MATHEMATICAL THINKING FROM THE PERSPECTIVES OF PROBLEM SOLVING AND AREA OF LEARNING CONTENTS

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1. MATHEMATICS EDUCATION AND MATHEMATICAL THINKING

Mathematics education has two main purposes. One is to enable children to make use of what they acquire in the study of mathematics, understand phenomena in their daily life in a mathematical way, and examine and process phenomena through logical thinking. To attain this objective, not only are a basic knowledge of and skill in mathematics important but also the ability to think logically using inductive, analogical and deductive approaches.

The other is to enable children to enjoy intellectual pleasure through learning mathematics creatively. However, in today's mathematics classes, a teacher will often show a problem to children, solve it as a matter of formality, and make the children repeat the practice. For children, this is a passive activity even though in terms of mathematics education, this method does produce some effect. However, as is clear from surveys conducted by institutions such as the International Association for the Evaluation of Educational Achievement (IEA) and the OECD Program for International Student Assessment (PISA), Japanese children are not learning mathematics in an environment as enjoyable as that of other countries. The attainment of the second purpose of mathematics education calls for a change in approach to allow children to take part in a more creative way, have them experience intellectual pleasure and make them feel that they are "creating" mathematics.

Children can make use of what they learn or engage in though creative learning activities only when they take a mathematical approach and learn mathematical thinking. Hereinafter, I will examine mathematical thinking using two approaches, one that uses the problem solving and another based on the area of learning contents.

2. FROM THE PERSPECTIVE OF PROBLEM SOLVING

The present "Course of Study" (school curriculum guideline) is aimed at fostering the growth of people who take initiative in their lives in the 21st century. A report by the Central Education Council submitted before its revision, calls for "education with less pressure through content selection" and "subjective learning." Regarding one purpose of the revision, the council mentions "the development of the ability to learn voluntarily and think independently." If making children acquire such ability is a major purpose of education, mathematics has a very important role to perform in view of its nature.

A major objective of mathematics education at present is to foster the ability to solve problems. The Ministry of Education, Culture, Sports, Science and Technology, intends to attain the above objective through curriculum guidance that meets this objective.

Curriculum guidance of the problem-solving type consists of the following five stages based on the four-stage approach of G. Polya:

1. The stage of posing a problem;
2. The stage of understanding the problem;
3. The stage of devising a plan for solution;
4. The stage of carrying out the plan for solution;
5. The stage of examining the solution

Polya's four stages seem to have been derived focusing on problem solving as an activity of individuals. Here, however, I consider the above five stages as the stages of problem solving instruction. On this assumption, I will discuss the mathematical thinking that children acquire in each stage.

<Posing a problem>

As already described, one purpose of mathematics education is to make children obtain mathematical knowledge and enable them to make use of it. For this purpose, it is important to provide children with experience in various situations where mathematics is used. Also important are the various types of activities in which children find a problem (pose a problem) and their ability to solve the problem.

For example, in the first class of learning a unit, children would understand a situation where the addition and subtraction of fractions is used, as well as grasp the overall meaning of the learning contents in the whole unit. When I was observing a class of forth-graders where the children had to find something in their surroundings that equaled 1/4 of something, one child asked: "Does 1/4 multiplied by 4, which is 4/4, equal 1?" and "Does 1/4 multiplied by 5 equal 5/4?" In a class of fifth-graders where average values were being taught, one of the children asked: "I understand the answer to (5+6+4)/3 is 5, which was obtained by removing 1 from 6 and adding it to 4. But what should we do when the numbers do not result in the same way by removing part of the numbers, as in (5+5+7)/3?"

It is an important activity from the viewpoint of developing mathematical thinking to expand the presented phenomenon mathematically or generalize it by taking questions asked in the class and using them as part of the contents to be learned in the next class.

Here I will describe using the following question as the target subject of a class:

Question: Here is a 2/5m steel bar that weighs 3/4 kg. How much does 1m of the steel bar weigh?

<Understanding a problem>

To understand this question, the children must consider the phenomenon mathematically. In this activity they pay attention to the numbers, quantities, and figures included in the phenomenon. In order to establish a mathematical formula necessary to answer this question, the children must understand the following:

1. The mathematical conditions related to the question (the relations between the length and weight of the steel bar);
2. What must be found (the weight of the 1m steel bar);
3. Remembering previously learned related questions (recalling a problem in which a quantity per unit had to be found);
4. Establishing a formula by selecting division as a procedure, which was previously learned (total quantity divided by number of units equals quantity per unit)

The above-mentioned concerns ability related to the first of the two purposes of mathematics education. It is necessary for children, in learning how to understand daily phenomena numerically, to practice such activities. In a lesson in which the children are required to comprehend a question, the teacher should extend guidance to help them develop the ability to practice using a mathematical way of thinking such as that mentioned above. The teacher should also use ingenuity to present a concrete phenomenon and a concrete situation related to the question shown to them.

