HANS FREUDENTHAL
University of Utrecht

REVISITING MATHEMATICS EDUCATION

China Lectures

KLUWER ACADEMIC PUBLISHERS
DORDRECHT / BOSTON / LONDON

REVISITING MATHEMATICS EDUCATION

Mathematics Education Library

VOLUME 9

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eBook ISBN: 0-306-47202-3
Print ISBN: 0-7293-1299-6

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New York, Boston, Dordrecht, London, Moscow

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PREFACE

This book is a product of love and respect. If that sounds rather odd I initially apologise, but let me explain why I use those words. The original manuscript was of course Freudenthal's, but his colleagues have carried the project through to its conclusion with love for the man, and his ideas, and with a respect developed over years of communal effort. Their invitation to me to write this Preface enables me to pay my respects to the great man, although I am probably incurring his wrath for writing a Preface for his book without his permission! I just hope he understands the feelings of all colleagues engaged in this particular project.

Hans Freudenthal died on October 13th, 1990 when this book project was well in hand. In fact he wrote to me in April 1988, saying "I am thinking about a new book. I have got the sub-title (China Lectures) though I still lack a title". I was astonished. He had retired in 1975, but of course he kept working. Then in 1985 we had been helping him celebrate his 80th birthday, and although I said in an Editorial Statement in *Educational Studies in Mathematics* (ESM) at the time "we look forward to him enjoying many more years of non-retirement" I did not expect to see another lengthy manuscript. Most people in their 80s do not aspire to write yet another book (particularly if they have already written nearly 200 previous publications in mathematics education -- see the list at the end of this book). But of course Freudenthal was not like most people, and that is one reason why this book is so important. Although it wasn't his last piece of academic writing, it was his last major contribution to our field and it offers the reader a kind of synopsis of his perspectives. It is the definitive Freudenthal.

Not that there is anything final about the book nor about its messages. As Leen Streefland said in his tribute in ESM 21/6 "he himself already prepared the first milestone of the post-Freudenthal era", namely this book. In particular the book poses the kinds of questions and issues which Freudenthal himself enjoyed grappling with, and which in another way show the love and respect he had for children. He rarely spoke overtly about this, but shines through all his writing as a beacon to all of us in mathematics education, and in education generally. His writing focuses as always on the essential concerns of mathematics education, as he saw them, and children are our principal concern.

The book is sprinkled with anecdotes and stories, with references to colleagues in the pursuit of understanding and wisdom, both adult and children, with respect to the predecessors on whose shoulders he stood, and with criticisms of those whose ideas he did not approve of. The language is "Freudenthal-English" and I do not say this in any unkind way. He enjoyed languages so much, and he was so aware of the dangers of words controlling his thoughts, that he tried to be creative craftsman with his languages. He said in his letter of May 1989, when he sent me his draft manuscript: "I did it by text processor -- a time-consuming activity because the easiness of correcting makes it difficult to stop correcting".

If you have read Freudenthal before, you will know something of what to expect -- the insights, the reflections, the charming and apposite examples, the scathing criticisms, the amusing asides, the wisdom -- they are all here. If you haven't read any of his writings before, then I have only one piece of advice -- don't try to skim-read, or to read too quickly. You need to *engage* with his words, in order to relate to his ideas. If you can manage to do that, I am sure that you, like everyone else who has engaged with them, will never be quite the same person again. That is, I believe, what he would have wanted.

Alan J. Bishop

APOLOGY AND EXPLICATION

This is neither preface nor introduction but rather an apology and a warning: the present book adds nothing but itself to work that I have published in the past in various places. I have renounced originality, I have refrained from offering the reader any new experiences, aspects, or ideas he might feel entitled to expect -- reason to apologise. Resuming old ideas, my aim is now one of comprehensiveness, not by compiling but by selecting and streamlining, by including essentials and eliminating contradictions. Views, even when supported by evidence, are susceptible to change. My present viewpoint is "here and now", that of review rather than of overview. I felt prompted by two experiences. Negative: reading dated quotations from my own work where I did not recognise myself. Positive: a visit to China, where in lectures, seminars, and discussions I tried to display and illustrate my view on mathematics education in all its aspects. The present book somehow reflects these talks, albeit in no way textually. Therefore the subtitle, meant as an homage to my listeners who, by their critical curiosity, induced me to improve on clarity and concreteness.

Hans Freudenthal

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CHAPTER 1

MATHEMATICS PHENOMENOLOGICALLY

1.1 WHAT IS MATHEMATICS?

1.1.1 *Sure and certain*

What is mathematics? A thorny question. Don't look it up in a dictionary! Whenever I did the answer was wrong.

"Mathematics" looks like a plural as it still is in French *"Les Mathématiques".* Indeed, long ago it *meant* a plural: four arts (liberal ones worth being pursued by free men). Mathematics was the *quadrivium,* the sum of arithmetic, geometry, astronomy, and music, held in higher esteem than the (more trivial) *trivium:* grammar, rhetoric, and dialectic. Only when this plural was no longer understood, was its final 's' usurped by sciences, like physics and economics, that historically did not deserve it.

On closer inspection it appears that I have answered the question "What *was* mathematics?", to wit in Greek-Roman era, in the Middle Ages, and even somewhat thereafter.

As far as I am familiar with languages, Dutch is the only one in which the term for mathematics is neither derived from nor resembles the internationally sanctioned *Mathematica.* The Dutch term was virtually coined by Simon Stevin (1548-1620): *Wiskunde,* the science of what is certain. *Wis en zeker,* sure and certain, is that which does not yield to any doubt; and *kunde* means knowledge, science, theory. Indeed, since Stevin, and even before, mathematicians have behaved as though this were the proper definition of mathematics; but since it seemed to be pretentious rather than a definition it could not find favour in the eyes of dictionary authors.

Even though it does Seem to be pretentious it was a lucky strike of Stevin's to name a science (and quite a few others) after what he grasped to be its most characteristic property, rather than after its subject. The most characteristic property of mathematics was certainty, or so he believed. But how certain is "certain"? Isn't common Sense the first, the nearest certainty, and is there anything as remote from common sense as mathematics? Common sense takes things for granted, for good reasons or for bad ones. Mathematics asks for good reasons, as does any science, maybe even for better ones than any other. The need for certainty in science is not satisfied by taking things for granted; certainty has to be pursued, and in mathematics this is done by a quite peculiar mental activity. It is this mental activity rather than its subject matter that characterises mathematics as the field where this activity can be exercised most adequately and most efficiently. We shall keep this in mind in order to deal with it more specifically in due course: that is, mathematical method as a mental activity which, for some mysterious reason, creates certainty and does so, it would seem, beyond compare.

Mathematics as the field of the most adequate and efficient use of the mathematical method? So there are fields outside mathematics where mathematical method does apply, maybe less adequately and efficiently. Or are these really outside mathematics? How far does mathematics as a field extend? Let us turn back to Stevin! He did not, in fact, use the word *wiskunde* but *wisconst,* or as we would spell it nowadays, *wiskunst. Kunde* means *science* while *kunst* means *art.* At our medical schools they teach *geneeskunde,* the *science* of curing, while the medical doctor's practice is *geneeskunst,* the art of *curing.* As a name-giver for domains of research Stevin preferred the noun *kunst* above *kunde,* art above science. "Art" in "Arts and Sciences" sounds different from "art" in "Arts and Crafts". With which second noun did Stevin combine "art"? Stevin was an engineer, but I am sure he understood it both ways, as many before and after him had done and would do.

Mathematics as an art, a mental art to be sure, which for most people will be closer to crafts than to sciences, a tool rather than an aim in itself, more relevant because it works than because it is certain. But why does it work? Because it is certain? Although many people trust mathematics more than it deserves, it works only when it is rightly applied. But what is right or wrong? Is there any way to verify it and if so, isn't such an verification again mathematics and -- if it is -- to which degree? Once one has admitted that mathematics is an art, one cannot shirk the responsibility ofjudging whether, in particular cases, it is being properly used or rather being abused; while trying to decide, one behaves once again like a mathematician. Let us illustrate this with a well-known historical example:

In spite of precursors, probability began with the discussion of two problems. The second was the *problème des partis.* Let me present it in a simplified version: Two persons A and B have agreed on a series of games (with equal chance of winning for either): the first to earn five points will collect all the stakes. It so happens that the series must be broken off with the score standing at

A: 4 B: 3.

How should the stakes be divided? People could not agree. Some said: in the ratio 4 : 3; others said: (5 - 3) : (5 - 4). Indeed, there is something to be said for each view. But what does mathematics say? Or rather, what did the mathematician Pascal say when he was asked his opinion? Nowadays we would say: Let us auction the state of affairs! How much would A' and B' pay in order to continue? If B' evens the score the stake must be divided fifty-fifty but the chance that he would do so is in fact 50%. Therefore he can claim just one-fourth of the stake and so the fair ratio of sharing is 3 : 1.

Behold the mathematician called to decide upon the right or wrong way to apply the art of arithmetic! Indeed, it was in no way obvious at the start that the decision, rather than being a matter of opinion or taste or of weighing pros and cons, was again one of arithmetic, that is of mathematics; and a mathematician was sorely needed to reveal this perspective.

As a useful tool, mathematics has conquered a rich variety of fast expanding areas of science and society, and as a tool it has proved indispensable for a rapidly growing host of people, who use mathematics because they cannot do without it. Yet again: how sure and certain is this mathematics?

Voyager 2 arrived one second late at its meeting with Neptune -- after having travelled for 12 years, it is true. So sure and certain is mathematics. Or is it? Astronomers count by millions of years, and on the scale of millions of years the one second would be one day -- at least if one admits proportionality. But sure and certain? When they predicted the moment of closest approach, they added some '±' to account for the "mean error". Mathematics knows precise numbers only -- I once heard a purist affirm. And what about error calculus? -- I asked him. That is physics, geodesy, astronomy -- he retorted; contrived, indeed, by mathematicians involved with mathematics "outside" mathematics. Judging how sure and certain some applied mathematics may be is again mathematics -- formal mathematics in error calculus; as such it is sure and certain as long as it is rightly used, that is, with insight rather than as a recipe, which unfortunately happens more often than not in its educational version. Nowadays, streamlined error theory no longer suffices for judging how sure and certain mathematics is "outside" mathematics. How about the numerous "mathematical models"? Mathematics is sure and certain to the extent that one faces the question *how* sure and certain it is.

Yet let us not forget mathematics as an aim in itself, which allured adepts as early as Babylonian Antiquity, and which historically has proved indispensable as a forceful motor for its own long-term development. It is an important aspect, although of less concern to us here, since our subject of mathematics education embraces a much larger group than only future professionals of whom once again only a small minority choose mathematics as an aim in itself.

Even if the "inside" certainty of mathematics is taken for granted, there still remains something to be said about its "outside" certainty. The answer to this is predesigned: it depends on the mathematical behaviour of the one who ventures out -- a point for further discussion. Anyway, Simon Stevin may have had good reasons to characterise mathematics as what is sure and certain, and his predecessors and successors had to behave accordingly, even if they failed to reflect on it. But what about the great majority? Are we too bold in assuming that, even to the man in the street certainty is mathematics' most significant feature, far beyond the limits of his ability to handle it? If this assumption is correct, one may ask how this confidence in the certainty of mathematics has come about. The answer is determined by the extent to which and the way in which people have become acquainted with mathematics.

Most people have been taught mathematics as a set of rules of processing or, as we call it in mathematics, of algorithms -- an agreeable experience where they have learned to master them, and a disagreeable one if they have failed, One reason why teachers teach it this way is tradition: it is the way they learned it themselves, while they have forgotten that it was not the way they really understood mathematics if ever they did. The other reason is the structure of mathematics which, unlike any other art or science, can be mounted in rules and algorithms. It is an astonishing fact that these rules never fail or -- if they do fail -- that it is the user who feels guilty, looks for errors of his own and tries to repair them. But to whom is this fact astonishing? Or is it simply an unfounded belief -- subjectively unfounded? Let me close this subsection by telling a little story:

A 75 year-old woman whose help is regularly invoked by children and students stuck in their homework, met -- perhaps for the first time in her life -- a professional mathematician, just when she had taught a college student the (Euclidean) algorithm for finding the greatest common divisor of two whole numbers. She asked the mathematician why this works or rather, whether there were any reason why. The mathematician helped her to answer the question on her own. Maybe it is worth mentioning how he did this. He gave a mere hint: If a and b are two numbers and $a > b$ (in fact he chose a pair of such numbers), compare the greatest common divisor of a and b with that of a and a-b*!*

The next time she asked him a question it was about an old-fashioned algorithm for finding the square root, in which she had believed all her life. Now the mathematician refused and instead taught her a more efficient algorithm, which she accepted comprehendingly.

Still another time she asked him why under addition, subtraction, and multiplication of two numbers the sum of digits behaved in correspondence with the original numbers -- a phenomenon that was easily explained.

Never did she ask the mathematician why, performing arithmetic, the inverse operation "proves the sum", why the algorithms of the various arithmetical operations work, why operations are commutative, why prime factorisation is unique. Why did she doubt some rules (or did she?) and accepted others? Later on we will try to answer this question.

1.1.2 *Mathematics as common sense*

Earlier on I asked whether common sense isn't the primordial certainty, the most abundant and reliable source of certainty. In a sense it is, although from olden times onwards philosophers and explorers have sounded warning notes against that very common sense, which is easily deceived by cunning Nature.

Iron is colder than wood -- this is common sense, at least as long as one uses the common sense thermometer of one's fingers. Unfortunately, sometimes this statement ceases to be true; for instance, when exposed to the blazing sun, iron feels hotter than wood. "Conduction of heat" helps us out of this dilemma. This is "uncommon" sense, yet common to physicists and to people who learned and didn't forget their physics lessons. It is the weakness of school lessons to be forgotten, and physics is not exempt from this rule. Whenever this happens, it is good luck if the matter learned wasn't worth remembering, and ill luck if it was worthwhile. In due course we shall reconsider and discuss the poor permeability of the membrane separating classroom and school experience from life experience and we will ask for measures on how to improve it.

My example confronted the physicist's "uncommon" sense with the "commoner's" common sense. But is it really that simple? Whether something is common sense is witnessed by how it is verbalised in common language. According to common sense the sun rises and sets. On authority it has been accepted that this statement is amended, if necessary, by the adverb "apparently". Moreover, the arguments substantiating why reality is different are accepted on authority, as is apparent from the almost verbal way in which they are repeated. I wonder how many people would be able to tell which phenomena and arguments in this particular case moved astronomers to distinguish between appearance and reality. But many distinctions between appearances and realities have likewise moved to the state of common sense and are verbalised in common, even daily language. However, sunrise and sunset occur in this context of appearance and reality only as far as astronomy is concerned, and even astronomers aren't astronomers all the time.

According to common sense the Earth used to be flat. This common sense statement differs essentially from the preceding one because in (what mathematicians call) a first approximation the Earth is indeed locally flat and allows us even now to behave as though it were so. However, not only as astronomers are we engaged by its roundness, but also as travellers, and as early as we are shown globes and maps. Nevertheless, even adults sometimes wonder why the sun is the hottest at noon rather than in the morning and evening when it is "closer to earth". Yet this common sense conflict is nothing but a low level example of a behaviour which is not unusual even on higher cognitive levels; in fact, it is an example, and at that a striking one, of applying simplistic rather than readily available sophisticated models.

If things have not changed in the mean time, it is the didactical principle in physics and chemistry to fight common sense by exorcising it: common sense ideas obstruct scientific ones. Researchers wonder why people -- even those who had been taught physics at school -- still adhere to common sense ideas, which should long ago have been expelled. Take the following example:

> Again and again subjects are given tests of the following kind: A cyclist, two opposite arrows on the ground, and the question of whether the forward force is stronger than the backward one (that of friction). Or the parabolic path of a ball tossed up, and the question of at which point it experiences the maximum force. Researchers search again and again for new explanations of why most people fail these tests. They forget that more often than not force is associated with velocity (rather than with acceleration). Descartes and Leibniz mathematised force as a function of speed (though they disagreed about the kind of function) and even in the past century conservation of force meant conservation of energy. Even today (and even in technical language) the Newtonian definition of force has not at all displaced other uses. Although the physicist on the merry-go-round feels the centrifugal force, the physicist outside calls it apparent. Would it not be wise to recognise this fact in instruction and to deal compara-

tively with various concepts of force and appreciate them according to their merits?

In general I believe that in instruction it would be more recommendable to start with common sense ideas rather than to reject them as outdated and better being suppressed. This belief is supported in any case by the fact of the more or less spontaneous development of mathematics.
Among mathematics rooted in common sense the most striking example is, of course, whole number. (Similarity is another, though superficially less striking, example.) Children acquire number in the stream of their physical and mental activities, which makes it difficult for researchers to find out how this happens in detail. The acquisition of number is strongly supported (if not made possible) by the corresponding numerals in spoken language, the acquisition of which can be more easily traced through observational research. As the child acquires the *syntactical* structural means to build new sentences so it occurs with the *morphological* structural means to build spoken numerals, beyond necessity and need, and far beyond the physical and (at least initial) mental grasp of number. Going on and on in this sequence is the first verbalised expression of the mathematical mind, an astonishing feature which can hardly be overestimated, though it has not drawn much attention, due to being simply a matter of fact, and experienced as common sense, which is not to be questioned. The morphology of spoken numerals is, moreover, the first algorithm of mathematical character the child acquires: a common sense algorithm acquired by means of common language.

My statement that researchers have paid little attention to children's first and spontaneous arithmetical activities would seem to be contradicted by the title of Piaget's work (with Szeminska) *"La génèse du nombre chez l'enfant"* (1941). The contradiction, however, lies only in the title, since the work itself is not at all concerned with the *genesis* of number. In fact, the genesis of mind had been thoroughly studied by Piaget in his earlier work. This genetic approach should be sharply distinguished from the epistemological one he chose in his later work. In order to make clear what this switch involved, let me briefly sketch the history of the number concept and its teaching.
In contrast to his geometry, Euclid's arithmetic is rooted in the whole number of common sense and, up the middle of the 19th century, the scientific approach to number remained common sense. As soon as infinities are to be dealt with *arithmetically,* this common sense approach becomes blemished by paradox. Cantor removed this blemish by abandoning the then common sense approach in favour of a more sophisticated one, later on extended by Frege-Russell in an even more refined way from *infinities* to *finite* natural number and its foundation.
In Piaget's epistemology, *episteme* means a state of knowledge as advanced as accessible to himself, which in the case of number happened to be the Frege-Russell ap proach, at least such as he interpreted it (and at a later stage, for mathematics in general, Bourbaki's system). This highly sophisticated number concept is Piaget's touchstone, his tool for assessing children's understanding of number at various ages, in far from common sense situations, and using an artificial language, both created for the sake of assessment. Obviously, this is no way to trace the genesis of number. It proved instead to be the source of such oddities as non-conservation, never observed in the unfortunately rare examples of *genetic* research. It is a pity that Piaget's monumental work has not yet been appreciated as a whole. I even wonder whether his switch from

genesiology to epistemology has ever been noticed.
Later on I will return to this question.

Let me add the remark that the common sense roots of number extend further (or are extended) alongside learning language, which proceeds from speaking to reading and writing. Isn't it noteworthy that in written and printed language the mathematical symbols for natural numbers precede their phonetically constructed alphabetical equivalents of spoken numerals ("3" precedes "three", "100" precedes "one hundred")? Anyway the mathematical symbols for natural numbers are integrated into written language. Literacy and numeracy overlap in this region, and even beyond. The place value principle, though deviating from the seriation principle prevailing in common written language, participates in the integration as do -- to a higher or lower degree in various languages -- the basic operations on natural numbers.

I have adduced these striking examples of the overlapping and integration of mathematical and common language as witnesses of the common sense origin of early arithmetic. The tree of knowledge, rooted in common sense, has, as it were, sprouted arithmetic as a branch in its own right -- historically as well as in individual life histories. In the course of life, common sense generates common habits, in particular, where arithmetic is concerned, algorithms and patterns of actions and thoughts, initially supported by paradigms, which in the long run are superseded by abstractions. These products of common sense acquire in turn the behavioural status of common sense, while their common sense ancestry may have even been forgotten. Historically viewed, there have been stages, or even levels of common sense, and the same holds for individual development: what is common sense may depend on the community that shares it.

Is common sense then something like the greatest common divisor of insight, shared by the members of a certain community? Indeed, but it is then a quite large common divisor with relatively small mutual divergences, at least as long as emotion is checked by cognition. In the case of the *problème des partis* the parties quarrelled about how to divide the stakes until a judge resolved the quarrel, albeit not on the strength of common law but by common sense. This was a sounder common sense than that of the quarrelling parties, which was spoiled by badly understood law-like algorithms. The judge's decision created a new paradigm of common sense.

Whereas historians are able to trace the cognitive development of mankind as a development of common sense, it is difficult to attempt the same with individuals, whether by observation or reflection; wrong application of rules, however, and wrong transfer of patterns may provide indications. At any developmental stage of common sense it may be significant how much the learner has contributed to this progress. One extreme is learning without being intentionally taught, and the other is learning what has been bluntly imposed; and since the first is more deeply rooted in previous common sense, it may matter some time in the future how the development took place in the past. It matters whether, for instance, an arithmetic algorithm was acquired as an abridged and streamlined version of former common sense activities, or whether abridging and streamlining (or even the algorithm itself) were imposed. Sure, some, algorithmically gifted people, learn to apply even imposed algorithms adequately; others -- perhaps the majority -- fail to identify the new algorithmic procedures with the common-sensical ones from which they should have originated through abridging and streamlining. They fail because some time in the past they were asked to take mental leaps which exceeded their mental powers. Even though they flawlessly learned the algorithm, they will fail to use it in true life situations where common sense counts; they will instead depend on less efficient lower level operations.

The following is a well-known example: We are taking a trip of 215 km; how much do we still have to drive after 88 km?

Children reason: 88 + 12 = 100, +100 = 200, +15= 215, The subtraction is not recognised (or not ventured); to such pupils the minus key on the calculator is of no use.

