \urcorner)))].$$

(*Extension of* β *from* (n, l) *to* (n, l').)

(5) $\forall l\, \exists \varphi\, E(\alpha, \varphi, n, l)$ *in* $\mathbf{S}^1$. (*Existence of an extension of* α, *for fixed* n, *for every* l.)

PROOF OF (5). By induction on l applied to $\exists \varphi\, E(\alpha, \varphi, n, l)$. Use (3) and (4) of this lemma and the arithmetical comprehension axiom.

DEFINITION 19.6. $G(\alpha, n, x)$: $\exists \varphi\, (E(\alpha, \varphi, n, \mathrm{ls}(x)) \wedge \varphi(x))$. $G(\alpha, n, x)$ shall be abbreviated to $G(x)$.

LEMMA 19.7. (1) $F'(\alpha, n), \mathrm{st}(\ulcorner A \urcorner) \rightarrow G(\ulcorner A \urcorner) \equiv T(\ulcorner A \urcorner)$ *in* $\mathbf{PA}'$

(2) $E(\alpha, \beta, n, l), \mathrm{st2}(\ulcorner A \urcorner), \mathrm{q2}(\ulcorner A \urcorner) \leqslant n \vee (\mathrm{q2}(\ulcorner A \urcorner) = n' \wedge \mathrm{ls}(\ulcorner A \urcorner) \leqslant l) \rightarrow$
$\rightarrow G(\ulcorner A \urcorner) \equiv \beta(\ulcorner A \urcorner)$ *in* $\mathbf{PA}'$.

(3) $\mathrm{st2}(\ulcorner A \urcorner), F'(\alpha, n), \mathrm{q2}(\ulcorner \neg A \urcorner) \leqslant n' \rightarrow G(\ulcorner \neg A \urcorner) \equiv \neg G(\ulcorner A \urcorner)$ *in* $\mathbf{S}^1$.

(4) $\mathrm{st2}(\ulcorner A \wedge B \urcorner), F'(\alpha, n), \mathrm{q2}(\ulcorner A \wedge B \urcorner) \leqslant n' \rightarrow$
$\rightarrow G(\ulcorner A \wedge B \urcorner) \equiv G(\ulcorner A \urcorner) \wedge G(\ulcorner B \urcorner)$ *in* $\mathbf{S}^1$.

(5) $\mathrm{st2}(\ulcorner \forall x_i\, A(x_i) \urcorner), F'(\alpha, n), \mathrm{q2}(\ulcorner \forall x_i\, A(x_i) \urcorner) \leqslant n' \rightarrow$
$\rightarrow G(\ulcorner \forall x_i\, A(x_i) \urcorner) \equiv \forall x\, G(\ulcorner A(\nu(x)) \urcorner)$ *in* $\mathbf{S}^1$.

(6) $\mathrm{st2}(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner), F'(\alpha, n), \mathrm{q2}(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) \leqslant n' \rightarrow$
$\rightarrow G(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) \equiv \forall \ulcorner V \urcorner\; (\mathrm{cab}(\ulcorner V \urcorner) \supset G(\ulcorner A(V) \urcorner))$ *in* $\mathbf{S}^1$.

PROOF. (1) From (1) of Lemma 19.3 and (1) of Lemma of 19.5.

(2) From (2) of Lemma 19.5.

(3) By (2) above and (5) of Lemma 19.5.

(4)–(6) are proved like (3) above. Notice that in (6)

$$\mathrm{q2}(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) \leqslant n' \text{ implies } \mathrm{q2}(\ulcorner A(V) \urcorner) \leqslant n.$$

LEMMA 19.8. (1) $F'(\alpha, n) \to F'(\{x\}G(x), n')$ *in* $\mathbf{S}^1$.
(2) $\to F'(\{x\}T(x), 0)$ *in* **PA'**.

PROOF. By (1) and (3)–(6) of Lemma 19.7, and the definition of F'.

PROPOSITION 10.9. $\forall n\ \exists \psi\ F'(\psi, n)$ *in* $\mathbf{S}^3$.

PROOF. By the comprehension axiom applied to $\{x\}G(x)$ and also to $\{x\}T(x)$, and induction applied to $\exists \psi\ F'(\psi, n)$, which are all Π^1_1 (in wider sense), and Lemma 19.8.

LEMMA 19.10. $F'(\alpha, n)$, $\mathrm{st2}(\ulcorner A \urcorner)$, $\mathrm{q2}(\ulcorner A \urcorner) \leqslant n \to \alpha(\ulcorner A \urcorner) \equiv I(\ulcorner A \urcorner)$ *in* $\mathbf{S}^1$.

PROOF. Use (2) and (3) of Lemma 19.3.

PROPOSITION 19.11. $\mathrm{st2}(\ulcorner A \urcorner)$, $\mathrm{q2}(\ulcorner A \urcorner) = 0 \to I(\ulcorner A \urcorner) \equiv T(\ulcorner A \urcorner)$ *in* $\mathbf{S}^3$.

PROOF. By Lemma 19.10, (1) of Lemma 19.3, and (2) of Lemma 19.8.

The following proposition asserts that I commutes with logical connectives. It is proved by Lemma 19.10 and Proposition 19.9: hence in $\mathbf{S}^3$.

PROPOSITION 19.12. (1) $\mathrm{st2}(\ulcorner A \urcorner) \to I(\ulcorner \neg A \urcorner) \equiv \neg I(\ulcorner A \urcorner)$.
(2) $\mathrm{st2}(\ulcorner A \wedge B \urcorner) \to I(\ulcorner A \wedge B \urcorner) \equiv I(\ulcorner A \urcorner) \wedge I(\ulcorner B \urcorner)$.
(3) $\mathrm{st2}(\ulcorner \forall x_i\, A(x_i) \urcorner) \to I(\ulcorner \forall x_i\, A(x_i) \urcorner) \equiv \forall x\, I(\ulcorner A(\nu(x)) \urcorner)$.
(4) $\mathrm{st2}(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) \to I(\ulcorner \forall \varphi_i\, A(\varphi_i) \urcorner) \equiv \forall \ulcorner V \urcorner\ (\mathrm{cab}(\ulcorner V \urcorner) \supset I(\ulcorner A(V) \urcorner))$.

We can now proceed to the consistency proof of $\mathbf{S}^2$.

DEFINITION 19.13. (1) $i(p)$ is the number of inferences in (the proof with Gödel number) p.

(2) $i(b, a)$ means: "b is a closed instantiation of a", viz. $i(b, a)$ if and only if $a = \ulcorner A(\beta, \ldots, c, \ldots) \urcorner$ and $b = \ulcorner A(V, \ldots, \nu(n), \ldots) \urcorner$ for some closed arithmetical abstracts $V, \ldots$, and numbers $n, \ldots$, where $\beta, \ldots, c, \ldots$ are all the free variables in A.

(3) $j(b, p)$ means: if p is a Gödel number of a proof in $\mathbf{S}^2$ of $A_1, \ldots, A_m \to B_1, \ldots, B_n$, then $i(b, \ulcorner A_1 \wedge \ldots \wedge A_m \supset B_1 \vee \ldots \vee B_n \urcorner)$.

(4) We write Prov_2 for $\mathrm{Prov}_{\mathbf{S}^2}$, and $\mathrm{Pr}_2(a)$ for $\exists x\ \mathrm{Prov}_2(x, a)$.

LEMMA 19.14. $\mathrm{Pr}_2(a) \rightarrow \forall x\, (i(x, a) \supset I(x))$ *in* $\mathbf{S}^3$.

PROOF. Define $H(m)$ as:

$$\forall p, x\, [i(p) \leqslant m \wedge j(x, p) \supset I(x)].$$

The proof is carried out by induction on m applied to $H(m)$, which is Π_1^1 (in wider sense). The argument proceeds according to the "last inference of p". The case where the last inference is an induction causes no problem, since the induction step is then proved by applying induction (on k) to a formula of the form $I(\ulcorner A(\nu(k), V, \ldots, \nu(n), \ldots)\urcorner)$ (where $A(a, \beta, \ldots, c, \ldots)$ is the induction formula of this inference, with eigenvariable a).

PROPOSITION 19.15. $\mathrm{st2}(\ulcorner A \urcorner), \mathrm{Pr}_2(\ulcorner A \urcorner) \rightarrow I(\ulcorner A \urcorner)$ *in* $\mathbf{S}^3$.

PROOF. By Lemma 19.14.

THEOREM 19.16. $\mathrm{Consis}(\mathbf{S}^2)$ *in* $\mathbf{S}^3$.

PROOF. A corollary of Propositions 19.11 and 19.15, and Theorem 8.13.

§20. Simple type theory

In this section we present the higher (finite) order predicate calculus. We shall formulate it in the sequential calculus. It is a simplification of a system called **GLC**, which was defined by the author. Now we restrict ourselves to predicate variables only. Following common practice, the word "type" is used instead of "order" (as in §15), and types start with 0 rather than 1. Thus the individual objects are of type 0.

DEFINITION 20.1. (1) We define *types* inductively as follows: 0 is a type; if $\tau_1, \ldots, \tau_k$ are types ($k \geqslant 1$), then so is $[\tau_1, \ldots, \tau_k]$; types are only as required by the above.

(2) The symbols of our language are classified as follows:

1) *Constants*:

1.1) individual constants: $c_0, c_1, \ldots$;

1.2) function constants with i argument-places ($i = 1, 2, \ldots$): $f_0^i, f_1^i, \ldots$;

1.3) predicate constants of type $\tau \neq 0$: $R_0^\tau, R_1^\tau, \ldots$.

2) *Variables*:

2.1) free variables: $a_0^\tau, a_1^\tau, \ldots$ of each type τ,

2.2) bound variables: $x_0^\tau, x_1^\tau, \ldots$ of each type τ.

3) *Logical symbols*: $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\supset$ (implies), $\forall$ (for all), $\exists$ (there exists).

4) *Auxiliary symbols*: (,), {, }, [,].

A higher-order language (a language of simple type theory) is given when all the constants are given. A predicate variable means a variable (free or bound) of some type $\neq 0$. We shall use the symbols of the language also as meta-variables. Type superscripts are sometimes omitted. We shall, further, take over all the appropriate notational conventions in §1.

Intuitively, variables of type 0 range over individual objects while variables of type $[\tau_1, \ldots, \tau_k]$ range over predicates which are associated with the subsets of $T_1 \times \ldots \times T_k$, where T_i is the set of objects of type τ_i.

DEFINITION 20.2. *Terms* (of given types), *formulas* and outermost *logical symbols* are defined inductively (and simultaneously).

1) Individual constants are terms of type 0.

2) Free variables of type τ are terms of type τ.

3) If f is a function constant with i argument-places and $t_1, \ldots, t_i$ are terms of type 0, then $f(t_1, \ldots, t_i)$ is a term of type 0.

4) Predicate constants of type τ are terms of type τ.

5) If A is a formula, $a_0^{\tau_0}, \ldots, a_k^{\tau_k}$ are distinct free variables of the indicated types, $x_0^{\tau_0}, \ldots, x_k^{\tau_k}$ are distinct bound variables of the indicated types not occurring in A, and A' is the result of simultaneously replacing, in A, a_0 by $x_0, \ldots, a_k$ by x_k, then $\{x_0, \ldots, x_k\}\ A'$ is a term (called an abstract) of type $[\tau_0, \ldots, \tau_k]$.

6) If α is a predicate constant or a free variable of type $[\tau_1, \ldots, \tau_k]$ and $t_1, \ldots, t_k$ are terms of type $\tau_1, \ldots, \tau_k$, then $\alpha[t_1, \ldots, t_k]$ is a formula, which is called atomic. There is no outermost logical symbol in this case.

7) If A and B are formulas, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$ are formulas, and their outermost logical symbols are $\neg$, $\wedge$, $\vee$, $\supset$, respectively.

8) If A is a formula, a^τ is a free variable, x^τ is a bound variable of the same type which does not occur in A, and A' is obtained from A by replacing all occurrences of a^τ by x^τ, then $\forall x^\tau\, A'$ and $\exists x^\tau\, A'$ are formulas, and their outermost logical symbols are $\forall$ and $\exists$, respectively.

These formation-rules may result in an excessive number of parentheses: if no ambiguity results, we may omit some of these as we did in the preceding sections.

Notice that here, unlike the preceding sections, abstracts are taken as formal objects. The notion of *alphabetical variant* is defined as before: for two expressions A and B, A is said to be an alphabetical variant of B (and vice versa) if A and B differ only in the names of some bound variables.

DEFINITION 20.3. The *height* of a type τ, $h(\tau)$, is defined inductively as follows:

$$h(0) = 0; \qquad h([\tau_1, \ldots, \tau_k]) = \max(h(\tau_1), \ldots, h(\tau_k)) + 1.$$

By the height $h(t)$ of a term (abstract) t, we mean the height of its type.

The *(logical) complexity* of a formula or abstract A is defined to be the total number of logical symbols and pairs of abstraction symbols $\{, \}$ in A.

Substitution of a term t of type τ for a free variable a of type τ in a formula or an abstract A is now defined by double induction on the height of τ and the complexity of A.

1) Basis: the height of τ is 0, i.e., the type τ is 0.

Then $A\binom{a}{t}$ can simply be defined as $(A\frac{a}{t})$, in accordance with Definition 1.4, or an alphabetical variant of this.

Let a and b be free variables of type 0 and let t and s be terms of type 0.

We can easily prove the following.

(i) If A is a formula (term of type τ) then $A\binom{a}{t}$ is a formula (term of type τ).

(ii) If A is an alphabetical variant of B then $A\binom{a}{t}$ is an alphabetical variant of $B\binom{a}{t}$.

(iii) $A\binom{a}{t}$ contains only those free variables contained in A or t.

(iv) If s does not contain a, then

$$A\binom{a}{t}\binom{b}{s} \quad \text{is} \quad A\binom{b}{s}\left(\begin{matrix} a \\ t\binom{b}{s} \end{matrix}\right)$$

2) Induction step: suppose $h(t) = n \neq 0$ and for any $m < n$, substitution of a term t of type σ (with $h(\sigma) = m$) for a free variable a of type σ has been defined so as to satisfy the following properties:

(1) If A is a formula (term of type σ), then $A\binom{a}{t}$ is a formula (term of type σ).

(2) If A is an alphabetical variant of B and s is an alphabetical variant of t, then $A\binom{a}{s}$ is an alphabetical variant of $B\binom{a}{t}$.

(3) $A\binom{a}{t}$ contains only free variables contained in A or t.

(4) $A\left(\begin{matrix} a \\ \{x_1, \ldots, x_k\}a[x_1, \ldots, x_k] \end{matrix}\right)$ is an alphabetical variant of A.

(5) $\{x_1, \ldots, x_k\}a[x_1, \ldots, x_k]\binom{a}{t}$ is an alphabetical variant of t.

(6) If s does not contain a, and the height of b is less than n, then

$$A\binom{a}{t}\binom{b}{s}$$

is an alphabetical variant of

$$A\binom{b}{s}\left(\begin{matrix} a \\ t\binom{b}{s}\end{matrix}\right).$$

(7) If A does not contain a, then $A\binom{a}{s}$ is an alphabetical variant of A.

Let a and t be a free variable and a term, respectively, of type τ such that $h(\tau) = n$. If t is a free variable or predicate constant, then $A\binom{a}{t}$ is defined again to be (an alphabetical variant of) $(A\frac{a}{t})$. So suppose t is an abstract. We define $A\binom{a}{t}$ by induction on the complexity of A. Let t be $\{x_1, \ldots, x_k\}U(x_1, \ldots, x_k)$, where x_i is of type τ_i, a is of type $\tau = [\tau_1, \ldots, \tau_k]$ and $\max(h(\tau_1), \ldots, h(\tau_k)) + 1 = n$. First note that for any term s of type 0, $s\binom{a}{t}$ is defined to be s.

2.1) If A is of the form $b[t_1, \ldots, t_k]$, where b is a predicate constant or variable other than a, then

$$A\binom{a}{t} \quad \text{is} \quad b\left[t_1\binom{a}{t}, \ldots, t_k\binom{a}{t}\right].$$

2.2) If A is $a[t_1, \ldots, t_k]$, then

$$A\binom{a}{t} \quad \text{is} \quad U(b_1, \ldots, b_k)\left(\begin{matrix} b_1 \\ t_1\binom{a}{t}\end{matrix}\right) \cdots \left(\begin{matrix} b_k \\ t_k\binom{a}{t}\end{matrix}\right),$$

where $b_1, \ldots, b_k$ are different from any free variable in A and

$$U(b_1, \ldots, b_k) \quad \text{is} \quad U(x_1, \ldots, x_k)\left(\frac{x_1, \ldots, x_k}{b_1, \ldots, b_k}\right).$$

$h(b_i) < n$ and t_i is "simpler" than A, and hence

$$B\left(\begin{matrix} b_i \\ t_i\binom{a}{t}\end{matrix}\right)$$

has been defined for arbitrary B.

2.3) If A is of the form $\neg B$, $B \wedge C$, $B \vee C$ or $B \supset C$, then $A\binom{a}{t}$ is

$$\neg B\begin{pmatrix}a\\t\end{pmatrix}, \quad B\begin{pmatrix}a\\t\end{pmatrix} \wedge C\begin{pmatrix}a\\t\end{pmatrix}, \quad B\begin{pmatrix}a\\t\end{pmatrix} \vee C\begin{pmatrix}a\\t\end{pmatrix} \quad \text{or} \quad B\begin{pmatrix}a\\t\end{pmatrix} \supset C\begin{pmatrix}a\\t\end{pmatrix}.$$

2.4) If A is of the form $\forall x\, F(x)$ or $\exists x\, F(x)$, then $A\binom{a}{t}$ is $\forall y\, G(y)$ or $\exists y\, G(y)$, respectively, where $G(b)$ is $F(b)\binom{a}{t}$, b is different from a and does not occur in A, and y does not occur in $G(b)$.

2.5) If A is of the form $\{x_1, \ldots, x_k\}B(x_1, \ldots, x_k)$, then

$$A\begin{pmatrix}a\\t\end{pmatrix} \quad \text{is} \quad \{y_1, \ldots, y_k\}C(y_1, \ldots, y_k),$$

where $C(b_1, \ldots, b_k)$ is $B(b_1, \ldots, b_k)\ \binom{a}{t}$, $b_1, \ldots, b_k$ are different from a and do not occur in A, and none of the y_i's occur in $C(b_1, \ldots, b_k)$.

Then we can prove (1)–(7) for $h(a) = n$. (The proof is omitted.) We often denote $A\binom{a}{t}$ by $A(t)$.

Here and henceforth we use $U, V, \ldots$ with or without type-superscripts, as meta-variables for abstracts. Also, $\alpha, \beta, \ldots$ are often used for free variables instead of $a, b, \ldots$, and $\varphi, \psi, \ldots$ for bound variables instead of $x, y, \ldots$, usually when we are thinking of variables of type $\neq 0$.

EXAMPLE 20.4. Let A be $\forall\varphi\ \exists x\ (\alpha[a] \equiv \varphi[x])$, where φ and α are of type $[0]$ and x and a are of type 0. Let V be $\{z\}\ \forall\varphi\ \exists x\ (\varphi[x] \wedge \beta[z])$, where φ and β are of type $[0]$ and x and z are of type 0. V is an abstract of type $[0]$. Consider $A\binom{\alpha}{V}$. The substitution is carried out step by step according to Definition 20.3. Since α is of type $[0]$, we start with clause 2) of the definition and are led repeatedly back to 1). By 2.2) and 1), $\alpha[a]\ \binom{\alpha}{V}$ is $\forall\varphi\ \exists x\ (\varphi[x] \wedge \beta[a])$. Using this, by 2.1) and 2.3), $(\alpha[a] \equiv \gamma[b])\ \binom{\alpha}{V}$ (for some b, and γ different from a and α, respectively) is $\forall\varphi\ \exists x\ (\varphi[x] \wedge \beta[a]) \equiv \gamma[b]$. From this and 2.4), we obtain

$$\forall\psi\ \exists y\ (\forall\varphi\ \exists x\ (\varphi[x] \wedge \beta[a]) \equiv \psi[y]).$$

EXERCISE 20.5. Let A be

$$\{y\}\alpha^2[\{z\}(\alpha^1[z] \equiv \beta^1[y])],$$

and let V^2 be

$$\{\psi^1\}\ \forall\varphi^1\ \exists x\ (\psi^1[a] \equiv \varphi^1[x]),$$

where $1 = [0]$ and $2 = [1]$. Compute $A\ \binom{\alpha^2}{V^2}$.

DEFINITION 20.6. Let V be an abstract of the form

$$\{x_1^{\tau_1} \dots x_n^{\tau_n}\} A(x_1^{\tau_1}, \dots, x_n^{\tau_n}),$$

and let $V_1, \dots, V_n$ be terms of types $\tau_1, \dots, \tau_n$, respectively. Then $V[V_1, \dots, V_n]$ is defined to be

$$A(a_1^{\tau_1}, \dots, a_n^{\tau_n}) \begin{pmatrix} a_1^{\tau_1} \\ V_1 \end{pmatrix} \dots \begin{pmatrix} a_n^{\tau_n} \\ V_n \end{pmatrix},$$

where $a_1^{\tau_1}, \dots, a_n^{\tau_n}$ are free variables of the indicated types which do not occur in any of $V, V_1, \dots, V_n$ (and $A(a_1^{\tau_1}, \dots, a_n^{\tau_n})$ is

$$A(x_1^{\tau_1}, \dots, x_n^{\tau_n}) \left(\frac{x_1^{\tau_1}, \dots, x_n^{\tau_n}}{a_1^{\tau_1}, \dots, a_n^{\tau_n}} \right)).$$

DEFINITION 20.7. The formal system of simple type theory is defined like $\mathbf{G^1LC}$ in §15. The sequents are, as usual, of the form $\Gamma \to \Theta$, where Γ and Θ consist of finitely many formulas. The rules of inference are those of $\mathbf{G^1LC}$ (cf. Definition 15.15) with the following generalization (to higher types). (We take over all relevant notions and terminology from the previous sections.)

$$\forall : \text{left:} \quad \frac{F(V), \Gamma \to \Theta}{\forall \varphi\, F(\varphi), \Gamma \to \Theta},$$

where V is an arbitrary term (of any type) and φ is a bound variable of the same type as V (and if $F(V)$ is $F(\frac{\alpha}{V})$ then $F(\varphi)$ is $(F\frac{\alpha}{\varphi})$).

$$\forall : \text{right:} \quad \frac{\Gamma \to \Theta, F(\alpha)}{\Gamma \to \Theta, \forall \varphi\, F(\varphi)},$$

where α does not occur in the lower sequent and φ is of the same type as α. Here a is called the eigenvariable of the inference. We define $\exists$: left and $\exists$: right similarly.

A sentence of the form

$$\forall y_1, \dots, y_m\, \exists \varphi\, \forall x_1, \dots, x_n\, (\varphi(x_1, \dots, x_n) \equiv A(x_1, \dots, x_n, y_1, \dots, y_m)),$$

where $A(a_1, \dots, a_n, b_1, \dots, b_m)$ is an arbitrary formula (and $x_1, \dots, x_n, y_1, \dots, y_m$ are of arbitrary types), is called a comprehension axiom (cf. Definition 15.14).

The following is analogous to Proposition 15.16.

PROPOSITION 20.8. *The comprehension axioms (for arbitrary A) are provable in simple type theory. Conversely, consider the subsystem of simple type theory in which $\forall$: left and $\exists$: right are restricted to the case that V is a free variable or predicate constant. In this subsystem, augmented by the comprehension axioms for all A, the general $\forall$: left and $\exists$: right are admissible.*

DEFINITION 20.9. (1) A *semi-formula* is an expression like a formula, except that it may contain free occurrences of bound variables. A semi-term is defined likewise.

(2) The *logical complexity* of a semi-formula or semi-term is defined as for formulas and abstracts (Definition 20.3).

§21. The cut-elimination theorem for simple type theory

About 25 years ago the author conjectured that the cut-elimination theorem holds for simple type theory as formulated in Definition 20.7. This was known as Takeuti's conjecture and it remained unresolved for many years. W. W. Tait provided support for my conjecture by proving the cut-elimination theorem for second order logic. The full conjecture was then resolved positively by Takahashi, and independently by Prawitz. In this section we will present a proof of the cut-elimination theorem for simple type theory, using the method of Takahashi and Prawitz. We, however, wish to point out that in 1971 J.-Y. Girard made significant improvements on several of the results of this section including the proof of the cut-elimination theorem. (See the *Proceedings of the Second Scandinavian Logic Symposium*, ed., J. E. Fenstad (North Holland, Amsterdam, 1971).) Girard's basic idea was then used by Martin-Löf and Prawitz, independently to produce a variant and somewhat more elegant form of cut-elimination.

Throughout this section we shall deal with variables and abstracts of one argument-place only and restrict the logical symbols to $\neg$, $\wedge$ and $\forall$ in order to simplify the argument. Thus the types are 0, $[0]$, $[[0]]$,... which may be called $0, 1, 2, \ldots$. We shall also omit the constants. The results can easily be extended to the general case.

DEFINITION 21.1. An *axiom of extensionality* is a formula of the form

$$\forall x\,(V_1(x) \equiv V_2(x)) \supset \forall \varphi\,(\varphi[V_1] \equiv \varphi[V_2]),$$

where V_1 and V_2 are arbitrary abstracts of the same type. In simple type theory this is equivalent to the following rule of inference (the extensionality rule):

$$\frac{V_1(a), \Gamma \to \Delta, V_2(a) \qquad V_2(a), \Gamma \to \Delta, V_1(a)}{\alpha[V_1], \Gamma \to \Delta, \alpha[V_2]}$$

where a does not occur in the lower sequent.

PROPOSITION 21.2. *The following is an admissible inference in simple type theory augmented by the extensionality rule*:

$$\frac{V_1(a), \Gamma \to \Delta, V_2(a) \qquad V_2(a), \Gamma \to \Delta, V_1(a)}{A(V_1), \Gamma \to \Delta, A(V_2)}$$

where $A(V_1)$ is obtained from an (arbitrary) formula $A(\beta)$ by substitution of V_1 for β and a does not occur in the lower sequent.

PROOF. By mathematical induction on the complexity of A. We shall deal with the case where $A(\beta)$ is of the form $\forall\varphi\, B(\varphi, \beta)$. Assume $V_1(a), \Gamma \to \Delta, V_2(a)$ and $V_2(a), \Gamma \to \Delta, V_1(a)$. By the induction hypothesis, $B(\gamma, V_1), \Gamma \to \Delta, B(\gamma, V_2)$ is provable, where γ is a free variable of appropriate type which does not occur elsewhere in this sequent; hence by introducing $\forall\varphi$ on both sides, $\forall\varphi\, A(\varphi, V_1), \Gamma \to \Delta, \forall\varphi\, A(\varphi, V_2)$ is provable.

THEOREM 21.3 (the cut-elimination theorem for simple type theory with extensionality: Takahashi). *Let* **S** *be simple type theory augmented by the extensionality rule. Then the cut-elimination theorem holds for* **S**.

The proof is obtained by modifying the original Takahashi-Prawitz method. The proof is presented stage by stage, introducing certain notions and notations as needed.

DEFINITION 21.4. (1) A *structure* (for simple type theory) is an ω-sequence of sets, say $\mathscr{S} = (S_0, S_1, \ldots, S_i, \ldots)$, where

1.1) S_0 is a non-empty set,

1.2) S_{i+1} is a subset of $P(S_i)$, the power set of S_i.

(2) An *assignment* ϕ (from $\mathscr{S}$) is a mapping from all (free and bound) variables such that to every variable of type i, ϕ assigns an element of S_i. An *interpretation* $\mathfrak{I}$ is a pair $(\mathscr{S}, \phi)$ consisting of a structure $\mathscr{S}$ and an assignment from $\mathscr{S}$.

(3) Given an interpretation $\mathfrak{I} = (\mathscr{S}, \phi)$, we will define the interpretation (by $\mathfrak{I}$) of semi-formulas and semi-terms. We use the notation $\phi(^x_S)$ in order

to express the assignment which agrees with ϕ except at x, where its value is S.

If A is a semi-formula or a semi-term, then its interpretation (by $\mathfrak{I}$) is denoted by $\phi(A)$. It is defined in such a way that for every semi-formula A, exactly one of $\phi(A) = \mathsf{T}$ and $\phi(A) = \mathsf{F}$ holds (where T stands for "truth" and F for "falsehood"), and for a semi-term A of type i, $\phi(A)$ is a subset of S_i. The definition is by induction on the complexity of A. (If A is a free or bound variable then $\phi(A)$ is already defined as the value of ϕ at A.)

3.1) $\phi(\alpha[W]) = \mathsf{T}$ if and only if $\phi(W) \in \phi(\alpha)$.

3.2) $\phi(\forall x\, A(x)) = \mathsf{T}$ (for x of any type) if and only if, for every ϕ' which agrees with ϕ except perhaps at x, $\phi'(A(x)) = \mathsf{T}$.

3.3) $\phi(\{x\}A(x)) = \{S \mid S \in S_i \text{ and } \phi\,\binom{x}{s}\,(A(x)) = \mathsf{T}\}$, where x is of type i.

3.4) $\phi(A \wedge B) = \mathsf{T}$ if and only if $\phi(A) = \mathsf{T}$ and $\phi(B) = \mathsf{T}$.

3.5) $\phi(\neg A) = \mathsf{T}$ if and only if $\phi(A) = \mathsf{F}$.

Let $S : A_1, \ldots, A_m \to B_1, \ldots, B_n$ be a sequent. Then

$$\phi(S) = \phi(\neg(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n)$$

(where $\vee$ is defined in terms of $\neg$ and $\wedge$).

(4) A structure $\mathscr{S}$ is called a Henkin structure if for every assignment ϕ from $\mathscr{S}$ and every abstract U^i of type i (for $i = 1, 2, \ldots$), $\phi(U^i)$ is a member of S_i.

PROPOSITION 21.5. *Suppose $\mathscr{S}$ is a Henkin structure and ϕ is an assignment from $\mathscr{S}$. If a sequent S is provable in* **S**, *then $\phi(S) = \mathsf{T}$.*

This is proved simply by examining each rule of inference.

DEFINITION 21.6. A semi-valuation with *extensionality* is an assignment v of at most one of the values T and F to formulas, which satisfies the following.

1) If $v(\neg A) = \mathsf{T}$, then $v(A) = \mathsf{F}$; if $v(\neg A) = \mathsf{F}$, then $v(A) = \mathsf{T}$.

2) If $v(A \wedge B) = \mathsf{T}$, then $v(A) = \mathsf{T}$ and $v(B) = \mathsf{T}$; if $v(A \wedge B) = \mathsf{F}$, then $v(A) = \mathsf{F}$ or $v(B) = \mathsf{F}$.

3) If $v(\forall x\, A(x)) = \mathsf{T}$, then $v(A(t)) = \mathsf{T}$ for every term t of the same type as x; if $v(\forall x\, A(x)) = \mathsf{F}$, then there is a free variable a of the same type as x such that $v(A(a)) = \mathsf{F}$.

4) If A is an alphabetical variant of B, then $v(A) = v(B)$.

5) Let α be a free variable of type > 1. If $v(\alpha[U_1]) = \mathsf{T}$ and $v(\alpha[U_2]) = \mathsf{F}$, then there is a free variable a of appropriate type such that either $v(U_1[a]) = \mathsf{T}$ and $v(U_2[a]) = \mathsf{F}$ or $v(U_1[a]) = \mathsf{F}$ and $v(U_2[a]) = \mathsf{T}$.

If v is a semi-valuation with extensionality, and S is a sequent

$$A_1, \ldots, A_m \to B_1, \ldots, B_n,$$

then we define

$$v(S) =_{\mathrm{df}} v(\neg(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n),$$

if the latter is defined.

PROPOSITION 21.7. *If S is not cut-free provable, then there is a semi-valuation with extensionality, say v, such that $v(S) = \mathsf{F}$.*

PROOF. This is proved like the completeness theorem; construct a reduction tree for S (cf. §8). We shall only outline the definition of the immediate successors (i.e., the sequents written down immediately above those being considered) when "extensionality" and formulas of the form $\forall\varphi\, F(\varphi)$ come under attention. For the former let $\Gamma \to \Delta$ be one of the uppermost sequents in the tree which has been constructed so far. Let

$$\langle \alpha_1[U_{111}], \alpha_1[U_{112}] \rangle, \ldots, \langle \alpha_1[U_{1k_11}], \alpha_1[U_{1k_12}] \rangle, \ldots$$
$$\ldots, \langle \alpha_m[U_{m11}], \alpha_m[U_{m12}] \rangle, \ldots, \langle \alpha_m[U_{mk_m1}], \alpha_m[U_{mk_m2}] \rangle$$

be all the pairs of atomic formulas in Γ, Δ such that $\alpha_i[U_{ij1}]$ occurs in Γ and $\alpha_i[U_{ij2}]$ occurs in Δ for $j = 1, \ldots, k_i$. Let $b_{11}, \ldots, b_{1k_1}, \ldots, b_{m1}, \ldots, b_{mk_m}$ be distinct new variables of appropriate types. Write all sequents of the form $U_{ijl}[b_{ij}], \Gamma \to \Delta, U_{ijl'}[b_{ij}]$, for $i = 1, \ldots, m$, $j = 1, \ldots, k_i$, $l = 1$ or 2 and $l' = 2$ or 1 (according as $l = 1$ or 2) immediately above $\Gamma \to \Delta$.

For the stages when the higher type quantifiers come to attention, we proceed as follows. We define the notion of available free variable (at a given stage) as in §8, but in such a way that at least one free variable of each type is always available. Now, for a (higher type) $\forall$: left reduction: let $\{\forall\varphi_i F_i(\varphi_i)\}_{i=1}^{m}$ be a sequence of all the formulas in the antecedent of a sequent $\Gamma \to \Delta$ which start with higher order quantifiers. Suppose it is the kth stage. For all i, $1 \leqslant i \leqslant m$, let $V_1^i, \ldots, V_k^i$ be the first k abstracts in some predetermined list of abstracts of the same type as φ_i. Then the immediate successor of $\Gamma \to \Delta$ is

$$F_1(V_1^1), \ldots, F_1(V_k^1), \ldots, F_m(V_1^m), \ldots, F_m(V_k^m), \Gamma \to \Delta.$$

Next, for a (higher type) $\forall$: right reduction, proceed as before (i.e., case II, 9) in the proof of Lemma 8.3, replacing bound variables by free variables of the same type).

In this way we can complete the prescription for constructing the tree. As in Lemma 8.3, if each branch of this tree is finite, and (hence) ending with a sequent whose antecedent and succedent contain a formula in common, then it is easy to convert this tree to a cut-free proof in $\mathbf{S}$ of the sequent S. So if S is not cut-free provable, then there is an infinite branch. Take one such infinite branch and define a semi-valuation v as follows: $v(A) = \mathsf{T}$ if A occurs in the antecedent of a sequent in the branch and $= \mathsf{F}$ if it occurs in the succedent. It is not difficult to see that v satisfies the conditions for a semi-valuation. From the definition, $v(S) = \mathsf{F}$. We shall show only that v satisfies conditions 5) and 3) of Definition 21.6.

Suppose $v(\alpha[U_1]) = \mathsf{T}$ and $v(\alpha[U_2]) = \mathsf{F}$. Then $\alpha[U_1]$ occurs in the antecedent and $\alpha[U_2]$ in the succedent of the branch under consideration. From the construction of the tree, it follows that once $\alpha[U_1]$ occurs in the antecedent of a sequent, then it occurs in the antecedents of all the sequents above it, and likewise with $\alpha[U_2]$. Thus there is a sequent in which $\alpha[U_1]$ occurs in the antecedent and $\alpha[U_2]$ occurs in the succedent and to which the "extensionality" stage applies; thus there is a free variable a such that its immediate successor contains $U_1[a]$ in the antecedent and $U_2[a]$ in the succedent. Thus, by the definition of v, $v(U_1[a]) = \mathsf{T}$ and $v((U_2[a]) = \mathsf{F}$.

For the case of a formula $\forall\varphi\, F(\varphi)$, suppose $v(\forall\varphi\, F(\varphi)) = \mathsf{T}$. This means that $\forall\varphi\, F(\varphi)$ occurs in the antecedent of a sequent (and hence of all sequents above it). By the construction of the tree, for every abstract V of the same type as φ, $F(V)$ occurs in the antecedent of some sequent: hence $v(F(V)) = \mathsf{T}$.

DEFINITION 21.8. Given a semi-valuation with extensionality v, we define the *Henkin structure* $\mathcal{S} = (S_0, S_1, \ldots)$ induced by v, as follows. The sets $S_0, S_1, \ldots$ and relations $U^{n+1} < S$ for $S \subseteq S_n$ are defined simultaneously.

1) S_0 is the set of all terms of type 0 (in our simplified case these are only free variables). For any terms t_1 and t_2, $t_1 < t_2$ means that t_1 is identical with t_2.

2) Suppose $S_0, \ldots, S_n$ and $<$ for these types have been defined. Suppose $S \subseteq S_n$. Then $U^{n+1} < S$ if and only if for every abstract U_0^n of type n and every S^n which belongs to S_n, if $U_0^n < S^n$ and $v(U^{n+1}[U_0^n]) = \mathsf{T}$, then S^n belongs to S, and if $U_0^n < S^n$ and $v(U^{n+1}[U_0^n]) = \mathsf{F}$, then S^n does not belong to S. S_{n+1} is defined by

$$S_{n+1} =_{\mathrm{df}} \{S \mid S \subseteq S_n \text{ and there exists a } U^{n+1} \text{ such that } U^{n+1} < S\}.$$

From the definition it is obvious that $S_{n+1} \subseteq P(S_n)$.

We can think of $U < S$ as meaning: "U is a possible name for S (under the semi-valuation v)".

For convenience, we use the word "abstract" below to mean (also) a free variable of type 0. Then with any free variable a, we associate an abstract, also written a, namely $\{x\}a[x]$ if a has type > 0, and a itself if its type is 0.

PROPOSITION 21.9. *For the structure $\mathscr{S}$ defined as in Definition 21.8, it holds that given a free variable of type n, say α, there exists an element of S_n, say S, such that $\alpha < S$ (α denoting an abstract as defined above).*

PROOF. By induction on n.

Basis: $n = 0$. For every free variable a of type 0, a belongs to S_0 and $a < a$.

Induction step: let $n > 0$ and suppose the proposition holds for $0, 1, \ldots, n-1$. Let S be the set defined by

$$S =_{\mathrm{df}} \{S^{n-1} \mid S^{n-1} \text{ is in } S_{n-1} \text{ and there exists a } U^{n-1} \text{ such that}$$

$$U^{n-1} < S^{n-1} \text{ and } v(\alpha[U^{n-1}] = \mathsf{T}\}.$$

Then, by definition, $S \subseteq S_{n-1}$. We claim that $\alpha < S$. For take arbitrary U^{n-1} and S^{n-1} such that $U^{n-1} < S^{n-1}$. We show the following.

(1) If $v(\alpha[U^{n-1}]) = \mathsf{T}$, then S^{n-1} belongs to S.

(2) If $v(\alpha[U^{n-1}]) = \mathsf{F}$, then S^{n-1} does not belong to S.

(1) is obvious by definition of S. (2) is proved as follows. Suppose not (2): $v(\alpha[U^{n-1}]) = \mathsf{F}$ and S^{n-1} belongs to S. Then there is a W^{n-1} such that $W^{n-1} < S^{n-1}$, $v(\alpha[W^{n-1}]) = \mathsf{T}$.

Case 1. $n = 1$. $W^{n-1} = S^{n-1} = U^{n-1}$, yielding a contradiction.

Case 2. $n > 1$. Since $v(\alpha[U^{n-1}]) = \mathsf{F}$ and $v(\alpha[W^{n-1}]) = \mathsf{T}$, by condition 5) in Definition 21.6, there is an a such that either $v(U^{n-1}[a]) = \mathsf{F}$ and $v(W^{n-1}[a]) = \mathsf{T}$, or $v(U^{n-1}[a]) = \mathsf{T}$ and $v(W^{n-1}[a]) = \mathsf{F}$. By the induction hypothesis, there is an S^{n-2} in S_{n-2} such that $a < S^{n-2}$. If $v(U^{n-1}[a]) = \mathsf{F}$ and $v(W^{n-1}[a]) = \mathsf{T}$, then (since $v(U^{n-1}[a]) = \mathsf{F}$) S^{n-2} does not belong to S^{n-1}, since $U^{n-1} < S^{n-1}$. On the other hand, $v(W^{n-1}[a]) = \mathsf{T}$ implies that S^{n-2} belongs to S^{n-1}, since $W^{n-1} < S^{n-1}$. Thus we have a contradiction.

Similarly, if $v(U^{n-1}[a]) = \mathsf{T}$ and $v(W^{n-1}[a]) = \mathsf{F}$, we obtain a contradiction. So (2) must hold.

From (1) and (2), $\alpha < S$ by definition.

DEFINITION 21.10. We shall extend the relation $<$ to formulas and truth values as follows.

1) $A < \mathsf{T}$ if and only if $v(A) \neq \mathsf{F}$.

2) $A < \mathsf{F}$ if and only if $v(A) \neq \mathsf{T}$.

As immediate consequences of this, the following hold:

(1) if $A < *$ (where $*$ stands for F or T) and $v(A) = \mathsf{T}$, then $* = \mathsf{T}$,

(2) if $A < *$ and $v(A) = \mathsf{F}$, then $* = \mathsf{F}$;

since if $v(A) = \mathsf{T}$, then $v(A) \neq \mathsf{F}$ by the definition of v; so by 1), $A < \mathsf{T}$. $*$ cannot be F for this case by virtue of 2). (2) is proved similarly.

PROPOSITION 21.11. *Let $\mathscr{S}$ be the structure which was defined in Definition 21.8, and let ϕ be an assignment from $\mathscr{S}$. Then for any abstract or formula $U(\alpha_1, \ldots, \alpha_n)$, where all the free variables in U are among $\alpha_1, \ldots, \alpha_n$, and for any abstracts $U_1, \ldots, U_n$ (of appropriate types), if $U_i < \phi(\alpha_i)$ for $i = 1, \ldots, n$, then $U(U_1, \ldots, U_n) < \phi(U(\alpha_1, \ldots, \alpha_n))$.*

PROOF. By induction on the complexity of $U(\alpha_1, \ldots, \alpha_n)$.

(1) U is α_i. By the hypothesis, $U_i < \phi(\alpha_i)$.

The following are the induction steps.

(2) U is $\alpha_i[W(\alpha_1, \ldots, \alpha_n)]$. Let α_i be of type n_i. $U_i < \phi(\alpha_i)$ by hypothesis, which implies (by the definition of $<$) that for every $U_0^{n_i-1}$ and S^{n_i-1} in S_{n_i-1}:

1) if $U_0^{n_i-1} < S^{n_i-1}$ and $v(U_i[U_0^{n_i-1}]) = \mathsf{T}$, then $S^{n_i-1} \in \phi(\alpha_i)$,

2) if $U_0^{n_i-1} < S^{n_i-1}$ and $v(U_i[U_0^{n_i-1}]) = \mathsf{F}$, then $S^{n_i-1} \notin \phi(\alpha_i)$.

Now take $W(U_1, \ldots, U_n)$ as $U_0^{n_i-1}$ and $\phi(W(\alpha_1, \ldots, \alpha_n))$ as S^{n_i-1}. By the induction hypothesis, $W(U_1, \ldots, U_n) < \phi(W(\alpha_1, \ldots, \alpha_n))$, hence the first premiss in 1) and 2) holds. If $v(U_i[W(U_1, \ldots, U_n)]) = \mathsf{T}$, then by 1) $\phi(W(\alpha_1, \ldots, \alpha_n)) \in \phi(\alpha_i)$, and if it $= \mathsf{F}$, then by 2) $\phi(W(\alpha_1, \ldots, \alpha_n)) \notin \phi(\alpha_i)$. The first case implies $U_i[W(U_1, \ldots, U_n)] < \mathsf{T}$ by Definition 21.10, and by Definition 21.4, part 3.1), $\phi(\alpha_i[W(\alpha_1, \ldots, \alpha_n)]) = \mathsf{T}$; hence $U_i[W(U_1, \ldots, U_n)] < \phi(\alpha_i[W(\alpha_i, \ldots, \alpha_n)])$. Similarly, the second case implies $U_i[W(U_1, \ldots, U_n)] < \mathsf{F} = \phi(\alpha_i[W(\alpha_1, \ldots, \alpha_n)])$. (Note that if $v(U(U_1, \ldots, U_n))$ is defined then (trivially) $U(U_1, \ldots, U_n) < \phi(U(\alpha_1, \ldots, \alpha_n))$, by definition of $<$ in Definition 21.10.)

(3) $U(\alpha_1, \ldots, \alpha_n)$ is $\forall \varphi\, A(\varphi, \alpha_1, \ldots, \alpha_n)$.

Case 1. $v(\forall \varphi\, A(\varphi, U_1, \ldots, U_n)) = \mathsf{F}$. There is a β such that $v(A(\beta, U_1, \ldots, U_n)) = \mathsf{F}$. For this β, take the S which was constructed in Proposition 21.9 so that $\beta < S$. Let ϕ' be $\phi(^{\beta}_{S})$. Then $\beta < \phi'(\beta)$ and $U_i < \phi'(\alpha_i)$, $1 \leqslant i \leqslant n$. By the induction hypothesis

$$A(\beta, U_1, \ldots, U_n) < \phi'(A(\beta, \alpha_1, \ldots, \alpha_n)),$$

so $\phi'(A(\beta, \alpha_1, \ldots, \alpha_n)) = \mathsf{F}$ by (2) of Definition 21.10, which means that $\phi(\forall\varphi\, A(\varphi, \alpha_1, \ldots, \alpha_n)) = \mathsf{F}$, so $U(U_1, \ldots, U_n) < \phi(U(\alpha_1, \ldots, \alpha_n))$.

Case 2. $v(\forall\varphi\, A(\varphi, U_1, \ldots, U_n)) = \mathsf{T}$. Let β be a new free variable of the same type as φ. For an arbitrary S in S_n, define $\phi' = \phi\binom{\beta}{S}$. There is an abstract V such that $V < S$, so $V < \phi'(\beta)$. By the induction hypothesis,

$$A(V, U_1, \ldots, U_n) < \phi'(A(\beta, \alpha_1, \ldots, \alpha_n)).$$

From the assumption, $v(A(V, U_1, \ldots, U_n)) = \mathsf{T}$, which implies

$$\phi'(A(\beta, \alpha_1, \ldots, \alpha_n)) = \mathsf{T}.$$

This is true for every S in S_n, i.e., for every ϕ' which agrees with ϕ except at β. Thus $\phi(\forall\varphi\, A(\varphi, \alpha_1, \ldots, \alpha_n)) = \mathsf{T}$, hence $U(U_1, \ldots, U_n) < \phi(U(\alpha_1, \ldots, \alpha_n))$.

(4) $U(\alpha_1, \ldots, \alpha_n)$ is $\{x\}A(x, \alpha_1, \ldots, \alpha_n)$. Let β be a new free variable and

$$Q = \left\{S \mid \phi'(A(\beta, \alpha_1, \ldots, \alpha_n)) = T, \text{ where } \phi' = \phi\begin{pmatrix}\beta\\ S\end{pmatrix}\right\} = \phi(\{x\}\, A(x, \alpha_1, \ldots, \alpha_n)).$$

For arbitrary U_0 and S_0 of appropriate type which satisfy $U_0 < S_0$, consider $A(U_0, U_1, \ldots, U_n)$ and $\phi' = \phi\binom{\beta}{S_0}$; so $U_0 < \phi'(\beta)$. By the induction hypothesis, $A(U_0, U_1, \ldots, U_n) < \phi'(A(\beta, \alpha_1, \ldots, \alpha_n))$. Suppose $v(A(U_0, U_1, \ldots, U_n)) = \mathsf{T}$. Then

$$\phi'(A(\beta, \alpha_1, \ldots, \alpha_n)) = \mathsf{T}$$

by Definition 21.10, so $S_0 \in Q$. If $v(A(U_0, U_1, \ldots, U_n)) = \mathsf{F}$, then

$$\phi'(A(\beta, \alpha_1, \ldots, \alpha_n)) = \mathsf{F},$$

and hence $S_0 \notin Q$. Therefore, by the definition of $<$, $U(U_1, \ldots, U_n) < Q$.

Other cases are left to the reader.

PROPOSITION 21.12. $\mathscr{S}$ *(as defined in Definition* 21.5) *is a Henkin structure.*

PROOF. Let ϕ be an arbitrary assignment from $\mathscr{S}$. We have only to show that for every U, $U' < \phi(U)$ for some U' of the same type as U. Suppose all the free variables in U are among $\alpha_1, \ldots, \alpha_n$. Since $\phi(\alpha_i)$ belongs to S_{n_i}, where n_i is the type of α_i, there exists a U_i of type n_i such that $U_i < \phi(\alpha_i)$. Hence, by Proposition 21.11, $U(U_1, \ldots, U_n) < \phi(U(\alpha_1, \ldots, \alpha_n))$. So take U' to be $U(U_1, \ldots, U_n)$.

PROPOSITION 21.13. *Let $\mathscr{S}$ be the structure we have been dealing with and let ϕ_0 be an assignment from $\mathscr{S}$ which satisfies the following*:

(i) $\phi_0(a) = a$ *if a is a free variable of type* 0;

(ii) $\phi_0(\alpha) =$ *the S which was defined in Proposition* 21.9, *if α is a free variable of type* > 0 (*and $\phi_0(x)$, for bound variables x, is arbitrary*).

Let A be any formula. Then $v(A) = \mathsf{T}$ implies $\phi_0(A) = \mathsf{T}$, and $v(A) = \mathsf{F}$ implies $\phi_0(A) = \mathsf{F}$.

PROOF. For ϕ_0 as above, $\alpha < \phi_0(\alpha)$ for every free variable α. Thus, by Proposition 21.11, $A < \phi_0(A)$ (by taking U_i to be α_i). Then the proposition is a consequence of Definition 21.10.

PROPOSITION 21.14. *If a sequent S is not cut-free provable* (*in* $\mathbf{S}$), *then there exists a Henkin structure $\mathscr{S}$ and an assignment from $\mathscr{S}$, say ϕ_0, such that $\phi_0(S) = \mathsf{F}$.*

PROOF. By Proposition 21.7, there is a semi-valuation with extensionality, say v, such that $v(S) = \mathsf{F}$. Let $\mathscr{S}$ be the Henkin structure induced by v (cf. Definition 21.8 and Proposition 21.12). Let ϕ_0 be the assignment from $\mathscr{S}$ defined in Proposition 21.13. Then $v(S) = \mathsf{F}$ implies $\phi_0(S) = \mathsf{F}$, again by Proposition 21.13.

PROOF OF THEOREM 21.3. By Proposition 21.14, if S is not cut-free provable (in $\mathbf{S}$), then there is a Henkin structure $\mathscr{S}$ and an assignment ϕ_0 from $\mathscr{S}$ such that $\phi_0(S) = \mathsf{F}$. But this and Proposition 21.5 imply that S is not provable in $\mathbf{S}$ at all. In other words, if S is provable in $\mathbf{S}$, then it is provable without a cut.

REMARK. By Proposition 21.14, we have proved not only cut-elimination for $\mathbf{S}$, but also *completeness* of $\mathbf{S}$ *without the cut rule* (relative to the semantics of Henkin structures). (*Soundness* of $\mathbf{S}$ follows from Proposition 21.5.)

Next we shall prove the same theorem for the system without the extensionality rule. The method is quite similar to the proof of Theorem 21.3.

THEOREM 21.15 (the cut-elimination theorem for simple type theory without extensionality: Takahashi-Prawitz). *Let $\mathbf{S}^-$ be the system of simple type theory given in* §20. *Then the cut-elimination theorem holds for $\mathbf{S}^-$.*

PROOF. We follow the proof of Theorem 21.3, pointing out the corresponding items of §21.

DEFINITION 21.16 (cf. Definition 21.4). (1) A structure (for simple type theory without extensionality) is an ω-sequence of sets, say $\mathscr{P} = (P_0, P_1, \ldots, P_i, \ldots)$, with a relation $\in$; where

1.1) P_0 is a non-empty set,

1.2) P_{i+1} is a set of pairs of the form $\langle U^{i+1}, S\rangle$, where U^{i+1} is an abstract of type $i+1$ and S is a subset of P_i. Let $P^{i+1} = \langle U^{i+1}, S\rangle$ be an element of P_{i+1}, and let P^i be an element of P_i.
Then $P^i \in P^{i+1}$ if and only if P^i belongs to S.

(2) An assignment from $\mathscr{P}$ is a map ϕ from variables such that to every variable of type i, ϕ assigns an element of P_i. An interpretation is a pair $\mathfrak{J} = (\mathscr{P}, \phi)$.

(3) For each semi-formula or semi-term A, $\phi(A)$ is defined as in Definition 21.4 except for the following cases: $\phi(\alpha[W]) = \mathsf{T}$ if and only if $\phi(\alpha) \in \phi(W)$;

$$\phi(\{x^n\}A(x^n, x_1, \ldots, x_m)) =$$

$$= \left\langle \{x^n\}A(x^n, U_1, \ldots, U_m), \left\{P^n \,\middle|\, P^n \in P_n \wedge \phi\begin{pmatrix} x^n \\ P^n \end{pmatrix}(A(x, x_1, \ldots, x_m)) = \mathsf{T}\right\}\right\rangle,$$

where $\phi(x_i) = \langle U_i, S_i\rangle$ and all the bound variables occurring free in $\{x\}A(x)$ are among $x_1, \ldots, x_m$.

(4) A structure is called a pre-Henkin structure if for every assignment ϕ from P and every abstract U of type i, $\phi(U)$ belongs to P_i.

REMARK. The reason why we must take pairs $\langle U, S\rangle$ instead of just S in defining P^{i+1} is that $\mathscr{P}$ is a model of the comprehension axioms for which the axiom of extensionality may not hold. Thus, we cannot always identify two objects whenever they have the same extension; in order to distinguish two objects with the same extension, we consider pairs so that the names (of the extension) are explicitly expressed.

PROPOSITION 21.17 (cf. Proposition 21.5). *Suppose $\mathscr{P}$ is a pre-Henkin structure and ϕ is an assignment from $\mathscr{P}$. If a sequent S is provable in* $\mathbf{S}^-$, *then* $\phi(S) = \mathsf{T}$.

DEFINITION 21.18 (cf. Definition 21.6). Semi-valuations are defined as in Definition 21.6, omitting 5).

PROPOSITION 21.19 (cf. Proposition 21.7). *If S is not cut-free provable in* $\mathbf{S}^-$, *then there is a semi-valuation, say v, such that $v(S) = \mathsf{F}$.*

DEFINITION 21.20 (cf. Definition 21.8). Given a semi-valuation v, we define the structure $\mathscr{P}$ induced by v: the sets $P_0, P_1, \ldots$ and relations $U^i < S$ and $U^i < P^i$ (for abstracts $U^i, P^i \in P_i$ and $S \subseteq P^i$) are defined simultaneously by induction on i.

1) P_0 is the set of all free variables of type 0. $t_1 < t_2$ if t_1 is t_2.

2) Suppose $P_0, \ldots, P_i$ and $<$ for those sets have been defined. Let S be a subset of P_i. $U^{i+1} < S$ is defined to be true if and only if for every U_0^i and every P^i in P_i with $U_0^i < P^i$: $v(U^{i+1}[U_0^i]) = \mathsf{T}$ implies $P^i \in S$, and $v(U^{i+1}[U_0^i]) = \mathsf{F}$ implies $P^i \notin S$.

3) $P_{i+1} =_{\mathrm{df}} \{\langle U^{i+1}, S\rangle \mid S \subseteq P_i \text{ and } U^{i+1} < S\}$.

4) Let $P^{i+1} = \langle U^{i+1}, S\rangle$ be an element of P_{i+1} (so $U^{i+1} < S$). Then $U < P^{i+1}$ if and only if U is U^{i+1}.

PROPOSITION 21.21 (cf. Proposition 21.9). *For an arbitrary α of type n there exists a P in P_n such that $\alpha < P$.*

PROOF. There are two cases.

1) $n = 0$. For every a in P_0, $a < a$ by definition.

2) $n > 0$. Define P as

$$P =_{\mathrm{df}} \langle \alpha, S\rangle,$$

where

$$S = \{P^{n-1} \mid P^{n-1} \in P_{n-1} \text{ and there exists a } U^{n-1} \text{ such that}$$
$$U^{n-1} < P^{n-1} \text{ and } v(\alpha[U^{n-1}]) = \mathsf{T}\}.$$

We have only to show that $\alpha < S$ for this S. In order to prove $\alpha < S$ it suffices to show that for any U^{n-1} and P^{n-1} with $U^{n-1} < P^{n-1}$:

(1) If $v(\alpha[U^{n-1}]) = \mathsf{T}$, then $P^{n-1} \in S$;

(2) if $v(\alpha[U^{n-1}]) = \mathsf{F}$, then $P^{n-1} \notin S$.

The proof of (2) in this case is trivial, since $U^{n-1} < P^{n-1}$ here means that $P^{n-1} = \langle U^{n-1}, \tilde{S}\rangle$ for an appropriate $\tilde{S}$.

DEFINITION 21.22. We can extend $<$ to formulas as in Definition 21.10.

PROPOSITION 21.23 (cf. Proposition 21.11). *Let $\mathscr{P}$ be the structure defined as above. Given an assignment ϕ, define ϕ_1 as follows: If $\phi(\alpha) = \langle U_1, S\rangle$, then*

$\phi_1(\alpha) = U_1$, and for free variables a of type 0, $\phi_1(a) = \phi(a) = a$. Let U be a formula or abstract whose free variables are $\alpha_1, \ldots, \alpha_n$. If

$$\phi_1(\alpha_1) = U_1, \ldots, \phi_1(\alpha_n) = U_n,$$

then

$$U(U_1, \ldots, U_n) < \phi(U(\alpha_1, \ldots, \alpha_n)).$$

PROOF. By induction on the complexity of $U(\alpha_1, \ldots, \alpha_n)$. The argument is very much the same as the proof of Proposition 21.11. We shall give one example for the induction step. Suppose U is $\alpha_i[W(\alpha_1, \ldots, \alpha_n)]$. Let α_i be of type n_i and let $\phi(\alpha_i)$ be $\langle U_i, S_i \rangle$ (this is the case for $n_i > 0$, since $\phi_1(\alpha_i) = U_i$; for $n_i = 0$, $\phi(\alpha_i) = \alpha_i = U_i$). Since $U_i < S_i$, for every $U_0^{n_i-1}$ and P^{n_i-1} in P_{n_i-1}:

1) If $U_0^{n_i-1} < P^{n_i-1}$ and $v(U_i[U_0^{n_i}]) = \mathsf{T}$, then $P^{n_i-1} \in S_i$;

2) if $U_0^{n_i-1} < P^{n_i-1}$ and $v(U_i[U_0^{n_i-1}]) = \mathsf{F}$, then $P^{n_i-1} \notin S_i$.

Now take $W(U_1, \ldots, U_n)$ as $U_0^{n_i-1}$ and $\phi(W(\alpha_1, \ldots, \alpha_n))$ as P^{n_i-1}. By the induction hypothesis, $W(U_1, \ldots, U_n) < \phi(W(\alpha_1, \ldots, \alpha_n))$, hence the first premiss in 1) and 2) holds. If $v(U_i[W(U_1, \ldots, U_n)]) = \mathsf{T}$, then by 1), $\phi(W(\alpha_1, \ldots, \alpha_n)) \in \phi(\alpha_i)$ (since $\phi(\alpha_i) = \langle U_i, S_i \rangle$ and $\phi(W(\alpha_1, \ldots, \alpha_n))$ belongs to S_i), so $\phi(\alpha_i[W(\alpha_1, \ldots, \alpha_n)]) = \mathsf{T}$, which implies $U_i[W(U_1, \ldots, U_n)] < \phi(\alpha_i[W(\alpha_1, \ldots, \alpha_n)])$. Similarly, if $v(U_i[W(U_1, \ldots, U_n)]) = \mathsf{F}$, then by 2), $\phi(\alpha_i[W(\alpha_1, \ldots, \alpha_n)]) = \mathsf{F}$, and hence the desired relation holds.

PROPOSITION 21.24 (cf. Proposition 21.12). *$\mathscr{P}$ is pre-Henkin structure.*

PROOF. Let ϕ be an arbitrary assignment from $\mathscr{P}$. We have only to show that if $\phi(U) = \langle V, S \rangle$, then $V < S$. Suppose all the free variables in U are among $\alpha_1, \ldots, \alpha_n$. Let $U_i = \phi(\alpha_i)$. Then $U(U_1, \ldots, U_n) < \phi(U(\alpha_1, \ldots, \alpha_n))$ by Proposition 21.23. By the definition of $<$, this means that $U(U_1, \ldots, U_n)$ is V and hence $V < S$, again by definition of $<$.

PROPOSITION 21.25 (cf. Proposition 21.13). *Let $\mathscr{P}$ be the structure we have been dealing with and let ϕ be an assignment from $\mathscr{P}$ which satisfies the following:*

(i) *$\phi(a) = a$ if a is a free variable of type 0;*

(ii) *$\phi(\alpha) =$ the P which was defined in Proposition 21.21, if α is a free variable of type > 0 (and $\phi(x)$, for bound variables x, is arbitrary).*

Let A be any formula. Then $v(A) = \mathsf{T}$ implies $\phi(A) = \mathsf{T}$, and $v(A) = \mathsf{F}$ implies $\phi(A) = \mathsf{F}$.

PROOF. For ϕ as above and ϕ_1 as in Proposition 21.23, $\alpha = \phi_1(\alpha)$ for every free variable α. Thus, by Proposition 21.23, $A < \phi(A)$ (by taking U_i to be α_i). Then the proposition is a consequence of Definition 21.22.

PROPOSITION 21.26 (cf. Proposition 21.14). *If a sequent S is not cut-free provable in $\mathbf{S}^-$, then there exists a pre-Henkin structure and an assignment from $\mathscr{P}$, say ϕ, such that $\phi(S) = \mathsf{F}$.*

PROOF. The proof is parallel to Proposition 21.14.

PROOF OF THEOREM 21.15. By Propositions 21.26 and 21.17. Follow the proof of Theorem 21.3.

CHAPTER 4

INFINITARY LOGIC

In this chapter we will deal with a proof-theoretic development of infinitary logic. One reason for our interest in infinitary logic is that it enables us to establish a stronger link between model theory and proof theory. Model theory and proof theory are related to each other in many respects. For example, Craig's theorem, Beth's theorem and Tarski's theorem, stated in Chapter I, can be regarded as theorems of both model theory and proof theory. On the other hand proof theory is somewhat narrower than model theory in the sense that one cannot always express a model-theoretic result in proof-theoretic terms although the converse is usually possible. For example, although there are several proof-theoretic results containing part of the Löwenheim-Skolem theorem, one of the most fundamental theorems in model theory, we do not have a proof-theoretic version of the full theorem in ordinary (finitary) proof theory. However, if we introduce infinitary logic with an appropriate notion of proof, then the Löwenheim-Skolem theorem can be stated syntactically (see Problem 22.20).

Let α be an ordinal number, let f be a mapping from α into $\{\forall, \exists\}$ and let $\boldsymbol{x}_{<\alpha}$ denote the sequence $\{x_\xi\}_{\xi<\alpha}$. Then $\mathsf{Q}^f\boldsymbol{x}_{<\alpha}$ is a quantifier of "arity" α. If all the values of f are $\forall$ or all the values of f are $\exists$, then $\mathsf{Q}^f\boldsymbol{x}_{<\alpha}$ is a *homogeneous* quantifier that we denote by $\forall\boldsymbol{x}_{<\alpha}$ or $\exists\boldsymbol{x}_{<\alpha}$, respectively. A quantifier that is not homogeneous is called *heterogeneous*.

Heterogeneous quantifiers can occur in more general forms (Henkin). Let X and Y be disjoint sets of bound variables and let T be a function that maps Y onto a subset S of $P(X)$. We associate with T, X, Y a quantifier $\mathsf{Q}(T, X, Y)$. For simplicity let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences composed of all the elements of X and Y, respectively, ordered by some well-orderings of X and Y. Then for a formula $A(\boldsymbol{a}, \boldsymbol{b})$, $\mathsf{Q}(T, \boldsymbol{x}, \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y})$ (denoted $\mathsf{Q}^T\boldsymbol{x}\boldsymbol{y}\, A(\boldsymbol{x}, \boldsymbol{y})$) is a formula having the following meaning. Given any values of the variables $\boldsymbol{x}$ there exist values of the variables $\boldsymbol{y}$ such that (1) for each η, the value of y_η is dependent on the values of those x_ξ's that are in $T(y_\eta)$, (2) for each η, the value of y_η

is independent of the values of those x_ξ's that are not in $T(y_\eta)$, and (3) $A(\boldsymbol{x}, \boldsymbol{y})$. In other words $\mathbf{Q}^T\boldsymbol{xy}\, A(\boldsymbol{x}, \boldsymbol{y})$ is equivalent to the second order formula.

$$(\exists f_0, \ldots, f_\eta, \ldots)\ (\forall \boldsymbol{x})\ (A(\boldsymbol{x}, f_0(x_{0_0}, x_{0_1}, \ldots), \ldots, f_\eta(x_{\eta_0}, x_{\eta_1}, \ldots), \ldots),$$

where $x_{\eta_0}, x_{\eta_1}, \ldots$ are the elements of $T(y_\eta)$. For example, if $X = \{x, y\}$, $Y = \{u, v\}$ and T is defined by

$$T(u) = \{x\}, \qquad T(v) = \{y\},$$

then we have the formula $\mathbf{Q}(T, X, Y)\ A(X, Y)$ that we denote by

$$\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y; u, v).$$

It is known (Mostowski) that this quantifier cannot be defined in terms of ordinary quantifiers $\forall$ and $\exists$. Other examples of this kind will be given below.

We shall consider both homogeneous and heterogeneous quantifiers. Were we to restrict ourselves to homogeneous quantifiers, the theory obtained would be more or less like a finitary first order theory, whose nature is well-understood. The situation with regard to heterogeneous quantifiers is more interesting. One of our objectives will be to determine whether logics with heterogeneous quantifiers are like finite first order logics or finite second order logics.

An infinitary logic, with heterogeneous quantifiers $\mathbf{Q}^f\boldsymbol{x}_{<\alpha}$ such that $f(\beta) = \forall$ if β is even and $f(\beta) = \exists$ if β is odd, is of particular interest in connection with the *axiom of determinateness*, an axiom that implies many interesting theorems in set theory. The axiom of determinateness asserts that for each quantifier $\mathbf{Q}^f$, and for every formula ψ, exactly one of the two formulas

$$\mathbf{Q}^f\boldsymbol{x}_{<\alpha}\, \psi(\boldsymbol{x}_{<\alpha}, \boldsymbol{a}_{<\beta}),$$

or

$$\mathbf{Q}^{\bar{f}}\boldsymbol{x}_{<\alpha} \neg\psi(\boldsymbol{x}_{<\alpha}, \boldsymbol{a}_{<\beta})$$

is true, where $\bar{f}$ is the dual of f, that is, $\bar{f}(\gamma) = \forall$ if $f(\gamma) = \exists$, and $\bar{f}(\gamma) = \exists$ if $f(\gamma) = \forall$.

Through the axiom of determinateness we can see connections between proof theory and set theory. For example, one of the important properties of rules of inference is that they come in symmetrically related pairs. This property was essential in the proof of the cut-elimination theorem of **LK** but apparently cannot be preserved when we introduce heterogeneous quantifiers. So it seems rather hopeless to expect that the cut-elimination theorem holds in infinitary logic with heterogeneous quantifiers. However, in a determinate logic (to be defined in §23, roughly as an infinitary logic in which the axiom of determinateness holds) the rules of inference are symmetric. This offers hope that the cut-elimination theorem might hold in such a logic. It is known, however, that the cut-elimination theorem fails for a determinate logic that has disjunction and conjunction symbols of arity $2^{\aleph_0}$, that is, disjunction symbols and conjunction symbols that operate on sequences of length $2^{\aleph_0}$. Is this also the case for a determinate logic in which disjunction and conjunction are only of arity ω?

There are two approaches to the study of determinate logic, one assuming the axiom of choice, and the other without it. Without the axiom of choice, some proofs turn upon very delicate arguments. Nevertheless we can prove the following without this axiom.

Let M be a transitive model of ZF + DC, that is, Zermelo-Fraenkel set theory augmented by the axiom of dependent choice, and let the power set of ω belong to M. Then the axiom of determinateness, AD, holds in M if and only if the cut-elimination theorem holds for every determinate logic of M, i.e., every determinate logic that is M-definable.

This theorem suggests that there is a close relationship between the cut-elimination theorem and the axiom of determinateness. Furthermore there is a natural reduction in **LK** that provides a basis for proving the cut-elimination theorem. This suggests that by extending the notion of reduction to infinitary languages we may be able to prove the cut-elimination theorem and thereby learn more about the axiom of determinateness. We shall, therefore, generalize the cut rule so that a natural reduction exists for infinitary languages.

The simplest cases of infinitary logic are those systems with propositional connectives of countable arity, but quantifiers only of finite arity. Although these are very interesting logics we will give only one result concerning such systems (cf. Problem 22.21: Lopez-Escobar). For more information the reader should see: J. Barwise, Infinitary logic and admissible sets, Journal of Symbolic Logic 34(1969).

An infinitary logic can be regarded as a subsystem of a second order logic simply because one can formulate the truth definition of any significant infinitary system in a reasonable second order system. An example is given as Problem 22.26.

In defining an infinitary language, the basic idea is to determine a set of variables, a set of constants, and formation rules for formulas. There are various ways of defining the formulas of the language:

a) Accept all the formulas that are inductively defined from the constants and the variables.
b) Restrict the "admissible" formulas to some subsets of all the formulas defined as in a), with the provision that the set of admissible formulas must be closed with respect to subformulas.

Unless we state otherwise, the systems we will study are ones in which the formulas are defined as in a). Although it is common practice to set an upper bound on the cardinality of the various sets of language symbols we will not always do so.

By an infinitary language we mean the following:
1) a set of bound variables;
2) a set of free variables;
3) a set of predicate constants each with its own arity, i.e., "number" of argument places;
4) a set of individual constants;
5) a set of logical symbols.

The set of logical symbols consists of the usual unary negation sign $\neg$, and the binary implication sign $\supset$, together with a collection of disjunction, conjunction, universal quantification, and existential quantification signs each with its own arity. However, we will not use different symbols for signs with different arity. We will use only one symbol for disjunction $\vee$, one for conjunction $\wedge$, one for universal quantification $\forall$, and one for existential quantification $\exists$. We will then rely upon the context to make clear which symbols are "distinct", for example, two $\forall$'s followed by sequences of bound variables are different symbols if the lengths of the sequences are different, i.e., the $\forall$'s in $\forall \boldsymbol{x}_{<\alpha}$ and $\forall \boldsymbol{x}_{<\beta}$ are different if $\alpha \neq \beta$. The same is true of $\exists$, $\vee$, and $\wedge$. For example, the $\wedge$'s in $\wedge_{\gamma<\alpha} A_\gamma$ and $\wedge_{\gamma<\beta} A_\gamma$ are different if $\alpha \neq \beta$.

In the case of formulas defined by b) the logical symbols of the language are determined by the admissible formulas. That is, a particular symbol $\wedge$ is a symbol of the language if it occurs in some admissible formula.

§22. Infinitary logic with homogeneous quantifiers

In this section we shall formulate an infinitary logic with homogeneous quantifiers by extending the Gentzen-style calculi. Although the treatment of languages with function constants is not difficult, we will, for simplicity, consider only languages without function constants.

DEFINITION 22.1. (1) The language L consists of the following:
1) *Logical symbols*:
$\neg$ (not),
$\bigwedge$ (conjunction of arity α for certain α's),
$\bigvee$ (disjunction of arity α for certain α's),
$\forall$ (universal quantifier of arity α for certain α's),
$\exists$ (existential quantifier of arity α for certain α's).
We will sometimes write $\bigwedge_\beta$ and $\bigvee_\beta$ for $\bigwedge_{\beta<\alpha}$ and $\bigvee_{\beta<\alpha}$, when the meaning is clear from the context (and especially in the case $\alpha = \omega$).
2) *Auxiliary symbols*: (,) and , (comma).
3) *Constants*:
3.1) Individual constants; $c_0, c_1, \ldots, c_\xi, \ldots, \xi < \mu$ for some μ.
3.2) Predicate constants of arity α; $p_0^\alpha, \ldots, p_\xi^\alpha, \ldots, \xi < \gamma$ for some γ and certain α's.
4) *Variables*:
4.1) Bound variables: $x_0, x_1, \ldots, x_\eta, \ldots, \eta < K_1$.
4.2) Free variables: $a_0, a_1, \ldots, a_\xi, \ldots, \xi < K_2$.
Here K_1 and K_2 are ordinals but they are not arbitrary. We must have a sufficiently large supply of bound and free variables.

We proceed in the following way. First we fix the number of constants and logical symbols. We then add a sufficiently large supply of bound variables. We then need a very large collection of free variables. Indeed the cardinality of the set of free variables must be the same as the cardinality of the set of all formulas.

Of course the number of free variables we have influences the number of formulas. Nevertheless, in set theory, we can show that if the number of language symbols is fixed, except for the free variables, then for a sufficiently large collection of free variables, the number of free variables will be the same as the number of formulas.

(2) A *term* is either a free variable or an individual constant.

(3) *Formulas* and their outermost logical symbols we define in the following way.

(3.1) If p is a predicate constant with arity α and $\{t_\beta\}_{\beta<\alpha}$, is a sequence of terms, then $p(t_0, \ldots, t_\beta, \ldots)$ is an *atomic formula*. An atomic formula does not have an outermost logical symbol.

(3.2) If A is a formula, then $\neg A$ is a formula and its outermost logical symbol is $\neg$.

(3.3) If $\bigwedge$ ($\bigvee$) of arity α belongs to our language and $\{A_\xi\}_{\xi<\alpha}$ is a sequence of formulas then $\bigwedge_{\xi<\alpha} A_\xi$ ($\bigvee_{\xi<\alpha} A_\xi$) is a formula and its outermost logical symbol is $\wedge$ ($\vee$).

(3.4) If $\forall$ ($\exists$) of arity β belongs to our language, if A is a formula, if $\boldsymbol{a}$ is a sequence of free variables of length β, and if $\boldsymbol{x}$ is a sequence of bound variables of length β none of whose terms occur in A, then $(\forall \boldsymbol{x})\, A(\boldsymbol{x})$ $((\exists \boldsymbol{x})\, A(\boldsymbol{x}))$ is a formula whose outermost logical symbol is $\forall$ ($\exists$), where $A(\boldsymbol{x})$ is the expression obtained from A by writing x's for the corresponding a's at all occurrences of a's in A.