<Devising a plan for solution>

From the viewpoint of instruction, this problem has been presented with two aims. One is to teach the children that the solution can be found through the formula "3/4 divided by 2/5" using division. The other is to teach them how to obtain the answer to "3/4 divided by 2/5." The ability to translate such a problem into a formula of division depends on how much they understand the meaning of division from past lessons.

When the children try to find the answer to a division calculation, it is important to have an estimate of the answer (the size of the value of the answer) and the ability to estimate which procedure should be used to find the answer. Having an estimated answer is important in the sense that they have their own viewpoint and respond using their own way of thinking. The following is an example of the importance of estimating:

In view of the meaning of division, the presented problem is one that requires the children to find the weight per meter. Because 2/5m weighs 3/4kg, it is important to foresee that the weight per meter is greater than 3/4kg. Moreover, it is hoped that the children are able to tell whether a given fraction as a weight is larger or smaller than 1kg, and can explain the reason for their answer. For example, 2/5m is smaller than half of 1m, but 3/4kg is larger than half of 1m. Therefore, a 1-meter steel bar is heavier than 1kg.

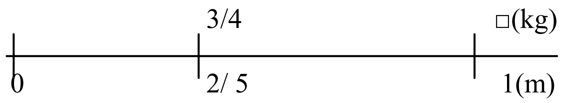
Furthermore, it is expected that the children, by sensing how much heavier than 3/4kg a 1-meter steel bar is, can find the correct procedure for calculating. Choosing the correct procedure means to have the ability to foresee the process that leads to the result. By considering which procedure should be used to find the result before taking the steps to solve the problem, one can reasonably and efficiently find the solution. In the case of this class, consideration should be given to the following when trying to judge which procedure should be applied:

(1) Using concrete items as a tool, such as paper tape;
(2) Describing the situation using a number line or a line segment figure;
(3) Thinking in the same way as one does when dividing a whole number by a fraction;
(4) Calculating by converting the fraction into a decimal number;
(5) Thinking in terms of a surface diagram;
(6) Correlating with the calculation of whole numbers.

It is expected that these points of consideration are not newly introduced at this point, but have already been learned.

<Carrying out the plan for solution>

When the children estimate a solution, they begin to find the solution. When calculating “3/4 divided by 2/5,” it is expected that the children will think in accordance with point (2) mentioned above.



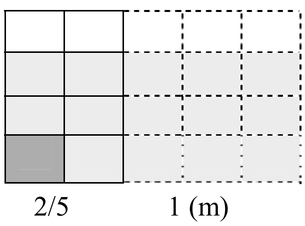
Based on this drawing, the children can think using the following logic:

(a) Because 1 meter is two 1/5 meter and 1/5 meter, the total weight can be found as 3/4+3/4+3/8 (3/8 is half of 3/4).
(b) Because the weight of 1/5 meter is 3/8kg and 1meter is five times that, we have “3/4 ÷2 x 5 (or 3/8 x 5) ”
(c) Because the weights of 2 meter is 3/4kg x 5, and 1meter is half of that, we have “3/4 x 5 ÷ 2.”
(d) Because 5/2 x 2/5 is 1, we can find the weight of 1meter is by calculating “3/4 x 5/2.”

The children have learned in (3) above that “a whole number divided by a fraction” equals “a whole number multiplied by the inverse number of the fraction.” Because the part “divided by fraction” is the same in this case, it can be applied to the “fraction divided by fraction” calculation.

The method given in (4) is a procedure used to find the solution to “3/4 ÷ 2/5” as “0.75 ÷ 0.4.”

The method given in (4) is a procedure used to find the weight of 1m using the following surface diagram:



(a) As in (2), the weight of 1meter is found after finding the weight of 1/5 meter.

(b) Because each small cell means 3/4 ÷ 6, and there are 3 x 5 in total (1 meter), the weight of 1meter can be found by calculating "3/4 ÷ 6 x 5."

What is mentioned in (6) corresponds to the way of thinking in (b) and (d) in (2).