Researchers have signalled this “relapse” and marvelled on it. Rarely, however, has it been diagnosed as a consequence of instruction, since no alternative instruction was envisaged alongside the imposition of the new algorithm (which was sometimes embroidered with explanations for conscience’s sake). The new algorithm, however, never did have the opportunity to reach the state of common sense; afraid of applying a wrong algorithm or the right algorithm wrongly, the learner instead relies on what has remained common sense to his mind.

Let us now answer the question posed at the end of the story of the 75 year-old algorithmic performer! The fact that after more than half a century she still masters the various algorithms well enough to teach them, characterises her as an algorithmically gifted person. Why did she worry about one kind of rule but take others for granted? Well, *common sense* is taken for granted. Rules learned by imposition will be taught the same way, at least if the learner who became a teacher is able to teach. But having been imposed they never had a real chance to develop into common sense of a higher order. Is it a privilege of old and wise people to doubt what looks like common sense, or can you not teach this behaviour to the young?

Why “sure and certain”, I asked earlier on, and I announced that this might be a question of behaviour. Of common sense? Yes, but also -- I would add now -- of a tendency to doubt common sense.

Where common sense as the root of mathematics is concerned, up until now I kept silent about geometry (except for one clause in parentheses). Yet I am convinced that geometry even precedes arithmetic in individual development. One of the earliest symptoms -- I believe -- is awareness of similarity of figures; this is found so early that it looks like innate (or might it even be so?)

“Similarity” is even characteristic for what happens with geometry by the token that every day language has no word for it. (This was already the case in Greek, and Greek mathematicians had to think up a word for it, which they did by restricting to geometry a word that in everyday language means any Similarity whatsoever.) It also characterises the individual development of geometry towards mathematics: Until the higher grades a term for the striking feature of similarity is withheld from the pupils -- obviously because one does not dare to speak about geometric similarity until one feels able to define it formally. This parsimony with regard to geometrical language, which is more difficult, indeed, heavily contrasts with the way in arithmetical instruction, where the number sequence is handed to the learner, as it were, on a plate.

Geometry is doomed to fall behind arithmetic in the development of verbal expression. Geometrical arguing is kept on the level of "I can see it", and when an attempt is made to raise it to a higher one, it may be too late in most cases. Common sense does not get the opportunity to develop into more common sense -- a consequence of lacking instruction.

1.1.3 *Why mathematics is different*

Let us be clear about one thing: to the view and mind of most people, mathematics, though deeply rooted in common sense, is more remote from it than anything else. What made it move so far away? The deeper the root the higher the top?

Mathematics is different, indeed, and we will take notice of this fact when we will turn to education. From the very beginning it has been different. It is the oldest among the sciences, even preceding astronomy by more than two millennia. Mathematics was more easily invented, as it was simply a question of common sense -- only better organised. And it developed in this way, becoming more and more organised, according to a pattern I am going to describe.

You know that 3+2=5 and the area of a rectangle by common sense. But as soon as Nature gets involved common sense becomes misleading as everybody knows. For common sense is even more urgently in need of expression and enrichment by transfer of knowledge than is mathematics -- and ever more urgently the farther removed a science from mathematics; and even where common sense is a nuisance it may persist alongside more educated views. Unlike the law of inertia or Newton's theory of gravity, the elements of mathematics have been invented independently of one another at various places in the world. While sciences underwent revolutions mathematics evolved, even during these revolutions and under their influence. How to explain this difference, what dynamic granted mathematics this apparent continuity of development?

Common sense, in order to become genuine mathematics and in order to progress, had to be systematised and organised. Common sense experiences, as it were, coalesced into rules (such as the commutativity of addition), and these rules again became common sense, say of a higher order, as a basis of even higher order mathematics -- a tremendous hierarchy, built thanks to a remarkable interplay of forces.

I have stressed earlier and I stress again that our first verbalised mathematics -- that of number -- distinguishes itself from any other verbalised knowledge by the fact that its mere form, detached from its contents, foreshadows (if not radiates) its mature form and contents. Its tremendous structure -- strictly regular after a short archaic start --stands unrivalled by any other linguistic phenomenon: and in this respect the phenomenon is interlinguistic as far as languages have been created in need of number. By the same token, number is even more common sense than any other human idea. Moreover, this fact itself is common sense to such a degree that it is hardly noticed and rarely made explicit, although it has greatly influenced mathematical thought as well as thinking *about* mathematics. The formal isomorphism between the mental object "number" and its numeral expression, which looks like substantial identity, has been the source of thinking about the relationship between mathematics and language, of identifying mathematical language with the language of mathematics, and of viewing mathematics as a mere language.

Properly said, it has not been as simple as that: what is considered to be *true* mathematics started by impoverishing that primordial structure of whole number, depriving it of the positional strait-jacket (the primordial fall as it were) by which mathematics escaped and outgrew common language. It was New Math's historical fault not to notice the depth of this fall, which eventually became a jump, from the positionally structured numeral to the more abstract free whole number. Number is the first realm where symbolism acquires a reality of its own, a reality apparently independent of its creator, who in turn tries to reorganise his creation, and by means of it his environment. But even this organisation would seem to have existed for eternity, at least as long as the view is restricted to whole number, which Kronecker believed to have been created by God, only to be spoiled by human efforts to transcend it.

Structuring, whether applied to products or to processes, means emphasising form. The first non-trivial structure as such, i.e. whole number as the product of the process of counting, begot rich process and product content which, organised by ever new structures, in turn begot new contents -- a never ending cyclic process.

Through reflecting on his own activity man discovers paradigms, which are abstracted into patterns of mental action, and made conscious as schemes by which thought is organised on behalf of new progress -- adaptable schemes, that is, which allow for varieties of use, as well as, in the same right, rigid single-purpose schemes which, thanks to their rigidity, can lead a life of their own, called algorithms. These forms in turn can become subject matter, a nucleus, as it were, of higher order contents -- again a repeating process, an interplay of form and content, which characterises mathematical thought. This interplay includes the linguistic expression by means of which mathematics is communicated as a product and a process. New content and form require new terminology which, in order to be efficient, has to be streamlined, both symbolically and notationally. Symbols and notations, in the course of reflection, again become subject matter handled as such.

Content is the result of primordial discovery as form is that of organisation, although of course organisation is also a matter of discovery (albeit secondary).

Historically, discoverers incessantly switched duties. In fact, each discoverer behaves as an organiser as soon as he sets out to make his discoveries known, although this is not a specific feature of mathematics. In mathematics, however, organising and reorganising is a continuing affair, and the newly acquired organisation forms may become content in the sense of subject matter to be examined as such. As far as longevity is concerned, no system of mathematics can equal Euclid's, which came near to suffocating its subject. Yet only a brief breathing space was allowed to Bourbaki's: it became outmoded almost at the same pace as it was developed; outmoded as a system, that is, while giving birth to new content.

This is why mathematics is different and why it looks even more different to more people. Schemes of thought can be imposed, algorithms can be taught as rigidly as computers are programmed, and, to be sure, such efforts are not lost on algorithmically gifted people; concepts can be taught by linguistic definitions, and this again works very well with people who are good verbalisers themselves. Last but not least, the interplay between form and content includes what is commonly called applied mathematics, provided the environment to which mathematics is applied is already being seen through the spectacles of mathematics. "Mathematical model" is the fashionable term for form, abstracted from paradigmatical application, and again models and their use become the subject matter of mathematical reflection, as though they were part of the environment. Modelling too can be taught by imposition -- successfully or to no avail -- depending on the preparedness of the learner.

All in all, this explains why mathematics is different, both objectively and subjectively, and it also explains why people do not grasp the reason *why* it is different. Content must be assimilated, while form can be imitated for the purpose of reproduction. It seems to be easier to teach and learn structured form than structureless content. It is as easy to yield to this temptation as it is difficult to resist it. This, then, is the familiar image of mathematics: a set of algorithms, as worthless as it is strict if one does not understand how and why it works. It is a misleading view, which is given the lie as soon as less strict schemes are to be applied.

We will recall the reason why mathematics is different when we discuss education. There we will explain why mathematics is learned differently and therefore should be taught differently, that is, neither as form nor as content but while maintaining respect for the interplay between them, acted out in the teaching/learning process! Learning is progress in knowledge and ability. Their interplay does not sound like that of instruments and voices in a concerto. It is rather a change of viewpoint from content to form, and conversely, leading to ever higher levels, by jumps as high as the learner can perform, and guided but not lifted by the teacher.

1.1.3.1 Examples

Let us dwell upon the interplay between form and content! As stressed above, the origin of the number sequence is form, even linguistic form. While used for counting it acquires content -- a rich variety of content. When in turn abstracted from this variety of counted phenomena, it acquires the status of the mental object that is: more or less formal whole number, though still laced in the formal strait-jacket of the decimal system, and still attached by short strings to counting *something,* which aims at content. Addition and subtraction are meaningful as content operations in preparation for their formal practice, which for a certain time remains supported by models which arose as content. Commutativity of addition is suggested by what addition meant as content before it acquired the -- at least implicit -- state of a formal rule; it can then, on a higher level, become a matter of mathematical content, when studied, for instance, with its consequences, in the context of operative laws. The relation between addition and subtraction arises as a matter of content before it is formally applied, in order to become once again subject matter and content in the context of algebraic structures.

Let us skip multiplication, which shows similar features, and turn to divisibility. This arises in the context of distributing, which is content, but more rapidly than anything else in arithmetic it acquires a most surprising formal status, only to recover that of content in number theory. There it becomes a matter of arithmetic scrutiny, in particular stimulated by the phenomenon of prime number, which means, as far as content is concerned, number resisting any arrangement in rectangular patterns. The search for prime numbers and research on prime numbers is largely one of mathematical content. Yet the convention not to admit the number 1 as a prime number, is formally motivated, namely by the preference for an easy formulation of the theorem on unique prime factorisation. Extending divisibility to other domains (such as polynomial rings) is a formal procedure, where the part played by 1 is taken over by the inversible elements.

The sum of digits (in decimal number notation) provides a formal criterion for divisibility by 9, although its proof, at least at a low level, requires content reasoning, for instance, by displacing balls on the bows of the abacus. The clock dial suggests arithmetic modulo 12. Operating modulo a whole number is a formal activity, from which such mathematical objects arise as finite rings and fields, which in their turn become mathematical contents. Manipulating formally $\sqrt{2}$, “as though its square were 2”, acquires mathematical content in the polynomial domain modulo $x^2 - 2$.

Positive number owes its existence to the geometric context of distances. Positive and negative numbers (together with 0) form what is called real number, beyond which is imaginary or complex number. In brief, historically (for more of it, see [146], Chap. XV) as early as Babylonian Antiquity there were methods -- one might even say, formulas -- to solve linear and quadratic equations arising in geometry or in puzzles. Was it not a pity -- in particular viewing the fine formulas for the quadratic equation -- that frequently one of its “roots” or even both of them had to be rejected as being “false”, and then merely because one only knew positive numbers? Eventually, around 1500, courageous mathematicians took the plunge. In spite of their being “false”, one could marvellously reckon with these numbers by preserving and extending the old arithmetical laws. And lo and behold, one century later, the “negative” numbers were assigned a -- geometrical -- place, left of 0 on the number array pointing at the right. Formal number acquired a content, thanks to which the whole plane could be co-ordinatised. The formal requirement that all algebraic equations can be solved brought forth a content-rich geometric-algebraic activity on the entire plane. We hardly need to add that soon this content again turned formal -- a new link in the long chain of form and content. Thanks to Gauss' geometric representation complex numbers (which had come into being as formally required solutions of algebraic equations and had long been repudiated as ontological monsters) acquired respectability and content status.

Common formal properties of certain algebraic structures led to define the field concept through formal requirements, with the obvious preconception of adding a new object -- the field -- to mathematical contents. Efforts to lend content to formal divisions by 0, and to formal (divergent) series such as 1-1+1-1+. . . are other examples of the dramatic historical interplay between form and content. In ancient history formal patterns for solving quadratic equations geometrically by the so-called application of areas led to the discovery of curves, afterwards recognised as conic sections and redefined as such by Apollonios, only to be redefined anew by means of formal equations in Cartesian geometry.

The term “variable”, as it is used today, embodies the symbiosis of form and content in the most striking way. Originally letters, whether indicating arbitrary points as in Greek geometry or arbitrary numbers as in Vieta's algebra, were nothing but easy generic *names* for these kinds of objects; yet, in the kinetic imagination of the inventors of Calculus and their followers, as well as in analytical mechanics, letters became symbols for mathematical and physical *objects* which varied time-dependently or due to mutual dependence. Initially the law of this dependence was not made explicit but implicitly given by the surrounding mathematical or physical context, which could be a curve or a mechanical event, involving these mathematical or mechanical objects. When the first literal function symbols appeared, that is, for solving the differential equation of the vibrating taut string, they were again no more than generic names of mathematical objects; as soon as necessity required it, they then became variable objects themselves. In the course of history as well as in mathematics acted out day by day, we may notice this incessant switch back and forth between letters as symbols for *“empty places”* (indeterminates or unknowns) and as symbols for *variable mathematical objects.*

As if out of necessity our exposition led us to the role of language in the interplay of content and form. Linguistic form should be well distinguished from abstract (mental) form. Language serves to express both (abstract) form and content. Historically, addition, subtraction, and equality were formal mental objects, presented in clumsy colloquial language, long before shorthand use of the plus, minus, and equality symbols appeared (which later even crossed the frontiers of mathematics and penetrated into colloquial language); and this is the way it still happens nowadays in individual histories. On the other hand, reasoning about such symbols, for instance about meaning and use of the equality sign, involves dealing with them as subject matter, that is, lending them the status of content. The same holds true for brackets as tools for structuring formal expressions and formulas. The need to linguistically express x^2 as a function of x requires the creation of linguistic tools, which resemble content. It may be a matter of content whether in a group structure the group operation itself is called addition or multiplication; but this very choice formally decides whether the neutral element is then called 0 or 1. Interpreting (in highbrow style) a group as a quadruple, consisting of (1) a set G, (2) a mapping of G x G on G (multiplication), (3) a (neutral) element of G, (4) a mapping of G on G (inverse), and thereby fulfilling certain (group axiom) conditions, means that mathematical language is being dealt with as if it were content.

1.1.4 *Mathematics as an activity*

When we distinguished stages or levels of common sense we noticed that attaining them depends on the individual's own contribution, on his activity. Common sense takes things for granted, as I mentioned, while mathematics asks for good reasons. Certainty has to be pursued, and the way this is done, characterises mathematics as an activity, leading to ever improved versions of common sense.

Most of the Dutch names Simon Stevin proposed for the various sciences involve activities rather than objects and instruments. Mathematics is a quite peculiar activity. Thinking logically? It depends on what "logic" is intended to mean. Logic as an established science or as common sense? Let us postpone tackling this question!

Mathematics as an activity is a point of view quite distinct from mathematics as printed in books and imprinted in minds. To be sure, mathematics is a precious treasure-chest of tools, precious that is for those who can put them to good use. This is true for any instrument, many of which can be handled by an inexperienced child: a switch to light a room, a key to open a door, a bell to ring; and many more, which require more sophisticated, yet narrowly streamlined programs in order to be handled, such as algorithms in mathematics.

Every researcher, every producer of mathematics will readily admit that mathematics is an activity -- his private activity, the product of which may or may not be published. Indeed, any author is entitled to have his privacy respected. Moreover, why should he annoy the public with the tale of the production process as it took place? Indeed, the author should not lead the reader of his work along all the wrong trails and into the blind alleys explored by him and eventually abandoned. But would it not contribute to the reader's understanding if he were allowed to watch the process leading to the result as it would have taken place if the author had somehow suspected all along what he finally came to know for sure? In the past some textbooks and even research papers were written in this way, but the fraction of mathematicians who prefer this style of editing has been declining steadily.

Let us avoid any source of misunderstanding! If I speak about products of mathematical activity, I do not do it in the narrow sense of new propositions and theorems. I include proofs, even definitions and notations, as well as the layout, in print and thought. Reading mathematics and listening to it is also mathematics, the mathematical activity of reproducing what is being offered, as though it were the receiver's own production. Reproducing should be an easier task than producing. Unfortunately, the ever more common style of presentation frustrates rather than favours this opportunity. The style of modern presentation creates the illusion that what is sure and certain, has been so from eternity rather than since its discovery was completed and its presentation polished. Yet our certainties are incessantly menaced, if not by confutation -- which has become rare -- then by a longing for more certainty, deeper insight and broader understanding which has become quite common. In this respect at any rate there is no reasonable certainty about what will be considered sure and certain in the future.

Mathematicians, even when reproducers rather than producers, tend to restrict the right to practice mathematics as a mental activity to an elite -- to those who enrich mathematics as a subject matter by new discoveries. Those who do not belong to this elite are required to purchase a stock of knowledge and abilities from the shelves of the mathematics supermarket. If some choice is indeed granted, is this at least left to the consumer, or does somebody else even do the shopping for him?

Does this profusion of metaphoric language bother you? Mathematicians are organisers, albeit in other fields and with other aims than those who are active in industry, trade, traffic and administration. A similar, if not identical mental make-up is characteristic of all of them. But in no other field does organising display itself in such purity, impose itself with such force and infiltrate so profoundly as it does in mathematics. Mathematics grows, as it were, by its self-organising momentum.

No wonder mathematicians like to present their subject in a well-organised state, and textbooks are frequently judged according to their degree of organisation. But for the great majority neither mathematics nor its organisation is an aim in itself. As a useful tool mathematics has conquered and covered a rich variety of fast expanding fields of science and society, and as a tool it has become indispensable to a quickly growing crowd of people. People increasingly use mathematics more often than they are aware of. They use mathematics because they cannot do without it.

That is mathematics as an activity of discovering and organising in an interplay of content and form; later on we will have to ask the question of whether mathematics teaching has really caught up with this development.

1.1.5 *Mathematics and reality*

If I were to continue in the same way, that is, by focusing on the mathematical process, I would imprudently be neglecting the medium in which this process takes place, which provokes this process and which is affected by it. Let us have a closer look at it!

Common sense reveals itself in those actions -- physical and mental -- which are common to people who share common realities. There is no reason, were it even possible, to restrict "reality" to the mere experience of sensual impressions. Even at the lowest level, reality is an inextricable mix-up of interpretation and of what a purist would call sensual experience. If somebody says

> when I heard the doorbell ring, I looked out the window and saw Alice and smelled her perfume

one may believe him and take for granted that this is the report on something that, in the reporter's eye, really happened. There is no reason why one should listen to vulgar positivists who object that "in reality" the ear of the person calling himself "I" was struck by noise of a certain pitch, interpreted as that of a doorbell, his eye by a flood of light waves, interpreted as issued by some source called Alice, and his nostrils were invaded by a stream of molecules, interpreted as Alice's scent. In fact, the "I" could have been a dog, and the above sentence could have been part of a story on the animal's behaviour, told in the first person's style. The doorbell's ring, and Alice's look and scent are primitives, hardly suitable for further analysis. Questions like "how do you know that it was the *doorbell's* ring and *Alice's* look and scent?" may or may not elicit reflection. "It is windy" may be a primitive indoor observation, which only afterwards is justified by references to moving branches and clouds. "A bang" and "a short-circuit" may be descriptions of the Same event on various cognitive and linguistic levels. A number of atomic events read by a Geiger counter, a passing K-meson, and a supernova burst are primitives to the physicist or astronomer who observed and reported these events, although to be reduced to more pedestrian ones, they need profound theories. "Cheerful" and "sad" may be irreducible predications (whether pronounced with regard to other people or oneself), and reducible only by reflection. To some readers, Homer's characters may be real persons, and their adventures real facts, as real as the world on the stage may be to the spectator. There can be little doubt that, as early as the lowest cognitive level, the species *canis* is as real as any particular dog, with whom it even shares a common name. I once told the story of a little boy to whom at an early stage "Utrecht" meant any place he spotted which resembled the neighbourhood of his home; a few years later this became a vague territory, later on he was able to localise it on a map, and after a few more years it became an administrative-political entity, which finally acquired a new, and to him even more vigorous dimension -- that of history. This all happened within an ever expanding reality, which included such institutions as The University, The Railways, The Government. As an example of one object within a variety of realities let us ask what a micro-computer means to various people: a commodity with a certain price to the seller, a weight in kilograms to the deliverer, unexplored land to the novice, and to the expert a part of his life's world, which has long since been abstracted from the tangible instrument.

How real concepts are depends on the conceiver, and under given circumstances cognitive grasps can be more vigorous than manual and sensual ones, which are in fact always mixed up with cognition. Mathematical realities are early phenomena in individual development, that is, not only geometrical realities but arithmetical ones as well. Numbers which are represented by observed and imagined quantities, named by spoken and written numerals, and mutually connected by actual, imagined and symbolised relations, belong to a realm which can extend from the nucleus of everyday life experience to the far frontiers of mathematical research, depending on the involvement of who is concerned. In spite of hundreds of years of resistance -- even on the part of mathematicians -- negative and complex numbers and their operations have become as real to mathematicians as positive ones had been for centuries, and whole numbers for millennia, and at they now belong to the reality of most people who have learned some mathematics. Reality is historically, culturally, environmentally, individually, and subjectively determined. Lawler's "micro-worlds" are less objective and Bauersfeld's "domains of subjective experience" less subjective than these terms may suggest. Solipsism is as artificial a construct as is all-embracing objectivism. I prefer to apply the term "reality" to that which at a certain stage common sense experiences as real.

"Real" is not intended here to be understood ontologically (whatever ontology may mean), therefore neither metaphysically (Plato) nor physically (Aristotle); not even, I would even say, psychologically, but instead commonsensically as when one uses is meant by the one who uses the term unreflectingly. It is not bound to the space-time world. It includes mental objects and mental activities. What I called "expanding reality" is accounted for on ever higher levels of common sense and witnessed by levels of everyday language or various technical languages.

The farther one's reality extends, the more often and more sharply one needs to zero in on more or less manifest fragments, isolated by aspects of reality. This may turn out to be an advantage or a disadvantage. Transfer of electricity is served by insulation, yet transfer of ideas by short-circuits. In between there is the cellular structure of biological life, defined by membranes with a highly selective and effective permeability, which is the amazing result of long evolution. The permeability of the membranes dividing *individual realities* may depend on the active or passive origin of these fragments: whether they are due to internal evolution or imposition from outside.