Subformulas are defined as for first order finite languages: If $A = \bigwedge_{\beta<\alpha} A_\beta$ is a formula of L (L-formula), then each A_β is a subformula of A, if $A : \forall \boldsymbol{x}\, A(\boldsymbol{x})$ is an L-formula, then $A(\boldsymbol{s})$ is a subformula of A for an arbitrary sequence of terms $\boldsymbol{s}$.

Of course, L must be so defined that each subformula of an L-formula is an L-formula. Since formulas are defined inductively, properties of formulas are normally proved by transfinite induction on the construction of formulas.

(4) In order to introduce the notion of proof we use auxiliary symbols $\rightarrow$ and — as before. In the following $\Gamma, \Delta, \Pi, \Lambda, \Gamma_0, \Gamma_1, \ldots$ denote sequences of formulas of length $< K^+$, where K is the cardinality of the set of all formulas in L.

$\Gamma \rightarrow \Delta$ is called a *sequent*. Γ and Δ are called the *antecedent* and *succedent* of the sequent, respectively.

The rules of inference of L are as follows:

(4.1) (Weak) structural rule of inference:

$$\frac{\Gamma \rightarrow \Delta}{\Gamma' \rightarrow \Delta'},$$

where every formula occurring in Γ occurs in Γ', and every formula occurring in Δ occurs in Δ'.

(4.2) Logical rule of inference:

$$\neg : \text{left}: \quad \frac{\Gamma \rightarrow \Delta, \{A_\lambda\}_{\lambda<\gamma}}{\{\neg A_\lambda\}_{\lambda<\gamma} \Gamma \rightarrow \Delta}$$

for some $\gamma < K^+$.

$$\neg : \text{right}: \quad \frac{\{A_\lambda\}_{\lambda<\gamma}, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \{\neg A_\lambda\}_{\lambda<\gamma}}$$

for some $\gamma < K^+$.

$$\wedge : \text{left}: \quad \frac{\{A_{\lambda,\mu}\}_{\mu<\beta_\lambda, \lambda<\gamma}, \Gamma \rightarrow \Delta}{\{\bigwedge_{\mu<\beta_\lambda} A_{\lambda,\mu}\}_{\lambda<\gamma}, \Gamma \rightarrow \Delta}$$

for some $\gamma < K^+$, where $\bigwedge_{\mu<\beta_\lambda}$ belongs to L for every $\lambda < \gamma$.

$$\bigwedge : \text{right}: \quad \frac{\Gamma \to \Delta, \{A_{\lambda,\mu_\lambda}\}_{\lambda<\gamma} \text{ for all } \{\mu_\lambda\}_{\lambda<\gamma} \text{ such that } \mu_\lambda < \beta_\lambda(\lambda < \gamma)}{\Gamma \to \Delta, \{\bigwedge_{\mu<\beta_\lambda} A_{\lambda,\mu}\}_{\lambda<\gamma}}$$

for some $\gamma < K^+$, where $\bigwedge_{\mu<\beta_\lambda}$ belongs to L for every $\lambda < \gamma$.

$$\bigvee : \text{left}: \quad \frac{\{A_{\lambda,\mu_\lambda}\}_{\lambda<\gamma}, \Gamma \to \Delta \text{ for all } \{\mu_\lambda\}_{\lambda<\gamma} \text{ such that } \mu_\lambda < \beta_\lambda(\lambda < \gamma)}{\{\bigvee_{\mu<\beta_\lambda} A_{\lambda',\mu}\}_{\lambda<\gamma}, \Gamma \to \Delta}$$

for some $\gamma < K^+$, where $\bigvee_{\mu<\beta_\lambda}$ belongs to L for each $\lambda < \gamma$.

$$\bigvee : \text{right}: \quad \frac{\Gamma \to \Delta, \{A_{\lambda,\mu}\}_{\mu<\beta_\lambda,\lambda<\gamma}}{\Gamma \to \Delta, \{\bigvee_{\mu<\beta_\lambda} A_{\lambda,\mu}\}_{\lambda<\gamma}}$$

for some $\gamma < K^+$, where $\bigvee_{\mu<\beta_\lambda}$ belongs to L for each $\lambda < \gamma$.

$$\forall : \text{left}: \quad \frac{\{A_\lambda(\boldsymbol{t}_\lambda)\}_{\lambda<\gamma}, \Gamma \to \Delta}{\{\forall \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\gamma}, \Gamma \to \Delta}$$

for some $\gamma < K^+$, where the $\boldsymbol{t}$'s are sequences of arbitrary terms of appropriate length.

$$\forall : \text{right}: \quad \frac{\Gamma \to \Delta, \{A_\lambda(\boldsymbol{a}_\lambda)\}_{\lambda<\gamma}}{\Gamma \to \Delta, \{\forall \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\gamma}},$$

for some $\gamma < K^+$, where the $\boldsymbol{a}$'s are sequences of distinct free variables of appropriate length. Each variable occurring in the $\boldsymbol{a}$'s is called an eigenvariable of the inference. When an eigenvariable a, of such an inference occurs in $\boldsymbol{a}_\lambda$, then $\forall \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)$ is called the principal formula of a and $A_\lambda(\boldsymbol{a}_\lambda)$ is called the auxiliary formula of both a and of the principal formula. The μth variable $a_{\lambda,\mu}$ in $\boldsymbol{a}_\lambda$ is said to be of order μ with respect to the principal formula $\forall \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)$.

$$\exists : \text{left}: \quad \frac{\{A_\lambda(\boldsymbol{a}_\lambda)\}_{\lambda<\gamma}, \Gamma \to \Delta}{\{\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\gamma}, \Gamma \to \Delta}$$

for some $\gamma < K^+$, where the $\boldsymbol{a}$'s are sequences of distinct free variables of appropriate length. Each of the $\boldsymbol{a}$'s is called an eigenvariable of the inference. When an eigenvariable a of such an inference occurs in $\boldsymbol{a}_\lambda$, then $\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)$ is called the principal formula of the eigenvariable and $A_\lambda(\boldsymbol{a}_\lambda)$ is called the auxiliary formula of a and of the principal formula. The μth variable $a_{\lambda,\mu}$ in $\boldsymbol{a}_\lambda$ is said to be of order μ with respect to the principal formula $\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)$.

$$\exists : \text{right}: \quad \frac{\Gamma \to \Delta, \{A_\lambda(\boldsymbol{t}_\lambda)\}_{\lambda<\gamma}}{\Gamma \to \Delta, \{\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\gamma}}$$

for some $\gamma < K^+$, where the $\boldsymbol{t}$'s are sequences of arbitrary terms of appropriate length.

(4.3) Cut rule:

$$\frac{\Gamma \to \Delta, A_1;\ \Gamma \to \Delta, A_2; \ldots ;\ \Gamma \to \Delta, A_\lambda; \ldots (\lambda < \gamma);\ \{A_\lambda\}_{\lambda<\gamma}, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}$$

for some $\gamma < K^+$.

A semi-proof P is a finite or infinite tree of sequents defined as follows: The topmost, or initial, sequents are of the form $D \to D$. Each sequent in P, but one, is an upper sequent of an inference followed by its lower sequent. The exceptional sequent is called the end sequent. A more precise definition of semi-proof is formulated inductively as follows:

1) A sequent of the form $D \to D$ alone is a semi-proof.

2) If each P_α is a semi-proof with end-sequent $\Gamma_\alpha \to \Delta_\alpha$ and

$$\frac{\ldots ; \Gamma_\alpha \to \Delta_\alpha ; \ldots}{\Gamma \to \Delta}$$

is an inference, then

$$\frac{\ldots ; P_\alpha ; \ldots}{\Gamma \to \Delta}$$

is a semi-proof.

3) Every semi-proof is obtained by 1) or 2).

Since semi-proofs are defined inductively, one can assign ordinals to sequents in a semi-proof, so that the ordinal assigned to S_1 is smaller than the ordinal assigned to S_2 if S_1 is an ancestor of S_2. Therefore it is also important to note that although a semi-proof may be an infinite figure, that is, the tree form may have infinitely many branches, each string of sequents traced up from the end-sequent or down from an initial sequent through the tree figure will be of finite length.

A semi-proof P is called a proof if P satisfies the following eigenvariable conditions.

(I) If a free variable occurs in two or more places as an eigenvariable, the principal formulas of these eigenvariables must be identical and the order of this eigenvariable with respect to each principal formula is the same in each of the inferences.

(II) For each free variable a in a proof, an ordinal number, $h(a)$ called its height, can be defined so that the height of a free variable occurring in an inference as an eigenvariable is larger than any of the heights of the free variables contained in the principal formula of that eigenvariable.

(III) No variable occurring in an inference as an eigenvariable can occur in the end sequent.

REMARK. The following is an alternate form of the eigenvariable conditions in the presence of the cut rule.

If J is an inference

$$\frac{\Gamma \to \Delta, \{A_\xi(\boldsymbol{a}_\xi)\}_{\xi<\alpha}}{\Gamma \to \Delta, \{\forall \boldsymbol{x}_\xi\, A(\boldsymbol{x}_\xi)\}_{\xi<\alpha}}$$

or

$$\frac{\{A_\xi(\boldsymbol{a}_\xi)\}_{\xi<\alpha}, \Gamma \to \Delta}{\{\exists \boldsymbol{x}_\xi\, A_\xi(\boldsymbol{x}_\xi)\}_{\xi<\alpha}, \Gamma \to \Delta}$$

then the following conditions must be satisfied:

(i) any member of $\boldsymbol{a}_\xi$ does not occur as an eigenvariable or in a principal formula of a $\forall$: right or $\exists$: left under J,

(ii) for any pair ξ, η such that $\xi < \eta$, each member of $\boldsymbol{a}_\xi$ cannot occur in $A_\eta(\boldsymbol{a}_\eta)$,

(iii) each member of $\boldsymbol{a}_\xi$ does not occur in $\forall \boldsymbol{x}_\xi\, A_\xi(\boldsymbol{x}_\xi)$ or $\exists \boldsymbol{x}_\xi\, A_\xi(\boldsymbol{x}_\xi)$,

(iv) no variable occurring in an inference as an eigenvariable can occur in the end-sequent.

It is evident that if the conditions (i)–(iv) are satisfied, then one can define "height" to satisfy (II), after renaming variables if necessary, and hence the original eigenvariable conditions hold. The converse can be proved by a method similar to that used in the last half of the proof of Proposition 22.25.

EXAMPLE 22.2. (1) A cut-free proof of the axiom of dependent choice in an infinitary logic with homogeneous quantifiers:

$$\begin{array}{r@{\;}l}
F(a_n, a_{n+1}) & \to F(a_n, a_{n+1}) \\ \hline
\{F(a_m, a_{m+1})\}_{m<\omega} & \to F(a_n, a_{n+1}) \text{ for each } n < \omega \\ \hline
\{F(a_m, a_{m+1})\}_{m<\omega} & \to \bigwedge_{n<\omega} F(a_n, a_{n+1}) \\ \hline
\{F(a_m, a_{m+1})\}_{m<\omega} & \to \exists \boldsymbol{x}_{<\omega} \bigwedge_{n<\omega} F_n \\ \hline
\{F(a_m, a_{m+1})\}_{m<\omega} & \to \forall x_0\, \exists \boldsymbol{x}_{<\omega} \bigwedge_{n<\omega} F(x_n, x_{n+1}) \\ \hline
\{\exists y\, F(a_m, y)\}_{m<\omega} & \to \forall x_0\, \exists \boldsymbol{x}_{<\omega} \bigwedge_{n<\omega} F(x_n, x_{n+1}) \\ \hline
\{\forall x\, \exists y\, F(x, y)\}_{m<\omega} & \to \forall x_0\, \exists \boldsymbol{x}_{<\omega} \bigwedge_{n<\omega} F(x_n, x_{n+1}) \\ \hline
\forall x\, \exists y\, F(x, y) & \to \forall x_0\, \exists \boldsymbol{x}_{<\omega} \bigwedge_{n<\omega} F(x_n, x_{n+1})
\end{array}$$

Here F_0 is $F(a_0, x_1)$ and F_{i+1} is $F(x_{i+1}, x_{i+2})$ for every $i < \omega$, and $x = (x_1, x_2, \ldots)$. The *heights* are defined by $h(a_m) = m$, $m < \omega$.

(2) A proof of

$$\forall x_0 \ldots (\neg \bigwedge_n x_{n+1} \in x_n), \forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_0),$$

where $\forall y \in x\, A(y)$ is an abbreviation of $\forall y\, (y \in x \supset A(y))$:

1) $\forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_0),\, a_1 \in a_0$.

PROOF.

$$\dfrac{A(a_0) \rightarrow A(a_0) \qquad \dfrac{\dfrac{a_1 \in a_0 \rightarrow a_1 \in a_0}{\rightarrow a_1 \in a_0,\, a_1 \in a_0 \supset A(a_1)}}{\rightarrow a_1 \in a_0,\, \forall y \in a_0\, A(y)}}{\dfrac{\forall y \in a_0\, A(y) \supset A(a_0) \rightarrow A(a_0),\, a_1 \in a_0}{\forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_0),\, a_1 \in a_0.}}$$

2) $\forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_n),\, a_{n+1} \in a_n$.

PROOF. Similar to that of 1).

3) $\forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_{n-k}),\, a_{n+1} \in a_n$ for $k = 0, 1, \ldots, n$.

PROOF. By induction on k we construct a figure ending with the sequent 3). Since the sequent 2) is the case $k = 0$ we need only show how to proceed from k to $k+1$:

$$\dfrac{\dfrac{A(a_{n-(k+1)}) \rightarrow A(a_{n-(k+1)}) \qquad \dfrac{\forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_{n-k}),\, a_{n+1} \in a_n}{\forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow \forall y \in a_{n-(k+1)}\, A(y),\, a_{n+1} \in a_n}}{\forall y \in a_{n-(k+1)}\, A(y) \supset A(a_{n-(k+1)}), \forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_{n-(k+1)}),\, a_{n+1} \in a_n}}{\forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_{n-(k+1)}),\, a_{n+1} \in a_n}$$

4) $\forall x (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_0), \bigwedge_n a_{n+1} \in a_n$.

PROOF. From 3) with $k = n$ we have

$$\dfrac{\forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_0),\, a_{n+1} \in a_n}{\forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_0), \bigwedge_n a_{n+1} \in a_n}.$$

From 4) we then conclude

5) $\forall x_0 \ldots (\neg \bigwedge_n x_{n+1} \in x_n), \forall x\, (\forall y \in x\, A(y) \supset A(x)) \rightarrow A(a_0)$.

In this proof $a_1, a_2, \ldots, a_n, \ldots$ are eigenvariables and $h(a_n) = n$.

(3) Malitz's example. Malitz found a counterexample to the interpolation theorem for homogeneous infinitary languages. His example is the following. Let A and B be two well-ordered sets with the same order type, and let F be a predicate that defines the order preserving map from A one-to-one onto B. That there is exactly one such map is easily proved. If the interpolation theorem held then this order preserving map could be defined in the homogeneous infinitary language without using the predicate F. This, however, is impossible because the length of the defining formula would set an upper bound on the order type of A, but that order type is not bounded. Let $\mathrm{Ln}(=, <)$ be a formula which expresses that $<$ together with $=$ is a linear ordering relation. Let Γ be the following sequence of formulas.

$$\mathrm{Ln}(\overset{1}{=}, \overset{1}{<}), \mathrm{Ln}(\overset{2}{=}, \overset{2}{<}).$$

$$\forall x\, \forall y\, \forall u\, \forall v\, (x \overset{1}{=} y \wedge u \overset{2}{=} v \supset (F(x, u) \equiv F(y, v))),$$

$$\forall x\, \forall y\, \forall u\, \forall v\, (x \overset{1}{=} y \wedge u \overset{2}{=} v \supset (G(x, u) \equiv G(y, v))),$$

$$\forall x\, \forall y\, \forall u\, \forall v\, (F(x, u) \wedge F(y, v) \supset (x \overset{1}{<} y \equiv u \overset{2}{<} v) \wedge (x \overset{1}{=} y \equiv u \overset{2}{=} v))$$

$$\forall x\, \forall y\, \forall u\, \forall v\, (G(x, u) \wedge G(y, v) \supset (x \overset{1}{<} y \equiv u \overset{2}{<} v) \wedge (x \overset{1}{=} y \equiv u \overset{2}{=} v))$$

It should be remarked that all the quantifiers in Γ are universal and at the front of a formula. The following sequent is easily proved to be valid.

$$\begin{array}{l} \Gamma, \forall x\, \exists y\, F(x, y), \forall x\, \exists y\, G(x, y) \\ \qquad \forall x\, \exists y\, F(y, x), \forall x\, \exists y\, G(y, x), F(a, b), \\ \qquad \forall x_0\, x_1 \ldots \neg \bigwedge\limits_{n} (x_{n+1} \overset{1}{<} x_n) \qquad \rightarrow G(a, b) \end{array}$$

We are going to present a cut-free proof of this sequent. Let T be the set of all finite sequences of 1's and 2's. It is understood that the empty sequence is a member of T. We use τ as a variable on T. The set D of free variables is defined as follows.

1) $a \in D$. (a is a^τ, where τ is an empty-sequence.)
2) If $a^\tau \in D$, then $b^{\tau 1}$ and $b^{\tau 2}$ are members of D.
3) If $b^\tau \in D$, then $a^{\tau 1}$ and $a^{\tau 2}$ are members of D.
4) All members of D are obtained by 1), 2) and 3).

The members of D are $a, b^1, b^2, a^{11}, a^{12}, a^{21}, a^{22}, b^{111}, b^{112}, \ldots$. Γ' is a sequence of all the formulas which are obtained from a formula in Γ by deleting all the universal quantifiers and replacing bound variables by the members of D. (From one formula, infinitely many formulas will be obtained. Of course, in one instance of substitution, the same member of D should be substituted for the same bound variable in a formula.) Δ' is a sequence of all the formulas of the form

$$F(a^\tau, b^{\tau 1}),\ F(a^{\tau 1}, b^\tau),\ G(a^\tau, b^{\tau 2}),\ G(a^{\tau 2}, b^\tau),\ \tau \in T.$$

In the following lemmas, we state several sequents which are provable in the ordinary first order predicate calculus and hence cut-free provable in Gentzen's **LK**.

We define "$\Gamma', \Delta' \to b^{\tau 11} \stackrel{2}{=} b^\tau$ is provable" to mean that $\Gamma^*, \Delta^* \to b^{\tau 11} \stackrel{2}{=} b^\tau$ is provable for some Γ^* that is a finite subsequence of Γ' and some Δ^* that is a finite subsequence of Δ'.

LEMMA 22.3. *The following are* **LK**-*provable.*

1) $\Gamma', \Delta' \to b^{\tau 11} = b^\tau$, *where* $b^{\tau 1} = b^{\tau 2}$ *is an abbreviation for* $b^{\tau 1} \stackrel{2}{=} b^{\tau 2}$. *In the same way,* $a^{\tau 1} = a^{\tau 2}$ *is an abbreviation for* $a^{\tau 1} \stackrel{1}{=} a^{\tau 2}$.
2) $\Gamma', \Delta' \to b^{\tau 22} = b^\tau$.
3) $\Gamma', \Delta' \to a^{\tau 11} = a^\tau$.
4) $\Gamma', \Delta' \to a^{\tau 22} = a^\tau$.

PROOF. Obviously, $\Gamma', F(a^{\tau 1}, b^{\tau 11}), F(a^{\tau 1}, b^\tau) \to b^{\tau 11} = b^\tau$. From this, 1) follows trivially. The proofs of 2), 3) and 4) are similar.

LEMMA 22.4. *The following are* **LK**-*provable.*

1) $\Gamma', \Delta', b^\tau = b^{\tau 12} \to a^{\tau 1} = a^{\tau 2}$.
2) $\Gamma', \Delta', a^\tau = a^{\tau 12} \to b^{\tau 1} = b^{\tau 2}$.

PROOF. 1) From $G(a^{\tau 2}, b^\tau), G(a^{\tau 1}, b^{\tau 12})$, the fourth formula of Γ with $a^{\tau 2}, a^{\tau 1}, b^{\tau 1}, b^{\tau 12}$ as x, y, u, v respectively and from $b^\tau = b^{\tau 12}$ it follows that $a^{\tau 1} = a^{\tau 2}$.

LEMMA 22.5. *The following are provable in* **LK**.

1) $\Gamma', \Delta', b^{\tau i 1} = b^{\tau i 2} \to a^{\tau 1} = a^{\tau 2}$ $(i = 1, 2)$.
2) $\Gamma', \Delta', a^{\tau i 1} = a^{\tau i 2} \to b^{\tau 1} = b^{\tau 2}$ $(i = 1, 2)$.

PROOF. Under the hypotheses of Γ', and Δ', $b^{\tau 11} = b^{\tau 12} \to b^\tau = b^{\tau 12} \to a^{\tau 1} = a^{\tau 2}$ (Lemmas 22.3 and 22.4). The other cases are proved similarly.

LEMMA 22.6. *The following is provable in* **LK**.

1) $\Gamma', \Delta', b^{\tau 1} = b^{\tau 2} \to b^1 = b^2$.

PROOF. By induction on the length of τ, using Lemma 22.5.

LEMMA 22.7. *The following are provable in* **LK**.

1) $\Gamma', \Delta', b^1 = b^2 \to G(a, b^1)$.
2) $\Gamma', \Delta', b^1 < b^2 \to a^{12} < a$, *where* $b^{\tau 1} < b^{\tau 2}$ *and* $a^{\tau 1} < a^{\tau 2}$ *are abbreviations for* $b^{\tau 1} \overset{2}{<} b^{\tau 2}$ *and* $a^{\tau 1} \overset{1}{<} a^{\tau 2}$, *respectively*.
3) $\Gamma', \Delta', b^2 < b^1 \to a^{21} < a$.

PROOF. 1) $\Gamma', G(a, b^2), b^1 = b^2 \to G(a, b^1)$.

2) $\Gamma', F(a, b^1), b^1 < b^2, G(a, b^2), G(a^{12}, b^1) \to a^{12} < a$.

3) $\Gamma', F(a, b^1), b^2 < b^1, F(a^{21}, b^2) \to a^{21} < a$.

LEMMA 22.8. *The following are provable in* **LK**.

1) $\Gamma', \Delta', b^{\tau 1} = b^{\tau 2} \to G(a, b^1)$.
2) $\Gamma', \Delta', b^{\tau 1} = b^{\tau 2} \to a^{\tau 12} < a^{\tau}$.
3) $\Gamma', \Delta', b^{\tau 2} < b^{\tau 1} \to a^{\tau 21} < a^{\tau}$.

PROOF. 1) follows from Lemma 22.6 and 1) of Lemma 22.7. The proofs of 2) and 3) are similar to the proof of Lemma 22.7.

DEFINITION 22.9. (i) $R^0(\tau)$ iff $b^{\tau 1} = b^{\tau 2}$.

(ii) $R^1(\tau)$ iff $b^{\tau 1} < b^{\tau 2}$.

(iii) $R^2(\tau)$ iff $b^{\tau 2} < b^{\tau 1}$.

(iv) $T_0 = \{\tau \in T \mid \text{the length of } \tau \text{ is odd}\}$.

LEMMA 22.10. *The following is cut-free provable for each* $i : T_0 \to \{0, 1, 2\}$

$$\{R^{i_\tau}(\tau)\}_{\tau \in T_0} \Gamma', \Delta' \to \bigwedge_n t_{n+1} \overset{1}{<} t_n, G(a, b^1),$$

where t_n *is a member of* D *whose length is* $2n$.

PROOF. Obvious from Lemma 22.8.

LEMMA 22.11. *The following is cut-free provable.*

$$\Gamma, \Delta', \forall x_0\, x_1 \ldots \neg \bigwedge_n (x_{n+1} \overset{1}{<} x_n) \to G(a, b^1).$$

PROOF. This follows from Lemma 22.10, since $\forall x\, \forall y\, (x \overset{2}{<} y \vee x \overset{2}{=} y \vee y \overset{2}{<} x)$ is contained in Γ.

THEOREM 22.12. *The following is cut-free provable.*

$$\Gamma, \Delta, \forall x_0\, x_1 \ldots \neg \bigwedge_n (x_{n+1} \overset{1}{<} x_n), F(a, b) \to G(a, b),$$

where Δ consists of $\forall x\, \exists y\, F(x, y)$, $\forall x\, \exists y\, F(y, x)$, $\forall x\, \exists y\, G(x, y)$ and $\forall x\, \exists y\, G(y, x)$.

PROOF. Take b to be b^1, and define $h(a^\tau)$ and $h(b^{\tau'})$ to be the length of τ and τ' respectively. The conclusion then follows from Lemma 22.11.

We now introduce a new cut rule, one we will find more convenient in infinitary languages than the old one. As we will prove, the new rule is a generalization of the old one.

DEFINITION 22.13 (the generalized cut rule). Let $\Gamma \to \Delta$ be a sequent and $\mathscr{F}$ be a set of formulas. Let $(\mathscr{F}_1, \mathscr{F}_2)$ denote a partition of $\mathscr{F}$ (i.e., $\mathscr{F}_1 \cup \mathscr{F}_2 = \mathscr{F}$ and $\mathscr{F}_1 \cap \mathscr{F}_2$ is empty). Suppose for an arbitrary partition of $\mathscr{F}$, $(\mathscr{F}_1, \mathscr{F}_2)$, there exists a pair of sets of formulas, say $\Phi \subseteq \mathscr{F}_1$, and $\Psi \subseteq \mathscr{F}_2$, such that there exists a semi-proof of $\Phi, \Gamma \to \Delta, \Psi$. Then the generalized cut rule allows us to infer $\Gamma \to \Delta$. This may be expressed as follows:

$$\text{g.c.}\quad \frac{\Phi, \Gamma \to \Delta, \Psi \text{ for all } (\mathscr{F}_1, \mathscr{F}_2)}{\Gamma \to \Delta}$$

PROPOSITION 22.14. (1) *The usual cut rule is a special case of the g.c. rule.*

(2) *The following is an admissible rule of inference:*

$$\frac{\Gamma \to \Delta}{\tilde{\Gamma} \to \tilde{\Delta}},$$

where $\tilde{\Gamma}$ is obtained from Γ by replacing some of the formulas by alphabetical variants. Similarly with $\tilde{\Delta}$.

(3) *Suppose that some (possibly all) of the upper sequents of a g.c. are obtained by applications of the g.c. rule. Then we can change the proof so that the lower sequent will be obtained by one application of the g.c. rule.*

(4) *In homogeneous systems the g.c. rule is an admissible rule of inference.*

(5) *The g.c. rule can be equivalently expressed as*

$$\frac{\Phi, \Gamma' \to \Delta', \Psi \text{ for all } (\mathscr{F}_1, \mathscr{F}_2)}{\Gamma \to \Delta},$$

where $\Gamma' \subseteq \Gamma$ *and* $\Delta' \subseteq \Delta$ *are determined by* $(\mathscr{F}_1, \mathscr{F}_2)$ *and* Φ *and* Ψ *have the same meaning as before.*

PROOF. (1) Consider a cut:

$$\frac{\Gamma \to \Delta, D_\mu \text{ for all } \mu < \lambda;\ \{D_\mu\}_{\mu<\lambda}, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}.$$

First we obtain $\Gamma, \Pi \to \Delta, \Lambda, D_\mu$ and $\{D_\mu\}, \Gamma, \Pi \to \Delta, \Lambda$ by applications of weakening. Let $\mathscr{F}$ be $\{D_\mu\}_{\mu<\lambda}$ and let $(\mathscr{F}_1, \mathscr{F}_2)$ be a partition of $\mathscr{F}$.

Case 1. $\mathscr{F}_2$ is not empty. Then take Φ to be the empty set and Ψ to be $\{D_\mu\}$, where D_μ is the first formula in $\mathscr{F}_2$.

Case 2. $\mathscr{F}_2$ is empty. Then take Φ to be $\mathscr{F}_1$, which is $\mathscr{F}$, and Ψ to be the empty set.

For any (Φ, Ψ) above, $\Phi, \Gamma \to \Delta, \Psi$ is an upper sequent of the cut in consideration. By the g.c. rule

$$\frac{\Phi, \Gamma \to \Delta, \Psi, \text{ for all } (\Phi, \Psi) \text{ as above}}{\Gamma \to \Delta}.$$

(2) We shall show that if a sequent $\Gamma \to \Delta$ is provable, then another sequent $\tilde{\Gamma} \to \tilde{\Delta}$ can be deduced, where $\tilde{\Gamma}$ is obtained from Γ by simply renaming some of the bound variables. Similarly with $\tilde{\Delta}$. For any formula A, if $\tilde{A}$ is an alphabetical variant of A, then $A \equiv \tilde{A}$ is easily proved. If $\Gamma \to \Delta$ is provable, then $\{A_\lambda \equiv \tilde{A}_\lambda\}_{\lambda<\mu}, \tilde{\Gamma} \to \tilde{\Delta}$ is provable for some A_λ's and $\tilde{A}_\lambda$'s. Using $\to A_\lambda \equiv \tilde{A}_\lambda$ for all $\lambda < \mu$, we obtain $\tilde{\Gamma} \to \tilde{\Delta}$ by the g.c. rule.

(3) Let I be the cut under consideration:

$$\frac{\Phi, \Gamma \to \Delta, \Psi \text{ for all appropriate } (\Phi, \Psi)}{\Gamma \to \Delta}.$$

The proof is by transfinite induction on the complexity of the subproof ending with $\Gamma \to \Delta$. Suppose, as the inductive hypothesis, that there is at most one cut along any string of sequents above I. Let $\{(\Phi_\mu, \Psi_\mu)\}_{\mu<\mu_0}$ be an enumeration of the (Φ, Ψ)'s in I and let $\mathscr{F}$ be the set of cut formulas. Let S_μ denote the sequent $\Phi_\mu, \Gamma \to \Delta, \Psi_\mu$. Let $\{I^\mu_\iota\}_{\iota<\nu_\mu}$ be an enumeration of all the cuts above S_μ and let $\mathscr{F}^\mu_\iota$ be the set of cut formulas of I^μ_ι for each (μ, ι). Let $\{\Phi^{\mu,\iota}_\gamma, \Psi^{\mu,\iota}_\gamma)\}_{\gamma<\delta^\mu_\iota}$

be an enumeration of the pairs of formulas which are related to I_ι^μ and hence to $\mathscr{F}_\iota^\mu$.

For each I_ι^μ, consider

$$\frac{\Phi_\gamma^{\mu,\iota}\Pi_\iota^\mu \rightarrow \Lambda_\iota^\mu \Psi_\gamma^{\mu,\iota}}{\Pi_\iota^\mu \Phi_\gamma^{\mu,\iota} \rightarrow \Psi_\gamma^{\mu,\iota}\Lambda_\iota^\mu}$$

for every $\gamma < \delta_\iota^\mu$. For every combination of γ's, i.e., $(\gamma^0, \gamma^1, \ldots, \gamma^\iota, \ldots)$, $\gamma^\iota < \delta_\iota^\mu$, copy the part of the original proof from $\Pi_\iota^\mu \rightarrow \Lambda_\iota^\mu$ to S_μ, starting with $\Pi_\iota^\mu, \Phi_\gamma^{\mu,\iota} \rightarrow \Psi_\gamma^{\mu,\iota}, \Lambda_\iota^\mu$ obtained as above, in place of $\Pi_\iota^\mu \rightarrow \Lambda_\iota^\mu$. Thus we obtain

$$(*) \qquad \Phi_\mu, \Gamma, \{\Phi_{\gamma^\iota}^{\mu,\iota}\}_{\iota<\nu_\mu} \rightarrow \{\Psi_{\gamma^\iota}^{\mu,\iota}\}_{\iota<\nu_\mu}, \Delta, \Psi_\mu$$

for every μ and $(\gamma^0, \gamma^1, \ldots, \gamma^\iota, \ldots)$. Call such a sequent $S_\mu(\{\gamma^\iota\}_{\iota<\nu_\mu})$.

Now consider the set of formulas $\mathscr{F}_0 = \mathscr{F} \cup \bigcup_{\mu,\iota} \mathscr{F}_\iota^\mu$ and an arbitrary partition of $\mathscr{F}_0$, say $\mathscr{F}_1$ and $\mathscr{F}_2$. There exist $\mu < \mu_0$ and $\{\gamma^\iota\}_{\iota<\nu_\mu}$ such that $\Phi_\mu \subseteq \mathscr{F}_1 \cap \mathscr{F}$, $\Psi_\mu \subseteq \mathscr{F}_2 \cap \mathscr{F}$, $\Phi_{\gamma^\iota}^{\mu,\iota} \subseteq \mathscr{F}_\iota^\mu \cap \mathscr{F}_1$ and $\Psi_{\gamma^\iota}^{\mu,\iota} \subseteq \mathscr{F}_\iota^\mu \cap \mathscr{F}_2$, for $\mathscr{F}_1$ and $\mathscr{F}_2$ determine partitions of $\mathscr{F}$ and $\mathscr{F}_\iota^\mu$. Define

$$\Phi = \Phi_\mu \cup \bigcup_{\iota<\nu_\mu} \Phi_{\gamma^\iota}^{\mu,\iota}, \qquad \Psi = \Psi_\mu \cup \bigcup_{\iota<\nu_\mu} \Psi_{\gamma^\iota}^{\mu,\iota}.$$

It is obvious that $\Phi \subseteq \mathscr{F}_1$, $\Psi \subseteq \mathscr{F}_2$ and $\Phi, \Gamma \rightarrow \Delta, \Psi$ is one of the sequents in (*). There is no cut above it. Since this holds for every partition of $\mathscr{F}_0$, we obtain

$$\text{g.c.} \quad \frac{\Phi, \Gamma \rightarrow \Delta, \Psi \text{ for all appropriate } (\Phi, \Psi)}{\Gamma \rightarrow \Delta}.$$

(4) As will be proved in Theorem 22.17, this follows easily from the completeness of the homogeneous systems and the fact that the g.c. rule preserves the validity of sequents.

(5) Obvious.

DEFINITION 22.15. Let L be an infinitary language. We define a structure for L, $\langle D, \phi \rangle$, an interpretation $\mathfrak{I} = (\langle D, \phi \rangle, \phi_0)$, the relation that an interpretation $\mathfrak{I}$ satisfies a formula A of L, the validity of a formula, the satisfaction relation for sequents and the validity of a sequent, as in Definition 8.1.

PROPOSITION 22.16 (consistency; Maehara-Takeuti). *Let $\mathscr{A}$ be an arbitrary structure for* L. *Then every provable sequent is valid in $\mathscr{A}$.*

PROOF. For each formula of the form $\exists \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{a})$ in which $\boldsymbol{x}$ is of length α, and $\boldsymbol{a}$ are exactly the free variables in A, we introduce a Skolem function $g^{\gamma}_{A}(\boldsymbol{a})$ for each $\gamma < \alpha$, and define the following interpretation of g^{γ}_{A} in $\mathscr{A}$:

If $\forall \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{a})$ is satisfied in $\mathscr{A}$ for an assignment ϕ_0, then the values of the g^{γ}_{A}'s are those satisfying

$$A(g^{<\alpha}_{A}(\boldsymbol{a}), \boldsymbol{a}).$$

Let 0 be an element of the domain of $\mathscr{A}$. If $\exists \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{a})$ is not satisfied by ϕ_0 in $\mathscr{A}$, then the $g^{\gamma}_{A}(\boldsymbol{a})$'s are interpreted to be 0.

Let P be a proof. We well-order all the eigenvariables in P, arranging the well-ordered sequences $a_0, a_1, \ldots, a_\beta, \ldots$, in such a way that $h(a_\beta) \leqslant h(a_\gamma)$ if $\beta < \gamma$. We define terms t_β by transfinite induction on β. Assuming that $\boldsymbol{t}_{<\beta}$ has been defined, we define t_β in the following way. Let $\forall \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{b})$ (or $\exists \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{b})$) and $A(\boldsymbol{d}, \boldsymbol{b})$ be the principal formula and an auxiliary formula of a_β and let the order of a_β with respect to this principal formula be γ, i.e., let a_β be d_γ. For each b_ν let s_ν be either the already defined t_γ for which b_ν is a_γ, if b_ν is an eigenvariable; or else b_ν itself. Then t_β is defined to be $g^{\gamma}_{\neg A}(\boldsymbol{s})$ (or $g^{\gamma}_{A}(\boldsymbol{s})$). By (I) of the eigenvariable condition, this definition does not depend on the choice of $A(\boldsymbol{a}, \boldsymbol{b})$.

Let P' be the result obtained from P by substituting t_β for a_β for every β. The bottom sequent of P' is the end sequent of P since it contains no eigenvariables.

For an arbitrary assignment of members of D to the free variables, any sequent S in P' is satisfied in A, where the g^{γ}_{A}'s are interpreted as above. This can be proved by transfinite induction on the complexity of the figure in P above S. As a consequence, the end-sequent of P is valid, since it does not involve eigenvariables. The other cases being obvious, we only consider $\exists$: left and $\forall$: right.

1) $\exists$: left. The corresponding part of P' is

$$\frac{\ldots, A(\boldsymbol{u}, \boldsymbol{s}), \ldots, \Gamma \to \Delta}{\ldots, \exists \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s}), \ldots, \Gamma \to \Delta},$$

where u_γ is $g^{\gamma}_{A}(\boldsymbol{s})$. It suffices to show that

$$\exists \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s}) \to A(\boldsymbol{u}, \boldsymbol{s}),$$

is satisfied in $\mathscr{A}'$; but this follows from the definition of the g^{γ}_{A}'s.

2) $\forall$: right. The corresponding part of P' is

$$\frac{\Gamma \to \Delta, \ldots, A(\boldsymbol{v}, \boldsymbol{s}), \ldots}{\Gamma \to \Delta, \ldots, \forall \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s}), \ldots}$$

where v_γ is $g^\gamma_{\neg A}(s)$. So it suffices to show that

$$A(\boldsymbol{v}, \boldsymbol{s}) \to \forall \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s}),$$

is satisfied in $\mathscr{A}$. This follows from

$$\exists \boldsymbol{x}\, \neg A(\boldsymbol{x}, \boldsymbol{s}) \to A(\boldsymbol{v}, \boldsymbol{s}),$$

which follows from the definition of the $g^\gamma_{\neg A}$'s.

We shall now prove the completeness theorem in combination with the cut-elimination theorem for an infinitary logic with homogeneous quantifiers. The method is basically the same as that for the proof of the completeness theorem in Chapter I.

As a corollary to the completeness theorem (Theorem 22.17) and Proposition 22.16 we have the cut-elimination theorem: Every provable sequent is provable without the cut rule.

THEOREM 22.17 (Maehara-Takeuti). *Every sequent valid in any non-empty domain is provable without the cut rule.*

PROOF. Let S be an arbitrary sequent and let D_0 be an arbitrary non-empty set containing all free variables and individual constants in S. Let D be the closure of D_0 with respect to all the function symbols g^γ_A and $\bar{g}^\gamma_A$ for all formulas A, i.e., let D be generated by all g^γ_A's and $\bar{g}^\gamma_A$'s from D_0. Here $\bar{g}^\gamma_A$ is $g^\gamma_{\neg A}$.

We define the tree $T(S)$ step by step.

Stage 0. We write S.

Stage $n+1$. (1) $n + 1 \equiv 1 \pmod 5$. When a sequent $\Pi \to \Lambda$ contains a formula whose outermost logical symbol is $\neg$, we write above $\Pi \to \Lambda$

$$\{D_\mu\}_{\mu<\delta}, \Pi' \to \Lambda' \{C_\lambda\}_{\lambda<\gamma},$$

where $\{\neg C_\lambda\}_{\lambda<\gamma}$ and $\{\neg D_\mu\}_{\mu<\delta}$ are the sequences of all formulas in Π and in Λ, respectively, whose outermost logical symbol is $\neg$, and Π' and Λ' are obtained from Π and Λ respectively by omitting the $\neg C_\lambda$'s and the $\neg D_\mu$'s.

(2) $n + 1 \equiv 2 \pmod 5$. When a sequent $\Pi \to \Lambda$ contains a formula whose outermost logical symbol is $\bigwedge$, we write above $\Gamma \to \Lambda$

$$\{C_{\mu,\lambda}\}_{\lambda<\gamma_\mu, \mu<\gamma}, \Pi' \to \Lambda', \{D_{\sigma,\rho_\sigma}\}_{\sigma<\delta}$$

for all sequences $\{\rho_\sigma\}_{\sigma<\delta}$ such that $\rho_\sigma < \nu_\sigma$, where $\{\bigwedge_{\lambda<\gamma_\mu} C_{\mu,\lambda}\}_{\mu<\gamma}$ and $\{\bigwedge_{\rho<\nu_\sigma} D_{\sigma,\rho}\}_{\sigma<\delta}$ are the sequences of all formulas in Π and in Λ, respectively, whose outermost logical symbol is $\wedge$, and Γ' and Δ' are obtained from Π and Δ, respectively, by omitting the $\bigwedge_{\lambda<\gamma_\mu} C_{\mu,\lambda}$'s and the $\bigwedge_{\rho<\nu_\sigma} D_{\sigma,\rho}$'s.

(3) $n + 1 \equiv 3 \pmod 5$. When a sequent $\Pi \to \Delta$ contains a formula whose outermost logical symbol is $\vee$, we write above $\Pi \to \Delta$

$$\{C_{\mu,\lambda_\mu}\}_{\mu<\gamma}, \Pi' \to \Delta', \{D_{\sigma,\rho}\}_{\rho<\nu_\sigma,\sigma<\delta}$$

for all sequences $\{\lambda_\mu\}_{\mu<\gamma}$ such that $\lambda_\mu < \gamma_\mu$, where $\{\bigvee_{\lambda<\gamma_\mu} C_{\mu,\lambda}\}_{\mu<\gamma}$ and $\{\bigvee_{\rho<\nu_\sigma} D_{\sigma,\rho}\}_{\sigma<\delta}$ are the sequences of all formulas in Π and Δ, respectively, whose outermost logical symbol is $\vee$, and Π' and Δ' are obtained from Π and Δ, respectively, by omitting the $\bigvee_{\lambda<\gamma_\mu} C_{\mu,\lambda}$'s and the $\bigvee_{\rho<\delta_\sigma} D_{\sigma,\rho}$'s.

(4) $n + 1 \equiv 4 \pmod 5$. When a sequent $\Pi \to \Delta$ contains a formula whose outermost logical symbol is $\forall$, we write above $\Pi \to \Delta$

$$\{A_\lambda(\boldsymbol{t}_{\lambda,\mu})\}_{\lambda<\gamma,\mu} \Pi' \to \Delta', \{B_\rho(\boldsymbol{u}_{\rho,\sigma})\}_{\sigma<\delta,\rho},$$

where $\{\forall \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\gamma}$ and $\{\forall \boldsymbol{y}_\rho\, B_\rho(\boldsymbol{y}_\rho)\}_{\rho<\delta}$ are the sequences of all formulas in Π and Δ, respectively, whose outermost logical symbol is $\forall$, and Π' and Δ' are obtained from Π and Δ respectively by omitting the $\forall \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)$'s and the $\forall \boldsymbol{y}_\rho\, B_\rho(\boldsymbol{y}_\rho)$'s. Furthermore, $\boldsymbol{t}_{\lambda,\mu}$ runs over all sequences of members of D that are the same length as $\boldsymbol{x}_\lambda$.

If $\boldsymbol{u}_{\rho,\sigma}$ is $u_{\rho,\sigma,0}, u_{\rho,\sigma,1}, \ldots, u_{\rho,\sigma,\nu}$, ν being the length of $\boldsymbol{y}_\rho$, then $u_{\rho,\sigma,\xi}$ is $g^\xi_{A_\rho}(\boldsymbol{v}_\rho)$, where $\xi < \nu$ and $\boldsymbol{v}_\rho$ is the sequence of free variables in $B_\rho(\boldsymbol{y}_\rho)$.

(5) $n + 1 \equiv 0 \pmod 5$. When a sequent $\Pi \to \Delta$ contains a formula whose outermost logical symbol is $\exists$, we write above $\Pi \to \Delta$

$$\{A_\lambda(\boldsymbol{t}_{\lambda,\mu})\}_{\gamma<\lambda,\mu}, \Pi' \to \Delta', \{B_\rho(\boldsymbol{u}_{\rho,\sigma})\}_{\sigma<\delta,\rho},$$

where $\{\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\gamma}$ and $\{\exists \boldsymbol{y}_\rho\, B_\rho(\boldsymbol{y}_\rho)\}_{\rho<\delta}$ are the sequences of all formulas in Π and in Δ, respectively, whose outermost logical symbol is $\exists$, and Π' and Δ' are obtained from Π and Δ, respectively, by omitting the $\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)$'s and the $\exists \boldsymbol{y}_\rho\, B_\rho(\boldsymbol{y}_\rho)$'s. Here $\boldsymbol{u}_{\rho,\sigma}$ runs over all sequences of the same length as that of $\boldsymbol{y}_\rho$, and, if $\boldsymbol{t}_{\lambda,\mu}$ is $t_{\lambda,\mu,0}, t_{\lambda,\mu,1}, \ldots, t_{\lambda,\mu,\eta}$, η being the length of $\boldsymbol{x}_\lambda$, then $t_{\lambda,\mu,\xi}$ is $g^\xi_{A_\lambda}(\boldsymbol{s}_\lambda)$ for $\xi < \eta$, where $\boldsymbol{s}_\lambda$ is the sequence of free variables in $A_\lambda(\boldsymbol{x}_\lambda)$.

Let S_1 and S_2 be sequents in $T(S)$. S_2 is called an immediate ancestor of S_1 if S_2 is one of the sequents written above S_1 by applying one of (1)–(5) to S_1. A branch of $T(S)$ is a sequence $S = S_0, S_1, \ldots$, possibly infinite, such that S_{n+1} is always an immediate ancestor of S_n.

For any sequent $\Gamma \to \Delta$ only one of two cases is possible:

Case 1. In every branch of $T(\Gamma \to \Delta)$ there exists at least one sequent of the form

$$\Gamma_1, D, \Gamma_2 \to \Delta_1, D, \Delta_2.$$

In this case we can obtain a proof of $\Gamma \to \Delta$ without the cut rule by modifying $T(S)$, and regarding the elements of $D - D_0$ as free variables. (The proof is left to the reader).

Case 2. There exists a branch B of $T(\Gamma \to \Delta)$ in which no sequent is of the form

$$\Gamma_1, D, \Gamma_2 \to \Delta_1, D, \Delta_2.$$

In this case we claim that there is an interpretation in which every formula occurring in Γ is true and every formula occurring in Δ is false. In the remainder of this proof we fix such a branch B and consider only the formulas and sequents occurring in B, i.e., "sequent" means "sequent in B".

First observe the following lemmas:

LEMMA 22.18. (1) *If a formula $\neg A$ occurs in the antecedent (succedent) of a sequent, then the formula A occurs in the succedent (antecedent) of a sequent.*

(2) *If a formula $\bigwedge_{\lambda<\beta} A_\lambda$ occurs in the antecedent (succedent) of a sequent, then for every (some) $\lambda < \beta$, A_λ occurs in the antecedent (succedent) of a sequent.*

(3) *If a formula $\bigvee_{\lambda<\beta} A_\lambda$ occurs in the antecedent (succedent) of a sequent, then for some (every) $\lambda < \beta$, A_λ occurs in the antecedent (succedent) of a sequent.*

(4) *If $\forall \boldsymbol{x}\, A(\boldsymbol{x})$ occurs in the antecedent of a sequent, then for every sequence $\boldsymbol{t}$ of elements of D whose length is the same as that of $\boldsymbol{x}$, the formula $A(\boldsymbol{t})$ occurs in the antecedent of a sequent. If $\forall \boldsymbol{x}\, A(\boldsymbol{x})$ occurs in the succedent of a sequent, then the formula $A(\boldsymbol{t})$ occurs in the succedent of a sequent, where t_γ is $\bar{g}^\gamma_A(\boldsymbol{s})$, $\boldsymbol{s}$ being the sequence of the free variables in $A(\boldsymbol{x})$.*

(5) *If $\exists \boldsymbol{x}\, A(\boldsymbol{x})$ occurs in the antecedent of a sequent, then the formula $A(\boldsymbol{t})$ occurs in the antecedent of a sequent, where t_γ is $g^\gamma_A(\boldsymbol{s})$, $\boldsymbol{s}$ being the sequence of the free variables in $A(\boldsymbol{x})$. If $\exists \boldsymbol{x}\, A(\boldsymbol{x})$ occurs in the succedent of a sequent, then for an arbitrary sequence $\boldsymbol{t}$ of elements of D whose length is the same as that of $\boldsymbol{x}$, the formula $A(\boldsymbol{t})$ occurs in the succedent of a sequent.*

(6) *If a formula occurs in the antecedent of a sequent, then it does not occur in the succedent of any sequent.*

PROOF. (1)–(5) are obvious from the definition of $T(S)$. (6) can be proved by transfinite induction on the complexity of the formula using (1)–(5).

It is now evident how to define ϕ: For each term $t \in D$ $\phi t = t$. For any predicate constant R, $R(\boldsymbol{t})$ holds in $\langle D, \phi \rangle$ if and only if it occurs in the antecedent of a sequent. This completes the proof of Theorem 22.17.

Note that in the proof of the completeness theorem we need a sequence of new free variables $\boldsymbol{a}$ for every subformula $\exists \boldsymbol{x}\, A(\boldsymbol{x})$ of the end sequent. Moreover for each such sequence $\boldsymbol{a}$ we need another free variable for each instance $A(\boldsymbol{a})$. We then see why we must have a very large supply of free variables available or we must be able to rename the variables that are present.

Briefly we shall consider systems with equality.

DEFINITION 22.19. We define an infinitary logic with homogeneous quantifiers with equality by specifying a binary predicate constant $=$ and adjoining the following rules of inference to those of L:

1) First rules for equality: Let $\Gamma^{(\boldsymbol{a})}$ stand for a sequence of formulas Γ in which some occurrences of $\boldsymbol{a}$ are indicated.

$$\frac{\Gamma^{(\boldsymbol{a})} \to \Delta^{(\boldsymbol{b})}}{\boldsymbol{a} = \boldsymbol{b},\, \Gamma^{(\boldsymbol{b})} \to \Delta^{(\boldsymbol{a})}}\,; \qquad \frac{\Gamma^{(\boldsymbol{a})} \to \Delta^{(\boldsymbol{a})}}{\boldsymbol{b} = \boldsymbol{a},\, \Gamma^{(\boldsymbol{b})} \to \Delta^{(\boldsymbol{b})}}\,.$$

Here $\boldsymbol{a} = \boldsymbol{b}$ denotes the sequence $\{a_\lambda = b_\lambda\}_{\lambda < \gamma}$ and $\Gamma^{(\boldsymbol{b})} \to \Delta^{(\boldsymbol{b})}$ denotes the result obtained from $\Gamma^{(\boldsymbol{a})} \to \Delta^{(\boldsymbol{a})}$ by replacing the indicated occurrences of a_λ by b_λ for each $\lambda < \gamma$.

2) Second rule for equality: Let Σ be an arbitrary set of free variables and let $\tilde{\Sigma}$ be the set of all atomic formulas $a = b$ such that a and b belong to Σ. $(\Phi | \Psi)$ is called a *decomposition* of $\tilde{\Sigma}$ if $\Phi \cup \Psi = \tilde{\Sigma}$ and $\Phi \cap \Psi = 0$.

$$\frac{\Phi, \Gamma \to \Delta, \Psi \text{ for all decompositions } (\Phi|\Psi) \text{ of } \tilde{\Sigma}}{\Gamma \to \Delta}\,.$$

Theorems corresponding to Proposition 22.16 and Theorem 22.17 hold for this system. Proofs can be obtained as special cases of the proofs of the corresponding theorems in the following section.

PROBLEM 22.20. Consider a finite, first order language L with K individual constants, where K is a cardinal. The Löwenheim-Skolem theorem is stated as follows: Let $\mathscr{F}$ be a set of L-formulas. If $\mathscr{F}$ has a model then there exists a model of cardinality K. Let L′ be an infinitary homogeneous language which is an extension of L (hence L′ has at least K individual constants). It is easily seen that the Löwenheim-Skolem theorem can be stated syntactically as follows: Let $\Gamma \to \Delta$ be a sequent of L, where the lengths of Γ and Δ can be any ordinal less than K^+. If the sequent

$$\exists x_0 \, \exists x_1 \, \ldots \, \exists x_\xi \, \ldots \, \forall y \, (\bigvee_{\xi<K} y = x_\xi), \, \Gamma \to \Delta$$

is provable in the homogeneous system, then so is $\Gamma \to \Delta$, where $=$ is not singled out in L or in L$'$.

Give a proof-theoretical proof of the Skolem-Löwenheim theorem in the syntactical form.

[*Hint*: 1) Introduce new constants $\{w_\alpha\}_{\alpha<K}$. Let $\mathrm{L}_0 = \mathrm{L} \cup \{w_\alpha\}_{\alpha<K}$ and consider the closed L_0-formulas of the form $\exists x \, F(x)$. We can define an enumeration (with repetition) of such formulas, $\{\exists x \, F_\alpha(x)\}_{\alpha<K}$, in such a manner that

(i) in $F_\alpha(x)$ no w_γ with $\gamma \geqslant \alpha$ occurs.

2) Let $\tilde{\mathrm{L}} = \mathrm{L}' \cup \{w_\alpha\}_{\alpha<K}$ and let $R(a)$ be $\bigvee_{\alpha<K} (a = w_\alpha)$. The relativization of formulas (of $\tilde{\mathrm{L}}$) to R, (the relativization of A to R is denoted by A^R), is defined as in §17: $(\exists \boldsymbol{y} \, A(\boldsymbol{y}))^R$ is $\exists \boldsymbol{y} \, (\bigvee_{\gamma<\delta} R(y_\gamma) \wedge A^R(y))$, where $\boldsymbol{y}$ is $\boldsymbol{y}_{<\delta}$.

3) It is obvious that $R(w_\gamma)$ is provable for every $\gamma < K$; hence $(\exists \boldsymbol{x} \, \forall y \, (\bigvee_{\beta<K} y = x_\beta))^R$ is provable. With the same method as in the theory of relativization in §17, we can prove the following:

Let $\Pi \to \Lambda$ be a sequent of L$'$. Let $\{b_i\}_{i<\beta}$ be the sequent of all free variables in $\Pi \to \Lambda$. If $\Pi \to \Lambda$ is provable in the homogeneous system (with language L$'$), then

$$\{R(b_i)\}_{i<\beta}, \, \Pi^R \to \Lambda^R$$

is provable in the homogeneous system with language L_0, where Π^R is obtained from Π by replacing each of its formula, say A, by A^R; similarly with Λ^R.

4) A proof-like figure is called a quasi-proof (of the homogeneous system) if it satisfies all the conditions of the proofs, except (II) and (III) of the eigenvariable conditions.

Besides the condition (i) in 1), we may require furthermore that for the enumeration of $\exists x \, F_\alpha(x)$'s the following holds.

(ii) There is an ω-type subset of $\{w_\alpha\}$, say $\Sigma = \{w_{\nu_0}, w_{\nu_1}, \ldots\}$ such that if Γ^* consists of all the $\exists x \, F_\alpha(x) \supset F_\alpha(w_\alpha)$, except those with $w_\alpha \in \Sigma$, then for every closed formula A of L_0 there is a quasi-proof ending with

$$\Gamma^* \to A \equiv A^R.$$

5) Suppose now that

$$\exists \boldsymbol{x} \, \forall y \, (\bigvee_{\alpha<K} y = x_\alpha), \, \Gamma \to \Delta$$

is provable (in the homogeneous system). Then by 3)

$$\{R(b_i)\}_{i<\mu}, \Gamma^R \to \Delta^R$$

is provable, where $\{b_i\}_{i<\mu}$ is the sequence of the free variables in Γ and Δ. Then $\mu \leqslant \omega$. We may identify b_i with w_{ν_i} in Σ; thus we may assume that

$$\{R(w_{\nu_i})\}_{i<\mu}, \Gamma^R\{w_{\nu_i}\}_{i<\mu} \to \Delta^R\{w_{\nu_i}\}_{i<\mu},$$

is provable where $\Gamma^R\{w_{\nu_i}\}_{i<\mu}$ is obtained from Γ^R by replacing b_i by w_{ν_i}, and similarly with Δ^R.

6) Finally, 3), 4) and 5) imply

$$\Gamma^*, \Gamma\{w_{\nu_i}\} \to \Delta\{w_{\nu_i}\} \quad \text{or} \quad \Gamma^*, \Gamma \to \Delta$$

has a quasi-proof. Recall that if $\exists x\, F(x) \supset F(w)$ belongs to Γ^*, then w does not belong to Σ. Regarding these w's as free variables, we obtain

$$\{\exists y\, (\exists x\, F(x) \supset F(y))\}, \Gamma \to \Delta,$$

whereas $\exists y\, (\exists x\, F(x) \supset F(y))$ is provable for each F. Therefore, by the cut rule, we have $\Gamma \to \Delta$. Assuming that we have carefully chosen the free variables, we may claim that the eigenvariable conditions are satisfied except for II, on heights. In the quasi-proof of $\{\exists y\, (\exists x\, F(x) \supset F(y))\}, \Gamma \to \Delta$, the w where $\exists x\, F(x) \supset F(w)$ is the rth formula in Γ^*, is assigned the height r; each eigenvariable in the quasi-proof in (ii) of 4) is assigned the height K, and any eigenvariable in the proof ending with $\{R(b_i)\}, \Gamma^R \to \Delta^R$ is assigned the height K^+.]

If one wishes to study an infinitary logic which is closer to first order logic, he may restrict the quantifiers to those that operate as in the finite case. Lopez-Escobar has defined such a system, called $L_{\omega_1,\omega}$ and proved the completeness and the interpolation theorem for it. The version of these theorems for $L_{\omega_1,\omega}$ is the same as that of **LK**. We shall present the results in the form of a problem.

PROBLEM 22.21 (Lopez-Escobar). The language $L_{\omega_1,\omega}$ is an extension of that of **LK**, and is defined as follows. There are arbitrarily many constants but the arity of each predicate and each function constant is finite. The number of variables is countable. For simplicity, we take only $\neg$, $\wedge$ and $\forall$ as logical symbols. The formulas are defined as usual: If A_i, $i < \omega$, is a sequence of formulas, then $\bigwedge_{i<\omega} A_i$ is a formula. Notice that $\forall$ behaves as in the finite case. A sequent consists of at most countably many formulas. The rules of inference, as well as the initial sequents, are those of **LK** except the following:

$$\text{Weak inference:}\quad \frac{\Gamma \rightarrow \Delta}{\Pi \rightarrow \Lambda},$$

where every formula in Γ occurs in Π and every formula in Δ occurs in Λ.

$$\bigwedge : \text{left}\quad \frac{A_i, \Gamma \rightarrow \Delta \text{ for some } i}{\bigwedge_{i<\omega} A_i, \Gamma \rightarrow \Delta}$$

$$\bigwedge : \text{right}\quad \frac{\Gamma \rightarrow \Delta, A_i \text{ for all } i}{\Gamma \rightarrow \Delta, \bigwedge_{i<\omega} A_i}$$

(1) Prove the completeness of the system.

(2) Prove the interpolation theorem for this system; viz. if $A \supset B$ is provable and A and B have at least one predicate symbol in common, then there exists a C of $\mathrm{L}_{\omega_1,\omega}$ such that $A \supset C$ and $C \supset B$ are provable.

(3) Show that the following is an admissible rule of inference:

$$\frac{\Gamma \rightarrow \Delta}{\tilde{\Gamma} \rightarrow \tilde{\Delta}}$$

where $\tilde{\Gamma}$ is obtained from Γ by replacing each formula of Γ by one of its alphabetical variants (possibly the formula itself); similarly with $\tilde{\Delta}$.

[*Hint*: (1) Consistency is obvious. For the opposite direction proceed in the following way.

1) Given a sequent of $\mathrm{L}_{\omega_1,\omega}$, say S, there are countably many terms which are obtained from the constants which occur in S and all the free variables.

2) Given a sequent S, the S-subformulas are defined as the ordinary subformulas of the formulas of S except for the following case: If $\forall x\, A(x)$ is an S-subformula, then for every term s which satisfies the condition in 1) $A(s)$ is an S-subformula.

3) There are countably many S-subformulas.

4) Given a sequent S, construct a tree $T(S)$. We may assume that there are countably many free variables which do not occur in S. This and the construction of $T(S)$ guarantee that at each step there will still be countably many free variables unused. From 3) we may assume that all the S-subformulas are indexed in ω. We define the tree step by step.

Stage 0. We write the sequent S.

Stage $n+1$. Let $\Gamma \to \Delta$ be a topmost sequent.

Case 1. $n+1 \equiv 1 \pmod 5$. Let $\{\neg A_i\}_{i=1}^m$ and $\{\neg B_j\}_{j=1}^l$ be all the formulas in Γ and Δ, respectively, whose outermost logical symbol is $\neg$ and whose indices in the fixed enumeration of subformulas are $\leqslant n+1$. Then write $\{B_j\}_{j=1}^l, \Gamma' \to \Delta', \{A_i\}_{i=1}^m$ above $\Gamma \to \Delta$, where Γ' is obtained from Γ by deleting $\{\neg A_i\}_{i=1}^m$ and Δ' is obtained from Δ by deleting $\{\neg B_j\}_{j=1}^l$.

Case 2. $n+1 \equiv 2 \pmod 5$. Let $\{\bigwedge_{i<\omega} A_i^j\}_{j=1}^m$ be all the formulas in Γ whose outermost logical symbol is $\bigwedge$ and whose indices are $\leqslant n+1$. Then write

$$\{A_i^1\}_{i\leqslant n}, \ldots, \{A_i^m\}_{i\leqslant n}, \Gamma' \to \Delta,$$

above $\Gamma \to \Delta$, where Γ' is obtained from Γ by deleting $\{\bigwedge_{i<\omega} A_i^j\}_{j=1}^m$.

Case 3. $n+1 \equiv 3 \pmod 5$. Let $\{\bigwedge_{i<\omega} A_i^j\}_{j=1}^m$ be all the formulas in Γ whose outermost logical symbol is $\bigwedge$ and whose indices are $\leqslant n+1$. Then write

$$\Gamma \to \Delta', \{A_{i_j}^j\}_{j=1}^m$$

for all combinations of $\{i_1, \ldots, i_m\}$ above $\Gamma \to \Delta$.

Case 4. $n+1 \equiv 4 \pmod 5$. Let $\{\forall x_i\, A_i(x_i)\}_{i=1}^m$ be all the formulas in Γ whose outermost logical symbol is $\forall$ and whose indices are $\leqslant n+1$. Let $A_i(s_i^1), \ldots, A_i(s_i^{n+1})$ be the first $n+1$ formulas in the enumeration that are S-subformulas of $\forall x_i\, A_i(x_i)$. Write

$$\{A_i(s_i^{j_i})\}_{i\leqslant m}^{j_i \leqslant n+1}, \Gamma' \to \Delta'$$

above $\Gamma \to \Delta$.

Case 5. $n+1 \equiv 0 \pmod 5$. Let $\{\forall x_i\, A_i(x_i)\}_{i=1}^m$ be all the formulas in Δ whose outermost logical symbol is $\forall$ and whose indices are $\leqslant n+1$. Let $a_{j_1}, \ldots, a_{j_m}$ be the first m free variables which have not occurred so far. Write

$$\Gamma \to \Delta', \{A_i(a_{j_i})\}_{i=1}^m$$

above $\Gamma \to \Delta$.

At any stage, if some formula occurs both in the antecedent and the succedent then stop.

5) Let $T(S)$ be the tree defined in 4).

Case 1. All branches are finite. Then S is provable without the cut rule.

Case 2. There is an infinite branch, say B. Let D be the set of all terms which satisfy the condition in 1). We define the structure with the domain D and the interpretation of formulas in the usual way. Then the formulas in the antecedent of B are true, while those in the succedent are false. This completes the proof of (1).

(2) From the proof of (1) above, any provable sequent is cut-free provable. Restate the interpolation theorem for sequents. Consider only cut-free proofs and show the described result by induction on the complexities of the proofs. The procedure is exactly the same as the corresponding theorem for **LK**.

(3) Obvious from the completeness.]

PROBLEM 22.22 (corollary to the Lopez-Escobar theorem). Suppose that $\Gamma \to \Delta$ is a provable sequent of $\mathrm{L}_{\omega_1,\omega}$ and Γ and Δ are finite sequences. Then there exists a cut-free proof of $\Gamma \to \Delta$ in which every sequent consists of finitely many formulas. (Such a proof may be infinite.)

PROBLEM 22.23. Consider a language consisting of the following:

Predicate symbol: $\in$.

Variables: $x_0, x_1, \ldots, x_\mu, \ldots, \mu \in \mathrm{On}$.

Logical symbols: $\neg$, $\bigwedge$, $\forall$.

Formulas are defined as usual. The atomic formulas are of the form $x \in y$. If A is a formula, $\neg A$ is a formula. If A_i, $i \leqslant \lambda$, is a sequence of formula for λ an ordinal, then $\bigwedge_{i<\lambda} A_i$ is a formula. If $A(\boldsymbol{y})$ is a formula, where $\boldsymbol{y}$ is a sequence of variables none of which is in the scope of a quantifier, then $\forall \boldsymbol{y}\, A(\boldsymbol{y})$ is a formula. Show that the truth definition of this language can be developed in a system of second order set theory, i.e., ZF augmented by second order quantifiers and some comprehension axioms.

[*Hint*: The method is similar to the truth definition of **PA** in a second order system. First assign sets to the formal objects of the language. The set assigned to a formal symbol we call the gödelization of that symbol. If A is a formal expression, its gödelization is denoted by $\ulcorner A \urcorner$. For example, $\ulcorner \in \urcorner = \langle 0, 0\rangle$, $\ulcorner x_i \urcorner = \langle 1, i\rangle$, $\ulcorner \neg \urcorner = \langle 3, 0\rangle$, $\ulcorner x \urcorner = \langle 5, x\rangle$, where $\ulcorner x \urcorner$ is the name of a set x, $\ulcorner x \in y \urcorner = \langle \ulcorner \in \urcorner, \ulcorner x \urcorner, \ulcorner y \urcorner\rangle$, $\ulcorner \bigwedge_{i<\lambda} A_i \urcorner = \langle \ulcorner \bigwedge \urcorner, \langle \ulcorner A_0 \urcorner, \ulcorner A_i \urcorner, \ldots\rangle\rangle$. We can then formally define "A is a closed formula" (cf($\ulcorner A \urcorner$)) and "the complexity of a formula A" (cm($\ulcorner A \urcorner$), which is an ordinal). Let α be a second order free variable and let μ be a variable which ranges over ordinals. Define $F(\alpha, \mu)$ as we defined $F(\alpha, n)$ in the case of **PA** to state that α is the truth definition of formulas whose complexities are $\leqslant \mu$. The clause for $\forall \boldsymbol{x}\, A(\boldsymbol{x})$ is expressed as:

$$\forall \ulcorner \boldsymbol{x}\, A(\boldsymbol{x}) \urcorner (\mathrm{cf}(\ulcorner \forall \boldsymbol{x}\, A(\boldsymbol{x}) \urcorner) \wedge \mathrm{cm}(\ulcorner \forall \boldsymbol{x}\, A(\boldsymbol{x}) \urcorner) \leqslant \mu \supset$$

$$\supset \forall x\, (x \text{ is a sequence of order type } \lambda, \text{ say } \langle x_0, x_1, \ldots\rangle) \supset$$

$$\supset \alpha(\ulcorner A(x_0, x_1, \ldots) \urcorner).$$

Then define $T(a) \Leftrightarrow_{\text{df}} \text{cf}(a) \wedge \exists\phi\,(F(\phi, \text{cm}(a)) \wedge \phi(a))$. Now prove $T(\ulcorner A \urcorner) \equiv A$ for all closed formulas A.]

REMARK. We can generalize the proposition in Problem 22.23 to the cases where there are predicate constants $p_0, p_1, \ldots$ and where the quantifiers are not homogeneous.

Next we will show that for any homogeneous system there is an equivalent homogeneous system whose eigenvariable conditions are "ordinary" ones, that is, the eigenvariable conditions are conditions on inferences and not on proofs. In order to simplify the argument we take only the logical symbols $\neg$, $\vee$, $\exists$, and regard others as a combination of these.

DEFINITION 22.24. The $\forall\exists$-calculus is defined as the homogeneous system with the following alteration: Replace $\exists$: left by

$$\forall\exists \text{ rule:}\quad \frac{\{A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{b}_\lambda)\}_{\lambda<\alpha}, \Gamma \to \Delta}{\{\forall \boldsymbol{x}_\lambda\, \exists \boldsymbol{y}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{y}_\lambda)\}_{\lambda<\alpha}, \Gamma \to \Delta},$$

where none of the free variables contained in $\boldsymbol{b}_\lambda$'s can occur in the lower sequent.

Each variable in $\boldsymbol{b}_\lambda$ is an eigenvariable. All of the variables of $\boldsymbol{b}_\lambda$ must be distinct and none of them can occur in $A_{\lambda'}(\boldsymbol{a}_{\lambda'}, \boldsymbol{b}_{\lambda'})$ for $\lambda' < \lambda$. There are no other eigenvariable conditions.

Note that $\forall \boldsymbol{x}_\lambda$ can be empty.

PROPOSITION 22.25 (Maehara-Takeuti). *The $\forall\exists$-calculus is equivalent to the homogeneous system (for the same language).*

PROOF. Let P be a proof in the $\forall\exists$-calculus. We may assign a height to every free variable; if b is the μth variable of $\boldsymbol{b}_\lambda$ in the $\forall\exists$-rule, then the height of b is $\sup_a(\text{height of } a) + (1 + \mu)$, where a ranges over all the free variables in A_λ other than $\boldsymbol{b}_\lambda$ and by sup we mean the strict supremum. If b is not used as eigenvariable, then the height of b is 0. Transform P as follows. If there is an application of the $\forall\exists$-rule, then replace it by:

$$\cfrac{\cfrac{\cfrac{\cfrac{\{A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{b}_\lambda)\}_{\lambda<\alpha}, \Gamma \to \Delta}{\{\exists \boldsymbol{y}_x\, A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{y}_x)\}_{\lambda<\alpha}, \Gamma \to \Delta}}{\Gamma \to \Delta, \{\neg\exists \boldsymbol{y}_x\, A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{y}_x)\}_{\lambda<\alpha}}}{\Gamma \to \Delta, \{\exists \boldsymbol{x}_\lambda\, \neg\exists \boldsymbol{y}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{y}_x)\}_{\lambda<\alpha}}}{\{\neg\exists \boldsymbol{x}_\lambda\, \neg\exists \boldsymbol{y}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{y}_\lambda)\}_{\lambda<\alpha}, \Gamma \to \Delta.}$$

It can be easily seen that the resulting figure is a proof in the homogeneous system with the same heights as P.

The opposite direction is proved as follows: Let P be a proof in the homogeneous system ending with $\Gamma \to \Delta$. Let $\{\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{x}_\lambda)\}_{\lambda<\alpha}$ be an enumeration of all the principal formulas of the $\exists$: left in P, where $\boldsymbol{a}_\lambda$ is the sequence of all free variables in A_λ. Then eliminate every application of the $\exists$: left as follows: for simplicity we demonstrate a case where there is only one auxiliary formula:

$$\exists : \text{left} \quad \frac{A(\boldsymbol{a}, \boldsymbol{b}), \Pi \to \Lambda}{\exists \boldsymbol{x}\, A(\boldsymbol{a}, \boldsymbol{x}), \Pi \to \Lambda}$$

is changed to

$$\frac{\dfrac{A(\boldsymbol{a}, \boldsymbol{b}), \Pi \to \Lambda}{A(\boldsymbol{a}, \boldsymbol{b}), \exists \boldsymbol{x}\, A(\boldsymbol{a}, \boldsymbol{b}), \Pi \to \Lambda} \qquad \neg\exists \boldsymbol{x}\, A(\boldsymbol{a}, \boldsymbol{x}), \exists \boldsymbol{x}\, A(\boldsymbol{a}, \boldsymbol{x}), \Pi \to \Lambda}{\neg\exists \boldsymbol{x}\, A(\boldsymbol{a}, \boldsymbol{x}) \vee A(\boldsymbol{a}, \boldsymbol{b}), \quad \exists \boldsymbol{x}\, A(\boldsymbol{a}, \boldsymbol{x}), \Pi \to \Lambda}.$$

Since no eigenvariables are involved we obtain a proof in the homogeneous system as well as in the $\forall\exists$-calculus. From this proof, which ends with

$$\{\neg\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{x}_\lambda) \vee A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{b}_\lambda)\}_{\lambda<\alpha}, \Gamma \to \Delta,$$

we obtain, by applying the $\forall\exists$-rule,

$$\{\forall \boldsymbol{y}_\lambda\, \exists \boldsymbol{z}_\lambda\, [\neg\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{y}_\lambda, \boldsymbol{x}_\lambda) \vee A(\boldsymbol{y}_\lambda, \boldsymbol{z}_\lambda)]\}_{\lambda<\alpha}, \Gamma \to \Delta.$$

On the other hand,

$$\forall \boldsymbol{y}_\lambda\, \exists \boldsymbol{z}_\lambda\, [\neg\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{y}_\lambda, \boldsymbol{x}_\lambda) \vee A(\boldsymbol{y}_\lambda, \boldsymbol{z}_\lambda)]$$

is provable with the $\forall\exists$-rule. Hence $\Gamma \to \Delta$ is provable in the $\forall\exists$-calculus.

PROBLEM 22.26. The compactness of **LK** (the first order predicate calculus) can be syntactically expressed as follows. Let $\Gamma \to \Delta$ denote a sequent consisting of formulas of **LK** with cardinality $\leqslant K$, where K is the cardinality of the set of formulas of **LK** (for a given language). For any such sequent that is provable in the homogeneous system (of an appropriate language), there exist finite subsets Γ_0 and Δ_0 of Γ and Δ, respectively, for which $\Gamma_0 \to \Delta_0$ is **LK**-provable.

Prove the compactness of the first order predicate calculus in this syntactic form.

[*Hint*: Let $\Gamma \to \Delta$ be a sequent as above and let P be a proof of $\Gamma \to \Delta$.

(*) For each sequent, say $\Pi \to \Lambda$, in P, we can select a finite subsequent $\Pi_0 \to \Lambda_0$, i.e., $\Pi_0 \subseteq \Pi$, $\Lambda_0 \subseteq \Lambda$ and Π_0 and Δ_0 are finite. If $\Pi \to \Lambda$ is the lower sequent of an inference, we can select finitely many upper sequents corresponding to it in such a manner that that part of P which consists of all the selected sequents corresponding to $\Pi \to \Lambda$ is a quasi-proof of $\Pi \to \Lambda$. In particular, $\Pi \to \Lambda$ can be $\Gamma \to \Delta$; hence there is a finite subsequent $\Gamma_0 \to \Delta_0$, which is provable.

Then applying Proposition 22.25, we can construct an **LK**-proof of $\Gamma_0 \to \Delta_0$. (*) is proved by transfinite induction on the construction of the subproof of P ending with $\Pi \to \Lambda$. For $\wedge$: right and $\vee$: left, use the generalized König's lemma. (Cf. the proof of Proposition 8.16.)]