<Examination of the solution>

As described by "Carrying out the plan for solution," each child can think using his/her own way of thinking. There are various procedures for finding the solution. The children can deepen their understanding by exchanging their opinions among themselves, finding similarities and differences and correlating them. Instead of simply comparing their methods of solution with those of others, it is important to have them examine and understand which procedure is more intelligible and easier to understand, or have them find a more general means of solving the problem, if necessary (which procedure would seem to be most suitable for future development).

When one ascertains that one's way of thinking is correct, or one shows that it is correct and conveys it to others, it is important to think logically, that is, inductively, analogously and deductively.

3. FROM THE PERSPECTIVE OF AREA OF LEARNING CONTENTS

The "Course of Study" guidelines mention four areas, "numbers and calculations," "graphic figures," "quantity and measurement" and "quantitative relations," for elementary schools, and also four areas, "numbers and formulas," "graphic figures," "functions" and "quantitative relations" for junior high schools.

A characteristic of school mathematics is that the learning contents in each grade are based on what the children learn in the previous year or earlier in the same year, which involves a new stage of learning. Consequently, sufficiently understanding the contents and procedures of the subject is vital for the children to work on problem-solving activities.

[Example 1] The area of "numbers and calculations"

Let's assume that the children are going to solve a problem mentioned in (2) in the area of numbers and calculations. What kind of mathematical ability is necessary to solve this problem? There is no need to mention that the ability to read and understand the word problem is necessary. Accordingly, this ability is excluded from our discussion.

First, the children must have the ability to understand the concept and expression method of fractions, such as 2/5 and 3/4. Second, it is important to understand that this problem can be solved using division ("3/4 ÷ 2/5"). In other words, it is important to understand the meaning of calculation, and what type of calculation should be used in a certain situation. With regard to learning the procedure of calculation, as already mentioned, it is vital that the children understand the meaning of the procedure previously introduced. Also vital is the manner in which they learned to understand it.

In that sense, the teacher should clearly understand what the children need to learn and how this will be used in the future.

For example, when teaching the meaning of division, the teacher usually introduces the concept of degression in connection with what the children have already learned. The divisor gradually expands to decimal numbers and fractions. In such a situation, the explanation that division is merely a set of repetitions of subtraction becomes impossible, and an expansion of the meaning becomes necessary. The teacher must carry out instruction that takes the future into consideration so that the children can recognize that division can also be applied as it was used in similar cases in the past.

For example, multiplication is understood as follows:

(Basic quantity) x (proportion) = (total quantity)

If sufficient instruction is provided to the children on using division to find a basic quantity or proportion, that is, to help them understand division as the reverse of multiplication, setting up the formula "3/4 divided by 2/5," should not be very difficult.

The problem is solved by applying the following learning steps:

Understanding the meaning of numbers → Understanding the meaning of calculation → Method of calculation → Familiarizing oneself with calculation

[Example 2] The area of "quantity and measurement"

When children seek a quadrature formula in the area of quantity and measurement, they study the following process to set up a quadrature of two-dimensional figures such as rectangles, parallelograms, and triangles. This can be used to create a formula for trapezoids:

Figure → quadrature → Formula

(Figure → quadrature): It becomes possible to calculate how many cells of 1 cm^2 there are if the figure is converted into a previously learned shape.

(quadrature → Formula): Regarding the necessary length, it appears to be possible to always apply a formula in the same place.

As in Example 1 and Example 2, there are peculiar ways of learning that are used repeatedly in the study of mathematics. If children learn such peculiar ways of learning as the basics, it serves as a great strength when they proceed to the next step. The way to study mathematics is an important part of mathematical thinking.

[Example 3] The area of "figures"

In the area of "figures," let us suppose that the children are studying parallelograms. In the study of figures, the teacher should direct the children's attention to the components of figures (the sides and the vertex in the case of a two-dimensional figure), the number of components, correlations between sides and angles, and the parallel and vertical relations between sides, in order to deepen their understanding of the concept of the figures.

When teaching parallelograms, the teacher should help the children understand by making comparisons with other figures, explaining that a parallelogram is a figure whose two opposite sides are parallel to each other (the step of forming a concept), and by making them practice drawing a parallelogram (the step of composing a figure). This will help them understand the properties of a parallelogram.

To roughly conclude, learning in the area of figures progresses in the following steps:

(1) Forming a concept of the figure concerned;
(2) Composing (drawing) the figure;
(3) Analyzing the properties of the figure;
(4) Comparing the figure with other figures.

As already mentioned, sufficiently understanding the contents and procedures of the subject is vital for children when taking part in problem-solving activities.

Reference

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