This is the key, in my opinion, to understanding the poor permeability of the membrane separating classroom and school experience from life experience, which I alluded to and promised to analyse in due course when I will deal with mathematics education; in spite of the fact that mathematics, as well as its various domains and aspects, is already a striking example of the poor, or even the lack of, permeability of membranes. The specific relation between content and form -- the emphasis on form, reinforced by the existence of a particularly efficient language -- favours the growth of watertight membranes between mathematical and "outside" contents, between mathematical language and everyday life or more technical languages. And this takes place in spite of the fact that mathematics could be an outstanding example of broad-minded integration into reality. Indeed, as pointed out earlier, mathematics, unlike any other science, arises at an early stage of development in the then "common sense reality" and its language in the common language of everyday life. Why does it not continue in this way? How can one strengthen this integrated existence in order to resist the temptations of separation and isolation? "Mathematics starting and staying in reality" was the title of a lecture I once gave on several occasions to answer these questions. In due course I will repeat these answers, which are of an educational character.

1.1.6 *Concepts or mental objects?*

What is the difference between number and number concept, between ratio and the concept of ratio, between triangle and the concept of triangle, between similarity and the concept of similarity, between X (an object) and the concept of X? This is a meaningful question for non-mathematical objects X as well, as are: force, mass, heat, evolution, content, form, science, government, race, humanity, poetry, pop art. The above sequence looks like one of increasing distance between *"X"* and "concept of X . There is at any rate a difference between both of them. "Concept of X" seems to mean how one conceives of an object X in a certain perspective, say, by inspection, reflection, analysis, scrutiny, or whichever you wish.

There would be no cause to ask that question were there not a tendency to decorate terms for objects, for dignity's sake, with the prefix "concept of"; even this would hardly matter if one could disregard the consequences, in particular for teaching where cognitive learning is concerned. Unfortunately teaching *concepts* looks more dignified than mere teaching. Teaching *concepts* is likely to create the illusion of adding more understanding to what is learned.

There have been various number concepts -- Euclid's, Cantor's, Dedekind's, Peano's, Frege's, Russell-Whitehead's, Kleene's -- which differ formally as well as substantially from each other. This difference, however, is a matter of expert sophistication, and of more concern to researchers in Foundations of Mathematics than to the great majority, even among mathematicians. There have been various concepts of force -- Aristotle's, Descartes', Leibniz's, Newton's. Physicists decided in favour of Newton, but they have so far failed to prevent people from nourishing less educated ideas on force.

Cognition does not start with concepts, but rather the other way around: concepts are the result of cognitive processes. Mathematics allows explicit definitions at an earlier stage than any other field of knowledge. For instance, "odd" and "even" can be defined on the basis of "whole number". Indeed, the first mathematical science, that of the Pythagoreans, was about *Odd and Even.* But what about "whole number"? It is generated by a process, that of counting, rather than by an explicit definition, only to become a matter of common sense rather than a concept. At least since Euclid, mathematicians have searched for deeper foundations, and they have succeeded. New Math adherents believed that children should not be denied this benefit. They fought common sense number, without success. Back to Basics eventually meant back to common sense.

Specimens of circles suffice to explain what are common sense circles, and nobody who is shown a circle doubts that it has a centre (which may be found by trial and error, or in a more sophisticated way). But at least since Euclid, circles were defined as locuses, and in Cartesian geometry by algebraic equations. New Math adherents fought the common sense approach, and in a sense they succeeded, at the price of expelling geometry entirely from mathematics instruction.

In the mean time it has become clear to increasingly more people that, where non-mathematicians are concerned, teaching the concept of X is not the appropriate way to teach X. Cautious researchers now admit that concepts are preceded by something less formal, by initiations, preconcepts, or whatever they call it, which in the long run means that the proper goal is still that of teaching concepts. In my view, the primordial and -- in most cases for most people -- the final goal of teaching and learning is *mental objects.* I particularly like this term because it can be extrapolated to a term that describes how these objects are handled, namely, by *mental operations.*

Whole number, the number line (even if drawn on the blackboard), geometrical shapes (even if materialised): all are mental objects, so far as it is understood (and it is understood) that visual images are rough representations of mental objects. How do mental objects develop to become concepts, and what criteria reveal whether this has indeed happened? In the case of whole number, shedding the strait-jacket of the positional system may be the symptom. Most often verbalisation and formalisation may indicate concept formation. It is worth mentioning that historically the *mental object* of group preceded the *group concept* by about half a century. Leibniz and John Bernoulli used the word "function" for something that was no more than a *mental object,* and only upon the first appearance of a letter symbol for a function in papers of d'Alembert and Euler was the road paved for the *function concept.*

The distance between mental object and concept will depend on the subject matter, but even more on the individual and his particular situation. This is the reason why it must be respected in instruction. When we deal with the teaching/learning process the question "Concepts or mental objects?" will be parallelled by the didactical one: "Concept attainment or constitution of mental objects (by mental operations)?"

1.2 STRUCTURE AND STRUCTURES

Structure is form abstracted from its linguistic expression. We will focus on structure with regard to mathematics, although much of what we have to say applies to a broader context. In mathematics the relationship between form and content is reflected by that between something *having* or *being* a structure. Structuring is a means of organising phenomena, physical and mathematical, and even mathematics as a whole.

1.2.1 *Structures --poor and rich*

A few examples will give us a clearer view of what structure means in mathematics.

A tetrahedron may be taken as a structure consisting of four vertices, six edges, and four faces in their mutual relation of "this vertex lying on that edge and in that face, this edge containing that vertex and being contained in that face, this face containing that vertex and that edge"; it is what one calls a combinatoric structure, which in this particular example entails restricting oneself to the relations of containing and being contained in, and forgetting about what points, edges, and faces actually are, i.e. that edges are straight and faces are flat. What is the use of so poor a structure? Well, with *sets* of such combinatoric tetrahedra one can build abstract polyhedra by identifying faces of different tetrahedra with each other -- sticking them together as it were. Tetrahedra can thus be used as building blocks for larger structures, which may become the subject of profound research known as combinatoric topology.

We need not, however, content ourselves with the combinatoric structure of the tetrahedron. It is more natural *to* consider it as a solid in space, with true points, edges, and faces. This is again a structure, a *geometric* structure. As a geometric structure a tetrahedron is a *richer* structure than combinatorically. There is more one can say about it; within this structure one can, for instance, measure distances, edges, angles, surfaces and volume.

A tetrahedron is also called a triangular pyramid. Or is it the other way round, is the triangular pyramid a tetrahedron? No. If this were true, how could it happen that a *regular* triangular pyramid need not be a *regular* tetrahedron? Well, triangular pyramids and tetrahedra are *different* structures. By definition, a pyramid has a bottom and a top (even though one is allowed to turn the pyramid upside down). A tetrahedron can be converted into a triangular pyramid in four different ways, as a matter of fact, by appointing one of its vertices to its top, and the opposite face to its bottom. The triangular pyramid is a *richer* structure than the tetrahedron.

Let us now knead the solid tetrahedron as though it were clay or dough, let us press and deform it but refrain from tearing apart what is connected, and connecting what is disconnected. The result maybe a nice ball, or a potato or, say, a dumb-bell, The vertices, edges, faces have disappeared. The thing is still connected, and is so in a very special manner, spherelike (and certainly not like a ring). It is a *topological* structure, poorer than the previous one of the geometrical tetrahedron: no role has been assigned to distances, angles, rectilinearity, and so forth.

1.2.2 *Structures defined by relations*

So far we could have contented ourselves with calling our objects figures rather than structures. In order to become structures they must be described structurally. Figures can be *constructed* out of their parts. Something that is finished can be described *structurally*. One focuses on certain relations between the elements of the figure, and chooses among them a set that characterises the figure in mind. The easiest case is the *combinatoric* tetrahedron: four arbitrary objects are called vertices, each pair of them is assigned to be an edge, and each trio to be a face, with the mutual relations of containing and being contained. The *geometric* tetrahedron, on the other hand, can be described as a *metric* structure, by assigning a mutual "distance" to each pair of its points (or vertices only). It is less easy to explain how a *topological* structure is described: by telling what is near to each other, with no appeal to quantitative distance, that is, in a qualitative way; indeed, topology forbids tearing and glueing, which would affect qualitative distance. As a topological structure the border of a disk is the same as a whimsical closed curve, but it is different from a straight or curved line segment, which on their part are topologically both the same.
More -- as well as more stringent -- relations increase the wealth of a structure. Dropping relations impoverishes and appending relations enriches structures in the same medium.

1.2.3 *Algebraic structures*

The system of whole numbers 1,2,3,... can be interpreted as a structure in various ways. While postponing the cardinal aspect, I start with the ordinal one, the counting sequence, which is an *order* structure of the kind "somehow it starts and each has its successor", while the names of the particular things do not matter. An addition can be derived from the order in such a structure: to each pair of things it assigns a third, its sum. Now imagine you have forgotten how this addition has arisen! Then all that is left is an infinite system of things, distinguished by symbols, and a table with two entries which, for every pair of these things, tells what you want to be their sum. The relations in this system are of the form $a + b = c$. It is what one calls an *addition* structure. The present one has some remarkable properties, for instance, that $a + b$ is always the same as $b + a$. Maybe, next to this table you allow for a second one (which may or may not be dependent upon the first) which of every pair of things tells what you want to be its product. This enriches the structure with relations of the kind $u\,.v=w$.

Such systems are called *algebraic* structures. There are plenty of them, with two or even more operations, which may satisfy widely different conditions.

1.2.4 *Structures ---from smaller to larger*

The best known among the algebraic structures are the number structures

- $\mathbb{N}$, of the natural numbers 0,1,2, ...
- $\mathbb{Z}$, of the integers. ..,-2,-1,0, 1,2,. ..
- $\mathbb{Q}$, of the rational numbers, represented by fractions of integers, such as $\frac{1}{3}$, $-\frac{5}{3}$, 5.33
- $\mathbb{R}$, of the real numbers, among which, for instance, $\sqrt{2}$, e, π
- $\mathbb{C}$, of the complex numbers $a + b$ i where i = $\sqrt{-1}$

It is a sequence of structures of increasing *size*, although not primarily -- and not even necessarily -- of increasing *richness*.

The operations of addition and multiplication are the same in all these examples, as are structuring relations of the kind

$$a + b = c, \quad u\,.\,v = w,$$

whereas some of their properties may change. The equation

$$a + x = c$$

is in general solvable for x only if negative numbers are admitted, the equation

$$u\,.\,x = v \quad (u \neq 0)$$

only if rational numbers are involved, the equation

$$x\,.\,x = k \quad (k>0)$$

only in the domain of non-negative real numbers. For instance, the solution of $x^2=2$ cannot be obtained as a fraction of integers, although it can be *approximated* by such fractions. $\frac{3}{2}$, $\frac{17}{12}$, $\frac{577}{408}$, …are increasingly better approximations of $\sqrt{2}$. "Approximating" is a topological concept. Beside with the algebraic structure, the domain of real (as well as that of complex numbers) is provided with a topological structure. In this sense it is not only larger but also richer than that of the integers. In another sense $\mathbb{N}$, the structure of natural numbers, is richer than $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ .Indeed, divisibility is a rich concept in $\mathbb{N}$, but a poor one in $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

1.2.5 *Generating the number system*

The system of real numbers is geometrically represented by the *number* line, where each point, by its distance from the point *0,* represents a number, positive at one side and negative at the other. In this model the algebraic operations are visualised by simple geometric mappings.
In the Greek tradition this was the common sense approach to (positive) real number, which was inundated by the "arithmetisation" wave in the past century. Arithmetisation meant disavowing the traditional geometrical foundation, and founding conversely geometry on number. The real numbers were to be defied by sequences of (approximating) rational numbers, the rationals as fractions of integers, and the integers as natural numbers preceded by a plus or minus sign. One long step backwards was taken when inatural number in turn was founded on nude sets, deprived of any structure (and ultimately all *sets* on the empty set). Sets were to be compared with each other by mapping, that is, by laying them mentally elementwise along each other. If the sets A and B fitted each other they were said to have the *same cardinal;* if they did not, even though A could be fitted to a part of B, then A was said to have a *smaller* cardinal than B.
The common sense counting number, and the common sense measuring number, as visualised on the numberline ruler, were derived from one common root by this profound analysis: the structureless set. Lack of structure may be an advantage in Foundations of Mathematics, which is high level mathematics, but much too high to start with. Indeed, the proper beginning is or should be common sense. At present most people will agree that New Math's contempt for common sense has been a historical mistake. But has everybody really learned this lesson?

1.2.6 *Geometric structures*

The geometric structure most familiar to us from early childhood onward is the space we live in -- *euclidean* geometry as it is called. Solid bodies allow us to compare, or rather to define, distances, and according to this system of distances the straight vision lines are experienced to be the shortest. From early childhood onwards we are familiar with faithful pictures of objects, obtained by shrinking and enlarging, called *similar* in mathematics. No doubt similarity even precedes number in cognitive development.
During the 19th century poorer geometric structures than the euclidean one began drawing the attention of geometers. Impoverishing a structure -- provided this means removing what under certain conditions look like scrolls and curls -- can lead to greater profundity. It is an art creative mathematicians have learned to practise and master during the past century.
In euclidean space we are dealing with distances, angles, straight lines, circles, planes, spheres. A first step towards impoverishing this structure is to forget about the general comparability of distances and angles, while preserving recti-

linearity and parallelism. This leads to *affine* geometry. In affine geometry all parallelograms are the same: rectangles and squares cannot be distinguished from other parallelograms, nor can circles be distinguished from ellipses.
A next step is to forget about parallelism, while preserving rectilinearity. This produces *projective* geometry, where all quadrilaterals are the same and all conics are the same.
One more step, and then even rectilinearity disappears as a structuring property. Space is being impoverished to become what we formerly called a topological structure, where we can still distinguish open curves from closed ones, the interior of a closed surface from its exterior, see whether or not a closed curve in space is knotted, and whether or not a pair of closed curves is linked.
More than a century ago this kind of classification was introduced into geometry by Felix Klein, when proclaiming his *Erlanger* program. This was the first modem attempt at structuring mathematics itself, or rather a part of it: geometry.

1.2.7 *Structure of mathematics*

It is one thing to interpret a small or large object as a structure, and quite another one to attempt this with an entire area of scientific cognition. When compared with creating, organising scientific cognition seems to be an inferior activity. Yet, as stressed before, in no science are these two activities so densely interwoven as they are in mathematics. A century ago such an organisation was attempted in geometry and for half a century the much broader task of organising the whole of mathematics remained dormant. After the second World War this task was undertaken by the French group Bourbaki. Although their work required a large amount of creativity, its eventual result could not in principle be anything but a deductive codification of *existing* mathematics. To be sure, Bourbaki's work is a monumental system of mathematics, which -- although now out of date on essential points -- has contributed enormously to the growth of mathematics. Even so, it has in the mean time been overtaken by mathematics itself. I doubt whether a fresh attempt can or will be tried in the future.
Bourbaki's hierarchy is primarily oriented *from poorer to richer* structures and -- accidentally, and certainly not as a matter of principle -- from smaller to larger ones. As a matter of fact this is indeed the most natural strategy for a systematic build-up: from poorer to richer. It starts, as it were, with a *tabula rasa,* with that which lacks any structure, i.e. the clean set. But after all, how can rich structures be created from this poor start?
If A is a set, one may consider the set P(A) of its subsets, which in fact has a remarkable structure, effected by the structuring relation of containing and being contained in.
Or, starting with two sets A and B, one gets their *product,* the set of pairs; it is obtained by steadfastly taking one element of A together with one of B -- a construction which can be visualised by a rectangular drawing where, for instance,

each of three boys (set A) is paired with each of four girls (set B).

Fig. 1

This product set is heavily structured, both horizontally and vertically. Similarly, the whole plane can be structured as the product of two straight lines, where each of its points is represented by a number pair -- the way this is done in coordinate geometry.
These are mere examples of how to impose structures on given or *ad hoc* constructed sets -- order structures, algebraic structures, topological structures, and combinations of structures. They can be built into a hierarchy, in order to impose a hierarchy upon mathematics as such -- one hierarchy among many conceivable possibilities. There is not, and never will be anything like the hierarchy of mathematics. This fact is evident from the variety of textbooks, which differ remarkably from another. There is certainly an inclination, at university level, in particular, to teach mathematics as a hierarchy, although one teacher will put more water into the hierarchic wine than the other, depending on the degree of immediate utility of the mathematics taught.
Even so, consumers of mathematics should be introduced to mathematical structures wherever it matters, and maybe even be made aware of their structural aspects to a certain degree. This does not, however, mean that mathematics as such should be presented to them as a structure, either according to Bourbaki's or any other conception. It would be a futile attempt, to be sure.
Moreover, a system like Bourbaki's, or any variant whatsoever, does not do justice to mathematics as a *servant* rather than a master. The most striking shortcoming of any system of mathematics may be its treatment of natural number: most often a disregard for the numeration structure, i.e. the decimal system. From the most pedestrian to the most sophisticated practice of numbers, the decimal structure is its *dominating* feature. From the (merely linguistic) learning of numerals to operating efficiently with numbers, this structure is indispensable. Yet in no system of mathematics is it ever mentioned. Indeed, high level mathematics has been objectified and stripped of such human rudiments like the decimality of our

fingers and toes. True mathematics behaves universally rather than anthropomorphically.

This, then, is the most striking feature of the epistemological and practical inadequacy of Bourbaki-like systems. There are more misfits. Visual geometry has no place in them. Similarity, which in cognitive development even precedes number, is far remote in this hierarchy, and even then is detached from its visual origin. No place is assigned to computers, which are functional structures. Mathematical modelling of structures for the benefit of even the simplest applications is incompatible with the hierarchic rigidity of such systems. No attempts have ever been made to structure mathematics as seen from reality as epistemological source and domain of applying mathematics, although such attempts would be extremely desirable -- psychologically, pedagogically, educationally and didactically. By their mere existence the deductive systems prevent these attempts from being undertaken.

1.2.8 *Structures as viewed from reality*

This is not to belittle mathematical structures. By discovering structures within mathematics we have learned to better understand the inherent organisation of our knowledge. Many psychologists and educationalists still perceive cognitive development as *concept attainment.* Superficially this is evident from the numerous titles of research papers which contain the word "concept", and in a more profound way from innumerable experiments where the possession or development of concepts is said to be tested.

Knowledge as a hierarchy of concepts is a prominent feature of the Aristotelian theory of science and epistemology. Its methodological inadequacy has become manifest, mainly under the influence of mathematics. Aristotelian hierarchy is oriented from the general to the particular, the general embracing the particular. It is most perfectly realised in biological taxonomies, where one descends along the lines of groups, divisions, classes, subclasses, families, species, varieties. Yet outside botany and zoology concept formation by classification has proved unsatisfactory. Knowledge is not just more than but quite a different category from a reference flora or fauna, as is structuring, when compared with classifying. Classes contain each other, or are contained within each other; in this way generality and particularity are expressed by mere size. If, in the case of structures, one speaks of general versus particular -- which I do not -- the poorer structure would be the more general, and particularity would develop by enrichment.

By structuring rather than forming concepts we get a grip on reality. We do this with rich or with poor structures according to our needs. Impoverishing may mean generalisation in the sense of wider applicability. Among the poorest scientific structures, those of mathematics are distinguished by their large measure of applicability as exemplified by numbers and geometrical objects. Mathematical structures are also more easily discerned than structures in general.

As I stressed before, the identification of learning with concept attainment is a superficial view, even though it is not unusual with psychologists and educationalists. It is to Piaget's merit that he stressed structure, at least in theory, though not as consequentially in practice.

1.2.8.1 *Structure of science, and development*

I wish to stress once more the distinction between structures within a science -- in particular mathematics -- and the structure of a science. It is an old adage that cognitive development proceeds from the particular to the general, and this direction is therefore considered compulsory for the educational process. Yet it lacks sophistication as an adage, mainly because of the multifarious interpretation of "general versus particular". It may be taken for granted that, for a child, being acquainted with a particular dog (or with a few of them) precedes acquaintance with the species "dog", but classbuilding is only one aspect of cognitive development and, for that matter, quite a modest one. Generalisation starts with ***situations*** rather than with *objects* and the function of a particular situation is paradigmatic rather than classbuilding.

Piaget's view is more sophisticated than the traditional adage, if not its direct converse. According to Piaget, development takes place along epistemological lines, where episteme is not individual cognition but, with regard to its contents, independent of the developing individual. As a consequence, the structure that is manifest in the present state of science is at the same time the pattern of the individual's cognitive development. Experiments serve to corroborate this parallelism, and they were designed with this purpose in mind, or at least this was their intention.

Piaget nourished this conviction before Bourbaki built his system, or at least before he could have become aware of it. He fully practised it in his work on geometry'. He chose the structure of geometry familiar to him as the canvas on which the child's development must be recorded. This structure was Felix Klein's, although geometry had by that time already outgrown it, if ever it had fitted at all. According to Piaget development progresses from the poorer to the richer structures as he found them in Felix Klein's hierarchy; it goes from topology via projective and affine to euclidean geometry and, according to him, this **is** true as well for the perceptive, the representative, and the cognitive aspects of development. Piaget's view betrays a high degree of confidence in -- or should I say, obedience to -- mathematics' hierarchies, although this has been belied by his own experiments as well as by those of others. At an early stage a child who is unable to draw neat squares and circles, can easily distinguish neat copies of squares and circles from bad ones.

When Piaget was confronted with Bourbaki's system of mathematics, he accepted it as an epistemology which can be interpreted genetically, or so it would seem. In fact, I believe this to be a misinterpretation of Piaget's reaction. I rather think that Piaget's own conception of cognitive structure had been entirely completed by that time. His acknowledgment of Bourbaki's structure of mathematics[2], rather than being a recognition, was an attempt to adapt the Bourbaki system to his own conception. This has, however, often been interpreted as a support lent by psychology to the enterprise of teaching mathematics according to the structure of this science, an enterprise which became famous under the name of New Math and eventually proved to be a failure.

Piaget can hardly be blamed for this. Even though he believed in the genetic-epistemological value of a structure of science, there is no proof that he advocated teaching according to curricula patterned on the structure of a science.