PROBLEM 22.27. First we shall define a formal infinitary language in set theoretical terms.

A basic language is an ordered triple $\langle C, P, S\rangle$, where C is a set of individual constants, P is a set of predicate constants, and S is a set of logical symbols. Each element of P is an ordered pair $\langle A, \alpha\rangle$ where α is an ordinal called the arity of $\langle A, \alpha\rangle$. An element of S is either $\neg$ or of the form $\langle \wedge, \alpha\rangle$, $\langle \vee, \alpha\rangle$, $\langle \forall, \alpha\rangle$ or $\langle \exists, \alpha\rangle$, where α is an ordinal called the arity of $\langle \wedge, \alpha\rangle$, $\langle \vee, \alpha\rangle$, $\langle \forall, \alpha\rangle$ or $\langle \exists, \alpha\rangle$, respectively. A basic language $\langle C, P, S\rangle$, also satisfies the following conditions.

1) The sets C, P, and S are mutually disjoint.

2) The symbols $\neg$, $\wedge$, $\vee$, $\forall$, and $\exists$ are different.

A language L is an ordered set $\langle C, P, S, B, F\rangle$, where $\langle C, P, S\rangle$ is a basic language, B and F are a set of bound variables and a set of free variables respectively, and C, P, S, B, and F are mutually disjoint.

Since the terms, formulas, etc. of L are what we commonly, in logic, understand them to be, we skip their formal set theoretical definitions. We however make the following deviations from our previous treatment. We call $\Gamma \to \Delta$ a sequent in L if Γ and Δ are sets of formulas in L. This change is useful when we wish to avoid the use of the axiom of choice as much as possible.

A tree is an ordered pair $\langle T, <\rangle$ satisfying the following conditions.

1. $T \neq \emptyset$.

2. The relation $<$ is a partial ordering on T.

We read $s_1 < s_2$ as "s_1 is below s_2" or "s_2 is above s_1". There is a unique lowermost point s_0 in T, i.e., there is a unique s_0 in T such that

$$\forall s \in T\ (s_0 < s \vee s_0 = s).$$

This lowermost point s_0 is called the end point of the tree. Every point, except the end point, has a unique point below it, i.e.,

$$\forall s \in T\ (s \neq s_0 \supset \exists ! t \in T\ \forall u\ (u < s \equiv u = t \vee u < t)).$$

If $s_1 < s_2$ and $\neg \exists s\ (s_1 < s \wedge s < s_2)$, then s_2 is said to be immediately above s_1. If $s_1 < s$, then there exists a unique s_2 such that $s_1 < s_2 \leqslant s$ and s_2 is immediately above s_1. A topmost point is called an initial point.

3. Any linearly ordered subset of T (with respect to $<$) is finite. A semi-proof P in L is a function f from a tree into a set of sequents in L satisfying the following conditions.

1) If S is an initial point and $f(s)$ is of the form $\Gamma \to \Delta$, then

$$\Gamma \cap \Delta \neq \emptyset.$$

Note that here an initial sequent need not be of the form $D \to D$. This change enables us to prove the completeness theorem with a minimal use of the axiom of choice.

2) Let $\ldots, s_\alpha, \ldots$ be the collection of all points immediately above s. Then

$$\frac{\ldots, f(s_\alpha), \ldots}{f(s)}$$

is an inference in L.

A proof P in L is an ordered pair $\langle P_0, \prec \rangle$, where P_0 is a semi-proof in L and $\prec$ is a well-founded partial ordering on the free variables in P_0 which satisfies our eigenvariable conditions.

A structure for L is defined in the usual manner.

Let M be a transitive set which needs not satisfy the axiom of choice. Let S and A be a structure for L in M and a sentence in L, respectively. Then "S satisfies A in M" denoted by $S \models^{M} A$ is defined as usual except for $\exists$ and $\forall$. Since $\forall$ is defined as $\neg\exists\neg$, we give only the definition for $\exists$.

$$S \models^{M} \exists x_0\, x_1 \ldots A(x_0, x_1, \ldots)$$

is defined to be

$$\exists f \in M\ (S \models^{M} A(f(0), f(1), \ldots)).$$

A sentence F is called M-valid, if for every structure S in M, $S \models^{M} F$.

THEOREM A. *Let* L *be a basic language and suppose that* $\forall a \in M$ $(a^\alpha \in M)$ *for every arity* α *in* L. *If an* L*-proof* P *is an element of* M, *then the end-sequent of* P *is* M*-valid.*

[*Hint*: Follow the proof of the validity theorem with the following modification. Define Skolem functions using the axiom of choice. The Skolem functions are possibly outside M but the sequences of terms made by these Skolem functions are members of M by the hypotheses of the theorem, provided their lengths are arities of L. Therefore the proof can be carried out as before.]

THEOREM B. *Let* L *be a basic language and let* λ *be the first regular cardinal greater than all the arities in* L. *We assume* $\lambda > \omega$. *Let* S *be an* L*-sequent. If* M *satisfies the following conditions, then either* S *has a cut free proof in* M *or there exists a counter model of* S *in* M.

1) $\mathrm{L} \in M$, $S \in M$ *and* $\lambda \in M$.

2) $\forall a, b \in M (\{a, b\} \in M)$ *and* $\forall a \in M$ $(\bigcup(a) \in M)$, *where* $\bigcup(a)$ *is the union of* a.

3) M *satisfies the axiom of replacement.*

4) $\forall a \in M$ $\forall \alpha \in \lambda$ $(a^\alpha \in M)$.

In case that L *has* $=$, *the condition* $P(D \times D) \in M$ *is added, where* $P(D \times D)$ *is the power set of* $D \times D$, D *is an adequate set of free variables, and* $D \in M$ *is a consequence of* 1)–4).

[*Hint*: Follow the proof of the completeness theorem in the following manner.

1) Introduce λ-many bound variables in M.

2) Construct all atomic semi-formulas in M.

3) Construct all semi-formulas without free variables in M. Here we use a definition by transfinite induction up to λ. This can be done in M since M satisfies the axiom of replacement.

4) In M, introduce Skolem function letters corresponding to each semi-formula without free variables.

5) Construct the set D of all free variables as the set of all possible function combinations of Skolem function letters in M. Here we again use a definition by transfinite induction up to λ.

6) Construct a reduction tree in M. We define a reduction tree up to the nth step in M by mathematical induction. Then it is easily shown that the whole reduction tree is in M.

7) Construct a countermodel or a cut free proof in M. This can be done as usual.

8) If L has $=$, we need $P(D \times D) \in M$ since all possible partitions of free variables are taken into consideration in the equality axiom.]

Without using $\in$, we can express the axiom of choice in the form

$$\forall x\, \exists y\, A(x, y) \rightarrow \exists f\, \forall x\, A(x, f(x)).$$

Since $\exists f$ occurs in this form, this is a second order form. Since some second order notions can be expressed in an infinitary language, it is natural to ask whether the axiom of choice can be expressed in an infinitary language. Actually, a weak form of the axiom of choice is elegantly expressed in an infinitary language: The axiom of dependent choice can be expressed as

$$\forall x\, \exists y\, A(x, y) \rightarrow \forall x_0\, \exists x_1\, x_2 \ldots \bigwedge_{i<\omega} A(x_i, x_{i+1}).$$

COROLLARY. *The axiom of choice is not expressible in an infinitary language.*

[*Hint*: Suppose the axiom of choice can be expressed in an infinitary language L. Then since the axiom of choice is true, there must be a proof P of the axiom. Let α be a large ordinal so that $\mathrm{L} \in R(\alpha)$, $P \in R(\alpha)$ and $\lambda < \alpha$, where $R(\alpha) = \{a \mid \operatorname{rank}(a) < \alpha\}$. This is a contradiction since it is very easy to prove the existence of a transitive set M with the following properties:

1) The axiom of choice is not M-valid.

2) M satisfies the conditions of Theorem A. (For example, take M to be the smallest transitive set satisfying the conditions in Theorem A and $R(\alpha) \in M$.)]

§23. Determinate logic

In this section we will discuss determinate logic with equality ($=$) as a special case of infinitary logic with heterogeneous quantifiers. In order to simplify the discussion, we will only consider languages that have no individual constants.

DEFINITION 23.1. (1) By a *heterogeneous quantifier of arity* α we mean a symbol $\mathbf{Q}^f$, where f is a map from α into $\{\forall, \exists\}$. For such a map f, the map $\bar{f}$, called the dual of f, is defined in the following way.

(i) The domain of $\bar{f}$ is the same as that of f,

(ii) $\bar{f}(\beta) = \forall$ or $\exists$ according as $f(\beta) = \exists$ or $\forall$ respectively.

If f and g are dual, then $\mathbf{Q}^f$ and $\mathbf{Q}^g$ are called dual quantifiers.

(2) By the language L_D we mean the language obtained from the language in §22 by replacing the quantifiers $\forall$ and $\exists$, of arity α, by heterogeneous quantifiers Q^f of the same arity.

(3) Let $\mathscr{A}$ be a structure for L_D. We define satisfaction and validity in $\mathscr{A}$ as in Definition 22.15. The structure is said to be *determinate* if for each formula A in L_D exactly one of the two formulas

$$Q^f \boldsymbol{x}_{<\alpha}\, A(\boldsymbol{x})$$

and

$$Q^{\bar{f}} \boldsymbol{x}_{<\alpha}\, \neg A(\boldsymbol{x})$$

is valid in $\mathscr{A}$.

(4) A logical system **S** with language L_D is called a determinate logic if for every closed formula A in L_D, "A is provable in **S**" is equivalent to "A is valid in every determinate structure".

In this section we will define a logical system **DL** and prove that (i) **DL** is a determinate logic, (ii) if a formula A is provable in **DL** by using heterogeneous quantifier introduction only once at the end of the proof, then A is valid, and (iii) in **DL** the completeness theorem, the cut-elimination theorem, and the interpolation theorem hold in a certain form.

The language of our formal system **DL** is L_D with equality. Consequently, we will develop the theory of determinate logic with equality. First we modify the notion of proof as defined in §22.

DEFINITION 23.2. (1) The rules for $=$ are as in Definition 22.19.

(2) The rules for $\forall$ and $\exists$ in Definition 22.1 are replaced by the following.

$$Q : \text{left:} \quad \frac{\{A_\lambda(\boldsymbol{a}_\lambda)\}_{\lambda<\gamma},\ \Gamma \to \Delta}{\{Q^{f_\lambda}\boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\gamma},\ \Gamma \to \Delta},$$

where $\boldsymbol{a}_\lambda$ denotes a sequence $a_{\lambda,0}, \ldots, a_{\lambda,\alpha}, \ldots$ $(\alpha < \mu_\lambda)$ for some μ_λ. The μth variable of this sequence, $a_{\lambda,\mu}$, we call the variable of $\boldsymbol{a}_\lambda$ of order μ. If $f_\lambda(\mu) = \exists$, then $a_{\lambda,\mu}$ is called an eigenvariable of the inference

$$Q : \text{right:} \quad \frac{\Gamma \to \Delta,\ \{A_\lambda(\boldsymbol{a}_\lambda)\}_{\lambda<\gamma}}{\Gamma \to \Delta,\ \{Q^{f_\lambda}\boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\gamma}}.$$

If $f_\lambda(\mu) = \forall$, then $a_{\lambda,\mu}$ is called an eigenvariable of the inference.

If $a_{\lambda,\mu}$ is an eigenvariable of either inference, $Q^{f_\lambda}\boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)$ is called a *principal formula* of $a_{\lambda,\mu}$ and also a principal formula of the inference. Furthermore, $A_\lambda(\boldsymbol{a}_\lambda)$ is called an *auxiliary formula* of $Q^{f_\lambda}\boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)$, of the eigenvariable $a_{\lambda,\mu}$, and of the inference.

If two different variables a and b have the same principal formula then a is said to *precede* b with respect to that principal formula if the order of a is less than the order of b.

(3) Every proof must satisfy the following eigenvariable conditions.

1) If a free variable a occurs in two or more places as an eigenvariable, then for each occurrence a must have the same principal formula and a must have the same order. Moreover, if a occurs in two different auxiliary formulas $A(\boldsymbol{a}_1)$ and $A(\boldsymbol{a}_2)$ as an eigenvariable of order μ then $a_{1,\nu}$ and $a_{2,\nu}$ must be the same variable for all $\nu < \mu$.

2) To each free variable a, we assign an ordinal number $h(a)$, called the height of a, which has the following properties:

2.1) The height, $h(a)$, of an eigenvariable a, is greater than the height, $h(b)$, of every free variable b in the principal formula of the eigenvariable a.

2.2) The height of an eigenvariable a is greater than the height of b if b precedes a with respect to a principal formula of a.

3) No variable occurring in an inference as an eigenvariable may occur in the end sequent.

Remark. The following weaker modification of the foregoing eigenvariable conditions is enough to assure that a logic is determinate.

Replace the last half of 1) by the following: If $A(\boldsymbol{a})$ is an auxiliary formula of a principal formula $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x})$ and a_ν and a_μ are eigenvariables of $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x})$ with $\nu \neq \mu$, then a_ν and a_μ are different. If a occurs in two different auxiliary formulas $A(\boldsymbol{a}_1)$ and $A(\boldsymbol{a}_2)$ as an eigenvariable of a principal formula $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x})$ then $a_{1,\nu}$ and $a_{2,\nu}$ are the same for each non-eigenvariable $a_{1,\nu}$ of $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x})$ for each ν less than the order of a.

Replace 2.2) by the following: If a is an eigenvariable with principal formula $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x})$ then the height of a is greater than the height of b if b precedes a with respect to the principal formula $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x})$ but b is not an eigenvariable of this principal formula.

We will use either the original form of 2.2) or the latter version choosing whichever is more convenient for our purposes.

Example 23.3. Proof of the axiom of determinateness: Let $\boldsymbol{a}$ be $a_{\lambda<\alpha}$ and $\boldsymbol{b}$ be $b_{\mu<\beta}$.

$$\cfrac{\cfrac{\cfrac{A(\boldsymbol{a}, \boldsymbol{b}) \to A(\boldsymbol{a}, \boldsymbol{b})}{\to A(\boldsymbol{a}, \boldsymbol{b}), \neg A(\boldsymbol{a}, \boldsymbol{b})}}{\to \mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{b}), \mathsf{Q}^f\boldsymbol{x}\, \neg A(\boldsymbol{x}, \boldsymbol{b})}}{\to \mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{b})\ \mathsf{Q}^f\boldsymbol{x}\, \neg A(\boldsymbol{x}, \boldsymbol{b}).}$$

In this proof, $h(a_\lambda) = 1 + \lambda$ and $h(b_\mu) = 0$.

THEOREM 23.4 (validity for determinate structures). *Let $\mathscr{A}$ be a determinate structure and $\Gamma \to \Delta$ be provable in the determinate logic* **DL**. *Then $\Gamma \to \Delta$ is satisfied in $\mathscr{A}$.*

PROOF. Take an arbitrary formula with a quantifier at the beginning, say

$$\mathrm{Q}^f \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{a}),$$

where $\boldsymbol{a}$ is the sequence of all free variables in this formula and the length of $\boldsymbol{x}$ is α. For each $\gamma < \alpha$, we introduce a Skolem function

$$g_A^{f,\gamma}(x_{\xi_0}, \ldots, x_{\xi_\mu}, \ldots, \boldsymbol{a}) \quad \text{or} \quad \bar{g}_A^{f,\gamma}(x_{\eta_0}, \ldots, x_{\eta_\mu}, \ldots, \boldsymbol{a})$$

according as $f(\gamma) = \exists$ or $f(\gamma) = \forall$, where $\xi_0, \ldots, \xi_\mu, \ldots$ are all the ordinals $< \gamma$ for which $f(\xi) = \forall$ and $\eta_0, \ldots, \eta_\mu, \ldots$ are all the ordinals $< \gamma$ for which $f(\eta) = \exists$. We define the following interpretation of $g_A^{f,\gamma}$ and $\bar{g}_A^{f,\gamma}$ with respect to $\mathscr{A}$.

If $\mathrm{Q}^f \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{a})$ is satisfied in $\mathscr{A}$, then

1) $\forall x_{\xi_0} x_{\xi_1} \ldots A(\tilde{x}_0, \ldots, \boldsymbol{a})$, where $\tilde{x}_\gamma$ is x_{ξ_γ} if $f(\gamma) = \forall$ and $\tilde{x}_\gamma$ is $g_A^{f,\gamma}(x_{\xi_0}, \ldots, \boldsymbol{a})$ if $f(\gamma) = \exists$.

Let D be the universe of $\mathscr{A}$ and 0 be a member of D. Here $\boldsymbol{a}$ is understood to be a sequence of members of D. If $\mathrm{Q}^f \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{a})$ is not satisfied in $\mathscr{A}$, then the $g_A^{f,\gamma}$'s are interpreted to be the constant function 0 in $\mathscr{A}$.

If $\mathrm{Q}^f \boldsymbol{x}\, \neg A(\boldsymbol{x}, \boldsymbol{a})$ is satisfied in $\mathscr{A}$, then

2) $\forall x_{\xi_0} x_{\xi_1} \ldots A(\tilde{x}_0, \ldots, \boldsymbol{a})$, where $\tilde{x}_\gamma$ is x_{η_γ} if $f(\gamma) = \exists$, and $\tilde{x}_\gamma$ is $\bar{g}_A^{f,\gamma}(x_{\xi_0}, \ldots, \boldsymbol{a})$ if $f(\gamma) = \forall$.

If $\mathrm{Q}^f \boldsymbol{x}\, \neg A(\boldsymbol{x}, \boldsymbol{a})$ is not satisfied in $\mathscr{A}$, then the $\bar{g}_A^{f,\gamma}$'s are interpreted to be the constant function 0 in $\mathscr{A}$.

Now let P be a proof in our system. Let

$$a_0, a_1, \ldots, a_\beta, \ldots$$

be a list without repetition, of all the eigenvariables in P with $h(a_\beta) \leqslant h(a_\gamma)$ if $\beta < \gamma$, h being the height function. By transfinite induction on β we will define terms $t_0, t_1, \ldots, t_\beta, \ldots$ corresponding to the above list of eigenvariables. Assuming that $\boldsymbol{t}_{<\beta}$ have been defined, we define t_β in the following way.

Suppose the principal formula of a_β is $\mathrm{Q}^f \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{b})$ and δ is the order of a_β. Let d be a free variable that precedes a_β with respect to the principal formula.

With the variable d we associate a variable u in the following way. If d is not an eigenvariable, let u be d itself. Otherwise, since $h(d) < h(a_\beta)$ by our eigenvariable conditions, d occurs in the above list of eigenvariables as a_γ for some $\gamma < \beta$. By the induction hypothesis t_γ has been defined. Let the u associated with d be this t_γ.

Let b be a free variable in $\boldsymbol{b}$. A variable s associated with b is defined in the same manner as the u associated with d; recall that $h(b) < h(a_\beta)$. It should be noted that these d's and b's are the same for all auxiliary formulas of a_β by virtue of the eigenvariable conditions. Thus t_β can be defined to be $g_A^{f,\delta}(\boldsymbol{u}_1, \boldsymbol{s})$ if the order of a_β is δ and $f(\delta)$ is $\exists$, where $\boldsymbol{u}_1$ is the sequence of the u's corresponding to appropriate d's as defined above. Similarly, t_β is defined to be $\bar{g}_A^{f,\delta}(\boldsymbol{u}_2; \boldsymbol{s})$ if the order of a_β is δ and $f(\delta)$ is $\forall$, where $\boldsymbol{u}_2$ has the same meaning as $\boldsymbol{u}_1$.

Now substitute $t_0, t_1, \ldots, t_\beta, \ldots$ for $a_0, a_1, \ldots, a_\beta, \ldots$ respectively in P. Let P' be the figure thus obtained from P. The end-sequents of P' and P are the same because the end-sequent of P has no eigenvariables. We shall show that every sequent of P' is satisfied in $\mathscr{A}$; this will imply that the end sequent of P is satisfied in $\mathscr{A}$. We have only to show that if the upper sequents of an inference in P' are satisfied in $\mathscr{A}$, then the lower sequent of this inference is also satisfied in $\mathscr{A}$. Since the other cases are obvious, we only consider the inferences on quantifiers.

An introduction of Q : left in P' is of the following form

$$3)\quad \frac{\ldots, A(\boldsymbol{u}, s), \ldots, \Gamma \to \Delta}{\ldots, \mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s}), \ldots, \Gamma \to \Delta},$$

where u_γ is of the form $g_A^{f,\gamma}(u_{\xi_0}, \ldots, \boldsymbol{s})$ if $f(\gamma) = \exists$.

An introduction of Q : right in P' is of the following form

$$4)\quad \frac{\Gamma \to \Delta, \ldots, A(\boldsymbol{u}', \boldsymbol{s}), \ldots}{\Gamma \to \Delta, \mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s}), \ldots},$$

where u'_γ is of the form $\bar{g}_A^{f,\gamma}(u'_{\eta_0}, \ldots, s)$, if $f(\gamma) = \forall$.

For 3) we have to show that

5) $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s}) \to A(\boldsymbol{u}, \boldsymbol{s})$.

But this is immediate from 1). For 4) we must show that

6) $A(\boldsymbol{u}', \boldsymbol{s}) \to \mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s})$.

Assume that $\neg\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s})$ holds in $\mathscr{A}$. Since $\mathscr{A}$ is determinate, $\mathsf{Q}^f\boldsymbol{x}\, \neg A(\boldsymbol{x}, \boldsymbol{s})$ holds in $\mathscr{A}$. Therefore what we have to show is that

$$Q^f \boldsymbol{x} \neg A(\boldsymbol{x}, \boldsymbol{s}) \to \neg A(\boldsymbol{u}', \boldsymbol{s}).$$

But this follows from 2). This completes the proof of Theorem 23.4.

Since in this proof the determinateness of $\mathscr{A}$ was used only for 6) and since the axiom of determinateness always holds for a homogeneous quantifier, we have the following.

PROPOSITION 23.5. *Let P be a proof in our determinate logic in which every quantifier, introduced in a succedent in P, is homogeneous. Then the end-sequent of P is valid.*

Next we shall prove two versions of completeness.

THEOREM 23.6. *Let $\Gamma \to \Delta$ be a sequent. Then either there exists a cut-free proof of $\Gamma \to \Delta$ in our determinate logic or else there exists a structure $\mathscr{A}$ (possibly not determinate) such that every formula in Γ is satisfied in $\mathscr{A}$ and no formula in Δ is satisfied in $\mathscr{A}$.*

PROOF. Let D_0 be an arbitrary non-empty set containing all free variables in Γ and Δ. Let D be the closure of D_0 with respect to all the functions $g_A^{f,\gamma}$ and $\bar{g}_A^{f,\gamma}$ for all formulas A in our language, i.e., D is generated by all $g_A^{f,\gamma}$'s and $\bar{g}_A^{f,\gamma}$'s from D_0. (Actually it is sufficient if D is closed under all the functions $g_A^{f,\gamma}$ and $\bar{g}_A^{f,\gamma}$ for all subformulas A of formulas in Γ and Δ). In this proof, a member of $D - D_0$ is treated as a free variable and a member of D_0 is treated as an individual constant. Let E be the set of all formulas of the form $s = t$, where s and t are members of D. Let $(\Phi|\Psi)$ be an arbitrary decomposition of E and consider the following sequent:

0) $\Phi, \Gamma \to \Delta, \Psi$.

If all the sequents of the form 0) are provable without the cut rule, then $\Gamma \to \Delta$ is also provable without the cut rule.

Let S be $\Gamma \to \Delta$. We shall define a tree $T(S)$ by considering the following eight cases.

1) The lowest sequent is S.

2) Immediate ancestors of S are all the sequents of the form 0).

3) When a sequent $\Pi \to \Lambda$ is

$$\{\neg C_\lambda\}_{\lambda<\gamma}, \Gamma' \to \Delta', \{\neg D_\mu\}_{\mu<\delta},$$

where Γ' and Δ' have no formulas whose outermost logical symbol is $\neg$, and $\Pi \to \Lambda$ is constructed by 2) or 8) (which is to be defined) the immediate ancestor of $\Pi \to \Lambda$ is

$$\{D_\mu\}_{\mu<\delta}, \Gamma' \to \Delta', \{C_\lambda\}_{\lambda<\gamma}.$$

4) When a sequent $\Pi \to \Lambda$ is

$$\{\bigvee_{\lambda<\alpha_\mu} C_{\mu,\lambda}\}_{\mu<\gamma}, \Gamma' \to \Delta', \{\bigvee_{\rho<\beta_\sigma} D_{\sigma,\rho}\}_{\sigma<\delta},$$

where Γ' and Δ' have no formulas whose outermost logical symbol is $\vee$, and when $\Pi \to \Lambda$ is constructed by 3), the immediate ancestors of $\Pi \to \Lambda$ are

$$\{C_{\lambda_\mu,\mu}\}_{\mu<\gamma}, \Gamma' \to \Delta', \{D_{\sigma,\rho}\}_{\rho<\delta_\sigma,\sigma<\delta}$$

for all sequences $\{\lambda_\mu\}_{\mu<\gamma}$ such that $\lambda_\mu < \alpha_\mu$.

5) When a sequent $\Pi \to \Lambda$ is

$$\{\bigwedge_{\lambda<\alpha_\mu} C_{\mu,\lambda}\}_{\mu<\gamma}, \Gamma' \to \Delta', \{\bigwedge_{\rho<\beta_\sigma} D_{\rho,\sigma}\}_{\sigma<\delta},$$

where Γ' and Δ' have no formulas whose outermost logical symbol is $\wedge$, and when $\Pi \to \Lambda$ is constructed by 4), then the immediate ancestors of $\Pi \to \Lambda$ are

$$\{C_{\mu,\lambda}\}_{\lambda<\gamma_\mu,\mu<\gamma}, \Gamma' \to \Delta', \{D_{\rho_\sigma,\sigma}\}_{\sigma<\delta}$$

for all sequences $\{\rho_\sigma\}_{\sigma<\delta}$ such that $\rho_\sigma < \beta_\sigma$.

6) When a sequent $\Pi \to \Lambda$ is

$$\{\mathsf{Q}^{f\lambda}\boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{s}_\lambda)\}_{\lambda<\delta}, \Gamma' \to \Delta',$$

where Γ' has no formulas whose outermost logical symbol is Q, and when $\Pi \to \Lambda$ is constructed by 5), then the immediate ancestor of $\Pi \to \Lambda$ is

$$\{A_\lambda(\boldsymbol{t}_{\lambda,\mu}\boldsymbol{s}_\lambda\}_{\mu,\lambda<\delta}\, \Gamma' \to \Delta'$$

for all $\boldsymbol{t}_{\lambda,\mu}$ satisfying the following:

$$\boldsymbol{t}_{\lambda,\mu} \quad \text{is} \quad \{t_{\lambda,\mu,0}, \ldots, t_{\lambda,\mu,\nu}, \ldots\}_{\nu<\gamma},$$

where γ is the length of $\boldsymbol{x}_\lambda$ if $\xi_0, \xi_1, \ldots,$ are all the ordinals $<\gamma$ such that $f(\xi) = \forall$ and $\eta_0, \eta_1, \ldots$ are all the ordinals $<\gamma$ such that $f(\xi) = \exists$, then $t_{\lambda,\mu,\xi_0}t_{\lambda,\mu,\xi_1}, \ldots$ is an arbitrary sequence of members of D and

$$t_{\lambda,\mu,\eta} = g^{f,\eta}_{A_\lambda}(t_{\lambda,\mu,\xi_0}, \ldots, s_\lambda),$$

for each $\eta = \eta_0, \eta_1, \ldots$.

7) When a sequent $\Pi \to \Lambda$ is

$$\Gamma' \to \Delta', \{Q^{f\lambda} x_\lambda\, A_\lambda(x_\lambda, s_\lambda)\}_{\lambda<\delta},$$

where Δ' has no formulas whose outermost logical symbol is Q, and when $\Pi \to \Lambda$ is constructed by 6), then the immediate ancestor of $\Pi \to \Lambda$ is

$$\Gamma' \to \Delta', \{A_\lambda(t_{\lambda,\mu}, s_\lambda)\}_{\mu,\lambda<\delta}$$

for all $t_{\lambda,\mu}$ satisfying the following:

$$t_{\lambda,\mu} \quad \text{is} \quad \{t_{\lambda,\mu,0}, \ldots, t_{\lambda,\mu,\nu}, \ldots\}_{\nu<\gamma},$$

where γ is the length of x_λ; if $\xi_0, \xi_1, \ldots$ are all the ordinals $< \gamma$ such that $f(\xi) = \forall$, and, if $\eta_0, \eta_1, \ldots$ are all the ordinals $< \gamma$ such that $f(\eta) = \exists$, then $t_{\lambda,\mu,\eta_0}, t_{\lambda,\mu,\eta_1}, \ldots$ are arbitrary members of D and $t_{\lambda,\mu,\xi} = \bar{g}^{f,\xi}_{A_\lambda}(t_{\lambda,\mu,\eta_0}, \ldots, s_\lambda)$.

8) When a sequent $\Pi \to \Lambda$ is

$$\{s_\lambda = t_\lambda\}_{\lambda<\beta}, \Gamma' \to \Delta',$$

where Γ' has no formulas of the form $s = t$ and when $\Pi \to \Lambda$ is constructed by 7), then the immediate ancestor of $\Pi \to \Lambda$ is the sequent $\Pi' \to \Lambda'$, where Π' and Λ' are sequences of all the formulas obtained from a formula in Π and Λ, respectively, by arbitrary interchange of s_λ and t_λ ($\lambda < \beta$). (So Π' and Λ' obviously include Π and Λ, respectively.)

This completes the description of $T(S)$.

A branch of $T(S)$ is an infinite sequence $S = S_0, S_1, S_2, \ldots$ such that S_{n+1} is an immediate ancestor of S_n. We have two cases.

Case 1. In every branch of $T(S)$, there exists at least one sequent of the form

$$\Gamma_1, D, \Gamma_2 \to \Delta_1, D, \Delta_2 \quad \text{or} \quad \Gamma \to \Delta_1, s = s, \Delta_2.$$

Case 2. There exists at least one branch of $T(S)$, in which there are no sequents of the form

$$\Gamma_1, D, \Gamma_2 \to \Gamma_1, D, \Delta_2 \quad \text{or} \quad \Gamma \to \Delta_1, s = s, \Delta_2.$$

For case 1, S is provable without the cut rule. In order to prove this we define the height of the free variables as follows.

(1) If a belongs to D_0, then $h(a) = 0$.

(2) If a is $g_A^{f,\gamma}(b_0, \ldots, b_\xi, \ldots)$ or $\bar{g}_A^{f,\gamma}(b_0, \ldots, b_\xi, \ldots)$, then $h(a)$ is the supremum of all $h(b_\xi) + 1$'s.

It is easily shown that $T(S)$ satisfies the conditions 1) and 3) in (3) of Definition 23.2.

In the remainder of this proof we will refer to a figure P as a semi-proof if P satisfies all the conditions of a proof except 4) of Definition 23.2. P is said to be a quasi-proof if P satisfies all the conditions of a proof except 3) in (3) of Definition 23.2.

We now consider the following conditions on P.

(3) P is a cut-free semi-proof.

(4) Every free variable in P occurs in $T(S)$ and every inference on Q in P occurs in $T(S)$.

(5) The end sequent of P is S.

If P satisfies (3), (4) and (5) then P obviously satisfies 1) and 3) in (3) of Definition 23.2 and therefore P is a cut-free quasi-proof. Now consider the condition C on a sequent S', that S' has a quasi-proof P satisfying (3), (4) and (5). Let S' be in $T(S)$. It is easily seen that if every ancestor of S' satisfies C, then S' satisfies C. Suppose that S is not provable without the cut rule. Then S does not satisfy C. (Recall that the height is defined). Then some ancestor of S, say S_1, does not satisfy C. Continuing this argument, we obtain a sequence $S, S_1, S_2, \ldots$, where S_{n+1} is an ancestor of S_n and does not satisfy C for each n. This contradicts the hypothesis of case 1.

For case 2, we will show that there exists a structure $\mathscr{A}$ in which every formula in Γ is true and every formula in Δ is false. In the rest of this proof, we fix one branch, whose existence is assumed in the hypothesis of case 2, and consider only the formulas and sequents in this branch, that is, throughout this discussion a sequent always means a sequent in this branch. We only have to define an interpretation which makes all the sequents in this branch false with respect to D.

LEMMA 23.7. (1) *If a formula* $\neg A$ *occurs in the antecedent (succedent) of a sequent, then the formula* A *occurs in the succedent (antecedent) of a sequent.*

(2) *If a formula* $\bigvee_{\lambda<\beta} A_\lambda$ *occurs in the antecedent (succedent) of a sequent, then a formula* A_λ *for some (every)* $\lambda < \beta$ *occurs in the antecedent (succedent) of a sequent.*

(3) *If a formula* $\bigwedge_{\lambda<\beta} A_\lambda$ *occurs in the antecedent (succedent) of a sequent, then a formula* A_λ *for every (some)* $\lambda < \beta$ *occurs in the antecedent (succedent) of a sequent.*

(4) *If* $\mathrm{Q}^f \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s})$ *occurs in the antecedent of a sequent and* $\xi_0, \xi_1, \ldots$ *are all ordinals such that* $f(\xi) = \forall$ *and* $\eta_0, \eta_1, \ldots$ *are all ordinals such that* $f(\eta) = \exists$, *then for an arbitrary sequence* $t_{\xi_0}, t_{\xi_1}, \ldots$ *of members of* D, *the formula* $A(\boldsymbol{t})$ *occurs in the antecedent of a sequent, where* $t_\eta = g_A^{f,\eta}(t_{\xi_0}, \ldots, \boldsymbol{s})$ for each $\eta = \eta_0, \eta_1, \ldots$.

(5) *If* $\mathrm{Q}^f \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s})$ *occurs in the succedent of a sequent and* $\xi_0, \xi_1, \ldots$ *are all ordinals such that* $f(\xi) = \forall$ *and* $\eta_0, \eta_1, \ldots$ *are all ordinals such that* $f(\eta) = \exists$, *then for an arbitrary sequence* $t_{\eta_0}, t_{\eta_1}, \ldots$ *of members of* D, *the formula* $A(\boldsymbol{t})$ *occurs in the succedent of a sequent, where* $t_\xi = \bar{g}_A^{f,\xi}(t_{\eta_0}, \ldots, \boldsymbol{s})$ *for each* $\xi = \xi_0, \xi_1, \ldots$.

PROOF. Obvious.

LEMMA 23.8. *If a formula occurs in the antecedent of a sequent, then it does not occur in the succedent of any sequent.*

PROOF. By transfinite induction on the complexity of formulas using Lemma 23.7.

LEMMA 23.9. (1) *For every member* t *of* D, *the formula* $t = t$ *occurs in the antecedent of a sequent.*

(2) *Let* s *and* t *be members of* D. *If* $s = t$ *occurs in the antecedent of a sequent, then* $t = s$ *occurs in the antecedent of a sequent.*

(3) *Let* t_1, t_2 *and* t_3 *be members of* D. *If* $t_1 = t_2$ *and* $t_2 = t_3$ *occur in the antecedent of a sequent, then the formula* $t_1 = t_3$ *occurs in an antecedent of a sequent.*

(4) *Let* $s_\lambda, t_\lambda, \lambda < \beta$, *be members of* D. *If* $A(s_0, \ldots, s_\lambda, \ldots)$ *and* $\{s_\lambda = t_\lambda\}_{\lambda<\beta}$ *occur in the antecedent of a sequent, then* $A(u_0, \ldots, u_\lambda, \ldots)$ *occurs in the antecedent of a sequent for each sequence* $u_0, \ldots, u_\lambda, \ldots$ *such that* u_λ *is* s_λ *or* t_λ.

PROOF. (1) $t = t$ must be contained in Φ or Ψ in 2) of the tree construction. Since $t = t$ cannot be contained in Ψ because of the hypothesis of case 2, $t = t$ must be contained in Φ.

(2) Let $s = t$ occur in the antecedent of a sequent and $t = s$ occur in the succedent of a sequent, then there is a sequent which contains $s = t$ in the antecedent and $t = s$ in the succedent. By the construction 8) of $T(S)$, there must be a sequent of the form $\Gamma_1 \to \Delta_1, s = s, \Delta_2$. This is a contradiction.

(3) and (4) can be proved similarly.

According to Lemma 23.9, D can be decomposed into equivalence classes by $=$. Let $D_=$ be the set of equivalence classes so obtained; from now on we will denote a class of $D_=$ by a representative of it. We define a structure $\mathscr{A}$ over $D_=$ as follows. Let s be a variable in D. Then the value of s with respect to $\mathscr{A}$ is defined to be the class represented by s. If P is a predicate constant, then $P(t_0, \ldots, t_\lambda, \ldots)$ is defined to be true with respect to $\mathscr{A}$ if $P(t_0, \ldots, t_\lambda, \ldots)$ is in the antecedent of a sequent and is defined to be false with respect to $\mathscr{A}$ otherwise. By transfinite induction on the complexity of A, we shall prove that A is true with respect to $\mathscr{A}$ if A is in the antecedent of a sequent and A is false with respect to $\mathscr{A}$ if A is in the succedent of a sequent. Since the other cases are easy, we only consider the cases where A is $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s})$.

Case 1. $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s})$ occurs in the antecedent of a sequent. In this case, it follows from the induction hypothesis and 6) of the construction of $T(S)$, that $A(\boldsymbol{t}, \boldsymbol{s})$ is true with respect to $\mathscr{A}$ for every $\boldsymbol{t}$ satisfying the following condition. If $\xi_0, \xi_1, \ldots$ are all the ordinals such that $f(\xi) = \forall$ and $\eta_0, \eta_1, \ldots$ are all the ordinals such that $f(\eta) = \exists$, then $t_\eta = g_A^{f,\eta}(t_{\xi_0}, \ldots, \boldsymbol{s})$ for every η. This implies that $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s})$ is true with respect to $\mathscr{A}$.

Case 2. $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s})$ is in the succedent of a sequent. In this case, it follows from the induction hypothesis and 7) of the construction of $T(S)$, that $A(\boldsymbol{t}, \boldsymbol{s})$ is false with respect to $\mathscr{A}$ for every $\boldsymbol{t}$ satisfying the following condition. If $\xi_0, \xi_1, \ldots$ are all the ordinals such that $f(\xi) = \forall$ and $\eta_0, \eta_1, \ldots$ are all the ordinals such that $f(\eta) = \exists$, then $t_\xi = \bar{g}_A^{f,\xi}(t_{\eta_0}, \ldots, \boldsymbol{s})$. This implies that $\neg A(\boldsymbol{t}, \boldsymbol{s})$ is true with respect to $\mathscr{A}$ for every such $\boldsymbol{t}$. Therefore $\mathsf{Q}^f\boldsymbol{x}\, \neg A(\boldsymbol{x}, \boldsymbol{s})$ is true with respect to $\mathscr{A}$. Since $\mathsf{Q}^f\boldsymbol{x}\, \neg A(\boldsymbol{x}, \boldsymbol{s}) \to \neg \mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s})$ is satisfied in all the structures, $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s})$ is false with respect to $\mathscr{A}$.

This completes the proof of our first version of completeness.

Before we proceed to the second version of completeness, we shall first prove the following.

PROPOSITION 23.10. *Let D and D_0 be the same as in the proof of Theorem* 23.6. *Γ_0 is defined to be the sequence consisting of all formulas of the form*

$$\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s}) \vee \mathsf{Q}^f\boldsymbol{x}\, \neg A(\boldsymbol{x}, \boldsymbol{s})$$

where $A(\boldsymbol{x}, \boldsymbol{s})$ is an arbitrary formula in our language and $\boldsymbol{s}$ is an arbitrary sequence of members of D. Without loss of generality, we may assume that no member of D_0 is ever used as an eigenvariable in any quasi-proof. Now let $\Gamma \to \Delta$ be a sequent of the original language and let $\tilde{\Gamma}$ be Γ_0, Γ. Then either there is a cut-free quasi-proof whose end-sequent is $\tilde{\Gamma} \to \Delta$ or else there exists a deter-

minate structure $\mathscr{A}$ such that every formula in $\tilde{\Gamma}$ is satisfied in $\mathscr{A}$ and no formula in Δ is satisfied in $\mathscr{A}$.

PROOF. This is proved similarly to the proof of Theorem 23.6 by replacing "proof" and "Γ" by "quasi-proof" and "$\tilde{\Gamma}$", respectively. Since $\tilde{\Gamma}$ includes Γ_0, it is easily shown that $\mathscr{A}$ is determinate.

THEOREM 23.11. *Let $\Gamma \to \Delta$ be a sequent. Then either $\Gamma \to \Delta$ is provable in our determinate logic or there exists a determinate structure $\mathscr{A}$ such that every formula in Γ is satisfied in $\mathscr{A}$ and no formula in Δ is satisfied in $\mathscr{A}$.*

PROOF. Since every formula in Γ_0 is provable in our determinate logic, (cf. Example 23.3) $\Gamma \to \Delta$ is obtained from $\tilde{\Gamma} \to \Delta$ by the cut rule as follows.

$$\frac{\to B_0, \ldots \qquad \to B_\beta, \ldots \qquad B_0, \ldots, B_\beta, \ldots, \Gamma \to \Delta}{\Gamma \to \Delta},$$

where $\{B_0, \ldots, B_\beta, \ldots\}$ is Γ_0. Thus, if $\tilde{\Gamma} \to \Delta$ has a quasi-proof, then from this quasi-proof we can obtain a proof of $\Gamma \to \Delta$, since $\Gamma \to \Delta$ is a sequent of the original language. Otherwise Proposition 23.10 guarantees that there is a determinate structure $\mathscr{A}$ in which every formula of $\tilde{\Gamma}$, and hence every formula of Γ, is satisfied, while no formula of Δ is satisfied.

REMARK. We cannot improve Theorem 23.11 by replacing "provable" by "provable without the cut rule". This is clear from the following example by Gale and Stewart. Let α_0 be the cardinal number of 2^ω, the set of functions from ω to 2. Let $f \in 2^\omega$. Then $\psi(f)$ is defined to be $a_0 = i_0, a_1 = i_1, \ldots$, where $i_k = 0$ or 1 according as $f(k) = 0$ or 1. The formula $\psi(f)$ implicitly defines the function f. If $A \subseteq 2^\omega$, then A is implicitly defined by the formula $\bigvee_{f \in A} \psi(f)$ where $\bigvee_{f \in A}$ is defined in terms of $\bigvee_{\alpha_0}$. It can be shown that there exists a set $A \subseteq 2^\omega$ such that the axiom of determinateness fails for the game defined by A. (The proof is given below.) If a formula ψ implicitly defines A, then

$$\forall x\, (x = 0 \vee x = 1) \to 0 = 1,$$

$$\neg(\forall x_0\, \exists x_1\, \forall x_2 \ldots \psi(x_0, x_1, \ldots) \vee \exists x_0\, \forall x_1\, \exists x_2 \ldots \neg\psi(x_0, x_1, \ldots))$$

is provable in our determinate logic, where ψ is constructed from $0, 1, =, \bigwedge_{\alpha_0}$ and $\bigvee_{\alpha_0}$. This means that $\forall x\, (x = 0 \vee x = 1) \to 0 = 1$ is provable in our determinate logic if our language has $\bigvee_{\alpha_0}$, since the negation of the last formula is an instance of the axiom of determinateness. However, this is not provable without the cut rule even if our language has $\bigvee_{\alpha_0}$.

The proof of the existence of A goes as follows. We shall show that there is a subset A of 2^ω for which there is no winning strategy.

DEFINITION 23.12. (1) For any subset of 2^ω, say A, $G(A)$, a game for A, is defined as follows: A first player I and a second player II alternately chooses a 0 or 1; thus

I: $x_0\ x_2\ x_4\ \ldots\ x_{2i}\ \ldots$

II: $x_1\ x_3\ x_5\ \ldots\ x_{2i+1}\ \ldots$

for $i < \omega$.

(2) the sequence $\langle x_0, x_1, x_2, \ldots \rangle$ generated in this manner, called a play of the game, determines the winner, that is, if $\langle x_0, x_1, x_2, \ldots \rangle \in A$, then I wins, otherwise II wins.

(3) A sequence $\langle x_0, f_2, f_4, \ldots, f_{2i}, \ldots \rangle$ $i < \omega$ is called a strategy for I if $x_0 \in 2$ and f_{2i} is a function from all i-tuples of 0's and 1's to 2.

(4) Let $\sigma = \langle x_0, f_2, f_4, \ldots \rangle$ be a strategy for I and let $x = \langle x_1, x_3, \ldots, x_{2i+1}, \ldots \rangle$ be a function from odd numbers to 2. Then $\sigma(x)$ is defined by

$$\sigma(x) = \langle x_0, x_1, f_2(x_1), x_3, f_4(x_1, x_3), \ldots \rangle.$$

(5) A strategy σ for I is called a winning strategy for I if

$$\forall x \in 2^{\{2i+1 \mid i<\omega\}}\ \sigma(x) \in A.$$

(6) A sequence $\langle f_1, f_3, \ldots, f_{2i+1}, \ldots \rangle$ is called a strategy for II if f_{2i+1} is a function from all $(i+1)$-tuples of 0's and 1's for every $i < \omega$.

(7) Let $\tau = \langle f_1, f_3, \ldots \rangle$ be a strategy for II and let $x = \langle x_0, x_2, \ldots, x_{2i}, \ldots \rangle$ be a function from even numbers to 2. Then $\tau(x)$ is defined by

$$\tau(x) = \langle x_0, f_1(x_0), x_2, f_3(x_0, x_2), \ldots \rangle.$$

(8) A strategy τ for II is called a winning strategy for II if $\forall x \in 2^{\{2i \mid i<\omega\}}\ \tau(x) \notin A$.

THEOREM 23.13 (Gale-Stewart). *In* ZF, *we can show that if* 2^ω *is well-ordered, then there exists a subset of* 2^ω, *say* A, *for which neither* I *nor* II *has a winning strategy in the game* $G(S)$.

PROOF. It is easy to see that

1. The cardinality of all strategies for I is α_0, and

2. The cardinality of all strategies for II is α_0.

3. If α is a strategy for I, then the cardinality of the set $\{(\sigma, \tau) \mid \tau$ is a strategy for II$\}$ is α_0.

4. If τ is a strategy for II, then the cardinality of $\{(\sigma, \tau) \mid \sigma$ is a strategy for I$\}$ is α_0.

Let $\sigma_0, \sigma_1, \ldots, \sigma_\alpha, \ldots, \alpha < \alpha_0$ and $\tau_0, \tau_1, \ldots, \tau_\alpha, \ldots, \alpha < \alpha_0$, be enumerations of all strategies for I and II, respectively. By transfinite induction we define plays

$$x^\alpha = \langle x_0^\alpha, x_1^\alpha, \ldots, x_i^\alpha, \ldots \rangle, \qquad y^\alpha = \langle y_0^\alpha, y_1^\alpha, \ldots, y_i^\alpha, \ldots \rangle,$$

where x_i^α, $y_i^\alpha = 0$ or 1:

(1) $x^0 = (\sigma_0, \tau_0)$.

(2) $y^0 = (\sigma_0, \tau_\beta)$, where β is the smallest ordinal such that $(\sigma_0, \tau_\beta) \neq x^0$.

(3) $S_\alpha = \{x^\beta \mid \beta < \alpha\}$ and $T_\alpha = \{y^\beta \mid \beta < \alpha\}$.

(4) $x^\alpha = (\sigma_\beta, \tau_\beta)$, where β is the smallest ordinal such that $(\sigma_\beta, \tau_\alpha), \notin S_\alpha \cup T_\alpha$.

(5) $y^\alpha = (\sigma_\alpha, \tau_\beta)$, where β is the smallest ordinal such that $(\sigma_\alpha, \tau_\beta) \notin S_\alpha \cup T_\alpha$ and $(\sigma_\alpha, \tau_\beta) \neq x^\alpha$. It is obvious that if $\alpha < \alpha_0$, then $S_\alpha \cap T_\alpha = 0$, $\overline{\overline{S}}_\alpha < \alpha_0$, and $\overline{\overline{T}}_\alpha < \alpha_0$.

(6) $A = \bigcup_{\alpha < \alpha_0} S_\alpha$.

We claim that for this A neither I nor II has a winning strategy. Suppose that I has a winning strategy, say σ_α. Let β be the smallest ordinal such that

$$(\sigma_\alpha, \tau_\beta) \notin S_\alpha \cup T_\alpha \wedge (\sigma_\alpha, \tau_\beta) \neq x^\alpha.$$

Then $(\sigma_\alpha, \tau_\beta) = y^\alpha \notin A$, which means that II has a winning strategy, yielding a contradiction.

In order to prove the interpolation theorem, we need the following proof-theoretic notion.

DEFINITION 23.14. Let P be a cut-free semi-proof and let I be an inference in P. Let A be a formula in an upper sequent of I and B be a formula in the lower sequent of I. B is said to be the immediate successor of A if the following is satisfied.

Case 1. If I is a structural inference

$$\frac{\Gamma \to \Delta}{\Pi \to \Lambda},$$

and A is a formula of $\Gamma(\Delta)$, then B is the first formula in $\Pi(\Lambda)$ which is identical with A.

Case 2. If I is a logical inference

$$\frac{\Pi, \Gamma \to \Delta, \Lambda}{\Pi', \Gamma \to \Delta, \Lambda'},$$

where I applies to the formulas of Π and Λ, and A is in $\Gamma(\Lambda)$, then B is the corresponding formula in $\Gamma(\Lambda)$.

Case 3. If I is a logical inference and A is an auxiliary formula of I, then B is the corresponding principal formula.

Case 4. If I is the first equality rule (cf. Definition 22.19) and A is a formula in $\Gamma^{(\boldsymbol{a})}$ $(\Delta^{(\boldsymbol{a})})$ then B is the corresponding formula in $\Gamma^{(\boldsymbol{b})}$ $(\Delta^{(\boldsymbol{b})})$.

Case 5. If I is the second equality rule, (cf. Definition 22.19) and A is a formula in $\Gamma(\Delta)$, then B is the corresponding formula in $\Gamma(\Delta)$.

Our interpolation theorem is then stated in the following form.

THEOREM 23.15 (an interpolation theorem for homogeneous languages). *If a sequent $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$ is valid and has no heterogeneous quantifiers, then there exists a formula C such that both the sequents*

$$\Gamma_1 \to \Delta_1, C \quad \textit{and} \quad C, \Gamma_2 \to \Delta_2$$

are valid and every free variable or predicate constant in C, except $=$, occurs in both Γ_1, Δ_1 and Γ_2, Δ_2. (C may have heterogeneous quantifiers and also logical connectives or quantifiers that are longer than the logical symbols in the original language).

PROOF. The proof will be divided into several parts.

1. First we shall introduce two auxiliary systems.

DEFINITION 23.16. A proof P in our determinate logic is said to satisfy condition (Q) if every inference I in P of Q : right is either homogeneous or is of the form

$$\text{(Q)} \qquad \frac{\Gamma \to \Delta, A(\boldsymbol{d})}{\Gamma \to \Delta, Q^f\boldsymbol{x}\, A(\boldsymbol{x})},$$

where no eigenvariable in P used above $\Gamma \to \Delta$, $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x})$ occurs in $\Gamma \to \Delta$, $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x})$.

PROPOSITION 23.17. *If a sequent S is provable with a proof which satisfies* (Q), *then S is valid.*

PROOF. Define $g_A^{f,\gamma}$ and $\bar{g}_A^{f,\gamma}$ as in the proof of Theorem 23.4 except that $\bar{g}_A^{f,\gamma}$ is defined only for homogeneous f. Then define substitution also as in the proof of Theorem 23.4 except that all eigenvariables in the inference of (Q) remain unsubstituted. Then P will be transformed into P'. What we have to show is that every sequent S' in P' is satisfied in $\mathscr{A}$. This is shown by transfinite induction on the complexity of the semi-proof of S. We can repeat the proof of Theorem 23.4 except in the following case. S is inferred by the inference I:

$$\frac{\Gamma \to \Delta,\, A(\boldsymbol{d}, \boldsymbol{b})}{\Gamma \to \Delta,\, \mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{b})},$$

where Q^f is not homogeneous. In order to illustrate the proof, we assume that $\mathsf{Q}^f\boldsymbol{x}$ is $\forall x_0\, \exists x_1\, \forall x_2\, \exists x_3 \ldots$ and $\boldsymbol{d}$ is $d_0, d_1, d_2, \ldots$. Since I satisfies (Q) and $h(d_0) < h(d_1) < h(d_2) < \ldots$, $(\Gamma \to \Delta, A(\boldsymbol{d}, \boldsymbol{b}))'$ is of the form

$$(*) \qquad \Gamma' \to \Delta',\, A(d_0, t_1(d_0, \boldsymbol{s}), d_2, t_3(d_0, d_2, \boldsymbol{s}), \ldots, \boldsymbol{s}).$$

It follows from the induction hypothesis that (*) is satisfied in $\mathscr{A}$ for every sequence $d_0, d_2, d_4, \ldots$ of members of $\mathscr{A}$. Therefore $\Gamma' \to \Delta'$, $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{s})$ is satisfied in $\mathscr{A}$.

Next we shall consider another logical system, the restricted homogeneous system **RHS**.

DEFINITION 23.18. A figure P is said to be a proof in **RHS** if P satisfies the following conditions:

1) All quantifiers in P are $\exists$.

2) P satisfies all conditions of a proof of determinate logic except (3) of Definition 23.2.

3) Every inference in P on the introduction of Q in the antecedent is of the following form

$$\frac{\{A_\lambda(\boldsymbol{a}_\lambda)\}_{\lambda<\gamma},\, \Gamma \to \Delta}{\{\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\gamma},\, \Gamma \to \Delta},$$

where no variable in $\boldsymbol{a}_\lambda$ occurs in the lower sequent.

Then we have the following proposition.

PROPOSITION 23.19. *If $\Gamma \to \Delta$ is provable in* **RHS** *and height (see Definition 23.2) is defined for all free variables in $\Gamma \to \Delta$, then there exists a proof P' in* **RHS** *ending with $\Gamma \to \Delta$ for which the heights are defined in such a way that the free variables in $\Gamma \to \Delta$ have the same heights as the original ones.*

PROOF. We may assume that the same eigenvariable is never used in two different places. (Otherwise, we can reletter some eigenvariables.) Then it is easy to define heights of free variables from the bottom.

2. Next we prove the following lemma.

LEMMA 23.20. *Let P be a cut-free proof of $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$ in the homogeneous system (see Definition 22.1), a proof satisfying the following conditions:*

(1) *Every quantifier in P is $\exists$.*

(2) *Every* Q*-introduction inference in P is a $\exists$-introduction inference in the succedent.*

Then there exist cut-free proofs P_1 and P_2 in **RHS** *and a formula C satisfying the following conditions.*

(2.1) *The end-sequent of P_1 is $C, \Gamma_1 \to \Delta_1$ and the end-sequent of P_2 is $\Gamma_2 \to \Delta_2, C$.*

(2.2) *Every free variable or predicate constant in C, except $=$, occurs in both Γ_1, Δ_1 and Γ_2, Δ_2.*

PROOF. The proof is by transfinite induction on the complexity of P.

Case 1. P consists of a single initial sequent. The theorem is obvious.

Case 2. The last inference of P is of the form

$$\frac{\Gamma_1, \Gamma_2 \to \Delta_1', \{A_\lambda(\boldsymbol{a}_\lambda)\}_{\lambda<\beta_1}, \Delta_2', \{B_\mu(\boldsymbol{b}_\mu)\}_{\mu<\beta_2}}{\Gamma_1, \Gamma_2 \to \Delta_1', \{\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\beta_1}, \Delta_2', \{\exists \boldsymbol{y}_\mu\, B_\mu(\boldsymbol{y}_\mu)\}_{\mu<\beta_2}},$$

where Δ_1 is $\Delta_1'\{\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\beta_1}$ and Δ_2 is $\Delta_2'\{\exists \boldsymbol{y}_\mu\, B_\mu(\boldsymbol{y}_\mu)\}_{\mu<\beta_2}$.

By the induction hypothesis, there exists a $C'(\boldsymbol{a}, \boldsymbol{b})$ for which

$$C'(\boldsymbol{a}, \boldsymbol{b}), \Gamma_1 \to \Delta_1', \{A_\lambda(\boldsymbol{a}_\lambda)\}_{\lambda<\beta_1}$$

and

$$\Gamma_2 \to \Delta_2', \{B_\mu(\boldsymbol{b}_\mu)\}_{\mu<\beta_2}, C'(\boldsymbol{a}, \boldsymbol{b})$$

are provable in **RHS**.

Moreover, every free variable and predicate constant in $C'(\boldsymbol{a}, \boldsymbol{b})$ is either $=$ or contained in both $\Gamma_1, \Delta_1', \{A_\lambda(\boldsymbol{a}_\lambda)\}_{\lambda<\beta_1}$ and $\Gamma_2, \Delta_2', \{B_\mu(\boldsymbol{b}_\mu)\}_{\mu<\beta_2}$. Here $\boldsymbol{a}$ is a sequence of all the variables in $C'(\boldsymbol{a}, \boldsymbol{b})$ which are not in Γ_1, Δ_1 and $\boldsymbol{b}$ is a sequence of all the variables in $C'(\boldsymbol{a}, \boldsymbol{b})$ which are not in Γ_2, Δ_2. Then the required formula C is $\exists \boldsymbol{x}\, \forall \boldsymbol{y}\, C'(\boldsymbol{x}, \boldsymbol{y})$, where $\forall$ is considered as an abbreviation of $\neg\exists\neg$.

Case 3. The last inference of P is of the form

$$\frac{\Gamma_1'^{(\boldsymbol{a})}, \Gamma_2'^{(\boldsymbol{a})} \to \Delta_1^{(\boldsymbol{a})}, \Delta_2^{(\boldsymbol{a})}}{\boldsymbol{a}_1 = \boldsymbol{b}_1, \boldsymbol{a}_2 = \boldsymbol{b}_2, \Gamma_1'^{(\boldsymbol{b})}, \Gamma_2'^{(\boldsymbol{b})} \to \Delta_1^{(\boldsymbol{b})}, \Delta_2^{(\boldsymbol{b})}}$$

where Γ_1 is $\boldsymbol{a}_1 = \boldsymbol{b}_1, \Gamma_1'^{(\boldsymbol{b})}$ and Γ_2 is $\boldsymbol{a}_2 = \boldsymbol{b}_2, \Gamma_2'^{(\boldsymbol{b})}$. This can be divided into two steps; first, the substitution of $\boldsymbol{a}_1$ for $\boldsymbol{b}_1$; then the substitution of $\boldsymbol{a}_2$ for $\boldsymbol{b}_2$. So we may assume that $\boldsymbol{a}_1 = \boldsymbol{b}_1$ is empty. By the induction hypothesis, there exists a formula $C'(\boldsymbol{a}, \boldsymbol{b})$ which satisfies the lemma for $\Gamma_1'^{(\boldsymbol{a})}, \Gamma_2'^{(\boldsymbol{a})} \to \Delta_1^{(\boldsymbol{a})}, \Delta_2^{(\boldsymbol{a})}$, where $\boldsymbol{a}$ is a sequence consisting of all variables in $C'(\boldsymbol{a}, \boldsymbol{b})$ which are not in Γ_1, Δ_1 and $\boldsymbol{b}$ is a sequence of all the variables in $C'(\boldsymbol{a}, \boldsymbol{b})$ which are not in Γ_2, Δ_2. If there exists a unique i such that $a_{2,\mu}$ is the ith variable of $\boldsymbol{a}$ then we define $\breve{a}_{2,\mu}$ to be the ith variable in $\boldsymbol{x}$. Otherwise we define $\breve{a}_{2,\mu}$ to be $a_{2,\mu}$. Then take C to be $\exists \boldsymbol{x}\, \forall \boldsymbol{y}\, (\bigwedge_\mu \breve{a}_{2,\mu} = \breve{b}_{2,\mu} \wedge C'(\boldsymbol{x}, \boldsymbol{y}))$.

Case 4. The last inference of P is of the form

$$\frac{\Phi, \Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, \Psi}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2} \quad \text{for all } (\Phi|\Psi).$$

By the induction hypothesis, there exist formulas $C_{(\Phi|\Psi)}$ such that $C_{(\Phi|\Psi)}, \Gamma_1 \to \Delta_1$ and $\Phi, \Gamma_2 \to \Delta_2, \Psi, C_{(\Phi|\Psi)}$ are provable in **RHS**. So

$$\bigvee_{(\Phi|\Psi)} C_{(\Phi|\Psi)}, \Gamma_1 \to \Delta_1$$

and

$$\Gamma_2 \to \Delta_2, \bigvee_{(\Phi|\Psi)} C_{(\Phi|\Psi)}$$

are provable in **RHS**. Let $\boldsymbol{a}$ be a sequence of all the free variables in $\bigvee_{(\Phi|\Psi)} C_{(\Phi|\Psi)}$ which do not appear in Γ_2, Δ_2 and let $\boldsymbol{b}$ be those which do not occur in Γ_1, Δ_1. We rewrite $\bigvee_{(\Phi|\Psi)} C_{(\Phi|\Psi)}$ as $C'(\boldsymbol{a}, \boldsymbol{b})$. Then take C to be $\forall \boldsymbol{x}\, \exists \boldsymbol{y}\, C'(\boldsymbol{x}, \boldsymbol{y})$.

Other cases. The proof is similar to the one above.

REMARK. In Lemma 23.20, note that (1) is not an essential restriction on P because $\forall$ can be expressed by $\neg$ and $\exists$. Note also that any sub-proof of P, i.e., any part of P consisting of all sequents above and including a given sequent, is a proof in **RHS** because there are no eigenvariables in P.

3. Let $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$ be as in the statement of the theorem. There exists a cut-free proof P of $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$ in the homogeneous system. For the completeness proof, we may assume that P satisfies the following condition.

3.1. If a variable occurs in two different auxiliary formulas as an eigenvariable, then these two formulas are the same.

Moreover, without loss of generality we may assume the following for P.

3.2. Every quantifier in P is $\exists$.

3.3. The height of a free variable in $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$ is less than the height of any eigenvariable in P.

3.4. The heights of two different variables in P are different.

Let $\Gamma_1' \to \Delta_1'$ be a sequent in P. Let $\Phi(\Gamma_1', \Delta_1')$ be the sequence $A_0, A_1, \ldots, A_\mu, \ldots$ of all A_μ's such that A_μ is of the form $\neg\exists \boldsymbol{x}\, A(\boldsymbol{x}) \vee A(\boldsymbol{a})$, where $\exists \boldsymbol{x}\, A(\boldsymbol{x})$ is a principal formula of a $\exists$: left above $\Gamma_1' \to \Delta_1'$ and $A(\boldsymbol{a})$ is its auxiliary formula. Replacing $\Gamma_1' \to \Delta_1'$ by $\Phi(\Gamma_1', \Delta_1'), \Gamma_1' \to \Delta_1'$ and inserting some appropriate structural inferences, we obtain a new figure P' satisfying the following conditions:

1) P' satisfies (1) and (2) of Lemma 23.20.

2) The end-sequent of P' is of the form (cf. the proof of Proposition 22.25)

$$\{\neg\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{c}_\lambda) \vee A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{c}_\lambda)\}, \Gamma_1,$$

$$\{\neg\exists \boldsymbol{y}_\mu\, B_\mu(\boldsymbol{y}_\mu, \boldsymbol{d}_\mu) \vee B_\mu(\boldsymbol{b}_\mu, \boldsymbol{d}_\mu)\}, \Gamma_2 \to \Delta_1, \Delta_2.$$

3) The height of any $c_{\lambda,\alpha}$ is less than the height of any $a_{\lambda,\beta}$. The height of any $d_{\mu,\alpha}$ is less than the height of any $b_{\mu,\beta}$.

4) Every free variable or predicate constant, except $=$, in $\exists \boldsymbol{z}_\lambda\, \exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{z}_\lambda)$ occurs in Γ_1, Δ_1 and every free variable or predicate constant, except $=$, occurring in $\exists \boldsymbol{z}_\mu\, \exists \boldsymbol{x}_\mu\, B_\mu(\boldsymbol{y}_\mu, \boldsymbol{z}_\mu)$ occurs in Γ_2, Δ_2.

5) Any $a_{\lambda,\alpha}$ and $b_{\mu,\beta}$ are different. (Otherwise we can modify P' so that P' satisfies 5) because P satisfies 3.1.)

Applying Lemma 23.20, there is a formula $C(\boldsymbol{a})$ such that

$$\text{(a)} \qquad C(\boldsymbol{a}), \{\neg\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{c}_\lambda) \vee A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{c}_\lambda)\}, \Gamma_1 \to \Delta_1$$

and

$$\{\neg\exists \boldsymbol{y}_\mu\, B_\mu(\boldsymbol{y}_\mu, \boldsymbol{d}_\mu) \vee B_\mu(\boldsymbol{b}_\mu, \boldsymbol{d}_\mu)\}, \Gamma_2 \to \Delta_2, C(\boldsymbol{a})$$

are provable in **RHS**. Let Q_1 and Q_2 be proofs of these sequents in **RHS**.

(b) Every free variable or predicate constant, except $=$, occurring in $C(\boldsymbol{a})$ is in both $\{A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{c}_\lambda)\}, \Gamma_1, \Delta_1$ and $\{B_\mu(\boldsymbol{b}_\mu, \boldsymbol{d}_\mu)\}, \Gamma_2, \Delta_2$.

(c) $\boldsymbol{a}$ is the sequence of all the free variables in $C(\boldsymbol{a})$ which are not in both Γ_1, Δ_1 and Γ_2, Δ_2 and well-ordered according to heights.

4. Then consider the following figure

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{Q}_1}{C(\boldsymbol{a}), (\neg\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{c}_\lambda) \vee A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{c}_\lambda)),\ \Gamma_1 \to \Delta_1}}{C(\boldsymbol{a}), \{\exists \boldsymbol{x}'_\lambda\, (\neg\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{c}_\lambda) \vee A_\lambda(\boldsymbol{x}'_\lambda, \boldsymbol{c}_\lambda))\}, \Gamma_1 \to \Delta_1}}{C(\boldsymbol{a}), \{\forall \boldsymbol{z}_\lambda\, \exists \boldsymbol{x}'_\lambda\, (\neg\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{z}_\lambda) \vee A_\lambda(\boldsymbol{x}'_\lambda, \boldsymbol{z}_\lambda))\}, \Gamma_1 \to \Delta_1}}{\mathsf{Q}^f\boldsymbol{x}\, C(\boldsymbol{x}), \{\forall \boldsymbol{z}_\lambda\, \exists \boldsymbol{x}_\lambda\, (\neg\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{z}_\lambda) \vee A_\lambda(\boldsymbol{x}'_\lambda, \boldsymbol{z}_\lambda))\}, \Gamma_1 \to \Delta_1}$$

where f is defined as follows.

(e) If a_α is $b_{\mu,\gamma}$ for some γ, then $f(\alpha) = \exists$.

(f) If a_α is $a_{\lambda,\gamma}$, for some γ, then $f(\alpha) = \forall$.

(g) If a_α is contained in Γ_1, Δ_1 but not in Γ_2, Δ_2, then $f(\alpha) = \forall$.

(h) If a_α is contained in Γ_2, Δ_2, but not in Γ_1, Δ_1, then $f(\alpha) = \exists$.

(i) If (e)–(h) are not the case, then $f(\alpha) = \exists$.

The heights for the free variables in $\boldsymbol{a}_\lambda$, $\boldsymbol{c}_\lambda$, $C(\boldsymbol{a})$, Γ_1, Δ_1 are defined to be the heights in P. The heights of all other variables in $\mathfrak{Q}_1$ can be so defined, according to Proposition 23.19, that the whole figure will become a proof in determinate logic. This means that $\mathsf{Q}^f\boldsymbol{x}\, C(\boldsymbol{x}), \Gamma_1 \to \Delta_1$ is valid. The validity of $\Gamma_2 \to \Delta_2, \mathsf{Q}^f\boldsymbol{x}\, C(\boldsymbol{x})$ is also easily seen from the following proof which satisfies (Q) (cf. Definition 23.14).

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{Q}_2}{\{\neg\exists \boldsymbol{y}_\mu\, B_\mu(\boldsymbol{y}_\mu, \boldsymbol{d}_\mu) \vee B_\mu(\boldsymbol{b}_\mu, \boldsymbol{d}_\mu)\},\ \Gamma_2 \to \Delta_2, C(\boldsymbol{a})}}{\{\exists \boldsymbol{y}'_\mu\, (\neg\exists \boldsymbol{y}_\mu\, B_\mu(\boldsymbol{y}_\mu, \boldsymbol{d}_\mu) \vee B_\mu(\boldsymbol{y}'_\mu, \boldsymbol{d}_\mu))\}, \Gamma_2 \to \Delta_2, C(\boldsymbol{a})}}{\{\forall \boldsymbol{z}_\mu\, \exists \boldsymbol{y}'_\mu\, (\neg\exists \boldsymbol{y}_\mu\, B_\mu(\boldsymbol{y}_\mu, \boldsymbol{z}_\mu) \vee B_\mu(\boldsymbol{y}'_\mu, \boldsymbol{z}_\mu))\}, \Gamma_2 \to \Delta_2, C(\boldsymbol{a})}}{\{\forall \boldsymbol{z}_\mu\, \exists \boldsymbol{y}'_\mu\, (\neg\exists \boldsymbol{y}_\mu\, B_\mu(\boldsymbol{y}_\mu, \boldsymbol{z}_\mu) \vee B_\mu(\boldsymbol{y}_\mu, \boldsymbol{z}_\mu))\}, \Gamma_2 \to \Delta_2\, \mathsf{Q}^f\boldsymbol{x}\, C(\boldsymbol{x}).}$$

This completes the proof of Theorem 23.15.

Using the same method we can prove the following theorem.

THEOREM 23.21 (cf. Theorem 23.15). *If every quantifier in $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$ is homogeneous, if $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$ is valid and does not contain $=$, and if Γ_1, Δ_1 and Γ_2, Δ_2 have at least one predicate constant in common, then there exists a formula C such that both $C, \Gamma_1 \to \Delta_1$ and $\Gamma_2 \to \Delta_2, C$ are valid and every free variable or predicate constant in C is contained in both Γ_1, Δ_1 and Γ_2, Δ_2.*

REMARK 23.22. In Theorems 23.15 and 23.21, we may add the condition that C contains only one heterogeneous quantifier in the front of C.

REMARK 23.23. For Malitz's example (cf. §22) we can construct an isomorphism between $\overset{1}{<}$ and $\overset{2}{<}$ by the following formula.

$$\begin{aligned}&\forall x_1\, \exists y_1\, \forall x_2\, \exists y_2 \ldots \Big(\bigwedge_i x_i \overset{1}{<} a \to \bigwedge_i y_i \overset{2}{<} b \wedge \bigwedge_i \bigwedge_j (x_i \overset{1}{<} x_j \leftrightarrow y_i \overset{2}{<} y_j)\\ &\qquad\qquad \wedge\ (x_i \overset{1}{=} x_j \leftrightarrow y_i \overset{2}{=} y_j)\Big)\\ &\wedge \forall y_1\, \exists x_1\, \forall y_2\, \exists x_2 \ldots \Big(\bigwedge_i y_i \overset{2}{<} b \to \bigwedge_i x_i \overset{1}{<} a \wedge \bigwedge_i \bigwedge_j (x_i \overset{1}{<} x_j \leftrightarrow y_i \overset{2}{<} y_j)\\ &\qquad\qquad \wedge\ (x_i \overset{1}{=} x_j \leftrightarrow y_i \overset{2}{=} y_j)\Big).\end{aligned}$$

The order type of a in $(\overset{1}{=}, \overset{1}{<})$ is denoted by $|a|_1$ and the order type of b in $(\overset{2}{=}, \overset{2}{<})$ is denoted by $|b|_2$. Then $|a|_1 \leqslant |b|_2$ is equivalent to

$$(*)\qquad \begin{aligned}&\forall x_1\, \exists y_1\, \forall x_2\, \exists y_2 \ldots \Big(\bigwedge_i x_i \overset{1}{<} a \to \bigwedge_i y_i \overset{2}{<} b \wedge \bigwedge_i \bigwedge_j (x_i \overset{1}{<} x_j \leftrightarrow y_i \overset{2}{<} y_j)\\ &\qquad\qquad \wedge\ (x_i \overset{1}{=} x_j \leftrightarrow y_i \overset{2}{=} y_j)\Big).\end{aligned}$$

This is easily shown by transfinite induction on $|a|_1$, as follows.

Let the formula (*) be denoted by $A(a, b)$. Suppose that for each $c \overset{1}{<} a$ and each d, $A(c, d)$ is equivalent to $|c|_1 \leqslant |d|_2$. Suppose also that $A(a, b)$ holds. Then for each $a_1 \overset{1}{<} a$, there exists a b_1 such that $A(a_1, b_1)$ holds because $\bigwedge_{i\geqslant 2} (x_i \overset{1}{<} a_i)$ implies $\bigwedge_{i\geqslant 2} (x_i \overset{1}{<} a)$ and hence for $x_2, y_2, \ldots$ selected for (a_1, b_1) in $A(a, b)$ we have $\bigwedge_{i\geqslant 2} y_i \overset{2}{<} b_1$. Then making the appropriate substitutions into $A(a, b)$ we obtain $A(a_1, b_1)$. Since $a_1 \overset{1}{<} a$ weh ave, by the induction hypothesis $|a_1|_1 \leqslant |b_1|_2 < |b|_2$. Therefore $|a|_1 \leqslant |b|_2$.

The converse is obvious.

The axiom of determinateness, AD, is a very powerful axiom that has numerous interesting and important applications. Augmented by the axiom of dependent choice

$$\text{DC}\qquad \forall x\, \exists y\, R(x, y) \to \forall x_0\, \exists x_1\, \exists x_2 \ldots \bigwedge_i R(x_i, x_{i+1})$$

the AD has even more implications for mathematics.

Unlike the axiom of choice, AC, which also has important implications for mathematics, the status of the AD is as yet unsettled. We do not know whether the AD is consistent with set theory. Neither do we know whether the AD and DC are consistent. We do know that although the AD implies the axiom of countable choice it is incompatible with the AC.

If it should develop that the AD is inconsistent with set theory we would, of course, cease to be interested in it. But even a proof of consistency would not be sufficient for our purposes, for in order to reap the benefits of the AD for mathematics we must have a transitive model of ZF + AD that contains the power set of ω, $P(\omega)$, as an element. Indeed we would like to have a transitive model of ZF + AD + DC that contains $P(\omega)$ as an element.

Concerning the existence of such models we know the following. Let $\mathrm{L}_\beta(P(\omega))$ be the set obtained from $P(\omega)$ by a β-fold transfinite iteration of Gödel's eight fundamental operations and let α be the smallest β such that $\mathrm{L}_\beta(P(\omega))$ is a model of ZF. Then we know that if there exists a transitive model of ZF + AD that contains $P(\omega)$, $\mathrm{L}_\alpha(P(\omega))$ is a model of the AD. But we also know that $\mathrm{L}_\alpha(P(\omega))$ satisfies DC.