This brings us to the next point, which will anticipate later discussion.

1.2.8.2 *Structure of science, and instruction*

In spite of their failure in mathematics, curricula patterned on the structure of a science, are becoming fashionable in other areas. As a pure mathematician I am able to view more clearly the relativity of what is offered as a structure of mathematics. If I place mathematics within the larger context of knowledge and abilities, I am struck by the large gaps in what is often propagated as structure of mathematics; for example, the lack of the numeration structure and the disdain for visual geometry and for even the simplest applications. As a mathematician I also feel obliged to turn against the structure of science as a means to structure education, because my personal experience has shown me how easily mathematicians yield to this temptation.

Piaget's work supplied me with no argument from developmental psychology in favour of a science structure curriculum, nor can this be justified by educational theory. A science structure codifies systematically (and in the case of mathematics this means deductively) the state of that science at a given moment and, for that matter, of a science that is not even the subject of the instruction envisaged. Moreover, there are serious arguments against science structure curricula. Systems of mathematics reveal a hierarchy oriented from poorer to richer structures which the curriculum designer tries to imitate. "From poor to rich" is, viewed mathematically-didactically, a questionable principle. Poor structures are utterly abstract as is evident from the poorest of all, the structureless set. Didactically one cannot come to grips with it, or it should be by concretising, by filling out the abstract form, and in practice this is done by creating artificial, and often even false concretisations. In genuine mathematics, sets as well as all of these structures serve goals; they are because they are operational. At the level where science structure curricula start, however, there is nothing mathematical one can do with sets. So, as a curriculum designer, one arbitrarily invents things to do with sets, which have nothing to do with the need for sets in mathematics -- false concretisation and operationalisation; nor do these have anything to do with the need for learning -- false didactisation. At best there will be no effect at all -- sets and

other poor structures in elementary instruction are introduced with no other aim than to pay lipservice to the science structure.
If these are but unfortunate slips, there are also more profound arguments against the precedence of poor over rich structures, even from the viewpoint of developmental psychology. What is usually called abstracting, is most often nothing but impoverishing a structure. Mathematical structures have arisen in richer contexts and be created in order to be applied where they have arisen and beyond. The orientation from poor to rich has been suggested by ready to hand mathematics. In due course we will deal with learning "mathematics in action", which is re-creating mathematics, not aimlessly of course, but under guidance. If this is agreed upon, the didactically recommendable direction will be the Same as that in which mathematics arises, that is, from rich to poor.

1.2.8.3 *Structuring rich context mathematically*

Didactically I have opposed rich to poor *mathematical* structure. This, however, is not enough. One should not be satisfied by staying within mathematics. The rich structures to be offered should also be sought for outside mathematics, albeit with a mathematical-didactical afterthought. After the failure of science structure curricula, teams of developers chose another way: that of offering non-mathematical rich structures in order to familiarise the learner with discovering structure, structuring, impoverishing structures and mathematising. By this means he may discover the powerful poor structures in the context of the rich ones in the hope that, by this approach, they will also function in other (mathematical as well as non-mathematical) contexts. Starting with poor mathematical structures may mean that one will never reach the rich non-mathematical ones, which are in fact the proper goal.
Let us illustrate this by an example!
Who is not familiar with the so-called logical blocks, 24 in number, available in a rich variety of models, for instance, red/blue, circle/square/ triangle, big/small, thick/thin? They are a paragon of an entirely prestructured world: one piece for each combination of the four criteria. Abstract operations on sets can be concretised marvellously by means of such a model. I have opposed to this system another one which I have called a little world, although I have never insisted on teaching it: it is, as it were, a toy shop containing cars, animals, dolls, buildings, and so on, of various sizes, kinds of material, colours, grades of mobility, and so on. The criteria of classification are not imposed a *priori* but are discovered and developed by the learner himself -- characteristics of colour, size and so on, which need not be sharply determined but may be susceptible to shades. In this little world, combinations of characteristics may be represented by *many* objects as well as by *none,* for instance, no big black wooden house but many little red tin cars. In brief, it is not a prestructured world but rather a world to be structured. In this case, structuring is performed by classifying. I chose this example not be-

cause I may believe in classifying as an important cognitive activity, but rather as an example that, thanks to its simplicity, sharply features the difference between *poor and structured* on the one hand, and *rich and to be structured* on the other. Logical blocks are a striking example of the implementation successes which can be reaped with sharply structured material -- cheap successes obtained thanks to love of ease. Rich material, open to structuring, which provides for more didactical opportunities, is more demanding and therefore less easy to implement.

Let us finish here! A large number of rich contexts for mathematics instruction is now available, more than anybody can imagine. The main problem is that of implementation, which requires a fundamental change in teaching attitudes before it can be solved.

1.3 MATHEMATISING

1.3.1 *The term*

After this discussion on the context and the internal and external structure of mathematics, we turn again to mathematics as an activity and look at one of its main characteristics: *mathematising*. Who was the first to use this term, which describes the process by which reality is trimmed to the mathematician's needs and preferences? Such terms usually emerge during informal talk and discussion before they enter the literature, and nobody can tell who invented them. In any case, mathematising is a process that continues as long as reality is changing, broadening and deepening under a variety of influences, including that of mathematics, which in turn is absorbed by that changing reality.
Mathematising as a term was very likely preceded and suggested by terms such as axiomatising, formalising, schematising, among which axiomatising may have been the very first to occur in mathematical contexts. Axioms and formulas are an old heritage, although the meaning of "axiom" (or "postulate"), and the form of formulas has changed in the course of time. Euclid's Elements are no paragon of flawless deductivity, as people believed for centuries, nor were they ever intended as such, as some people still seem to believe. Axiomatics, as we now use this term, is a modem idea, and ascribing it to the ancient Greeks is, in spite of precursors, an anachronism. Nevertheless, reshuffling an area of knowledge so that ends are chosen as starting points, and conversely, using proven properties as definitions to prove what was a definition originally -- this structuring "upside down" is a mathematical activity of long standing. It is as old as Greek mathematics, or perhaps even older, although never until modem times has it as consciously, as systematically, as passionately been exercised as it was by the Greeks who greatly enjoyed organising and reorganising knowledge. Nowadays, mush-

rooming axiomatic systems are the result of attempts to reorganise fields of mathematical research. This technique is called *axiomatising,* and is well understood and mastered by modem mathematicians. Its first striking example was groups. From the turn of the 18th century onwards, mathematicians were confronted with mappings of sets upon themselves, often singled out by invariance properties, and were led to compose such mappings. In this way they became acquainted with sets of transformations which, under composition, automatically satisfied the well-known postulates, required later on for groups. Cayley, in 1854, took the unifying step to define, by means of these postulates, the (finite) object he called a *group;* yet not before 1870 did this new conception become whole-heartedly accepted by leading creative mathematicians, and then also in infinite substrates.
Formulas, in everyday life as well as in symbolic language, are as old a feature as axioms, or even older. Improving the linguistic expression by increasingly effective symbols and symbolisms has been a long process, which first concerned mathematical subject matter and only later also affected the language in which this subject matter was expressed. This process of trimming, adjusting, and transforming language is called formalising.
To be sure, as popular as axioms and axiomatising might be in modem mathematics, they are only the highlights and the finishing touches in the course of an activity where the stress is on *form* rather than on *content.* The same holds true with regard to formulas and formalising. Axioms arise from paradigms or sets of paradigms, and axiomatising means generalising experienced paradigms. It is an old human habit to make one's experiences and actions paradigmatical, to generalise them by abstracting them into laws and rules, to create schemes to fit reality. This last activity is called *schematising,* which is the counterpart to axiomatising and formalising insofar as contents rather than abstract form and language are concerned.
The preceding exposition serves to explain the origin of the term *mathematising* as an analogue to axiomatising, formalising, schematising. I pay so much attention to it since it is not unusual, in particular in education, to restrict the term to one of its components. I myself insist on including in this one term the entire organising activity of the mathematician, whether it affects mathematical content and expression, or more naive, intuitive, say lived experience, expressed in everyday language. But let us not forget about the individual and the environmental dependence of "lived" and "everyday life" on expanding reality and progressing linguistic sophistication!

1.3.2 *Some aspects*

Modelling

Moreover, with regard to schematising there is a tendency to identify schemas with such things as solving formulas and procedures within formalised mathematics. Nowadays the term "schema", in the broad sense, seems to have been superseded by more topical "model" -- a valuable term, but unfortunately devaluated by thoughtless use and misuse. I have castigated this practice often enough, or so it seems to me. [87]

Mathematics has always been applied in nature and society, but for a long time it was too tightly entangled with its applications for it to stimulate thinking on the way it is applied and the reason why this works. Counting was indeed, common sense; surveyors acted as though their pegs and rods were geometrical points and lines, and money changers, merchants, and ointment mixers behaved as though proportionality were a self-evident feature of nature and society. Even the astronomers in Babylonian Antiquity stuck as long as feasible, or even longer, to linear inter- and extrapolations in their attempts to describe celestial phenomena numerically; which meant, by means of piecewise linear and by zigzag functions, which their Greek heiresses eventually changed into goniometric ones. But goniometric functions did not drop from the sky they were studying. The underlying theory was that celestial movements ought to be circular. Efforts to save both this postulate and the contradicting phenomena gave birth to what we now would call a model describing the celestial movements -- a contraption of circles, epicycles and excenters which, geometrically and numerically processed, required goniometric functions. This model survived for almost two millennia. Kepler, rather than offering a new model, formulated three general mathematical laws of planetary motion, derived afterwards by Newton as consequences of his theory of gravitation. Newton himself refused to contrive primitive mechanic models for explaining gravitation. So, as time went on, physicists had to accept, quite often grudgingly, attraction by the force of gravitation as a model in its own right which, second only to Huygens' wave theory of light, was the first model in modem times to transcend common sense experience. History repeats itself: When elastic models of light propagation, suggested by 19th century's common sense mechanics, to refine Huygens' wave theory failed, physicists had to accept Maxwell's electro-magnetic light model.

Modelling is a modern feature. Until modem times the application of rigorous mathematics to fuzzy nature and environment boiled down to more or less consciously ignoring all of what appeared to be inessential perturbations spoiling the *ideal case*. For a long period, simple geometry and arithmetic had sufficed to strip such ideals of their disguises. But what is ideal and what perturbation? The first example of a non-commonsensical split between ideal and perturbation was delivered by Galileo: uniform motion as the ideal, perturbed by resistance or, as Newton put it, by force in general. In a way this methodology has survived up to the present day. Even if a "rigorous" theory is available, it is almost never applied as such but instead simplified in order to be accessible to actual processing, which afterwards may be refined by better approximations, or immediately by feedback models. The first great example of non-trivial idealisation of this kind is d'Alembert's vibrating taut string: by neglecting, as it were, the string curvature, he succeeds "linearising" the differential equation which, once made linear, is easily solved. Indeed, physical remodelling with the aim of linearisation has become a standard tool in applied mathematics.

In natural sciences the first use of the word "model" is probably related to the well-known orrery models of the solar system, where the interplay of planetary and lunar movements caused by gravitation is rendered (coarsely simplified) by means of a mechanical device; being merely a model, justice is done to the kinematics though not to the dynamics of the celestial motions, while for practical reasons the radii of the spheres representing the celestial bodies are disproportionate to each other as well as when compared with the sizes of the orbits. Familiar as the Rutherford-Bohr atom model may be, it describes the atom and its manifestations as a little solar system, with strange restrictions imposed on the possible orbits; its model character stems from the *ad hoc* conditions to which the orbits are subjected, and the *ad hoc* assumptions regarding jumps from one orbit to another which are incompatible with the laws of classical physics. A more recent model is the drop model of the nucleus, in which protons and neutrons are smeared out as a fluid -- an idea typical of a simplifying model. A particularly revealing example is the cosmological model of the expanding universe. It was originally contrived as a purely kinetic explanation of the all-sided flight of the galaxies yet, as time went on, it became enriched by numerous features of dynamics and of elementary particle physics, although it was still understood as a grossly simplified model of the evolution of the cosmos.

All these are idealising models, which introduce mathematical precision into a coarser physical reality or simplify a reality that is tacitly agreed to be more complex than its so-called model. Strangely enough the first use of the term "model" in *mathematics* aims at just the contrary: concrete models of abstract geometrical shapes, in plaster or wire and cardboard. The same mathematician who had a large collection of such geometrical models made -- Felix Klein -- was also, if I am not mistaken, the first to apply the term "model" *within* mathematics itself. I mean here the image of non-Euclidean within projective geometry -- Cayley's invention which was interpreted as a model by Klein in order to concretise abstract looking non-Euclidean geometry within the frame of the more concrete looking projective geometry. Although not as palpable as gypsum models, this model is indeed more easily accessible to visualisation than its pre-image. Klein's example was the root of the present model concept with regard to axiomatic systems: what is *implicitly* given by formal axioms is made *explicit* by means of a suitable mathematical object which, as it were, fills the axiomatic form with what looks like substantial content. For instance, a particular group or transformation groups in general function as models of the general, axiomatically defined, group concept. Or euclidean, in particular three-dimensional space may serve as a model of axiomatically defined linear or metric spaces. On the scale of concreteness, however, one can progress even further, by stepping out of pure mathematics, and consider physical, or merely experienced space as a model of its axiomatically defined pre-image.

Only for the sake of completeness did I mention this use of the term "model", which is actually the inverse of the term we started with. As a matter of fact, in the present context we are not concerned with models of axiomatic systems, which are lavishly used in foundational research, but rather with modelling in the sense of idealising. By this means we are able to simplify complex situations which are dominated by mathematical theories of too great a complexity to be practical, or which are only accessible by *ad hoc* mathematical theories.

Since our subject is modelling as an aspect of mathematising, I would like to stress that in the present context I should readily include tangibly concrete models such as wind-tunnels where aeroplane models are tested and laboratory simulations of hydrodynamic theories. In other words, models that are evaluated by observation rather than by mathematics, even though their mere construction may require more mathematics than the processing of many less tangible models. I would not even exclude computer simulations of such tangible models, where the factual evaluation requires less mathematics than does the simulating activity. On the other hand I strongly oppose the fashion of slapping the label "model" on any system of algebraic, differential or integral equations (related to or arising in applied research) -- "*mathematical* model", as they like to call it. According to my terminology, a model is just the -- often indispensable -- intermediary by which a complex reality or theory is idealised or simplified in order to become accessible to more formal mathematical treatment. I therefore do not like the term "mathematical model'' in a context where it wrongly suggests that mathematics directly or almost directly applies to the environment. As a matter of fact, this only remained true as long as mathematics was tightly entangled with the environment. I lay so much stress on the role of the model as an intermediary because people are all too often unaware of its indispensability. Much too often mathematical formulas are applied like recipes in a complex reality that lacks any intermediate model to justify their use.

Probability and statistics are particularly striking examples. In probability the urn from which lots are drawn is, along with other random devices, the model by means of which one attempts to mathematise everything in the world that seems to be conditioned by chance: pollination of plant by another of the same species, marriages and deaths in a population, which are viewed -- rightly or not -- as if mating and dying were decided by casting lots. This model suffices for simple applications of probability in statistics. As far as I know, however, no models exist for mediating the conventional -- or should I say, ritual -- use of correlation and regression coefficients and factor analysis in a certain kind of social, in particular, educational research: these tools have simply been copied from other sciences where they are reasonably justified by intermediate models.

Looking back I realise that I have dwelt more on models than on modelling, and in rather general terms at that. Maybe suspicion that this might happen was the proper reason why I hesitated so long to tackle the subject. Of course, I could have overwhelmed the reader with a catalogue of streamlined models, such as harmonic oscillators, electric networks, transition matrices, diffusion processes, games, steering devices, population dynamics, queuing, etc. Some of these models, with a large domain of transfer, are certainly worth acquiring by students who are expected to put them to good use. On the other hand, however vaguely I defined modelling -- as idealising and simplifying -- it did hit the nail on the head: to grasp the essentials of a (static or dynamic) situation and to focus on them within what I earlier called a rich context and, as things progress, within ever richer ones. This, then, is the viewpoint from which I will continue to look for aspects of mathematising.

Looking for essentials,
that is looking within a context, which can mean

- within a situation and across situations
- within a problem and across problems
- within a procedure and across procedures
- within an organisation and across organisations
- within a scheme and across schemes
- within an algorithm and across algorithms
- within a structure and across structures

  within a formulation and across formulations
  within a symbolisation and across symbolisations
  within an axiomatic system and across axiomatic systems.

Why this multiplicity of "and across"? Because discovering common features, similarities, analogies, isomorphisms is the way towards
  generalising
as a subconscious habit or as a more or less conscious activity. More often than one tends to believe, generality is achieved by the *aha* experience of one single
  paradigm
only to be reinforced by a few (albeit not necessarily many) more of them. Now,
  generalising paradigms
is the inverse of
  exemplifying general ideas
which, if imposed, is an instance of what I have tend to call the "antididactical inversion", which I will deal with later on. However,
  approaching problematic generality paradigmatically
is a valuable
  heuristic activity
to be distinguished from fashionable *heuristics,* which is understood as a kit of prefabricated tools.
When stressing the efficiency of the unique paradigm, I had in mind the sudden emergence of fresh mental objects and operations. Yet objects and operations can become routine through daily practice, and eventually tiring enough to provoke
  streamlining and short-cutting,
which may lead to
  progressive
    organising
    schematising

structuring

and, in particular, as far as clumsy language and symbolics are concerned, to

- progressive
  formalising
  algorithmising
  symbolising.

A particularly important aspect of mathematising is that of

- reflecting on one own's activities,

which may instigate a

- change of perspective

with the possible local result of

- turning things upside down

and the global one of

- axiomatising

which again, if imposed, are instances of antididactical inversion.

1.3.3 *Examples*

(to be continued in 1.3.5)

1. Find the middle between 16 and 72 on the number line! Children I observed shifted the two points uniformly towards each other, first by units, and then by larger steps, in particular by steps of 10. Short-cuts led to halving the difference, the half being added to the smaller number. In general terms this means the expression
   $a+\frac{1}{2}(b-a)$,
   which by algebra equals the more usual one
   $\frac{1}{2}(a+b)$.
   After I suggested that shifting away from each other the two numbers would as well keep the middle, the children eventually shifted the smaller one to 0, and consequentially the larger to $a+b$ which, as it were, proves the usual expression for the middle.
   Rather than imposing a method of finding the middle between two numbers, it is allowed to gradually develop through progressive schematising. To achieve generality of this scheme, one paradigm seems to suffice, even beyond the domain of whole numbers. If verbalised, the general solution "I add the two given numbers and divide by 2" may be reformulated via the stage "half the sum of the two given numbers" in algebraic idiom and thereby contribute to motivating the creation and the use of algebraic language.
   Another generalising streak may be by asking the original question with regard to more than two numbers, with the aim of establishing the mental object of average and the scheme of averaging. Only if one is satisfied by formal generalisationsthis can be done by "the sum of the given numbers divided by how many there are given", or its algebraic equivalent. On the other hand, as

soon as contents are aimed at, one should look for situations where the envisaged addition imposes itself in a natural way or is even implicit to the situation. I.e: *not* the adding of ages, sizes, prices, and so on, but *rather* daily consumptions of a certain food, working hours, one person's receipts or expenses during a week or a month: or, given the total consumption of some food or other commodity by some population, ask for the consumption per capita, or as for speed per second, as derived from speed per hour.

It would be too much to go into more detail in dealing with the concept of average only on behalf of schematising and formalising. I will, however, mention one more generalisation of the concept of "middle", namely, the "middle" of a plane figure or a solid body. Many aspects of mathematising are required for answering the questions which may be asked in this context.

2. If one tap fills a basin in one hour and the other takes two hours, how long will it take both of them together to fill it?

   This problem of venerable age (together with venerable ones such as two labourers working together, two people surviving together on a certain quantity of food, and so on) looks ridiculous only as long as it is not integrated into a broader context of mathematising and is expected to be solved according to imposed traditional schemes. Children to whom I posed the problem divided the filled basin into two parts, each of which was assumed to be filled separately by one of the taps: two-thirds of the basin by the "larger" and one-third by the "smaller" one, so both of them were filled in two-thirds of an hour. Even when larger numbers were given children stuck to this visualised *proportionality* reasoning, supported, for instance, by the substitution of a number of "slow" taps for a "fast" tap. This strongly diverges from the traditionally sanctioned scheme of *reduction to one hour:* If the two taps fill the basin in a and b hours, respectively, then in *one hour* the first fills $\frac{1}{a}$ and the second $\frac{1}{b}$ of the basin: therefore both of them together manage $\frac{1}{a}+\frac{1}{b}$ of it, and the whole basin in

$$\frac{1}{\frac{1}{a}+\frac{1}{b}} = \frac{ab}{a+b}$$

   hours. The children's reasoning, however, corresponds to dividing the whole basin in the ratio

   b : a

   to be separately filled by the two taps, thus $\frac{b}{a+b}$ of the basin by the first, which is then the factor by which the original a hours have to be reduced.

   Surprisingly, as soon as this kind of problem takes the guise of two people walking towards each other with different speeds, adults familiar with this type of problem usually do not notice its isomorphism with the other ones but solve it by linear path-time graphs. It seems that distances to be distributed among two persons are more likely to elicit geometrical strategies than are quantities, such as tap water, work and food.

Mental objects such as "speed" are basically schematised and formalised in two -- opposite -- ways:
path per time and *time per path;*
the latter is particularly preferred when comparing sport achievements. Gasoline consumption is another example of the twin schematisation: the driver figures out how far a full tank can take him, in order to know whether he can cover a certain distance with one tank.
If one is aware of the manifold phenomena related to this twin schematisation and the important interrelatedness of its components, the basin and taps problem and its companions begin to look less ridiculous. *Harmonic* adding and averaging (that is, after transformation into reciprocals) is indeed a precious scheme which, to be acquired, requires carefully guided schematising.

3. The criterion for divisibility by 9, as learned at school, is hardly mathematics, but challenging its validity is. The positional system, modelled by the abacus, can be a source of schematising: If the given number is represented by balls on the bows of the abacus, than transferring a ball from one bow to another means changing the number by a multiple of 9; so if all of them are transferred to the units bow, it appears that modulo 9 the number itself equals its sum of digits. This reasoning extends by generalising to other positional systems.
4. With regard to schematising, percentages is too far-reaching an instrument to be dealt with at length here. We will only indicate one feature as it is of paramount importance: the restructuring switch
from p percent more or less
to $(1 + \frac{p}{100})$ or $(1 - \frac{p}{100})$ times.
5. When do the two dial-hands of the clock cover one another? Infinite series, simple algebra, or linear graphs can answer the question but, once answered, a short-cut produces the very scheme: during one full turn of the short hand the long hand turns 12 times, thus overtaking the short hand 11 times within 12 hours, at equal intervals. This is a far-reaching scheme, applying among others to explain a few astronomic phenomena.
6. Ten children are at a birthday party; there are two more boys than girls.
A milk-can with milk weighs 10 kg. The milk left in it weighs 2 kg more than the empty can.
Chickens and rabbits on a farm: 13 heads and 36 feet.
Starting by trial and error, which becomes less efficient with larger numbers, children begin to use a rich variety of preferably visual schemes to answer the related questions. Reasonings starting with If: If each girl takes a boy, . . . If each rabbit was a chicken, . . . Again the end of generalising is reached in algebra.
7. In case you are not yet familiar with it, stop to think about the following problem: In a crowd of people among any five people there are always two of the same age. Show that among 17 of them there are always 5 of the same age! Perhaps you will tackle it by various visualised schemes, only to finally

change the perspective: In fact there are no more than four year-classes.

8. Two players, and a pile of 100 matches, from which they shall in turn remove at least one and at most ten. The player who takes the last matches, wins the game. Since the winner cannot but reveal the trick to the loser, almost all people know it. So let the couple play another game:
   A pile of matches, from which each may in turn remove a power of 2. Again the player who takes the last matches, wins the game.
   If the pile is 1 or 2, the player whose turn it is wins; if it is 3, he loses; if 4 he wins, and so he does with 5 by removing 2 and thus putting the other in the losing position 3. If 6, however, whether he removes 1,2 or 4 he cannot but yield the other a winning position, and therefore loses himself. 7 and 8 are saved by removing 1 or 2 respectively, but 9 is again on the wrong side. Going on one would guess that multiples of 3, and no others are losing positions for the player whose turn it is. How can you prove it? The result suggests arithmetic modulo 3. Powers of 2 modulo 3 equal 1 or 2. So all those high-brow powers of 2 do not matter at all but instead it all comes down to removing one match or two -- a slight variant of our old game.
   Another variant: only prime numbers (1 included) are allowed to be removed. Again we draw up a list of winning and losing positions for the player whose turn it is:
   Clearly
   1,2,3,5,6,7,9, 10, 11,. . . are winning positions
   4,8,12, . . . are losing positions.
   Indeed, whichever prime you subtract from 12, for instance, you grant the other a winning position; however, subtracting 1, 2, or 3 from a number on the upper line puts the other player into the lower line. This suggests arithmetic modulo 4, which degrades the game to the old one with simply replacing 10 by 3.
   Still another version: The numbers you may remove are 1 and 4. Then the positions
   1,3,4,6,8,9, 11, . . . are winning positions,
   2,5,7,10, 12,. . . are losing positions.
   Modulo 5,
   1,3,4 are winning,
   and 0,2 are losing.
   Indeed, the player whose turn it is in the first kind of state can answer any move -1 by -4 and conversely, and thus preserve his state.
   The games proposed here show similar features to each other. What is the deeper nature of their similarity? Are they mere paradigms of a more general one and, if so, how to formulate this?
   Rather than starting paradigmatically in each case as we did, the usual way of presentation is to start with the final result in order to prove it -- the antididactical inversion. We presented an open end rather than a final result.
9. A sequence of disks 1,2,3, . . ., one side black, the other side white. To start

with, all have the black side up. All even numbered disks are turned upside down, then all divisible by 3, then all divisible by 4, and so on. Which ones will in the long run again be black side up? People start experimentally, then look for prime divisors and suchlike, only to finish with the short-cut that any non-trivial divisor k of n has its counterpart divisor $\frac{n}{k}$, both of which coincide only in the case where n is a square. A neat exposition would start just at this point.

10. Schematising experiences like the following leads to the idea of multiple counting: three edges meeting at each of the eight comers of the cube would Seem to result in eight times three edges, whereas in fact there are no more than twelve of them.
11. Draw one broken line through the five points (fig.2) of the Five of

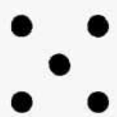


Fig. 2

the die, touching each point once and only once! How many different figures can be obtained?
First, the idea of "different figures" must be schematised, which is done by what is called congruence. Then the counting procedure must be structured by classifying suitably, for instance, with respect to the midpoint of the Five: take it as the start, as first stop, as second stop, and continue this way with a comer.
12. Quite another aspect of mathematising but that of the preceding problems is exemplified by the famous "grains on the chessboard: In order to estimate 2^{64}, one replaces 2^{10} by 10^3 It is an example of numerical schematising.
13. So far I have neglected the linguistic features of mathematising. So in order to make a choice, I consulted [87, IV, 15]. Choosing means losing, which I do not like. So please, do as I did!
14. I also didn't pay enough attention to change of perspective. This is a particularly rich subject as appears from the examples in [87, IV, 16]. Meanwhile, I could add many more but I won't. The subject deserves a more systematic treatment, which I do not dare to undertake.
15. A barrel, closed on the upper side, except for four holes, which are arranged in a square (fig.3). Right beneath the holes there are four disks, one side

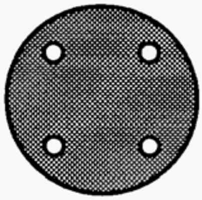


Fig. 3

black and the other side white, although their colour is invisible. The player is allowed to choose one or two holes and turn the corresponding disks upside down. After this operation the barrel is rotated at random around its vertical axis into a new position, yet in a way that the player cannot identify the holes he chose before. The operation is repeated *ad libitum.* As soon as all the disks show the same colour on their upper side, a bell rings and the game is finished.
Look for a strategy which guarantees that eventually all the disks will show the same colour!
This problem is a rich mine of mathematising features. Since I should not like to disappoint readers who would enjoy solving the problem by themselves, I have relegated the solution to an Appendix.

1.3.4 *Mathematising -- horizontally and vertically*

Treffers, in his thesis[3] of 1978, distinguished *horizontal* and *vertical* mathematising -- not sharply but with due reservations: Horizontal mathematising, which makes a problem field accessible to mathematical treatment (mathematical in the narrow formal sense) versus vertical mathematising, which effects the more or less sophisticated mathematical processing. For a long time I have hesitated to accept this distinction. I was concerned about the theoretical equivalence of both kind of activities and, as a consequence, their equal status in practice, which I was afraid would be endangered by this distinction. How often haven't I been disappointed by mathematicians interested in education who narrowed mathematising to its vertical component, as well as by educationalists turning to mathematics instruction who restricted it to the horizontal one (to use Treffers' terminology)! Eventually I have reconciled myself with the idea of this distinction, even to the point of appreciating it positively; I do add certain nuances to its formulation, but in a way that still respects Treffers' intentions, I believe. I have accepted the distinction because of its consequences for mathematics education, and in particular, for characterising educational styles. I will explain this in detail when I will deal with theoretic frameworks of mathematics education (3.1.2).
Let us characterise the distinction as follows: Horizontal mathematisation leads from the world of *life* to the world of *symbols.* In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated,

mechanically, comprehendingly, reflectingly; this is vertical mathematisation. The world of life is what is experienced as reality (in the sense I used the word before), as is symbol world with regard to its abstraction. To be sure, the frontiers of these worlds are rather vaguely marked. The worlds can expand and shrink -- also at one another's expense. Something may belong in one instance to the world of life and in another to the world of symbols (road-systems, geographical maps, geometrical figures, bills, tables, forms to be filled out, and so on). Natural number can already belong to the world of life, while abstract addition still requires symbolic schemes. Abstract addition may have been incorporated into the world of life, while the cognition of its commutativity (or multiplication based on it) still need models which are processed and the equivalence of which is understood in the world of symbols. For the expert mathematician, mathematical objects can be part of his life in quite a different way but for the novice. The distinction between horizontal and vertical mathematising depends on the specific situation, the person involved and his environment. Apart from these generalities, examples on various levels are the best way to explain the difference between horizontal and vertical mathematising.

1.3.5 *Examples*

1. *Counting:* In order to be counted an *unstructured* set of objects or events must be structured -- manually, visually, acoustically or mentally -- while in a more or less *structured* set the available structure must be uncovered or reinforced. This requires horizontal mathematising. Applying the counting sequence to this (created or uncovered) structure, on the other hand, is vertical mathematisation, which, depending on the structure, can take place in a more or les sophisticated way: by using multiplication, for instance, to count a set presented or interpreted in a rectangular structure.
2. *More or less:* Structuring two given sets simultaneously may be horizontal mathematisation, while finding out which is part of which may be vertical. Or, at another level, counting both of them may be horizontal mathematising, while in this case reciting the number sequence, and listening to which number precedes which, vertical.
3. *Adding: A* problem requiring the addition of five and three imagined marbles to be added to each other may be mathematised horizontally by the "fingers schema", while counting the fingers may still be vertical mathematisation. Or, at another level, the preceding problem is mathematised horizontally by the arithmetical sum 5+3, which is solved vertically by counting forth, or by the replacement 4+4, or by memory.
4. *Adding:* If the number realm up to 10 belongs to the world of life, then solving 8+5 by way of (10-2)+(5+2)=10+5 may be vertical mathematisation, while the structures of both summands have been obtained horizontally.
5. *Commutativity:* Replacing 2+9 by 9+2 may be due to horizontal mathematis-

ing if 2 and 9 are visually or mentally combined as linearly structured sets and their combination is read backwards. It may be vertically interpreted as soon as the law of commutativity is generally applied.

6. *Addition:* It is a sign of vertical mathematisation when addition is used in situations like the following: After the distances from A to B and from B to C have been paced off, the distance from A via B to C, rather than being paced off anew, is obtained by adding.
7. *Multiplication:* Five times eight (things) may be mathematised horizontally by the rectangular scheme of 5 rows of 8 each. In vertical mathematisation it is, for instance, read as the sequence 8,16,24,32,40.
8. *Multiplication:* In the long run, addition of equal summands is recognised and dealt with as an operation in its own right -- a process that starts horizontally and finishes vertically.
9. *Division:* When dividing a number of objects among a number of persons (dealing playing-cards to players around a table, for instance), one can start by distributing the objects one by one, or by distributing an equal number of objects to each person, continuing until the objects are exhausted; this is horizontal mathematisation of the distribution problem. Vertical mathematising can be seen in the search for increasingly larger shares (eventually as large as is convenient) in order to shorten the process. This process is a conspicuous example of *progressive schematising* (in the present case, *progressive algorithmising,* eventually directed towards the standard algorithm of long division).
10. *Combinatorics:* If A and B are joined by 3 roads and B and C by 4 roads, then how many ways lead from A via B to C? Horizontal mathematisation is recognising the structure of the problem, which may start with clever counting, in order to finish with vertical mathematisation by means of the product. Applying this "roads scheme" in other situations may be either horizontal or vertical mathematising, depending on the case. Replacing 3 and 4 by letter symbols is vertical mathematisation.
11. *Ratio:* Mathematising of visual similarity geometrically and arithmetically may take place along a path where horizontal and vertical tracks alternate with each other, starting with statements like: what is double in size here, must also be double in size there.
12. *Ratio:* Putting the football scores 2 to 1, and 3 to 2 on a par with each other may be refuted by comparing them with 4 to 3, 5 to 4, and so on, which is a trick of vertical mathematisation. Trying to find a fair method of comparison may require the use of horizontally introduced and vertically processed geometric schemes or proportionality tables.
13. *Linearity:* Ratio can be further mathematised vertically by the scheme and the straight line graph of the linear function, as can many everyday situations involving ratio be mathematised horizontally. The relation between constant ratio and straightness is a feat of vertical mathematisation, as is the relation between the value of ratio and the steepness of a graph. The horizontal mathe-

matisation of commercial transactions involving both a fixed and a proportionally determined rate is followed by a vertical one of relating fa· mes of the transaction to features of the graph.

14. *Figurate numbers:* Sizes and relations between figurate numbers may be a matter of horizontal mathematisation as long as they are geometrically represented. For instance (fig. 4): the sum of the first n odd numbers equals the *n*-th square. Or (fig. 5): the sum of the *(n* -1)-th and the n-th triangular number equals the n-th square. As soon as such expressions and relations are put into formulas to be processed, vertical mathematisation takes over which, in the long run, is experienced horizontally. The inductional steps required for proving such relations are again of a vertical character, even though in the long run they will be experienced as being horizontal. Verbalising complete induction such as what is used in the proof is again an occurrence of vertical mathematisation.

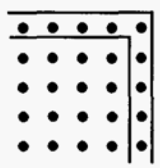


Fig. 4

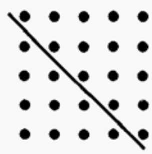


Fig. 5

15. *Pascal triangle:* This situation is similar to the preceding one. As long as the triangle is given as such, then numerous relations between its elements are obtained by horizontal mathematisation. The usual algebraic expression of the binomial coefficients requires vertical mathematisation, as do well-known combinatorial problems related to the Pascal triangle.

16. *Area:* Areas delimited by rubber strings on the geoboard are numerically obtained by horizontal mathematisation. In this vein the discovery that triangles having the same horizontal basis and equal heights also have equal areas appears to surprise even quite a few adults who had learned geometry at school. Relating this experience to the formula for the triangular area requires vertical mathematising.

CHAPTER 2

DIDACTICAL PRINCIPLES

Rather than being a didactical phenomenology of mathematical structures [146] the first chapter was one of *mathematics* itself, although I left "didactical" out of the title. Here it reappears. As far as the relation between teaching and learning is concerned I like "didactics" better than "education" -- influenced as I am by continental terminology. I therefore chose it also for the title of the second chapter, together with "principles", which means trying to avoid details, certainly with respect to subject matter. The principles I will deal with are my choice. In fact I was led by one particular principle: doing didactical justice to the results of the phenomenological analysis.

Didactics of a subject area means the organisation of the teaching/learning processes relevant to this area. Didacticians are organisers: educational developers, textbook authors, teachers of any sort, maybe even students, who organise their individual or group learning processes. The present work is not primarily concerned with didactics, and not at all with ready-made didactics as opposed to didactics as an activity. Our view on *didactics* will reflect the one on *mathematics* presented in the first chapter: *mathematics* arising by *mathematising* is mirrored by *didactics* arising from *didactising.* Notice that the intended parallelism even extends to distinguishing *horizontal* and *vertical* didactising: from the didactical reality to becoming conscious of it on the one hand and to paradigmatising it on the other.

Yet there is a difference: the view on mathematics was the result of a phenomenological analysis, while mathematical didactics viewed as an activity is a postulate, inspired by the character of the subject area. When we discuss the practice of mathematics education will be discussed (3.3), this parallelism will be elaborated in more detail; meanwhile it will remain implicit to the exposition of ideas, which were conceived and are to be reborn by didactising. So, by didactising, justice shall be done to the results of the first chapter in the broader sense of this parallelism.

2.1 GUIDED REINVENTION

"Problem Solving" and "Discovery Learning" have become catchwords. I never liked them as mere slogans, and I like them even less since I first time saw them exemplified. Problem solving: solving the teacher's or the textbook author's or the researcher's problems according to methods they had in mind, rather than the

learner himself grasping something as a problem. Discovery learning: i.e. uncovering what was covered up by somebody else -- hidden Easter eggs.
I shall not start an idle discussion on whether mathematics arises by discovery or invention, or to what degree it does so as a science or as an art. With regard to the subject matter to be organised I preferred the term "discovery"; in the context of teaching, however, my choice of long ago was "invention", which embraces both content and form, fresh discovery and organisation. Inventions, as understood here, are steps in learning processes, which is accounted for by the "re" in reinvention, while the instructional environment of the learning process is pointed to by the adjective "guided". In the first chapter I analysed mathematics as an activity. Indeed, mathematics in individual lives starts in this way. But is the learner allowed to continue like this? Curious children will not ask for permission; indifferent and lazy ones prefer to be guided. So in order to explain how I imagined mathematics would be learned I long ago[4] chose the term "guided reinvention". It didn't catch. Should I have done otherwise? Fortunately I did not.
Why not "heuristic", for instance? Old Polya, in his marvellous books, guided young Polya, according to the heuristic method, to reinvent mathematics. *Heureka* -- I found it -- is what Archimedes exclaimed after an unexpected discovery.

> Some time ago, upon consulting English dictionaries, I could not locate the adjective "heuristic" between "hetman" and "hew"; only in a Dutch-English dictionary did I find it as the translation of its Dutch counterpart. So I went to a larger library. Neither the adjective nor the noun "heuristic" were recorded in any English encyclopedia, whereas they occurred in the oldest Dutch, French and German ones I consulted. With the dictionaries I fared a little better but I had to resort to the biggest ones to encounter at least the adjective "heuristic"; not until the nineteen-fifties and sixties of this century did either "heuristic" or "heuristics" emerge as a noun. Needless to say, in the continental vocabularies the corresponding adjectives and nouns have long been terms in good standing. Of course, the lack of terms does not at all imply the lack of the concepts they cover. Believe it or not, up to a quarter of a century ago the geometric term "congruent" did not exist in standard English, although we may take it for granted that British teachers and students knew as much about congruent triangles as did their continental peers.

No doubt "heuristic" entered the English vocabulary via mathematics education. Unfortunately, it did so in the form "heuristics" where the additional "s" did not reflect that of physics and economics but was indeed meant as a plural. Unfamiliar with the continental tradition, the anglophone authors did not mean by heuristics a didactical method but rather a set of tricks and wrinkles, helpful for solving mathematical problems. When I made my choice, I could not possibly have foreseen this development. So it was mere good luck that I chose "reinvention" rather than "heuristic method". I would not have liked to see ideas on learning mathematics which I tried to convey by "reinvention", drowned by what is now fashionably known as "heuristics".
I also failed to choose another term of good standing -- the term "genetic method" -- because it does not allude to any activity of the learner; moreover, although it is not intended to do so, it does seem to describe a narrow-minded approach.

Learning as a genesis, as opposed to teaching as implementation would be a good metaphor, were it not for the biological associations of the adjective “genetic” as opposed to “societal”.

On the other hand it can hardly be denied that the first appearance of mathematics in cognitive development does look like an almost biological genesis -- I mean here the emergence of similarity and of whole number. At the same time, the unfolding of number is driven by the learner’s dynamics, which manifests itself (as I pointed out earlier) in a constructive linguistic activity. The learner reinvents the number sequence more or less consciously and this reinventing activity may even extend to arithmetical operations. It is well-known that some children reinvent arithmetic on their own, to various degrees indeed which depends on the children’s individual characteristics as well as on their environment. Is it therefore too far-fetched to assume that. with some support, every normal child might be able to reinvent as much mathematics as needed in one’s future daily life? This does not in fact happen and it is difficult to ascertain whether it would be possible or not, because in general, after a promising start, the child is not given the opportunity to reinvent anything whatsoever, at least in institutionalised learning. Instead, knowledge and action patterns are more or less imposed, as is most of the knowledge the child acquires by instruction. This is not the place to ask to what degree this is justifiable in general, since we are concerned here with mathematics which, as I explained earlier on, is different. One of the reasons why it is different is history. Can the historical learning process of mankind somehow be repeated by individual learners? A clever youth can reinvent quite a lot of mathematics on his own, I said. So why should less clever ones not be able to do so with the help and under the guidance of others -- adults as well as their peers? Why should they not be able to continue the way they began?

Not all the way, to be sure. No individual needs to run through the whole historical pedigree and conceptual hierarchy of knowledge and abilities which grew and were built through the incessant interplay of form and content. But why should people not get the chance to aspire, to climb and to dive to heights and depths as steep and as deep as they can reach and afford? Aside from this there must be, in whatever direction, some level within everybody’s reach. I confess that the horizon of these levels is the one thing in mathematics education that, over the years, I have become ever more interested in -- its natural extension and the opportunities and possibilities to extend it as far as feasible.

This is a view at variance with that of prescribing to people *a priori* the mathematics they should learn. Learners should be allowed to find their own levels and explore the paths leading there with as much and as little guidance as each particular case requires. There are sound pedagogical arguments in favour of this policy. First knowledge and ability, when acquired by one’s own activity, stick better and are more readily available than when imposed by others. Second discovery can be enjoyable and so learning by reinvention may be motivating. Third it fosters the attitude of experiencing mathematics as a human activity.

Traditionally, mathematics is taught as a ready made subject. Students are given definitions, rules and algorithms, according to which they are expected to proceed. Only a small minority learn mathematics in this way. If you ask mathematicians how they read papers, most of them will answer that they try to reinvent their contents. I believe the young learner may claim the same privilege.

To do justice to the phenomenological analysis I include definitions into the realm of things students are entitled to reinvent. A formal theory of ready-made mathematics starts with definitions and notations. Once upon a time each of them was coined. Gradually and in connection with each other they have been improved upon. Indeed, knowledge (however acquired) must be organised and systematised in order to be efficient and effective, and fit to be communicated to others and to oneself (that is, to remember it). If the student is granted the Same opportunity we may trust him to better grasp the need of definitions and notations. But definitions and notations are only the start of ready-made mathematics. I chose these as particularly striking examples, but all I said about them holds for the entire course of the more or less deductive ready-made system.

History teaches us how mathematics was invented. I asked the question of whether the learner should repeat the learning process of mankind. Of course not. Throughout the ages history has, as it were, corrected itself, by avoiding blind alleys, by cutting short numerous circuitous paths, by rearranging the road-system itself. We know nearly nothing about how thinking develops in individuals, but we can learn a great deal from the development of mankind. Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now.