There are then three possibilities:

1) The AD is inconsistent with set theory.

2) The AD is consistent with set theory but no transitive model of ZF + AD exists that contains $P(\omega)$.

3) $\mathrm{L}_\alpha(P(\omega))$ is a model of the AD.

If alternative 1) or 2) should be the case, we would have no further interest in the AD. Our hopes center around alternative 3) which we conjecture to be true. We are, however, unable to prove that $\mathrm{L}_\alpha(P(\omega))$ is the model we conjecture it to be. Moreover, at the present time no one appears to have a method that might resolve the question. In view of the implications of this conjecture for mathematics it is important that a thorough study of the AD be made. As a contribution to this study we will prove a relation between the AD and the cut-elimination theorem.

Let M be a transitive model of ZF + DC that contains $P(\omega)$ as an element. M may be a set or a proper class. Although we cannot assume the AC in M we will assume it in V, the universe of all sets. Using the AC in V we will prove that the AD holds in a model M if and only if the cut-elimination theorem holds in M-definable determinate logic. For the proof we need the following definitions.

DEFINITION 23.24. A set A is *at most the continuum* in M iff $A \in M$, $A \neq 0$ and there is a function f in M such that f maps $P(\omega)$ onto A.

Clearly, if A and B are nonempty sets in M and $B \subseteq A$, then B is at most the continuum in M if A is at most the continuum in M.

Since M is a model that contains $P(\omega)$ as an element it follows that for any language L, having not more than $\aleph_1$ symbols, we can assign to each symbol and to each formula of L a gödelization in M. This enables us to identify collections of language symbols and formulas with sets in M. Since we know that these identifications can be made we will follow the convention of speaking simply of sets of language symbols as being in M. With this convention in mind we define M-definable determinate logic.

DEFINITION 23.25. A language L for *M-definable determinate logic* consists of the following:

1) *Free variables*: A free variable a_s for each s in $P(\omega)$.
2) *Bound variables*: $x_0, x_1, \ldots, x_\alpha, \ldots, \alpha < \omega_1$.
3) *Individual constants*: A set of individual constants that is at most the continuum in M and which contains $0, 1, 2, \ldots$.
4) *Predicate constants*: A set of predicate constants that is at most the con tinuum in M. The arity of each predicate constant is at most ω.
5) *Logical symbols*:
   $=$ (equality),
   $\neg$ (not),
   $\bigwedge$ (conjunctions of arity α for $\alpha \leqslant \omega$),
   $\bigvee$ (disjunctions of arity α for $\alpha \leqslant \omega$),
   Q^f (heterogeneous quantifiers of arity α for $1 \leqslant \alpha \leqslant \omega$).

Note that the set of free variables is at most the continuum in M. Furthermore, since $P(\omega)$ is in M, $\omega_1 = \omega_1^M$, that is, ω_1 is M-absolute.

The formulas of L we define in the following way:

Let R be a predicate constant or $=$ and let the arity of R be α. Let $\{t_i\}_{i<\alpha}$ be a sequence of terms. Then $R(t_0, t_1, \ldots)$ is an atomic formula.

Let A be a formula. Then $\neg A$ is a formula.

Let $\{A_i\}_{i<\alpha}$ be a sequence of formulas. Then $\bigwedge_{i<\alpha} A_i$ and $\bigvee_{i<\alpha} A_i$ are formulas.

Let $A(\boldsymbol{a})$ be a formula where $\boldsymbol{a}$ is the sequence of free variables $\{a_i\}_{i<\alpha}$, with $\alpha \leqslant \omega$. Let $\boldsymbol{x}$ be the sequence of bound variables $\{x_i\}_{i<\alpha}$ and let f map α into $\{\forall, \exists\}$. Then $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x})$ is a formula, where $A(\boldsymbol{x})$ is obtained from $A(\boldsymbol{a})$ by replacing some occurrences of a_i by x_i for each i.

Let Γ and Δ be sets of formulas of at most the continuum. Then $\Gamma \to \Delta$ is called a sequent.

Notice that we cannot assume the well-ordering of Γ and Δ, since the axiom of choice is not assumed in M.

COROLLARY 23.26. (1) *If a language* L *is fixed, then the set of* L-*formulas is at most the continuum.*

(2) *Given an* L-*formula, the number of variables, constants and logical symbols which occur in it is at most countable.*

When we consider a set of formulas $\{A_\lambda\}_\lambda$ or a set of free variables, we must remember that they are just sets; they may not be well-ordered.

LEMMA 23.27. (1) *Consider a tree of length* ω *which has* ω *branches extending from each node. We may identify this with* ω^ω, *which is at most the continuum.*

(2) *Let* α *be an ordinal which is at most the continuum. Consider a tree of length* ω *which has* α *branches extending from each node. We may identify this with* α^ω, *which is at most the continuum.*

DEFINITION 23.28. The notion of proof and the rules of inference for M-definable determinate logic are defined as follows.

1) The initial sequents are the logical initial sequents; $\to t = t$, where t is an arbitrary term; those sequents of the form $i = j \to$, where $i \neq j$ and $i, j < \omega$; and those sequents of the form $\to t = 0, t = 1, t = 2, \ldots$, where t is an arbitrary term.

2) The rules of inference are those rules of determinate logic, which we have already presented. One should keep in mind that in the sequents the formulas form sets that are not necessarily well-ordered. As an example, $\bigvee$: right looks like this:

$$\frac{\Gamma \to \Delta, \{A_{\lambda,i_\lambda}\}_{i_\lambda < \alpha_\lambda}}{\Gamma \to \Delta, \{\bigvee_{i<\alpha_\lambda} A_{\lambda,i}\}}$$

where $\alpha_\lambda \leqslant \omega$ and λ ranges over a set of at most the continuum.

3) A proof in M, say P, is a member of M which is a proof in the ordinary sense except that the notion of height must be replaced by a relation $\prec$:

3.1) Suppose a is an eigenvariable in P, and b is a free variable which occurs in its principal formula. Then $b \prec a$.

3.2) Suppose a is an eigenvariable, $A(\boldsymbol{a})$ is its auxiliary formula, $\mathsf{Q}^f\boldsymbol{x}\, A(\boldsymbol{x})$ is its principal formula, and suppose a corresponds to $f(i)$. If b also occurs in $\boldsymbol{a}$ and b corresponds to $f(j)$, where $j < i$, then $b \prec a$.

The eigenvariable condition is simply that $\prec$ is well-founded. If $b \prec a$, let us say that a depends on b.

REMARK. "$\prec$ is well-founded" is M-absolute because a countable subset of free variables (in V) is a countable subset of $P(\omega)$ and $P(\omega)$ belongs to M, hence this set of free variables is a countable subset of M.

Before we get into the next argument, we should remark that we may and will restrict the indices of free variables, the s in a_s, where $s \in P(\omega)$, to subsets of even numbers. This way, we will be free to introduce new free variables.

LEMMA 23.29. *Consider a countable set of free variables, say* $\tilde{A} = \{a_{s_1}, a_{s_2}, \ldots\}$ *where* $\tilde{A}$ *belongs to* M *and* $\ldots \prec a_{s_2} \prec a_{s_1}$ *(in* V*). Define* $R(a, b)$ *by*

$$R(a, b) \Leftrightarrow_{\mathrm{df}} a \in \tilde{A} \supset (b \in \tilde{A} \wedge b \prec a).$$

Then $\forall x_0 \, \exists x_1 \, x_2 \ldots \bigwedge_i R(x_i, x_{i+1})$ *in* M.

PROOF. It is easily seen that $\forall x \, \exists y \, R(x, y)$ in M; hence by DC the desired formula is obtained.

DEFINITION 23.30. A quantifier $\mathbf{Q}$ in a formula A is said to be essentially succedent in A if it is in the scope of an even number of $\neg$'s. A sequent $\Gamma \to \Delta$ is said to be succedent-homogeneous if every quantifier in a formula of Δ which is essentially succedent is homogeneous and every quantifier in a formula of Γ which is not essentially succedent is homogeneous.

For the following proposition, we assume that $0, 1, 2, \ldots$ are the only individual constants in L. This simplifies the discussion.

PROPOSITION 23.31. (1) *If the* AD *holds in* M, *then all the provable sequents of* M*-definable determinate logic are* M*-valid, that is, valid in every* M*-definable structure.*

(2) *If a sequent is provable with a proof in which all the sequents are succedent-homogeneous, then it is* M*-valid.*

Note that in (2) the AD is not assumed for M.

PROOF. Suppose P is a proof for $\Gamma \to \Delta$. Let $\prec$ be the well-founded relation defined for free variables of P. We can assign ordinals to these free variables in such a manner that if $a \prec b$, then the ordinal of a is less than the ordinal of b. Start with those variables that do not depend on any other variables and assign them the value 0. Next, assign the ordinal 1 to those variables that depend only on variables whose ordinals are 0. Continuing in this way we will assign ordinals to all variables in P for the following reason. Suppose there are variables in P which are not assigned ordinals by this process. Let $\tilde{A}$ be the set of those free variables. Then by Lemma 23.29,

$$\forall x_0\, \exists x_1\, x_2 \ldots \bigwedge_i (x_i \in \tilde{A} \supset (x_{i+1} \in \tilde{A} \wedge x_{i+1} \prec x_i)).$$

Since $\tilde{A}$ is not empty, this means there is an infinite sequence $\{a_i\}_i$ from $\tilde{A}$ such that $a_{i+1} \prec a_i$, contradicting the well-foundedness of $\prec$.

The ordinals assigned to the free variables of P as above will be called heights. It is easy to see that they satisfy the conditions of heights in the previous sense.

Consider an M-definable structure $\mathscr{A}$. Notice that the natural numbers of $\mathscr{A}$ are not necessarily the natural numbers in the absolute sense. They are, however, in one-to-one correspondence with the actual natural numbers. Therefore we may assume, without loss of generality, that the universe of $\mathscr{A}$ is ω and the constants $0, 1, 2, \ldots$ in the language are interpreted in the obvious way; thus $\mathscr{A} = \langle \omega, 0, 1, 2, \ldots \rangle$.

Then consider all the formulas and subformulas in P and their Skolem functions, $g_A^{f,\gamma}$ and $\bar{g}_A^{f,\gamma}$, defined as before. Let $\mathsf{Q}^f \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{a})$ be a formula and suppose $\boldsymbol{a}$ exhausts all the free variables in this formula. If $g_A^{f,\gamma}$ is regarded as a function of $\boldsymbol{x}$ or of some variables of $\boldsymbol{x}$, while $\boldsymbol{a}$ is held fixed, then such a function is a member of M. If $g_A^{f,\gamma}$ is regarded as a function of $\boldsymbol{a}$ as well as some variables of $\boldsymbol{x}$, it is not guaranteed that the function belongs to M. In spite of this, we can carry out the subsequent argument entirely in M, for once the values of $\boldsymbol{a}$ are assigned, $g_A^{f,\gamma}$ is an element of M, and $g_A^{f,\gamma}$ occurs only in this context. What we will do is to construct such functions and substitute them for eigenvariables. The resulting figure P' may not be an element of M, but each formula in P' becomes a formula of M once those functions are computed.

The process of obtaining P' and determining the interpretation of the $g_A^{f,\gamma}$'s and $\bar{g}_A^{f,\gamma}$'s parallels the proof of Theorem 23.4 for (1), and the proof of Proposition 23.5 for (2). In a similar manner we can show that for an arbitrary sequent in P', say $\Gamma \to \Delta$, either there exists a formula of Γ which is false in $\mathscr{A}$ or there is a formula of Δ which is true in $\mathscr{A}$. For 6) in the proof of Theorem 23.4 we need the determinateness of $\mathscr{A}$. That $\mathscr{A}$ is a determinate structure is a consequence of the fact that the AD holds in M. Since substitution for eigenvariables does not change the end-sequent, this means that the end-sequent is true in $\mathscr{A}$.

DEFINITION 23.32. A generalized cut is called inessential if all its cut formulas are equalities, i.e., of the form $t_1 = t_2$; it is called essential otherwise.

Throughout the remainder of this chapter we call a proof cut-free if it has only inessential generalized cuts.

PROPOSITION 23.33. (1) *If the* AD *holds in M, then all the valid M-sequents are cut-free provable.*

(2) *If $\Gamma \to \Delta$ is succedent-homogeneous and valid, then $\Gamma \to \Delta$ is cut-free provable.*

PROOF. We follow the proof of Theorem 23.6. As was mentioned before, we may assume that the indices of the free variables are subsets of even numbers; this way we can introduce new free variables when necessary. Let

$$\tilde{\Gamma} = \Gamma \cup \{\forall x\,(x = 0 \vee x = 1 \vee \ldots)\}$$

and let $\mathsf{Q}^f\boldsymbol{x}\,A(\boldsymbol{x}, \boldsymbol{a})$ be an arbitrary formula in $\tilde{\Gamma} \to \Delta$. For each such formula we introduce a function symbol $g_A^{f,\gamma}$ (interpreted as a Skolem function) for each x_γ in $\boldsymbol{x}$ if $\mathsf{Q}^f\boldsymbol{x}\,A(\boldsymbol{x}, \boldsymbol{a})$ is essentially antecedent, and we introduce $\bar{g}_A^{f,\gamma}$ if it is essentially succedent. Let D be the set of terms which are generated from the individual constants and free variables by these Skolem functions. By Lemma 23.27, the set of those subformulas is at most the continuum, hence D is at most the continuum, because the individual constants and free variables form a set, D_0, of at most the continuum, and each stage of applying the Skolem functions increases the set by at most the continuum and we need to repeat the application of Skolem functions ω_1 times, more precisely α times for all $\alpha < \omega_1$.

We regard the terms in $D - D_0$ as free variables and identify them with the free variables which have been saved. (So it may happen that more than one such free variable corresponds to one term in $D - D_0$.) A natural partial ordering $\prec$ can be defined for the free variables from $D - D_0$. If s occurs in t, then $s \prec t$. It can be easily shown that $\prec$ is a well-founded relation and $\prec$ is a member of M.

We are now prepared for the completeness proof of Theorem 23.6. In this proof we use appropriate terms from $D - D_0$ in the reduction of quantifiers.

Since all the terms in D are members of M, it is obvious that the sequents thus obtained (in forming a tree) are members of M, and that they consist of at most the continuum of formulas. For example, in part 6) in the proof of Theorem 23.6 the possibility for $\boldsymbol{t}_{\lambda,\mu}$ is at most $(2^{\omega})^{\omega} = 2^{\omega}$, which is at most the continuum.

Case 1. Every branch of $T(\tilde{\Gamma} \to \Delta)$ has a sequent of the form

$$\ldots D \ldots \to \ldots D \ldots \quad \text{or} \quad \ldots \to \ldots s = s \ldots .$$

As before, consider the condition C. In order to show that $\tilde{\Gamma} \to \Delta$ satisfies C, we take the following step. Let $S', S'', \ldots$ denote sequents in $T(\tilde{\Gamma} \to \Delta)$ and define $R(S', S'')$ by

$$\begin{aligned} R(S', S'') \Leftrightarrow_{\mathrm{df}} (S' \in T(\tilde{\Gamma} \to \Delta)) \wedge (S' \text{ does not satisfy } C) \supset \\ \supset S'' \in T(\tilde{\Gamma} \to \Delta) \wedge (S'' \text{ does not satisfy } C) \\ \wedge (S'' \text{ is an immediate ancestor of } S'). \end{aligned}$$

If we assume that $\tilde{\Gamma} \to \Delta$ does not satisfy C, then $\forall S' \, \exists S'' \, R(S', S'')$ is true in M', hence by DC,

$$\forall S_0 \, \exists S_1 \, S_2 \ldots \bigwedge_i R(S_i, S_{i+1}).$$

Letting S_0 be $\Gamma \to \Delta$, we conclude that there is an infinite branch which does not satisfy C, contradicting the assumption of case 1.

Case 2. In this case we can construct a counterexample for $\tilde{\Gamma} \to \Delta$ in the same manner as before. Recall that the domain D belongs to M. Consequently, the fact that a formula occurs in the antecedent or in the succedent can be expressed in M.

In proving (1), we need to show that $\mathsf{Q}^{f}\boldsymbol{x} \, \neg A(\boldsymbol{x}, \boldsymbol{s}) \to \neg \mathsf{Q}^{\prime}\boldsymbol{x} \, A(\boldsymbol{x}, \boldsymbol{s})$ is true in any M-structure, (cf. 2 in the proof of Theorem 23.6). This holds since the AD holds in M. In proving (2), this case does not arise, since the given sequent is succedent-homogeneous.

We now have a cut-free proof of

$$\forall x \, (x = 0 \vee x = 1 \vee \ldots), \Gamma \to \Delta$$

and since

$$\cfrac{\cfrac{\to a = 0, a = 1, \ldots}{\to a = 0 \vee a = 1 \vee \ldots}}{\to \forall x \, (x = 0 \vee x = 1 \vee \ldots)}$$

we have, from the cut rule

$$\Gamma \to \Delta.$$

This last cut however is easily eliminated.

THEOREM 23.34. *The* AD *holds in* M *if and only if the cut-elimination theorem holds for* M*-definable determinate logic.*

PROOF. 1. Suppose that the cut-elimination theorem holds for M-definable determinate logic but the AD does not hold in M, namely, there is a set of M, say A, such that $A \subset \omega^\omega$ and the AD fails for A. Use A as a predicate symbol and let $i_0, i_1, \ldots$ be individual constants corresponding to $0, 1, \ldots$. Then consider two sets of atomic sentences:

$$\Gamma_0 = \{A(i_0, i_1, \ldots) \mid A(i_0, i_1, \ldots) \text{ is true when } A \text{ and } i_0, i_1, \ldots \text{ are interpreted as above}\}$$

and

$$\Delta_0 = \{A(i_0, i_1, \ldots) \mid A(i_0, i_1, \ldots) \text{ is false}\}.$$

We claim that

(1) $$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots A(x_0, y_0, x_1, y_1, \ldots), \Gamma_0 \to \Delta_0$$

and

(2) $$\exists x_0 \forall y_0 \exists x_1 \forall y_1 \ldots \neg A(x_0, y_0, x_1, y_1, \ldots), \Gamma_0 \to \Delta_0$$

are both valid. Then, since both are homogeneous-succedent, 2) of Proposition 23.33 implies that those sequents are cut-free provable.

Case 1. A is interpreted differently from the given set A. Then, since $\Gamma_0 \cup \Delta_0$ exhausts all the possibilities for $A(i_0, i_1, \ldots)$, there is at least one $(i_0, i_1, \ldots)$ such that either $A(i_0, i_1, \ldots)$ is in Γ_0 and it is false or $A(i_0, i_1, \ldots)$ is in Δ_0 and it is true. Hence $\Gamma_0 \to \Delta_0$ is true, which implies that both sequents are true.

Case 2. A is interpreted as the given set A. Then neither

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots A(x_0, y_0, x_1, y_1, \ldots)$$

nor

$$\exists x_0 \forall y_0 \exists x_1 \forall y_1 \ldots \neg A(x_0, y_0, x_1, y_1, \ldots)$$

is true; hence both sequents are true.

From (1) and (2) we obtain

$$\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \ldots A(x_0, y_0, x_1, y_1, \ldots)$$
$$\vee\ \exists x_1\, \forall y_0\, \exists x_1\, \forall y_1 \ldots \neg A(x_0, y_0, x_1, y_1, \ldots),\ \Gamma_0 \rightarrow \Delta_0$$

is provable in M-definable determinate logic. On the other hand

$$\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \ldots A(x_0, y_0, x_1, y_1, \ldots)$$
$$\vee\ \exists x_0\, \forall y_1\, \exists x_1\, \forall y_1 \ldots \neg A(x_0, y_0, x_1, y_1, \ldots)$$

is provable in the same system. Hence by the cut rule $\Gamma_0 \rightarrow \Delta_0$ is provable; but this is impossible. Therefore the AD must hold in M.

2. If AD holds in M, then by (1) of Proposition 23.33 together with 1) of Proposition 23.19, the cut-elimination theorem holds.

This completes the proof of the theorem.

Next we point out a relation between the cut-elimination theorem and the infinitary propositional calculus **IPC** which is the quantifier-free part of infinitary logic. **IPC** is common to determinate logic and ordinary infinitary logic. Consequently, provable sequents in **IPC** are valid.

Let Γ_0 be a set of quantifier-free sentences. It is well known that if Γ_0 is consistent (with **IPC**), then Γ_0 has a model.

PROPOSITION 23.35. *Let M be as above. Then the following two conditions are equivalent.*

(1) *The cut-elimination theorem holds in M-definable determinate logic.*

(2) *Let Γ_0 be a set of quantifier-free sentences and suppose Γ_0 belongs to M. If Γ_0 is consistent with* **IPC**, *then Γ_0 is consistent with M-definable determinate logic.*

PROOF. Obviously (1) implies (2). Suppose the cut-elimination theorem does not hold. Then by Theorem 23.34 there exists a counterexample for the AD in M, say $A \subseteq \omega^\omega$. In the proof of the theorem, the set of formulas $\Gamma_0 \cup \neg\Delta_0$, where $\neg\Delta_0$ consists of all the formulas of the form $\neg B$ for B in Δ_0, is consistent with **IPC**, since an A as above can be a model. On the other hand, $\Gamma_0 \rightarrow \Delta_0$ is provable in M-definable determinate logic, hence $\Gamma_0 \cup \neg\Delta_0$ is not consistent with determinate logic.

PROBLEM 23.36. Let us again assume the AC and assume that the antecedents and succedents of sequents are well-ordered. Consider the following language.

1. Individual constants: 0, 1, 2, ...
2. Bound variables: $x_0, x_1, \ldots, x_\alpha, \ldots$ $(\alpha < \omega_1)$.
3. Free variables: $a_0, a_1, \ldots, a_\alpha, \ldots$ $(\alpha < 2^{\aleph_0})$.
4. Predicate constants: $=$ and $R_1, R_2, R_3, \ldots$, where R_i has i argument places.
5. Logical symbols: $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$, Q_1, Q_2.

By a term, we mean an individual constant or a free variable.

Formulas and their orders are defined simultaneously as follows. The order of a formula is a natural number.

1. If t_i is a term for $i < n$, then $t_0 = t_1$ and $R_n(t_0, \ldots, t_{n-1})$ are atomic formulas and an order of an atomic formula is zero.
2. If A is a formula, so is $\neg A$. The order of $\neg A$ is the same with the order of A.
3. If A_i is a formula for each $i < \omega$ and the maximum of orders of A_i $(i < \omega)$ exists, then $\bigwedge_{i<\omega} A_i$ and $\bigvee_{i<\omega} A_i$ are formulas and the order of these formulas are the same with the maximum of orders of A_i $(i < \omega)$.
4. If $A(a_0, a_1, \ldots, a_i, \ldots)$ $(i < \omega)$ is a formula and $x_0, x_1, \ldots, x_i, \ldots$ $(i < \omega)$ are distinct bound variables not occurring in $A(a_0, a_1, \ldots)$, then

$$\forall x_0\, x_1 \ldots A(x_0, x_1, \ldots) \quad \text{and} \quad \exists x_0\, x_1 \ldots A(x_0, x_1, \ldots)$$

are formulas and the order of these formulas is $n + 1$, where n is the order of $A(a_0, a_1, \ldots)$.

5. If $A(a_0, a_1, \ldots, a_i, \ldots)$ $(i < \omega)$ is a formula without any occurrence of Q_1 or Q_2 and $x_0, x_1, \ldots, x_i, \ldots$ $(i < \omega)$ are distinct bound variables not occurring in $A(a_0, a_1, \ldots)$, then $\mathsf{Q}_i x_0\, x_1 \ldots A(x_0, x_1, \ldots)$ $(i = 1, 2)$ are formulas and the order of these formulas is $n + 1$, where n is the order of $A(a_0, a_1, \ldots)$. $\mathsf{Q}_1 x_0\, x_1 \ldots$ and $\mathsf{Q}_2 x_0\, x_1 \ldots$ are also denoted by $\exists x_0\, \forall x_1\, \exists x_2 \ldots$ and $\forall x_0\, \exists x_1\, \forall x_2 \ldots$, respectively.

A sequent is of the form

$$A_0, A_1, \ldots, A_\alpha, \ldots \to B_0, B_1, \ldots, B_\beta, \ldots,$$

where A_α's and B_β's are formulas and α and β range over ordinals less than α_0 and β_0, respectively, where α_0 and β_0 are some countable ordinals.

Inference rules are the same as in §22, except that the length of a sequent is restricted to be countable. Of course, $\forall$, $\exists$, Q_1, Q_2 should be expressed by an adequate form Q^f.

Non-logical initial sequents are of the following form.
1. $\rightarrow t = t$, where t is an arbitrary term.
2. $i = j \rightarrow$, where i and j are distinct individual constants.
3. $\rightarrow t = 0, t = 1, t = 2, \ldots$, where t is an arbitrary term.
Now prove the following theorem.

THEOREM 23.37. *The projective determinacy holds if and only if every provable sequent in the system defined above is provable without essential cuts.*

The proof goes as in Theorem 23.34.

§24. A general theory of heterogeneous quantifiers

The problem of the completeness of logical systems is an interesting and important one. While much is known, open questions still exist. We know, for example, that first order logic is complete and second order logic is incomplete. For infinitary languages we know that homogeneous systems are complete but whether heterogeneous systems are complete is an open question.

Incompleteness is an inherent weakness in any logical system. In second and higher order systems we can partially compensate for this weakness by a heavy dependence on comprehension axioms and the axiom of choice. In the infinitary languages however, we do not have the comprehension axioms and indeed in determinate logic we are even denied the axiom of choice. This raises the very practical question of whether there exist useful alternatives to the comprehension axioms and the axiom of choice for infinitary languages. In this section we will explore such alternatives. In order to do this we will develop a very general theory of heterogeneous quantifiers, a theory that encompasses the quantifiers $\mathbf{Q}^f$ of determinate logic as a special form (well ordered) of heterogeneous quantifiers.

The system we will present is a very useful one. How to extend it to a complete system is an open question. But before we take up the definition we would like to point out a few things about the system. For one thing, in the right and left quantifier introduction rules we do not have the duality that exists in finite languages and in determinate logic. Although we will assume that the formal objects of our system are well ordered, that assumption is only for convenience and is not essential for the theory. For further simplification we will always omit the individual and function constants unless otherwise stated. Finally we point out that we will not specify the number of bound and free variables. It is understood that we first adjoin a sufficient supply of bound variables. Having fixed the number of bound variables we then adjoin a sufficient supply of free variables. For an explanation of what constitutes a "sufficient supply" we refer the reader to the discussion after Definition 22.1.

DEFINITION 24.1. (1) By a *language with heterogeneous quantifiers*, $\mathrm{L}(J)$, where J is a set of mapping such that each T in J is a mapping from β to $P(\alpha)$ for some α and β, we mean the following collection of symbols.

1) Variables:
   1.1) Free variables: $a_0, a_1, \ldots, a_\mu, \ldots$.
   1.2) Bound variables: $x_0, x_1, \ldots, x_\nu, \ldots$.
2) Predicate constants of arity α for certain α's:

$$P_0^\alpha, P_1^\alpha, \ldots, P_\xi^\alpha, \ldots .$$

3) Logical symbols:
   $\neg$ (not),
   $\supset$ (implication),
   $\bigvee$ (disjunctions of arity α for certain α's),
   $\bigwedge$ (conjunctions of arity α for certain α's),
   $\forall$ (universal quantifiers of arity α for certain α's),
   $\exists$ (existential quantifiers of arity α for certain α's).
4) Heterogeneous quantifiers: We have a quantifier $\mathbf{Q}^T$ for each T in the set J.
5) Auxiliary symbols: (,).

(2) The formulas of $\mathrm{L}(J)$ are defined in the usual way with the following modifications.

(2.1) If $\bigvee$ ($\bigwedge$) of arity α belongs to $\mathrm{L}(J)$ and A_μ, $\mu < \alpha$ is a sequence of formulas then $\bigvee_{\mu<\alpha} A_\mu$ ($\bigwedge_{\mu<\alpha} A_\mu$) is a formula.

(2.2) Let T be a function in J. Then for some α and β, T maps β into $P(\alpha)$. Let $A(\boldsymbol{a}, \boldsymbol{b})$ be a formula, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are sequences of free variables of length α and β, respectively. We assume that some (possibly none) of the occurrences of $\boldsymbol{a}$ and $\boldsymbol{b}$ in A are indicated. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences of bound variables, of length α and β respectively which do not occur in $A(\boldsymbol{a}, \boldsymbol{b})$. Then $\mathbf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y})$ is a formula.

(3) As before we assume that the collection of formulas of the language $\mathrm{L}(J)$ is closed under subformulas.

(4) Let K be the cardinality of the formulas of the language. A sequent $\Gamma \to \Delta$ is defined as usual, where the lengths of Γ and Δ are less than K^+.

EXAMPLE 24.2. Consider a language $L(J)$ where there are countably many free variables (arranged in the order type of ω), the logical symbols are of finite arity and J is arbitrary. This means that the propositional connectives are defined as for the usual, finite languages and there are ω-many free variables. We shall assume that there are ω-many bound variables and a single predicate constant $=$. Let T be a function from 2 to $P(2)$ such that $T(0) = \{0\}$ and $T(1) = \{1\}$. Then

$$(a_0 = a_1 \equiv b_0 = b_1) \wedge b_0 \neq c$$

is a formula of the language which we denote by $A(a_0, a_1, b_0, b_1)$, where all the occurrences of a_0, a_1, b_0, b_1 are supposed to be indicated. Then

$$\mathrm{Q}^T(x_0, x_1; y_0, y_1)\, A(x_0, x_1, y_0, y)$$

is a formula. We are going to define a system in which this formula will have the meaning that for every x_0 there exists a y_0, depending on x_0 only, and for every x_1 there exists a y_1, depending on x_1 only, such that $A(x_0, x_1, y_0, y_1)$ holds. Such quantifiers, called dependent quantifiers, were first proposed by Henkin.

DEFINITION 24.3. The rules of inference of our intended system are those of Definition 22.1 with some alterations. We shall remark only on the crucial changes.

1) The $\wedge$: left, $\wedge$: right, $\vee$: left and $\vee$: right rules in (4.2) of Definition 22.1 are admitted only for conjunctions and disjunctions that belong to the language $L(J)$, that is, only for values of β_λ that are arities of conjunctions and disjunctions that belong to $L(J)$.

2) The rules for quantifiers are quite different here.

2.1) Q : left:

$$\frac{\{A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{b}_\lambda)\}_{\lambda<\gamma}, \Gamma \to \Delta}{\{\mathrm{Q}^{T_\lambda}(\boldsymbol{x}_\lambda; \boldsymbol{y}_\lambda)\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{y}_\lambda)\}_{\lambda<\gamma}, \Gamma \to \Delta},$$

where the variables of $\boldsymbol{b}_\lambda$ do not occur in $A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{y}_\lambda)$. If $\boldsymbol{a}_\lambda$ and $\boldsymbol{b}_\lambda$ are of types α_λ and β_λ respectively, then T_λ is a function in J from β_λ to $P(\alpha_\lambda)$.

Any variable of $\boldsymbol{b}_\lambda$, say b, is called an eigenvariable of the inference, $A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{b}_\lambda)$ is called an auxiliary formula of b, and $\mathrm{Q}^{T_\lambda}(\boldsymbol{x}_\lambda; \boldsymbol{y}_\lambda)\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{y}_\lambda)$ is called the principal formula of b.

2.2) Q : right:

$$\frac{\Gamma \to \Delta, \{A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{b}_\lambda)\}_{\lambda<\gamma}}{\Gamma \to \Delta, \{\mathrm{Q}^{T_\lambda}(\boldsymbol{x}_\lambda; \boldsymbol{y}_\lambda)\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{y}_\lambda)\}_{\lambda<\gamma}},$$

where if $\boldsymbol{a}_\lambda$ and $\boldsymbol{b}_\lambda$ are of types α_λ and β_λ respectively, then T_λ is a function in J from β_λ to $P(\alpha_\lambda)$.

Every variable of $\boldsymbol{a}_\lambda$, say a, is called an eigenvariable of the inference, $A_\lambda(\boldsymbol{a}_\lambda, \boldsymbol{b}_\lambda)$ is called an auxiliary formula of a, and $\mathsf{Q}^{T_\lambda}(\boldsymbol{x}_\lambda; \boldsymbol{y}_\lambda)\, A_\lambda(\boldsymbol{x}_\lambda, \boldsymbol{y}_\lambda)$ is called the principal formula of a.

3) The cut rule is replaced by the generalized cut rule (g.c.).

DEFINITION 24.4. A proof P in the system is defined in the usual way, as a tree consisting of sequents, where the following eigenvariable conditions must hold in P.

1) If a free variable b is used as an eigenvariable in more than one place, then the principal formulas of b must be the same.

2) Suppose $A(\boldsymbol{a}_1, \boldsymbol{b}_1)$ and $A(\boldsymbol{a}_2, \boldsymbol{b}_2)$ are auxiliary formulas of applications of Q : left in which b is an eigenvariable.

2.1) If b occurs as the αth variable of $\boldsymbol{b}_1$ then b occurs as the αth variable of $\boldsymbol{b}_2$.

2.2) Let $a_{1,\lambda}$ and $a_{2,\lambda}$ denote the λth variable in $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, respectively. Suppose b is the αth variable in $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. Then for any λ in $T(\alpha)$, $a_{1,\lambda}$ and $a_{2,\lambda}$ are the same.

Let a be a free variable which is either an $a_{1,\lambda}$ or $a_{2,\lambda}$ where $\lambda \in T(\alpha)$, or it is a free variable in the principal formula of b. Then we say that b depends on a, and we write $a \prec b$.

REMARK 24.5. The above conditions do not imply that, in the notation of 2), the sequences $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are identical. It is not guaranteed either that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are the same. Even if $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ happened to be the same, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ may not be the same. For example, it is possible to have different $a_{1,\lambda}$ and $a_{2,\lambda}$ if none of the variables of $\boldsymbol{b}_1$ depend on $a_{1,\lambda}$ and none of the variables of $\boldsymbol{b}_2$ depend on $a_{2,\lambda}$.

3) All the auxiliary formulas of an eigenvariable of the Q : right rule are identical.

For this case the dependence is defined only between the eigenvariables and the free variables in the principal formula: Let a be an eigenvariable in $\boldsymbol{a}$ and let c be a free variable in $A(\boldsymbol{x}, \boldsymbol{y})$. Then a depends on c, i.e., $c \prec a$.

4) No eigenvariable occurs in the end-sequent of P.

5) We shall relate a and b by $a \prec b$ if there is a finite sequence of free variables $a_0, \ldots, a_n$, where $a_0 = a$, $a_n = b$, and $a_i \prec a_{i+1}$ for $0 \leqslant i \leqslant n-1$ in the sense of 2) and 3). Then $\prec$ is a partial well-ordering; that is there is no

cycle $a_1 \prec a_2 \prec \ldots \prec a_1$ and there is no infinite decreasing sequence of variables $\{a_i\}_{i<\omega}$ such that $a_{i+1} \prec a_i$.

6) We define $a_1 \preceq a_2$ to mean that $a_1 \prec a_2$ or a_1 and a_2 are the same. There exists a well ordering of all the auxiliary formulas of occurrences of the $\mathbf{Q}$: right rule in the proof, say $\{A_\xi(\boldsymbol{a}_\xi, \boldsymbol{b}_\xi)\}_\xi$, which satisfies the following, where the same $A_\xi(\boldsymbol{a}_\xi, \boldsymbol{b}_\xi)$ may appear in several different places in P.

Let $\mathbf{Q}^{T_\xi}(\boldsymbol{x}_\xi : \boldsymbol{y}_\xi)\ A_\xi(\boldsymbol{x}_\xi, \boldsymbol{y}_\xi)$ be the principal formula of an eigenvariable and let $A_\xi(\boldsymbol{a}_\xi, \boldsymbol{b}_\xi)$ be the corresponding auxiliary formula.

6.0) If $b_{0,\sigma}$ is the σ^{th} variable of $\boldsymbol{b}_0$, then

$$\{e \mid e \preceq b_{0,\sigma} \text{ and } e \text{ is an eigenvariable of a } \mathbf{Q} : \text{right}\}$$

is a subset of

$$\{a_{0,\lambda} \mid a_{0,\lambda} \text{ is the } \lambda^{\text{th}} \text{ variable of } \boldsymbol{a}_0 \text{ and } \lambda \in T_0(\sigma)\}$$

and if c is a free variable in A_0 which is neither in $\boldsymbol{a}_0$ nor $\boldsymbol{b}_0$, then for no eigenvariable e of a $\mathbf{Q}$: right, is $e \preceq c$.

6.ξ) If $b_{\xi,\alpha}$ is the α^{th} variable of $\boldsymbol{b}_\xi$, then

$$\{e \mid e \preceq b_{\xi,\alpha} \text{ and } e \text{ is an eigenvariable of a } \mathbf{Q} : \text{right}\}$$

is a subset of

$$\bigcup_{\eta<\xi} \boldsymbol{a}_\eta \cup \{a_{\xi,\lambda} \mid \lambda \in T_\xi(\alpha)\}.$$

Also, if c is a free variable in A_ξ which does not belong to $\boldsymbol{a}_\xi$, or $\boldsymbol{b}_\xi$, then

$$\{e \mid e \preceq c \text{ and } e \text{ is an eigenvariable of a } \mathbf{Q} : \text{right}\}$$

is a subset of $\bigcup_{\eta<\xi} \boldsymbol{a}_\eta$.

NOTATIONAL CONVENTION. For quantifiers that are relatively simple, we shall use more intuitive notation. For example

$$\begin{pmatrix} \forall x\ \exists u \\ \forall y\ \exists w \end{pmatrix} A(x, y, u, w)$$

can express $\mathbf{Q}^T(xy; uw)\ A(x, y, u, w)$, where $T(0) = \{0\}$ and $T(1) = \{1\}$. $\mathbf{Q}^T(x_0x_1 \ldots ;)$ can be expressed with the usual notation $\forall x_0\, x_1 \ldots$ and $\mathbf{Q}^T(; x_0x_1 \ldots)$ with $\exists x_0\, x_1 \ldots$.

EXAMPLE 24.6. Examples of proofs with heterogeneous quantifiers.

1) $\begin{pmatrix} \forall x \; \exists u \\ \forall y \; \exists w \end{pmatrix} ((x = y \equiv u = w) \wedge u \neq a) \rightarrow \exists z_0 \, z_1 \ldots \bigwedge_{i \neq j} (z_i \neq z_j)$

is provable.

PROOF.

$$(1) \qquad \{(c_{k+j} = c_k \equiv c_{k+j+1} = a_k) \wedge c_{k+j+1} \neq c_0\}_{k,j<\omega} \rightarrow \bigwedge_{i \neq j} (c_i \neq c_j)$$

is obvious. From (1), by a $\mathbf{Q}$: right, where there are no eigenvariables involved,

$$(2) \quad \{(c_{h+j} = c_h \equiv c_{h+j+1} = a_h) \wedge c_{h+j+1} \neq c_0\}_{h,j<\omega} \rightarrow \exists z_0 \, z_1 \ldots \bigwedge_{i \neq j} (z_i \neq z_j).$$

Consider $(c_{k+j} = c_k \equiv c_{k+j+1} = a_k) \wedge c_{k+j+1} \neq c_0$, or $A(c_{k+j}, c_{k+j+1}, a_k)$, leaving out c_0. Define $T(0) = \{0\}$ and $T(1) = \{1\}$. Then $\mathbf{Q}^T(xy; uw)\, A(x, y, u, w)$ means

$$\begin{pmatrix} \forall x \; \exists u \\ \forall y \; \exists w \end{pmatrix} ((x = y \equiv u = w) \wedge u \neq c_0).$$

This applies to all pairs (k, j); hence by a $\mathbf{Q}^T$: left applied to all the formulas in the antecedent of (2), followed by contraction, we obtain

$$\mathbf{Q}^T(xy; uw)\, A(x, y, u, w) \rightarrow \exists z_0 \, z_1 \ldots \bigwedge_{i \neq j} (z_i \neq z_j).$$

By renaming c_0 as a, we obtain the required sequent. In order to see that we have given a proper proof, we examine the conditions in Definition 24.4. Since all the principal formulas are the same, 1) is obvious. Also 2.1) is obvious, since the c's are the first and the a's are the second eigenvariables in any auxiliary formulas. For 2.2), let $b = c_{k+j+1}$. Then $T(0) = \{0\}$ and the 0th variable in (c_{k+j}, c_k), i.e., c_{k+j} is uniquely determined by b. Similarly with a_k. Since there is no eigenvariable for a $\mathbf{Q}$: right, we do not have to worry about 3). Since the eigenvariables are c_{k+j+1} and a_k and not c_0 4) is obvious. As for 5), $c_{k+j} \prec c_{k+j+1}$, $c_k \prec a_k$, $c_0 \prec c_{k+j+1}$, $c_0 \prec a_k$ exhaust all the dependence relations. It is then easily seen that $\prec$ is a partial well-ordering. Clearly 6) is irrelevant.

2) $$\forall x\,\exists y\,A(x, y),$$

$$\forall x\,\forall y\,(A(x, y) \supset y \neq c_0),$$

$$\forall x\,\forall y\,\forall u\,\forall v\,(A(x, u) \wedge A(y, v) \supset (x = y \equiv u = v))$$

$$\to \begin{pmatrix} \forall x\ \exists u \\ \forall y\ \exists v \end{pmatrix} ((x = y \equiv u = v) \wedge u \neq c_0).$$

PROOF.

$$A(a, c),\ A(b, d),\ A(a, c) \supset c \neq c_0,$$

$$A(a, c) \wedge A(b, d) \supset (a = b \equiv c = d) \to$$

$$\to (a = b \equiv c = d) \wedge c \neq c_0$$

is obvious. We can introduce $\forall$'s to all the variables in the antecedent except c_0. Let $A(a, b, c, d)$ denote $(a = b \equiv c = d) \wedge c \neq c_0$. Let $T(0) = \{0\}$ and $T(1) = \{1\}$. Then $\mathbf{Q}^T(xy : uv)\ A(x, y, u, v)$ is the formula in the succedent.

Furthermore, 5) of the eigenvariable conditions is obviously satisfied. Since there is only one auxiliary formula of a $\mathbf{Q}$: right introduction, 6) is also easy to see.

DEFINITION 24.7. Let $\mathscr{A}$ be a structure for our language. Let Φ be an assignment from $\mathscr{A}$. The relation that a formula A is satisfied in $\mathscr{A}$ by Φ is defined as usual. $\mathbf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\ A(\boldsymbol{x}, \boldsymbol{y})$ is satisfied if and only if the following holds. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be of lengths α and β and let $\boldsymbol{a}$ and $\boldsymbol{b}$ be new free variables corresponding to $\boldsymbol{x}$ and $\boldsymbol{y}$. There exists a sequence of functions $\boldsymbol{f}$ corresponding to $\boldsymbol{b}$ such that for every sequence $\boldsymbol{d}$ of elements of $\mathscr{A}$ of length α, if

$$\Phi' = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{d} & \boldsymbol{f}(\bar{\boldsymbol{d}}) \end{pmatrix},$$

then $A(\boldsymbol{a}, \boldsymbol{b})$ is satisfied in $\mathscr{A}$ by Φ', where $\boldsymbol{f}(\bar{\boldsymbol{d}})$ is a sequence of terms such that if

$$T(\gamma) = \{\xi_0, \xi_1, \ldots, \xi_i, \ldots\},$$

then the γth expression is $f_\gamma(d_{\xi_0}, d_{\xi_1}, \ldots, d_{\xi_i}, \ldots)$.

THEOREM 24.8 (validity for heterogeneous quantifiers). *Every theorem of our system of heterogeneous quantifiers is valid.*

PROOF. The proof is similar to the proof of Theorem 23.4. Given a proof P in the system, a structure $\mathscr{A}$ and an assignment from $\mathscr{A}$ we first take an arbitrary formula in the antecedent of a sequent with a quantifier at the beginning, say

$$\mathbf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}),$$

where $\boldsymbol{c}$ is the sequence of all the free variables in this formula and the lengths of $\boldsymbol{x}$ and $\boldsymbol{y}$ are α and β respectively. For each $\gamma < \beta$ we introduce the Skolem function

$$g_A^{T,\gamma}(x_{\xi_0}, x_{\xi_1}, \ldots, \boldsymbol{c}),$$

where $x_{\xi_0}, x_{\xi_1}, \ldots$ are all the variables of $\boldsymbol{x}$ such that $\xi_i \in T(\gamma)$. $g_A^{T,\gamma}$ is interpreted as follows with respect to $\mathscr{A}$. If $\mathbf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c})$ is satisfied in $\mathscr{A}$, then the $g_A^{T,\gamma}$'s are the functions satisfying

$$(+) \qquad \forall \boldsymbol{x}\, A(\boldsymbol{x}, \boldsymbol{y}', \boldsymbol{c}),$$

where the γth expression of $\boldsymbol{y}'$ is $g_A^{T,\gamma}(x_{\xi_0}, x_{\xi_1}, \ldots, \boldsymbol{c})$. If $\mathbf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y})$ is not satisfied in $\mathscr{A}$, then the $g_A^{T,\gamma}$'s are interpreted to be constant functions for a distinguished element of the universe of $\mathscr{A}$.

Well-order all the eigenvariables in P for $\mathbf{Q}$: left introductions in such a way that if $a \prec b$ (b depends on a), then a precedes b in the ordering:

$$b_0, b_1, \ldots, b_\delta, \ldots .$$

We shall define terms $t_0, t_1, \ldots, t_\delta, \ldots$ by transfinite induction on β. Assuming that $\boldsymbol{t}_{<\delta}$ have been defined, we show how to define t_δ.

Suppose the principal formula of b_δ is $\mathbf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c})$ and let d be a variable in an auxiliary formula of b_δ which corresponds to a variable in $\boldsymbol{x}$ with $d \prec b$. If d is not used as an eigenvariable of any $\mathbf{Q}$: left, then define u corresponding to d to be d itself; otherwise d occurs in the above list of eigenvariables, hence is a b_κ, $\kappa < \delta$, since $d \prec b$. Therefore t_κ has been defined and we take u to be this t_κ. Let c be a free variable in $\boldsymbol{c}$. A term s corresponding to c is defined as the u corresponding to d; recall that $c \prec b$ by the eigenvariable condition. It should be noticed that those d's and c's are the same for all auxiliary formulas of b_δ by virtue of the eigenvariable condition. Thus t_δ can be defined to be $g_A^{T,\lambda}(\boldsymbol{u}, \boldsymbol{s})$. From the definition, a free variable in t_δ, say d, satisfies $d \prec b$. Now substitute $t_0, t_1, \ldots, t_\delta, \ldots$ for $b_0, b_1, \ldots, b_\delta, \ldots$, respectively, in P. Let P' be the figure thus obtained from P.

Let $\{A_\xi(\boldsymbol{a}_\xi, \boldsymbol{b}_\xi, \boldsymbol{c}_\xi)\}_\xi$ be the well-ordering of the auxiliary formulas of the Q : right in P satisfying the conditions in 6) of Definition 24.3, where $\boldsymbol{c}_\xi$ is the sequence of all the free variables in A_ξ which do not occur in $\boldsymbol{a}_\xi$ or $\boldsymbol{b}_\xi$.

We shall define substitutions of terms for eigenvariables of the Q : right introductions in P' in the following manner. Suppose the substitution has been completed for the ξ^{th} stage, giving us a figure P_ξ.

Applying this substitution to $\{A_\xi(\boldsymbol{a}_\xi, \boldsymbol{b}_\xi, \boldsymbol{c}_\xi)\}_\xi$ we obtain formulas in P' that we will denote by $\{A_\xi(\boldsymbol{a}_\xi, \boldsymbol{b}'_\xi, \boldsymbol{c}'_\xi)\}_\xi$. Here $\boldsymbol{b}'_\xi$ and $\boldsymbol{c}'_\xi$ are terms that may contain many free variables. Because of (6) of Definition 24.3, $b'_{\xi,\sigma}$ and $c'_{\xi,\sigma}$ satisfy the following condition.

(*) The eigenvariables of the Q : right that are contained in $b'_{\xi,\sigma}$ form a subset of

$$\bigcup_{\eta<\xi} \boldsymbol{a}_\eta \cup \{a_{\xi,\lambda_0}, a_{\xi,\lambda_1}, \ldots\}$$

where

$$\{\lambda_0, \lambda_1, \ldots\} = T_\xi(\sigma)$$

and any eigenvariable of a Q : right that is contained in $c'_{\xi,\sigma}$ is an eigenvariable of an $\boldsymbol{a}_\eta$ for some $\eta < \xi$.

Next we define, by transfinite induction on ξ, substitutions of members of $\mathscr{A}$ for $\boldsymbol{a}_\xi$.

Suppose we have completed the definition of substitutions for eigenvariables of $\boldsymbol{a}_\eta$, $\eta < \xi$. Consider the stage ξ. We shall define $a^*_{\xi,\gamma}$ corresponding to each $a_{\xi,\gamma}$ in $\boldsymbol{a}_\xi$.

Case 1. $\mathsf{Q}^{T_\xi}(\boldsymbol{x}_\xi; \boldsymbol{y}_\xi)\, \tilde{A}_\xi(\boldsymbol{x}_\xi, \boldsymbol{y}_\xi)$ is true, where $\tilde{A}_\xi(\boldsymbol{x}_\xi, \boldsymbol{y}_\xi)$ is obtained from $A_\xi(\boldsymbol{x}_\xi, \boldsymbol{y}_\xi)$ by substitutions up to the ξ^{th} stage. Let k_0 be a distinguished element of the given structure $\mathscr{A}$. Then $a^*_{\xi,\gamma}$ is interpreted to be k_0.

Case 2. $\mathsf{Q}^{T_\xi}(\boldsymbol{x}_\xi; \boldsymbol{y}_\xi)\, A_\xi(\boldsymbol{x}_\xi, \boldsymbol{y}_\xi)$ is false. Henceforth we omit the subscript ξ unless needed.

By hypothesis, $\neg\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y})$ is true, or $\exists \boldsymbol{x}\, \neg A(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}))$ is true no matter what the interpretation of $\boldsymbol{f}$ is, where $\boldsymbol{f}(\boldsymbol{x})$ stands for the sequence

$$f_0(\boldsymbol{x}_0), f_1(\boldsymbol{x}_1), \ldots, f_\sigma(\boldsymbol{x}_\sigma), \ldots \ (\sigma < \beta),$$

$\boldsymbol{x}_\sigma$ being $x_{\sigma_0}, x_{\sigma_1}, \ldots, x_{\sigma_i}, \ldots$, with $T(\sigma) = \{\sigma_0, \sigma_1, \ldots, \sigma_i, \ldots\}$. $A(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ has become $A(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v})$ where μ_σ is a term that may contain unsubstituted a's as free variables. By (*), the free variables in u_σ, form a subset of $\boldsymbol{a}_\sigma = \{a_{\sigma_0}, a_{\sigma_1}, \ldots, a_{\sigma_i}, \ldots\}$, where

$$\{\sigma_0, \sigma_1, \ldots, \sigma_i, \ldots\} = T(\sigma).$$

Therefore, we may put $f_\sigma(\boldsymbol{a}_\sigma) = u_\sigma$. Since for this interpretation of the f_σ's, $\exists \boldsymbol{x} \neg A(\boldsymbol{x}, f(\boldsymbol{x}))$ is true, there are values of $\boldsymbol{a}$, say $\boldsymbol{a}^*$, for which $\neg A(\boldsymbol{a}^*, f(\boldsymbol{a}^*))$. When these substitutions are completed, all the eigenvariables are replaced by the g_A's and the $\boldsymbol{a}^*$'s. We shall call the resulting figure P^*.

Now we can show that every sequent in P^* is true. Since the end-sequent does not contain any eigenvariable, this implies that the end-sequent of P is true.

For the proof that every sequent in P^* is true in $\mathscr{A}$, under the given assignment, we shall deal with three crucial cases only.

1) g.c.:

$$\frac{\Phi, \Gamma \to \Delta, \Psi \text{ for all appropriate } (\Phi, \Psi)}{\Gamma \to \Delta},$$

where $\mathscr{F}$ is the set of cut formulas.

Let $\mathscr{F}_1$ be the set of all formulas of $\mathscr{F}$ which are true and let $\mathscr{F}_2$ be the rest of the formulas. If for $\Phi \subseteq \mathscr{F}_1$ and $\Psi \subseteq \mathscr{F}_2$, $\Phi, \Gamma \to \Delta, \Psi$ is provable, then by the induction hypothesis this sequent is true, where all the formulas in Φ are true and those in Ψ are false. Therefore $\Gamma \to \Delta$ must be true.

2) Q : left:

$$\frac{\{\tilde{A}'(w, u)\}, \Gamma \to \Delta}{\{\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A'(\boldsymbol{x}, \boldsymbol{y})\}, \Gamma \to \Delta}.$$

It suffices to show that if $\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y}) A'(\boldsymbol{x}, \boldsymbol{y})$ is true then so is $A'(\boldsymbol{w}, \boldsymbol{u})$. However, this is obvious since the $g_A^{T,\gamma}$'s are so chosen (cf. (+) in the definition of $g_A^{T,\gamma}$).

3) Q : right:

$$\frac{\Gamma \to \Delta, \{A(\boldsymbol{w}, \boldsymbol{u})\}}{\Gamma \to \Delta, \{\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y})\}}.$$

It suffices to show that if $\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y})$ is false, then so is $A(\boldsymbol{w}, \boldsymbol{u})$, but this too is obvious, since the a_γ^*'s are so chosen.

Since a homogeneous system is a subsystem of a system which satisfies (Q), Proposition 22.14 implies that a homogeneous system is a subsystem of a heterogeneous system.

Heterogeneous quantifiers, even the finite ones, are stronger than homogeneous quantifiers.

PROPOSITION 24.9. *The heterogeneous quantifier*

$$\begin{pmatrix} \forall x\ \exists u \\ \forall y\ \exists v \end{pmatrix}$$

(*cf. Example* 24.6) *cannot be expressed by* (*finite*) *first order quantifiers.*

PROOF. Consider a formula of the form

$$\begin{pmatrix}\forall x\ \exists u\\ \forall y\ \exists v\end{pmatrix}((x = y \equiv u = v) \wedge A(x, y, u, v)), \tag{1}$$

in which

$$\begin{pmatrix}\forall x\ \exists u\\ \forall y\ \exists v\end{pmatrix}$$

is the only quantifier that occurs. In a second order system with function quantifiers and the axiom of choice, (1) is equivalent to the following formulas:

$$\exists f\ \exists g\ \forall x\ \forall y\ ((x = y \equiv f(x) = g(y)) \wedge A(x, y, f(x), g(y)));$$

$$\exists f\ \exists g\ (\forall x\ \forall y\ (x = y \equiv f(x) = g(y)) \wedge \forall x\ \forall y\ A(x, y, f(x), g(y)));$$

$$\exists f\ \forall x\ \forall y\ A(x, y, f(x), f(y)).$$

Define $A(x, y, f(x), f(y))$ to be

$$y = x + 1 \supset f(y) < f(x).$$

Then $\exists f\ \forall x\ \forall y\ A(x, y, f(x), f(y))$ expresses "$<$ is not well-founded". Although the set of natural numbers is well-founded its non-standard enlargement is not well-founded. Since both of them satisfy the same first order sentences, we conclude that (1) cannot be expressed in terms of homogeneous, first order quantifiers.

Proposition 24.9 explains why we have to place different eigenvariable conditions according as a quantifier is introduced in the antecedent or the succedent. If we were to use the same conditions on eigenvariables in the succedent as those in the antecedent, we would have

$$\neg\begin{pmatrix}\forall x\ \exists u\\ \forall y\ \exists v\end{pmatrix} A(x, y, u, v) \leftrightarrow \begin{pmatrix}\exists x\ \forall u\\ \exists y\ \forall v\end{pmatrix} \neg A(x, y, u, v).$$

On the other hand,

$$\begin{pmatrix}\exists x\ \forall u\\ \exists y\ \forall v\end{pmatrix} \neg A(x, y, u, v) \leftrightarrow \exists x\ \exists y\ \forall u\ \forall v\ \neg A(x, u, y, v).$$

So it would follow that

$$\begin{pmatrix}\forall x\ \exists u\\ \forall y\ \exists v\end{pmatrix} A(x, y, u, v) \leftrightarrow \forall x\ \forall y\ \exists u\ \exists v\ A(x, y, u, v),$$

contradicting Proposition 24.9.

EXAMPLE 24.10. Let $S(a_1, b_1, \ldots, a_n, b_n, \ldots)$ be a formula with free variables $a_1, b_1, \ldots, a_n, b_n, \ldots$ and let $S_n(a_1, b_1, \ldots, a_n, b_n)$ be short for

$$\exists x_{n+1}\ \forall y_{n+1}\ \ldots\ S(a_1, b_1, \ldots, a_n, b_n, x_{n+1}, y_{n+1}, \ldots).$$

Then

$$\exists u_1\ \forall v_1\ \ldots\ (\bigvee_n S_n(u_1, v_1, \ldots, u_n, v_n)) \rightarrow$$

$$\rightarrow \exists x_1\ \forall y_1\ \ldots\ \exists x_n\ \forall y_n\ \ldots\ S(x_1, y_1, \ldots, x_n, y_n, \ldots).$$

First of all, since

$$S_{n-1}(a_1, b_1, \ldots, a_{n-1}, b_{n-1}) \equiv \exists x_n\ \forall y_n\ S_n(a_1, \ldots, b_{n-1}, x_n, y_n)$$

is easily proved, we can identify

$$S_{n-1}(a_1, \ldots, b_{n-1}) \quad \text{and} \quad \exists x_n\ \forall y_n\ S_n(a_1, \ldots, b_{n-1}, x_n, y_n)$$

by using cuts.

For every n, we first consider a figure P_n ending with

$$S_n(a_1, b_1, \ldots, a_n, b_n) \rightarrow S_0.$$

We shall demonstrate P_3 as an example:

$$\cfrac{\cfrac{\cfrac{S_3(a_1, b_1, a_2, b_2, \ldots) \rightarrow S_3(a_1, b_1, a_2, b_2, a_3, b_3)}{S_3(\ldots) \rightarrow \exists x_3\ \forall y_3\ S_3(a_1, b_1, a_2, b_2, x_3, y_3)}}{S_3(\ldots) \rightarrow \exists x_2\ \forall y_2\ \exists x_3\ \forall y_3\ S_3(a_1, b_1, x_2, y_2, x_3, y_3)}}{S_3(\ldots) \rightarrow \exists x_1\ \forall y_1\ \exists x_2\ \forall y_2\ \exists x_3\ \forall y_3\ S_3(x_1, y_1, x_2, y_2, x_3, y_3).}$$

It is important to note that we do not introduce $S_3(\ldots) \rightarrow S_0$ in one step. Now,

$$\cfrac{\cfrac{P_n,\ n < \omega}{\bigvee_n S_n(a_1, b_1, \ldots, a_n, b_n) \rightarrow S_0}}{\exists u_1\ \forall v_1\ \ldots\ (\bigvee_n S_n(u_1, v_1, \ldots, u_n, v_n)) \rightarrow S_0.}$$

It is easy to see that the eigenvariable conditions are satisfied: The P_n's are carefully constructed so that the auxiliary formulas of an eigenvariable b_i are all identical. $\{S_n(a_1, b_1, \ldots, a_n, b_n)\}_{n>0}$ is the required enumeration of the auxiliary formulas of the Q : right introductions.

EXAMPLE 24.11. In order to state the next example we need some auxiliary definitions. Consider a quantifier of the form $\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})$, where the lengths of $\boldsymbol{x}$ and $\boldsymbol{y}$ are α and β, respectively. Suppose α and β can each be decomposed into two sets $\tilde{\alpha}_1, \tilde{\alpha}_2$ and $\tilde{\beta}_1, \tilde{\beta}_2$ respectively, i.e., $\alpha = \tilde{\alpha}_1 \cup \tilde{\alpha}_2$ and $\beta = \tilde{\beta}_1 \cup \tilde{\beta}_2$ where $\tilde{\alpha}_1 \cap \tilde{\alpha}_2 = \emptyset$ and $\tilde{\beta}_1 \cap \tilde{\beta}_2 = \emptyset$, and in addition,

(1) $T(\gamma) \subseteq \tilde{\alpha}_1$ if $\gamma \in \tilde{\beta}_1$,

(2) $T(\gamma) \subseteq \tilde{\alpha}_2$ if $\gamma \in \tilde{\beta}_2$.

If we well-order $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta}_1, \tilde{\beta}_2$ each and restrict T to $\tilde{\beta}_1$ and $\tilde{\beta}_2$ respectively, then we obtain T_1 and T_2 such that

$$\forall\gamma \in \tilde{\beta}_1\,(T_1(\gamma) = T(\gamma) \subseteq \tilde{\alpha}_1) \qquad \forall\gamma \in \tilde{\beta}_2\,(T_2(\gamma) = T(\gamma) \subseteq \tilde{\alpha}_2).$$

Suppose $A(\boldsymbol{x}, \boldsymbol{y})$ can be expressed as $A'(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2)$, where $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is the partition of $\boldsymbol{x}$ determined by $(\tilde{\alpha}_1, \tilde{\alpha}_2)$; and similarly with $(\boldsymbol{y}_1, \boldsymbol{y}_2)$.

We now show that under those circumstances

$$\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow \mathsf{Q}^{T_1}(\boldsymbol{x}_1; \boldsymbol{y}_1)\, \mathsf{Q}^{T_2}(\boldsymbol{x}_2; \boldsymbol{y}_2)\, A'(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2)$$

is provable.

Suppose that $A(\boldsymbol{a}, \boldsymbol{b})$ can be written as $A'(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{b}_1, \boldsymbol{b}_2)$ corresponding to $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta}_1, \tilde{\beta}_2$.

1)
$$\begin{array}{r}
A'(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{b}_1, \boldsymbol{b}_2) \to A(\boldsymbol{a}, \boldsymbol{b}) \\
\hline
\mathsf{Q}^{T_2}(\boldsymbol{x}_2; \boldsymbol{y}_2)\, A'(\boldsymbol{a}_1, \boldsymbol{x}_2, \boldsymbol{b}_1, \boldsymbol{y}_2) \to A(\boldsymbol{a}, \boldsymbol{b}) \\
\hline
\mathsf{Q}^{T_1}(\boldsymbol{x}_1; \boldsymbol{y}_1)\, \mathsf{Q}^{T_2}(\boldsymbol{x}_2; \boldsymbol{y}_2)\, A'(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2) \to A(\boldsymbol{a}, \boldsymbol{b}) \\
\hline
\mathsf{Q}^{T_1}(\boldsymbol{x}_1; \boldsymbol{y}_1)\, \mathsf{Q}^{T_2}(\boldsymbol{x}_2; \boldsymbol{y}_2)\, A'(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2) \to \mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y})
\end{array}$$

In introducing Q^{T_2}, the variables of $\boldsymbol{b}_2$ depend on all the variables of $\boldsymbol{a}_1$ as well as those of $\boldsymbol{b}_1$ and some of the variables of $\boldsymbol{a}_2$ (determined by T_2).

2)
$$\begin{array}{r}
A(\boldsymbol{c}, \boldsymbol{d}) \to A'(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{d}_1, \boldsymbol{d}_2) \\
\hline
A(\boldsymbol{c}, \boldsymbol{d}) \to \mathsf{Q}^{T_2}(\boldsymbol{x}_2; \boldsymbol{y}_2)\, A'(\boldsymbol{c}_1, \boldsymbol{x}_2, \boldsymbol{d}_1, \boldsymbol{y}_2) \\
\hline
A(\boldsymbol{c}, \boldsymbol{d}) \to \mathsf{Q}^{T_1}(\boldsymbol{x}_1; \boldsymbol{y}_1)\, \mathsf{Q}^{T_2}(\boldsymbol{x}_2; \boldsymbol{y}_2)\, A'(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2) \\
\hline
\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y}) A(\boldsymbol{x}, \boldsymbol{y}) \to \mathsf{Q}^{T_1}(\boldsymbol{x}_1; \boldsymbol{y}_1)\, \mathsf{Q}^{T_2}(\boldsymbol{x}_2; \boldsymbol{y}_2)\, A'(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2)
\end{array}$$

From the partition of the variables, it is evident that the variables of $\boldsymbol{d}_1$ do not depend on the variables of $\boldsymbol{c}_1$, hence a cycle can be avoided.

It is easy to see that 6) of the eigenvariable conditions (Definition 24.4) is satisfied. The other conditions are obvious.

EXERCISE 24.12. Show that the following is provable

$$\exists x_1\, P_1(x_1),\ \forall x_1\, \forall y_1\, \exists x_2\, (P_1(x) \supset P_2(x_1, y_1, x_2)), \ldots,$$
$$\forall x_1 \ldots y_{n-1}\, \exists x_n\, (P(x_1) \wedge P_2(x_1, y_1, x_2) \wedge \ldots \wedge P_{n-1}(x_1, y_1, \ldots, x_{n-1}) \supset$$
$$\supset P_n(x_1, \ldots, y_{n-1}, x_n)),$$
$$\forall x_1\, y_1 \ldots (\bigwedge_n P_n(x_1, \ldots, x_n) \supset S(x_1, y_1, \ldots)) \to$$
$$\to \exists x_1\, \forall y_1\, \exists x_n\, \forall y_n \ldots S(x_1, y_1, x_2, y_2, \ldots).$$

[*Hint*: The sequent

$$P_1(a_1),\ P_1(a_1) \supset P_2(a_1, b_1, a_2), \ldots,$$
$$\bigwedge_n P_n(a_1, \ldots, b_{n-1}, a_n) \supset S(a_1, b_1, \ldots) \to S(a_1, b_1, \ldots)$$

can easily be seen to be provable. From this the desired sequent follows.]

We can improve our system of heterogeneous quantifiers as follows.

DEFINITION 24.13 (cf. Definitions 24.3 and 24.4). (1) Add to 2) of Definition 24.4 the following. If $A(\boldsymbol{a}_1)$ and $A(\boldsymbol{a}_2)$ are auxiliary formulas of an eigenvariable a of an application of a homogeneous Q : right and a is the α^{th} variable of $\boldsymbol{a}_1$, then a is the α^{th} variable of $\boldsymbol{a}_2$.

To the definition of $\prec$, the following is added. If a is an eigenvariable of a homogeneous Q : right and b is a free variable which occurs in the principal formula of a, then $b \prec a$, i.e., a depends on b.

(2) Part 3) of Definition 24.4 should read: All the auxiliary formulas of an eigenvariable of a heterogeneous Q : right are identical.

(3) In 6) of Definition 24.4, read "heterogeneous Q : right" in place of "Q : right".

(4) Suppose $\mathsf{Q}^{T}(\boldsymbol{x}; \boldsymbol{y})$ can be split into $\mathsf{Q}^{T_1}(\boldsymbol{x}_1; \boldsymbol{y}_1)$ and $\mathsf{Q}^{T_2}(\boldsymbol{x}_2; \boldsymbol{y}_2)$ in the sense of Example 24.15. We shall abbreviate those quantifiers as Q, Q_1 and Q_2.

Then introduce a new rule of inference:

$$\frac{\Gamma_1, \{\mathsf{Q}_1^{\alpha}\mathsf{Q}_2^{\alpha}\, A_{\alpha}(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\alpha})\}_{\alpha}, \Gamma_2 \to \Delta_1, \{\mathsf{Q}_1^{\beta}\mathsf{Q}_2^{\beta}\, B_{\beta}(\boldsymbol{u}^{\beta}, \boldsymbol{v}^{\beta})\}_{\beta}, \Delta_2}{\Gamma_1, \{\mathsf{Q}^{\alpha}(\boldsymbol{x}^{\alpha}; \boldsymbol{y}^{\alpha})\, A(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\alpha})\}_{\alpha}, \Gamma_2 \to \Delta_1, \{\mathsf{Q}^{\beta}(\boldsymbol{u}^{\beta}; \boldsymbol{v}^{\beta})\, B(\boldsymbol{u}^{\beta}, \boldsymbol{v}^{\beta})\}_{\beta}, \Delta_2}\,.$$

This new system has the advantage that some proofs become much simpler, even cut-free.

EXAMPLE 24.14. A game is called an open game if the winner is determined after a finite number of steps. Let $Y(a_1, b_1, a_2, b_2, \ldots)$ denote the game in which the players I and II choose the terms of the sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ alternately and the winner of the game is determined. Then a game is open if

$$(1)\quad Y(a_1, b_1, \ldots) \to \bigvee_n (\forall x_{n+1}\, y_{n+1} \ldots)\, Y(a_1, b_1, \ldots, a_n, b_n, x_{n+1}, y_{n+1}, \ldots).$$

In our new system, there is a simple proof of the fact that an open game is determinate, i.e., (1) implies

$$\forall x_1\, \exists y_1\, \forall x_2\, \exists y_2 \ldots\, Y(x_1, y_1, \ldots) \vee \exists x_1\, \forall y_1\, \exists x_2\, \forall y_2 \ldots\, \neg Y(x_1, y_1, \ldots).$$

$$(2)\qquad \forall x_{n+1}\, y_{n+1} \ldots\, Y(a_1, \ldots, b_n, x_{n+1}, y_{n+1}, \ldots) \to \\ \to \forall x_1\, \exists y_1 \ldots \forall x_n\, \exists y_n \ldots\, Y(x_1, y_1, \ldots)$$

for each n:

$$\begin{array}{c}
Y(a_1, \ldots, b_n, a^n_{n+1}, b^n_{n+1}, \ldots) \to Y(a_1, \ldots, b_n, a^n_{n+1}, \ldots) \\
\hline
\forall x_{n+1} y_{n+1} \ldots\, Y(a_1, \ldots, b_n x_{n+1}, y_{n+1}, \ldots) \to \\
\to \forall x_{n+1}\, \exists y_{n+1} \ldots\, Y(a_1, \ldots, b_n, x_{n+1}, y_{n+1}, \ldots) \\
\hline
\forall x_{n+1} y_{n+1} \ldots\, Y(a_1, \ldots, b_n, x_{n+1}, \ldots) \to \\
\to \exists y_n\, (\forall x_{n+1}\, \exists y_{n+1} \ldots)\, Y(a_1, \ldots, a_n, y_n, x_{n+1}, \ldots) \\
\hline
\forall x_{n+1} y_{n+1} \ldots\, Y(a_1, \ldots, b_n, x_{n+1}, \ldots) \to \\
\to \exists y_n\, \forall x_{n+1}\, \exists y_{n+1} \ldots\, Y(a_1, \ldots, a_n, y_n, x_{n+1}, \ldots) \\
\hline
\forall x_{n+1} y_{n+1} \ldots\, Y(a_1, \ldots, b_n, x_{n+1}, \ldots) \to \\
\to \forall x_n\, (\exists y_n\, \forall x_{n+1}\, \exists y_{n+1} \ldots)\, Y(a_1, \ldots, b_{n-1}, x_n, y_n \ldots) \\
\hline
\forall x_{n+1} y_{n+1} \ldots\, Y(a_1, \ldots, b_n, x_{n+1}, \ldots) \to \\
\to \forall x_n\, \exists y_n\, \forall x_{n+1}\, \exists y_{n+1} \ldots\, Y(a_1, \ldots, b_{n-1}, x_n, y_n, \ldots) \\
\hline
\vdots \\
\forall x_{n+1} y_{n+1} \ldots\, Y(a_1, \ldots, b_n, x_{n+1}, \ldots) \to \\
\to \forall x_1\, \exists y_1 \ldots \forall x_n\, \exists y_n \ldots\, Y(x_1, \ldots, x_n, y_n, \ldots).
\end{array}$$

Notice that our new rule of inference is applied repeatedly. From (2), by $\bigvee$: left,

$$(3)\quad \bigvee_n \forall x_{n+1}\, y_{n+1} \ldots\, Y(a_1, \ldots, b_n, x_{n+1}, y_{n+1}, \ldots) \to \forall x_1\, \exists y_1 \ldots\, Y(x_1, y_1, \ldots).$$

Hence,

$$\frac{\begin{array}{r} Y(a_1, b_1, \ldots) \supset \bigvee_n \forall x_{n+1} y_{n+1} \ldots\ Y(a_1, \ldots, b_n, x_{n+1}, y_{n+1}, \ldots) \rightarrow \\ \rightarrow \forall x_1\, \exists y_1 \ldots\ Y(x_1, y_1, \ldots), \neg Y(a_1, b_1, \ldots) \end{array}}{\begin{array}{r} \forall v_1 w_1 \ldots (Y(v_1, w_1, \ldots) \supset \bigvee_n (\forall x_{n+1} y_{n+1} \ldots\ Y(u_1, w_1, \ldots, x_{n+1}, y_{n+1}, \ldots))) \rightarrow \\ \rightarrow \forall x_1\, \exists y_1 \ldots\ Y(x_1, y_1, \ldots) \vee \exists x_1\, \forall y_1 \ldots \neg Y(x_1, y_1, \ldots). \end{array}}$$

In this proof, the auxiliary formulas of the heterogeneous Q : right are

$$Y(a_1, \ldots, b_n, a^n_{n+1}, \ldots), \qquad \neg Y(a_1, \ldots, b_n, a_{n+1}, \ldots)$$

with eigenvariables $a^n_{n+1}, a^n_{n+2}, \ldots$ and $b_1, b_2, \ldots$, respectively. The eigenvariables of homogeneous quantifications are $a_1, a_2, \ldots$. Enumerate the auxiliary formulas of applications of Q : right introductions in the order of

$$\neg Y(a_1, b_1, \ldots, a_n, b_n, \ldots), \{Y(a_1, \ldots, b_n, a^n_{n+1}, \ldots)\}_n.$$

Then one can check the eigenvariable conditions (cf. Definition 24.4). We shall examine only condition 6). Here A_0 is $\neg Y(a_1, b_1, \ldots)$, each a_i satisfies the condition, and in 6.0) a_i is used as an eigenvariable of a homogeneous inference.

If e is an eigenvariable of a heterogeneous Q : right and $e \prec a_i$, then e is one of $b_1, \ldots, b_{i-1}$. If $\exists x_1\, \forall y_1 \ldots \exists x_i\, \forall y_i \ldots$ is denoted by $\mathbf{Q}^{T_0}(\boldsymbol{y}; \boldsymbol{x})$, then b_j $(1 \leqslant j \leqslant i-1)$ is the jth eigenvariable of the Q : right applied to A_0 and $j \in T_0(i)$. Consider 6.n), where $n \geqslant 1$. Let A_n be $Y(a_1, \ldots, b_n, a^n_{n+1}, \ldots)$. Suppose d is one of

$$a_1, b_1, \ldots, a_n, b_n, b^n_{n+1}, b^n_{n+2}, \ldots .$$

If d is used as an eigenvariable of a heterogeneous Q : right then it is one of $b_1, \ldots, b_n$, its auxiliary formula is A_0 and $0 < n$. Since b^n_j, $j = n+1, n+2, \ldots$ is not used as an eigenvariable, a relation $e \prec b^n_j$ never happens. Next, let c be one of the variables $a_1, b_1, \ldots, a_n, b_n$. If c is an a_i, then the first possibility of an eigenvariable e of a heterogeneous Q : right, such that $e \prec a_i$ is one of the variables $b_1, \ldots, b_{i-1}$. These are eigenvariables of A_0. Since the $b_1, \ldots, b_{i-1}$ do not depend on any variable, this completes the discussion of eigenvariables.