New generations continue what their forbears wrought but they do not step in at the same level reached by their elders. They are consigned to much lower levels from which they resume the learning process of mankind, albeit in a modified way. Educators are charged with the task of helping them, not by prescribing but by allowing them to reinvent the mathematics they should learn. I agree this is far from easy, yet it is even more difficult to properly understand how far and why. The first and foremost thing is to become aware of this challenge and to prepare oneself to meet it as well as preparing those one wants to guide on the way of guiding their guided ones. “Teacher training” will give us the opportunity to resume this question (albeit entangled with the problem of reinventing didactics by didactising) but in the mean time I can say that common sense and guided reinvention of mathematics will again pave the road to the answer.

It will not be a simple answer since guiding reinvention means striking a subtle balance between the freedom of inventing and the force of guiding, between allowing the learner to please himself and asking him to please the teacher. Moreover, the learner’s free choice is already restricted by the “re” of “reinvention”. The learner shall invent something that is new to him but well-known to the guide.

In any case reinventing is at least reinventing *something.* Let us discuss this first!

2.1.1 *Guiding --where to?*

Governments, who consider themselves responsible for assessment, want educational objectives to be formulated in terms of subject matter and neatly tailored abilities, the mastery of which can be tested "objectively" (as they call it) -- that is, by computers. It is for the teachers' own good, our objectivists stress; teachers should be free to arrange their teaching at will, as long as their students are deemed likely to attain the prescribed goals. Educational authorities are supported by educators who promise to shape these kinds of instruments of assessment, and opposed by others who doubt their validity in advance. A justifiable doubt, it seems to me, but this doubt does not discharge them of their responsibility to look for better tools of assessment. This is a serious problem, which we must tackle in due course.

Meanwhile we cannot evade the question of where to guide the reinventor. Not surprisingly, I shall not answer this question with catalogues of subject matter and neatly tailored abilities. To be sure, they are indispensable, that is, as indispensable as is the table of contents in works like the present one. But table of contents and indexes are not useful unless one is familiar with the content itself -- globally familiar, I mean.

Whatever the importance of subject matter and neatly tailored abilities, they are, they are considerably less so in mathematics than in any other teaching. Since I stressed mathematics as an activity my answer to the question "where to?" will be: "to an activity". In other words, the learner should reinvent mathematising rather than mathematics; abstracting rather than abstractions; schematising rather than schemes; formalising rather than formulas; algorithmising rather than algorithms; verbalising rather than language -- let us stop here, now that it is obvious what is meant.

If the learner is guided to reinvent all this, then valuable knowledge and abilities will more easily be learned retained, and transferred than if imposed.

> For almost a century it has been known through experimentation that meaningful matter is more easily learned and retained than meaningless matter. Whereas those experiments primarily involved verbal matter, more recent research focuses on mathematics, on arithmetic, in particular. Multiplication tables are a striking example – at least as long as they are considered worth learning and memorising, which I still think they are. Didactical methods have changed. The oldest method was to have children build, say, the table of 7 in the natural order 1x7,2x7,3x7,..., perhaps acoustically supported by unison recitation; a student who had memorised it in this way, was expected to answer "6x7=" by whispering the table up to the stop "6x7=42". This method has now yielded to methods of unguided incidental learning by unsystematic exercises of great frequency, much like memorising phone numbers and addresses. The newer method seems to be less successful than the old, but both of them are unacceptably wasteful. More recently, attention has been paid to what is called "children's informal methods", such as doubling the better known 3x7 to get 6x7. Such examples of vertical mathematising should not only be permitted (which is not at all a trivial matter if seen from the children's viewpoint) but indeed reinforced and encouraged, either directly or indirectly, that is, through the learners' interaction among each other[5].

> There are also indications that transfer is favoured by guided reinvention, in particular if word problems are to be solved by column arithmetic. We will present the evidence now available in due course.

The third member of this triplet deserves closer scrutiny. Once a more definite stage in the process of learning by reinvention has been attained, intermediate stages are likely to be wiped out of the reinventor's memory; and as time goes on, the whole learning process may even be obfuscated as though it never took place. This is not necessarily a loss as long as its essentials can be restored in case of need. If not, there is nothing left to be transferred. What kind of measures can be taken to prevent this? We will try to answer this question when we deal with "retrospective learning" (2.4.5,3).

When asked "where to" the reinventing learner should be guided, my answer was: to mathematising and its various aspects. The lack of more substantial objectives can be made up for by asking *what* the learner is expected to mathematise. This can be answered in one word: Reality. What kind of reality? The learner's own reality as laid open to him by his guide. This leads us to the next question.

2.1.2 *Guiding -- where?*

1. Counting is the child's first verbalised mathematics; verbal counting may even precede counting something. So many kinds of things are present in the child's world to be counted, so many are waiting to be counted, and so many new ones can be offered to be counted. Take the number of people around the table, the number of noses, eyes and ears, even feet, invisible under the table. Applying the sequence of numerals to such sets is horizontal mathematising. Wondering why some among these numbers are equal is asking a vertically mathematising question, which is answered by the extrapolating transfer of the "as many as" from one's own body to the bodies in a group. This is a valid answer, even if the underlying one-to-one relation is not made explicit, and asking to provide it would be schoolmarmish. The same holds if the "earlier" and "later" of the sequence of numerals is horizontally transferred to the "less" and "more" of sets counted or viewed as being counted. Transitivity may evolve from a horizontal mathematising experience into a vertical mathematising activity, only to become part of reality itself in the long run. For a long time in the long run, if not forever, the cardinal aspect may remain an implicit feature of number, which needs no verbalising, as it has been for millennia in the history of mankind; the same is true for transitivity of order.

The things counted are structured sets, rather than structureless ones. Such a set structure may be more or less distinctly visible in the given situation, or else, the task of structuring can be left to the learner.

The set of guests around the table or of children sitting in a circle on the floor is structured by its cyclic order -- "didn't you forget anybody"? Or the children may count themselves, each adding 1 to the number of his neighbour at the right. Or number one, as soon as it is his turn, may continue numbering himself, one more round, two, three, in infinitum -- unwinding, as it were, the cyclic order. In a group of seven, which numbers does the first one get? And what kind of numbers? And the seventh, the third? And who gets number 100? Have it marked on the number line -- unwinding the circle as it were! This is a marvellous example of prospective learning (2.4.5,3) of multiplication and its tables, as well as of division and, if reflected upon, of vertical mathematising. Strangely enough I have never seen this most natural example of guidance applied.

The set of eyes in a company is structured by the set of people. Or is it structured as a set of pairs? If counted by means of 2,4,6, . . . it is vertical mathematising, at least the first time it happens. How many seats in a theatre, how many occupied? How many places in the parking lot, how many cars? How many days until the holidays? How many pages left to read? To be sure, one can count all of them one by one, or more cleverly, by looking for structures or creating them. The basic arithmetic operations are likely to be applied in counting; they may even be given new meanings by the counting activity -- examples of horizontal as well as vertical mathematising.

Conversely, counting can serve to structure a situation: take the third street comer from here! How many blocks are needed to build a tower as high as that? How many layers of bricks to build this house? A lot.

How to count large quantities? Or is there any need to count them? It depends on what is worth knowing. On December 31,1974, there were

13,599,082

people living in The Netherlands. How for heaven's sake did they count them? Elsewhere their number is given in thousands of people,

6772 men and 6827 women.

Except for 82 people the sum is correct. How far off could it have been?

2. Mathematics education developers, who believed in mathematics arising in the reality, and in having this reality organised and mathematised by the learner, looked first for primordial sources. There are plenty of them. Make your choice! But isn't it a pity to choose one source, say of addition or subtraction, and to forget about the others? Well, the others may still get their turn. Once the said operations are mastered, they are to be applied. This is what they call *word problems,* disliked by students as well as by their teachers. Indeed, would you like to leave the safe harbour of routines and venture into the open sea? Each word problem is a new one, and if you are not told how to solve it, you must then remember what addition and subtraction mean and return to their sources -- which might have been clogged by the algorithmic routines.

There is an alternative. When mathematics education developers looked for primordial sources, they were nearly drowned by the variety they found. They could not make a choice or else they did not want to; if they did, they preferred not to make a single choice, but to have a rich variety. They did not want to narrow the wide real world on display, nor to degrade world problems to word problems.

So an all-round approach is needed, or so it seems: at the very beginning a broad display of situations to be mathematised with a view to learning addition and subtraction, telling and drawing little stories, played by the teacher or invented by the children themselves. Wins and losses, spending and saving, growth and decay, jumps on the numberline (there is a big one in any classroom), bus stops with people getting on and off, signposts suggesting distances to be added and subtracted, block towers to be built and compared -- let us stop here!

3. *Addition and subtraction* are learned as activities, and there is no need to obliterate this origin, to freeze the dynamics of adding 3 to 4 into the static interpretation, prescribed by New Math, where 4 + 3 is merely another name for 7; even having a try makes little sense as long as no situations are available where such ideas can be reinvented rather than being imposed upon the learner.

4. When addition is incorporated into the learner's reality, it may serve as a source for *multiplication:* adding a sequence of equal summands. Yet it should not be the only source, as was in traditional teaching. It should not even be a primordial source but rather an auxiliary one. Indeed, why should one add sequences of equal summands? This "why?" is readily answered by numerous situations where summands are equal, and all of them are sources of multiplication, and for that matter, are richer than the more formal source of merely adding equal *numerical* summands. (Notice that this procedure may, thanks to the richer approach, afterwards even resemble vertical rather than horizontal mathematising.)

5. *Division* is rooted both in subtraction and in multiplication, more formally in the latter, that is, as its inversion. Elementary divisions are most easily performed by making use of just this fact. But this does not matter here. The root of division in subtraction is twofold: *exhausting* one measure copying a smaller one, and *distributing.* Formally, both look the same, even with regard to the numerical procedures applied. But division, like multiplication, requires the use of quite different sources: real-world exhaustion and distribution problems. Experimental research[9] has proved that disguising long division problems as "word problems" consistently and for the entire length of the learning period not only solves the old didactical "word problems" problem with regard to division, but also favours the process of learning long division: much better results being obtained in a much shorter time. We will try to explain this experience when we will discuss learning algorithms in general.

6. When the wealth of sources in reality is concerned, *fractions* is second only to ratio, and both of them are closely connected with each other in real-world situations. In cognitive development -- as I have often stressed -- similarity, thanks to its visual expression in proportionality, even precedes number, which in turn is much more easily verbalised than ratio. Although situations involving ratio and proportionality are as easily and as early accessible (in particular visual and visualised ones) as are those involving number, they are hardly considered worth paying attention in traditional instruction. Why? Number is so much more easily verbalised than ratio, and verbal introduction of concepts is in general preferred to non-verbal formation of mental objects. Even at present few mathematics ed-

ucation developers are aware of the educational opportunities lost by the lack of attention to ratio, and research done in this field is negligible compared with that on number and separated as it is from development, hardly to the point. One of the few exceptions is L. Streefland's Thesis[6], in which the learning of fractions and ratio are interrelated. Although the emphasis, even in Streefland's work, is on the one traditional didactical access to fractions -- cake divided into equal portions -- it is distinguished by the much more abundant exploitation of this source. In fact, in Streefland's approach not one cake -- or rather pizza -- is divided by fair sharing but a varying number of them, which moreover takes place at a varying number of tables. By this simple artifice this single source of ratio and fractions is explored and exploited to such a great depth that a whole world of fractions and ratio is displayed.

7. I must admit that one important source (at the same time application) of fractions and ratio has amply drawn the attention of developers: probability. But where it did so this source has most often been tapped on behalf of probability rather than fractions and ratio. I don't feel that much progress has been made since the first attempts. We know nearly nothing about the individual learning processes by which probability as a mental object is constituted and eventually mathematised. I would guess that by necessity they are unusually long and depend on an unusually great variety of facts being experienced in reality. In spite of this, the examples of *teaching* probability I saw developed, were aimed at relatively short term and rather compact learning processes. As soon as long term learning processes will are considered, probability will have to be discussed anew.

8. It seems to me that the teaching of probability suffers from the emphasis on *combinatorics*, which through the feedback of the teaching/learning process is likely to be transfigurated from a tool serving probability into a safe algorithmic shelter, *protecting against* it. Of course, this does not mean we should reject combinatorics as such as a matter to be reinvented. On the contrary, combinatorics -- and figurate numbers in particular -- is a most appropriate matter for reinvention: Starting with numerical paradigms, guessing general relations, experiencing and satisfying needs for good definitions and convincing proofs, encountering mathematical induction thanks to these efforts, and using mathematical induction, first instinctively, then intentionally, and eventually in a more or less formally verbalised manner -- all this together appears to be a most efficient course in reinvention.

9. Let us pass to less elementary mathematics! Nowadays exponential growth is on the verge of becoming an everyday idea if it has not already. On the other hand, mathematising the idea of growth has been a rather recent feature in mathematics history. Logarithm tables and the logarithmic function (the integral of $\frac{1}{x}$ as a function of x) preceded exponential functions. If history is allowed to be reinvented, emphasis must be shifted from the exponential as the inverse of the logarithm to the exponential as a growth function. As a matter of fact, compound interest as an instance of *discreet* exponential growth had been cultivated much

longer; extending it to *continuous* growth is a historically rather recent acquisition, which only afterwards was formalised by means of the well-known exponential functions. In teaching, however, supported by a rich variety of examples, it should be a source for and a guide to reinventing exponential (and in its slipstream, logarithmic) functions.

10. As a matter of fact, "growth" as meant here, includes shrinking --negative growth as it were. "As it were"? It has become a linguistic habit (even far outside mathematics) when speaking about values, magnitudes, forces, trends, or whatever, to quasi-mathematise *opposite of* -- in any sense whatsoever -- by *negative of.* This transfer of an arithmetical relation is a most striking example of formal transfer which carries momentous consequences for contents. Acquired as it is much earlier in the learning process, it certainly deserves to be made conscious, and growth and shrinking is perhaps the best opportunity to accomplish this.

11. At a more formal level, the *isomorphisms* mediated by exponential and logarithmic functions between addition and multiplication are a matter of constructive reinvention. Indeed, it starts operationally in the interpolation process leading from discreet to continuous growth, only to become paradigmatical, and if subjected to reflection, a generally formulated feature of real number.

12. Another case of history which must be revised by reinvention is the *sine* (and other goniometric functions).

It so happened that the 75 year-old woman I introduced earlier on once asked me what sines were good for. Though for many years 1 had already cherished ideas on this subject at variance with the traditional ones, I was for a moment in danger of relapsing into the old habit of citing trigonometry. Fortunately I did not. Instead I told her about oscillations, vibrations, and superposing sine functions. This is a more convincing answer, in particular if trigonometry as such is no longer remembered.

The sine (as well as other goniometric functions) was once upon a time invented for the computation of mangles -- first astronomical and later also survey-geodetic ones. In the law of refraction it was still related to right triangles. Not until the cycloide was studied and its equation discovered, did the sine occur as a function. Its importance greatly increased when it reappeared in the mathematical description of the oscillations of the pendulum, and of the shape and the vibrations of the taut string. At present, vibration phenomena of all kinds is the vast domain of application of sine and cosine.

Triangles are not the first and certainly not the only source for reinventing sines. They should rather be reinvented on the level of *functions,* more precisely, as graphically represented functions of time: While a wheel is uniformly spinning on its axle, a luminous point on its rim moves periodically up and down. The graph of its variable height is easily drawn and can be shown on a TV screen; The graph's dependence on the wheel's radius, on its speed, on the spinning sense, on the point observed (with consequences for the relation between sine and cosine) can be studied in this context, long before triangles are computed. When this comes up for discussion, the right triangles in question will emerge as though produced by *instantaneous* exposures of the spinning wheel. The occurrence of sines in the description of phenomena of oscillation and vibration should at this stage simply be accepted as an empirical matter of fact, which nevertheless adds much to their importance.

(Note that I used the plural "sines" more often than the singular "sine". The plural sounds like one is speaking about exponential and logarithmic functions alongside with *the* exponential and *the* logarithmic function. This is a mere consequence of the spinning wheel approach, with all its parameters. As soon as one has settled on the unit for both the radius and the circular speed, on a counterclockwise spin, on the luminous point at time 0 at the "right" end of the "horizontal" diameter, and its height measured "above" the "horizontal" axis, one has got the sine. Is this mathematical slang? I believe it is rather terminological flexibility.)

13. Neither Calculus nor Analysis is included in this overview, not because I have renounced comprehensiveness, but because both of them, if taught at all, should be preceded didactically by something I propose to call *Differential and Integral Methods.* This topic deserves a place in an early stage of the learning process where algorithmisation has not yet been developed far enough as to allow teaching Calculus or even Analysis. It is an approach (in principle by graphic representations) initially merely qualitative and later on quantitatively refined (if possible). It aims at understanding and interpreting such ideas as the steepness of a graph and areas covered by the moving ordinate segment, maybe even curvature, in contexts where the drawing of the curve mathematises a given situation or occurrence in primordial reality -- we hardly need stress the vast opportunities here for reinvention. In our exposition, *Calculus* is better discussed alongside with algorithms. *Analysis* was once invented as a safeguard against false Calculus. It still waits for justifying as a subject at Highschool level. By justification I mean guiding the learner to avoid the pitfalls of automatically applied Calculus by reinventing more critical Analysis. As long as I have yet to see any didactical attempt at repeating this very historical origin of Analysis, I think that "Differential and Integral Methods" is a didactically more trustworthy safeguard than Analysis can be trusted to be.

14. In my overview I did not strive for comprehensiveness. Two serious gaps, however, are still left: Algorithms, and Geometry. There are reasons to delay filling them.

2.1.3 *Guiding -- how?*

Guiding means striking a delicate balance between the force of teaching and the freedom of learning. It depends on such a perplexing manifold of hardly retrievable and only vaguely discernible variables that it seems inaccessible to any general approach. Observational reports on guiding may be a source of understanding and a help for teaching guidance. Unfortunately, most of the reports available are concerned with single lessons or short sequences, and little is known about long term learning processes.

My own teaching experiences, on which I have reported incidentally and unsystematically, are concerned with individual children. Let me mention two more systematic reports [73;154]. The first, though concerned with a 5 $\frac{1}{2}$ year-old boy's experiments on a *physical* subject, is essentially *mathematical* reinvention. The second publication is unfortunately only a short summary of a larger report on the long-term teaching arithmetic and mathematics to a 12-14 year-old fractions; negative numbers and vectors; linear graphs, functions, equations and inequalities; powers. Let me quote myself from the above publication:

> It was my instructional principle to put her in concrete, if possible, visual situations and to let her work intuitively. Never did I explain anything to her. never did I formulate any rule, nor did I ask her to formulate rules. When I had the impression that she was still behaving as acting intuitively and insightfully, whereas she had already algorithmised the type of problem, I exaggerated the numerical data in the problem in order to make the algorithmic procedure visible. I asked the question "why?" only if I was sure she knew the answer.

The method was conditioned by the learner. With other children, for instance, I insisted much more on reflection. With others I did on occasion suggest an answer, but when this happened with her I inserted this symptom of impatience into the report and blamed myself for it.

Never explain anything -- how is it possible? Well, the situations this girl was placed in were self-explanatory, and the only thing I added was something resembling the caption to a picture. For instance:

That is the x-axis;
that is a vector;
that is putting into brackets;
$2.2.2.2.2 = 2^5$ reads 2 to the power 5.

I can extract two principles from my report:

a) Choosing learning situations within the learner's current reality, appropriate for horizontal mathematising.

b) Offering means and tools for vertical mathematising.

It may rightly be argued that individual instruction like the above cannot be normative for the reality of instruction in general. This is correct, not because of the mere fact that it is *individual* instruction but because of the unlimited opportunity the teacher for improvisation granted in this setting, by which he can reinforce the learner's attempts at reinventing. In the average classroom – one would say -- only pedagogical geniuses can act this way, but the average teacher needs instructional plans, predesigned by able developers. Yes, indeed, but then predesigned in a style that does not unnecessarily restrict the teacher's freedom to take advantage of the class situation as it presents itself at any given moment, so that the situation can be grasped by the teacher's intuition and experience, and fitted according to the teacher's principles. This requires a system of

c) Interactive instruction, interactive, not only in the sense of a class-teacher relation, but -- maybe even more so -- in that of a mutual relation between the learners, leaving the teacher in the background more room and time for efficient improvising. The teacher's apparent withdrawal is manifest in allowing and stimulating

d) The learner's own production, which includes the reinvention not only of solutions but also of problems -- a most effective kind of training. Viewed from the curricular perspective, improvising is facilitated by intertwining teaching strands, which should result in

e) Intertwining learning strands, an idea that shall be elucidated in connection with long-term learning processes (2.4.5,4).

The Five Tenets a-e are from A.Treffers[7]. I would not be able to formulate more concisely what, in my view, guidance means in "guided reinvention". This brief exposition covers the principles and at the same time leaves plenty of room for further elaboration.

2.1.4 *Algorithmising*

1. Algorithmising aims at the mastery of *particular* algorithms to be acquired. Or rather, let us consider it for a moment from this viewpoint: Mastery of algorithms is as crucial for individual progress as it has been historically for that of mankind. Algorithms allow us to act automatically for long stretches of time, avoiding the perturbing or delaying interference of insightful thought. But algorithms are exacting; mastery means either complete mastery or none. Less than 100% mastery can mean that everything is wrong. Of course, nobody is infallible -- not even computers. Mastery includes the ability to identify and to correct one's mistakes, casual ones such as computational slips, and fundamental ones such as applying an algorithm where it does not fit. Mastery, moreover, includes lost mastery to be recovered.

Algorithms can be taught. The learner is offered a paradigm to be imitated. One paradigm may suffice. The paradigm *donner* suffices to teach the first conjugation of French verbs (except the irregular ones). One paradigm may suffice to teach an algorithm. If not, a few more are offered, or as many as are needed. Usually it works. If not, the pupil is considered a failure.

Learn first, understand afterwards! For a long time this was the adage for teaching, and maybe it still is to many teachers. Was the "first" ever justified by an "afterwards"? Anyway, instruction has proved to be a success story. Otherwise how, for heaven's sake, would you explain the undeniable fact that so many mathematicians are ready to produce so much mathematics as is published today? Well, there are people who can learn mathematics this way. I am one of this sort. Or rather, I was -- up to the moment when I discovered that what I had learned was no mathematics at all. (Let me forget geometry for a moment, which is our next case.)

There are algorithmically gifted people who can learn a lot of algorithms in this

way. Moreover some can apply them if needed, and a few can remember even those they never applied. Still fewer somehow get the opportunity of experiencing mathematics to be quite another thing than they had thought it to be.
Algorithmically gifted people seem to form a small minority. Nevertheless, even those outside this group still had to learn algorithms and to apply them. This is still true in the age of computers and calculators, perhaps even more so. One has to know which button to push, and hardware buttons are no easier than brain buttons.
Learn first, understand afterwards. I don't remember when I myself eventually *understood.* Rather late, I guess. Too late, at any rate. I mention it, by the way, in order to argue that the medicine I am going to prescribe is also meant for those who will understand eventually.
As a matter of fact, "understanding *first*" isn't a new medicine. At the beginning of our century (or even earlier) textbook authors started justifying the algorithms they were going to teach (or have taught by the teachers) in advance, and so did teacher trainers, or at least they pretended to. Did teachers follow suit? It is a useless question. "Justify beforehand" is not an effective medicine. "Justify beforehand" implies "forget afterwards". It is a tenet of the logic of separating form from content which, if at all, works after rather than before understanding. Even if argued *beforehand,* an algorithm is not better learned than if argued not at all. Algorithmising means that arguing is left to the learner, even if it remains implicit to the learning process for some time or forever. If argued, algorithms create the illusion of simplicity because they were tailored to look simple. Reinventing algorithms, however, can be a tedious and time-consuming activity, and it sophisticated strategies are necessary to convince teachers, textbook authors, developers and researchers that the final result is worth the labour and time spent. Reinventing algorithms involves a progression of schematisings, which is shortened again and again by the reinventor, who is allowed to approach the standard algorithms as closely as learning needs and abilities urge and permit. Let me illustrate this by a few classic examples from teaching arithmetic[8].
2. Palpable material, at present available to learners to fill the linguistically acquired formal number sequence with contents, is intended to stimulate schematising by packing units (fig. 6)

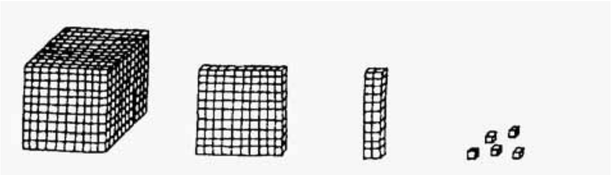


Fig. 6

shortened by the scheme of the positional abacus, where each bead is worth the rank of its position (fig. 7).

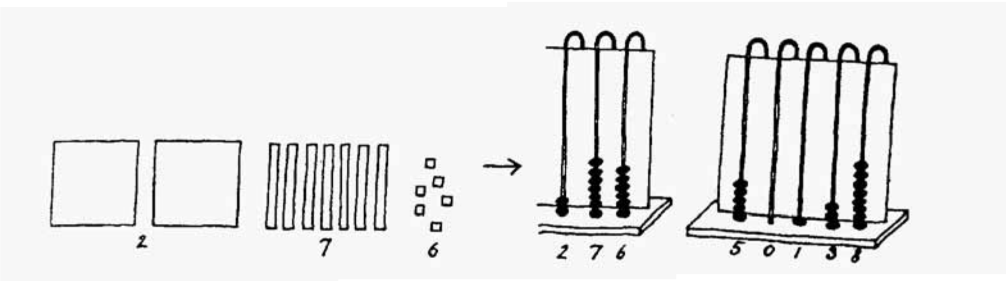


Fig. 7

and tens of bead units are readily exchanged for higher ones (fig. 8).

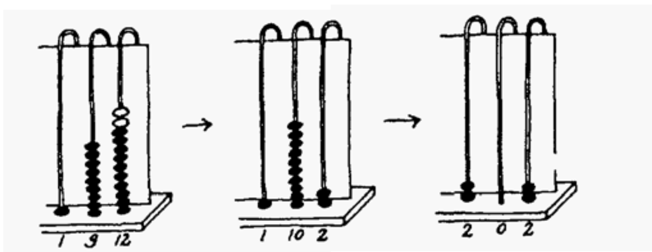


Fig. 8

Addition is schematised on the drawn and written abacus (fig. 9),

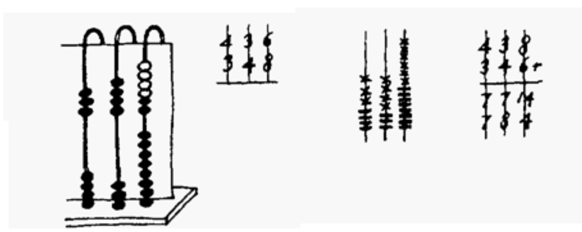


Fig. 9

again schematised by the position card (fig. 10)

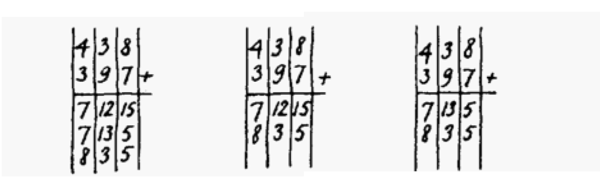


Fig. 10

which is the last step before the final schematisation of column addition.
Actual reinvention is less streamlined than the above example. For instance,

choosing the order of processing the particular columns is left to the learner. The latitude is even greater in reinventing column subtraction, where heterodox tools, such as (implicit) negative numbers, are allowed:

43
-19
36
= 24

where the underlined 6 means 3-9 = - 6.

Column multiplication allows for an even larger number and greater variety of reinvention paths. A problem such as

> A directory of 62 pages has 45 names per page; how many names are there?

may first be interpreted as 62 times adding 45 -- a process that is gradually shortened and schematised by using (and at the same time improving) the knowledge of multiplication tables, and by taking ever larger groups of tens in the multiplier (fig. 11). The students are allowed to work on different levels of shortening and schematising and to proceed at their own pace towards the standard pattern of multiplication.

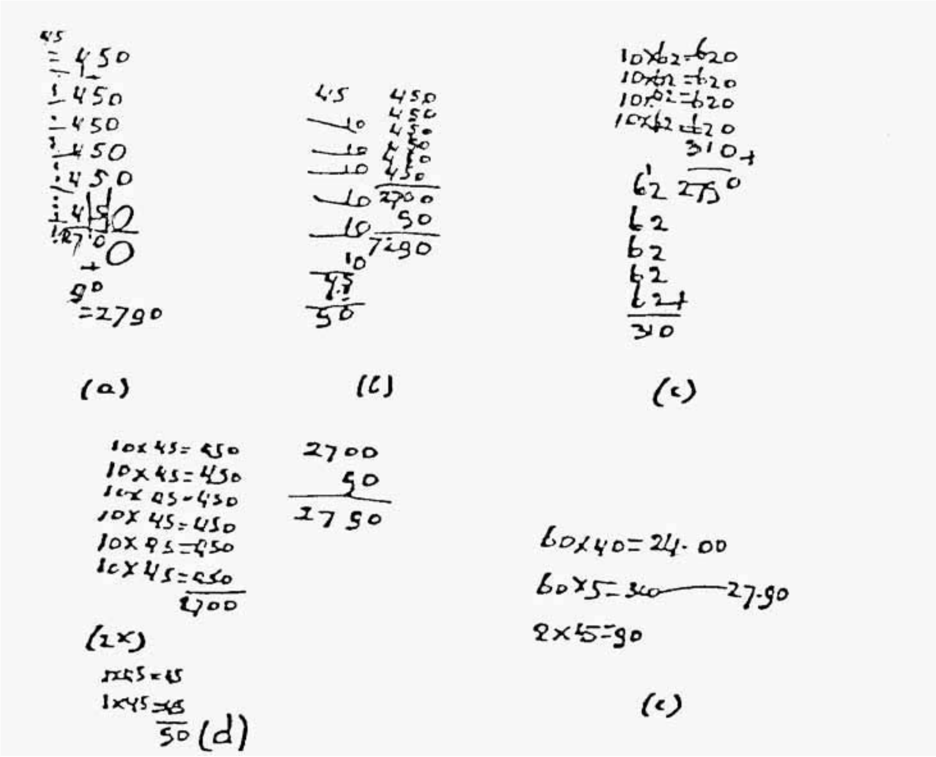


Fig. 11

An even greater diversity of learning processes may be observed in

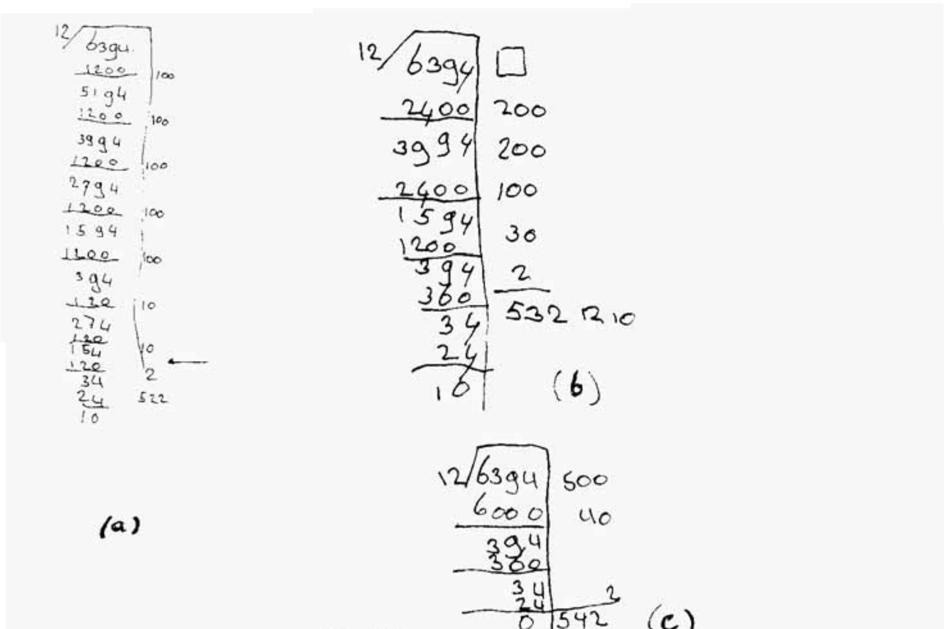


Fig. 12

the progressive algorithmising of long division (fig. 12), which has already been mentioned in (1.3.5, 9). The progression of shortening consists of subtracting ever larger chunks of the divisor from the dividend.

During the entire process of learning column multiplication and division, all exercises are formulated by way of "word problems", which aid rather than impede learning. Even when column addition has become part of the learner's reality, the redundant support from a reality acquired earlier is still valuable, or so it seems; and the same is true for long division with respect to subtraction and multiplication. When we put Reality in its didactical context, we will deal with this question once more. Using Treffers' terminology we call this teaching strategy *integrated* progressive algorithmisation. Experiments[9] have proved it to be much superior to the traditional "isolated progressive complication", as labelled by Treffers. In spite of the integrated learning, almost all children reach the stage where the algorithm can be detached from the learning context and attains the status of having a reality of its own, although in the case of long division, however, not everyone gets as far as the standard form, which can hardly be considered a "must" in the age of calculators.

3. Fractions are a case apart, at least in the eyes of L. Streefland as set out in his Thesis[6]. Fractions are a subject of schematising, not only to a greater degree than the arithmetic of whole number, but also in a more sophisticated way, even though this is less so with regard to algorithmising. Mastering the algorithms for

fractions is not aspired to, and perhaps in many cases not even desirable, lest it be degraded to blind mastery. There is nothing lost if the goal of algorithmic fractions is relegated to those parts of algebra where they are really needed, although I judge it feasible even in arithmetic, provided the teacher is allowed to observe the utmost restraint in guiding the learner.
This view is based on the teaching experiences [154] I mentioned above (2.1.3), where algorithmic rules were never explicitly formulated, but where the learner acquired them by tacit generalising. For instance, after a large sequence of exercises such as

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \ldots$$

made on the numberline,

$$\frac{7}{11} = \ldots$$

and suchlike were made without any visual support. One operation after the other was tacitly detached from its visual source. When, after a time the child's teacher noticed her progress and taught her the algorithmic rules for fractions, she needed three weeks to recover from this intervention, which had seriously damaged her mastery of fractions. I must confess that I did not teach her division by fractions until it occurred automatically in algebra. Although she never explicitly learned the famous rule on the multiplicative behaviour of the decimal point, she did apply it correctly.
4. In the same vein I had her reinventing algebra, that is, by algorithmising. Indeed, algebra offers a profusion of algorithms, less standardised than those of arithmetic, and for each particular problem a greater choice. One may question whether negative number properly belongs to arithmetic or to algebra. In one respect -- computations with letters -- it does not. On the other hand, the algebraic permanence principle is didactically crucial for operating on and with negative numbers. I applied it in the form I have called the geometric-algebraical permanence principle in [146, Chap. 15], from which I also borrowed the teaching strategy: first the "thermometer", then vectors in the coordinatised plane, in order to step finally with linear equations and inequalities into algebra. Let me give a typical example of progressive algorithmising: Up to a certain day the same girl had the habit of solving equations like

$$\frac{2}{3}x = a$$

by first multiplying by 3, and then dividing by 2. On that particular day it so happened that she drew the direct conclusion

from $\frac{5}{3}x = a$ to $x = \frac{3}{5}a$

which then became a tacitly applied rule. I never intervened to suggest such short-cuts. She had, in fact, discovered them long before but didn't dare apply them overtly because she was unsure of my approval; indeed, sometimes when she took short-cuts, she cast a playful glance as though asking for my permission. (I attach great value to the above experience, although people usually shrug about the fuss I make of it. "People" -- I mean non-mathematicians. Creative mathematicians have experienced similar developments by themselves)
Another example of short-cuttings: Exercises like

$$a^3 . a^5 =$$

were initially dealt with by way of

$$a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a = a^8$$

until larger exponents provoked short-cuts. Never was any rule formulated on the behaviour of exponents under multiplication and division.
Still another example: For a long time she clumsily calculated expressions like

$$(a+b)^2, \quad (a-b)^2 \quad (a+b)\ (a-b)$$

before applying short-cuts toward the usual formulas.
How fast do algorithms learned in this way stick? I will try to answer this question later on (2.4).
5. I promised to deal with Calculus from the viewpoint of algorithmising. As a matter of fact, Calculus was invented as an algorithm, or rather, one of its inventors, Leibniz, was obsessed by a passion for algorithms. To his regret and our satisfaction, this passion was satisfied once and only once, by the invention of Calculus. The advantage of Leibniz' notation above Newton's is that, with its differentials, differential quotients, and integral symbols, it candidly exposes its origin. I know very well that Leibniz' suggestive notations are in the black book of authors who fear the wrath of the rigorous mathematician and I agree that teaching differential and integral methods while doing justice to Calculus as an algorithm requires more sensibility than imparted in teacher training. On the other hand I firmly believe we are not allowed to deprive the learner of such great values. In [40, chap. 17], I tried a synthesis of Calculus as an algorithm and as Differential and Integral Methods. Reinvention is here a bigger problem than in the domains I have dealt with so far. Reinventing something that since Archimedes has waited for about two millennia to be invented the first time is not that easy. It requires stronger but nevertheless more subtle guidance. It seems to me that we are just beginning to understand and tackle this problem.
6. At the beginning of this subsection I announced a more comprehensive view on algorithms and algorithmising. Algorithms are, as it were, the shop-windows of mathematics, that is, of ready-made mathematics. For many, if not for most

people, the result of mathematics instruction is a view of mathematics as a basket of formulas and recipes, which at a closer glance may be as full or as empty as suits its owner. Algorithms are indispensable. One can get quite far with algorithms -- also far astray. Algorithms are tempting. Mastering an algorithm may make one feel happy. Algorithms are easily taught and when taught as such, they are likely to be remembered in the same way. But how to rectify this picture of mathematics as a basket of algorithms? How to cure this aberration?

An ounce of prevention is better worth a pound of cure. The treatment I propose is: algorithmising. Reinventing algorithms may be a means of preventing them from becoming calcified, and mathematics from being identified with algorithms. It may indeed be a means but will it suffice as such? Probably not, but perhaps it will, provided the algorithm once acquired does not wipe out the memory of its acquisition. Keeping this alive, at least in a dormant state, is a subject we will discuss when dealing with learning processes.

2.1.5 *Reinventing geometry*

Viewed developmentally, geometry is the direct opposite of arithmetic. Space and the bodies around us are early mental objects, the results of structuring and being structured. To what degree are they mathematical objects? To what degree are distances viewed as lengths, chunks of matter as volumes, bodies as figures? Name-giving is a first step toward consciousness. But a name for geometric similarity, for instance (which is one of the earliest geometric experiences), is still far away at the time it is experienced. It has to be invented. The Greeks did this by lending the general word for similarity a specific geometrical meaning, but even nowadays this has hardly influenced everyday language. One says: copying, reducing, enlarging. “They are the same, uh, this one is big, that one small, uh, the same shape.” Geometry, though omnipresent, too often lacks the linguistic means for expression.

Arithmetic, however, starts as it were as a language, of counting, which eventually becomes the language of counting something. It is an algorithmic language, which provides the first algorithm -- an automatism producing the number sequence. And so it goes on in arithmetic: Every idea is as quickly verbalised as it arises, by existing as well as by fresh vocabulary. Verbalisation propels abstraction.

But abstraction starts earlier, as early as language or even earlier. It is implicit to the processing of sensual and motor experience, and not until linguistic support is needed, is abstraction adopted or created. I think that this is the reason for the lack of geometric vocabulary in everyday language: it is simply not needed. In our world there is a need for words like chair and table, and for words that help us to distinguish different chairs and tables from each other. There is a need for topographical terms like above and below, left and right, inside and outside. Geometrical ideas, however, are formally and intuitively suggested with a visual and palpable force that delays, if not impedes, their verbalisation. Moreover, various modes of geometrical abstraction are competing with one another and, in particular, with the topographical mode. Pieces of a jigsaw puzzle are freely moveable; all that matters is their congruence class, or so it would seem. But the picture they are expected to compose may not be hung upside down, and this information imparts such stigmas (edges, sky blue, feet, smoke) to some of the pieces as may support the work of composing the picture.

Geometrical abstractions are much more stringent than those leading to chairs and tables. They depend on contexts, which as a mere fact is trivial. They depend on a geometrical context, which as such is not at all trivial. I say a geometrical context because there are many, although which one is usually implicitly understood.

Verbalisation propels abstraction. This cannot be less true in geometry than it is in arithmetic. Anyway it need not, but this need has been awfully wronged by traditional instruction. Indeed, all means of verbalising geometrical ideas and operations have always been available in little children's language. They were simply not used because teaching geometry did not start until the age of 12 to 13 when children were judged mature enough for Euclid -- forgive me this venerable British instructional terminology! They were judged mature enough, although almost everybody knew they were not. The need for propaedeutic courses was felt; in [40] I reported on some marvellous ones, but they did not solve the genuine problem. Propaedeutics as such is a preposterous view on mathematics learning. If it is true that learning is a discontinuous process, then it is so by jumps rather than by cuts: jumps of reinvention rather than cuts of indoctrination. There is as much difference between schooling and preschooling as there is between suppositions and presuppositions, that is to say, none at all. But let us not anticipate too much what we still have to say about the unity of learning processes.

If it is true that arithmetic and geometry develop mentally as direct opposites of one another, this does not imply that one of them should be dropped in favour of the other. Instruction should instead continue this development, that is, help geometry to stride at a pace comparable to that of arithmetic, while respecting their opposite origins. Instruction should pay as much attention to geometry as to arithmetic, though a different kind of attention: broader to arithmetic and more profound to geometry. Whereas in arithmetic algorithms should be pursued watchfully so as not to loosen the bonds with reality -- the primordial one included. Indeed, reality is omnipresent in geometry, both visually and palpably, waiting for non-algorithmic schematising, and supported by badly needed verbalisation.

The variety of situations where arithmetic can start and stay, although much richer than traditionally displayed, is utterly poor if compared with those in which geometry arises -- nay, can be made to arise: rather than gazing at "Euclid", let us, in order to find out, pass in review the impressive row of more recent proposals and attempts. It is a precarious wealth, however. Choices have to be made, not as in arithmetic, of sources and approaches, but of subject matter. There is so *much* available, and the danger looms large of choosing too *many.* Or even if there are

not too many things to choose from, the choice may be too incidental: nice problems, one nicer than the other, and each of ready to stimulate reinvention but as well ready to be lost in the stream of new experiences, a *hapax legomenon,* a nonce experience. Cautiously applied, the operational structure of arithmetic may provide a global guide-line for reinvention. Geometry can be structured deductively, although this is of no use unless deductivity itself is reinvented. Little deductive steps -- why not? -- it is common sense; perhaps even a bit of local deductivity. But how to link together nice pieces of geometrical reinvention, to get chains of long-term learning processes, rather than leaving the learners with heaps of loose ends? I think that this is the very problem in teaching young children a kind of geometry that is likely to come too late for older children.
Posing nice geometrical problems is particularly gratifying when little hints suffice to solve them Take, for instance, the following three:
Find the centre of a given circle!
The diagonals of a parallelogram bisect each other -- why?
The angle inscribed over the diameter of a circle is a right angle -- why?
The hint I have in mind is quite simple: cut out the figure! When cut out, a figure can be manipulated, folded, moved, turned around. Symmetry proves a lot of things, and as a principle it can concatenate quite a number of geometric reinventions. But this does not suffice to solve our problem. Or more pointedly, problem solving does not solve the problem.
Let me recall the tenets d and e from (2.1.3)! One of them is the children's own production, which includes reinventing not only solutions but also problems. This tenet has successfully been applied in arithmetic; so why should it not be applied in geometry? The wealth of situations to be geometrised is an ever-lasting guarantee.
The other tenet was the intertwining of learning strands. The strand "numberline" is long enough, and it can be broadened to the co-ordinatised plane, which in turn can be restricted to the geoboard. Ratio and measurement have geometrical components, which should be taken seriously: similarity and congruence.
But don't forget about it the very first key. I mean tenet b: offering means and tools for vertical mathematising. In the case of geometry this means, as emphasised above: linguistic means and tools to geometrise spatial experience.