EXAMPLE 24.15. Let us consider another formulation of determinateness of an open game. An open game is expressed as $\bigvee_m Y_m(a_1, b_1, \ldots, a_m, b_m)$. Let us prove that this is determinate. Let $Y(a_1, \ldots, b_m, a^m_{m+1}, \ldots, b_m)$ be $Y_k(a_1, \ldots, b_k)$ if $1 \leqslant k \leqslant m$

$$
\begin{array}{l}
Y_m(a_1, \ldots, b_m) \to Y_m(a_1, \ldots, b_m) \\
\hline
Y_m(a_1, \ldots, b_m) \to \bigvee_k Y_k(a_1, \ldots, b_m, a^m_{m+1}, \ldots, b^m_k) \\
\hline
Y_m(a_1, \ldots, b_m) \to \forall x_{m+1}\, \exists y_{m+1} \ldots \bigvee_k Y_k(a_1, \ldots, b_m, x_{m+1}, \ldots, y_k) \\
\hline
Y_m(a_1, \ldots, b_m) \to \exists y_m\, (\forall x_{m+1}\, \exists y_{m+1} \ldots) \bigvee_k Y_k(a_1, \ldots, y_m, x_{m+1}, \ldots, y_k) \\
\hline
Y_m(a_1, \ldots, b_m) \to \exists y_m\, \forall x_{m+1}\, \exists y_{m+1} \ldots \bigvee_k Y_k(a_1, \ldots, y_m, x_{m+1}, \ldots, y_k) \\
\hline
Y_m(a_1, \ldots, b_m) \to \forall x_m (\exists y_m \forall x_{m+1} \exists y_{m+1} \ldots \bigvee_k Y_k(a_1, \ldots, x_m, y_m, x_{m+1}, \ldots, y_k)) \\
\hline
Y_m(a_1, \ldots, b_m) \to \forall x_m \exists y_m \forall x_{m+1} \exists y_{m+1} \ldots \bigvee_k Y_k(a_1, \ldots, x_m, y_m, x_{m+1}, \ldots, y_k) \\
\hline
\vdots \\
Y_m(a_1, \ldots, b_m) \to \forall x_1\, \exists y_1 \ldots \bigvee_k Y_k(x_1, \ldots, y_k) \\
\hline
\bigvee_m Y_m(a_1, \ldots, b_m) \to \forall x_1\, \exists y_1 \ldots \bigvee_k Y_k(x_1, y_1, \ldots, y_k) \\
\hline
\to \forall x_1\, \exists y_1 \ldots \bigvee_k Y_k(x_1, y_1, \ldots, y_k), \neg \bigvee_m Y_m(a_1, b_1, \ldots, b_m) \\
\hline
\to \forall x_1\, \exists y_1 \ldots \bigvee_m Y_m(x_1, y_1, \ldots, y_m), \exists x_1\, \forall y_1 \ldots \neg \bigvee_m Y_m(x_1, y_1, \ldots, y_m)
\end{array}
$$

The auxiliary formulas of applications of the heterogeneous $\mathbf{Q}$: right can be listed in the order of

$$\neg \bigvee_m Y_m(a_1, b_1, \ldots, a_m, b_m), \{\bigvee_n Y_n(a_1, \ldots, b_m, a^m_{m+1}, \ldots, b^m_n)\}_m$$

with eigenvariables $b_1, b_2, \ldots$, for the first formula, and

$$a^m_{m+1}, a^m_{m+2}, \ldots \quad \text{for} \quad \bigvee_k Y_k(a_1, \ldots, b_m, a^m_{m+1}, \ldots).$$

The reader should go over the eigenvariable conditions.

EXAMPLE 24.16.

$$
\begin{array}{c}
A(a_1, b_1, a_2, b_2, \ldots) \to A(a_1, b_1, a_2, b_2, \ldots) \\
\hline
\exists x_2\, \forall y_2 \ldots A(a_1, b_1, x_2, y_2, \ldots) \to \exists x_1\, \forall y_1 \ldots A(x_1, y_1, \ldots) \\
\hline
\neg\exists x_1\, \forall y_1 \ldots A(x_1, y_1, \ldots) \to \neg\exists x_2\, \forall y_2 \ldots A(a_1, b_1, x_2, y_2, \ldots) \\
\hline
\neg\exists x_1\, \forall y_1 \ldots A(x_1, y_1, \ldots) \to \exists y_1\, \neg\exists x_2\, \forall y_2 \ldots A(a_1, y_1, x_2, y_2, \ldots) \\
\hline
\neg\exists x_1\, \forall y_1 \ldots A(x_1, y_1, \ldots) \to \forall x_1\, \exists y_1\, \neg\exists x_2\, \forall y_2 \ldots A(x_1, y_1, \ldots)
\end{array}
$$

REMARK. There are examples which are cut-free provable in our new system, but which require the cut rule in the old system. For example, consider

$$\to (\forall x\, \exists y)\, A(x, y), (\exists x\, \forall y)\, \neg A(x, y).$$

Here $(\forall x\, \exists y)$ and $(\exists x\, \forall y)$ are regarded as heterogeneous, while $\forall x\, \exists y$ and $\exists x\, \forall y$ are considered two applications of homogeneous quantifiers.

(1) This is provable with a cut in the old system:

$$\dfrac{\dfrac{\dfrac{A(a, b) \to A(a, b)}{\forall x\, \exists y\, A(x, y) \to (\forall x\, \exists y)\, A(x, y)}}{\to \forall x\, \exists y\, A(x, y) \vee \exists x\, \forall y\, \neg A(x, y)} \qquad \dfrac{\dfrac{\neg A(c, d) \to \neg A(c, d)}{\exists x\, \forall y\, \neg A(x, y) \to (\exists x\, \forall y)\, \neg A(x, y)}}{\forall x\, \exists y\, A(x, y) \vee \exists x\, \forall y\, \neg A(x, y) \to \; \to (\forall x\, \exists y)\, A(x, y), (\exists x\, \forall y)\, \neg A(x, y)}}{\to (\forall x\, \exists y)\, A(x, y), (\exists x\, \forall y)\, \neg A(x, y).}$$

(2) This cannot be proved without the cut rule in the old system, for if we assume a cut-free proof, then

$$\to \ldots A(a_\alpha, b_\alpha), \ldots \neg A(c_\beta, d_\beta) \ldots$$

is provable for some free variables. Then for some α and β, a_α is c_β and b_α is d_β. Therefore in

$$\to \ldots A(a, b), \ldots, \neg A(a, b), \ldots$$

both a and b are eigenvariables of a heterogeneous $\mathbf{Q}$: right. But this implies that $A(a, b)$ and $\neg A(a, b)$ cannot be ordered.

(3) This sequent is provable in our new system without the cut rule:

$$\dfrac{\dfrac{\vdots \atop \to A(a, b), \neg A(a, b)}{\to \forall x\, \exists y\, A(x, y), \exists x\, \forall y\, \neg A(x, y)}}{\to (\forall x\, \exists y)\, A(x, y), (\exists x\, \forall y)\, \neg A(x, y).}$$

PROPOSITION 24.17. *A proof in determinate logic that satisfies the condition* (Q) *in Definition* 23.16 *is a proof in the* (*extended*) *heterogeneous system.*

PROOF. Suppose a proof P in a given language, say L, of determinate logic satisfies (Q). Define a language L(J) by admitting logical symbols of precisely the same arities as those of L and defining J as follows. Let $\mathbf{Q}^f\boldsymbol{z}$ be a quantifier of the determinate logic.

Let $\boldsymbol{x}$ be the sequence of all variables of $\boldsymbol{z}$ for which f assumes the value $\forall$ and let $\boldsymbol{y}$ be the sequence of all the variables of $\boldsymbol{z}$ for which f assures the value $\exists$. Let y be the β^{th} variable of $\boldsymbol{y}$. Then $\alpha \in T(\beta)$ if and only if the α^{th} variable of $\boldsymbol{x}$ precedes y in $\boldsymbol{z}$. Such a T belongs to J. Therefore we can translate $\mathbf{Q}^f\boldsymbol{z}$ into $\mathbf{Q}^T(\boldsymbol{x}; \boldsymbol{y})$. Thus the formulas of P are regarded as those of the language L(J).

By renaming variables in P (if necessary), we can assume the following condition because of (Q).

(*) If the inference

$$\frac{\Gamma \to \Delta,\ A(\boldsymbol{a}, \boldsymbol{b})}{\Gamma \to \Delta,\ \mathsf{Q}^T(\boldsymbol{x}, \boldsymbol{y})\ A(\boldsymbol{x}, \boldsymbol{y})}$$

is a heterogeneous Q : right in P, then no eigenvariables in P used above $\Gamma \to \Delta,\ \mathsf{Q}^T(\boldsymbol{x}, \boldsymbol{y})\ A(\boldsymbol{x}, \boldsymbol{y})$ occur below $\Gamma \to \Delta,\ \mathsf{Q}^T(\boldsymbol{x}, \boldsymbol{y})\ A(\boldsymbol{x}, \boldsymbol{y})$.

In order to see that P is a proof in this heterogeneous system, it suffices to examine the eigenvariable conditions in Definitions 24.4 and 24.14. There 1), 2) and 4) are exactly conditions on P. By virtue of (Q), 3) is satisfied. Suppose $c \prec a$ holds in P. Then the height of c is less than the height of a; hence 5) is obvious.

We can define an enumeration of the auxiliary formulas of the heterogeneous Q : right in P in such a manner that it satisfies the eigenvariable condition 6). Let J_1 and J_2 be heterogeneous Q : right in P, with J_1 above J_2 and with J_1 and J_2 having the form

$$J_1\quad \frac{\Gamma_1 \to \Delta_1,\ A(\boldsymbol{a}, \boldsymbol{b})}{\Gamma_1 \to \Delta_1,\ \mathsf{Q}^{T_1}(\boldsymbol{x}, \boldsymbol{y})\ A(\boldsymbol{x}, \boldsymbol{y})}$$

$$J_2\quad \frac{\Gamma_2 \to \Delta_2,\ B(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e})}{\Gamma_2 \to \Delta_2,\ \mathsf{Q}^{T_2}(\boldsymbol{x}', \boldsymbol{y}')\ B(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{e})}$$

where $\boldsymbol{e}$ is the sequence of variables that are neither in $\boldsymbol{c}$ nor in $\boldsymbol{d}$. Suppose that d is in $\boldsymbol{d}$ (or $\boldsymbol{e}$), a' is an eigenvariable in P, and $a' \prec d$ (or $a' \prec e$). Since (*) implies that d (or e) is not an eigenvariable above $\Gamma_1 \to \Delta_1,\ \mathsf{Q}^{T_1}(\boldsymbol{x}, \boldsymbol{y})\ A(\boldsymbol{x}, \boldsymbol{y})$, a' cannot be in $\boldsymbol{a}$. An appropriate enumeration of the auxiliary formulas in P is obtained if we enumerate them from the bottom.

We shall now investigate the interpolation theorem for subsystems of heterogeneous systems.

DEFINITION 24.18. Let q be an arbitrary symbol and let A be a formula. We say that an occurrence of q is *positive* or *negative* according as q lies in the scope of an even or odd number of $\neg$'s. $A \supset B$ is understood to be $\neg A \vee B$. We say that q is positive in $\Gamma \to \Delta$ if it is positive in Δ or negative in Γ; and that q is negative in $\Gamma \to \Delta$ if it is negative in Δ or positive in Γ.

A sequent $\Gamma \to \Delta$ is said to be negative if all the heterogeneous quantifiers in it are negative.

PROPOSITION 24.19. (1) *In our old system, every negative sequent is either cut-free provable or has a counter-model.*

(2) *The interpolation theorem holds for negative sequents. Suppose* $\Gamma \to \Delta$ *is a valid negative sequent and* $[\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$ *is a decomposition of* $\Gamma \to \Delta$ *such that* $\{\Gamma_1; \Delta_1\}$ *and* $\{\Gamma_2; \Delta_2\}$ *have at least one predicate symbol in common. Then there exists a formula* C *(not necessarily negative) such that* $\Gamma_1 \to \Delta_1, C$ *and* $C, \Gamma_2 \to \Delta_2$ *are valid, and all the predicate symbols and free variables in* C *occur both in* $\{\Gamma_1; \Delta_1\}$ *and* $\{\Gamma_2; \Delta_2\}$.

PROOF. (1) This can be proved similarly to other completeness proofs, by constructing a tree for a given sequent. Notice that in a stage which concerns heterogeneous quantifiers the reduction is done in the antecedent only.

(2) For technical reasons, we assume that all the homogeneous quantifiers are $\exists$. This restriction is not essential. Following Definition 23.18, we say that a figure P is a proof in **RHS**$'$ if P satisfies the following:

(i) P satisfies all the conditions of a proof in our system except the eigenvariable conditions.

(ii) The only inferences which introduce quantifiers are $\exists$: left introductions,

$$\frac{\{A_\lambda(\boldsymbol{a}_\lambda)\}_{\lambda<\gamma}, \Gamma \to \Delta}{\{\exists \boldsymbol{x}_\lambda\, A_\lambda(\boldsymbol{x}_\lambda)\}_{\lambda<\gamma}, \Gamma \to \Delta},$$

where no variable in $\boldsymbol{a}_\lambda$ occurs in the lower sequent.

Notice that P may contain heterogeneous quantifiers, but they are introduced either by initial sequents or weakenings. Then in a manner similar to the proof of Proposition 23.19, we can prove the following: Let P be a proof in **RHS**$'$ ending with $\Gamma \to \Delta$, and suppose a well-founded relation $\prec_0$ is defined for the free variables in $\Gamma \to \Delta$. Then $\prec_0$ can be extended to a dependence relation for eigenvariables in P.

LEMMA 24.20. *Let* P *be a cut-free proof of* $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$ *in our system in which every homogeneous quantifier is* $\exists$, *no rule of heterogeneous quantifiers applies, and every inference of the introduction of a quantifier is an inference of the introduction of* $\exists$ *in the succedent. Suppose also that* $\{\Gamma_1; \Delta_1\}$ *and* $\{\Gamma_2; \Delta_2\}$ *have a predicate symbol in common. Then there exist cut-free proofs* P_1 *and* P_2 *in* **RHS**$'$ *and a formula* C *such that the end-sequent of* P_1 *is*

$$C, \Gamma_1 \to \Delta_1,$$

the end-sequent of P_2 *is*

$$\Gamma_2 \to \Delta_2, C,$$

and every free variable and predicate symbol in C *is common to* $\{\Gamma_1; \Delta_1\}$ *and* $\{\Gamma_2, \Delta_2\}$.

PROOF. Similar to that of Lemma 23.20.

We now return to the proof of Proposition 24.19 (2) which we will do in a manner similar to 3. of the proof of Theorem 23.15.

Consider a cut-free proof, P_1 of $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$. The eigenvariable conditions for such a proof can be expressed in terms of a well-founded relation $\prec$. Fix such a $\prec$ throughout. We may assume that every homogeneous quantifier in P is $\exists$, that heterogeneous quantifiers are introduced in the antecedent only, and that free variables which occur in $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$ do not depend on any eigenvariables in P. Let $\{\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d})\}$ be an enumeration of all principal formulas of the Q : left introductions (heterogeneous Q or homogeneous $\exists$: left) in the given proof whose descendents are in Γ_1 or Δ_1. We define $\{\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, B(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d}')\}$ similarly for Γ_2 and Δ_2. Then we can construct a proof P' of

$$\{\neg\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d}) \vee A(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{d})\}, \{\neg\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, B(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d}') \vee B(\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{d}')\},$$

$$\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2,$$

such that every homogeneous quantifier in P' is $\exists$; such that the only inference which introduces a quantifier is $\exists$: right; such that $c, d \prec a$ for any a in $\boldsymbol{a}$, c in $\boldsymbol{c}$ and d in $\boldsymbol{d}$; such that $e, d' \prec b$ for any b in $\boldsymbol{b}$, e in $\boldsymbol{e}$ and d' in $\boldsymbol{d}'$; such that every free variable in $A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ occurs in Γ_1 or Δ_1; such that every free variable in $B(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ occurs in Γ_2 or Δ_2; and such that all variables in $\boldsymbol{a}$ and $\boldsymbol{b}$ are different. (We have used ambiguous notation such as the same letters $\boldsymbol{x}$ and $\boldsymbol{y}$ for different quantifiers. The meaning of the above expressions should, however, be obvious.)

Then, by Lemma 24.20, there exists a formula C such that

$$C, \{\neg\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d}) \vee A(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{d})\}, \Gamma_1 \to \Delta_1$$

and

$$\{\neg\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, B(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d}) \vee B(\boldsymbol{c}, \boldsymbol{b}, \boldsymbol{d}), \Gamma_2 \to \Delta_2, C$$

are provable in $\mathbf{RHS}'$ and C satisfies some appropriate conditions. Let $\boldsymbol{f}$ be a sequence of all free variables in C which are not common to $\{\Gamma_1; \Delta_1\}$ and $\{\Gamma_2; \Delta_2\}$. We assume that the variables in $\boldsymbol{f}$ are arranged so that if f_1 precedes f_2, then it is not the case that $f_2 \prec f_1$. ($\prec$ is the relation which is defined for the original proof P.) Let P_1 and P_2 be proofs in $\mathbf{RHS}'$ of the above two sequents and let $\prec_1$ and $\prec_2$ be the dependence relations for the free variables in the end sequents of P_1 and P_2, respectively, which are induced from $\prec$ (for P). Then, we can extend these relations to all the free variables. Let us denote these extended relations by $\prec_{P_1}$ and $\prec_{P_2}$.

Next, consider the following quasi-proofs, P_1' and P_2':

$$\begin{array}{r} P_1 \\ C(\boldsymbol{f}), \{\forall \boldsymbol{z}\, \mathsf{Q}^T(\boldsymbol{u}; \boldsymbol{v})\, (\neg \mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \vee A(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}))\}, \Gamma_1 \rightarrow \Delta_1 \end{array}$$

and

$$\begin{array}{r} P_2 \\ \{\forall \boldsymbol{z}\, \mathsf{Q}^T(\boldsymbol{u}; \boldsymbol{v})\, (\neg \mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, B(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \vee B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}))\}, \Gamma_2 \rightarrow \Delta_2, C(\boldsymbol{f}). \end{array}$$

There are three kinds of variables in $\boldsymbol{f}$, those of $\boldsymbol{a}$, those of $\boldsymbol{b}$, and the rest. The first ones, denoted by $\boldsymbol{f}_1$, are eigenvariables in P_1'; the second ones, denoted by $\boldsymbol{f}_2$, are eigenvariables in P_2'; and the third ones are denoted by $\boldsymbol{f}_3$. Define T', a function from $\boldsymbol{f}_2$ to subsets of $\boldsymbol{f}_1$, so that it satisfies the dependence relation $\prec$. Note that

$$\forall \boldsymbol{z}\, (\mathsf{Q}^T(\boldsymbol{u}; \boldsymbol{v})\, (\neg \mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \vee A(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}))$$

is provable. Therefore, we obtain

$$\begin{array}{r} P_1' \\ \forall \boldsymbol{w}\, \mathsf{Q}^{T'}(\boldsymbol{w}_1; \boldsymbol{w}_2)\, C(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}), \Gamma_1 \rightarrow \Delta_1 \end{array}$$

where $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, and $\boldsymbol{w}$ replace $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ and $\boldsymbol{f}_3$, respectively. Similarly, we obtain

$$\begin{array}{l} P_2' \\ \Gamma_2 \rightarrow \Delta_2, \forall \boldsymbol{w}\, \mathsf{Q}^{T'}(\boldsymbol{w}_1; \boldsymbol{w}_2)\, C(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}). \end{array}$$

Since we may assume that the eigenvariables of the proofs of

$$\forall \boldsymbol{z}\, \mathsf{Q}^T(\boldsymbol{u}; \boldsymbol{v})\, (\neg \mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \vee A(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}))$$

are different from those of P_1', we naturally extend $\prec_{P_1}$ to the entire proof of

$$\forall \boldsymbol{w}\, \mathsf{Q}^{T'}(\boldsymbol{w}_1; \boldsymbol{w}_2)\, C(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}), \Gamma_1 \rightarrow \Delta_1.$$

It is easily shown that it satisfies the dependence relation; indeed, T' was so chosen. In a similar manner we extend $\prec_{P_2}$.

Even with our new inference, the cut-elimination theorem for the entire system can hardly be expected to hold. Defining a complete system for a heterogeneous language is a difficult problem. It is even more difficult if we wish to define a cut-free complete system. We can see this clearly from the following example.

EXAMPLE 24.21. Consider

$$\{c_i \neq c_j\}_{i \neq j, i, j < w} \to \begin{pmatrix} \forall a \, \exists u \\ \forall b \, \exists v \end{pmatrix} (a = b \equiv u = v \wedge u \neq c_0).$$

This sequent is provable in our system:

$$\{c_i \neq c_j\}_{i \neq j}, \bigwedge_i (a = c_i \supset u = c_{i+1}) \wedge (\bigwedge_i a \neq c_i \supset u = a),$$

$$\bigwedge_i (b = c_i \supset v = c_{i+1}) \wedge (\bigwedge_i b \neq c_i \supset v = b) \to (a = b \equiv u = v) \wedge u \neq c_0$$

can be easily proved. It then follows that

$$\forall a \, \exists u \, (\bigwedge_i (a = c_i \supset u = c_{i+1}) \wedge (\bigwedge_i a \neq c_i \supset u = a)),$$

$$\forall b \, \exists v \, (\bigwedge_i (b = c_i \supset v = c_{i+1}) \wedge (\bigwedge_i b \neq c_i \supset v = b)),$$

$$\{c_i \neq c_j\}_{i \neq j} \to \begin{pmatrix} \forall a \, \exists u \\ \forall b \, \exists v \end{pmatrix} ((a = b \equiv u = v) \wedge u \neq c_0).$$

On the other hand

$$\to \forall a \, \exists u \, (\bigwedge_i (a = c_i \supset u = c_{i+1}) \wedge (\bigwedge_i a \neq c_i \supset u = a))$$

and

$$\to \forall b \, \exists v \, (\bigwedge_i (b = c_i \supset v = c_{i+1}) \wedge (\bigwedge_i b \neq c_i \supset v = b))$$

are provable in our system. Therefore by the cut rule

$$\{c_i \neq c_j\}_{i \neq j} \to \begin{pmatrix} \forall a \, \exists u \\ \forall b \, \exists v \end{pmatrix} ((a = b \equiv u = v) \wedge u \neq c_0)$$

is provable.

If we apply Gentzen-type reduction, which was primarily defined for finite languages, to the proof given above, which contains a cut, then we will obtain a "proof-like" figure

$$\frac{\begin{array}{l}\{c_i \neq c_j\}_{i \neq j} \to \{(a = b \equiv c_{i+1} = c_{j+1}) \wedge c_{i+1} \neq c_0\}_{i,j},\\ \qquad\qquad\qquad \{(a = b \equiv a = c_{j+1}) \wedge a \neq c_0\}_j,\\ \qquad\qquad\qquad \{(a = b \equiv c_{i+1} = b) \wedge c_{i+1} \neq c_0\}_i,\\ \qquad\qquad\qquad (a = b \equiv a = b) \wedge a \neq c_0\end{array}}{\{c_i \neq c_j\}_{i \neq j} \to \begin{pmatrix}\forall a\ \exists u\\ \forall b\ \exists v\end{pmatrix} ((a = b \equiv u = v) \wedge u \neq c_0).}$$

It is obvious that in this figure the auxiliary formulas of the eigenvariables a and b are not unique. Therefore it cannot be a proof in our sense.

From this figure, we see that there is little hope of expanding our system so that this figure will fit into it and hence little hope of establishing a complete cut-free system.

Although our system is far from being complete, a weak completeness can be proved. For the proof, we employ a more general formulation of the generalized cut rule that we will call the strong generalized cut rule (s.g.c.): Let $\mathscr{F}$ be a non-empty set of formulas. Suppose, for an arbitrary decomposition of $\mathscr{F}$, say $(\mathscr{F}_1, \mathscr{F}_2)$, there are subsets of $\mathscr{F}_1$ and $\mathscr{F}_2$, say Φ and Ψ, respectively, and subsets of Γ and Δ, say Γ' and Δ', respectively, such that $\Phi, \Gamma' \to \Delta', \Psi$. Then $\Gamma \to \Delta$ can be inferred. We also allow the case where $\Gamma'(\Delta')$ has some repetition of some formulas of $\Gamma(\Delta)$.

It is obvious that the s.g.c. rule is a generalization of the g.c. rule that involves two types of inferences, inferences that are essential cuts and inferences that are basically weakenings.

PROPOSITION 24.22. *Consider a language with heterogeneous quantifiers* L *that contains individual constants* $c_0, c_1, \ldots, c_\alpha, \ldots, \alpha < K$, *and contains the logical symbols* $\bigwedge_{\alpha < K}$ *and* $\bigvee_{\alpha < K}$, *where* K *is an ordinal. Then this system, augmented by the axioms*

$$\to t = c_0, t = c_1, \ldots, t = c_\alpha, \ldots \ (\alpha < K)$$

for arbitrary terms t, *is complete. If a sequent is provable in this system it is also provable without an essential cut.*

PROOF. Let θ be the length of J and let τ denote the set $\{c_0, c_1, \ldots, c_\alpha, \ldots\}$. Consider a formula of L of the form $\mathrm{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y})$. Let α be an ordinal and suppose that $T(\alpha)$ is of type β_α, according to the natural ordering of ordinals.

Let f^α be a function from $\tau^{\beta\alpha}$, the cartesian product of τ, to τ; let $\boldsymbol{m}$, $\boldsymbol{n}$, etc., be sequences (of appropriate type) of elements of τ. Suppose $\boldsymbol{m}^\alpha$, a sequence $m_{\gamma_0}, m_{\gamma_1}, \ldots$, where $T(\alpha) = \{\gamma_0, \gamma_1, \ldots\}$, is given for each $\alpha < \theta$. Then $\boldsymbol{f}(\boldsymbol{m})$ will denote the sequence of sequences $f^0(\boldsymbol{m}^0), f^1(\boldsymbol{m}^1), \ldots, f^\alpha(\boldsymbol{m}^\alpha), \ldots$. Finally, let L′ be the extended language in which $\bigvee_{\boldsymbol{f}}$ and $\bigwedge_{\boldsymbol{m}}$ are allowed, where in $\bigvee_{\boldsymbol{f}}$, $\boldsymbol{f}$ ranges over all sequences of functions defined as above and in $\bigwedge_{\boldsymbol{m}}$, $\boldsymbol{m}$ ranges over all the sequences of elements of τ defined as above. When those symbols are involved, provability means provability in the system with language L′. Note that L′ is an extension of L.

Under these conventions we shall first show that

$$\text{(1)} \qquad \mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}) \to \bigvee_{\boldsymbol{f}} \bigwedge_{\boldsymbol{m}} A(\boldsymbol{m}, f(\boldsymbol{m}))$$

is provable. Let us abbreviate $\bigwedge_\beta (a_\beta = m_\beta)$ to $\boldsymbol{a} = \boldsymbol{m}$. Let $\boldsymbol{g}$ be an arbitrary, but fixed sequence of functions, defined as $\boldsymbol{f}$ above, and let $\boldsymbol{m_f}$ denote an arbitrary, but fixed sequence of elements of τ, of appropriate length, chosen for $\boldsymbol{f}$. Let

$$\text{(2)} \qquad \ldots, A(\boldsymbol{n}, \boldsymbol{g}(\boldsymbol{n})), \ldots \to \ldots, A(\boldsymbol{m_f}, \boldsymbol{f}(\boldsymbol{m_f})), \ldots$$

be a sequence, where $\boldsymbol{g}$ is fixed, $\boldsymbol{n}$ and $\boldsymbol{f}$ range over all possibilities and the $\boldsymbol{m_f}$'s are arbitrarily chosen. $\boldsymbol{f} = \boldsymbol{g}$, where $\boldsymbol{n} = \boldsymbol{m_f}$. Then (2) is provable.

Now we assume that L′ has an adequate number of free variables so we can carry out the subsequent argument. For each $\alpha < \theta$, and for each $\boldsymbol{n}$, choose a free variable $a_{\alpha,\boldsymbol{n}}$ of L′. Then

$$\text{(3)} \qquad \to a_{\alpha,\boldsymbol{n}} = c_0, a_{\alpha,\boldsymbol{n}} = c_1, \ldots$$

is an axiom. We assume that we can choose different variables for different $(\alpha, \boldsymbol{n})$. For each c, there is a $\boldsymbol{g}$ such that $g^\alpha(\boldsymbol{n}^\alpha) = c$. Therefore, for an arbitrary choice of $(c_{\sigma_0}, c_{\sigma_1}, \ldots)$ a sequence of constants of length θ, (2) for such a $\boldsymbol{g}$ implies

$$\text{(4)} \quad \ldots, a_{0,\boldsymbol{n}} = c_{\sigma_0}, a_{1,\boldsymbol{n}} = c_{\sigma_1}, \ldots, A(\boldsymbol{n}, \boldsymbol{a_n}), \ldots \to \ldots, A(\boldsymbol{m_f}, f(\boldsymbol{m_f})), \ldots,$$

where $\boldsymbol{a_n}$ is the sequence $a_{0,\boldsymbol{n}}, a_{1,\boldsymbol{n}}, \ldots$. Then by an application of the strong generalized cut rule to (3) and (4),

$$\text{(5)} \qquad \ldots, A(\boldsymbol{n}, \boldsymbol{a_n}), \ldots \to \ldots, A(\boldsymbol{m_f}, f(\boldsymbol{m_f})), \ldots$$

for all possible combinations of $\boldsymbol{m_f}$. Therefore from (5)

$$\ldots, A(\boldsymbol{n}, \boldsymbol{a_n}), \ldots \rightarrow \ldots, \bigwedge_{\boldsymbol{m}} A(\boldsymbol{m}, f(\boldsymbol{m})), \ldots$$

or

$$(6) \qquad \ldots, A(\boldsymbol{n}, \boldsymbol{a_n}), \ldots \rightarrow \bigvee_{f} \bigwedge_{\boldsymbol{m}} A(\boldsymbol{m}, f(\boldsymbol{m})).$$

Finally, introducing quantifiers in (6), we obtain

$$\mathrm{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}) \rightarrow \bigvee_{f} \bigwedge_{\boldsymbol{m}} A(\boldsymbol{m}, f(\boldsymbol{m})),$$

which is (1). Since we have chosen distinct free variables for different $(\alpha, \boldsymbol{n})$, it is obvious that the eigenvariable conditions are satisfied.

In the proof of (1), the strong generalized cut rule is applied only to atomic formulas.

Next, we want to prove the converse of (1) in the form:

$$(7) \quad \forall \boldsymbol{x}\, \exists \boldsymbol{y} \bigwedge_{\boldsymbol{n}} (\boldsymbol{x} = \boldsymbol{n} \supset \boldsymbol{y} = f(\boldsymbol{n})), \bigvee_{f} \bigwedge_{\boldsymbol{m}} A(\boldsymbol{m}, f(\boldsymbol{m})) \rightarrow \mathrm{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}; \boldsymbol{y}).$$

First, we have

$$\boldsymbol{a} = \boldsymbol{n}, \boldsymbol{a} = \boldsymbol{n} \supset \boldsymbol{b} = f(\boldsymbol{n}), A(\boldsymbol{n}, f(\boldsymbol{n})) \rightarrow A(\boldsymbol{a}, \boldsymbol{b})$$

for every f and $\boldsymbol{n}$, and

$$\rightarrow a_\alpha = c_0, a_\alpha = c_1, \ldots .$$

Therefore, by the strong generalized cut rule,

$$\bigwedge_{\boldsymbol{n}} (\boldsymbol{a} = \boldsymbol{n} \supset \boldsymbol{b} = f(\boldsymbol{n})), \bigwedge_{\boldsymbol{m}} A(\boldsymbol{m}, f(\boldsymbol{m})) \rightarrow A(\boldsymbol{a}, \boldsymbol{b}).$$

From this we obtain

$$\forall \boldsymbol{x}\, \exists \boldsymbol{y} \bigwedge_{\boldsymbol{n}} (\boldsymbol{x} = \boldsymbol{n} \supset \boldsymbol{y} = f(\boldsymbol{n})), \bigwedge_{\boldsymbol{m}} A(\boldsymbol{m}, f(\boldsymbol{m})) \rightarrow A(\boldsymbol{a}, \boldsymbol{b}),$$

and

$$(8) \quad \forall \boldsymbol{x}\, \exists \boldsymbol{y} \bigwedge_{\boldsymbol{n}} (\boldsymbol{x} = \boldsymbol{n} \supset \boldsymbol{y} = f(\boldsymbol{n})), \bigwedge_{\boldsymbol{m}} A(\boldsymbol{m}, f(\boldsymbol{m})) \rightarrow \mathrm{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}).$$

Since (8) holds for every f, it follows that

$$(9) \quad \forall \boldsymbol{x}\, \exists \boldsymbol{y} \bigwedge_{\boldsymbol{n}} (\boldsymbol{x} = \boldsymbol{n} \supset \boldsymbol{y} = f(\boldsymbol{n})), \bigvee_{f} \bigwedge_{\boldsymbol{m}} A(\boldsymbol{m}, f(\boldsymbol{m})) \rightarrow \mathrm{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}),$$

which is (7).

On the other hand

$$(10) \qquad \to \forall \boldsymbol{x}\, \exists \boldsymbol{y} \bigwedge_{\boldsymbol{n}} (\boldsymbol{x} = \boldsymbol{n} \supset \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{n}))$$

is provable for every $\boldsymbol{f}$ in the following way. For an arbitrary $\boldsymbol{n}$ and $\boldsymbol{m}$,

$$\boldsymbol{a} = \boldsymbol{n} \to \boldsymbol{a} = \boldsymbol{m} \supset \boldsymbol{f}(\boldsymbol{n}) = \boldsymbol{f}(\boldsymbol{m}),$$

hence

$$\boldsymbol{a} = \boldsymbol{n} \to \exists \boldsymbol{y} \bigwedge_{\boldsymbol{m}} (\boldsymbol{a} = \boldsymbol{m} \supset \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{m})).$$

Also,

$$\to a_\alpha = c_0, a_\alpha = c_1, \ldots \text{ for each } \alpha.$$

Therefore by the strong generalized cut rule,

$$\to \exists \boldsymbol{y} \bigwedge_{\boldsymbol{m}} (\boldsymbol{a} = \boldsymbol{m} \supset \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{m})).$$

From this, (10) follows.

The eigenvariable conditions in the proofs of (1) and (7) can be easily examined.

We can extend the above method. Let A be a formula of the form $\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(x, y)$. Then the set of formulas

$$\{\forall \boldsymbol{x}\, \exists \boldsymbol{y} \bigwedge_{\boldsymbol{n}} (\boldsymbol{x} = \boldsymbol{n} \supset \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{n}))\}_{\boldsymbol{f}}$$

in (7) corresponding to A will be denoted by $\Phi(A)$. Note that $\Phi(A)$ actually depends only on the length of $\boldsymbol{x}$; therefore it does not matter what free variables A may contain.

LEMMA 24.23. *Let $B_1, B_2, \ldots$ be all the subformulas of A of the above form. Then there is a quantifier free formula $\tilde{A}$ (in the extended language) such that $\tilde{A}$ contains exactly the same free variables as A and*

$$(11) \qquad \Phi(B_1), \Phi(B_2), \ldots, A \to \tilde{A}, \qquad \Phi(B_1), \Phi(B_2), \ldots, \tilde{A} \to A,$$

are provable without essential cuts.

PROOF (by transfinite induction on the complexity of A). Since other cases are obvious, we shall work on the case where A is of the form $\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y})$. To prove the first sequent, we proceed as follows. Consider $A(\boldsymbol{d}, \boldsymbol{e})$ where $\boldsymbol{d}$ and $\boldsymbol{e}$ are new free variables. Then, by the induction hypothesis, there is a quantifier free formula $\tilde{A}(\boldsymbol{d}, \boldsymbol{e})$ such that

$$\Phi(B_1), \ldots, A(\boldsymbol{d}, \boldsymbol{e}) \to \tilde{A}(\boldsymbol{d}, \boldsymbol{e})$$

is provable without cuts. Therefore with the same reasoning as for (2),

$$\Phi(B_1), \ldots, A(\boldsymbol{n}, \boldsymbol{g}(\boldsymbol{n})), \ldots \to \ldots, \tilde{A}(\boldsymbol{m_f}, \boldsymbol{f}(\boldsymbol{m_f})), \ldots$$

is provable without cuts (cf. (2) above). The proof of (1) then follows.

In proving the second sequent, start with

$$\Phi(B_1), \ldots, \tilde{A}(\boldsymbol{d}, \boldsymbol{e}) \to A(\boldsymbol{d}, \boldsymbol{e}).$$

From this we obtain

$$\Phi(B_1), \ldots, \boldsymbol{a} = \boldsymbol{n}, \boldsymbol{a} = \boldsymbol{n} \supset \boldsymbol{b} = \boldsymbol{f}(\boldsymbol{n}), \tilde{A}(\boldsymbol{n}, \boldsymbol{f}(\boldsymbol{n})) \to A(\boldsymbol{a}, \boldsymbol{b}),$$

without cuts. Then, by following the proof of (7), we obtain a cut-free proof of

$$\Phi(A), \Phi(B_1), \ldots, \tilde{A} \to A.$$

Now consider an arbitrary, valid sequent of L, say

$$A_0, A_1, \ldots \to B_0, B_1, \ldots,$$

and attempt to prove it. Since our system with language L′ is consistent, the validity of the given sequent and (11) imply that

$$\Phi, \tilde{A}_0, \tilde{A}_1, \ldots \to \tilde{B}_0, \tilde{B}_1, \ldots \tag{12}$$

is valid, where Φ consists of the formulas of the form

$$\forall \boldsymbol{x}\, \exists \boldsymbol{y} \bigwedge_{\boldsymbol{n}} (\boldsymbol{x} = \boldsymbol{n} \supset \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{n})).$$

Therefore (12) is provable (without essential cuts) in the homogeneous part of our extended system.

On the other hand, Φ, $A_\alpha \to \tilde{A}_\alpha$ and Φ, $\tilde{B}_\beta \to B_\beta$ are provable in the system with language L′ without essential cuts. Now by the strong generalized cut rule with the cut formulas

$$\Phi, \tilde{A}_0, \tilde{A}_1, \ldots, \tilde{B}_0, \tilde{B}_1, \ldots \tag{13}$$

we obtain the given sequent.

Therefore, what remains to be shown is that this essential cut in the proof of the given sequent can be eliminated (in the language of L′). If so, then the given sequent of the language L can be proved in the original system with language L and without essential cuts.

In proving the cut-elimination theorem, we shall make use of the proof given above.

The proof is carried out with a generalized Gentzen's reduction method. In order to simplify the discussion, we assume that the language L has no function symbols. It will also be assumed that the initial sequents consist of atomic formulas only. We shall prove the cut-elimination theorem in the following form.

(*) Let P be a proof (in L′) such that

(i) Along each branch of the sequents there is at most one essential g.c.,

(ii) the principal formulas of the quantifier introductions which are ancestors of cut formulas are of the form

$$\exists y \bigwedge_{\boldsymbol{m}} (a = \boldsymbol{m} \supset y = f(\boldsymbol{m}))$$

and

$$\forall x\, \exists y \bigwedge_{\boldsymbol{m}} (x = \boldsymbol{m} \supset y = f(\boldsymbol{m})),$$

while the auxiliary formulas are of the form

$$\bigwedge_{\boldsymbol{m}} (a = \boldsymbol{m} \supset b = f(\boldsymbol{m})) \qquad \text{(in the antecedent)},$$

$$\bigwedge_{\boldsymbol{m}} (a = \boldsymbol{m} \supset f(\boldsymbol{n}) = f(\boldsymbol{m})) \qquad \text{(in the succedent)},$$

$$\exists y \bigwedge_{\boldsymbol{m}} (x = \boldsymbol{m} \supset y = f(\boldsymbol{m}))$$

respectively (cf. (13) above).

Let $S: \Gamma \to \Delta$ be the lower sequent of a s.g.c. We then list some consequences of a reduction.

(iii) S as well as the descendents of S remain unchanged.

(iv) Either the s.g.c. by which S is obtained in P is eliminated, or sequents of the form $\ldots D \ldots \to \ldots D \ldots$ are eliminated, or some introductions of logical symbols above S are eliminated, or the essential g.c. is pushed up one step.

Let $\mathscr{F}$ be the set of cut formulas of the s.g.c. above S. The reduction is defined according to the stage number k (mod 10).

$k \equiv 0$ (mod 10). Look for a sequent of the form

$$\ldots, D, \ldots \rightarrow \ldots, D, \ldots \quad \text{or} \quad \ldots \rightarrow \ldots, a = a, \ldots$$

among the upper sequents of S. Suppose there is one. In the subsequent treatment, the cases with earlier numbers have priority over those with later numbers.

Case 1. For some D as above, D occurs both in Γ and Δ. Change the figure above S to

$$\frac{D \rightarrow D}{\Gamma \rightarrow \Delta} \quad \text{(no cuts)}.$$

Case 2. An equality $a = a$ occurs in Δ. Change the figure above S to

$$\frac{\rightarrow a = a}{\ldots \rightarrow \ldots, a = a, \ldots}.$$

Case 3. For any such D, D does not occur in either Γ or Δ. This case cannot happen, since then D must belong to both $\mathscr{F}_1$ and $\mathscr{F}_2$ for any partition $(\mathscr{F}_1, \mathscr{F}_2)$ of $\mathscr{F}$.

Case 4. D occurs in Γ but not in Δ. Then the D in the antecedent is not a cut formula but the D in the succedent is a cut formula. Eliminate all the upper sequents which contain D in the succedent. If D occurs in the antecedent in one of the remaining sequents (and such a sequent exists), then regard it as a formula in Γ. Do the same to all such D's. Thus the set of cut formulas will be $\mathscr{F} - \{$all such D's$\}$ and Case 4 can be eliminated.

Case 5. D occurs in Δ but not Γ. The reduction is defined as for Case 4.

$k \equiv 1$ (mod 10). There is an upper sequent of the form

$$\ldots \rightarrow \ldots, t = c_0, \ldots, t = c_1, \ldots, t = c_\alpha, \ldots \quad (\alpha < K).$$

If all the $t = c_\alpha$ occur in Δ, then $\Gamma \rightarrow \Delta$ can be obtained from

$$\frac{\rightarrow t = c_0, t = c_1, \ldots, t = c_\alpha, \ldots}{\Gamma \rightarrow \Delta}$$

and hence without cuts. Otherwise consider the following. Let G be the set of all formulas of the form $t = c_\alpha$ which are cut formulas and let H be the rest of the cut formulas. Let $(\mathscr{F}_1, \mathscr{F}_2)$ be a partition of $\mathscr{F}$. Then there is a $\Phi_1 \subseteq \mathscr{F}_1 \cap G$, a $\Psi_1 \subseteq \mathscr{F}_1 \cap H$, a $\Phi_2 \subseteq \mathscr{F}_2 \cap G$ and a $\Psi_2 \subseteq \mathscr{F}_2 \cap H$ such that

$$\Phi_1, \Psi_1, \Gamma' \to \Delta', \Phi_2, \Psi_2$$

is an upper sequent of S.

Consider all the partitions of $\mathscr{F}$ which leave $G_1 = \mathscr{F}_1 \cap G$ and $G_2 = \mathscr{F}_2 \cap G$ fixed. Then for each such partition there are Ψ_1 and Ψ_2 such that

$$G_1, \Psi_1, \Gamma' \to \Delta', G_2, \Psi_2$$

is provable without cuts, without increasing the number of inferences. Hence, by the s.g.c. rule applied to H, we obtain

$$G_1, \Gamma \to \Delta, G_2.$$

This is true for all possible partitions of G; hence by the s.g.c. rule applied to G, we obtain $\Gamma \to \Delta$. Here the last s.g.c. is regarded as a weak inference. Thus this case can be eliminated.

$k \equiv 2$. Suppose there are cut formulas whose outermost logical symbols are $\neg$. (If there is no such symbol, pass on to the next stage.) Take an arbitrary upper sequent S_0:

$$S: \ldots, \neg A, \ldots, \Gamma_0 \to \Delta_0, \ldots, \neg B, \ldots,$$

where $\neg A, \neg B, \ldots$ are cut formulas and $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$. Let Q_0 be the sub-proof of P above and including S_0. Then, by changing Q_0 slightly, we obtain a proof of

$$S_0': \ldots, B, \ldots, \Gamma_0 \to \Delta_0, \ldots, A, \ldots$$

without increasing the number of inferences. Recall that there is no essential cut in Q_0. Replace each formula of $\mathscr{F}$ whose outermost logical symbol is $\neg$, say $\neg A$, by A, thus obtaining $\mathscr{F}'$. Then an arbitrary partition of $\mathscr{F}$, say $(\mathscr{F}_1', \mathscr{F}_2')$, induces a partition of $\mathscr{F}$, say $(\mathscr{F}_1, \mathscr{F}_2)$, in a natural manner. If S_0, as above, is an upper sequent of S corresponding to $(\mathscr{F}_1, \mathscr{F}_2)$, then S_0' corresponds to $(\mathscr{F}_1', \mathscr{F}_2')$. Thus the assumption for the s.g.c. rule applied to $\mathscr{F}'$ is satisfied. By a s.g.c. we obtain S. In this case some inferences which introduce $\neg$ are eliminated.

$k \equiv 3$. Consider all the formulas in $\mathscr{F}$ whose outermost logical symbols are $\wedge$. Let S_0 be an upper sequent of S:

$$\ldots, \bigwedge_i C_i, \ldots, \Gamma_0 \to \Delta_0, \ldots, \bigwedge_j A_j, \ldots, \bigwedge_h B_h, \ldots,$$

where $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$ and $\bigwedge_i C_i$ etc., are only some of the cut formulas whose outermost logical symbols are $\wedge$. Let Q_0 be the figure which is above and including S_0. Then, changing Q_0 in an obvious manner, we can construct a quasi-proof of the sequent

$$\ldots, \{C_i\}_i, \ldots, \Gamma_0 \to \Delta_0, \ldots, A_j, \ldots, B_h, \ldots$$

for every combination of $(\ldots, j, \ldots, h, \ldots)$. Let $\mathscr{F}'$ be the set of formulas obtained from $\mathscr{F}$ by replacing all the formulas of the form $\bigwedge_i C_i$ by

$$\{C_0, C_1, \ldots, C_i, \ldots\}.$$

Let $(\mathscr{F}_1', \mathscr{F}_2')$ be a partition of $\mathscr{F}$ for which $\bigwedge_i C_i$ belongs to $\mathscr{F}_2$ if $\bigwedge_i C_i$ belongs to $\mathscr{F}$ and there is one C_i which belongs to $\mathscr{F}_2'$; $\bigwedge_i C_i$ belongs to $\mathscr{F}_1$ if all $C_0, C_1, \ldots, C_i, \ldots$ belong to $\mathscr{F}_1'$; and all other formulas belong to $\mathscr{F}_1(\mathscr{F}_2)$ if and only if they belong to $\mathscr{F}_1'(\mathscr{F}_2')$. There is an upper sequent of S corresponding to $(\mathscr{F}_1, \mathscr{F}_2)$:

$$\ldots, \bigwedge_i C_i, \ldots, \Gamma_0 \to \Delta_0, \ldots, \bigwedge_j A_j, \ldots, \bigwedge_h B_h, \ldots .$$

Then, as was shown above, we can change this to

$$\ldots, \{C_i\}_i, \ldots, \Gamma_0 \to \Delta_0, \ldots, A_j, \ldots, B_h, \ldots$$

for every combination of $\ldots, j, \ldots, h, \ldots$. Since $\bigwedge_i C_i$ belongs to $\mathscr{F}_1$ only if all C_i's belong to $\mathscr{F}_1'$,

$$\ldots, \{C_i\}_i, \ldots \subseteq \mathscr{F}_1' \quad \text{and} \quad \ldots, A_j, \ldots, B_h, \ldots, \subseteq \mathscr{F}_2'.$$

This argument goes through for any partition of $\mathscr{F}'$. Therefore the s.g.c. rule can apply to $\mathscr{F}'$.

$k \equiv 4$. The reduction for $\vee$ can be defined likewise.

$k \equiv 5$. Consider all the formulas in $\mathscr{F}$ whose outermost logical symbols are quantifiers. If such a formula occurs in the antecedent of a sequent, then the auxiliary formula of such qualifications is either of the form

$$\bigwedge_{\boldsymbol{m}} (\boldsymbol{a} = \boldsymbol{m} \supset \boldsymbol{b} = \boldsymbol{f}(\boldsymbol{m})) \tag{1)}$$

or

$$\exists \boldsymbol{y} \bigwedge_{\boldsymbol{m}} (\boldsymbol{a} = \boldsymbol{m} \supset \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{m})). \tag{2)}$$

If it occurs in the succedent, then the auxiliary formula is either of the form

$$\bigwedge_{m} (a = m \supset f(n) = f(m)) \tag{3}$$

or

$$\exists y \bigwedge_{m} (a = m \supset y = f(m)). \tag{4}$$

If 1) and 3) are the case, then substitute $f(n)$ for b with n arbitrary; if 2) and 4) are the case, then do nothing. For the first case, replace the original cut formula by 3) and for the second case replace it by 2). If there are other formulas with b, then they are also replaced by $f(n)$. Given a partition of the new set of cut formulas, it should be obvious how to induce a partition of the original set.

$k \equiv 6$. Suppose $\Phi, \Gamma_0 \to \Delta_0, \Psi$ is an upper sequent of S. Let $\neg A, \ldots$ be all the formulas in Γ_0 whose outermost logical symbols are $\neg$, and let $\tilde{\Gamma}$ be the rest of Γ_0. $\neg B, \ldots$ and $\tilde{\Delta}$ are defined likewise. Then we can construct a quasi-proof of

$$\Phi, \tilde{\Gamma}, B, \ldots \to \tilde{\Delta}, A, \ldots, \Psi$$

without increasing the number of inferences. By the s.g.c. rule applied to the same cut formulas, we obtain $\Gamma' \to \Delta'$, where Γ' is obtained from Γ by replacing some $\neg A$'s by B's, and Δ' is defined likewise. So we obtain

$$\begin{array}{c} \vdots \\ \Gamma' \to \Delta' \\ \hline\hline \Gamma \to \Delta \end{array}$$

where $=\!=\!=$ stands for two inferences, a $\neg$: left and a $\neg$: right. In this case some logical inferences (introduction of $\neg$) are eliminated from above the s.g.c.

$k \equiv 7$. Let $\Phi, \Gamma_0 \to \Delta_0, \Psi$ be an upper sequent of S. Consider all the formulas in Δ_0 whose outermost logical symbols are $\bigwedge$, say $\ldots \bigwedge_i A_i, \ldots, \bigwedge_j B_j, \ldots$ and let $\tilde{\Delta}_0$ denote the remaining formulas in Δ_0. We can construct a quasi-proof of

$$\Phi, \Gamma_0 \to \tilde{\Delta}_0, \ldots, A_i, \ldots, B_j, \ldots, \Psi$$

for every combination of $(\ldots, i, \ldots, j, \ldots)$. Hence by the s.g.c. rule we obtain

$$\Gamma' \to \tilde{\Delta}, \ldots, A_i, \ldots, B_j, \ldots,$$

from which we can infer $\Gamma \to \Delta$.

$k \equiv 8$. Consider the formulas in Γ_0 whose outermost logical symbol is $\wedge$.

$k \equiv 9$. Let $\Phi, \Gamma_0 \to \Delta_0, \Psi$ be an upper sequent of S. Let $\mathsf{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}), \ldots$ be all the formulas in Γ_0 whose outermost logical symbol is Q^T, and let $\tilde{\Gamma}_0$ be the rest of Γ_0. Let $\mathsf{Q}^{T'}(\boldsymbol{u}; \boldsymbol{v})\, B(\boldsymbol{u}, \boldsymbol{v}), \ldots$ be all the formulas in Δ_0 whose outermost logical symbol is $\mathsf{Q}^{T'}$, and let $\tilde{\Delta}_0$ be the rest of Δ_0. Then we can construct a quasi-proof of

$$\Phi, \tilde{\Gamma}_0, \{A(\boldsymbol{s}, \boldsymbol{a})\}_{\boldsymbol{a}, \boldsymbol{s}}, \ldots \to \{B(\boldsymbol{b}_i, \boldsymbol{t}_i)\}_i, \ldots, \tilde{\Delta}_0, \Psi$$

for some $\boldsymbol{a}$, $\boldsymbol{s}$ and i's. Applying the s.g.c. rule we obtain

$$\tilde{\Gamma}, \{A(\boldsymbol{s}, \boldsymbol{a})\}_{\boldsymbol{a}, \boldsymbol{s}}, \ldots \to \{B(\boldsymbol{b}_i, \boldsymbol{t}_i)\}_i, \ldots, \tilde{\Delta}.$$

Introducing quantifiers to both sides, we can infer $\Gamma \to \Delta$. Since the eigenvariable conditions are defined for an entire proof, it is obvious that those conditions, which are satisfied in P, are transferred to the new figure.

This completes the description of the reduction.

Take an arbitrary inference which is not one of the weak inferences, i.e., which is an introduction of a logical symbol. The principal formulas of such an inference are either cut formulas or formulas in $\Gamma \to \Delta$. Let $\mathscr{S}$ be a string of sequents to which the lower sequent of this inference belong. Since the number of inferences along a string is finite, and the reduction process reduces the number of logical inferences above the s.g.c., the inference under consideration will eventually be either eliminated or carried down under the s.g.c. Hence along any string of sequents, there will eventually be no s.g.c. within a finite number of stages. Therefore we will obtain a cut-free proof of the given sequent.

REMARK. Let us introduce a function symbol $\bar{f}$ corresponding to each function f from τ^β to τ. Let $A(\bar{f})$ be the set of sentences of the form $\bar{f}(\boldsymbol{n}) = \boldsymbol{m}$, where $f(\boldsymbol{n}) = \boldsymbol{m}$. From

$$A(\bar{f}), \boldsymbol{a} = \boldsymbol{n}, A(\boldsymbol{n}, f(\boldsymbol{n})) \to A(\boldsymbol{a}, \bar{f}(\boldsymbol{a})) \quad \text{for all } \boldsymbol{n}$$

and

$$\to a_\alpha = c_0, a_\alpha = c_1, \ldots \quad \text{for all } a_\alpha \text{ in } \boldsymbol{a},$$

we can infer, by the s.g.c. rule,

$$A(\bar{f}), \ldots, A(\boldsymbol{n}, f(\boldsymbol{n})) \ldots \to A(\boldsymbol{a}, \bar{f}(\boldsymbol{a})),$$

where $\boldsymbol{n}$ ranges over all possible sequences of constants. This is true for every $\boldsymbol{f}$; hence we obtain (7) in the form

$$\ldots, A(\bar{\boldsymbol{f}}), \ldots, \bigwedge_{\boldsymbol{n}} A(\boldsymbol{n}, \boldsymbol{f}(\boldsymbol{n})) \to \mathrm{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}),$$

or

$$\ldots, A(\bar{\boldsymbol{f}}), \ldots, \bigvee_{\boldsymbol{f}} \bigwedge_{\boldsymbol{n}} A(\boldsymbol{n}, \boldsymbol{f}(\boldsymbol{n})) \to \mathrm{Q}^T(\boldsymbol{x}; \boldsymbol{y})\, A(\boldsymbol{x}, \boldsymbol{y}).$$

Going through the same argument as before, we claim that, if $\Gamma \to \Delta$ is a valid sequent of the original language, then

$$\ldots, A(\bar{\boldsymbol{f}}), \ldots, \Gamma \to \Delta$$

is cut-free provable. This formulation may be more convenient for some purposes.

CHAPTER 5

CONSISTENCY PROOFS

§25. Introduction

This chapter is devoted to the consistency problems of systems of second order arithmetic. Before we take up these problems there are two points we would like to call to the reader's attention.

1. Mathematicians have an extremely good intuition about the world of the natural numbers as conceived by an infinite mind. Consequently, consistency for the natural numbers is not a particularly important question. In contrast, we can conceive of the world of sets only through our imagination and our mathematical experience. Consequently, the problem of the consistency of the comprehension axioms is a serious and important foundational question.

2. Mathematicians use the term "consistency" as a sort of foundational watchword. The first implication of the term is that no contradiction is derived. Of course this is the most important assurance for our imaginary world of the infinite mind. But sometimes we would like to know more. For example, the fact that no contradiction arises does not explain what it means to say that a theorem is provable from the comprehension axiom. Nonconstructive proofs provide no insight into this important question. On the other hand a constructive proof strengthens our intuition and adds meaning to the theorem. In particular, a constructive proof of the cut-elimination theorem would give us greater confidence in the comprehension axioms and hence strengthen our convictions about our imaginary world of sets.

In this chapter we will be interested in second order arithmetic. Here our comprehension axioms are

$$\forall : \text{left} \quad \frac{F(V), \Gamma \to \Delta}{\forall \phi\, F(\phi), \Gamma \to \Delta}, \qquad \exists : \text{right} \quad \frac{\Gamma \to \Delta, F(V)}{\Gamma \to \Delta, \exists \phi\, F(\phi)}.$$

In any cut free proof our comprehension axioms mean roughly that we can introduce a new variable ϕ to express a given abstract V, and that we later use an abbreviation $\forall\phi\, F(\phi)$ or $\exists\phi\, F(\phi)$ in place of

$$F(\phi_1) \wedge \ldots \wedge F(\phi_n) \quad \text{or} \quad F(\phi_1) \vee \ldots \vee F(\phi_n).$$

We will discuss this at greater length later. We point out that a similar interpretation holds in systems of higher order although there the situation is more complicated.

We begin our study with a theory of modulations, a theory that provides our basic argument against "practical" foundations. Given a formula A, we will define left modulations of A and right modulations of A. The definition is according to the outermost logical symbol of A. We assume these symbols to be $\neg$, $\wedge$, $\vee$, and $\forall$. For each case a left modulation of A is defined as a formula of the form $A_1 \wedge \ldots \wedge A_m$, where $A_1, \ldots, A_m$ are some left atomic modulations of A, while a right modulation of A is defined as a formula of the form $B_1 \vee \ldots \vee B_n$, where $B_1, \ldots, B_n$ are some right atomic modulations of A. It is also required that if A' is a modulation of A, then every free variable in A' occurs in A.

DEFINITION 25.1. (1) If A is an atomic formula, then the left and the right (atomic) modulations of A are A itself.

(2) If A is of the form $\neg B$, a left atomic modulation of A is of the form $\neg B'$, where B' is a right modulation of B; a right atomic modulation of A is of the form $\neg B''$, where B'' is a left modulation of B.

(3) If A is of the form $B \wedge C$, a left atomic modulation of A is of the form $B' \wedge C'$, where B' and C' are left modulations of B and C, respectively. A right atomic modulation of A is of the form $B'' \wedge C''$, where B'' and C'' are right modulations of B and C respectively. For $B \vee C$, the definition is similar.

(4) If A is of the form $\forall x\, F(x)$, and t is an arbitrary term such that no free variables in t occur in $\forall x\, F(x)$, then $\forall x_1 \ldots \forall x_n\, G$ is an atomic left modulation of A, where G is a left modulation of $F(t)$ and $\forall x_1 \ldots \forall x_n$ bind all free variables in t. A right atomic modulation of $\forall x\, F(x)$ is of the form $\forall x\, G(x)$, where a does not occur in $F(x)$ and $G(a)$ is a right modulation of $F(a)$.

(5) If A is of the form $\forall\phi\, F(\phi)$, and V is an arbitrary abstract such that no free variables in V occur in $\forall\phi\, F(\phi)$, then $\forall\phi_1 \ldots \forall\phi_m \forall x_1 \ldots \forall x_n\, G$ is a left atomic modulation of A, where G is a left modulation of $F(V)$ and $\forall\phi_1 \ldots \forall\phi_m \forall x_1 \ldots \forall x_n$ bind all free variables in V. A right atomic modulation

of $\forall\phi\, F(\phi)$ is of the form $\forall\phi\, F'(\phi)$, where α does not occur in $F(x)$ and $F'(\alpha)$ is a right modulation of $F(\alpha)$.

PROPOSITION 25.2. *Let α and β be either free variables or constants. If $A'(\alpha)$ is a modulation (right or left) of $A(\alpha)$, then $A'(\beta)$ is also a modulation (right or left) of $A(\beta)$.*

PROPOSITION 25.3. *If A' is a left modulation of A, then $A \to A'$ is provable. If A'' is a right modulation of A, then $A'' \to A$ is provable.*

PROOF. Since other cases are easy, we shall consider only the first part for the crucial case, (5): i.e., A is of the form $\forall\phi\, F(\phi)$. A' is of the form $A_1' \wedge \ldots \wedge A_n'$, where A_i' is an atomic left modulation of A. It suffices to show $A \to A_i'$ for each i. Let A_i' be of the form $\forall\phi_1 \ldots \forall\phi_l\, \forall x_1 \ldots \forall x_m\, G$, where G is a left modulation of $F(V)$. Suppose $F(V) \to G$ is provable. Then

$$\cfrac{\cfrac{F(V) \to G}{\forall\phi\, F(\phi) \to G}}{\forall\phi\, F(\phi) \to \forall\phi_1 \ldots \forall\phi_l\, \forall x_1 \ldots \forall x_m\, G.}$$

The eigenvariable condition is satisfied.

DEFINITION 25.4. (1) A sequent $A_1', \ldots, A_n' \to B_1', \ldots, B_m'$ is called a modulation of $A_1, \ldots, A_n \to B_1, \ldots, B_m$ if A_i' is a left modulation of A_i for each i and B_j' is a right modulation of B_j for each j.

(2) Let P be a cut free proof. For each sequent $\Pi \to \Lambda$ in P, we define a modulation $\Pi' \to \Lambda'$ of $\Pi \to \Lambda$ by induction on the number of inferences above $\Pi \to \Lambda$. $\Pi' \to \Lambda'$ is called the P-modulation of $\Pi \to \Lambda$ and is so defined that if $\Pi \to \Lambda$ is $A_1, \ldots, A_m \to B_1, \ldots, B_n$, then $\Pi' \to \Lambda'$ is $A_1', \ldots, A_m' \to B_1', \ldots, B_n'$, where A_i' is a left modulation of A_i and B_j is a right modulation of B_j. We shall give explicit definitions only for some exemplary cases.

1) If $\Pi \to \Lambda$ is an initial sequent in P, then $\Pi' \to \Lambda'$ is $\Pi \to \Lambda$ itself.

2) The last inference is an interchange : left.

$$\frac{\Gamma, C, D, \Xi \to \Delta}{\Gamma, D, C, \Xi \to \Delta}.$$

If $\Gamma', C', D', \Xi' \to \Delta'$ is the P-modulation of $\Gamma, C, D, \Xi \to \Delta$, then $\Gamma', D', C', \Xi' \to \Delta'$ is the P-modulation of $\Gamma, D, C, \Xi \to \Delta$.

3) The last inference is a weakening : left.

$$\frac{\Gamma \rightarrow \Delta}{C, \Gamma \rightarrow \Delta}.$$

If $\Gamma' \rightarrow \Delta'$ is the P-modulation of $\Gamma \rightarrow \Delta$, then $C, \Gamma' \rightarrow \Delta'$ is the P-modulation of $C, \Gamma \rightarrow \Delta$.

4) The last inference is a contraction : left.

$$\frac{C, C, \Gamma \rightarrow \Delta}{C, \Gamma \rightarrow \Delta}.$$

If $C_1, C_2, \Gamma' \rightarrow \Delta'$ is the P-modulation of $C, C, \Gamma \rightarrow \Delta$, then $C_1 \wedge C_2, \Gamma' \rightarrow \Delta'$ is the P-modulation of $C, \Gamma \rightarrow \Delta$.

The last inference is a contraction : right.

$$\frac{\Gamma \rightarrow \Delta, C, C}{\Gamma \rightarrow \Delta, C}.$$

If $\Gamma' \rightarrow \Delta', C_1, C_2$ is the P-modulation $\Gamma \rightarrow \Delta, C, C$ then $\Gamma' \rightarrow \Delta', C_1 \vee C_2$ is the P-modulation of $\Gamma \rightarrow \Delta, C$.

5) The last inference is a second order $\forall$: left.

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall\phi\, F(\phi), \Gamma \rightarrow \Delta}.$$

If $G, \Gamma' \rightarrow \Delta'$ is the P-modulation of $F(V), \Gamma \rightarrow \Delta$, then

$$\forall\phi_1 \ldots \forall\phi_n \forall x_1 \ldots \forall x_m\, G, \Gamma' \rightarrow \Delta'$$

is the P-modulation of $\forall\phi\, F(\phi), \Gamma \rightarrow \Delta$, where $\forall\phi_1 \ldots \forall\phi_n \forall x_1 \ldots \forall x_m$ binds all free variables in V. More precisely, let $V = V(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_m)$, where all the free variables are indicated, and let $\beta_1, \ldots, \beta_n, b_1, \ldots, b_m$ be free variables which are not contained in $F(V)$. Let $G'(\beta_1, \ldots, \beta_n, b_1, \ldots, b_m)$ be a left modulation of $F(V(\beta_1, \ldots, \beta_n, b_1, \ldots, b_m))$ so that $G'(\alpha_1, \ldots, \alpha_n, a_1, \ldots, a_m)$ is G. By $\forall\phi_1 \ldots \forall\phi_n \forall x_1 \ldots \forall x_m\, G$ we mean

$$\forall\phi_1 \ldots \forall\phi_n \forall x_1 \ldots \forall x_m\, G'(\phi_1, \ldots, \phi_n, x_1, \ldots, x_m).$$

6) The last inference is a second order $\forall$: right.

$$\frac{\Gamma \rightarrow \Delta, F(\alpha)}{\Gamma \rightarrow \Delta, \forall\phi\, F(\phi)}.$$

Suppose $\Gamma' \to \Delta', F'(\alpha)$ is the P-modulation of the upper sequent. Then $\Gamma' \to \Delta', \forall\phi\, F'(\phi)$ is the P-modulation of the lower sequent.

PROPOSITION 25.5. *Let P be a cut-free proof of $\Gamma \to \Delta$, and let $\Gamma' \to \Delta'$ be the P-modulation of $\Gamma \to \Delta$. Then there exists a cut-free predicative proof of $\Gamma' \to \Delta'$.*

PROOF. This is an immediate corollary of Definition 25.5. That is, in Definition 25.5, if S_2 is a lower sequent of S_1 (and S) then S_2', the P-modulation of S_2, can be derived from S_1' (and S') without a cut and without impredicative comprehension axioms. The latter fact is obvious from the definition of P-modulation in Case 5) of Definition 25.5. From Proposition 1 it is obvious that the eigenvariable condition is satisfied for any application of $\forall$: right.

Our theory of modulations, and in particular Proposition 25.6, makes clear one important reason for our interest in cut-free proofs, namely, a cut-free proof of a theorem enables us to give to that theorem an interpretation that avoids Russell's vicious circle. Of course, each theorem has as many vicious-circle-free interpretations as it has cut-free proofs, and each different interpretation of a theorem determines a different interpretation of the theory.

If we had a uniform "method" that transformed every formal proof into a cut-free proof, then that "method" would itself determine an interpretation of mathematics that avoids Russell's vicious circle. Consequently, producing such a method of transformation should be a matter of high priority.

We would, of course, prefer an interpretation that is as close to the original (natural) meaning as possible. Consequently, the cut-elimination theorem should be proved by an elimination procedure that preserves as much as possible of the meaning of the original proofs.

It is on precisely this issue that we oppose the recent trend of foundational studies in the direction of what we have chosen to call "quasi" foundations. Let us illustrate our point with an example.

One may view analysis in different ways. One view is that analysis is a theory. An alternate view is that analysis is not a system of axioms but a collection of results. It is this alternate view that gives rise to the problem of quasi foundations.

The job of quasi foundations is to develop a kind of quasi-analysis; to find a collection of theorems, that are similar to a given collection of theorems of analysis, but which are in fact weaker results than the given theorems.

There are two points we wish to make that explain why we do not view "quasi" foundations as the proper direction for foundational studies to take. First of all, when one develops a collection of results that are "similar" to a given set of theorems of analysis it is not at all clear what metamathematical conclusions can be drawn or what kind of theory has been founded. But, second, and of greater importance, our theory of modulations shows that in principle the problems of quasi foundations are solved: If we wish a "quasi foundation" for a certain collection of results we simply start with predicative comprehension axioms and develop the modulations of the results that we wish. Of course the results we obtain will not be classical theorems but modulations. However, as we have shown, modulations are stronger results than the theorems of classical mathematics.

We may, therefore, regard the predicative comprehension axioms as what is really important. The theorems of classical mathematics are simply approximations to stronger theorems. Consequently, so long as our concern is with practical foundations we need not prove the cut-elimination theorem; we only need to justify the predicative comprehension axioms. Having done this, our task is to construct sufficient and appropriate modulations for our purposes.

We, however, advocate a different viewpoint with respect to which proving the cut-elimination theorem constructively is a matter of paramount importance. Indeed, only through a constructive proof of cut-elimination can the theory of modulations become truly a significant theory in a world of predicative mathematics. The proof of the cut-elimination theorem as presented in Chapter 3 is set-theoretical, and therefore useless for our purpose.

§26. Ordinal diagrams

In this section we will develop a theory of ordinal diagrams which, as we pointed out earlier, will play a fundamental role in our study of consistency problems. For each pair of nonempty, well-ordered sets I and A we will define the set of ordinal diagrams based on I and A. At the same time we will also define the notion of connected ordinal diagrams.

DEFINITION 26.1. Let I and A be nonempty, well-ordered sets with 0 the smallest element in I. The system of *ordinal diagrams*, based on I and A, we define inductively in the following way.

1) 0 is a connected ordinal diagram.

2) Let i be an element of I, a an element of A, and let α be an ordinal diagram. Then (i, a, α) is a *connected* ordinal diagram.

3) Let $n \geqslant 2$ and let $\alpha_1, \ldots, \alpha_n$ be connected ordinal diagrams. Then $\alpha_1 \# \alpha_2 \# \ldots \# \alpha_n$ is a *nonconnected* ordinal diagram.

For convenience, in discussing ordinal diagrams we will use i, j, k etc., as variables on I; a, b, c, etc., as variables on A; and α, β, γ, etc., as variables on ordinal diagrams. Hereafter "ordinal diagram" will mean "ordinal diagram based on I and A".

Definition 26.1 defines an inductive procedure for constructing ordinal diagrams. If the ordinal diagram γ enters the construction of α we call γ a sub-ordinal diagram of α. But from the definition it is clear that an ordinal diagram γ could enter the construction of α at several different stages of the construction. In the work to follow it is sometimes important to identify a specific occurrence of a sub-ordinal diagram. For this purpose we will use the notation $\bar{\gamma}$, that is, the notation $\bar{\gamma}$ is to indicate that we are talking about a specific occurrence of γ and not just about the ordinal diagram γ itself. Thus $\bar{\gamma} = \bar{\beta}$ means not only that $\gamma = \beta$ but also that $\bar{\gamma}$ and $\bar{\beta}$ denote the same occurrence of this ordinal diagram.

DEFINITION 26.2. (1) Each connected ordinal diagram $\alpha_1, \ldots, \alpha_n$ is a component of the nonconnected ordinal diagram $\alpha_1 \# \ldots \# \alpha_n$.

(2) Each connected ordinal diagram α has only one component, namely itself.

DEFINITION 26.3. Let $l(\alpha)$ be the total number of ( )'s and $\#$'s in α. Then

$$l(\alpha, \beta, \ldots, \gamma) =_{\mathrm{df}} l(\alpha) + l(\beta) + \ldots + l(\gamma).$$

DEFINITION 26.4. *Equality* for ordinal diagrams we define inductively on $l(\alpha, \beta)$ in the following way.

(1) $0 = 0$.

(2) Let α be of the form (i, a, γ). Then $\alpha = \beta$ if and only if β is of the form (i, a, δ), where $\gamma = \delta$.

(3) Let α be $\alpha_1 \# \ldots \# \alpha_m$ and let β be $\beta_1 \# \ldots \# \beta_n$, where $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ are connected. Then $\alpha = \beta$ if and only if $m = n$ and there is a

permutation of $\{1, 2, \ldots, m\}$, say $\{j_1, \ldots, j_m\}$, such that

$$\alpha_1 = \beta_{j_1}, \alpha_2 = \beta_{j_2}, \ldots, \alpha_m = \beta_{j_m}.$$

It can be easily proved that $=$ is an equivalence relation. Note also that $\alpha = 0$ or $0 = \alpha$ if and only if α is 0.

DEFINITION 26.5. (1) Consider an occurrence $\overline{(i, a, \gamma)}$ in α. If $\bar{\beta}$ is an occurrence of β in γ, then the occurrence of i and of a in $\overline{(i, a, \gamma)}$ are said to be connected to $\bar{\beta}$ in α. We also say that the occurrence of the pair (i, a) in $\overline{(i, a, \gamma)}$ is connected to $\bar{\beta}$.

(2) Let $\bar{\beta}$ be an occurrence of β in α and let j be an element of I. If for every element i of I that occurs in α and is connected to $\bar{\beta}$ we have that $i \geqslant j$, then $\bar{\beta}$ is said to be j-active in α.

(3) A connected, j-active occurrence of a sub-ordinal diagram of α is called a j-subsection of α.

(4) Let $\overline{(i, a, \gamma)}$ be a j-subsection of α for some $j > i$. Then the occurrence $\bar{\gamma}$ in $\overline{(i, a, \gamma)}$ is called an i-section of α. If there is an i-section of α, then we say that i is an index of α.

Note that an i-section of α is a special case of an i-subsection. An i-section of α is an occurrence of a proper sub-ordinal diagram of α, but an i-subsection of α may be α itself.

For certain purposes ahead we introduce a special symbol ∞ that we adjoin to I and regard as the maximal element of the extended set.

DEFINITION 26.6. (1) $\tilde{I} = I \cup \{\infty\}$. The *ordering* of $\tilde{I}$ is that of I with ∞ the maximal element of $\tilde{I}$.

(2) If $i \in \tilde{I}$ and $j \in I$, then i is called a super index of j with respect to $\alpha, \beta, \ldots, \gamma$, if either i is ∞ or $i > j$ and i is an index of at least one of $\alpha, \beta, \ldots, \gamma$.

(3) $j_0(j, \alpha, \beta, \ldots, \gamma)$ is the least super index of j with respect to $\alpha, \beta, \ldots, \gamma$.

(4) $\iota(j, \alpha, \beta, \ldots, \gamma)$ denotes the number of super indices of j with respect to $\alpha, \beta, \ldots, \gamma$ when j is an element of I; it is defined to be 0 if j is ∞.

(5) The outermost index of the ordinal diagram (i, a, α) is i.

(6) A pair (i, a), where $i \in I$ and $a \in A$ is called a *value*. The set of values is ordered lexicographically.

(7) The value (i, a) is the outermost value of (i, a, α).

Note that a value is not an occurrence.

For each i in $\tilde{I}$ we now define an ordering $<_i$ of ordinal diagrams (based on I and A). The definition is by transfinite induction on $\omega \cdot l(\alpha, \beta) + \iota(i, \alpha, \beta)$.

DEFINITION 26.7. (1) $0 <_i \beta$ if $\beta \neq 0$.

(2) If α is $\alpha_1 \# \ldots \# \alpha_m$ and β is $\beta_1 \# \ldots \# \beta_n$, where $m + n > 2$, then $\alpha <_i \beta$ if one of the following conditions hold.

i) There exists a q such that $1 \leqslant q \leqslant n$ and $\alpha_p <_i \beta_q$ for all p with $1 \leqslant p \leqslant m$.

ii) $m = 1$, $n > 1$ and $\alpha_1 = \beta_q$ for some q with $1 \leqslant q \leqslant n$.

iii) $m > 1$, $n > 1$ and there exists a q and a p such that $1 \leqslant q \leqslant m$, $1 \leqslant p \leqslant n$, $\alpha_q = \beta_p$ and

$$\alpha_1 \# \ldots \# \alpha_{q-1} \# \alpha_{q+1} \# \ldots \# \alpha_m <_i \beta_1 \# \ldots \# \beta_{p-1} \# \beta_{p+1} \# \ldots \# \beta_n.$$

(3) If α and β are connected, if $i \neq \infty$ and if $j = j_0(i, \alpha, \beta)$, (cf. (2) of Definition 26.6) then $\alpha <_i \beta$ if one of the following holds.

i) There exists an i-section $\bar{\sigma}$ of β such that $\alpha \leqslant_i \sigma$, i.e., $\alpha <_i \sigma$ or $\alpha = \sigma$.

ii) $\alpha <_j \beta$ and for every i-section $\bar{\delta}$ of α, $\delta <_i \beta$.

(4) If $\alpha = (j, a, \gamma)$ and $\beta = (k, b, \delta)$, then $\alpha <_\infty \beta$ if

i) $j < k$ (in I) or

ii) $j = k$ and $a < b$ (in A) or

iii) $j = k$, $a = b$, and $\gamma <_j \delta$.

The ordering $<_\infty$ is slightly different from the original version in which a and b were compared first.

PROPOSITION 26.8. *For each i in $\tilde{I}$ the definition of $<_i$ is sound and $<_i$ is a linear ordering of the ordinal diagrams based on I and A.*

PROOF. By induction on $\omega \cdot l(\alpha, \beta, \gamma) + \iota(i, \alpha, \beta, \gamma)$ and $\omega \cdot l(\alpha, \beta) + \iota(i, \alpha, \beta)$, respectively, we can prove simultaneously that

I if $\alpha <_i \beta$ and $\beta <_i \gamma$, then $\alpha <_i \gamma$,

II if $\alpha = \beta$ and $\beta <_i \gamma$, then $\alpha <_i \gamma$ and if $\alpha <_i \beta$ and $\beta = \gamma$ then $\alpha <_i \gamma$,

III exactly one of $\alpha <_i \beta$, $\alpha = \beta$ and $\beta <_i \alpha$ holds.

If $l(\alpha, \beta, \gamma) = 0$ or $l(\alpha, \beta) = 0$, respectively, then $\alpha = \beta = \gamma = 0$ and hence I, II and III hold trivially. For $l(\alpha, \beta, \gamma) > 0$ and $l(\alpha, \beta) > 0$ we will present proofs only for some exemplary cases of I and III. We assume that α, β and γ are (j, a, α'), (k, b, β') and (m, c, γ'), respectively.

(1) $\iota(i, \alpha, \beta, \gamma) = \iota(i, \alpha, \beta) = 0$ and hence $i = \infty$.

I. If $\alpha <_\infty \beta$ and $\beta <_\infty \gamma$, it follows from the definition of $<_\infty$ that $j \leqslant m$ and $a \leqslant c$. If $j < m$ or $a < c$, then $\alpha <_\infty \beta$. If $j = m$ and $a = c$, then $j = k = m$ and $a = b = c$; hence $\alpha' <_j \beta'$ and $\beta' <_j \gamma'$. Then by the induction hypothesis $\alpha' <_j \gamma'$, hence $\alpha <_\infty \gamma$.

III. If $\alpha \neq \beta$, then $(j, a) \neq (k, b)$ or $(j, a) = (k, b)$ and $\alpha' \neq \beta'$. If $(j, a) \neq (k, b)$, then $\alpha <_\infty \beta$ or $\beta <_\infty \alpha$ by definition of $<_\infty$. If $(j, a) = (k, b)$, then by the induction hypothesis $\alpha' <_j \beta'$ or $\beta' <_j \alpha'$ and hence $\alpha <_\infty \beta$ or $\beta <_\infty \alpha$ as the case may be

(2) $\iota(i, \alpha, \beta, \gamma) > 0$ and $\iota(i, \alpha, \beta) > 0$, respectively.

I. We consider only one case: There exists an i-section of β, say $\bar{\sigma}$, such that $\alpha \leqslant_i \sigma$, and $\beta <_{i_0} \gamma$, where $i_0 = j_0(i, \beta, \gamma)$ and for each i-section of β, say $\bar{\delta}$, $\delta <_i \gamma$. Then $\alpha \leqslant_i \sigma$, $\sigma <_i \gamma$ and $l(\alpha, \sigma, \gamma) < l(\alpha, \beta, \gamma)$; hence by the induction hypothesis $\alpha <_i \gamma$.

III. Suppose $\alpha \not\leqslant_i \beta$, i.e., $\alpha \neq \beta$ and it is not the case that $\alpha <_i \beta$. Then

$$(*) \qquad \text{for every } i\text{-section of } \beta, \text{ say } \bar{\sigma}, \ \alpha \not\leqslant_i \sigma,$$

hence by the induction hypothesis $\sigma <_i \alpha$.

Let $i_0 = j_0(i, \alpha, \beta)$. If $\beta <_{i_0} \alpha$, then by (*) and Definition 26.7 $\beta <_i \alpha$. If $\beta \not<_{i_0} \alpha$ then since $\iota(i_0, \alpha, \beta) < \iota(i, \alpha, \beta)$ it follows from the induction hypothesis that $\alpha <_{i_0} \beta$. If for every i-section of α, say $\bar{\delta}$, $\delta <_i \beta$, then $\alpha <_i \beta$. But this contradicts our initial assumption. Therefore there is an i-section of α, say $\bar{\delta}$, such that $\beta \leqslant_i \delta$. But then by Definition 26.7 $\beta <_i \alpha$.

PROPOSITION 26.9. *If $\bar{\sigma}$ is an i-section of α, then $\sigma <_i \alpha$.*

PROOF. If σ is connected, then by (3)i) of Definition 26.7 applied to σ and the component of α in which $\bar{\sigma}$ is an i-section, $\sigma <_i \alpha$. If σ is not connected, then for each component of σ, say δ, $\delta <_i \sigma$ by (2)ii) of Definition 26.7 and hence by (3)i) of Definition 26.7, $\delta <_i \alpha$. Then from (2)i) it follows that $\sigma <_i \alpha$.

PROPOSITION 26.10. *Let α be a connected ordinal diagram and let $\bar{\beta}$ be a proper i-subsection of α. Then $\beta <_j \alpha$ for every $j \leqslant i$.*

PROOF. The proof is by induction on $\omega \cdot l(\alpha, \beta) + \iota(j, \alpha, \beta)$ for each $j \leqslant i$.

1. If $\beta = 0$, then $\beta <_j \alpha$ for all j.

2. Suppose $\alpha = (k, b, \gamma)$ and β is a component of γ. Then $i \leqslant k$ and since γ occurs as a k-section of α we have by Proposition 26.9 that $\beta \leqslant_k \gamma <_k \alpha$.

Consequently, $\beta <_k \alpha$. If $j < k$, then every j-section of β is a j-section of α. But this implies that for every $\bar{\sigma}$, a j-section of β, $\sigma <_j \alpha$, and hence $\beta <_j \alpha$ (by induction on $\iota(j, \alpha, \beta)$).

3. Suppose $\alpha = (k, b, \gamma)$ and β is not a component of γ. Then $k \geqslant i$ and there is a component of γ, say δ, such that β occurs as an i-subsection of δ. Therefore by the induction hypothesis $\beta <_j \delta$ for all $j \leqslant i$. But $\delta \leqslant_j \gamma$ by definition, and $\gamma <_j \alpha$ for every $j \leqslant k$; therefore, $j \leqslant i$ by 2. above and Definition 26.7(2). Consequently, $\beta <_j \delta$ if $j \leqslant i$.

PROPOSITION 26.11. *If β has an i-active occurrence as a proper subordinal diagram of α, then $\beta <_j \alpha$ for every $j \leqslant i$.*

PROOF. Apply Proposition 26.10 to each component of such an occurrence of β.

DEFINITION 26.12. Let α be an ordinal diagram and let i be an element of $\tilde{I}$. Then $[\alpha]_i = [\alpha_1, \ldots, \alpha_m]_i$ will mean that $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the components of α and

$$\alpha_1 \geqslant_i \alpha_2 \geqslant_i \cdots \geqslant_i \alpha_m.$$

DEFINITION 26.13. $O(I, A)$ will denote the structure consisting of the set of ordinal diagrams based on I and A and the orderings $<_i$ for all i in $\tilde{I}$.

We will follow the usual convention of using $O(I, A)$ to denote the universe of this structure, that is, the set of ordinal diagrams based on I and A.

Our next objective is to present a nonconstructive accessibility proof for $<_i$:

THEOREM 26.14. *For each i in $\tilde{I}$, $<_i$ well-orders $O(I, A)$.*

PROOF. This theorem will be proved as a sequence of lemmas using the notion of $<_i$-accessibility of ordinal diagrams.

DEFINITION 26.15. Let α be of the form $\alpha_1 \# \alpha_2 \# \ldots \# \alpha_m$ and β be of the form $\beta_1 \# \beta_2 \# \ldots \# \beta_n$. Then $\alpha \# \beta$ will denote the ordinal diagram

$$\alpha_1 \# \alpha_2 \# \ldots \# \alpha_m \# \beta_1 \# \beta_2 \# \ldots \# \beta_n.$$

DEFINITION 26.16. (1) Let B be a subset of $O(I, A)$ and let i be an element of $\tilde{I}$. An ordinal diagram α is $<_i$-accessible in B if α is an element of B and, with respect to $<_i$, there is no infinite (strictly) decreasing sequence of elements of B starting with α.

(2) α is $<_i$-accessible if α is $<_i$-accessible in $O(I, A)$.

DEFINITION 26.17. We define F_i a subset of $O(I, A)$ for every i in I.

(1) $F_0 = O(I, A)$.

(2) $F_{i+1} = \{\alpha \in F_i \mid \text{For every } \bar{\sigma} \text{ an } i\text{-section of } \alpha, \sigma \text{ is } <_i\text{-accessible in } F_i\}$, where $i + 1$ denotes the successor of i in I.

(3) $F_i = \bigcap_{j<i} F_j$ if i is a limit element of I.

From the definition it is obvious that if $\alpha \in F_i$, then $\alpha \in F_j$ for all $j \leqslant i$.

LEMMA 26.18. *Let B be a set of ordinal diagrams and let i be an element of $\tilde{I}$. If every element of B is $<_i$-accessible in B, then B is well-ordered by $<_i$.*

Lemma 26.18 assures us that for such a set B we may use transfinite induction on $<_i$.

LEMMA 26.19. *If every connected ordinal diagram is $<_i$-accessible, then every ordinal diagram is $<_i$-accessible.*

PROOF. It is sufficient to prove the following:

(*) If there is an infinite (strictly) $<_i$-decreasing sequence of ordinal diagrams starting with α, then there is an infinite $<_i$-decreasing sequence of connected ordinal diagrams starting with α_1 where $[\alpha]_i = [\alpha_1, \ldots, \alpha_m]_i$.

Let C be the set of all the connected ordinal diagrams. We will prove (*) by transfinite induction on α_1 along $<_i$ in C using Lemma 26.18. Let $\{\beta_n\}$ be a $<_i$-decreasing sequence, where $\beta_1 = \alpha$.

1) If all the components of α are 0, then all the components of each $\beta_1, \beta_2, \ldots$ are 0. Therefore the number of components must decrease from term to term in the infinite sequence $\{\beta_n\}$. Since this is impossible, α must have a nonzero component. Suppose $\alpha_1 > 0$.

2) Suppose $m = 1$. Then $\alpha_1 = \alpha = \beta_1 >_i \beta_2 >_i \ldots$. If α_1 is a limit element of C (with respect to $<_i$ relativized to C), then there is a connected ordinal diagram β such that $\alpha_1 >_i \beta >_i \beta_2 >_i \ldots$. Then by the induction hypothesis (applied to β) we can construct a $<_i$-decreasing sequence of connected ordinal diagrams starting with β. Adding α_1 to this sequence as a first term, we obtain a $<_i$-decreasing infinite sequence of connected ordinal diagrams starting with α_1.

3) Suppose $m = 1$ and α_1 is the successor of β in C (presuming that such a case is possible). Then $\sigma = \beta \# \beta \# \ldots \# \beta >_i \beta_2 >_i \ldots$ and $\beta <_i \alpha$. Consequently, by the induction hypothesis we can construct an infinite $<_i$-decreasing sequence of elements of C starting with β. Adding α_1 to the sequence as first term, we obtain the desired sequence.

4) Suppose $m > 1$ and there is an n such that β_n does not contain α_1 as a component. Let $[\beta_n]_i = [\beta_1^n, \ldots, \beta_l^n]_i$. Then $\beta_i^n <_i \alpha_1$. By the induction hypothesis there is a $<_i$-decreasing sequence of connected ordinal diagrams beginning with β_1^n. To this sequence we add α_1 as first term to obtain the desired sequence.

5) Suppose $m > 1$ and for every n, β_n contains α_1 as a component (hence as a maximum one). We prove (*) for this case by induction on the number of occurrences of α_1 in α. Let $\nu(\alpha)$ be the number of occurrences of α_1 in α. For each n, define β_n' to be the ordinal diagram obtained from β_n by deleting one occurrence of α_1. We define α' similarly: $\alpha' = \alpha_2 \# \ldots \# \alpha_m$. If $\nu(\alpha) = 1$, then α' does not have an occurrence of α_1 so by the induction hypothesis there is a $<_i$-decreasing sequence from C which starts with α_2. Adding α_1 to this sequence we obtain the desired sequence. If $\nu(\alpha) > 1$, then $0 < \nu(\alpha') < \nu(\alpha)$. If there is an n such that $\nu(\beta_n') = 0$ then 4) applies to $\{\beta_n'\}$. The resulting sequence starts with α_1. If there is no such n then 5) applies and by the induction hypothesis we can construct the desired sequence starting with $\alpha_2 (= \alpha_1)$. This completes the proof.

COROLLARY 26.20. *Let $[\alpha]_i = [\alpha_1, \alpha_2, \ldots, \alpha_m]_i$. Then α is $<_i$-accessible if and only if each $\alpha_1, \alpha_2, \ldots, \alpha_m$ is $<_i$-accessible in C.*

PROOF. The "only if" part is obvious. Suppose α is not $<_i$-accessible. Then by (*) in the proof of Lemma 26.19, α_1 is not $<_i$-accessible in C.

Lemma 26.19 assures us that in order to show that $<_i$ is a well-ordering of $O(I, A)$, it is sufficient to show that with respect to $<_i$, there is no infinite decreasing sequence of connected ordinal diagrams.

LEMMA 26.21. *An ordinal diagram α belongs to F_i if and only if each component of α belongs to F_i.*

PROOF (by transfinite induction on i). If $i = 0$ the result is obvious. Suppose $\alpha \in F_{i+1}$ but some component of α, say β, does not belong to F_{i+1}. If $\beta \notin F_i$ then by the induction hypothesis $\alpha \notin F_i$. But this is impossible since $\alpha \in F_{i+1}$

implies $\alpha \in F_i$. Therefore $\beta \in F_i$ and there is an i-section of β, say $\bar{\sigma}$, such that σ is not $<_i$-accessible in F_i. But such an i-section of β is also an i-section of α. Since this implies that $\alpha \notin F_{i+1}$ we have a contradiction.

Suppose all the components of α are in F_{i+1}. Let $\bar{\sigma}$ be an i-section of α. Then $\bar{\sigma}$ is an i-section of one of the components of α and σ is $<_i$-accessible in F_i. Therefore $\alpha \in F_{i+1}$.

For a limit element of I the conclusion is obvious from the induction hypothesis.

LEMMA 26.22. *If there exists an infinite sequence $\{\alpha_n\}$ of connected ordinal diagrams that is strictly decreasing with respect to $<_0$, then there exists an infinite sequence $\{\beta_n\}$ of connected ordinal diagrams such that*

1) *for all n, $\beta_n \in F_1$,*
2) *if $\alpha_1, \alpha_2, \ldots, \alpha_h \in F_1$, then $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_h = \beta_h$,*
3) *if $\alpha_1 = \beta_1, \ldots, \alpha_h = \beta_h$ but $\alpha_{h+1} \neq \beta_{h+1}$, then there is an occurrence of β_{h+1} in α_{h+1}, i.e., β_{h+1} is a subordinal diagram of α_{h+1},*
4) *$\{\beta_n\}$ is strictly decreasing with respect to $<_0$ and $<_1$.*

PROOF. Suppose $\alpha_1, \ldots, \alpha_h \in F_1$ and $\alpha_{h+1} \notin F_1$, where $h \geqslant 0$. Then let $\beta_1 = \alpha_1, \ldots, \beta_h = \alpha_h$. Since $\alpha_{h+1} \notin F_1$ there exists a 0-section of α_{h+1}, say $\bar{\gamma}$, for which γ is not $<_0$-accessible. Then by Corollary 26.20, γ has a component γ_0 that is not $<_0$-accessible. If $\gamma_0 \in F_1$, then let β_{h+1} be γ_0. If $\gamma_0 \notin F_1$, then there exists a 0-section of γ_0, say $\bar{\delta}$, for which δ is not $<_0$-accessible. After a finite number of such steps we will arrive at a β_{h+1} which is a connected sub-ordinal diagram of α_{h+1}, hence $\beta_{h+1} <_0 \alpha_{h+1}$, $\beta_{h+1} \in F_1$ and β_{h+1} is not $<_0$-accessible. This last property of β_{h+1} and Corollary 26.20 imply that there exists an infinite decreasing sequence of connected ordinal diagrams starting with β_{h+1}, say $\beta_{h+1} >_0 \beta'_{h+2} >_0 \ldots$. Then $\beta_1, \beta_2, \ldots, \beta_h, \beta_{h+1}, \beta_{h+2}, \ldots$ satisfies the same conditions as $\{\alpha_n\}$, namely it is an infinite decreasing sequence of connected ordinal diagrams.

We then apply the same process to the above sequence and produce $\beta_1, \ldots, \beta_h, \beta_{h+1}, \beta_{h+2}, \ldots$.

It is obvious that 1)–3) hold. Also $\{\beta_n\}$ is a decreasing sequence with respect to $<_0$. We next show that $\{\beta_n\}$ is decreasing with respect to $<_1$. Suppose for example $\beta_1 <_1 \beta_2$. Since β_1 and β_2 are connected it follows that if $\beta_2 <_0 \beta_1$, then there is a 0-section of β_1, say $\bar{\delta}$ such that $\beta_2 \leqslant_0 \delta$. But since $\beta_1 \in F_1$, δ is $<_0$-accessible while β_2 is not $<_0$-accessible. Since this is impossible, $\beta_2 <_0 \beta_1$ implies $\beta_2 <_1 \beta$.

In a similar manner we can show that $\beta_{l+1} <_0 \beta_l$ implies $\beta_{l+1} <_1 \beta_l$ for each l.

LEMMA 26.23. *If $\{\alpha_n\}$ is an infinite $<_i$-strictly decreasing sequence of connected elements of F_i, then there exists a sequence $\{\beta_n\}$ of connected ordinal diagrams for which*

1) *$\beta_n \in F_{i+1}$ for each n,*
2) *if $\alpha_1, \ldots, \alpha_h \in F_{i+1}$, then $\alpha_1 = \beta_1 \ldots \alpha_h = \beta_h$,*
3) *if $\alpha_1 = \beta_1, \ldots, \alpha_h = \beta_h$ but $\alpha_{h+1} \neq \beta_{h+1}$, then β_{h+1} is a (connected) sub-ordinal diagram of α_{h+1}.*
4) *$\{\beta_n\}$ is strictly decreasing with respect to $<_i$ and $<_{i+1}$.*

PROOF. Similar to that of Lemma 26.22.

LEMMA 26.24. *Suppose $\{\alpha_n\}$ is an infinite sequence of connected ordinal diagrams that is strictly decreasing with respect to $<_0$. Then for each i we can construct a sequence $\{\alpha_n^i\}_n$ of connected ordinal diagrams from F_i such that*

1) *for all n, $\alpha_n^0 = \alpha_n$.*
2) *$\{\alpha_n^i\}_n$ is strictly decreasing with respect to $<_i$.*
3) *if $\alpha_1^i = \alpha_1^{i+1}, \ldots, \alpha_h^i = \alpha_h^{i+1}$ and $\alpha_{h+1}^i \neq \alpha_{h+1}^{i+1}$, then α_{h+1}^{i+1} is a connected sub-ordinal diagram of α_{h+1}^i, hence $l(\alpha_{h+1}^{i+1}) < l(\alpha_{h+1}^i)$.*
4) *if i is a limit element of I, then*

$$\forall p\ \exists j \in I\ [j < i \wedge \forall l\ [j \leqslant l \leqslant i \rightarrow [\alpha_p^i = \alpha_p^l]]].$$

PROOF. If $\alpha_n^0 = \alpha_n$ for all n, then 1) and 2) obviously hold. Suppose we have constructed $\{\alpha_n^i\}_n$. Then, by Lemma 26.23, there exists a sequence of connected elements of F_{i+1}, $\{\alpha_n^{i+1}\}_n$ that is strictly decreasing with respect to $<_i$ and $<_{i+1}$; if $\alpha_1^i, \ldots, \alpha_h^i \in F_{i+1}$, then $\alpha_1^{i+1} = \alpha_1^i, \ldots, \alpha_h^{i+1} = \alpha_h^i$; and if $h+1$ is the first number such that $\alpha_{h+1}^{i+1} \neq \alpha_{h+1}^i$, then α_{h+1}^{i+1} is a (connected) sub-ordinal diagram of α_{n+1}^i.

We therefore only have to worry about the case where i is a limit element of I. Suppose for every $k < i$, $\{\alpha_n^k\}_n$ has been defined satisfying the condition. We claim that

$$(*) \qquad \forall h\ \exists j \in I\ [j < i \wedge \forall k\ [j \leqslant k < i \supset \alpha_h^k = \alpha_h^j]].$$

Suppose not. Then

$$\exists h\, \forall j \in I\, [j < i \supset \exists k\, [j \leqslant k < i \wedge \alpha_h^k \neq \alpha_h^j]].$$

There is then a smallest such h. We can then find an infinite sequence of j's, say $j_1, j_2, \ldots, j_p, \ldots$, such that

$$\alpha_h^{j_p} = \alpha_h^{j_p+1} = \alpha_h^{j_p+2} = \ldots = \alpha_h^{j_{p+1}} \neq \alpha_h^{j_{p+1}+1}.$$

This is possible because if we take $j = j_p$, then the least k such that $j < k < i$ and $\alpha_h^k \neq \alpha_h^j$ is a successor element (by 4) of the conditions of this lemma applied to any limit element which is less than i). Let $j_{p+1} + 1$ be this least k. Then by 3),

$$l(\alpha_h^{j_{p+1}+1}) < l(\alpha_h^{j_p+1}) = l(\alpha_h^{j_p+1}).$$

Thus we obtain a decreasing sequence of natural numbers. But this is impossible, consequently (*) must hold.

If we define $\{\alpha_h^i\}_h$ by $\alpha_h^i = \alpha_h^j$ for an appropriate j_h, then 4) holds. Furthermore, for each $k < i$, $\alpha_h^j \in F_k$ and $\alpha_h^i = \alpha_h^j \in \bigcap_{k<i} F_k = F_i$, that is, $\alpha_h^i \in F_i$.

Next we will prove that $\{\alpha_h^i\}_h$ is $<_i$-decreasing. For each $h = 1, 2, 3, \ldots$ it follows by the construction of $\{\alpha_h^i\}_h$ that there is a j_h and a j_{h+1} for which $\alpha_h^i = \alpha_h^k$ for every k such that $j_h \leqslant k < i$ and $\alpha_{h+1}^i = \alpha_{h+1}^k$ for every k such that $j_{h+1} \leqslant k < i$. Let $j_0 = \max(j_h, j_{h+1})$. Then $\alpha_h^i = \alpha_h^k$ and $\alpha_{h+1}^i = \alpha_{h+1}^k$ for each k such that $j_0 \leqslant k \leqslant i$. Consequently, $\alpha_{h+1}^i <_k \alpha_h^i$ for all such k, by the induction hypothesis. Suppose $\alpha_h^i <_i \alpha_{h+1}^i$. Then for each k such that $j_0 \leqslant k < i$ there must be a k' such that $k \leqslant k' < i$ and there is a k'-section of α_h^i, say $\tilde{\gamma}$, for which $\alpha_{h+1}^i \leqslant_{k'} \gamma$. But since the number of indices of α_0^i is finite, there is a maximum index in α_h^i, say k_0. Then if $k_0 < k < i$ (such a k exists) there is no way of getting $\alpha_{h+1}^i <_k \alpha_h^i$. But this is a contradiction; so $\alpha_h^i <_i \alpha_{h+1}^i$.

DEFINITION 26.25. $F_\infty = \bigcap_{j\in I} F_j$ if the order type of I is a limit ordinal.

$F_\infty = \{\alpha \in F_i \mid$ for each $\tilde{\delta}$ an i-section of α, δ is $<_i$-accessible in $F_i\}$ if i is the greatest element of I.

Clearly if $\alpha \in F_\infty$, then $\alpha \in F_i$ for each i.

LEMMA 26.26. *Suppose there is an infinite sequence of connected ordinal diagrams which is strictly decreasing with respect to* $<_0$. *Then there exists a sequence of connected ordinal diagrams from* F_∞ *that is strictly decreasing with respect to* $<_\infty$.

PROOF. Similar to that of Lemma 26.24 with ∞ instead of i. If I has a maximal element, then the first half of the proof goes through, if I does not have a maximal element, then the latter half of the proof goes through.

LEMMA 26.27. *If there is a sequence of connected ordinal diagrams from F_j that is strictly decreasing with respect to $<_j$, then there exists a sequence of connected ordinal diagrams from F_∞ that is strictly decreasing with respect to $<_\infty$.*

PROOF. Similar to Lemma 26.24 with F_j in the place of F_0.

LEMMA 26.28. *Every element of F_∞ is $<_\infty$-accessible in F_∞.*

PROOF. Suppose not, that is, suppose there is a sequence $\{\alpha_n\}$ of elements of F_∞ that is strictly decreasing with respect to $<_\infty$. We may assume that the α_n's are connected (Lemma 26.19). Recall that for $<_\infty$, ordinal diagrams are first compared by their outermost values (i, a). Since those values are well ordered (cf. (5) and (6) of Definition 26.6), a decreasing sequence of values must be finite. Thus after a certain stage, the outermost values of α_n will be constant, say (i, a). Then the α_n's are compared by their i-sections for all large n. If i is the maximal element in I, then $\alpha_n \in F_\infty$ means that if $\bar{\delta}$ is an i-section of α_n, then δ is $<_i$-accessible in F_i. If i is not the maximal element of I, then $\alpha_n \in F_{i+1}$ and hence if $\bar{\delta}$ is an i-section of α_n then δ is $<_i$-accessible in F_i. Therefore, the comparison of α_n with respect to $<_\infty$ is reduced to the comparison of i-sections of α_n which are $<_i$-accessible in F_i. Therefore, there cannot be an infinite decreasing sequence of such i-sections (cf. Lemma 26.18), hence $\{\alpha_n\}$ cannot be strictly decreasing with respect to $<_\infty$.

Next we will prove that, for every i, $F_i = F_0 = O(I, A)$ and that every ordinal diagram is $<_i$-accessible in F_i, hence is $<_i$-accessible.

LEMMA 26.29. *$F_i = O(I, A)$ and every ordinal diagram is $<_i$-accessible for all i in $\tilde{I}$.*

PROOF. $F_0 = O(I, A)$, by definition. Suppose there exists an infinite sequence of connected ordinal diagrams which is strictly decreasing with respect to $<_0$. Then by Lemma 26.26 there is an infinite sequence of connected elements of F_∞ which is decreasing with respect to $<_\infty$. But this contradicts Lemma 26.28. Consequently, there cannot be such a sequence. Therefore, by virtue of Lemma 26.19, all ordinal diagrams are $<_0$-accessible.

Suppose that $F_i = O(I, A)$ and the $<_i$-accessibility of all ordinal diagrams has been proved. Then for each element of F_i each of its i-sections are $<_i$-accessible, hence $F_{i+1} = F_i = O(I, A)$. Using Lemmas 26.27, 26.28 and

26.19, we can show, in a similar manner as above, that every ordinal diagram is $<_{i+1}$-accessible in F_{i+1}, hence is $<_{i+1}$-accessible. For a limit j, $F_j = \bigcap_{i<j} F_i = O(I, A)$, by the induction hypothesis. That every ordinal diagram is $<_j$-accessible can be proved as for $i + 1$. The proof is valid also when $i + 1$ or j is ∞.

Reversing the processes in the foregoing argument we obtain a somewhat constructive proof due to A. Kino which we outline below. First we shall list a few preliminary lemmas.

LEMMA 26.30. *If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are $<_i$-accessible in F_i, then $\alpha_1 \# \alpha_2 \# \ldots \# \alpha_n$ is $<_i$-accessible in F_i.*

PROOF. Relativize the proof of Lemma 26.18 to F_i.

LEMMA 26.31. *Let β be an element of F_i and let $\bar{\gamma}$ be an i-subsection of β. Then $\gamma \in F_i$.*

PROOF. (By transfinite induction on i.) For $i = 0$ the result is obvious. If i is a limit element then the result follows from the induction hypothesis since an i-subsection is a j-subsection if $j < i$. If $i = j + 1$, then since $\beta \in F_i$, $\beta \in F_j$ and for every j-section of β, say $\bar{\delta}$, δ is $<_j$-accessible in F_j which implies that $\delta \in F_j$. If $\bar{\gamma}$ is an i-subsection of β, then it is a j-subsection of β, and hence $\gamma \in F_j$. Let $\bar{\delta}$ be a j-section of γ. Then $\bar{\delta}$ is a j-section of β and hence δ is $<_j$-accessible in F_j. Therefore, by definition, $\gamma \in F_i$.

We now begin the proof of accessibility.

LEMMA 26.32. *For each α in F_{i+1} if α is $<_{i+1}$-accessible in F_{i+1}, then α is $<_i$-accessible in F_i.*

PROOF. (By transfinite induction on $<_{i+1}$ for those ordinal diagrams which are $<_{i+1}$-accessible in F_{i+1}. See Lemma 26.18.)

Consider any β in F_i for which $\beta <_i \alpha$. We will prove that β is $<_i$-accessible in F_i and hence α is $<_i$-accessible in F_i. Note that by definition $\alpha \in F_i$.

Let $\Gamma(= \Gamma_\beta)$ be the set of sub-ordinal diagrams of β defined inductively as follows:

1) Each component of β belongs to Γ.

2) If $\gamma \in \Gamma$, then every component of every i-section of γ belongs to Γ.

3) Only those ordinal diagrams satisfying 1) and 2) belong to Γ.

It is obvious from the definition and Lemma 26.31 that

$$(*) \qquad \Gamma \subseteq F_i.$$

We now wish to prove by induction on $l(\gamma)$ that

$$(**) \qquad \gamma \in F_{i+1} \text{ and } \gamma \text{ is } <_i\text{-accessible in } F_i \text{ for each } \gamma \text{ in } \Gamma.$$

Then, as a special case, every component of β is $<_i$-accessible in F_i. So, by Lemma 26.30, β is $<_i$-accessible in F_i. This will complete the proof of the lemma.

We now turn to the proof of (**).

If $\gamma \in \Gamma$ is minimal in the sense that γ has no i-section, then $\gamma \in F_i$ implies $\gamma \in F_{i+1}$. By (*), $\gamma \in F_i$.

Case 1. $\gamma <_{i+1} \alpha$. Then γ is $<_{i+1}$-accessible in F_{i+1}. Consequently, by the induction hypothesis, γ is $<_i$-accessible in F_i.

Case 2. $\alpha <_{i+1} \gamma$; $\gamma \leqslant_i \beta <_i \alpha$. Then there is an i-section of α, say $\bar{\sigma}$, such that $\gamma \leqslant_i \sigma$. But since $\alpha \in F_{i+1}$, σ is $<_i$-accessible in F_i, and hence so is γ.

If $\gamma \in \Gamma$ is not minimal, let δ be a component of an i-section of γ. Then by the induction hypothesis, $\delta \in \Gamma$, so δ is $<_i$-accessible in F_i. Therefore, by Lemma 26.30, if $\bar{\tau}$ is an i-section of γ then τ is $<_i$-accessible in F_i. But this means that $\gamma \in F_{i+1}$. Repeating cases 1 and 2 above we conclude that γ is $<_i$-accessible in F_i.

LEMMA 26.33. *Let i be a limit element. Then*

$$\forall j < i\, \forall \beta\, [(\beta \textit{ is } <_j\textit{-accessible in } F_j) \supset \forall k < j (\beta \textit{ is } <_k\textit{-accessible in } F_k)]$$

$$\supset \forall k < i\, \forall \alpha\, [(\alpha \textit{ is } <_i\textit{-accessible in } F_i) \supset (\alpha \textit{ is } <_k\textit{-accessible in } F_k)].$$

PROOF. (By transfinite induction on the elements of F_i which are $<_i$-accessible in F_i. See Lemma 26.18).

Let i_0 be the greatest index of α which is less than i (cf. (4) of Definition 26.5) assuming that such an index exists. Then assuming the premise of the lemma it suffices to show that

(*) for each α that is $<_i$-accessible in F_i, and for every k such that $i_0 < k < i$, α is $<_k$-accessible in F_k.

If no such i_0 exists, then we prove (*) for all $k < i$. Indeed, if (*) holds and $k \leqslant i_0$, then there is a j such that $k \leqslant i_0 < j < i$ (since i is a limit element) and α is $<_j$-accessible in F_j. So by the premise of our lemma, α is $<_k$-accessible in F_k. We can prove (*) by transfinite induction on such α's along $<_i$.

We may assume that i_0, as above, can be defined. Note that for each k and j for which $i_0 < k < i$ and $k \leqslant j \leqslant i$, if $\beta <_k \alpha$, then $\beta <_j \alpha$ because there is no h-section of α when $k \leqslant h < i$. In order to prove (*) for any α, let β be a member of F_k such that $\beta <_k \alpha$. If we can show that any such β is $<_k$-accessible in F_k we are done. We will prove that

(**) If $\bar{\gamma}$ is a k-subsection of β, then $\gamma \in F_i$ and γ is $<_j$-accessible in F_j for all j such that $k \leqslant j < i$.

As a special case of (**), β is $<_k$-accessible in F_k. We also know that for any $\bar{\gamma}$ for which (**) holds, $\gamma \in F_k$ because $\beta \in F_k$ (cf. Lemma 26.31).

Suppose $\bar{\gamma}$ is a minimal k-subsection of β, i.e., γ does not contain a k-subsection, then $\gamma \in F_i$ because $\gamma \in F_k$, and there is no j-section of γ for any j between k and i.

Case 1. $\gamma <_i \alpha$. Then γ is $<_i$-accessible in F_i since α is, and, by the induction hypothesis γ is $<_j$-accessible for all j such that $k \leqslant j < i$.

Case 2. $\alpha <_i \gamma$. Then $\alpha <_j \gamma$ for every j such that $i_0 < j < i$. In particular, $\alpha <_k \gamma$. But since $\bar{\gamma}$ is a k-subsection of β, $\gamma \leqslant_k \beta <_k \alpha$. Since this is a contradiction this case is impossible.

Next, suppose that $\bar{\gamma}$ is a k-subsection of β which is not a minimal one. Then $\gamma \in F_k$. Let $k \leqslant j < i$ and let $\bar{\delta}$ be a j-section of γ. Then $\bar{\delta}$ is a k-subsection of β and hence by the induction hypothesis ($\delta \in F_i$ and) δ is $<_j$-accessible in F_j. This is true for every such j. So $\gamma \in F_k$ implies $\gamma \in F_i$. Repeating Cases 1 and 2 above, we can prove that γ is $<_j$-accessible in F_j for all j such that $k \leqslant j < i$. This completes the proof.

LEMMA 26.34. *For every i in $\bar{I}$ and every α, if α is $<_i$-accessible in F_i, then $\forall j < i$ [α is $<_j$-accessible in F_j].*

PROOF. (By transfinite induction on i.) Suppose α is $<_i$-accessible in F_i. If $i = k + 1$, then by Lemma 26.32 α is $<_k$-accessible in F_k. So by the induction hypothesis α is $<_j$-accessible in F_j for all $j < k$.

Suppose i is a limit element. For every $k < i$ the lemma holds by the induction hypothesis. This means that the premise in Lemma 26.33 holds. Consequently, the conclusion of Lemma 26.33 holds. But this is the result we were to prove.

LEMMA 26.35 (cf. Lemma 26.28). *Every member of F_∞ is $<_\infty$-accessible in F_∞.*

LEMMA 26.36. *Every member of F_∞ is $<_i$-accessible in F_i for every i in I.*

PROOF. Every ordinal diagram of F_∞ is $<_\infty$-accessible in F_∞ by Lemma 26.35. Hence by Lemma 26.34 it is $<_i$-accessible in F_i for every i in I.

LEMMA 26.37. *Every ordinal diagram of F_j is $<_j$-accessible in F_j for every j, hence, in particular, F_j is well-ordered by $<_j$.*

PROOF. Let α be an element of F_j.

Case 1. I has no maximal element. Let i be an element of I which is greater than all the elements of I occurring in α. Then $(i, 0, 0) \in F_\infty$ and $\alpha <_j (i, 0, 0)$. By Lemma 26.36, $(i, 0, 0)$ is $<_j$-accessible in F_j, hence so is α.

Case 2. I has a maximal element but A does not. Let i be the greatest element of I occurring in α and let a be an element of A which is greater than any element of A that occurs in α. Then $(i, a, 0) \in F_\infty$ and $\alpha <_j (i, a, 0)$. By Lemma 26.36 $(i, a, 0)$ is $<_j$-accessible, hence so is α.

Case 3. Both I and A have maximal elements. Let i and a be the greatest elements in I and A, respectively. Then there is a β of the form

$$(i, a, (i, a, \ldots, (i, a, 0)\ldots))$$

such that $\alpha <_j \beta$. If we can show that $\beta \in F_\infty$, then it will follow from Lemma 26.36 that β is $<_j$-accessible in F_j. From this in turn it follows that α is $<_j$-accessible in F_j and this will complete the proof.

If $\beta = (i, a, 0)$, then obviously $\beta \in F_\infty$. Suppose

$$\beta_0 = (i, a, \ldots (i, a, 0)\ldots) \in F_\infty.$$

Then by Lemma 26.36, β_0 is $<_i$-accessible in F_i. Therefore, by definition, $\beta = (i, a, \beta_0) \in F_\infty$.

As a special case of Lemma 26.37 we have the following.

THEOREM 26.38. *Every ordinal diagram is $<_0$-accessible.*

REMARK. As an alternate proof, case 3 of Lemma 26.37 can take the following form. Let a_0 be a new symbol. Define $\tilde{A}$ to be $A \cup \{a_0\}$, where $\tilde{A}$ is ordered as A with a_0 greater than every element of A. We then define $O(I, \tilde{A})$ and its orderings $\precsim_i$, one for each i in $\tilde{I}$. Then $O(I, A)$ is a subsystem of $O(I, \tilde{A})$ and $\precsim_i$ is an extension of $<_i$.

Let i be the largest element of I occurring in α. Then $\alpha \gtrsim_0 (i, a_0, 0)$ and $(i, a_0, 0)$ is $\gtrsim_0$-accessible in $O(I, \tilde{A})$. Therefore, α is $\gtrsim_0$-accessible in $O(I, \tilde{A})$. But this implies that α is $<_0$-accessible (in $F_0 = O(I, A)$).

When this method is employed, we need only a weaker version of Lemmas 26.36 and 26.37:

LEMMA 26.39 (cf. Lemma 26.36). *Every member of F_∞ is $<_0$-accessible.*

LEMMA 26.40 (cf. Lemma 26.37). *Every ordinal diagram is $<_0$-accessible.*

The proof of accessibility depends strongly on F_i which is a highly abstract notion. If one wishes to justify this proof he should interpret the definition of F_i as follows: $\alpha \in F_{i+1}$ if and only if it can be concretely assured that $\alpha \in F_i$ and for every $\bar{\delta}$, an i-section of α, δ is $<_i$-accessible in F_i. (It can be concretely assured that δ is $<_i$-accessible in F_i.)

One problem with our system of ordinal diagrams is that the order relations $<_i$ are defined by induction on $\omega \cdot l(\alpha, \beta) + \iota(i, \alpha, \beta)$ and hence the structure of those order relations is not clearly understood. It is very important to be able to visualize the proof of accessibility if we are to claim that the system of ordinal diagrams provides a good basis for the study of foundations.

We would like to emphasize that neither the first nor the second proof of accessibility given above is not very constructive. We will present a more constructive proof in a soon to be published papers entitled "Fundamental sequences of ordinal diagrams" and "An accessibility proof of ordinal diagrams".

In order to explain the difficulty we would like to review from Chapter 2 some of the ideas in the accessibility proof for the ordinals up to ε_0. In Chapter 2, several useful devices were taken for granted. For example, we can assign a natural number n to each ordinal $\alpha < \varepsilon_0$, with $\omega_n \leqslant \alpha < \omega_{n+1}$. This number n, the *height* of α, gives a rough indication of the "size" of α. For two ordinals α, $\beta < \varepsilon_0$ we can define $\alpha < \beta$ by, firstly, comparing the height of α with the height of β, and, secondly, assuming that α and β have the same height and that $\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \ldots + \omega^{\beta_n}$ with

$$\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n \quad \text{and} \quad \beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_m,$$

by comparing α_1 with β_1, α_2 with β_2, etc. For ordinal diagrams the approximation theory corresponding to this approach is rather involved but we will develop it at the rest of this section, though they are not necessary after this section.

There is another difficulty. We defined in Chapter 2 $\alpha_1 < \alpha_2 < \dots$ to be a fundamental sequence of α if α is the limit of the sequence. Then for an $\alpha < \varepsilon_0$ we had a simple uniform method for constructing fundamental sequences for α. However for ordinal diagrams the construction of a fundamental sequence and the proof of its basic properties are very involved. Nevertheless, all these ideas are necessary for an accessibility proof in the style of Chapter 2.

The major problem of constructive mathematics lies in the complications of the expressions and the descriptions in the arguments. This is due to the fact that in constructive mathematics we must constantly take into account delicate distinctions.

Now we shall explain the theory of approximations. That is, given an element j of I and a connected ordinal diagram α, we are to define the (n, k)th j-approximation of α for $n, k = 0, 1, 2, \dots$, and see that they present good criteria for the comparison of two ordinal diagrams with respect to j.

We shall now define j-valuations and j-approximations of α for every $j \in I$ and every connected nonzero ordinal diagram α.

DEFINITION 26.41. (1) When (i, a) is the outermost value of a connected ordinal diagram, i is its *outermost index*.

(2) Let $\bar{\gamma}$ be a j-subsection of α where the outermost index of γ is $< j$. Then we say that $\bar{\gamma}$ is a *j-kernel* of α. We include $\overline{0}$ as a j-kernel.

DEFINITION 26.42. Let $v_0(j, \alpha)$ be the maximum of the outermost values of the ordinal diagrams represented by the j-subsections of α. Then $v_0(j, \alpha)$ is called the 0^{th} *j-valuation* of α.

Note that every non-zero connected ordinal diagram has a 0^{th} j-valuation.

PROPOSITION 26.43. *Let β and α be non-zero connected ordinal diagrams. If $v_0(j, \beta) < v_0(j, \alpha)$ then $\beta <_j \alpha$.*

PROOF. Let $v_0(j, \alpha) = (i, a)$. For any j-subsection of α, say $\bar{\delta}$, $\delta \leqslant_j \alpha$ (cf. Proposition 26.10). Therefore it suffices to show that $\beta <_j (i, a, \gamma)$ for any j-subsection of α of the form $\overline{(i, a, \gamma)}$ (i.e., the outermost value of the ordinal diagram represented by it is (i, a)). We shall, however, show that
(*) $\eta <_m (i, a, \gamma)$ for all $m \geqslant j$ and each $\bar{\eta}$ a j-subsection of β.

Then as a special case we have $\beta <_j (i, a, \gamma)$.

The proof of (*) is by induction on $l(\eta)$. Note that $v_0(j, \eta) \leqslant v_0(j, \beta) < (i, a)$.

(1) $\bar{\eta}$ is a j-kernel of β. Either η is 0 or of the form (k, b, η') where $k < j$. Then obviously $\eta <_m (i, a, \gamma)$ for every $m \geqslant j$.

(2) η is of the form (k, b, η'), where $k \geqslant j$. Since $(k, b) < (i, a)$, $\eta <_m (i, a, \gamma)$ if $m > k$. Let $j \leqslant m \leqslant k$ and let $\bar{\delta}$ be an m-section of $\bar{\eta}$. Let $\bar{\delta}_0$ be any component of $\bar{\delta}$. Then $\bar{\delta}_0$ is a j-subsection of β; hence by the induction hypothesis $\delta_0 <_m (i, a, \gamma)$. This implies that $\delta <_m (i, a, \gamma)$ so by induction on $\iota(m, \eta)$, $\eta <_m (i, a, \gamma)$ for all m for which $j \leqslant m \leqslant k$.

DEFINITION 26.44. (1) Let α be a connected ordinal diagram and let $(i, a) = v_0(j, \alpha)$. Consider any j-subsection of α whose outermost value is (i, a), say $\overline{(i, a, \gamma)}$. Let $\mathrm{apr}(0, j, \alpha)$ be the greatest such (i, a, γ) with respect to $<_i$. Any j-active occurrence of $\mathrm{apr}(0, j, \alpha)$, say $\overline{\mathrm{apr}(0, j, \alpha)}$, is called a 0th *$j$-approximation* of α.

We shall use α_0 as an abbreviation for $\mathrm{apr}(0, j, \alpha)$. There may, of course, be many occurrences of α_0 that are not 0th j-approximations of α. However, we are not interested in such occurrences. Therefore, for notational convenience we will hereafter use the symbols $\bar{\alpha}_0$ and $\overline{\mathrm{apr}(0, j, \alpha)}$ only for occurrences that are 0th j-approximations of α.

(2) If a j-subsection of α, say $\bar{\eta}$, does not contain an $\bar{\alpha}_0$, a j-active occurrence of $\mathrm{apr}(0, j, \alpha)$, and is not contained by any $\bar{\alpha}_0$, then we say that $\bar{\eta}$ j-omits $\bar{\alpha}_0$. When j is understood we will say simply that $\bar{\eta}$ omits $\bar{\alpha}_0$.

LEMMA 26.45. (1) $\overline{\mathrm{apr}(0, j, \alpha)}$ *is a j-subsection of* α.

(2) *Let* $\bar{\alpha}' = \overline{(i, a, \delta)}$ *be a j-subsection of α whose outermost value is (i, a) and which is different from α_0. Then $\alpha' <_i \alpha_0$, and hence $\delta <_i \gamma$ (where $\alpha_0 = (i, a, \gamma)$). This implies that $\alpha' <_m \alpha_0$ for all $m \geqslant i$.*

(3) *If $\bar{\eta}$ is a j-kernel of α which is not $\bar{\alpha}_0$, then $\eta <_m \alpha_0$ for all $m \geqslant j$.*

(4) *If $\bar{\eta}$, a j-subsection of α, j-omits $\bar{\alpha}_0$, then $\eta <_m \alpha_0$ for all $m \geqslant j$.*

(5) *Let $(i, a) = v_0(j, \alpha)$ and $\bar{\delta}_1, \bar{\delta}_2, \ldots, \bar{\delta}_m$ be all the j-subsections of α. Then*

$$\alpha_0 = \mathrm{apr}(0, j, \alpha) = \max_{<_{i+1}}(\delta_1, \delta_2, \ldots, \delta_m).$$

PROOF. (3) Let $\eta = (k, b, \eta')$ and $k < j$. If $j \leqslant i$, then evidently $\eta <_m \alpha_0$ for all $m \geqslant j$. Suppose $j > i$. Then $\alpha = \alpha_0$ and there is no j-kernel except α_0.

(4) (i) $\bar{\eta}$ is a j-kernel of α. Then $\eta <_m \alpha$ for all $m \geqslant j$ by (3).

(ii) Let $\eta = (k, b, \eta')$, where $k \geqslant j$. If $(k, b) = (i, a)$, then by (2) $\eta <_m \alpha_0$ for all $m \geqslant i$. If $j \geqslant i$ this will do. Suppose $j < i$. Let $j \leqslant m < i$ and let $\bar{\delta}$ be a component of an m-section of η. Then $\bar{\delta}$ is a j-subsection of α which j-omits $\bar{\alpha}_0$. Therefore by the induction hypothesis $\delta <_m \alpha_0$ for all $m \geqslant j$. By induction on $\iota(m, \eta)$ we can then prove that $\eta <_m \alpha_0$. If $(k, b) < (i, a)$, then $\eta <_m \alpha$, for all $m > k$. Let $j \leqslant m \leqslant k$, and let $\bar{\delta}$ be a component of an m-section of η. Then by the induction hypothesis $\delta <_m \alpha_0$. From this in turn it follows that $\eta <_m \alpha_0$.

PROPOSITION 26.46. *Let α and β be connected ordinal diagrams, where $v_0(j, \alpha) = v_0(j, \beta) = (i, a)$ and* $\mathrm{apr}(0, j, \beta) <_i \mathrm{apr}(0, j, \alpha)$. *Then $\beta <_j \alpha$.*

PROOF. Since $\bar{\alpha}_0$ is a j-subsection of α, it suffices to show that $\beta <_j \alpha_0$, for then $\beta <_j \alpha_0 \leqslant_j \alpha$. We can easily show that

(1) If $\bar{\eta}$ is a j-subsection of β, then $v_0(j, \eta) \leqslant (i, a)$.

Furthermore, if $v_0(j, \eta) = (i, a)$, then $\mathrm{apr}(0, j, \eta) <_i \alpha_0$, and

(2) for any two ordinal diagrams of the form (i, a, δ) and (i, a, γ), $(i, a, \delta) <_i (i, a, \gamma)$ implies $(i, a, \delta) <_m (i, a, \gamma)$ for all $m \geqslant i$.

Using (1) and (2) we shall prove by induction on $l(\eta)$

(3) for any j-subsection of β, say $\bar{\eta}, \eta <_m \alpha_0$ for all $m \geqslant j$.

As a special case of (3) we have $\beta <_j \alpha_0$.

1) $\bar{\eta}$ is a j-kernel of β. Then η is 0, or η is of the form (k, b, η'), where $k < j$. Then $(k, b) \leqslant (i, a)$. If $(k, b) < (i, a)$, then $\eta <_m \alpha_0$ for all $m > k$, hence for all $m \geqslant j$. If $(k, b) = (i, a)$, then $\eta \leqslant_i \beta_0 <_i \alpha_0$, hence by (2) above, $\eta <_m \alpha_0$ for all $m \geqslant i$. Since $k = i$ and $k < j$, we have $j > i$. So $\eta <_m \alpha_0$ for all $m \geqslant j$.

2) η is of the form (k, b, η'), where $k \geqslant j$ and $(k, b) < (i, a)$. It is obvious that $\eta <_m \alpha_0$ if $m > k$. Let $j \leqslant m \leqslant k$ and let $\bar{\delta}$ be a component of an m-section of η. Then by the induction hypothesis $\delta <_m \alpha_0$ for all $m \geqslant j$. Therefore, by induction on $\iota(m, \eta)$, $\eta <_m \alpha_0$ for all m such that $j \leqslant m \leqslant k$.

3) η is of the form (i, a, η'). Since $\eta \leqslant_i \beta_0 <_i \alpha_0$ it follows from (2) that $\eta <_m \alpha_0$ for all $m \geqslant i$. If $j \geqslant i$ this will do. Suppose $j < i$. Consider any m such that $j \leqslant m < i$ and suppose $\bar{\delta}$ is a component of an m-section of η. Then by the induction hypothesis $\delta <_m \alpha_0$ for all $m \geqslant j$. From this it follows that $\eta <_m \alpha_0$ for all such m.

PROPOSITION 26.47. *Let $\bar{\eta}$ be a j-subsection of α that contains an $\bar{\alpha}_0$. Suppose in addition that for each $\bar{\alpha}_0$ in $\bar{\eta}$ there is an occurrence of an element of I that is less than i and connected to $\bar{\alpha}_0$. Let $\bar{\alpha}_0^1, \bar{\alpha}_0^2, \ldots, \bar{\alpha}_0^m$ be all such occurrences of*

α_0 in $\bar{\eta}$ and let q_k be the least such element of I for $\bar{\alpha}_0^k$ as described above. Define $q = q(\eta) = \max(q_1, \ldots, q_m)$. Then $\eta <_p \alpha_0$ for every p such that $q < p \leqslant i$.

PROOF. First note that $q \geqslant j$. Since $\bar{\eta}$ is a j-subsection of α it can be easily shown that $v_0(p, \eta) \leqslant v_0(j, \alpha)$ $(= (i, a))$ for every $p \geqslant j$, and in particular for a p such that $q < p \leqslant i$. Also $v_0(p, \alpha_0) = (i, a)$ and $\mathrm{apr}(0, p, \alpha_0) = \alpha_0$. So if $v_0(p, \eta) < (i, a)$, then $\eta <_p \alpha_0$ by Proposition 26.43. Suppose $v_0(p, \eta) = (i, a)$. Then $\mathrm{apr}(0, p, \eta) <_i \alpha_0 = \mathrm{apr}(0, p, \alpha_0)$, for $\bar{\alpha}_0$ is not p-active in $\bar{\eta}$ (cf. (2) of Lemma 26.45). Therefore $\eta <_p \alpha_0$.

DEFINITION 26.48. Let $\bar{\gamma}$ be a j-subsection of α for which there is an $\bar{\alpha}_0$ in γ as a j-subsection of γ and such that i is the only element of I that occurs in γ and is connected to $\bar{\alpha}_0$. (Namely, such an $\bar{\alpha}_0$ is i-active in $\bar{\gamma}$.) Let $\mathrm{apr}(1, j, \alpha)$ be the greatest with respect to $<_i$ of such γ. Any such occurrence of $\mathrm{apr}(1, j, \alpha)$ is called a *first j-approximation* of α.

We will use the symbol α_1 as an abbreviation for $\mathrm{apr}(1, j, \alpha)$. Hereafter we will use the symbols $\bar{\alpha}_1$ and $\overline{\mathrm{apr}(1, j, \alpha)}$ only for occurrences that are first j-approximations of α.

Note that according to the definition, $\alpha_0 = \alpha_1$ is possible.

LEMMA 26.49. *Let $v_0(j, \alpha) = (i, a)$, $\alpha_0 = \mathrm{apr}(0, j, \alpha)$ and $\alpha_1 = \mathrm{apr}(1, j, \alpha)$.*

(1) *If $\bar{\alpha}_1$ properly contains an $\bar{\alpha}_0$, then $j \leqslant i$ and $\alpha_0 <_k \alpha_1$ for every $k \leqslant i$.*

(2) *If (i, b, δ) is a sub-ordinal diagram of α_1 such that δ contains an $\bar{\alpha}_0$ i-active, then $b < a$.*

(3) *$\bar{\alpha}_1$ is "maximal" in the sense that if $\overline{(k, c, \gamma)}$ is a j-subsection of α, where $\bar{\alpha}_1$ is a component of $\bar{\gamma}$, then $k < i$.*

DEFINITION 26.50. Let $\bar{\eta}$ be a j-subsection of α. If $\bar{\eta}$ neither contains an $\bar{\alpha}_1$ nor is contained by an $\bar{\alpha}_1$ and is not properly contained by any $\bar{\alpha}_0$, then $\bar{\eta}$ is said to *j-omit* $\bar{\alpha}_1$. When j is understood we will say simply that $\bar{\eta}$ omits $\bar{\alpha}_1$.

PROPOSITION 26.51. *If a j-subsection of α, say $\bar{\eta}$, omits $\bar{\alpha}_1$, then $\eta <_k \alpha_1$ for all k such that $j \leqslant k \leqslant i$.*

PROOF. (By induction on $l(\eta)$.)

1) $\bar{\eta}$ omits $\bar{\alpha}_0$. Then $\eta <_k \alpha_0$ for all $k \geqslant j$ by (4) of Lemma 26.45. If $j \leqslant k \leqslant i$, then $\alpha_0 <_k \alpha_1$ (see (1) of Lemma 26.49), hence $\eta <_k \alpha_1$.

2) $\eta = \alpha_0 <_k \alpha$ if $j \leqslant k \leqslant i$.

3) $\bar{\eta}$ contains $\bar{\alpha}_0$ and i is the only element of I that occurs in η and is connected to $\bar{\alpha}_0$. Then $\eta <_i \alpha_1$ by definition of α_1. Let $j \leqslant k < i$ and let $\bar{\delta}$ be a component of a k-section of $\bar{\eta}$. If $\bar{\delta}$ omits $\bar{\alpha}_0$ or contains $\bar{\alpha}_0$, then $\bar{\delta}$ omits $\bar{\alpha}_1$, so by the induction hypothesis $\delta <_k \alpha_1$, if $\bar{\delta}$ is contained in $\bar{\alpha}_0$, then $\bar{\delta}$ is a k-subsection of α_0, hence $\delta <_k \alpha_0 <_k \alpha_1$. So $\delta <_k \alpha_1$ in any case, and by induction on $\iota(k, \eta)$, $\eta <_k \alpha_1$.

4) $\bar{\eta}$ contains $\bar{\alpha}_0$ and for every occurrence of $\bar{\alpha}_0$ there is an element of I that has an occurrence in $\bar{\eta}$ connected to $\bar{\alpha}_0$ and which is less than i. By Proposition 26.47, $\eta <_i \alpha_0 <_i \alpha_1$. For a k satisfying $j \leqslant k < i$, refer to 3) above.

PROPOSITION 26.52. *Let α and β be connected ordinal diagrams and $v_0(j, \alpha) = v_0(j, \beta) = (i, a)$. Let $\alpha_0 = \mathrm{apr}(0, j, \alpha) = \mathrm{apr}(0, j, \beta) = \beta_0$, $\alpha_1 = \mathrm{apr}(1, j, \alpha)$ and $\beta_1 = \mathrm{apr}(1, j, \beta)$. If $\beta_1 <_i \alpha_1$, then $\beta <_j \alpha$.*

PROOF. In order that $\beta_1 <_i \alpha_1$ under the assumption, $\bar{\alpha}_1$ must properly contain $\bar{\alpha}_0$. Therefore $j \leqslant i$. We shall show that for any j-subsection of β, say $\bar{\eta}$, which either contains $\bar{\beta}_0$ or omits $\bar{\beta}_0$.

(*) $\eta <_k \alpha_1$ for all k such that $j \leqslant k \leqslant i$.

As a special case we have $\beta <_j \alpha_1 \leqslant_j \alpha$. The proof of (*) is by induction on $l(\eta)$.

1) $\eta = \beta_0$ or $\bar{\eta}$ omits $\bar{\beta}_0$. Then $\eta \leqslant_k \beta_0 = \alpha_0 <_k \alpha_1$ if $j \leqslant k \leqslant i$.

2) $\bar{\eta}$ properly contains $\bar{\beta}_0$ and there is a $\bar{\beta}_0$ such that the only element of I which has an occurrence in $\bar{\eta}$ connected to $\bar{\beta}_0$ is i. Then by definition of β_1, $\eta \leqslant_i \beta_1 <_i \alpha_1$. Let $j \leqslant k < i$ and let $\bar{\delta}$ be a component of a k-section of $\bar{\eta}$. Then $\bar{\delta}$ either omits $\bar{\beta}_0$ in which case $\delta <_k \alpha_1$ by 1) above; or $\bar{\delta}$ contains $\bar{\beta}_0$ and hence $\delta <_k \alpha_1$ by the induction hypothesis; or $\bar{\delta}$ is a k-subsection of $\bar{\beta}_0$ and hence there is a k-active occurrence of δ in $\bar{\alpha}_1$, so $\delta <_k \alpha_1$. In any case $\delta <_k \alpha_1$. So by induction on $\iota(k, \eta)$, $\eta <_k \alpha_1$ can be proved.

DEFINITION 26.53. Let $v_0(j, \alpha) = (i, a)$, $\mathrm{apr}(0, j, \alpha) = \alpha_0$ and $\mathrm{apr}(1, j, \alpha) = \alpha_1$. Define, as a matter of notational convenience $i_0 = i_1 = i$.

Suppose we have defined some pairs of j-subsections of α and elements of I which have occurrences in α, say $(\alpha_0, i_0), (\alpha_1, i_1), \ldots, (\alpha_n, i_n)$, which satisfy the following conditions:

(*) i) For every m, $1 \leqslant m < n$, $j \leqslant i_{m+1} < i_m$.

ii) For each m, $m \geqslant 1$, i_{m+1} is the maximum k for which there is a j-subsection of α, of the form $\overline{(k, b, \gamma)}$ such that $\bar{\alpha}_m$ is a component of $\bar{\gamma}$.

iii) Let $\bar{\eta}$ denote any j-subsection of α such that $\bar{\eta}$ contains $\bar{\alpha}_m$ and all the elements of I that occur in $\bar{\eta}$ and are connected to $\bar{\alpha}_m$ are $\geqslant i_{m+1}$. Then α_{m+1} is the maximum, with respect to $<_{i_{m+1}}$, among those η and $\bar{\alpha}_{m+1}$ denotes such an occurrence of α_{m+1}.

Now we define (α_{n+1}, i_{n+1}) as follows, provided that $\bar{\alpha}_n$ is not α. We define i_{n+1} as i_{m+1} in ii) of (*) reading n in place of m and we define $\bar{\alpha}_{n+1}$ as $\bar{\alpha}_{m+1}$ in iii) of (*) reading n in place of m.

We call $\bar{\alpha}_n$ an n^{th} j-approximation of α and denote it by $\overline{\operatorname{apr}(n, j, \alpha)}$, i.e., $\alpha_n = \operatorname{apr}(n, j, \alpha)$. Define v_n by $v_n(j, \alpha) = i_n$, $n = 1, 2, \ldots$.

If $\bar{\alpha}_n = \alpha$, then $\bar{\alpha}_{n+1}$ needs not be defined. We may, however, use the expression $v_{n+1}(j, \beta) < v_{n+1}(j, \alpha)$ to mean that $v_{n+1}(j, \beta)$ is not defined while $v_{n+1}(j, \alpha)$ is.

COROLLARY 26.54. (1) *Let $\overline{(p, e, \xi)}$ be a j-subsection of α in which $\bar{\alpha}_n$ is a component of $\bar{\xi}$. Then $p < i_n$.*

(2) $j \leqslant i_{n+1} < i_n$.

(3) *There is at least one $\bar{\alpha}_n$ which occurs as a component of γ in $\overline{(i_{n+1}, b, \gamma)}$, which is a j-subsection of α presuming that $\alpha_n \neq \alpha$.*

DEFINITION 26.55. Let $\bar{\eta}$ be a j-subsection of α. We say that $\bar{\eta}$ j-omits $\bar{\alpha}_n$ if $\bar{\eta}$ is not contained by any $\bar{\alpha}_n$, $\bar{\eta}$ does not contain any $\bar{\alpha}_n$, and $\bar{\eta}$ is not properly contained by any of $\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{n-1}$.

PROPOSITION 26.56. I^n. *Let $\bar{\eta}$ be a j-subsection of α that contains $\bar{\alpha}_n$. Suppose for each occurrence of $\bar{\alpha}_n$ in $\bar{\eta}$ there is an element of I that has an occurrence in $\bar{\eta}$ that is connected to $\bar{\alpha}_n$ and is less than i_{n+1}. Let $\alpha_n^1, \ldots, \alpha_n^m$ be all the occurrences of $\bar{\alpha}_n$ in $\bar{\eta}$ and let q_k be the least element of I that has an occurrence connected to α_n^k. Let $q = q_n(\bar{\eta}) = \max(q_1, \ldots, q_m)$. Then $\eta <_p \alpha_n$ for every p such that $q < p \leqslant i_n$. (Note that $j \leqslant q < i_{n+1}$.)*

II^{n+1}. *If $\bar{\eta}$ j-omits $\bar{\alpha}_{n+1}$, then $\eta <_p \alpha_{n+1}$ for any p such that $j \leqslant p \leqslant i_{n+1}$.*

III^{n+1}. *Let α and β be connected ordinal diagrams and suppose that* $\overline{\operatorname{apr}(n, j, \alpha)}$, $\overline{\operatorname{apr}(n, j, \beta)}$ *and* $\overline{\operatorname{apr}(n+1, j, \alpha)}$ *are defined. Suppose also that* $\alpha_n = \operatorname{apr}(n, j, \alpha) = \operatorname{apr}(n, j, \beta) = \beta_n$ *(hence* $\alpha_0 = \beta_0, \alpha_1 = \beta_1, \ldots, \alpha_{n-1} = \beta_{n-1}$*).*

(1) *If* $v_{n+1}(j, \beta) = i'_{n+1} < i_{n+1} = v_{n+1}(j, \alpha)$, *then* $\beta <_j \alpha$.

(2) *If*

$$v_{n+1}(j, \beta) = v_{n+1}(j, \alpha) = i_{n+1}$$

and

$$\beta_{n+1} = \operatorname{apr}(n+1, j, \beta) <_{i_{n+1}} \operatorname{apr}(n+1, j, \alpha) = \alpha_{n+1},$$

then $\beta <_j \alpha$.

PROOF. (By induction on n.) Note that I^0, II^0, II^1, III^0 and III^1 have been established (cf. Proposition 26.47, Lemma 26.45, Propositions 26.51, 26.43 and 26.46). First we will prove I^n by assuming III^r for all r such that $1 \leqslant r \leqslant n$. Then we will prove II^{n+1} from I^n and II^n. Finally we will prove III^{n+1} from II^n and II^{n+1}.

I^n. For any p, $j \leqslant p \leqslant i_n$ and any r, $0 \leqslant r \leqslant n$, $v_r(p, \alpha_n) = v_r(j, \alpha_n) = v_r(j, \alpha)$; also $\operatorname{apr}(r, p, \alpha_n) = \operatorname{apr}(r, j, \alpha_n) = \operatorname{apr}(r, j, \alpha)$. Now let $q < p \leqslant i_n$. Then for some $r, 0 \leqslant r \leqslant n, \operatorname{apr}(0, p, \eta) = \operatorname{apr}(0, p, \alpha_n), \ldots, \operatorname{apr}(r-1, p, \eta) = \operatorname{apr}(r-1, p, \alpha_n)$ and either $v_r(p, \eta) < v_r(p, \alpha_n)$ or $v_r(p, \eta) = v_r(p, \alpha_n) (= i_r)$ and $\operatorname{apr}(r, p, \eta) <_{i_r} \operatorname{apr}(r, p, \alpha_n) (= \alpha_r)$. So by III^r applied to p, $\eta <_p \alpha_n$ for every such p.

II^{n+1}. By induction on $l(\eta)$.

1) $\bar{\eta}$ is $\bar{\alpha}_n$ or $\bar{\eta}$ j-omits $\bar{\alpha}_n$. Then by II^n $\eta \leqslant_p \alpha_n <_p \alpha_{n+1}$ if $j \leqslant p \leqslant i_{n+1}$.

2) $\bar{\eta}$ contains $\bar{\alpha}_n$ properly and there is an occurrence of $\bar{\alpha}_n$ such that all the elements of I that have occurrences in $\bar{\eta}$ that are connected to it are $\geqslant i_{n+1}$. Then $\eta <_{i_{n+1}} \alpha_{n+1}$ by definition of α_{n+1}. Let $j \leqslant p < i_{n+1}$ and let $\bar{\delta}$ be a component of a p-section of $\bar{\eta}$. Then either $\bar{\delta}$ omits $\bar{\alpha}_n$ or $\bar{\delta}$ is a p-subsection of $\bar{\alpha}_n$. Therefore $\delta <_p \alpha_n <_p \alpha_{n+1}$. Then by induction on $\iota(p, \eta)$, $\eta <_p \alpha_{n+1}$ for all such p.

3) $\bar{\eta}$ properly contains $\bar{\alpha}_n$ and for every occurrence of $\bar{\alpha}_n$ in $\bar{\eta}$ there is an element of I which is less than i_{n+1} and has an occurrence in η that is connected to $\bar{\alpha}_n$. By I^n, $\eta <_p \alpha_n$ if $(q <) i_{n+1} \leqslant p \leqslant i_n$. In particular $\eta <_{i_{n+1}} \alpha_n <_{i_{n+1}} \alpha_{n+1}$. Let $j \leqslant p < i_{n+1}$ and let $\bar{\delta}$ be a component of a p-section of $\bar{\eta}$. Then, as in 2) incorporated with induction on $l(\eta)$, $\delta <_p \alpha_{n+1}$, which implies $\eta <_p \alpha_{n+1}$ for all such p.

III^{n+1}. (1) Let $\bar{\xi} = \overline{(i_{n+1}, b, \gamma)}$ be any j-subsection of α whose outermost index is i_{n+1} and $\bar{\gamma}$ contains $\bar{\alpha}_n$ as a component. We shall show that, for every j-subsection of β, say $\bar{\eta}$, which either j-omits $\bar{\beta}_n$ or contains $\bar{\beta}_n$, $\eta <_p \xi$ if $j \leqslant p \leqslant i_{n+1}$. As a special case $\beta <_j \xi \leqslant_j \alpha$.

1) $\bar{\eta}$ is $\bar{\beta}_n$ or $\bar{\eta}$ omits $\bar{\beta}_n$. Then $\eta \leqslant_p \beta_n$ by II^n. So $\eta \leqslant_p \beta_n = \alpha_n <_p \xi$.

2) $\bar{\eta}$ properly contains $\bar{\beta}_n$. Recall that $\bar{\beta}_n$ occurs in $\bar{\eta}$ in the following context. There is a j-subsection of $\bar{\eta}$, say $\overline{(k, c, \rho)}$, where $\bar{\beta}_n$ occurs in $\bar{\rho}$ as a component and $j \leqslant k \leqslant i'_{n+1} < i_{n+1}$.

We can show that

(*) There is a number r, $0 \leqslant r \leqslant n$, such that

$$\operatorname{apr}(r-1, i_{n+1}, \eta) = \alpha_{r-1} \; (= \operatorname{apr}(r-1, i_{n+1}, \xi))$$

and either

$$v_r(i_{n+1}, \eta) < i_r,$$

or

$$v_r(i_{n+1}, \eta) = i_r \quad \text{and} \quad \operatorname{apr}(r, i_{n+1}, \eta) <_{i_r} \alpha_r \; (= \operatorname{apr}(r, i_{n+1}, \xi)).$$

Applying (*) and III^r to η, ξ and i_{n+1} it follows that $\eta <_{i_{n+1}} \xi$. Let $j \leqslant p < i_{n+1}$ and let $\bar{\delta}$ be a component of a p-section of $\bar{\eta}$. If $\bar{\delta}$ satisfies the same condition as $\bar{\eta}$, then $\delta <_p \xi$ by the induction hypothesis. If $\bar{\delta}$ is a p-subsection of $\bar{\beta}_r$ for some r, $0 \leqslant r \leqslant n$, then $\delta <_p \beta_r \leqslant_p \beta_n = \alpha_n <_p \xi$. In any case, $\delta <_p \xi$, and by induction on $\iota(p, \eta)$, $\eta <_p \xi$.

The assertion (*) is a special case of the following.

(**) Let $\bar{\eta}$ be an arbitrary j-subsection of β that properly contains $\bar{\beta}_n$. Let m be any element of I such that $m > i'_{n+1}$ $(= v_{n+1}(j, \beta))$. Then there is a number r, $0 \leqslant r \leqslant n$, such that $\operatorname{apr}(r-1, m, \eta) = \beta_{r-1}$ $(= \alpha_{r-1})$ and either $v_r(m, \eta) < i_r = v_r(j, \beta)$ or $v_r(m, \eta) = i_r$ and $\operatorname{apr}(r, m, \eta) <_{i_r} \beta_r \; (= \alpha_r)$. (Note that $m > i_{n+1}$ includes $m = i_{n+1}$.) When $r = 0$, $v_r(m, \eta) < (i, a) = v_r(m, \beta)$ or $v_r(m, \eta) = (i, a)$ and $\operatorname{apr}(0, m, \eta) <_i \beta_0$.

We prove (**) in the following way. Since $m \geqslant j$, $v_0(m, \eta) \leqslant v_0(j, \beta)$ $(= (i, a))$. If this is a strict inequality we are done. If equality holds, then consider $r = 1$. Continuing the same argument, suppose that we have reached $\operatorname{apr}(n-1, m, \eta) = \beta_{n-1}$ and $v_n(m, \eta) = i_n$. Then $\operatorname{apr}(n, m, \eta)$ must contain $\bar{\beta}_{n-1}$. This, $m > i'_{n+1}$, and the fact that $\bar{\eta}$ properly contains $\bar{\beta}_n$ imply that $\operatorname{apr}(n, m, \eta) \neq \beta_n$. So by definition of $\overline{\operatorname{apr}(n, j, \eta)}(= \bar{\beta}_n)$, $\operatorname{apr}(n, m, \eta) <_{i_n} \beta_n$.

(2) We shall show that

(***) for any $\bar{\eta}$ a j-subsection of β which either omits $\bar{\beta}_{n+1}$, is $\bar{\beta}_{n+1}$ or properly contains $\bar{\beta}_{n+1}$, $\eta <_p \alpha_{n+1}$ for any p such that $j \leqslant p \leqslant i_{n+1}$.

As a special case of (***) we have $\beta <_j \alpha_{n+1} \leqslant_j \alpha$. The proof of (***) is by induction on $l(\eta)$.

1) $\bar{\eta}$ is $\bar{\beta}_{n+1}$ or $\bar{\eta}$ omits $\bar{\beta}_{n+1}$. By II^{n+1}, $\eta \leqslant_{i_{n+1}} \beta_{n+1} <_{i_{n+1}} \alpha_{n+1}$. Using an argument similar to one employed earlier we can prove by induction on $\iota(p, \eta)$ that $\eta <_p \alpha_{n+1}$ provided $j \leqslant p < i_{n+1}$.

2) $\bar{\eta}$ contains $\bar{\beta}_{n+1}$ properly and there is an occurrence of $\bar{\beta}_n$ in $\bar{\eta}$ such that every element of I that has an occurrence in $\bar{\eta}$ and is connected to it is $\geqslant_{i_{n+1}}$. Then, by definition of $\bar{\beta}_{n+1}$,

$$\eta <_{i_{n+1}} \beta_{n+1} <_{i_{n+1}} \alpha_{n+1}.$$

That $\eta <_p \alpha_{n+1}$ for $j \leqslant p < i_{n+1}$ can be shown as above.

3) $\bar{\eta}$ properly contains β_{n+1}, and, for every occurrence of $\bar{\beta}$ in $\bar{\eta}_n$, there is an element of I which has an occurrence connected to $\bar{\beta}_n$, and is $<_{i_{n+1}+1}$. Then, by I^n, $\eta <_p \beta_n$ if $q < p \leqslant i_n$, where $j \leqslant q < i_{n+1}$. Therefore, letting $p = i_{n+1}$ we obtain

$$\eta <_{i_{n+1}} \beta_n <_{i_{n+1}} \beta_{n+1} <_{i_{n+1}} \alpha_{n+1}.$$

If $j \leqslant p < i_{n+1}$, then $\eta <_p \alpha_{n+1}$ is proved as before.

We next define refinements of approximations. We shall define $\bar{\alpha}_{(n,k)}$ by induction on k in such a way that $\bar{\alpha}_{(n,k)}$ is an i_{n+1}-subsection of $\bar{\alpha}_{n+1}$.

DEFINITION 26.57. $\bar{\alpha}_{(n,0)}$ is any occurrence of $\bar{\alpha}_n$ that is i_{n+1}-active in $\bar{\alpha}_{n+1}$.

Suppose $\bar{\alpha}_{(n,k)}$ has been defined so that $\bar{\alpha}_{(n,k)}$ is an i_{n+1}-subsection of $\bar{\alpha}_{n+1}$. Suppose $\alpha_{(n,k)} \neq \alpha_{n+1}$. Let $\bar{\gamma}_1, \ldots, \bar{\gamma}_m$ be all the occurrences of i_{n+1}-subsections of $\bar{\alpha}_{n+1}$ that properly contain an occurrence of $\bar{\alpha}_{(n,k)}$. Let

$$\alpha_{(n,k+1)} = \max_{<_{i_{n+1}+1}} (\gamma_1, \ldots, \gamma_m).$$

Then $\bar{\alpha}_{(n,k+1)}$ denotes any such occurrence of $\alpha_{(n,k+1)}$ in $\bar{\alpha}_{n+1}$.

Note that although $\bar{\gamma}_1, \ldots, \bar{\gamma}_m$ above are determined relative to an occurrence of $\bar{\alpha}_{n+1}$, $\gamma_1, \ldots, \gamma_m$ (as ordinal diagrams) are determined uniquely from α_{n+1}. The same is true of $\alpha_{(n,k+1)}$.

$\bar{\alpha}_{(n,k)}$ is called the (n, k)th j-approximation of α and is denoted by $\mathrm{apr}((n, k), j, \alpha)$.

PROPOSITION 26.58. *Either $\bar{\alpha}_{(n,k)}$ is $\bar{\alpha}_{n+1}$ or it occurs in $\overline{(i_{n+1}, c, \delta)}$ as a component of $\bar{\delta}$.*

PROOF. Suppose $\bar{\alpha}_{(n,k)}$ occurs in $\overline{(p, c, \delta)}$ as a component of $\bar{\delta}$. Then by the definition of $\bar{\alpha}_{n+1}$, $p \geqslant i_{n+1}$. If $p > i_{n+1}$, then $p \geqslant i_{n+1} + 1$ hence $\bar{\alpha}_{(n,k)}$ in $\bar{\delta}$ is an $i_{n+1} + 1$-subsection of $\overline{(p, c, \delta)}$. So $\alpha_{(n,k)} <_{i_{n+1}+1} (p, c, \delta)$. Furthermore, (p, c, δ) contains some occurrence of $\bar{\alpha}_{(n,k-1)}$ as an i_{n+1}-subsection, hence, by definition of $\bar{\alpha}_{(n,k)}$, $(p, c, \delta) <_{i_{n+1}+1} \alpha_{(n,k)}$, which is a contradiction. Therefore, $p = i_{n+1}$.

DEFINITION 26.59. An i_{n+1}-subsection of $\bar{\alpha}_{n+1}$, say $\bar{\eta}$, j-omits $\bar{\alpha}_{(n,k)}$, if it does not contain any occurrence of $\bar{\alpha}_{(n,k)}$, is not contained by $\alpha_{(n,k)}$, and is not contained by any occurrence of $\bar{\alpha}_{(m,p)}$, where $(m, p) < (n, k)$.

Note that if $k = 0$, it is possible that $\bar{\eta}$ j-omits $\bar{\alpha}_{(n,0)}$ but not $\bar{\alpha}_n$.

PROPOSITION 26.60. *Suppose that $\bar{\eta}$ is an i_{n+1}-subsection of $\bar{\alpha}_{n+1}$ that j-omits $\bar{\alpha}_{(n,k)}$. Then $\eta <_{i_{n+1}} \alpha_{(n,k)}$.*

PROOF. (By induction on k within which by induction on $l(\eta)$.)

$k = 0$: $\bar{\eta}$ omits $\bar{\alpha}_{(n,0)}$.

1) $\bar{\eta}$ omits $\bar{\alpha}_n$. Then $\eta <_{i_{n+1}} \alpha_n$ $(= \alpha_{(n,0)})$ by II^n of Proposition 26.56.

2) $\bar{\eta}$ contains $\bar{\alpha}_n$. Then for each j-active $\bar{\alpha}_n$ in η there is an element of I that has an occurrence in $\bar{\eta}$ connected to $\bar{\alpha}_n$, and which is $< i_{n+1}$. So by I^n of Proposition 26.56, $\eta <_{i_{n+1}} \alpha_n = \alpha_{(n,0)}$.

$k > 0$: $\bar{\eta}$ omits $\bar{\alpha}_{(n,k)}$.

1) $\bar{\eta}$ omits $\bar{\alpha}_{(n,k-1)}$. Then $\eta <_{i_{n+1}} \alpha_{(n,k-1)}$ by the induction hypothesis. But $\bar{\alpha}_{(n,k-1)}$ is an i_{n+1}-subsection of $\bar{\alpha}_{(n,k)}$ (by definition). So $\bar{\eta} <_{i_{n+1}} \alpha_{(n,k)}$.

2) $\bar{\eta}$ is $\bar{\alpha}_{(n,k-1)}$. Then $\eta <_{i_{n+1}} \alpha_{(n,k)}$.

3) $\bar{\eta}$ properly contains $\bar{\alpha}_{(n,k-1)}$. Then $\eta <_{i_{n+1}+1} \alpha_{(n,k)}$ by the definition of $\bar{\alpha}_{(n,k)}$. Let $\bar{\delta}$ be a component of an i_{n+1}-section of $\bar{\eta}$. Since $\bar{\delta}$ omits $\bar{\alpha}_{(n,k)}$, $\delta <_{i_{n+1}} \alpha_{(n,k)}$ by the induction hypothesis.

PROPOSITION 26.61. *Suppose*

$$\bar{\alpha}_{n+1} = \overline{\operatorname{apr}(n+1, j, \alpha)}, \qquad \bar{\beta}_{n+1} = \overline{\operatorname{apr}(n+1, j, \beta)},$$

$$\bar{\alpha}_{(n,k)} = \overline{\operatorname{apr}((n,k), j, \alpha)}, \qquad \bar{\beta}_{(n,k)} = \overline{\operatorname{apr}((n,k), j, \beta)}$$

are defined for α and β. If

$$\beta_{(n,k-1)} = \alpha_{(n,k-1)}, \quad v_{n+1}(j, \beta) = v_{n+1}(j, \alpha) = i_{n+1} \quad \textit{and} \quad \beta_{(n,k)} <_{i_{n+1}+1} \alpha_{(n,k)},$$

then $\beta_{n+1} <_{i_{n+1}} \alpha_{n+1}$, hence $\beta <_j \alpha$.

PROOF. We first claim that

(1) $\beta_{(n,k)} <_{i_{n+1}} \alpha_{(n,k)}$.

But (1) is a special case of the following.

(2) For any $\bar{\gamma}$ an i_{n+1}-subsection of $\beta_{(n,k)}$ which either contains $\bar{\beta}_{(n,k-1)}$ or j-omits $\bar{\beta}_{(n,k-1)}$, $\gamma <_{i_{n+1}} \alpha_{(n,k)}$.

We prove (2) by induction on $l(\gamma)$.

1) $\bar{\gamma}$ is $\bar{\beta}_{(n,k-1)}$. Then $\gamma = \alpha_{(n,k-1)} <_{i_{n+1}} \alpha_{(n,k)}$.

2) $\bar{\gamma}$ omits $\bar{\beta}_{(n,k-1)}$. Then by Proposition 26.60 $\gamma <_{i_{n+1}} \beta_{(n,k-1)}$ hence $\gamma <_{i_{n+1}} \beta_{(n,k-1)} = \alpha_{(n,k-1)} <_{i_{n+1}} \alpha_{(n,k)}$.

3) $\bar{\gamma}$ properly contains $\bar{\beta}_{(n,k-1)}$. Let γ be of the form (p, c, γ'), where $p \geqslant i_{n+1}$. Then $\gamma \leqslant_{i_{n+1}+1} \beta_{(n,k)}$ by definition of $\bar{\beta}_{(n,k)}$, and hence

$$\gamma \leqslant_{i_{n+1}+1} \beta_{(n,k)} <_{i_{n+1}+1} \alpha_{(n,k)}$$

by the hypothesis of the proposition. Let $\bar{\delta}$ be a component of an i_{n+1}-section of $\bar{\gamma}$. If $\bar{\delta}$ satisfies the same condition as $\bar{\gamma}$, then $\delta <_{i_{n+1}} \alpha_{(n,k)}$ by the induction hypothesis. The remaining possibility is that $\bar{\delta}$ is a $\bar{\beta}_{(m,p)}$, where $(m, p) < (n, k-1)$. Therefore $\delta =_{i_{n+1}} \beta_{(m,p)} = \alpha_{(m,p)} <_{i_{n+1}} \alpha_{(n,k)}$. From this it follows that $\gamma <_{i_{n+1}} \alpha_{(n,k)}$.

In order to finish the proof of the proposition we need only prove that

(3) for any $\bar{\eta}$ an i_{n+1}-subsection of $\bar{\beta}_{n+1}$ which either omits $\bar{\beta}_{(n,k)}$ or contains $\bar{\beta}_{(n,k)}$, $\eta <_{i_{n+1}} \alpha_{(n,k)}$.

This we prove by induction on $l(\eta)$.

1) $\bar{\eta}$ is $\bar{\beta}_{(n,k)}$. By (1), $\eta <_{i_{n+1}} \alpha_{(n,k)}$.

2) $\bar{\eta}$ omits $\bar{\beta}_{(n,k)}$. By Proposition 26.60, $\eta <_{i_{n+1}} \beta_{(n,k)} <_{i_{n+1}} \alpha_{(n,k)}$.

3) $\bar{\eta}$ properly contains $\bar{\beta}_{(n,k)}$. Let η be (p, c, η'), where $p \geqslant i_{n+1}$. By definition of $\bar{\beta}_{(n,k)}$, $\eta <_{i_{n+1}+1} \beta_{(n,k)}$ and hence

$$\eta <_{i_{n+1}+1} \beta_{(n,k)} <_{i_{n+1}+1} \alpha_{(n,k)}.$$

From this it follows in the same manner as 3) in the proof of (2) above that $\eta <_{i_{n+1}} \alpha_{(n,k)}$.

This completes the theory of approximations. As we have seen, this theory supplies a criterion for the evaluation of orderings between two connected ordinal diagrams.

§27. A consistency proof of second order arithmetic with the Π_1^1-comprehension axiom

The following lemma, concerning the system of ordinal diagrams $O(\omega + 1, \omega^3)$ is essential for the consistency proof of this section.

LEMMA 27.1 (the Main Lemma). *Let p be a natural number and let γ and δ be ordinal diagrams for which there exist two finite sequences of ordinal diagrams $\gamma = \gamma_0, \ldots, \gamma_m$ and $\delta = \delta_0, \ldots, \delta_m$ which satisfy the following conditions* (1)–(4).

(1) *Each γ_i, $i < m$, is of the form $(k, 0, \gamma_{i+1})$ for some natural number $k \geqslant p$, or $(\omega, a+1, \gamma_{i+1} \# \eta)$.*

(2) *Each* δ_i, $i < m$, *is* $(k, 0, \delta_{i+1})$ *or* $(\omega, a + 1, \delta_{i+1} \# \eta)$ *according as* γ_i *is* $(k, 0, \gamma_{i+1})$ *or* $(\omega, a + 1, \gamma_{i+1}, \# \eta)$.

(3) $\delta_m <_j \gamma_m$ *for each* j *such that* $p \leqslant j \leqslant \omega$.

(4) *For each* j *such that* $p \leqslant j < \omega$ *and for each* j*-section* $\bar{\alpha}$ *of* δ_m, *there exists a* j*-section* $\bar{\beta}$ *of* γ_m *for which* $\alpha \leqslant_j \beta$.

Then $\delta <_j \gamma$ *for each* j *such that* $p \leqslant j \leqslant \omega$, *and for each* j *with* $p \leqslant j < \omega$ *and each* j*-section* $\bar{\alpha}$ *of* δ, *there exists a* j*-section* $\bar{\beta}$ *of* γ *such that* $\alpha \leqslant_j \beta$.

PROOF. (By double induction on m and $n = \iota(j, \gamma, \delta)$.)

1. If $m = 0$, then the result is obvious from (3) and (4).

2. Suppose $m > 0$, $\gamma = (k, 0, \gamma_1)$ and $\delta = (k, 0, \delta_1)$ where $k \geqslant p$.

2.1. Then $\delta_1 <_k \gamma_1$ by the induction hypothesis so that $\delta <_\infty \gamma$.

2.2. If $k < j \leqslant \omega$, then $\delta <_j \gamma$ since there are no q-sections of γ or δ for $q \geqslant j$, and since by 2.1 $\delta <_\infty \gamma$.

2.3. If $j = k$, then 2.2 implies that $\delta <_j k$ if $\delta_1 <_j \gamma$. Since $\delta_1 <_j \gamma_1$ by the induction hypothesis on m and since $\gamma_1 <_j \gamma$ because γ_1 is a j-section of γ it follows that $\delta_1 <_j \gamma$.

2.4. If $p \leqslant j < k$, then $\delta <_{j_1} \gamma$, where $j_1 = \iota(j, \gamma, \delta)$. Therefore, $\delta <_j \gamma$ if, for each j-section $\bar{\alpha}$ of δ, $\alpha <_j \gamma$. Suppose $\bar{\alpha}$ is a j-section of δ. Then $\bar{\alpha}$ is a j-section of δ, so by the induction hypothesis on m there exists a j-section $\bar{\beta}$ of γ_1 such that $\alpha \leqslant_j \beta$. Furthermore, β is a j-section of γ as well. Therefore, $\alpha \leqslant_j \beta <_j \gamma$.

3. Suppose $m > 0$, $\gamma = (\omega, c + 1, \gamma_1 \# \eta)$ and $\delta = (\omega, c + 1, \delta_1 \# \eta)$.

3.1. Since, by the induction hypothesis on m, $\delta_1 <_\omega \gamma_1$ it follows that $\delta <_\infty \gamma$.

3.2. If $j = \omega$, then it is sufficient to show that $\delta_1 \# \eta <_\omega \gamma$, that is, $\delta_1 <_\omega \gamma$ and $\eta <_\omega \gamma$. Since $\delta_1 <_\omega \gamma_1$ and $\overline{\gamma_1 \# \eta}$ is an ω-section of γ, it follows that $\delta_1 <_\omega \gamma$ and $\eta <_\omega \gamma$.

3.3. If $p \leqslant j < \omega$, then by the induction hypothesis on n, $\delta <_{j_1} \gamma$, where $j_1 = \iota(j, \gamma, \delta)$. Hence $\delta <_j \gamma$ if for any j-section $\bar{\alpha}$ of δ, $\alpha <_j \gamma$. Let $\bar{\alpha}$ be a j-section of δ. Then $\bar{\alpha}$ is either a j-section of δ_1 or a j-section of η. If the former is the case, then by the induction hypothesis on m, there exists a j-section $\bar{\beta}$ of γ_1 such that $\alpha \leqslant_j \beta$. Moreover, $\bar{\beta}$ is a j-section of γ. Therefore, $\alpha <_j \gamma$. If $\bar{\alpha}$ is a j-section of η, then $\bar{\alpha}$ is a j-section of γ, and hence $\alpha <_j \gamma$.

We now proceed to the consistency proof of a system of second order arithmetic. In order to simplify the discussion we will use $\neg$, $\wedge$, and $\forall$ as primitive logical symbols. Other symbols will be used as abbreviations.

DEFINITION 27.2. (1) The language, formulas, abstracts, sequents and proofs of second order arithmetic are as in Definition 18.1.

(2) A semi-formula or a semi-abstract is respectively a formula-like, or an abstract-like expression, where a bound variable may occur free. The outermost logical symbol of a semi-formula or a semi-abstract is defined naturally.

(3) Let A be a semi-formula or a semi-abstract, let $\forall \phi\, B$ be a semi-formula in A and let $\sharp$ be the outermost $\forall$ in $\forall \phi\, B$, i.e., the $\forall$ which precedes ϕ. Let G be an arbitrary symbol in B. Then we say that $\sharp$ ties G and G is tied by $\sharp$ in A. If G is a $\forall$ on a second order variable in B and G ties ϕ in B, then we say that $\sharp$ affects G in A.

(4) Let $\sharp$ be a $\forall$ on a second order variable in A. We say that $\sharp$ is isolated in A if the following conditions are satisfied.

(4.1) No $\forall$ on a second order variable in A affects $\sharp$.

(4.2) $\sharp$ does not affect any $\forall$ on a second order variable.

(5) A semi-formula or a semi-abstract A is called isolated if every $\forall$ on a second order variable in A is isolated. Originally we used "semi-isolated" instead of "isolated".

The following result is easily shown.

PROPOSITION 27.3. *The class of isolated formulas (abstracts) is Π^1_1-in-the-wider-sense* (cf. (3) of Definition 18.1). *Therefore, if V is an isolated abstract and $F(\alpha)$ is isolated, then so is $F(V)$.*

From Proposition 27.3 we see that when we study isolated formulas, we are essentially dealing with Π^1_1-formulas.

DEFINITION 27.4. By the *isolated system of natural numbers*, **INN**, we mean a system of second order arithmetic as in Definition 18.1, where the induction formulas are arbitrary, i.e., the system has full induction, and the abstracts for $\forall$: left, the $\forall$ in (1) of Definition 18.1, are restricted to the isolated ones, that is, the system has isolated comprehension.

Originally this system was called the semi-isolated system of natural numbers, **SINN**.

This section is devoted to the proof of the following.

THEOREM 27.5. **INN** *is consistent.*

PROOF. Theorem 27.5 will be proved using the system of ordinal diagrams $O(\omega + 1, \omega^3)$. The proof will be presented stage by stage. We shall take over much of the terminology from first order arithmetic, for example, explicit and implicit bundles and formulas, end piece, boundary inferences, etc.

DEFINITION 27.6. Let A be a formula. We define the *γ-degree* of A, denoted $\gamma(A)$, as follows:

1) $\gamma(A) = 0$ if A is isolated.

In the following we assume that A is not isolated.

2) If A is of the form $\neg B$, then $\gamma(A) = \gamma(B) + 1$.
3) If A is of the form $B \wedge C$, then $\gamma(A) = \max(\gamma(B), \gamma(C)) + 1$.
4) If A is of the form $\forall x\, G(x)$, then $\gamma(A) = \gamma(G(a)) + 1$.
5) If A is of the form $\forall \phi\, F(\phi)$, then $\gamma(A) = \gamma(F(\alpha)) + 1$.
6) The γ-degree of an abstract $\{x_1, \ldots, x_n\}\, H(x_1, \ldots, x_n)$ is defined to be $\gamma(H(a_1, \ldots, a_n))$.

PROPOSITION 27.7. *If V is an isolated abstract, then $\gamma(F(V)) = \gamma(F(\alpha))$.*

PROOF. If $\gamma(F(\alpha)) = 0$, the proposition is evident (cf. Proposition 27.3). If $\gamma(F(\alpha)) \neq 0$, we shall prove the proposition by mathematical induction on the number of logical symbols in $F(\alpha)$. Since other cases are treated similarly we shall consider only the case where $F(\alpha)$ is of the form $\forall \phi\, G(\phi, \alpha)$. By the induction hypothesis, $\gamma(G(\beta, V)) = \gamma(G(\beta, \alpha))$. This implies that

$$\gamma(F(V)) = \gamma(G(\beta, V)) + 1 = \gamma(G(\beta, \alpha)) + 1 = \gamma(F(\alpha)).$$

PROPOSITION 27.8. *If V is isolated and $\gamma(F(V)) > 0$, then $\gamma(\forall \phi\, F(\phi)) = \gamma(F(V)) + 1$.*

PROOF. Let V be isolated and $\gamma(F(V)) > 0$. By Proposition 27.7, $\gamma(F(\alpha)) > 0$, that is, $F(\alpha)$ is not isolated. Hence $\forall \phi\, F(\phi)$ is not isolated either. Then $\gamma(\forall \phi\, F(\phi)) = \gamma(F(\alpha)) + 1 = \gamma(F(V)) + 1$.

DEFINITION 27.9. Let A be an occurrence of a formula in a proof P, in **INN**. The *grade* of A with respect to P, denoted by $g(A; P)$ or simply $g(A)$, is defined to be $\omega^2 \cdot \gamma(A) + \omega \cdot m_1 + m_0$, where m_1 is the number of second order free variables used as eigenvariables of second order $\forall$: right under the sequent containing A, and m_0 is the number of logical symbols in A.

To prove Theorem 27.5, we shall modify the notion of proof in **INN**, by introducing the following rule of substitution:

DEFINITION 27.10. Rule of substitution in **INN**.

$$\frac{A_1, \ldots, A_n \rightarrow B_1, \ldots, B_m}{A_1\begin{pmatrix}\alpha\\V\end{pmatrix}, \ldots, A_n\begin{pmatrix}\alpha\\V\end{pmatrix} \rightarrow B_1\begin{pmatrix}\alpha\\V\end{pmatrix}, \ldots, B_m\begin{pmatrix}\alpha\\V\end{pmatrix}},$$

where α is a second order free variable and V is an isolated abstract with the same number of argument-places as α. Here α is called the eigenvariable of the substitution. This schema is essentially redundant in **INN**, but the introduction of it helps us in the reduction of proofs in **INN**.

DEFINITION 27.11. We say that an inference J, which is either a substitution or a second order $\forall$: right, *disturbs* a semi-formula A if the eigenvariable of J is tied by a second order $\forall$ in A.

DEFINITION 27.12. Let P be a proof in **INN**. We call P a *proof with degree* if the following conditions are satisfied.

1) Every substitution is in the end-piece and there is no ind under a substitution.

2) We can assign an ordinal number $\leqslant \omega$ to every semi-formula A or substitution J in P as follows. We denote this assigned number by $d(A; P)$ or $d(J; P)$, or, for short, $d(A)$ or $d(J)$ read "degree of A or J".

2.1) If A is explicit, then $d(A) = 0$. Suppose A is implicit.

2.2) If A is not isolated, then $d(A) = \omega$. Suppose A is isolated.

2.3) $d(A) = 0$ if A contains no logical symbol.

2.4) $d(A) = d(B) + 1$ if A is of the form $\neg B$.

2.5) $d(A) = \max(d(B), d(C)) + 1$ if A is of the form $B \wedge C$.

2.6) $d(A) = d(B(x)) + 1$ if A is of the form $\forall x\, B(x)$.

2.7) $d(A) = \max(d(F(\phi)), d(J_0)) + 1$ if A is of the form $\forall\phi\, F(\phi)$, where J_0 ranges over substitutions which disturb $\forall\phi\, F(\phi)$.

2.8) $d(B) < d(J)$ for every implicit formula B in the upper sequent of J.

2.9) $0 < d(J) < \omega$.

DEFINITION 27.13. Let P be a proof with degree and let S be a sequent in P. The *i-resolvent* of S is the upper sequent of the uppermost substitution under S whose degree is not greater than i, if such exists; otherwise, the i-resolvent of S is the end-sequent of P.

DEFINITION 27.14. Consider the system of ordinal diagrams $O(\omega + 1, \omega^3)$. We shall assign an ordinal diagram from $O(\omega + 1, \omega^3)$ to every sequent of a proof with degree, as follows:

1) The ordinal diagram of an initial sequent is 0.

2) If S_1 and S_2 are the upper sequent and the lower sequent, respectively, of a weak structural inference J, then the ordinal diagram of S_2 is equal to that of S_1.

3) If S_1 and S_2 are the upper sequent and the lower sequent respectively of $\neg$, $\wedge$: left, first order $\forall$, second order $\forall$: right or explicit second order $\forall$: left, then the ordinal diagram of S_2 is $(\omega, 0, \sigma)$, where σ is the ordinal diagram of S_1.

4) If S_1 and S_2 are the upper sequents and S is the lower sequent of an $\wedge$: right inference, then the ordinal diagram of S is $(\omega, 0, \sigma_1 \# \sigma_2)$, where σ_1 and σ_2 are the ordinal diagrams of S_1 and S_2, respectively.

5) If S_1 and S_2 are the upper sequent and the lower sequent respectively of an implicit, second order $\forall$: left of the form

$$\frac{F(V), \Gamma \to \Delta}{\forall\phi\, F(\phi), \Gamma \to \Delta},$$

then the ordinal diagram of S_2 is $(\omega, g(F(V)) + 2, \sigma)$, where σ is the ordinal diagram of S_1.

6) If S_1 and S_2 are the upper sequents and S is the lower sequent of a cut J, then the ordinal diagram of S is $(\omega, m + 1, \sigma_1 \# \sigma_2)$, where m is the grade of the cut formula and σ_1 and σ_2 are the ordinal diagrams of S_1 and S_2, respectively.

7) If S_1 and S_2 are the upper sequent and the lower sequent respectively of a substitution with degree i, then the ordinal diagram of S_2 is $(i, 0, \sigma)$, where σ is the ordinal diagram of S_1.

8) If S_1 and S_2 are the upper sequent and the lower sequent respectively of an application of induction, then the ordinal diagram of S_2 is $(\omega, m + 2, \sigma)$, where m is the grade of the induction formula and σ is the ordinal diagram of S_1.

9) The ordinal diagram assigned to the end-sequent of a proof P with degree is called the ordinal diagram of P.

The ordinal diagram of a sequent S in P will be denoted by $O(S; P)$ or simply $O(S)$; the ordinal diagram of P will be denoted by $O(P)$.

DEFINITION 27.15. We shall define the notion of *reduction of proofs*.

1) Let $S_1, \ldots, S_m$ and S be sequents. S is *reducible* to $S_1, \ldots, S_m$ if S is provable without a cut presuming that $S_1, \ldots, S_m$ are provable without a cut.

2) Let $P_1, \ldots, P_m$ and P be proofs with degree. We say P is *reduced* to $P_1, \ldots, P_m$ if the following conditions are satisfied:

2.1) The ordinal diagram of each P_i is less than that of P (in the sense of $<_0$).

2.2) The end-sequent of P is reducible to the end-sequents of $P_1, \ldots, P_m$.

(1) Preparation for the reduction. Suppose that the sequent $\rightarrow$ is provable in **INN**. In the following we shall reduce a proof P of $\rightarrow$ to another proof of $\rightarrow$. Then by transfinite induction on $<_0$, we can prove that there exists a proof in **INN** of $\rightarrow$ of which the entire part is the end-piece. Following the method of the consistency proof of **PA**, we can eliminate the cut inference from the proof of $\rightarrow$ so obtained. But this is impossible.

Without loss of generality we may assume that all free variables used as eigenvariables in a proof are distinct and are not contained in the sequents under the inference in which it is used as an eigenvariable.

Let P be a proof of $\rightarrow$.

1) We add the following rules of inference, called term-replacement.

$$\frac{\Gamma_1, F(s), \Gamma_2 \rightarrow \Delta}{\Gamma_1, F(t), \Gamma_2 \rightarrow \Delta}, \qquad \frac{\Gamma \rightarrow \Delta_1, F(s), \Delta_2}{\Gamma \rightarrow \Delta_1, F(t), \Delta_2},$$

where s and t are terms which do not contain any free variable and which express the same number. (These rules of inference are redundant in the original system.)

2) If S_1 and S_2 are the upper sequent and the lower sequent of an application of term-replacement, then the ordinal diagram of S_2 is equal to that of S_1.

3) We substitute 0 for every free variable of type 0 in P except if it is used as an eigenvariable. In this alteration the proof remains correct and neither the end-sequent of P nor the ordinal diagram of P changes.

(2) Suppose that P contains an application of ind in its end-piece. Because of 3), immediately above, P contains no first order free variables in its end-piece other than those used as eigenvariables. Let J be a lowermost induction in the end-piece of P:

$$\begin{array}{c} Q(a) \\ \vdots \\ J \quad \dfrac{A(a), \Gamma \xrightarrow{\mu} \Delta, A(a')}{A(0), \Gamma \rightarrow \Delta, A(t)}, \\ \vdots \\ \rightarrow \end{array}$$

where t contains no free variables and $Q(a)$ is the proof of the upper sequent of J. We obtain a proof P' from P by replacing J by the following:

Case 1. $t = 0$. Replace the part of P above $A(0), \Gamma \to \Delta, A(t)$ (inclusive) by

$$\begin{array}{c} A(0) \to A(0) \\ \hline \text{some weakenings and exchanges} \\ \hline A(0), \Gamma \to \Delta, A(0) \\ \hline A(0), \Gamma \to \Delta, A(t). \end{array}$$

Since the ordinal diagram of $A(0), \Gamma \to \Delta, A(t)$ is 0, it is obvious that $O(P') <_0 O(P)$.

Case 2. $t \neq 0$. Then $t = n$ for some numeral n. Consider the following proof P':

$$\begin{array}{c} \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}Q(0)\\ \vdots\\ A(0), \Gamma \overset{\mu}{\to} \Delta, A(0')\end{array} \quad \begin{array}{c}Q(0')\\ \vdots \\ A(0'), \Gamma \overset{\mu}{\to}\Delta, A(0'')\end{array}}{A(0), \Gamma, \Gamma \to \Delta, \Delta, A(0'')}}{\text{some exchanges and contractions}}}{A(0), \Gamma \to \Delta, A(0'')} \quad \begin{array}{c}Q(0'')\\ \vdots\\ A(0''), \Gamma \overset{\mu}{\to}\Delta, A(0''')\end{array}}{A(0), \Gamma, \Gamma \to \Delta, \Delta, A(0''')}}{\dfrac{\text{some exchanges and contractions}}{A(0), \Gamma \to A(0'''), \Delta}} \\ \vdots \\ \dfrac{A(0), \Gamma \to \Delta, A(n)}{A(0), \Gamma \to \Delta, A(t)} \\ \vdots \\ \to \end{array}$$

Every substitution in P' is assigned the same degree as the corresponding substitution in P. It is easily seen that P' is a proof with degree whose end-sequent is $\to$.

That $O(P') <_0 O(P)$ is shown as follows. First compare

$$\mu_0 = O(A(0), \Gamma \to \Delta, A(t); P) = (\omega, g(A(a); P) + 2, \mu)$$

and

$$\mu_1 = O(A(0), \Gamma \to \Delta, A(0''); P') = (\omega, g(A(0'); P') + 1, \mu \# \mu).$$

Since $g(A(a); P) = g(A(0); P')$, $\mu_1 <_\infty \mu_0$. The only ω-section of μ_1 is $\overline{\mu \# \mu}$, and since $\bar{\mu}$ is an ω-section of μ_0, $\mu \# \mu <_\omega \mu_0$. Thus $\mu_1 <_\omega \mu_0$. There is no substitution above an ind, so this implies

$$O(A(0), \Gamma \to \Delta, A(t); P') <_j O(A(0), \Gamma \to \Delta, A(t); P)$$

for every j, where j is ∞ or $j \leqslant \omega$. Let $\{\gamma_0, \ldots, \gamma_m\}$ and $\{\delta_0, \ldots, \delta_m\}$ be sequences of ordinal diagrams such that

i) $\gamma_m = O(A(0), \Gamma \rightarrow \Delta, A(t); P)$,

ii) $\delta_m = O(A(0), \Gamma \rightarrow \Delta, A(t); P')$,

iii) $\gamma_{m-1}, \ldots, \gamma_0$ are the ordinal diagrams of the sequences in P which are under $A(0), \Gamma \rightarrow \Delta, A(t)$, in that order, and

iv) $\delta_{m-1}, \ldots, \delta_0$ are the ordinal diagrams of the corresponding sequences in P'.

Then these sequences of ordinal diagrams satisfy the conditions in Lemma 27.1. Thus by this Main Lemma, taking p to be 0, $\delta_0 <_0 \gamma_0$, or $O(P') <_0 O(P)$.

(3) Because of the reduction in (2), we may now assume that there is no ind, hence no first order free variable, in the end-piece of P. Suppose that there occur axioms of the form $s = t, A(s) \rightarrow A(t)$ in the end-piece of P. Let $s = t, A(s) \rightarrow A(t)$ be one such. Then there are numerals m and n such that m and n are equal to s and t, respectively. Either $m = n \rightarrow$ or $\rightarrow m = n$ is a mathematical, initial sequent.

Case 1. If $m = n \rightarrow$ is an axiom, then replace that axiom by

$$\begin{array}{c} m = n \rightarrow \\ \hline \text{weakenings and an exchange} \\ \hline m = n, A(m) \rightarrow A(n) \\ \hline \text{term replacements} \\ \hline s = t, \ A(s) \ \rightarrow A(t). \end{array}$$

This does not change the ordinal diagram.

Case 2. If $m = n \rightarrow$ is not a mathematical, initial sequent, replace the initial sequent by:

$$\begin{array}{c} A(m) \rightarrow A(n) \\ \hline \text{term replacements} \\ \hline A(s) \rightarrow A(t) \\ \hline s = t, A(s) \rightarrow A(t). \end{array}$$

(4) By virtue of (3), we may assume that there are no applications of ind and no equality axioms as initial sequents in the end-piece of P. Suppose that the end-piece of P contains logical, initial sequents. Suppose P is of the following form and $D \rightarrow D$ is one of the initial sequents in the end-piece of P:

$$\begin{array}{c} \begin{array}{cc} & D \rightarrow D \\ \vdots & \vdots \\ \Gamma \overset{\mu}{\rightarrow} \Delta, \tilde{D} & \tilde{D}, \Pi \overset{\nu}{\rightarrow} \Lambda_1, \tilde{D}, \Lambda_2 \end{array} \\ \hline \Gamma, \Pi \rightarrow \Delta, \Lambda_1, \tilde{D}, \Lambda_2 \\ \vdots \\ \rightarrow \end{array}$$

where two $\tilde{D}$'s in the right upper sequent of the cut denote the descendants of the D's occurring in the initial sequent which is explicitly written.

We shall consider a proof P' of the following form:

$$\dfrac{\dfrac{\Gamma \xrightarrow{\mu} \Delta, \tilde{D}}{\text{some weakenings and exchanges}}}{\Gamma, \Pi \to \Delta, \Lambda_1, \tilde{D}, \Lambda_2}$$
$$\vdots$$
$$\to$$

where every substitution in P' is assigned the same degree as the corresponding one in P.

$$O(\Gamma, \Pi \to \Delta, \Lambda_1, \tilde{D}, \Lambda_2; P) = (\omega, g(\tilde{D}) + 1, \mu \# \nu),$$

while

$$O(\Gamma, \Pi \to \Delta, \Lambda_1, \tilde{D}, \Lambda_2; P') = \mu <_j (\omega, g(\tilde{D}) + 1, \mu \# \nu)$$

for all $j \leqslant \omega$ and, if $j < \omega$, for each $\bar{\beta}$ a j-section of μ, $\bar{\beta}$ is also a j-section of $(\omega, g(\tilde{D}) + 1, \mu \# \nu)$. Thus by Lemma 27.1, $O(P') <_0 O(P)$.

If P is of the form

$$\dfrac{\begin{array}{c} D \to D \\ \vdots \\ \Gamma_1, \tilde{D}, \Gamma_2 \to \Delta, \tilde{D} \end{array} \qquad \tilde{D}, \Pi \to \Lambda}{\Gamma_1, \tilde{D}, \Gamma_2, \Pi \to \Delta, \Lambda}$$

then the reduction is carried out similarly.

(5) We may assume besides the condition in (3) that the end-piece of P contains no logical, initial sequent. Let Q be a proof with degree whose end-sequent is not necessarily $\to$ but which satisfies the same conditions as those required for P. We can define Q^*, obtained from Q by eliminating weakenings in the end-piece of Q, by induction on the number of inferences in the end-piece of Q as for **PA**. We deal with the following case only: If the last inference of Q is a substitution, say

$$\frac{\Gamma \to \Delta}{\Gamma\binom{\alpha}{V} \to \Delta\binom{\alpha}{V}},$$

where Γ and Δ are $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$, respectively, and $\Gamma\,(\begin{smallmatrix}\alpha\\V\end{smallmatrix})$ and $\Delta\,(\begin{smallmatrix}\alpha\\V\end{smallmatrix})$ denote $A_1\,(\begin{smallmatrix}\alpha\\V\end{smallmatrix}), \ldots, A_m\,(\begin{smallmatrix}\alpha\\V\end{smallmatrix})$ and $B_1\,(\begin{smallmatrix}\alpha\\V\end{smallmatrix}), \ldots, B_n\,(\begin{smallmatrix}\alpha\\V\end{smallmatrix})$, respectively, and the end-sequent of Q_0^*, where Q_0 is the proof of the upper sequent, is $\Gamma^* \to \Delta^*$, then Q^* is

$$\begin{array}{c} Q^* \\ \vdots \\ \dfrac{\Gamma^* \to \Delta^*}{\Gamma^*\begin{pmatrix}\alpha \\ V\end{pmatrix} \to \Delta^*\begin{pmatrix}\alpha \\ V\end{pmatrix}} \end{array}.$$

If the end-piece of P contains a weakening, we can reduce P to P^*, where every substitution in P^* has the same degree as the corresponding substitution in P.

(6) In the following we shall assume that the end-piece of a proof with degree contains no logical inference, ind, weakening or axioms other than mathematical axioms. Moreover, we may assume that the proof is different from its end-piece, for if the entire proof is its end-piece, then we can eliminate cuts as mentioned at the beginning of (1).

Let P be a proof with degree. We repeat the definition of a suitable cut: A cut in the end-piece of a proof with degree, P, is called suitable if both of its cut formulas have ancestors which are principal formulas of boundary (hence logical) inferences. We can show, exactly the same way as for **PA**, that under those conditions there exists a suitable cut in the end-piece of P.

Now, let P be a proof with degree whose end-sequent is $\to$ and let J be a suitable cut in P. To define the essential reduction, we must treat separately several cases according to the form of the outermost logical symbol of the cut formulas of J.

(7) We shall first treat the case where the outermost logical symbol of J is second order ∀. Let P be of the following form:

$$\begin{array}{lll}
 & & \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
 & J_0 & \dfrac{\Gamma_1 \overset{\lambda}{\to} \Delta_1, F_1(\alpha)}{\Gamma_1 \overset{(\omega,0,\lambda)}{\to} \Delta_1, \forall\phi\, F_1(\phi)} \qquad \dfrac{F_2(V), \Pi_1 \overset{\mu}{\to} \Lambda_1}{\forall\phi\, F_2(\phi), \Pi_1 \overset{(\omega,n+2,\mu)}{\to} \Lambda_1} \\
S_1 & & \qquad\qquad\qquad\qquad\qquad\quad S_3 \\
 & & \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
S_2 & & \Gamma_2 \overset{\tau}{\to} \Delta_2, \forall\phi\, F(\phi) \qquad S_4 \qquad \forall\phi\, F(\phi), \Pi_2 \overset{\rho}{\to} \Lambda_2 \\
S_5 & J & \overline{\qquad\qquad \Gamma_2, \Pi_2 \overset{(\omega,m+1,\tau\#\rho)}{\to} \Delta_2, \Lambda_2 \qquad\qquad} \\
 & & \vdots \\
S_6 & & \Gamma_3 \overset{\nu}{\to} \Delta_3 \\
 & & \vdots \\
 & & \overset{\sigma}{\to}
\end{array}$$

where $m = g(\forall\phi\, F(\phi))$, $n = g(F_2(V))$, and S_6: $\Gamma_3 \to \Delta_3$ is the i-resolvent of

S_5: $\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2$, i being $d(\forall\phi\, F_1(\phi))$. Here we should remark that the i-resolvent $\Gamma_3 \rightarrow \Delta_3$ will be used only for the case when $\forall\phi\, F(\phi)$ is isolated.

Case 1. $\forall\phi\, F(\phi)$ is isolated.

Let, in the above figure,

$$S_1:\quad \Gamma_1 \rightarrow \Delta_1, \forall\phi\, F_1(\phi), \qquad S_2:\quad \Gamma_2 \rightarrow \Delta_2, \forall\phi\, F(\phi),$$

$$S_3:\quad \forall\phi\, F_2(\phi), \Pi_1 \rightarrow \Lambda_1, \qquad S_4:\quad \forall\phi\, F(\phi), \Pi_2 \rightarrow \Lambda_2.$$

Here we should remark that $\forall\phi\, F_1(\phi)$ and $\forall\phi\, F_2(\phi)$ are $\forall\phi\, F(\phi)$ itself up to term-replacement; that is, no substitution applies to those formulas, for if there were a substitution with degree k between S_1 and S_2 which applies to $\forall\phi\, F_1(\phi)$, then this substitution would disturb $\forall\phi\, F_1(\phi)$. But this implies that $k < i$, which contradicts 2.8) of Definition 27.12. Thus $\forall\phi\, F_1(\phi)$ is $\forall\phi\, F(\phi)$ up to term-replacement. By the same reasoning, $\forall\phi\, F_2(\phi)$ is $\forall\phi\, F(\phi)$ up to term-replacement. In the inference J_0, $d(F_1(\phi)) < i$ $(= d(\forall\phi\, F_1(\phi)))$. Let P' be the following:

$$S_1' \quad \frac{\Gamma_1 \xrightarrow{\lambda'} \Delta_1, F_1(\alpha)}{\Gamma_1 \rightarrow F_1(\alpha), \Delta_1, \forall\phi\, F_1(\phi)} \qquad S_3 \quad \frac{F_2(V), \Pi_1 \xrightarrow{\mu} \Lambda_1}{\forall\phi\, F_2(\phi), \Pi_1 \rightarrow \Lambda_1}$$

$$\frac{S_2' \quad \Gamma_2 \xrightarrow{\tau'} F(\alpha), \Delta_2, \forall\phi\, F(\phi) \qquad S_4 \quad \forall\phi\, F(\phi), \Pi_2 \xrightarrow{\rho} \Lambda_2}{S_5' \quad \Gamma_2, \Pi_2 \xrightarrow{(\omega, m+1, \tau'\#\rho)} F(\alpha), \Delta_2, \Lambda_2}$$

$$\frac{\dfrac{S_6' \quad \Gamma_3 \xrightarrow{\theta} \Delta_3, F(\alpha)}{S_7 \quad \Gamma_3 \xrightarrow{(i,0,\theta)} \Delta_3, F(V)}\, J_1 \qquad S_3' \quad F(V), \Pi_1 \xrightarrow{\mu} \Lambda_1}{\dfrac{\Gamma_3, \Pi_1 \xrightarrow{(\omega, n+1, (i,0,\theta)\#\mu)} \Delta_3, \Lambda_1}{S_8 \quad \forall\phi\, F(\phi), \Pi_1, \Gamma_3 \rightarrow \Delta_3, \Lambda_1}}$$

$$\frac{S_1 \quad \Gamma_2 \xrightarrow{\tau} \Delta_2, \forall\phi\, F(\phi) \qquad S_9 \quad \forall\phi\, F(\phi), \Pi_2, \Gamma_3 \xrightarrow{\rho'} \Delta_3, \Lambda_2}{\dfrac{\Gamma_2, \Pi_2, \Gamma_3 \xrightarrow{(\omega, m+1, \tau\#\rho')} \Delta_2, \Delta_3, \Lambda_2}{S_{10} \quad \Gamma_2, \Pi_2, \Gamma_3 \rightarrow \Delta_3, \Delta_2, \Lambda_2}}$$

$$\frac{\Gamma_3, \Gamma_3 \rightarrow \Delta_3, \Delta_3}{S_{11} \quad \Gamma_3 \xrightarrow{\nu'} \Delta_3}$$

$$\xrightarrow{\sigma'}$$

where J_1 is a substitution whose eigenvariable is α and whose degree is defined to be i. Every substitution in this proof other than J_1 is assigned the same degree as the corresponding substitution in P. Here we should remark that, in the upper sequent of J_1, the descendant of $F_1(\alpha)$ in $\Gamma_1 \to \Delta_1, F_1(\alpha)$ is $F(\alpha)$. As was remarked, no substitution disturbs $F(\alpha)$ between $\Gamma_1 \to \Delta_1, F_1(\alpha)$ and $\Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2$ in P. If there were such a substitution with degree k between $\Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2$ and $\Gamma_3 \to \Delta_3$, it would disturb $\forall\phi\, F(\phi)$, i.e., $k < i$. But this contradicts the fact that $\Gamma_3 \to \Delta_3$ is the i-resolvent of $\Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2$.

We shall show that P' is a proof with degree. For this it is sufficient to show that $d(F_1(\alpha); P') < i$. If there is an inference other than J_1 which disturbs $F_1(\alpha)$ (in P'), the corresponding substitution in P disturbs $F_1(\phi)$. J_1 does not disturb $F(\alpha)$ for otherwise the outermost $\forall$ of $\forall\phi\, F_1(\phi)$ affects another $\forall$ for a second order variable in $\forall\phi\, F_1(\phi)$. But this contradicts the fact that $\forall\phi\, F_1(\phi)$ is isolated. So $d(F_1(\alpha); P') = d(F_1(\alpha); P) < i$.

In order to prove $O(P') <_0 O(P)$, or $\sigma' <_0 \sigma$, we first prove $\nu' <_j \nu$, where $j \leqslant \omega$, and, for any j, $0 \leqslant j < i$, and a j-section of ν', say $\bar{\eta}$, there is a j-section of ν, say $\bar{\xi}$, such that $\eta \leqslant_j \xi$. This is shown below (cf. (7.5)).

(7.1) For any $j \leqslant \omega$, $\tau' <_j \tau$, where $\tau = O(S_2; P)$ and $\tau' = O(S_2; P')$. If $j < \omega$, and $\bar{\alpha}$ is any j-section of τ', then there exists a j-section of τ, say $\bar{\beta}$, such that $\alpha \leqslant_j \beta$.

PROOF. Since there is no substitution above S_1 we see by the Main Lemma, with $p = 0$, that it is sufficient to show that $\lambda' <_j (\omega, 0, \lambda)$ for all $j \leqslant \omega$.

1) $j = \omega$. Then $\lambda' \leqslant_\omega \lambda$, by the definition of P'. Obviously $\lambda <_\omega (\omega, 0, \lambda)$. Therefore $\lambda' <_\omega (\omega, 0, \lambda)$.

2) $j < \omega$. Since there is no j-section of λ', $\lambda' <_j (\omega, 0, \lambda)$ if $\lambda' <_\omega (\omega, 0, \lambda)$. But $\lambda' <_\omega (\omega, 0, \lambda)$ by 1).

(7.2) $\theta <_j \nu$ for each $j \leqslant \omega$. If $j < \omega$, then for each j-section $\bar{\alpha}$ of θ, there exists a j-section $\bar{\beta}$ of ν such that $\alpha \leqslant_j \beta$.

PROOF. By the Main Lemma, with $p = 0$, it is sufficient to prove

1) $(\omega, m+1, \tau' \# \rho) <_j (\omega, m+1, \tau \# \rho)$ for all $j \leqslant \omega$, and

2) for each $j < \omega$, and for each j-section $\bar{\alpha}$ of τ', there exists a j-section $\bar{\beta}$ of τ such that $\alpha \leqslant_j \beta$.

But 2) is part of (7.1). We therefore need only prove 1). This we will do by induction on the total number of indices greater than j (super indices of j) in $(\omega, m+1, \tau' \# \rho)$ and in $(\omega, m+1, \tau \# \rho)$.

i) Since $\tau' <_\omega \tau$ it follows from (7.1) that

$$(\omega, m+1, \tau' \# \rho) <_\infty (\omega, m+1, \tau \# \rho).$$

ii) If $j = \omega$, then $\tau' \# \rho <_\omega \tau \# \beta$ by (7.1). Since $\tau \# \rho <_\omega (\omega, m+1, \tau \# \rho)$, 1) follows from i).

iii) If $j < \omega$, then by the induction hypothesis and i),

$$(\omega, m + 1, \tau' \# \rho) <_{j_1} (\omega, m + 1, \tau \# \rho),$$

where $j_1 = j_0(j, (\omega, m + 1, \tau' \# \rho), (\omega, m + 1, \tau \# \rho))$. Let $\bar{\alpha}$ be a j-section of $(\omega, m + 1, \tau \# \rho)$. Then $\bar{\alpha}$ is also a j-section of $\tau' \# \rho$. If $\bar{\alpha}$ is a j-section of τ', then $\alpha <_j (\omega, m + 1, \tau \# \rho)$ by (7.1). If $\bar{\alpha}$ is a j-section of ρ, then $\bar{\alpha}$ is a j-section of $(\omega, m + 1, \tau \# \rho)$ and hence $\alpha <_j (\omega, m + 1, \tau \# \rho)$. Thus

$$(\omega, m + 1, \tau' \# \rho) <_j (\omega, m + 1, \tau \# \rho).$$

(7.3) $O(S_8; P') <_j O(S_3; P)$ for $i < j \leqslant \omega$ or $j = \infty$.

PROOF. $O(S_3) = (\omega, n + 2, \mu)$ and $O(S_8) = (\omega, n + 1, (i, 0, \nu) \# \mu)$. The proof is by induction on $\iota(j, O(S_3), O(S_8))$.

1) Since $n + 1 < n + 2$, $O(S_8) <_\infty O(S_3)$.

2) If $j = \omega$, then since $\mu <_\infty O(S_3)$ and $O(S_8) <_\infty O(S_3)$, it is sufficient to prove that $(i, 0, \theta) <_\omega O(S_3)$. But this is clearly the case since $(i, 0, \theta)$ has no ω-section and $(i, 0, \theta) <_\infty O(S_3)$.

3) If $i < j < \omega$, then $O(S_8) <_j O(S_3)$ because neither $(i, 0, \theta)$ nor μ has a j-section and from 2) $O(S_8) <_\omega O(S_3)$.

(7.4) If $i < j \leqslant \omega$, then $\rho' <_j \rho$. If $i < j < \omega$ then for each j-section $\bar{\alpha}$ of ρ' there exists a j-section $\bar{\beta}$ of ρ such that $\alpha \leqslant_j \beta$.

PROOF. Let us regard $i + 1$, $O(S_4)(= \rho)$, and $O(S_9)(= \rho')$ as ρ, γ, and δ, respectively, in the Main Lemma. Let γ_0 $(= O(S_4))$, $\gamma_1, \ldots, \gamma_m$ $(= O(S_3))$ be the sequence of distinct ordinal diagrams of sequents from S_4 to S_3 in P and let δ_0 $(= O(S_9))$, $\delta_1, \ldots, \delta_m$ $(= O(S_8))$ be the sequence of distinct ordinal diagrams of sequents from S_9 to S_8 in P'. The proposition then follows from the Main Lemma and (7.3). Here we should recall that $O(S_8)$ has no j-section if $i < j < \omega$.

(7.5) $\nu' <_j \nu$ for $j \leqslant \omega$.

PROOF. We first show that $\nu' <_j \nu$ for any j, $i < j \leqslant \omega$. Let p be any number, $i < p \leqslant \omega$ and let $p \leqslant j \leqslant \omega$. Take $O(S_6)(= \nu)$, $O(S_4)(= \nu')$ and O as γ, δ, p respectively in the Main Lemma. Let γ_0 $(= O(S_6)), \ldots, \gamma_m$ $(= O(S_5))$ be the sequence of distinct ordinal diagrams of sequents from S_6 to S_5 in P and let

$$\delta_0\ (= O(S_{11})), \ldots, \delta_m\ (= O(S_{10}))$$

be the sequence of distinct ordinal diagrams of sequents from S_{11} to S_{10} in P'. We then only have to prove that the conditions of the Main Lemma are satisfied for $O(S_5)$ and $O(S_{10})$. This we prove by induction on $\iota(j, O(S_5), O(S_{10}))$, where $O(S_5) = (\omega, m+1, \tau \,\#\, \rho)$ and $O(S_{10}) = (\omega, m+1, \tau \,\#\, \rho')$.

1) From (7.4) $\rho' <_\omega \rho$. Therefore, $O(S_{10}) <_\infty O(S_5)$.

In 2)–3) we assume that $O(S_{10}) <_j O(S_5)$, where $j_1 = j_0(j, O(S_5), O(S_{10}))$.

2) If $j = \omega$, then $O(S_{10}) <_\omega O(S_5)$ provided $\tau \,\#\, \rho' <_\omega O(S_5)$ and $O(S_{10}) <_\infty O(S_5)$. But this follows from (7.4) and 1).

3) If $p \leqslant j < \omega$, then $O(S_{10}) <_j O(S_5)$ provided for each j-section $\bar{\alpha}$ of $O(S_{10})$, $\alpha <_j O(S_5)$. Let $\bar{\alpha}$ be a j-section of $O(S_{10})$, i.e., of $\tau \,\#\, \rho'$. If $\bar{\alpha}$ is a j-section of τ, then $\bar{\alpha}$ is a j-section of $O(S_5)$ as well. Therefore, $\alpha <_j O(S_5)$. If $\bar{\alpha}$ is a j-section of ρ', then $\alpha <_j O(S_5)$ by (7.4).

Having established $\nu' <_j \nu$, $i < j \leqslant \omega$, now consider an i-section of ν'. If it is not $\bar{\theta}$, then it is an i-section of ν. If it is $\bar{\theta}$, then $\theta <_i \nu$ has been established in (7.2). For $j < i$, let $\bar{\alpha}$ be a j-section of ν'. It can be easily shown that there is a j-section of ν whose o. d. is α. So $\alpha <_j \nu$. Thus $\nu' <_j \nu$ for any $j \leqslant i$. This completes the first objective, $\nu' <_j \nu$ for all $j \leqslant \omega$.

Next, recall that either $\Gamma_3 \to \Delta_3$ is the end-sequent or $\Gamma_3 \to \Delta_3$ is the upper sequent of a substitution of degree ($= k_0$) $< i$. If the former is the case, then $\nu' <_0 \nu$ means $\alpha' <_0 \sigma$. Suppose the latter is the case. Then $\alpha' <_0 \sigma$ follows from (7.5) by virtue of the Main Lemma; notice that k_0 as above prevents $\bar{\theta}$ from being an i-section of an o. d. between ν' and α'.

Case 2. $\forall\phi\, F(\phi)$ is not isolated.

Let P'' have the following form:

$$\dfrac{\dfrac{\Gamma_1 \to \Delta_1, F_1(V)}{\text{some exchanges and a weakening}}}{\Gamma_1 \to F_1(V), \Delta_1, \forall\phi\, F_1(\phi)} \qquad\qquad \dfrac{\dfrac{F_2(V), \Pi_1 \to \Lambda_1}{\text{some exchanges and a weakening}}}{\forall\phi\, F_2(\phi), \Pi_1, F_2(V) \to \Lambda_1}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_2 \to F(V), \Delta_2, \forall\phi\, F(\phi) \quad \forall\phi\, F(\phi), \Pi_2 \to \Lambda_2}{\Gamma_2, \Pi_2 \to F(V), \Delta_2, \Lambda_2}}{\text{some exchanges}}}{\Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2, F(V)} \quad \dfrac{\dfrac{\dfrac{\Gamma_2 \to \Delta_2, \forall\phi\, F(\phi) \quad \forall\phi\, F(\phi), \Pi_2, F(V) \to \Lambda_2}{\Gamma_2, \Pi_2, F(V) \to \Delta_2, \Lambda_2}}{\text{some exchanges}}}{F(V), \Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2}}{\dfrac{\dfrac{\Gamma_2, \Pi_2, \Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2, \Delta_2, \Lambda_2}{\text{some exchanges and contractions}}}{\Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2}}$$

$$\sigma''$$

$$\to$$

where every substitution is assigned the same degree as the corresponding substitution in P, and the proof of $\Gamma_1 \to \Delta_1, F_1(V)$ is obtained from the proof of $\Gamma_1 \to \Delta_1, F_1(\alpha)$ by substituting V for α everywhere. Since V is isolated, $\gamma(G(V)) = \gamma(G(\alpha))$ by Proposition 27.7. Hence the ordinal diagram of $\Gamma_1 \to \Delta_1$, $F_1(V)$ is not greater than λ in the sense of $<_j$ for every j.

P'' is clearly a proof with degree. Since $g(F(V)) < g(\forall\phi\, F(\phi))$, we can easily see that $\sigma'' <_0 \sigma$.

(8) Next we treat the case in which the outermost logical symbol of the cut formula of J is $\wedge$. Let P be of the following form:

$$
\cfrac{\begin{array}{c} \cfrac{\overset{\vdots}{\Gamma_1 \to \Delta_1, A_1} \qquad \overset{\vdots}{\Gamma_2 \to \Delta_2, B_1}}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, A_1 \wedge B_1} \\ \vdots \\ \Gamma_3 \to \Delta_3, A \wedge B \end{array} \qquad \begin{array}{c} \cfrac{\overset{\vdots}{A_2, \Pi_2 \to \Lambda_1}}{A_2 \wedge B_2, \Pi_1 \to \Lambda_1} \\ \vdots \\ A \wedge B, \Pi_2 \to \Lambda_2 \end{array}}{\begin{array}{c} \Gamma_3, \Pi_2 \to \Delta_3, \Lambda_2 \\ \vdots \\ \to \end{array}}
$$

We see that P can be reduced to a P' of the following form:

$$
\cfrac{\begin{array}{c}\cfrac{\cfrac{\cfrac{\begin{array}{c} \cfrac{\overset{\vdots}{\Gamma_1 \to \Delta_1, A_1}}{\Gamma_1, \Gamma_2 \to A_1, \Delta_1, \Delta_2, A_1 \wedge B_1}\ \text{some exchanges and a weakening} \\ \vdots \\ \Gamma_3 \to A, \Delta_3, A \wedge B \end{array} \qquad \overset{\vdots}{A \wedge B, \Pi_2 \to \Lambda_2}}{\Gamma_3, \Pi_2 \to A, \Delta_3, \Lambda_2}}{\text{some exchanges}}}{\Gamma_3, \Pi_2 \to \Delta_3, \Lambda_2, A}\end{array} \qquad \begin{array}{c}\cfrac{\cfrac{\cfrac{\overset{\vdots}{\Gamma_3 \to \Delta_3, A \wedge B} \qquad \begin{array}{c} \cfrac{\overset{\vdots}{A_2, \Pi_1 \to \Lambda_1}}{A_2 \wedge B_2, \Pi_1, A_2 \to \Lambda_1}\ \text{some exchanges and a weakening} \\ \vdots \\ A \wedge B, \Pi_2, A \to \Lambda_2 \end{array}}{\Gamma_3, \Pi_2, A \to \Delta_3, \Lambda_2}}{\text{some exchanges}}}{A, \Gamma_3, \Pi_2 \to \Delta_3, \Lambda_2}\end{array}}{\cfrac{\cfrac{\Gamma_3, \Pi_2, \Gamma_3, \Pi_2 \to \Delta_3, \Lambda_2, \Delta_3, \Lambda_2}{\text{some exchanges and contractions}}}{\begin{array}{c}\Gamma_3, \Pi_2 \to \Delta_3, \Lambda_2 \\ \vdots \\ \to\end{array}}}
$$

Every substitution in P' is assigned the same degree as the corresponding substitution in P. Thus P' is a proof with degree whose ordinal diagram is less than that of P.

(9) The remaining cases, i.e., the case in which the outermost logical symbol of the cut-formula of J is $\neg$ and the case in which the outermost logical symbol of the cut-formula of J is $\forall$, for a first order variable, are treated in the same way as the above cases.

This completes the proof of Theorem 27.5.

§28. A consistency proof for a system with inductive definitions

In this section we will prove the consistency of a system obtained from **INN** by adding inductive definitions with Π_1^1-clauses. This system we call the system of isolated inductive definitions, **IID**.

DEFINITION 28.1. **IID** is the system **INN** with the following modifications.

1) **IID** contains a unary primitive recursive predicate I and a binary primitive recursive predicate $<^*$, where $<^*$ is a well-ordering of $\{a \mid I(a)\}$.

2) **IID** contains ternary predicate symbols $A_0, A_1, \ldots$ for which $A_n(s, t, V)$ is an atomic formula for s and t terms and V an abstract.

3) If $\forall\phi\, B$ is a semi-formula of **IID**, then the outermost quantifier $\forall$ affects A_n in B if there is a ϕ in an argument of A_n, i.e., if A_n occurs in B in the form $A_n(a, b, V)$ and ϕ occurs in V.

4) A semi-formula or abstract A of **IID** is isolated if no $\forall$ for a second order variable affects any other $\forall$ for a second order variable or $A_0, A_1, \ldots$, in A.

5) The initial sequents of **IID** are those of **INN**, extended to include formulas with A_n's, and the sequents of the following forms:

$$I(s),\, A_n(s, t, V) \rightarrow G_n(s, t, V, \{x, y\}(A_n(x, y, V) \wedge x <^* s))$$

$$I(s),\, G_n(s, t, V, \{x, y\})(A_n(x, y, V) \wedge x <^* s) \rightarrow A_n(s, t, V)$$

for $n = 0, 1, 2, \ldots$. Each $G_n(a, b, \alpha, \beta)$ is an arbitrary isolated formula containing none of $A_n, A_{n+1}, \ldots$, and V is an arbitrary abstract, which may contain $\forall$ for second order variables or $A_n, A_{n+1}, \ldots$.

6) The rules of inference for **IID** are those of **INN**.

The purpose of this section is to prove the consistency of **IID**:

THEOREM 28.2. **IID** *is consistent.*

PROOF. This we will prove using the system of ordinal diagrams

$$O(\omega^{I_\infty} + 1,\, \omega^{I_\infty} \cdot \omega \cdot \omega^{I_\omega}),$$

where $I_\infty = (2 \cdot |I| + 1) \cdot \omega$ and $|I|$ is the order-type of $<^*$. The proof is similar to the proof of Theorem 27.5, therefore we will present only new aspects of the proof.

PROPOSITION 28.3. *Let $F(\alpha)$ and V be an isolated formula and an isolated abstract, respectively. Then $F(V)$ is isolated.*

PROOF. (By induction on the number n of logical symbols contained in $F(\alpha)$.) If $n = 0$, the assertion is clear. Let $n > 0$. We shall treat several cases according to the outermost logical symbol of $F(\alpha)$. Since the other cases are easy, we shall consider the case where $F(\alpha)$ is of the form $\forall \phi\, G(\alpha, \phi)$. By the induction hypothesis $G(V, \beta)$ is isolated, where β is a free second order variable not contained in V. We have only to show that the outermost $\forall$ of $\forall \phi\, G(V, \phi)$ affects none of the $\forall$'s for second order variables or $A_0, A_1, \ldots$. But this is obvious since $\forall \phi\, G(\alpha, \phi)$ and $G(V, \beta)$ are isolated.

We next define several well-ordered systems.

DEFINITION 28.4. (1) Let $|I|$ be the ordinal of the well-ordering $<^*$. Let $\tilde{I}$ be $\{\tilde{\imath} \mid i \in I\}$ and let $I_* = I \cup \tilde{I}$. Then $<_*$ is the well-ordering of I_* defined as follows:

(1.1) If $i \in I$, then $i <_* \tilde{\imath}$.

(1.2) If $i <^* j$, then $i <_* j$.

(1.3) If $i <^* j$, then $\tilde{\imath} <_* \tilde{\jmath}$.

(1.4) If $i <^* j$, then $\tilde{\imath} <_* j$.

(1.5) If $i <^* j$, then $i <_* \tilde{\jmath}$.

The ordinal of $<_*$ is $2 \cdot |I|$.

(2) Let n be a natural number. Then $I_n = \{(i, n) \mid i \in I_*\} \cup \{\infty_n\}$ and $<_n$ is the well-ordering of I_n defined as follows:

(2.1) If $i <_* j$, then $(i, n) <_n (j, n)$.

(2.2) If $i \in I_*$, then $(i, n) <_n \infty_n$.

(3) $I_\infty = I_0 \cup I_1 \cup \ldots$ and $<_\infty$ is the well-ordering of I_∞ defined as follows:

(3.1) If $i \in I_n$, $j \in I_m$ and $n < m$, then $i <_\infty j$.

(3.2) If $i < j$ in I_n for some n, then $i <_\infty j$.

The order type of $<_\infty$ is $(2 \cdot |I| + 1) \cdot \omega$.

DEFINITION 28.5. Let A be a formula. The rank of A_n in A, denoted by $r(A_n : A)$, is an element of I_∞ defined as follows:

1) If $A_n(s, t, V) \wedge s <^* i$, occurs in A, where $I(i)$ is provable and either s is a variable or s is a numeral for which one of $\neg I(s)$ or $i \leqslant^* s$ is provable, then $r(A_n : A) = (i, n)$.

2) If $A_n(j, t, V)$ occurs in A, where $I(j)$ is provable, and 1) does not hold, then $r(A_n : A) = (\bar{j}, n)$.

3) If A_n occurs in A and neither 1) nor 2) applies, then $r(A_n : A) = \infty_n$.

PROPOSITION 28.6. *Let B and C be two arbitrary formulas in which A_m and A_n occur, respectively. Then $r(A_m : B) <_\infty r(A_n : C)$ if $m < n$.*

DEFINITION 28.7. The γ-degree of a formula or an abstract, $\gamma(A)$, is a number less than $\omega^{I\infty}$, defined in the following way. Here $<$ is the ordering of $\omega^{I\infty}$.

1) If A is isolated, then $\gamma(A) = 0$.

In 2)–6), A is assumed not to be isolated.

2) If A is of the form $\neg B$, then $\gamma(A) = \gamma(B) + 1$.

3) If A is of the form $A_n(s, t, V) \wedge s <^* i$, then $\gamma(A) = \gamma(V) + \omega^{r(A_n:A)+1}$. If A is of the form $B \wedge C$ and not of the form just mentioned, then $\gamma(A) = \max(\gamma(B), \gamma(C)) + 1$.

4) If A is of the form $\forall x\, (G(x))$, then $\gamma(A) = \gamma(G(a)) + 1$.

5) If A is of the form $\forall \phi\, F(\phi)$, then $\gamma(A) = \gamma(F(\alpha)) + 1$.

6) If A is of the form $A_n(s, t, V)$, then $\gamma(A) = \gamma(V) + \omega^{r(A_n:A)}$.

7) If A is of the form $\{x_1, \ldots, x_n\}B(x_1, \ldots, x_n)$, then $\gamma(A) = \gamma(B(a_1, \ldots, a_n))$.

PROPOSITION 28.8. *Let $\{x_1, \ldots, x_n\}H(x_1, \ldots, x_n)$ be an abstract and let $s_1, \ldots, s_n$ be arbitrary terms. Then*

$$\gamma(H(s_1, \ldots, s_n)) \leqslant \gamma(\{x_1, \ldots, x_n\}H(x_1, \ldots, x_n)).$$

LEMMA 28.9. *If $G(\beta, \alpha)$ is an isolated quasi-formula (allowing other second order free variables as well) which contains none of $A_n, A_{n+1}, \ldots$, if s is a constant for which $I(s)$ is provable, and if V is an arbitrary abstract which is not isolated, then*

$$\gamma(G(V, A_n^s(V))) \leqslant \gamma(V) + \sum_{l=1}^{k} \omega^{r(A_{j_l}:B_l)} + m$$

for some $j_1, \ldots, j_k \leqslant n$, for some formulas $B_1, \ldots, B_k$, and for a number m, where $A_n^s(V)$ is an abbreviation for

$$\{x, y\}(A_n(x, y, V) \wedge x <^* s), \quad \textit{and} \quad r(A_{j_l} : B_l) < r(A_n : A_n)$$

for $l \leqslant k$.

PROOF. By induction on the construction of G.

PROPOSITION 28.10. *If s is a constant for which $I(s)$ is provable, if V is not isolated and if $G_n(a, b, \alpha, \beta)$ is as in Definition* 28.1, *then*

$$\gamma(G_n(s, t, V, A_n^s(V))) < \gamma(A_n(s, t, V)).$$

PROOF. As a special case of Lemma 28.9,

$$\gamma(G(V, A_n^s(V))) \leqslant \gamma(V) + \sum_{l=1}^{k} \omega^{r(A_{j_l}:B_l)} + m,$$

where $r(A_{j_l} : B_l) < r(A_n : A_n)$ and $m < \omega$. On the other hand,

$$\gamma(A_n(s, t, V)) = \gamma(V) + \omega^{\gamma(A_n:A_n)}.$$

The proposition then follows.

Next we add the rule of substitution to the system **IID** (cf. Definition 27.10).

DEFINITION 28.11. A substitution or a $\forall$: right for a second order variable, say J, is said to *disturb* a semi-formula A if the eigenvariable of J occurs in the scope of $\forall$ for a second order variable or in an argument of an A_n occurring in A.

We define a proof with degree to be a proof satisfying the following conditions.

1) Every substitution is in the end-piece, and no ind occurs under a substitution.

2) We can assign an element of $\omega^{I_\infty} + 1$ to every semi-formula or abstract A and every substitution J in the end-piece, which is called the degree of A or of J (written $d(A)$ or $d(J)$), respectively, so as to satisfy the following conditions:

2.1) If A is explicit, then $d(A) = 0$.

2.2) If A is implicit and not isolated, then $d(A) = \omega^{I_\infty}$.

2.3) Let A be implicit and isolated.

2.3.1) $d(A) = 0$ if A contains no logical symbol or $A_0, A_1, \ldots$.

2.3.2) $d(A) = d(B) + 1$ if A is of the form $\neg B$.

2.3.3) $d(A) = \max_J(d(V), d(J)) + \omega^{r(A_n:A)} + 1$, where J ranges over all the substitutions which disturb A, if A is of the form $A_n(s, t, V) \wedge s <^* i$.

$d(A) = \max(d(B), d(C)) + 1$, if A is of the form $B \wedge C$ and not of the form just mentioned.

2.3.4) $d(A) = d(B(x)) + 1$, if A is of the form $\forall x\, B(x)$.

2.3.5) $d(A) = \max_J(d(F(\phi)), d(J)) + 1$, where J ranges over all the substitutions which disturb $\forall\phi\, F(\phi)$, if A is of the form $\forall\phi\, F(\phi)$.

2.3.6) $d(A) = \max_J(d(V), d(J)) + \omega^{r(A_n:A_n)}$, where J ranges over all the substitutions which disturb A, if A is of the form $A_n(s, t, V)$.

3) $d(A) = d(B)$, if A is an abstract of the form $\{x_1, \ldots, x_n\}B$.

4) If J is a substitution in the end-piece, then $d(B) < d(J)$ for every formula B in the upper sequent of J.

5) If J is a substitution, then $0 < d(J) < \omega^{I_\infty}$.

LEMMA 28.12. *Suppose $G(\beta, \alpha)$ is an isolated quasi-formula whose only second order free variables are β and α, and which contains none of $A_n, A_{n+1}, \ldots$. Assume also that i is a constant for which $I(i)$ is provable. If V is isolated, then*

$$d(G(V, A^i_n(V))) \leqslant \max_J(d(V), d(J)) + \sum_{l=1}^{k} \omega^{r(A_{j_l}:B_l)} + m,$$

for some $j_1, \ldots, j_k \leqslant n$, some $B_1, \ldots, B_k$ and a number $m < \omega$, where $r(A_{j_l} : B_l) <_\infty r(A_n : A_n)$ and J ranges over all substitutions which influence V.

PROPOSITION 28.13. *Suppose $A_n(i, t, V)$ is isolated, i.e., V is isolated, and i is a constant for which $I(i)$ is provable. If either*

$$I(i), A_n(i, t, V) \to G_n(i, t, V, A^i_n(V))$$

or

$$I(i), G_n(i, t, V, A^i_n(V)) \to A_n(i, t, V)$$

is an initial sequent in a proof with degree, in which $A_n(i, t, V)$ is implicit, then

$$d(G_n(i, t, V, A^i_n(V))) < d(A_n(i, t, V)).$$

PROOF. This is a special case of Lemma 28.12.

DEFINITION 28.14. Let A be a semi-formula or an abstract. We define the *norm* of A, $n(A)$, to be an element of ω^{I_∞} as follows:

1) If A contains no logical symbol or $A_0, A_1, A_2, \ldots$, then $n(A) = 0$.

2) If A is of the form $\neg B$, then $n(A) = n(B) + 1$.

3) If A is of the form $A_n(s, t, V) \wedge s <^* i$, then $n(A) = n(V) + \omega^{r(A_n:A)} + 1$. If A is of the form $B \wedge C$ and not of the above form, then

$$n(A) = \max(n(B), n(C)) + 1.$$

4) If A is of the form $\forall x\, B(x)$, then $n(A) = n(B(a)) + 1$.
5) If A is of the form $\forall \phi\, F(\phi)$, then $n(A) = n(F(\alpha)) + 1$.
6) If A is of the form $A_n(s, t, V)$, then $n(A) = n(V) + \omega^{r(A_n : A)}$.
7) If A is of the form $\{x_1, \ldots, x_m\}H(x_1, \ldots, x_m)$, then $n(A) = n(H(a_1, \ldots, a_m))$.

LEMMA 28.15. *If $G(\beta, \alpha)$ contains none of $A_n, A_{n+1}, \ldots$, if i is a constant for which $I(i)$ is provable and if V is an arbitrary abstract, then*

$$n(G(V, A_n^i(V))) \leqslant n(V) + \sum_{l=1}^{k} \omega^{r(A_{j_l} : B_l)} + m,$$

where $j_l \leqslant n$, $r(A_{j_l} : B_l) < r(A_n : A_n)$ and $m < \omega$.

PROPOSITION 28.16. *If*

$$I(i), G_n(i, t, V, A_n^i(V)) \rightarrow A_n(i, t, V)$$

or

$$I(i), A_n(i, t, V) \rightarrow G_n(i, t, V, A_n^i(V))$$

is an initial sequent of our system, and i is a constant for which $I(i)$ is provable, then

$$n(G_n(i, t, V, A_n^i(V))) < n(A_n(i, t, V)).$$

PROOF. A special case of Lemma 28.15.

DEFINITION 28.17. Let $N(I_\infty) = \omega^{I_\infty} \times \omega \times \omega^{I_\infty}$ and let $\prec$ be the lexicographical ordering of $N(I_\infty)$. The *grade* of a formula A, $g(A)$, is $\langle \gamma(A), a, n(A) \rangle$, where a is the number of eigenvariables in A for second order $\forall$: right under A, and $g(A)$ is an element of (I_∞).

PROPOSITION 28.18. *If*

$$I(i), A_n(i, t, V) \rightarrow G_n(i, t, V, A_n^i(V))$$

or

$$I(i), G_n(i, t, V, A_n^i(i, t, V)$$

is an initial sequent of a proof with degree, and i is a constant for which $I(i)$ is provable, then

$$g(G_n(i, t, V, A_n^i(V))) \prec g(A_n(i, t, V)).$$

DEFINITION 28.19. We shall assign an element of $O(\omega^{I_\infty} + 1, \omega^{I_\infty} \times \omega \times \omega^{I_\infty})$ to every sequent of a proof P with degree as follows. We denote ω^{I_∞}, the maximum element of $\omega^{I_\infty} + 1$, by ξ.

1.1) The ordinal diagram of an initial sequent of the form

$$D \rightarrow D, \; s = t, A(s) \rightarrow A(t)$$

or a mathematical initial sequent is $\langle 0, 0, 0 \rangle$.

1.2) The ordinal diagram of an initial sequent of the form

$$I(i), A_n(i, t, V) \rightarrow G_n(i, t, V, \{x, y\}(A_n(x, y, V) \wedge x <^* i))$$

or

$$I(i), G_n(i, t, V, \{x, y\}(A_n(x, y, V) \wedge x <^* i)) \rightarrow A_n(i, t, V)$$

is $g(A_n(i, t, V))$.

2) If S_1 and S_2 are the upper sequent and the lower sequent of a weak, structural inference, then the ordinal diagram of S_2 is equal to that of S_1.

3) If S_1 and S_2 are the upper sequent and the lower sequent of one of the inferences $\neg$, $\wedge$: left, $\forall$ for a first order variable, $\forall$: right for a second order variable and explicit $\forall$: left for a second order variable, then the ordinal diagram of S_2 is $(\xi, \langle 0, 0, 0 \rangle, \sigma)$, where σ is the ordinal diagram of S_1.

4) If S_1 and S_2 are the upper sequents and S is the lower sequent of $\wedge$: right, then the ordinal diagram of S is $(\xi, \langle 0, 0, 0 \rangle, \sigma_1 \# \sigma_2)$, where σ_1 and σ_2 are the ordinal diagrams of S_1 and S_2, respectively.

5) If S_1 and S_2 are the upper sequent and the lower sequent of an implicit $\forall$: left for a second order variable of the form

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall \phi\, F(\phi), \Gamma \rightarrow \Delta},$$

then the ordinal diagram of S_2 is $(\xi, \langle \mu, k, \nu \# 0 \# 0 \rangle, \sigma)$, where σ is the ordinal diagram of S_1 and $\langle \mu, k, \nu \rangle$ is $g(F(V))$.

6) If S_1 and S_2 are the upper sequents and S is the lower sequent of a cut, then the ordinal diagram of S is $(\xi, \langle \mu, k, \nu \# 0 \rangle, \sigma_1 \# \sigma_2)$, where $\langle \mu, k, \nu \rangle$ is the grade of the cut-formula and σ_1 and σ_2 are the ordinal diagrams of S_1 and S_2, respectively.

7) If S_1 and S_2 are the upper sequent and the lower sequent of a substitution J, then the ordinal diagram of S_2 is $(d(J), \langle 0, 0, 0 \rangle, \sigma)$, where σ is the ordinal diagram of S_1.

8) If S_1 and S_2 are the upper and the lower sequents of an ind, then the ordinal diagram of S_2 is $(\xi, \langle \mu, k, \nu \# 0 \# 0 \rangle, \sigma)$, where $\langle \mu, k, \nu \rangle$ is the grade of the induction formula and σ is the ordinal diagram of S_1.

9) The ordinal diagram of P is defined to be the ordinal diagram assigned to the end-sequent of P.

Suppose the sequent $\rightarrow$ is provable in this system. We shall reduce a proof P of $\rightarrow$ to another proof of $\rightarrow$. This reduction will be carried out in the same way as in §27. We can assume that the end-piece of P contains no first order free variable, ind, axiom of the form $m = n$, $A(m) \rightarrow A(n)$ or $D \rightarrow D$, or weakening and we assume that term-replacement has been introduced. Suppose that the end-piece of P contains an initial sequent of the form 5) of Definition 28.1, say

$$(*) \qquad I(i), A_n(i, t, V) \rightarrow G_n(i, t, V, \{x, y\}(A_n(x, y, V) \wedge x <^* i)),$$

where we can assume without loss of generality that i and t are numerals. By our assumption either $I(i) \rightarrow$ or $\rightarrow I(i)$ is an initial sequent. We shall abbreviate $\{x, y\}(A_n(x, y, V) \wedge x <^* i)$ as $A_n^i(V)$.

Case 1. $I(i) \rightarrow$ is an initial sequent. Replace (*) by the following:

$$\cfrac{\cfrac{I(i) \rightarrow}{\text{weakenings and an exchange}}}{I(i), A_n(i, t, V) \rightarrow G_n(i, t, V, A_n^i(V)).}$$

The ordinal diagram of the proof is less than that of (*). Hence evidently P is reduced to the proof obtained by replacement.

Case 2. $\rightarrow I(i)$ is an initial sequent. Since every formula in P is implicity, there exists a cut J where one of the cut-formulas is a descendant of $A_n(i, t, V)$ in (*). Let P be of the following form:

$$\begin{array}{c}
A_n(i, t, V) \rightarrow A_n(i, t, V) \qquad\qquad I(i), A_n(i, t, V) \rightarrow G_n(i, t, V, A_n^i(V)) \\
\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
J \; \cfrac{\Gamma \xrightarrow{\sigma_1} \Delta, A_n(i, t, V) \qquad A_n(i, t, V), \Pi \xrightarrow{\sigma_2} \Lambda}{\Gamma, \Pi \xrightarrow{\sigma} \Delta, \Lambda}, \\
\vdots \\
\rightarrow
\end{array}$$

where $A_n(i, t, V) \rightarrow A_n(i, t, V)$ need not appear. Here we should note that no substitution applies to $A_n(i, t, V)$: in fact, if there were such a substitution J_0, it would disturb $A_n(i, t, V)$, i.e., $d(J_0) < d(A_n(i, t, v))$. But this contradicts 4) of Definition 28.11.

Consider the following proof P':

$$
\dfrac{\dfrac{\dfrac{\dfrac{I(i), A_n(i,t,V) \to G_n(i,t,V,A^i_n(V))}{A_n(i,t,V), I(i) \to G_n(i,t,V,A^i_n(V))}}{\vdots}}{\Gamma, I(i) \xrightarrow{\sigma_1'} \Delta, G_n(i,t,V,A^i_n(V))} \quad \dfrac{\dfrac{G_n(i,t,V,A^i_n(V)) \to G_n(i,t,V,A^i_n(V))}{\vdots}}{G_n(i,t,V,A^i_n(V)), \Pi \xrightarrow{\sigma_2'} \Lambda}}{\Gamma, I(i), \Pi \to \Delta, \Lambda}
$$

$$
\dfrac{\to I(i) \qquad \dfrac{\Gamma, I(i), \Pi \to \Delta, \Lambda}{I(i), \Gamma, \Pi \to \Delta, \Lambda}\ \text{some exchanges}}{\dfrac{\Gamma, \Pi \xrightarrow{\sigma'} \Delta, \Lambda}{\dfrac{\vdots}{\to}}}
$$

Every substitution in P' has the same degree as the corresponding substitution in P. Then P' is a proof with degree by virtue of Proposition 28.10. Furthermore,

$$\sigma = (\xi, \langle \mu, j, \lambda \# 0 \rangle, \sigma_1 \# \sigma_2)$$

and

$$\sigma' = (\xi, \langle 0, 0, 0 \# 0 \rangle, \langle 0, 0, 0 \rangle \# (\xi, \langle \nu, k, \delta \# 0 \rangle, \sigma_1' \# \sigma_2')),$$

where $\langle \mu, j, \lambda \rangle = g(A_n(i, t, V))$, and $\langle \nu, k, \delta \rangle = g(G_n(i, t, V, A^i_n(V)))$. Proposition 6 implies that $\sigma' <_l \sigma (l \leqslant \xi)$, from which it follows that the ordinal diagram of P' is less than that of P. Thus P is reduced to P'. (For the computation of ordinal diagrams, one should refer to §27.)

Suppose that the end-piece of P does not contain a logical inference, ind, or initial sequents other than mathematical ones, or weakening. If P contains a logical symbol, we can find a suitable cut in P in the same way as in 26.16 and define an essential reduction in the same way as in §27.

As an addendum to this section, as well as the previous section, we shall explain the general theory of γ-degree. We consider a second order language.

DEFINITION 28.20. A function γ from semi-formulas and abstracts to ordinals is called *monotone* if it satisfies the following conditions.

1) $\gamma(\neg A) \geqslant \gamma(A)$.
2) $\gamma(A \wedge B) \geqslant \max(\gamma(A), \gamma(B))$.
3) $\gamma(\forall x\, G(x)) \geqslant \gamma(G(x))$.
4) $\gamma(\{x_1, \ldots, x_n\}H(x_1, \ldots, x_n)) = \gamma(H(x_1, \ldots, x_n))$.
5) $\gamma(\forall \phi\, F(\phi)) \geqslant \gamma(F(\phi))$.
6) If A is an alphabetical variant of B, then $\gamma(A) = \gamma(B)$.

7) If $\gamma(V) = 0$ and $\gamma(\forall\phi\, F(\phi)) > 0$, then $\gamma(\forall\phi\, F(\phi)) > \gamma(F(V))$.

We say that A is *γ-simple* if $\gamma(A) = 0$.

A second order $\forall$: left, say

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall\phi\, F(\phi), \Gamma \rightarrow \Delta},$$

is called *γ-simple* if V is γ-simple; it is called *strictly γ-simple* if both V and $\forall\phi\, F(\phi)$ are γ-simple.

A proof P in $\mathbf{G^1LC}$ is called (strictly) γ-simple if every implicit, second order $\forall$: left in P is (strictly) γ-simple.

PROPOSITION 28.21. *Suppose γ is monotone and for every strictly γ-simple proof the cut-elimination theorem holds. Then the cut-elimination theorem holds for every γ-simple proof.*

PROOF. The grade of a formula in a proof, say A, is defined as $\omega^2 \cdot \gamma(A) + \omega \cdot m + l$, where m is the number of eigenvariables of the second order $\forall$: right introductions which occur under A, and l is the number of logical symbols in A. The grade of A will be denoted by $g(A)$. Let P be a γ-simple proof and let J be a cut in P. J is called "γ-simple" if the cut formula of J is γ-simple. The grade of J, $g(J)$, is defined to be the grade of the cut formula of J. The grade of P, $g(P)$, is defined to be $\sum_J \omega^{g(J)}$, where J ranges over all the cuts in P which are not γ-simple, and we assume that $\omega^{g(J)}$ in $\sum$ are arranged in the decreasing order.

If $g(P) = 0$, then there is no implicit formula which is not γ-simple, in particular, the principal formula of every implicit $\forall$: left is γ-simple, which means that P is strictly γ-simple. Therefore, by the assumption of the proposition, the cut-elimination theorem holds for P. Suppose now that $g(P) > 0$; hence there is a cut J in P which is not γ-simple and such that every cut above J is γ-simple. Since other cases are easily treated, we shall deal with the case where the cut formula is of the form $\forall\phi\, F(\phi)$:

$$J \quad \frac{\Gamma \rightarrow \Delta, \forall\phi\, F(\phi) \qquad \forall\phi\, F(\phi), \Pi \rightarrow \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda}.$$

Let P_0 be the proof ending with $\Gamma, \Pi \rightarrow \Delta, \Lambda$. Let A be the left cut formula of J and let B be the right cut formula of J. We may assume that the uppermost ancestor of $A(B)$ which is identical with $A(B)$ is the principal formula of a logical inference and $F(\alpha)$ is the auxiliary formula of such inference related to A. By replacing the ancestors of A which are identical with A by $F(\alpha)$, we obtain a proof P_1 ending with $\Gamma \to \Delta, F(\alpha)$.

Let $\Pi_1 \to \Lambda_1$ be an arbitrary sequent which occurs above the right upper sequent of J. We can construct a proof ending with a sequent of the form $\Pi_1^*, \Gamma \to \Delta, \Lambda_1$, where Π_1^* is obtained from Π_1 by eliminating all the ancestors of B which are identical with B. This can be done by induction on the number of inferences in the proof ending with $\Pi_1 \to \Lambda_1$. As an example, suppose $\Pi_1 \to \Lambda_1$ is the lower sequent of a cut:

$$\dfrac{\Pi_2 \to \Lambda_2, D \qquad D, \Pi_3 \to \Lambda_3}{\Pi_2, \Pi_3 \to \Lambda_2, \Lambda_3}$$

where Π_2, Π_3 is Π_1 and Λ_2, Λ_3 is Λ_1. Define the following:

$$\dfrac{\dfrac{\Pi_2^*, \Gamma \to \Delta, \Lambda_2, D \qquad D, \Pi_3^*, \Gamma \to \Delta, \Lambda_3}{\Pi_2^*, \Gamma, \Pi_3^*, \Gamma \to \Delta, \Lambda_2, \Delta, \Lambda_3}}{\Pi_2^*, \Pi_3^*, \Gamma \to \Delta, \Lambda_2, \Lambda_3.}$$

As another example, let $\Pi_1 \to \Lambda_1$ be the lower sequent of a second order $\forall$: left whose principal formula is an ancestor of B which is identical with B:

$$\dfrac{F(V), \Pi_2 \to \Lambda_1}{\forall\phi\, F(\phi), \Pi_2 \to \Lambda_1},$$

where $\forall\phi\, F(\phi), \Pi_2$ is Π_1. Consider the following:

$$\dfrac{\dfrac{\Gamma \to \Delta, F(V) \qquad F(V), \Pi_2^*, \Gamma \to \Delta, \Lambda_1}{\Gamma, \Pi_2^* \to \Delta, \Delta, \Lambda}}{\Pi_2^*, \Gamma \to \Delta, \Lambda,}$$

where $\Gamma \to \Delta, F(V)$ is obtained from P_1 by substituting V for α everywhere.

By taking $\Pi_1 \to \Lambda_1$ to be $\forall\phi\, F(\phi), \Pi \to \Lambda$, we obtain $\Pi, \Gamma \to \Delta$, and hence $\Gamma, \Pi \to \Delta, \Lambda$. The grade of this proof, say Q, is less than $g(P_0)$, since $\gamma(F(V)) < \gamma(\forall\phi\, F(\phi))$ by assumption. Now replace P_0 by Q in P, obtaining a proof of the same end-sequent, but with a grade less than P. Then by the induction hypothesis the cuts can be eliminated.

DEFINITION 28.22. A set of semi-formulas and abstracts, say $\mathscr{F}$, is said to be closed if the following hold.

1) If A is atomic, then A belongs to $\mathscr{F}$.

2) If $\neg B$ belongs to $\mathscr{F}$, then B belongs to $\mathscr{F}$.

3) If $B \wedge C$ belongs to $\mathscr{F}$, then B and C belong to $\mathscr{F}$.

4) If $\forall x\, F(x)$ belongs to $\mathscr{F}$, then $F(s)$ belongs to $\mathscr{F}$ for every semi-term s.

5) If $\forall \phi\, F(\phi)$ belongs to $\mathscr{F}$, then $F(\alpha)$ belongs to $\mathscr{F}$ for every second order variable α.

6) If $\{x_1, \ldots, x_n\}H(x_1, \ldots, x_n)$ belongs to $\mathscr{F}$ then $H(a_1, \ldots, a_n)$ belongs to $\mathscr{F}$ for every $a_1, \ldots, a_n$; if $H(a_1, \ldots, a_n)$ belongs to $\mathscr{F}$ for some $a_1, \ldots, a_n$, then $\{x_1, \ldots, x_n\}H(x_1, \ldots, x_n)$ belongs to $\mathscr{F}$.

7) If B and C are alphabetical variants of one another, then B belongs to $\mathscr{F}$ if and only if C belongs to $\mathscr{F}$.

8) If $F(\alpha)$ and V belong to $\mathscr{F}$, then $F(V)$ belongs to $\mathscr{F}$.

We define a function γ relative to $\mathscr{F}$, which we call the γ determined by $\mathscr{F}$.

(1) $\gamma(A) = 0$ if A belongs to $\mathscr{F}$.

Assume A does not belong to $\mathscr{F}$.

(2) $\gamma(A) = \gamma(B) + 1$ if A is $\neg B$.

(3) $\gamma(A) = \max(\gamma(B), \gamma(C)) + 1$ if A is $B \wedge C$.

(4) $\gamma(A) = \gamma(F(x)) + 1$ if A is $\forall x\, F(x)$.

(5) $\gamma(A) = \gamma(F(\phi)) + 1$ if A is $\forall \phi\, F(\phi)$.

(6) $\gamma(\{x_1, \ldots, x_n\}H(x_1, \ldots, x_n)) = \gamma(H(x_1, \ldots, x_n))$.

In a manner similar to the proof of Proposition 27.7, we can easily prove the following.

PROPOSITION 28.23. *Suppose $\mathscr{F}$ is closed and γ is the function determined by $\mathscr{F}$. If V belongs to $\mathscr{F}$, then $\gamma(F(\alpha)) = \gamma(F(V))$.*

PROPOSITION 28.24. *Suppose $\mathscr{F}$ is closed and γ is the function determined by $\mathscr{F}$. Then γ is monotone.*

PROOF. Immediate from the definition of γ and Proposition 28.23.

CHAPTER 6

SOME APPLICATIONS OF CONSISTENCY PROOFS

§29. Provable well-orderings

We shall consider provable well-orderings of **INN** and show that any provable well-ordering of **INN** has order type less than that of the system of ordinal diagrams $O(\omega + 1, \omega^3)$, with respect to $<_0$. We will borrow much of the argument of §13. The results we will prove can be extended to **IID** with little modification.

DEFINITION 29.1. Let $<\cdot$ be a recursive linear ordering of the natural numbers which is actually a well-ordering. (Without loss of generality we may assume that $<\cdot$ is defined for all natural numbers and the least element with respect to $<\cdot$ is 0.) We use the same symbol $<\cdot$ to denote the formula in **INN** which expresses the ordering $<\cdot$.

Let $\mathbf{TI}(<\cdot)$ be a formula expressing the principle of transfinite induction along $<\cdot$:

$$\forall \phi\ (\forall x\ (\forall y\ (y <\cdot x \supset \phi(y)) \supset \phi(x)) \supset \forall x\ \phi(x)).$$

If $\mathbf{TI}(<\cdot)$ is **INN**-provable, then we say that $<\cdot$ is a provable well-ordering of **INN**.

We assume that an arithmetization of the system of ordinal diagrams $O(\omega + 1, \omega^3)$ has been carried out. We use the same notation to denote both an object and its arithmetization.

THEOREM 29.2. *Let $<_0$ be the well-ordering of the system of ordinal diagrams $O(\omega + 1, \omega^3)$ with respect to 0. (Recall that the consistency of* **INN** *was proved by using $<_0$.) If $<\cdot$ is a provable well-ordering of* **INN**, *then there exists a recursive function from natural numbers to an initial segment of $<_0$ which is $<\cdot$-$<_0$ order-preserving. That is to say, there is a recursive function f such that $a <\cdot b$ if and only if $f(a) <_0 f(b)$ and there is an ordinal diagram μ in $O(\omega + 1, \omega^3)$ such that for every a, $f(a) <_0 \mu$.*

PROOF. We will follow the proof of Theorem 13.4; and shall only point out how to modify that proof so that the arguments fit **INN**.

(1) **TJ**-proofs (for **INN**) are defined as in 13.1); in particular the **TJ** initial sequents have the form

$$\forall x\,(x <\cdot\, t \supset \varepsilon(x)) \rightarrow \varepsilon(t),$$

and the end-sequents have the form

$$\rightarrow \varepsilon(m_1), \ldots, \varepsilon(m_n).$$

(2) $|m|_{<\cdot}$ and the end-number of a **TJ**-proof are defined as in 13.1).

(3) For 13.2), 13.5) simply read **INN** in place of **PA**. We will, however, repeat the Fundamental Lemma:

LEMMA 29.3, the Fundamental Lemma (cf. Lemma 13.5). *The end-number of any* **TJ**-*proof is not greater than the order type of its ordinal diagram (with respect to* $<_0$*).*

(4) Ordinal diagrams are assigned to the sequents of the **TJ**-proofs as for **INN**: The ordinal diagram of a **TJ**-initial sequent is

$$(\omega, 0, (\omega, 0, (\omega, 0, (\omega, 0, (\omega, 0, 0))))).$$

See 13.6).

(5) The proofs of 13.7) through 13.11) go through as before.

(6) In 13.12), the ordinal diagram of the proof presented there is

$$(\omega, 0, (\omega, 0, (\omega, 0, 0 \# (\omega, 0, 0)))),$$

regarding $A \supset B$ as an abbreviation for $\neg(A \wedge \neg B)$. This is less than the ordinal diagram of a **TJ**-initial sequent. By this, and obvious changes, P becomes a **TJ**-proof P' whose end-sequent is

$$\rightarrow \varepsilon(m), \varepsilon(m_1), \ldots, \varepsilon(m_n),$$

where $\rightarrow \varepsilon(m_1), \ldots, \varepsilon(m_n)$ is the end-sequent of P. The ordinal diagram of P' is less than that of P and the end number of P' is $|m|_{<\cdot}$.

(7) As in 13.13), we obtain the Gentzen-type theorem:

THEOREM 29.4 (cf. Theorem 13.6). *The order type of* $<\cdot$ *is less than the order type of* $O(\omega + 1, \omega^3)$ *with respect to* $<_0$.

(8) As in 13.14), we can define a proof P_k for every k, where the end-number of P_k is $|k|_{<\cdot}$. Then we define a function h as in 13.15), where $+$ is the ordinal sum.

(9) In order to claim that h is recursive, and that 13.16) holds, we need the following.

1) The ordinal sum, $+$, of ordinal diagrams is recursive.

2) If two ordinal diagrams, μ and ν, are connected (i.e., the last operations used to form μ and ν are not $\sharp$) and $\mu <_0 \nu$, then $\mu + \nu = \nu$.

(10) From (9), we conclude that h is recursive and 13.16) holds. This implies that h is order-preserving.

§30. The Π_1^1-comprehension axiom and the ω-rule

An analogue to Problem 13.9 can be proved for **INN**, viz., the elimination of cuts in a system with the constructive ω-rule. We repeat some of the definitions which were given in Chapter 5.

DEFINITION 30.1. (1) We assume a standard Gödel numbering for axioms and for rules of finite inference. The ω-rule is expressed as follows:

$$\frac{\begin{array}{ccc} P_0 & & P_n \\ \Gamma \to \Delta, A(0) \ldots & & \ldots \Gamma \to \Delta, A(n) \ldots \end{array}}{\Gamma \to \Delta, \forall x\, A(x)}.$$

Here P_n is defined for every natural number n and is a proof of $\Gamma \to \Delta, A(n)$. To P_n assign a Gödel number of $\ulcorner P_n \urcorner$. If there exists a recursive function such that $f(n) = \ulcorner P_n \urcorner$ for every n, then the ω-rule is said to be constructive and $3 \cdot 5^e$ is assigned to the whole proof, where e is the Gödel number of f, i.e., $\{e\}(n) = \ulcorner P_n \urcorner$. Let **S** be any logical system. A proof, in the system obtained from **S** by adjoining the constructive ω-rule to it, is called an ω-proof in **S**.

(2) Let $S(a)$ and $a <\cdot\, b$ be primitive recursive predicates such that $<\cdot$ is a well-ordering of $\{a: S(a)\}$, whose first element is 0. A number-theoretic function Ψ is called $<\cdot$-recursive if it is defined by the following scheme which is a repetition of a previous definition.

(i) $\qquad f(a) = a + 1.$

(ii) $\qquad f(a_1, \ldots, a_n) = 0.$

(iii) $$f(a_1, \ldots, a_n) = a_i (1 \leqslant i \leqslant n).$$

(iv) $$f(a_1, \ldots, a_n) = g(\lambda_1(a_1, \ldots, a_n), \ldots, \lambda_m(a_1, \ldots, a_n)),$$

where g and λ_i $(1 \leqslant i \leqslant m)$ are $<^{\boldsymbol{\cdot}}$-recursive.

(v) $$f(0, a_2, \ldots, a_n) = g(a_2, \ldots, a_n)$$

$$f(a+1, a_2, \ldots, a_n) = \lambda(a, f(a, a_2, \ldots, a_n), a_2, \ldots, a_n),$$

where g and λ are $<^{\boldsymbol{\cdot}}$-recursive.

(vi) $$f(0, a_2, \ldots, a_n) = g(a_2, \ldots, a_n)$$

$$f(a+1, a_2, \ldots, a_n) = \lambda(a, f(\tau^*(a, a_2, \ldots, a_n), a_2, \ldots, a_n), a_2, \ldots, a_n),$$

where g, λ and τ are $<^{\boldsymbol{\cdot}}$-recursive and

$$\tau^*(a, a_2, \ldots, a_n) = \begin{cases} \tau(a, a_2, \ldots, a_n) & \text{if} \quad \tau(a, a_2, \ldots, a_n) <^{\boldsymbol{\cdot}} a+1, \\ 0 & \text{otherwise.} \end{cases}$$

We shall transform a proof in **INN** whose end-sequent contains no first order free variables, into a proof of the same end-sequent in the system with the constructive ω-rule. In proving the consistency of **INN** in §27 we defined reductions on a proof of $\rightarrow$. This notion, however, can easily be extended to any proof whose end-sequent has no first order free variables.

DEFINITION 30.2. (1) $O(\mathbf{INN}) = O(\omega + 1, \omega^3)$ is the system of ordinal diagrams used to prove the consistency of **INN** and $\prec$ is its well-ordering (namely $<_0$).

(2) For an ordinal diagram α, and natural number i, $\alpha^{(i)}$ is defined by $\alpha^{(0)} = \alpha$ and $\alpha^{(i+1)} = \alpha^{(i)} \# \alpha$.

(3) For an ordinal diagram μ and natural number m, $\langle \mu, m \rangle =_{\mathrm{df}} (0, 0, \mu \# 0)^{(m)}$.

By an abuse of notation, we shall use the notations for proofs, ordinal diagrams and $\prec$ both for formal objects and their Gödel numbers.

To the sequents in a proof we make the same assignment of ordinal diagrams as in §27 and we define the ordinal diagram of a proof P to be $\langle \mu, m \rangle$, where μ is the ordinal diagram assigned to its end-sequent and m is the number of first order free variables in its end-piece (denoted $m(P)$). If the ordinal diagram of P is less than the ordinal diagram of Q, then we write $\ulcorner P \urcorner \prec \ulcorner Q \urcorner$, or simply $P \prec Q$.

REMARK. In the definition of a reduction, we may be asked to take, say, a lowermost inference satisfying a certain condition. Such an inference may not be uniquely determined; however, we may suppose that the inferences are Gödel-numbered, and then take an inference as required with smallest Gödel number.

THEOREM 30.3. *There exists a $\prec$-recursive function f such that, for every proof P in* **INN** *whose end sequent contains no first order free variable, $f(\ulcorner P \urcorner)$ is the Gödel number of an ω-proof of the end-sequent of P which contains no cut and no application of mathematical induction or first order $\forall$: right.*

PROOF. Let P be a proof in **INN** whose end-sequent contains no first order free variables. We define reductions $r(P)$ and $q(i, P)$ for each $i < \omega$ and a transformation $f(\ulcorner P \urcorner)$ by transfinite induction on the ordinal diagram of P.

1) The end-piece of P contains an application of induction or an explicit logical inference.

1.1) The end-piece of P contains a first order free variable which is not used as an eigenvariable. We define $r(P)$ to be the Gödel number of the proof obtained from P by substituting 0 for each of such first order free variables. Obviously, $r(P) \prec P$. We define $f(P)$ to be $f(r(P))$.

1.2) The end-piece of P does not contain a first order free variable which is not used as an eigenvariable. Let J be a lowermost induction or (explicit) logical inference. We consider several cases.

1.2.1) J is an induction. Let $r(P)$ be the proof obtained from P by applying to J the reduction in (2) of §27 and let $f(P)$ be $f(r(P))$. Then $r(P) \prec P$.

1.2.2) J is an explicit logical inference.

1.2.2.1) J is not a first order $\forall$: right. Since all the cases are treated similarly, we consider the case where J is $\wedge$: left. Let P be

$$\begin{array}{c} \vdots \\ \dfrac{A, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta} \\ \vdots \\ \Gamma_0 \rightarrow \Delta_0. \end{array}$$

We define $r(P)$ to be the proof

$$\begin{array}{c} \vdots \\ A, \Gamma \rightarrow \Delta \\ \hline \text{some exchanges and weakening} \\ \hline A \wedge B, \Gamma, A \rightarrow \Delta \\ \vdots \\ \Gamma_0, A \rightarrow \Delta_0. \end{array}$$

Since $r(P) \prec P$, $f(r(P))$ has been defined by the induction hypothesis. We define $f(P)$ to be the following proof

$$
\dfrac{\dfrac{\dfrac{f(r(P))\left\{\begin{matrix}\vdots \\ \Gamma_0, A \rightarrow \Delta_0\end{matrix}\right.}{A, \Gamma_0 \rightarrow \Delta_0}\ \text{some exchanges}}{A \wedge B, \Gamma_0 \rightarrow \Delta_0}}{\Gamma_0 \rightarrow \Delta_0.}\ \text{some exchanges and a contraction}
$$

We shall refer to this figure as $g(f(r(p)))$.

1.2.2.2) I is a first order $\forall$: right. Let P be the following form:

$$
\begin{matrix}
\vdots \\
J\ \dfrac{\Gamma \rightarrow \Delta, A(a)}{\Gamma \rightarrow \Delta, \forall x\, A(x)} \\
\vdots \\
\Gamma_0 \rightarrow \Delta_0.
\end{matrix}
$$

For each i we consider the proof (referred to as $q(i, P)$):

$$
\begin{matrix}
\vdots \\
\dfrac{\Gamma \rightarrow \Delta, A(i)}{\Gamma \rightarrow A(i), \Delta, \forall x\, A(x)}\ \text{some exchanges and a weakening} \\
\vdots \\
\Gamma_0 \rightarrow A(i), \Delta_0
\end{matrix}
$$

where the proof of $\Gamma \rightarrow \Delta, A(i)$ is obtained from the proof of the upper sequent of J by substituting the numeral i for a. Obviously $q(i, P) \prec P$ for each numeral i. Thus $f(q(i, P))$ has been defined for each i. We define $f(P)$ to be the proof

$$
\dfrac{\omega\text{-rule}\ \dfrac{\cdots\quad \dfrac{f(q(i,P))\left\{\begin{matrix}\vdots \\ \Gamma_0 \rightarrow A(i), \Delta_0\end{matrix}\right.}{\Gamma_0 \rightarrow \Delta_0, A(i)}\ \text{some exchanges} \quad \cdots \text{ for each } i}{\Gamma_0 \rightarrow \Delta_0, \forall x\, A(x)}}{\Gamma_0 \rightarrow \Delta_0.}\ \text{some exchanges and a contraction}
$$

2) The endpiece of P contains no explicit logical inference or induction, but does contain an explicit, logical initial sequent. Then the end-sequent

of P is obtained from it by some weakenings and exchanges. Let $f(P)$ be one such proof.

3) The end-piece of P contains no induction or logical inference or explicit, logical sequent. We define $r(P)$ to be the proof obtained from P by applying the reductions in (1) through (9) of §27, retaining explicit weakenings. Then $r(P) \prec P$. Since the end sequent is unchanged by the reductions, we define $f(P)$ to be $f(r(P))$.

We have identified many notions with their Gödel numbers, e.g., a proof P sometimes means its Gödel number. Thus we can consider the functions r, q, g, f to be number-theoretic functions. We can obviously take r, q and g to be primitive recursive. Let $P(a)$ be a primitive recursive predicate stating that a is a proof in **INN** whose end sequent contains no first order free variables. Let P_0, P_1, P_2 and P_3 be defined by:

$P_0(m) \Leftrightarrow_{\text{df}} P(m)$ and one of the conditions 1.1), 1.2.1) or 2) applies.

$P_1(m) \Leftrightarrow_{\text{df}} P(m)$ and the end piece of m contains an explicit logical inference, other than first order $\forall$: right, to which the reduction applies.

$P_2(m) \Leftrightarrow_{\text{df}} P(m)$ and the reduction will apply to a first order $\forall$: right in the end piece of m.

$P_3(m) \Leftrightarrow_{\text{df}} \neg(P_0(m) \vee P_1(m) \vee P_2(m))$.

Obviously, P_0, P_1, P_2 and P_3 are primitive recursive and in the light of the consistency proof have the following properties:

$$\forall x\, \exists! i\, (i \leqslant 3 \text{ and } P_i(x));$$

$$P_0(m) \Rightarrow r(m) \prec m;$$

$$P_1(m) \Rightarrow r(m) \prec m;$$

$$P_2(m) \Rightarrow \forall n\, (q(n, m) \prec m).$$

With the help of recursion theory we shall show that f is recursive, in fact $\prec$-recursive. In fact,

$$f_0(e, m) \simeq \begin{cases} \{e\}(r(m)) & \text{if } P_0(m), \\ g(\{e\}(r(m))) & \text{if } P_1(m), \\ 3 \cdot 5^{S_1{}^2(c_0, e, m)} & \text{if } P_2(m), \\ m & \text{if } P_3(m), \end{cases}$$

where c_0 is the general recursive index $\Lambda n, e, m\{e\}(q(n, m))$ (i.e., an index for $\{e\}(q(n, m))$ as a function of n, e, m; see: Kleene, Introduction to Metamathematics (North-Holland, Amsterdam, 1967), p. 344. By the recursion theorem (op. cit., §66), there is a number c such that $f_0(c, m) \simeq \{c\}(m)$. Then define f by $f(m) \simeq \{c\}(m)$, i.e.,

$$f(m) \simeq \begin{cases} f(r(m)) & \text{if } P_0(m), \\ g(f(r(m))) & \text{if } P_1(m), \\ 3 \cdot 5^{S_1{}^2(c_0, c, m)} & \text{if } P_2(m), \\ m & \text{otherwise.} \end{cases}$$

Thus f is partial recursive. By transfinite induction on $\prec$ we can show that f is totally defined. It is also easy to see that f is $\prec$-recursive, that $f(P)$ has the same end-sequent as P, and that $f(P)$ has no cut or mathematical induction or first order free variable. This completes the proof.

DEFINITION 30.4. A number-theoretic function $f(a_1, \ldots, a_n)$ is called provably recursive in **INN** if the following sequent is provable in **INN**:

$$\rightarrow \forall x_1 \ldots \forall x_n \exists y\, T_n(e, x_1, \ldots, x_n, y),$$

where T_n expresses Kleene's primitive recursive predicate T_n (cf. §**13**; we can easily extend the definition in §13 to the case where there are more than one x) and e is a Gödel number of f.

As an application of our technique we can give an alternate proof of a theorem which was first proved by Kino. This is an analogue to Problem **13**.8.

THEOREM 30.5. *Let ψ be a provably recursive function in* **INN**. *Then we can find an ordinal diagram μ of O(***INN***) such that ψ is $\prec^\mu$-recursive, where $\prec^\mu$ is $\prec$ restricted to arguments $\prec \mu$.*

PROOF. Without loss of generality we may assume that ψ is a function of one argument. Let e be a Gödel number of ψ such that the sequent $\rightarrow \forall x \exists y\, T_1(e, x, y)$ is provable in **INN**. Let P be a proof of $\rightarrow \exists y\, T_1(e, a, y)$ whose ordinal diagram is μ. We define P_m to be the proof obtained from P by substituting the numeral m for a. The process of obtaining P_m from P is primitive recursive. To each P_m we apply the transformation f of the previous theorem. Then $f(P_m)$ is a proof without a cut. Since P does not contain any explicit $\forall$: right for a first order variable (which is the only inference which induces an application of the ω-rule in the transformation), it is easily proved by transfinite induction that $f(P_m)$ does not contain any application of the ω-rule. By checking the proof $f(P_m)$ we can find primitive recursively a numeral n satisfying $T_1(e, m, n)$. Since $n = \psi(m)$ and f is $\prec^{\mu}$-recursive by Theorem 30.3, we see that ψ is $\prec^{\mu}$-recursive.

In defense of the constructive infinite rule we submit the following argument. Many theorems in first order proof theory follow from the cut-elimination theorem. This is still true even for higher order proof theory in which the cut-elimination theorem is proved constructively. However, if one wishes to consider an extension of arithmetic, it is impossible to eliminate all cuts due to the fact that the formal proofs contain applications of mathematical induction. Schütte has introduced the ω-rule and eliminated all applications of the cut rule and ind in first order arithmetic. This is an excellent idea and can be considered an improved form of cut-elimination when ind is involved. However, since the main objective of our investigation is a finite proof, it is better if we can restrict the ω-rule so that the infinite proofs considered are possessed of some important properties of finite proofs. For this reason we consider the constructive ω-rule.

The adequacy of the constructive ω-rule has been proved by Shoenfield for first order arithmetic, and by Takahashi for second order arithmetic. Therefore, mathematically the constructive ω-rule is strong enough.

§31. Reflection principles

DEFINITION 31.1. (1) Let **P** be **PA** augmented with second order free variables which function as parameters.

(2) For the sake of technical convenience, we restrict the constants in **INN** to the individual constants 0, 1; function constants $+$, $\cdot$; predicate constants $=$, $<$; and we will use $\vee$, $\supset$ and $\exists$ as well as $\neg$, $\wedge$ and $\forall$ as logical symbols.

(3) A first order formula with second order parameters $\alpha_1, \ldots, \alpha_m$ is called rudimentary in $\alpha_1, \ldots, \alpha_m$ if every (first order) quantifier is bounded, that is, quantifiers occur in the form $\forall x\,(x < s \supset \ldots)$ or $\exists x\,(x < s \wedge \ldots)$ for a term s. These formulas will be denoted by $\forall x < s\,(\ldots)$ and $\exists x < s\,(\ldots)$, respectively.

We assume a standard Gödel numbering for expressions and notions concerning **INN**. Because of (2) of Definition 31.1, we may assume that the mathematical initial sequents are those of Definition 9.3. The first purpose of this section is to prove the reflection principle in the following form.

THEOREM 31.2 (Takeuti and Yasugi). *Let $R(\alpha, a, b)$ be rudimentary in α and let* $\mathrm{Ind}_1(O(\mathbf{INN}))$ *be the formula which expresses transfinite induction through* $O(\mathbf{INN})$ *for the Σ_1^0-formulas (i.e., the formulas of the form $\exists x\, R(x, a)$, R recursive without second order parameters). Then*

$$\mathrm{Ind}_1(O(\mathbf{INN})), \mathrm{Prov}(\ulcorner \forall x\, \exists y\, R(\alpha, x, y) \urcorner) \to \forall x\, \exists y\, R(\alpha, x, y)$$

is provable in **P**, *where $\ulcorner A \urcorner$ is the Gödel number of A and* $\mathrm{Prov}(\ulcorner A \urcorner)$ *means that "A is provable in* **INN**".

In order to prove this we first observe the following.

PROPOSITION 31.3. *Let $R(a, \alpha)$ be rudimentary in α with one first order free variable a, and let $\exists x\, R(x, \alpha)$ be provable in* **INN**. *Then there is a proof of $\exists x\, R(x, \alpha)$ in* **INN** *containing no essential cut or induction. Moreover, this can be proved with the system of ordinal diagrams* $O(\mathbf{INN})$.

The proposition could be stated for several parameters, $\alpha_1, \ldots, \alpha_m$, instead of just one α.

Let S be a sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ of **INN**. S is said to have the property (P) if the following conditions are satisfied:

p1. S contains no first order free variable.

p2. Every A_i, $1 \leqslant i \leqslant m$ is rudimentary in α.

p3. Every B_j, $1 \leqslant j \leqslant n$ is rudimentary in α or is of the form $\exists x\, R'(x, \alpha)$, where $R'(a, \alpha)$ is rudimentary in α.

We will prove the proposition in the following form:

PROPOSITION 31.4. *We can define a reduction, using* $O(\mathbf{INN})$, *in such a way that if a sequent S has the property* (P) *and is provable in* **INN**, *then its proof can be reduced to one with no essential cut or induction.*

Proposition 31.3 is only a special case of this proposition.

PROOF. The proof is for the most part the same as in §27. We introduce a new rule of inference bq, "bounded quantification":

$$\text{bq} \quad \frac{\Gamma \to \Delta, (0 < k \supset S(0)) \wedge \ldots \wedge (k-1 < k \supset S(k-1))}{\Gamma \to \Delta, \forall x\, (x < k \supset S(x))},$$

where k is a numeral, $S(a)$ is rudimentary, and in which the formulas

$$(0 < k \supset S(0)) \wedge \ldots \wedge (k-1 < k \supset S(k-1)), \qquad \forall x\, (x < k \supset S(x))$$

are called, respectively, the auxiliary formula and the principal formula of the inference.

This rule is not regarded as one of the logical rules of inference but as a structural rule. (It is easily seen that the lower sequent of bq can be proved from its upper sequent without an essential cut or induction.) The ordinal diagram of the lower sequent of bq is defined to be the same as that of the upper sequent.

A proof is called a proof with degree if it contains applications of bq only in its end-piece as explicit inferences and is a proof with degree in the sense of §27.

We shall define the reduction of a proof P of a sequent satisfying (P). By a reduction-step we mean a process which decreases the ordinal diagram of the proof together with one or more preceding auxiliary processes which preserve the ordinal diagram of the proof. See the proof of Theorem 30.3.

Case 1. P contains an application of explicit logical inference or induction in its end-piece. We treat the cases according to the bottom most such inference.

Subcase 1. Induction. As in §27.

Subcase 2. Explicit logical inference other than $\forall$: right for a first order variable. Since all the cases can be treated similarly, we give an example:

$$\frac{\Gamma \to \Delta, F(t)}{\Gamma \to \Delta, \exists x\, F(t)}$$
$$\vdots$$
$$\Gamma_0 \to \Delta_0.$$

We reduce this to P':

$$\vdots$$
$$\frac{\dfrac{\Gamma \to \Delta, F(t)}{\text{a weakening and some exchanges}}}{\Gamma \to F(t), \Delta, \exists x\, F(x)}$$
$$\vdots$$
$$\Gamma_0 \to F(t), \Delta_0.$$

The end-sequent of P' obviously satisfies (P) and P' has a smaller ordinal diagram than that of P, hence P' can be transformed to a proof without essential cuts and inductions. Then add some explicit inferences to obtain $\Gamma_0 \to \Delta_0$.

Subcase 3. Explicit $\forall$: right for a first order variable:

$$\begin{array}{c} \vdots \\ \dfrac{\Gamma \to \Delta, b < t \supset \tilde{R}(b, \alpha)}{\Gamma \to \Delta, \forall y\, (y < t \supset \tilde{R}(y, \alpha))} \\ \vdots \\ \Gamma_0 \to \Delta_1, \forall y\, (y < s \supset \tilde{R}'(y, \alpha)), \Delta_2, \end{array}$$

where $\tilde{R}(b, \alpha)$ is rudimentary in α, t contains no variable, and s and $\tilde{R}'(y, \alpha)$ are obtained from t and $\tilde{R}(y, \alpha)$, respectively, by zero or more term-replacements. Let $t = n$ for a numeral n. If $n = 0$, hence $s = 0$, then, P is reduced to

$$\dfrac{c < s \to}{\Gamma_0 \to \Delta_1, \forall y\, (y < s \supset \tilde{R}'(y, \alpha)), \Delta_2.}$$

If $n > 0$, then for each $k < n$, let P_k be

$$\begin{array}{c} \vdots \\ \dfrac{\Gamma \to \Delta, k < n \supset \tilde{R}(k, \alpha)}{\Gamma \to k < n \supset \tilde{R}(k, \alpha), \Delta, \forall y\, (y < n \supset \tilde{R}(y, \alpha))} \\ \vdots \\ \Gamma_0 \to k < n \supset \tilde{R}'(k, \alpha), \Delta_1, \forall y\, (y < s \supset \tilde{R}'(y, \alpha)), \Delta_2, \end{array}$$

where $\Gamma \to \Delta$, $k < n \supset \tilde{R}(k, \alpha)$ is the end-sequent of the proof obtained from that of $\Gamma \to \Delta$, $b < t \supset \tilde{R}(b, \alpha)$ by substituting k for b. Every substitution in P_k is assigned the same degree as the corresponding one in P. Then P is reduced to $P_0, P_1, \ldots, P_{n-1}$, for

bq
$$\dfrac{\dfrac{\dfrac{P_0\, P_1 \ldots P_{n-1}}{\Gamma_0 \to \Delta', (0 < n \supset \tilde{R}'(0,\alpha)) \wedge (1 < n \supset \tilde{R}'(1,\alpha)) \wedge \ldots \wedge (n-1 < n \supset \tilde{R}'(n-1,\alpha))}}{\Gamma_0 \to \Delta', \forall y\, (y < n \supset \tilde{R}'(y, \alpha))}}{\Gamma_0 \to \Delta_1, \forall y\, (y < s \supset \tilde{R}'(y, \alpha)), \Delta_2,}$$

where Δ' denotes $\Delta_1, \forall y\, (y < s \supset \tilde{R}'(y, \alpha)), \Delta_2$, is a proof of the end-sequent of P.

Case 2. P contains no explicit logical inference or induction but contains an axiom of the form $s = t, A(s) \to A(t)$ in its end-piece. Do the reduction as in §27.

Case 3. P contains no explicit logical inference or induction or axiom of the form $s = t$, $A(s) \to A(t)$, but contains either an explicit logical axiom or an implicit logical axiom of the form $D \to D$, e.g.,

$$\dfrac{\Gamma \to \Delta, \tilde{D} \qquad \overset{D \to D}{\tilde{D}, \Pi \to \Lambda_1, \tilde{\tilde{D}}, \Lambda_2}}{\dfrac{\Gamma, \Pi \to \Delta, \Lambda_1, \tilde{\tilde{D}}, \Lambda_2}{\Gamma_0 \to \Delta_0,}}$$

where $\tilde{D}$ and $\tilde{\tilde{D}}$ in the right upper sequent of the cut are the descendants of D's in the antecedent and succedent of $D \to D$, respectively. If the former, then the end-sequent of P is obtained from it by weakenings, exchanges and bq's. If the latter, and $\tilde{D}$ and $\tilde{\tilde{D}}$ are the same, up to term-replacement, we apply the corresponding reduction in §27. Otherwise, $\tilde{D}$ and $\tilde{\tilde{D}}$ are of the form

$$(s_0 < s \supset S(s_0)) \wedge \ldots \wedge (s_{n-1} < s \supset S(s_{n-1}))$$

and $\forall y\,(y < t \supset S'(y))$, respectively, where $s = n$ and $t = n$ for some numeral n, $s_i = i$ $(i < n)$ and $S'(y)$ is either $S(y)$ itself or else obtained from it by term replacements. Then P is reduced to

$$\dfrac{\mathrm{bq}\;\dfrac{\dfrac{\Gamma \to \Delta, \tilde{D}}{\Gamma \to \Delta, (0 < n \supset S'(0)) \wedge \ldots \wedge (n-1 < n \supset S'(n-1))}}{\Gamma \to \Delta, \forall y\,(y < n \supset S'(y))}}{\dfrac{\Gamma, \Pi \to \Delta, \Lambda_1, \tilde{\tilde{D}}, \Lambda_2}{\Gamma_0 \to \Delta_0}}$$

Case 4. Elimination of weakenings in the end-piece of P is defined as usual. If the last inference of a proof Q is a bq, say

$$Q \qquad \dfrac{Q_0 \left\{ \Gamma \to \Delta, (0 < k \supset S(0)) \wedge \ldots \wedge (k-1 < k \supset S(k-1)) \right.}{\Gamma \to \Delta, \forall y < k\, S(y)},$$

then the definition goes as follows.

If Q_0^* is $\Gamma^* \to \Delta^*$, then Q^* is Q_0^*. If Q_0^* is

$$\Gamma^* \to \Delta^*, (0 < k \supset S(0)) \wedge \ldots \wedge (k-1 < k \supset S(k-1)),$$

then Q^* is

$$\frac{Q_0^*}{\Gamma^* \to \Delta^*, \forall y\,(y < k\; S(y))}.$$

Case 5. In the following we assume that the end-piece does not contain any logical inference, induction, initial sequent other than mathematical initial sequents or weakening, while it may contain some applications of bq. We may also assume that the proof is different from its end-piece, for if the entire proof is the end-piece, then the end-sequent is provable from the mathematical initial sequents by bq, exchanges, contractions and non-essential cuts, and hence bq can be eliminated without use of an essential cut or induction. The existence of an essential cut and the essential reduction are carried out as usual, since applications of bq are all explicit.

This completes the proof of the proposition.

We consider an arithmetization of **INN** in **P**. Let us introduce the following notational conventions:

Pf($\ulcorner P \urcorner$) for "P is a proof in **INN**";

Prov($\ulcorner P \urcorner$, $\ulcorner S \urcorner$) for "P is a proof of a sequent S";

Prov($\ulcorner S \urcorner$) for "S is provable";

Prov($\ulcorner A \urcorner$) for "Prov($\ulcorner \to A \urcorner$)";

Pf*($\ulcorner P \urcorner$) for "P is a proof without an essential cut or induction";

Prov*($\ulcorner P \urcorner$, $\ulcorner S \urcorner$) for "P is a proof of S without an essential cut or induction";

Prov*($\ulcorner S \urcorner$) for "S is provable without an essential cut or induction";

Prov*($\ulcorner A \urcorner$) for "Prov*($\ulcorner \to A \urcorner$)".

It should be noted that under the assumption of this section **INN** is axiomatizable, i.e., the set of the schemata for mathematical initial sequents is finite.

PROPOSITION 31.5. *Let $R(a, \alpha)$ be rudimentary in α. Then*

$$\mathrm{Ind}_1(O(\mathbf{INN})), \mathrm{Prov}(\ulcorner \exists y\, R(y, \alpha) \urcorner) \to \mathrm{Prov}^*(\ulcorner \exists y\, R(y, \alpha) \urcorner)$$

is provable in **P**.

PROOF. This is proved by arithmetization of the proof of Proposition 31.3. We shall give only the outline of the proof that $\mathrm{Ind}_1(O(\mathbf{INN}))$ is adequate.

First let us introduce some notational conventions. Assume that p denotes the Gödel number of a proof P in **INN**. Then

ends(P) is the Gödel number of the end-sequent of P;

$Q(p)$ is true if and only if the end-sequent of P has the property (P);

$C(p)$ is true if and only if P is a proof which has no essential cut or induction;

$\tilde{o}(P)$ is defined by $\tilde{o}(p) = o(p) \,\sharp\, 0^{(p-1)}$, where $o(p)$ is the ordinal diagram of P and $0^{(p-1)}$ is as defined in Definition 30.2 (2). Note that $\tilde{o}(p)$ is an ordinal diagram of $O(\mathbf{INN})$ and all these predicates and functions are primitive recursive.

Now from the proof of Proposition 31.4, we can define a primitive recursive function r as follows. Let p be the Gödel number of a proof P. If $C(p) \vee \neg Q(P)$, then define $r(p) = p$. If $\neg C(p) \wedge Q(p)$, define $r(p)$ to be the Gödel number of the resulting proof of the reduction of P. Then r is primitive recursive and satisfies the following.

1) $\tilde{o}(r(p)) \prec \tilde{o}(p)$ if $\neg C(p) \wedge Q(p)$.

2) $\tilde{o}(r(p)) = \tilde{o}(p)$ if $C(p)$.

Define $\tilde{r}(a, b)$ by

$$\tilde{r}(0, p) = p; \qquad \tilde{r}(n+1, p) = r(\tilde{r}(n, p)).$$

Then $\tilde{r}(a, b)$ is primitive recursive. Finally, define

$$p <\cdot\, q \Leftrightarrow_{\mathrm{df}} \tilde{o}(p) \prec \tilde{o}(q).$$

Then $<\cdot$ is a primitive recursive well-ordering of the natural numbers. Furthermore, the order type of $<\cdot$ is that of $O(\mathbf{INN})$. So transfinite induction can be applied to the ordering $<\cdot$, with induction formula $Q(p) \supset \exists n\, C(r(n, p))$, or equivalently, $\exists n\, (Q(p) \supset C(r(n, p)))$, which is Σ^0_1.

DEFINITION 31.6. (1) A formula of **INN** is said to have the property (Q) if it contains no second order quantifiers or first order free variables.

For every formula A having the property (Q) we define the subformulas of A as follows: A is a subformula of A; if $B \wedge C$ is a subformula of A then so are B and C. If $\neg C$ is a subformula of A then so is C; if $\forall x\, B(x)$ is a subformula of A, then so is $B(n)$ for every numeral n. Evidently, every subformula of A has the property (Q).

(2) We can give a truth-definition T_A for the subformulas of A and also for sequents consisting only of such subformulas. The truth definition is an arithmetical formula with second order parameters (i.e., free variables).

PROPOSITION 31.7. *Let A be a formula having the property* (Q). *Then the following are provable in* **P**.

(1) $\mathrm{T}_A(\ulcorner \neg B \urcorner) \leftrightarrow \neg \mathrm{T}_A(\ulcorner B \urcorner)$ *for every subformula B of A.*

(2) $\mathrm{T}_A(\ulcorner B \vee C \urcorner) \leftrightarrow \mathrm{T}_A(\ulcorner B \urcorner) \vee \mathrm{T}_A(\ulcorner C \urcorner)$ *for every pair B and C subformulas of A.*

(3) $\mathrm{T}_A(\ulcorner \forall x_i\, B(x_i) \urcorner) \leftrightarrow \forall x\, \mathrm{T}_A(\ulcorner B(n(x)) \urcorner)$ *for every subformula $\forall x_i\, B(x)$ of A; here $n(a)$ denotes the a^{th} numeral.*

(4) $\mathrm{T}_A(\ulcorner B(n(b_1), \ldots, n(b_k)) \urcorner) \rightarrow B(b_1, \ldots, b_k)$, *where $B(0, \ldots, 0)$ is an arbitrary subformula of A such that originally $B(y_1, \ldots, y_k)$ for some bound variables $y_1, \ldots, y_k$ occurred in A.*

(5) $P_A(a) \wedge \mathrm{Prov}^*(a) \rightarrow \mathrm{T}_A(a)$, *where $P_A(a)$ means "a is (the Gödel number of) a sequent consisting of subformulas of A".*

PROOF. (1) through (4) can be proved in the same manner as for the truth definition of **PA**.

(5) Assume $P_A(a)$ and $\mathrm{Prov}^*(a)$, and let P be a proof such that $\mathrm{Prov}^*(\ulcorner P \urcorner, a)$. We can show by induction on the number of inferences in P, using (1)–(4), that

$$\mathrm{T}_A(\ulcorner \Gamma(n(c_1), \ldots, n(c_k)) \rightarrow \Delta(n(c_1), \ldots, n(c_k)) \urcorner)$$

is provable in **P**, where $n(c)$ denotes the c^{th} numeral and $\Gamma \rightarrow \Delta$ is a sequent in **P**.

PROOF OF THEOREM 31.2. Take $\forall x\, \exists y\, R(x, y, \alpha)$ as the A in Proposition 31.7 and let $\mathrm{T}(a)$ denote $\mathrm{T}_A(a)$. Then

(1) $\mathrm{Prov}(\ulcorner \forall x\, \exists y\, R(x, y, \alpha) \urcorner) \rightarrow \forall a\, \mathrm{Prov}(\ulcorner \exists y\, R(n(a), y, \alpha) \urcorner)$ is provable in **P**.

By Proposition 31.5,

(2) $\mathrm{Ind}_1(O(\mathbf{INN})), \mathrm{Prov}(\ulcorner \exists y\, R(n(a), y, \alpha) \urcorner) \rightarrow \mathrm{Prov}^*(\ulcorner \exists y\, R(n(a), y, \alpha) \urcorner)$ is provable in **P** for any free variable a.

By virtue of Proposition 31.7 the following are provable in **P**:

(3) $\mathrm{Prov}^*(\ulcorner \exists y\, R(n(a), y, \alpha) \urcorner) \rightarrow \mathrm{T}(\ulcorner \exists y\, R(n(a), y, \alpha) \urcorner)$, since $P_A(\ulcorner \exists y\, R(n(a), y, \alpha) \urcorner)$ is provable in **P**;

(4) $\forall a\, \mathrm{T}(\ulcorner \exists y\, R(n(a), y, \alpha) \urcorner) \rightarrow \mathrm{T}(\ulcorner \forall x\, \exists y\, R(x, y, \alpha) \urcorner)$;

(5) $\mathrm{T}(\ulcorner \forall x\, \exists y\, R(x, y, \alpha) \urcorner) \rightarrow \forall x\, \exists y\, R(x, y, \alpha)$.

The theorem follows from (1)–(5).

We can prove the uniform reflection principle by modifying the proof of Theorem 31.2.

THEOREM 31.8. *In* **P**:

$$\mathrm{Ind}_1(O(\mathbf{INN})) \to \forall m\,(\mathrm{Prov}(\ulcorner \forall x\,\exists y\, R(x, y, \alpha, n(m))\urcorner) \supset \forall x\,\exists y\, R(x, y, \alpha, m))$$

PROOF. We first note that a modified version of Proposition 31.5:

$$\mathrm{Ind}_1(O(\mathbf{INN})),\ \mathrm{Prov}(\ulcorner \exists x\, R'(x, \alpha, n(m)\urcorner) \to \mathrm{Prov}^*(\ulcorner \exists x\, R'(x, \alpha, n(m))\urcorner),$$

is provable, by replacing $\ulcorner \exists x\, R(x, \alpha)\urcorner$ by $\ulcorner \exists x\, R'(x, \alpha, n(m))\urcorner$ in Proposition 31.5. Taking $\forall z\, \forall x\, \exists y\, R'(x, y, \alpha, z)$ as A in Proposition 31.7, it follows that (1)–(5) in the proof of Theorem 31.2 are provable with $\ulcorner \forall x\, \exists y\, R'(x, y, \alpha, n(m))\urcorner$ instead of $\ulcorner \forall x\, \exists y\, R(x, y, \alpha)\urcorner$. With this observation, the theorem follows easily.

Now let $B(\alpha)$ be an arbitrary formula of **P** of the form

$$(*) \qquad \exists x_1\, \forall y_1 \ldots \exists x_n\, \forall y_n\, B_0(\alpha, x_1, y_1, \ldots, x_n, y_n),$$

where $B_0(\alpha, a_1, b_1, \ldots, a_n, b_n)$ is a quantifier-free formula whose only free variables are $\alpha, a_1, b_1, \ldots, a_n, b_n$.

The subformulas of $B(\alpha)$ are defined as in Definition 31.6.

PROPOSITION 31.9. *Given $B(\alpha)$ which satisfies (*), we can define the truth definition $\mathrm{T}_{B(\alpha)}$ for subformulas of $B(\alpha)$ in* **P** *with a Σ^0_{2n}-formula having the second order parameter α. It is obvious that $\mathrm{T}_{B(\alpha)}$ can be extended to sequents consisting of some subformulas of $B(\alpha)$.*

DEFINITION 31.10. Let $B(\alpha)$ be a formula satisfying (*). We define the condition $\mathscr{S}_{B(\alpha)}$ as follows; let $\ulcorner S\urcorner$ denote the Gödel number of the sequent S. We use quotes to mean that the quoted sentence is actually an arithmetized formula.

$\mathscr{S}_1(B(\alpha); \ulcorner S\urcorner)$: "Each formula of S is a subformula of $B(\alpha)$".

$\mathscr{S}_2(\ulcorner S\urcorner)$: "Each formula in the antecedent of S is quantifier-free".

$\mathscr{S}_3(B(\alpha); \ulcorner S\urcorner)$: $\neg \mathrm{T}_{B(\alpha)}(\ulcorner S\urcorner)$.

$\mathscr{S}_{B(\alpha)}(\ulcorner S\urcorner)$: $\mathscr{S}_1(B(\alpha); \ulcorner S\urcorner) \wedge \mathscr{S}_2(\ulcorner S\urcorner) \wedge \mathscr{S}_3(B(\alpha); \ulcorner S\urcorner)$.

From now throughout, $B(\alpha)$ shall be arbitrary but fixed so that it satisfies (*). For simplicity we shall abbreviate $\mathrm{T}_{B(\alpha)}$ and $\mathscr{S}_{B(\alpha)}$ as T and $\mathscr{S}$, respectively.

PROPOSITION 31.11.

$$\mathrm{Ind}_2(O(\mathbf{INN})),\ \mathrm{Prov}(p,\ \ulcorner B(\alpha)\urcorner),\ \neg\mathrm{T}(\ulcorner B(\alpha)\urcorner) \rightarrow \exists q \leqslant\cdot\, p\ (\mathrm{Pf}^*(q) \wedge \mathscr{S}(\mathrm{ends}(q)))$$

is **P**-*provable, where* $<\cdot$ *is the well-ordering of natural numbers defined in the proof of Proposition* 31.5 *and* $\mathrm{Ind}_2(O(\mathbf{INN}))$ *is the schema which allows transfinite induction along the order* $<\cdot$ *applied to* Σ^0_{2n+1}-*formulas.*

The proposition is an immediate consequence of the following:

(**) $\mathrm{Ind}_2(O(\mathbf{INN}))$, $\mathscr{S}(\ulcorner S\urcorner)$, $\mathrm{Prov}(p,\ \ulcorner S\urcorner) \rightarrow \exists q \leqslant\cdot\, p\ (\mathrm{Pf}^*(q) \wedge \mathscr{S}(\mathrm{ends}(q)))$ is **P**-provable.

Therefore, we shall prove (**). It is proved by applying $\mathrm{Ind}_2(O(\mathbf{INN}))$ to the following formula:

$$(1)\qquad \mathscr{S}(\mathrm{ends}(p)) \wedge \mathrm{Pf}(p) \supset \exists q \leqslant\cdot\, p\ (\mathrm{Pf}^*(q) \wedge \mathscr{S}(\mathrm{ends}(q))).$$

Since T and $\mathscr{S}$ are in Σ^0_{2n} and Π^0_{2n}, respectively, the induction formula is in Σ^0_{2n+1} with the parameter α.

It is now obvious that in order to prove (1) it suffices to show

$$(2)\quad \mathscr{S}(\mathrm{ends}(p)) \wedge \mathrm{Pf}(p) \wedge \neg\mathrm{Pf}^*(p) \supset \exists q\ (\mathscr{S}(\mathrm{ends}(q)) \wedge \mathrm{Pf}^*(q) \wedge q <\cdot\, p),$$

for if $\mathrm{Pf}^*(p)$, then we may take p itself as q in (1).

Assume $\mathscr{S}(\mathrm{ends}(p)) \wedge \mathrm{Pf}(p) \wedge \neg\mathrm{Pf}^*(p)$ and find a q which satisfies (2). This is done in the same manner as in the consistency proofs of **INN**, although, strictly speaking, the whole argument is developed in the arithmetized language.

Let P be the proof with Gödel number p.

1) Preparations for reduction as in §27 are applicable.

2) If there is an explicit logical inference or an induction in the end-piece of P, then the proof is carried out according to the bottom most such inference.

2.1) The last such inference is a first order $\exists$: right. Let P be of the form

$$\begin{array}{l}
\vdots \\
\underline{\underline{\Gamma \rightarrow \Delta,\ \forall y_i \ldots \exists x_n \forall y_n B_0(\alpha, t_1, s_1, \ldots, t_i, y_i, \ldots, x_n, y_n)}} \\
\Gamma \rightarrow \Delta,\ \exists x_i \forall y_i \ldots \exists x_n \forall y_n B_0(\alpha, t_1, s_1, \ldots, x_i, y_i, \ldots, x_n, y_n) \\
\vdots \\
\Pi \rightarrow \Lambda_1,\ \exists x_i \forall y_i \ldots \exists x_n \forall y_n B_0(\alpha, m_1, l_1, \ldots, x_i, y_i, \ldots, x_n, y_n),\ \Lambda_2.
\end{array}$$

Notice that t_i is a closed term which consists of 0, 1, $+$ and $\cdot$. Therefore, t_i can be computed and is equal to a numeral m_i. P is reduced to the following.

$$\frac{\Gamma \rightarrow \Delta, \forall y_i \ldots \exists x_n \forall y_n B_0(\alpha, t_1, s_1, \ldots, t_i, y_i, \ldots, x_n, y_n)}{\begin{array}{l}\Gamma \rightarrow \forall y_i \ldots \exists x_n \forall y_n B_0(\alpha, t_1, s_1, \ldots, m_i, y_i, \ldots, x_n, y_n), \Delta, \\ \qquad \exists x_i \forall y_i \ldots \exists x_n \forall y_n B_0(\alpha, t_1, s_1, \ldots, x_i, y_i, \ldots, x_n, y_n),\end{array}}$$

$$\begin{array}{l}\Pi \rightarrow \forall y_i \ldots \exists x_n \forall y_n B_0(\alpha, m_1, l_1, \ldots, m_i, y_i, \ldots, x_n, y_n), \Lambda_1, \\ \qquad \exists x_i \forall y_i \ldots \exists x_n \forall y_n B_0(\alpha, m_1, l_1, \ldots, x_i, y_i, \ldots, x_n, y_n), \Lambda_2\end{array}$$

$$\neg \mathrm{T}(\ulcorner \exists x_i \forall y_i \ldots \exists x_n \forall y_n B_0(\alpha, m_1, l_1, \ldots, x_i, y_i, \ldots, x_n, y_n) \urcorner)$$

implies

$$\neg \mathrm{T}(\ulcorner \forall y_i \ldots \exists x_n \forall y_n B_0(\alpha, m_1, l_1, \ldots, m_i, y_i, \ldots, x_n, y_n) \urcorner).$$

2.2) The last inference which satisfies the condition is a first order $\forall$: right. Let P be of the form

$$\frac{\Gamma \rightarrow \Delta, \exists x_{i+1} \ldots \exists x_n \forall y_n B_0(\alpha, t_1, s_1, \ldots, a, x_{i+1}, \ldots, x_n, y_n)}{\Gamma \rightarrow \Delta, \forall y_i \exists x_{i+1} \ldots \exists x_n \forall y_n B_0(\alpha, t_1, s_1, \ldots, y_i, x_{i+1}, \ldots, x_n, y_n)}$$

$$\Pi \rightarrow \Lambda_1, \forall y_i \exists x_{i+1} \ldots \exists x_n \forall y_n B_0(\alpha, m_1, l_1, \ldots, y_i, x_{i+1}, \ldots, x_n, y_n), \Lambda_2.$$

This is reduced to

$$(l_i)$$

$$\frac{\Gamma \rightarrow \Delta, \exists x_{i+1} \ldots \exists x_n \forall y_n B_0(\alpha, t_1, s_1, \ldots, l_i, x_{i+1}, \ldots, x_n, y_n)}{\begin{array}{l}\Gamma \rightarrow \exists x_{i+1} \ldots \exists x_n \forall y_n B_0(\alpha, t_1, s_1, \ldots, l_i, x_{i+1}, \ldots, x_n, y_n), \Delta, \\ \qquad \forall y_i \exists x_{i+1} \ldots \exists x_n \forall y_n B_0(\alpha, t_1, s_1, \ldots, y_i, x_{i+1}, \ldots, x_n, y_n)\end{array}}$$

$$\begin{array}{l}\Pi \rightarrow \exists x_{i+1} \ldots \exists x_n \forall y_n B_0(\alpha, m_1, l_1, \ldots, l_i, x_{i+1}, \ldots, x_n, y_n), \Lambda_1, \\ \qquad \forall y_i \exists x_{i+1} \ldots \forall y_n B_0(\alpha, m_1, l_1, \ldots, y_i, x_{i+1}, \ldots, x_n, y_n), \Lambda_2,\end{array}$$

where (l_i) means the substitution of the numeral l_i for the free variable a in the proof and l_i is chosen so that

$$\mathrm{T}(\ulcorner \forall x_{i+1} \ldots \forall x_n \exists y_n \neg B_0(\alpha, m_1, l_1, \ldots, l_i, x_{i+1}, \ldots, x_n, y_n) \urcorner)$$

holds, when

$$\neg \mathrm{T}(\ulcorner \forall y_i \exists x_{i+1} \ldots \exists x_n \forall y_n B_0(\alpha, m_1, l_1, \ldots, y_i, x_{i+1}, \ldots, x_n, y_n) \urcorner)$$

is assumed, or

$$\exists y\, \mathrm{T}(\ulcorner \forall x_{i+1} \ldots \forall x_n \exists y_n \neg B_0(\alpha, m_1, l_1, \ldots, n(y), x_{i+1}, \ldots, x_n, y_n) \urcorner),$$

where $n(y)$ is the y^{th} numeral.

2.3) All other cases of logical inferences are proved easily. By virtue of $\mathscr{S}_1(B(\alpha), \text{ends}(p))$ and $\mathscr{S}_2(\text{ends}(p))$, there is no first order $\forall$: left and no first order $\exists$: left.

2.4) The last inference which satisfies the condition is an ind. This case is proved as in §27.

3) Now we may assume that there is no explicit logical inference or induction in the end-piece of P. Hereafter we can follow exactly the consistency proof of §27. Thus we have proved (**).

PROPOSITION 31.12. $\text{Ind}_2(O(\mathbf{INN})), \text{Prov}(\ulcorner B(\alpha) \urcorner), \neg \text{T}(\ulcorner B(\alpha) \urcorner) \rightarrow$ *is* **P**-*provable, where* $B(\alpha)$ *satisfies* (*).

PROOF. From the definition of T, $\text{Pf}^*(q) \rightarrow \text{T}(\text{ends}(q))$. But this contradicts $\mathscr{S}_3(\text{ends}(q), B(\alpha))$. Thus the proposition follows from Proposition 31.11.

Now we can present another form of the reflection principle for **INN**.

THEOREM 31.13 (Takeuti and Yasugi).

$$\text{Ind}_2(O(\mathbf{INN})), \text{Prov}(\ulcorner A(\alpha) \urcorner) \rightarrow A(\alpha)$$

is **P**-*provable for an arbitrary arithmetical sentence* $A(\alpha)$ *with a second order parameter* α, *where* $\text{Ind}_2(O(\mathbf{INN}))$ *applies to the formulas of* **P**, *that is, to the formulas arithmetical in some second order parameters.*

PROOF. It is well known that

$$(1) \qquad A(\alpha) \leftrightarrow B(\alpha)$$

is **P**-provable for some $B(\alpha)$ which satisfies (*).

$$(2) \qquad \text{Ind}_2(O(\mathbf{INN})), \text{Prov}(\ulcorner B(\alpha) \urcorner) \rightarrow \text{T}_{B(\alpha)}(\ulcorner B(\alpha) \urcorner)$$

and

$$(3) \qquad \text{T}_{B(\alpha)}(\ulcorner B(\alpha) \urcorner) \rightarrow B(\alpha)$$

are **P**-provable from Propositions 31.12 and 31.7, respectively. It is also known that

$$(4) \qquad \text{Prov}(\ulcorner A(\alpha) \urcorner) \leftrightarrow \text{Prov}(\ulcorner B(\alpha) \urcorner)$$

is **P**-provable. Then (1)–(4) yields the theorem.

Here again we can prove the uniform reflection principle.

THEOREM 31.14.

$$\mathrm{Ind}_2(O(\mathbf{INN})) \to \forall m\, (\mathrm{Prov}(\ulcorner A(\alpha, n(m))\urcorner) \supset A(\alpha, m)),$$

where $\mathrm{Ind}_2(O(\mathbf{INN}))$ *applies to the formulas of* $\mathbf{P}$.

PROOF. This is proved with modifications similar to those that have been carried out in the proof of Theorem 31.8: First apply (**) in the proof of Proposition 31.11 to $\ulcorner B(\alpha, n(m))\urcorner$ in the place of $\ulcorner B(\alpha)\urcorner$. Then take $\forall z\, B(\alpha, z)$ as $B(\alpha)$ and define the truth definition for $B(\alpha)$. The rest of the proof of Theorem 31.13 goes through after this alteration.

We now present another formulation of the reflection principle for the formulas $\forall \phi\, A(\phi)$, where $A(\alpha)$ is arithmetical in α. We shall state it in the form of the uniform reflection principle.

THEOREM 31.15. *Let* $A(\alpha, a)$ *be arithmetical in* α *and let* α *and* a *be the only free variables of* A. *Then*

(1) $$\mathrm{Ind}'(O(\mathbf{INN})), \mathrm{Prov}(\ulcorner \forall\phi\, A(\phi, n(a))\urcorner) \to \forall\phi\, A(\phi, a)$$

is $\mathbf{INN}$*-provable, where* Ind' *applies to* Σ^0_3*-formulas with a second order parameter.*

PROOF. First, with a slight extension of the language of $\mathbf{INN}$ as specified in Definition 31.1, there exists a quantifier-free formula $R(\alpha, b, c, a)$ for which

(2) $$\forall\phi\, A(\phi, a) \leftrightarrow \forall\phi\, \exists x\, \forall y\, R(\phi, x, y, a)$$

is $\mathbf{INN}$-provable. Then (2) implies that

(3) $$\mathrm{Prov}(\ulcorner \forall\phi\, A(\phi, n(a))\urcorner) \leftrightarrow \mathrm{Prov}(\ulcorner \forall\phi\, \exists x\, \forall y\, R(\phi, x, y, n(a))\urcorner)$$

is $\mathbf{INN}$-provable. Finally, (2) and (3) guarantee that, in order to prove (1), we only have to prove

(4) $$\mathrm{Ind}'(O(\mathbf{INN})), \mathrm{Prov}(\ulcorner \exists x\, \forall y\, R(\alpha, x, y, n(a))\urcorner) \to \exists x\, \forall y\, R(\alpha, x, y, a)$$

in $\mathbf{INN}$. But (4) follows from

(5) $$\mathrm{Ind}'(O(\mathbf{INN})), \mathrm{Prov}(\ulcorner \exists x \forall y R(\alpha,x,y,n(a))\urcorner), \neg \mathrm{T}(\ulcorner \exists x \forall y R(\alpha,x,y,n(a))\urcorner) \to,$$

which is proved like Proposition 31.11.

Notice that T is the truth definition for $\exists x\, \forall y\, R(\alpha, x, y, a)$ so that we may assume it is a Σ^0_2-formula with the parameter α. But this implies that $\mathrm{Ind}'(O(\mathbf{INN}))$ applies to Σ^0_3-formulas with the parameter α.

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