2.2 BONDS WITH REALITY

2.2.1 *Primordial reality*

Mathematics has arisen and arises through mathematising. This phenomenological fact is didactically accounted for by the principle of guided reinvention. Mathematising is mathematising something -- something non-mathematical or something not yet mathematical enough, which needs more, better, more refined, more perspicuous mathematising.

Mathematising is mathematising reality, pieces of reality. But reality is not just one thing; it is as many things as there are people, and to one people it may be as many things as there are states of internal understanding and external circumstances. Anyway, as soon as mathematising is didactically translated into reinventing, the reality to be mathematised is that of the learner, the reality into which the learner has been guided, and mathematising is the learner's own activity.

What is reality to whom depends on many variables, as does what is mathematical, what is non-mathematical, what is mathematical enough or not enough, what asks for more mathematising. Our world, described in a vernacular that shares names for arithmetical and geometrical ideas with mathematics language, has already been mathematised so extensely that we are no longer aware of it unless it is brought to our attention. Much mathematics in our reality is common sense and taken for granted, and again, how much and how great its influence is depends both on the individual mind and its specific environment.

In the preceding section I spoke casually about *primordial reality.* Let us focus on the learning process for the multiplication algorithm! In our setting questions like

> A directory of 62 pages has 45 names per page; how many names are there?

are likely to be answered by clumsy procedures, to be shortened progressively, such as adding *62* times the number of *names* per page (while adding *62 numerical* summands 45 is already within the range of the learner's reality). What, then, is the rationale of this mode of instruction? Why, if at all, should progressive algorithmisation take place in an *integrated* way, rather than in splendid *isolation?* -- to use the former terminology. Why should clear essentials be disturbed by lot of noise?

Simple as they are, both this particular piece of instruction and the query it provokes is no more than a didactical paradigm for a principle intended to prevail on mathematics instruction in general. Noise is eliminated as soon as newly invented mathematics is published, and for it has long been a didactical principle to avoid noise as much as possible where basic abilities and concepts were to be taught. But when mathematics once learned is to be applied (even as schematically and straightforwardly as it is expected to happen in word problems); then noise could no longer be escaped, and its sound is likely to become the more disturbing however less noisy the previous learning process had been. The world is noisy; mathematising the world means looking for essentials, sensing the message within the noise. This, too, has to be learned, that is, reinvented by the learner, and the earlier the better; once the learner has fully been indoctrinated by ready-made schemes and algorithms it may be too late.

This explains why mathematising should not start in a necessarily adjacent reality but in what I called a primordial reality. In still another respect our example is paradigmatic. Is there anything more obvious than having multiplication reinvented as the addition of equal summands? But as one stops to think about it, one

asks oneself why, for whatever reason should one add equal summands? The answer is obvious, that is, to whoever knows it. But in fact there is not one answer, or rather, if one, it is an abstraction from many. Primordial reality yields a rich variety of answers to this question. Both are matters to be reinvented, the question and its many answers.
With Treffers I distinguished horizontal and vertical mathematising. Vertical mathematising is the most likely part of the learning process for the bonds with reality to be loosened and eventually cut. Indeed, in the long run algorithmic habits, say in learning long division, will be formed to enfeeble the operative association of division with distributing or exhausting and to relegate it to the subconscious or even unconscious. It is this danger that should be prevented by a long lasting insistence on formulating problems in the primordial reality. Not only the product of the learning process but also the process itself are worth remembering and should remain accessible to the memory if they are needed. We will raise the question of whether this policy is indeed efficient enough when we deal with long-term learning processes (2.4).

2.2.2 *Odd worlds*

Strange as it may seem, it cannot be concealed that the bonds with reality may be endangered as early as horizontal mathematising albeit in another way than during vertical mathematising arises. The imminent danger is that of the wrong bonds -- wrong in principle rather than due to grave or minor errors. In order to explain what I mean let me cite a most interesting research entitled *"Quel est l'âge du capitaine?"* by the *"Elémentaire"* team of the IREM *(Institut de la Recherche sur I'Enseignement Mathématique)* of Grenoble.

> The team had posed the question
> On a ship there are 26 sheep and 10 goats. How old is the captain?
> to 97 CE 1-2 pupils (7-9 olds). 76 among the pupils succeeded in figuring out the captain's age on the strength of these data.
> Encouraged by this "success" the team administered a battery of six similar tests to seven classes CE and six classes CM (9-11 years olds):
> I have 4 peppermints in my right pocket and 9 in my left. How old is my father?
> A shepherd has 125 sheep and 5 dogs. How old is he?
> A shepherd has 360 sheep and 10 dogs. How old is he?
> In a class are 12 girls and 13 boys. How old is the teacher?
> On a ship are 36 sheep; 10 fall overboard. How old is the captain?
> There are 7 rows of 4 tables in a classroom. How old is the teacher?
> Every problem was followed by the question: What do you think about this problem?
> I will spare the reader the statistical result. Anyway, the harvest was indeed reaped.
> (From: N 19, décembre 1979, or Bulletin de l'APMEP, no.323, avril 1980.)

For illustration, the Grenoble team has added a collection of word problems gathered from old and more recent French textbooks; these will give you the creeps. It is a welcome suggestion that instruction (and in particular instruction of a certain kind of word problems) is responsible for the non-mathematical or rather anti-mathematical attitude adopted by the tested pupils, but I am not sure whether one should accept it at face value. Is instruction the real culprit, or is it not too easy to accuse instruction of having sinned, albeit only by negligence? Aren't there more active causes that might have produced these failures, and if so which ones? Is the culprit really those word problems that give you the creeps? I am not that sure.

In my youth the Sunday supplements of most newspapers and most weeklies contained a kind of puzzles -- I don't know whether it still exists -- say, a drawing of an English garden -- a foliage vault with garden furniture and tools underneath -- with the caption question: "Where is the gardener?" By turning the drawing or one's head and looking closely, one could discern somewhere between the branches, leaves and sunspots the contours of a person who looks like a gardener hidden in the foliage. This then was the answer to the question.

Why do we accept such a solution and at the same time refuse similar answers in the case of the captain's age? A gardener, even when hiding, does not hang himself askew, feet up and head down, in treetops. This isn't normal. But our question deserves to be taken seriously and tackled thoughtfully. It touches, or so it seems to me, the fundamentals of mathematics and mathematics teaching. The word logic is likely to be dropped when discussing this matter. The problems, parodied by that of the captain's age, are indeed vested with a kind of logic that uniquely determines their solution, whereas problems like those of the gardener seem to assume nothing but visual abilities. I cautiously said "seem" since there are more sophisticated ones of this kind that do require intelligent reasoning combined with visual aptitudes. Anyway, the appeal to logic is too easy a solution. "Logic" and "lack of logic" are bromides to lull in nagging sceptics.

I have dubbed this *problématique:* text and context. "Context" refers to (1.2.8.3), where it is dealt with in particular in the sense of rich contexts. "Text" means a linguistic vehicle, in particular of word problems or text problems as they are occasionally called.

The context of the gardener problem is well-defined: the Sunday puzzle page -- the same kind of problems week after week, one generation telling the next how to solve them. The textbook problem "26 + 10 =" is first of all situated in the context of mathematics instruction, where one learns what these symbols mean, but its most important feature is that it remains meaningful outside any context, or if one prefers another formulation, that it fits or can be fitted into any context. This is, indeed, what characterises mathematics, or at least its formal aspect.

Word problems are different:

> John has 26 marbles; he wins 10 more; how many does he have now?

John is a boy, boys like to play marbles -- girls do too -- it is a game where you win and lose, and with each marble won or lost the number of marbles in your pocket increases or decreases by one unit. The context is self-evident: the problem stems from the plain reality of the child's life, and whoever does not grasp it is advised to stop playing marbles -- nothing prevents him from continuing to do arithmetic.

But let us now turn to butcher Smith whom I came across in a textbook (or was it on an achievement test?). Here the problem is not about 26 marbles but the 26 kg of ham Smith had in his shop and the 10 kg more he ordered, and the question is about how much he has now. Again a context, let us guess, from the plain reality of a butcher's life. Obviously there is some isomorphism assumed between two worlds, the child's and the butcher's world. "John" is replaced by "Smith", "marbles" by "kg of ham", "wins" by "orders", and so everything is settled. More closely viewed, however, this isomorphism is all but perfect. Marbles are won and lost, while ham is bought and sold -- all right. But marbles won are put in your pocket, whereas the ham ordered by phone does not fly instantly into the shop, and when it arrives some of the 26 kg of ham -- I don't know how much -- will have been sold. Indeed, this is why the butcher ordered more.

The context of the butcher problem, rather than the butcher's reality, is the textbook, or more precisely the vast domain of pseudo-isomorphic images of the marble problem, and everything I have said about the butcher problem applies as well to the whole treasure of word problems: the context is the textbook, rather than reality proper, or in other words, it portrays a world of pseudo-isomorphisms.

In the textbook context each problem has one and only one solution: There is no access for reality, with its unsolvable and multiply solvable problems. The pupil is supposed to discover the pseudo-isomorphisms envisaged by the textbook author and to solve problems, which look as though they were tied to reality, by means of these pseudo-isomorphisms. Wouldn't it be worthwhile investigating whether and how this didactic breeds an anti-mathematical attitude and why the children's immunity against this mental deformation is so varied?

Arithmetic applies because of true isomorphisms. In [146, p.58], besides the usual marbles added to each other in play I listed one under the other

5 steps and 3 steps (of stairs)
5 days and 3 days
5 km and 3km
5 florins and 3 florins
5 times and 3 times

a list I could have continued ad lib. Even though the mental process of adding 3 to 5 belongs to the learner's reality, these elementary additions still have to be performed and reflectingly understood in primordial reality in order to grasp their isomorphisms. Obviously this cannot be done in a mechanistic setting, which is likely to generate pseudo-isomorphisms, nor in the New Math fashion of cardinal interpretation of number and addition.

But I think there is more to it -- there is more to be analysed and more profundity is required. The tension between the contexts of reality proper and the textbook does not suffice to explain such phenomena as those signalled by the Grenoble team. There is still another context implicated in the evolution of that -- mathematical, or rather anti-mathematical -- attitude. Before giving it a suitable name I will introduce it by a few examples.

"How old are you?" The answer -- "four" -- is accompanied by the gesture of lifting four fingers. The child does not know what "age" means, nor what the fingers lifted have to do with the "age", with a number of years, nor with the question as such. According to children's folklore a lady-bird is as old as one can count dots on its wings. Counting the number of rings of a cross-section, one learns the age of a tree, even though one cannot explain why. One asks the cuckoo the number of years one may still expect to live. The picture of a birthday cake together with the question "how old is she?" provokes counting the candles. A fly-agaric pictured as a gnome's home betrays its owner's age by its white dots. And, taking a long step into the adult world, you come face to face with hosts of people, juggling the Number of the Beast of the Apocalypse and the dimensions of the Cheops pyramid and the Stonehenge monument. This is a venerably old preoccupation, called hermeneutics if concerned with the Holy Scripture, but as a matter of even older than the art of reading and writing. Looking for marks on flowers and leaves in order to know what ills they may cure, foretelling the future from the flight of birds and the livers of sacrificed animals, unveiling secrets of J.S.Bach's profundity by counting beats and bars in his compositions, and -- let us not skip it -- expecting discoveries in psychology and education by means of regression analysis -- it is all cut from the same cloth hermeneutics, or else the term I would propose, the *magic context.* Stories about Pythagoras and his school tell us they practised witchcraft through numbers. Is it too bold an assumption to credit children with this kind of mentality? From my own childhood I remember this context only too well, which I succeeded in eliminating from my mathematical activity. But what about many other people? To solve a problem they look for secret marks, for signals hidden by accident or intention, and in particular, for numbers that to put them on the right track.

How old is the captain? I gave this problem (with 18 sheep and 16 goats to make it less easy and thus more stimulating) to an 8 year-old girl who lived at that time in a world of fairy tales and sorcerers where she played her small and large roles, with gnomes sitting on each toadstool. She beamed with joy because she had discovered the secret, and had calculated the captain's age. Thanks to her enthusiasm she was unaware of the mockery of her two years older brother, who is as sober-minded as she is imaginative, and so no illusions were cracked. When I tried to explain to the boy how others reasoned when they calculated the captain's age by adding, subtracting, multiplying, and eventually by dividing, he shook his head: "I cannot understand what you mean; this yields at most the number of sheep per dog."

This brings me to tell of another experience with these two children I had half a year earlier. At the end of a stroll we came to a huge sandpit, bordered by a circle of thick poles -- actually cuts of tree-trunks. I asked the children how many poles there were. Resolutely the boy paced around the pit, counted 38 steps (each step being 7 poles according to his reckoning), and then concluded 38 x 7 = 266. (It is a nice example of mathematising but this does not matter now.) Meanwhile his sister had effortlessly solved the whole problem. She said "60", and when I asked "why?", she answered: "It looks like a clock-dial, does it not, and that is 60 minutes.

At the risk of being pronounced silly, I confess I like the girl's answer better than the boy's. It is the most beautiful nonsense, which promises more geometrical imagination than does her brother's solution. In her fairy tale world she did with her beaming eyes what he had done in his prosaic world with his feet. I am sure that, had she been two years older, she would have parcelled out at sight an arc corresponding to five minutes, counted with her eyes the number of poles per arc and multiplied it by 12 (the direct converse of her brother's kind of mathematisation).

How old is the captain? The 26 sheep and 10 goats on board are like the data used by the astrologer to foretell the future. Children's realities are worth studying, in particular the magic context as well as its influence on learning processes and the shaping of attitudes. Framing the children's world in rich contexts may be a preventative measure. This has been tried in various ways. Plain word problems, if disconnected, are probably not rich enough a context to prevent the pseudo-isomorphistic temptation; they should be interrelated by constructive and reflective measures. Streefland's approach (cf. 2.1.2,6, footnote) towards fractions, within one single but deeply excavated context, is promising, although it has not been continued as far as to reach formal fractions. IOWO's abundantly rich contexts have been tried out successfully but dissemination of these ideas strongly depends on the teacher's conception of which realities are accessible to the learner. We shall come back to this problem.

Let me illustrate it right now by an experience had by an IOWO staff member in a 4th grade class with a worksheet which included:

Railroad fares according to the number of kilometres travelled (half-price for the under ten years olds), a railroad map of distances, a story about Mum's train trip with John (10) and Mary (9) from Hilversum to Enkhuizen, and the question how much she must pay?

Annette could not solve it, even after she had correctly found the distance Hilversum-Enkhuizen.

"What is the matter?"

"I don't know how old is the mother."

"What do you think?"

"I would guess: 39." (Her own mother's age.)

"All right. It is a good guess. Continue!"

So she added 39 + 10 + 9 = 58, looked up the price in the 58 km row, which at that time was? 8.15.

"And if the mother were 50 years old, she would pay more?" our staff member asked.

"Of course she would."

"And the old people, the grand-dads and grannies, they pay even more, do they?"

"Sure, they do."

This example shows that the reasonably rich context did not work as it was expected to. The distance Hilversum-Enkhuizen, though figured out, was not used at all. What is the reason for this? My guess is that the reality to be conveyed by this context is nothing but an adult prejudice. One should ask oneself at what age the railroad trips, tickets sold at the windows, and the very thing travellers pay for, represent a valid context. Or rather, under what condition, since it is not the mere age that counts. Would practical experience be a better guide for answering such questions? Our knowledge of instruction is covered by a thick layer of prejudices, but I am afraid that this holds as well for the so-called practice. Moreover, people with much practical experience are prone to premature interventions, glossing over rather than revealing and bridging chasms between the child's and the adult's realities.

Our colleague did not get the opportunity to do this, or he simply didn't dare because the chasm looked too wide. Remedial teaching starts at the roots, which may lie unexpectedly deep. Yet let us ask what should be done in the Grenoble case! The remedy to prescribe, which for that matter, could have already been administered along with the tests, does seem readily available. I mean problems where the applied methods will not work, such as:

250 sheep and 120 goats aboard -- how old is the captain?

360 sheep and one dog (or two, rather than the improbable 10) -- how old is the shepherd?

I confess I am not sure whether this would be of much help. The children might object that there are not enough data available to answer the new questions while insisting that the previous ones did provide enough indications. This is no miracle. Magic sometimes works and sometimes does not. Or do the fresh examples provoke what is called a cognitive conflict? "Cognitive conflict" is an adult contraption. Cognitive conflicts have first to be experienced as conflicting realities. If there are no bonds with reality, then conflicting realities cannot provoke cognitive conflicts.

2.2.3 *Rich contexts*

Fraught with relations was the term I chose in [40] for the mathematics I wanted to be taught. In the mean time the term has become mathematics in *rich contexts.* I used the term several times: for instance in 1.2.9.3, and again in the last subsection, I opposed rich and poor[9a] contexts to one another. Let me briefly sketch the evolution of the terminology!

As I pointed out in 1.3 mathematics, unlike any other science, has arisen and still arises in common sense reality -- broad-minded common sense and broad-minded reality as I explained in 1.1.6. Where it was once invented, mathematics should now be reinvented. *Contexts* then means that domain of reality, which in some particular learning process is disclosed to the learner in order to be mathematised.

When in the early nineteen-seventies developers, in particular for primary education, looked within the reality they judged accessible to the learner, for sources of specific mathematics to be learned and taught, they were overwhelmed by so rich a variety that they saw their task shifted from finding sources to choosing among them. Not: choosing one and rejecting all others. But: allowing for a vast space and a continuous flow, creating what was called *context-rich* mathematics instruction. Language is more than mere communication; it influences thought, whether one is aware of it or not. Somehow, maybe under the influence of "rich structures" (see 1.2.1), *context-rich* became *rich contexts.* Rather than discussing the difference, I shall explain the situation, that is, what kind of wealth is meant by "rich"?

Bonds with reality is the title of this section, shortened from *Creating, strengthening, and maintaining bonds with reality.* This, then, is what rich contexts have to effect as domains of reality disclosed to the learner to be mathematised. We once tried to group our experiences with contexts. Let me copy that unassuming list!

First on the list, and at that time the most striking item, the *location:* a meaningful gathering of situations, which can be handled separately, or in more or less close connection with each other. Our best known example[10] was "Waterland", a Disney-like island-- its large picture hung in the first grade classroom -- where lots of things were happening: there were landings for ships to moor, bus stops with people getting on and off, roads leading hither and thither, road signs placed and to be placed, block towers and mountains to be climbed, networks of streets to be walked on, play grounds, windmills -- a profusion of situations, where mathematics could be discovered and acted out -- an incentive to stimulate the imagination of both teachers' and children.

A counterpart of this kind of rich context was the *story,* that is, rather than a gathering, something that, reeled off as a succession of worksheets, is structured in time. It may be a true story or fiction, a classic or invented *ad hoc* Examples[11]: "Gulliver in Lilliput", where one string -ratio- is played on over and over, or "Grains on the Chessboard, with its powers of 2, or "Around the World in 80 days", where one tone -- travelling around the world -- is coloured by many overtones, or "Ralph the Buccaneer" with the area of islands as the main theme, or "Ship Ahoy" (geometrical orientation implicated in the rescue of a ship), and many others.

The third example was the *project,* which means reality to be created, such as building a bungalow (7th grade), or a peep-show box (kindergarten), or collecting and classifying all kinds of commercial packing material, or constructing regular bodies.

The fourth was the *theme,* a mathematically oriented strand of subject matter with varying relations to reality, such as "Light and Shadow" (7th grade), "Flying" (thematising goniometric functions), "Exponential Functions" (wholly integrated into the context of growth), "Matrices" (in various contexts, among which transition processes), "Two Variables" (that is, surfaces as though abstracted from mountains).

At this moment I would close this list with a fifth item, which has in the mean time been tried out extensively and successfully. Let us call it: *clippings* -- mainly from newspapers and weeklies, but also from books and other media. Anyone intent on it has little trouble discovering a lot of mathematics in the printed and published reality -- meaningful as well as nonsense mathematics. Thanks to this large quantity, the small percentage that lends itself to mathematics lessons is still significant. Rethinking the author's thoughts, analysing whether they were right or wrong, pursuing his ideas in order to review, strengthen, weaken, modify them, may provide valuable mathematising activities in the upper grades and in teacher training. Let me mention that each year one of the problems at on our final mathematics school examinations is of this kind. But even in the middle and lower grades a variant -- let us call it pseudo-clippings -- may be useful. A pronouncement by somebody, or a quarrel in a group about some question, an explanation, or pairs of contradictory explanations, in particular, if the mathematics involved is still hidden, may be a useful context as soon as children are able to communicate. Should this happens in the classroom it deserves to be exploited, but inventing such stories can be an equally good idea. This then is what I would call pseudo-clippings.

Contexts were defined as domains of reality disclosed to the learner in order to be mathematised. In the cases of *location, story, project,* and *theme* such domains are purposefully -- and sometimes artificially -- delimited by the teacher or developer, who wants the learner to reinvent certain processes and products of mathematising. The case of *clippings* is a bit different. Here it is not a domain but a small piece that is cut out, although its paradigmatical value for mathematising and for acquiring a mathematical attitude may be enormous in comparison.

But in all cases it should be kept in mind that context is not a mere garment clothing nude mathematics, and mathematising is quite another thing than simply unbuttoning this garment. Or, to give a former metaphor a new twist: Viewing context as noise, apt to disturb the clear mathematical message, is wrong; the context itself is the message, and mathematics a means of decoding.

2.2.3.1 *Contexts versus "material"*

No doubt once it was real progress when developers and teachers offered learners tangible material in order to teach them arithmetic of whole number. (In the case of fractions it never did work properly.) Even now I would not like do without it. Whether unstructured, at the start of arithmetic, or structured, as on the abacus, it is indispensable as part of the primordial reality. It is a small part, to be sure, one source among many others of whole number arithmetic. Relying solely or too strongly on it is dangerous, and since this is the habit in some kinds of instruction I feel obliged to warn against it. Low achievers have great difficulty detaching arithmetic from the palpable material -- according to remedial teachers and developers of remedial teaching. This may unfortunately be true, that is, for a kind of inflexible instruction where from the start onwards learners have been tied too tightly to palpable material. Flexibility should be allowed, and if need be, taught rather than fought. Moreover, one may ask, what is the use of low achievers learning an arithmetic they are judged in advance unable to detach from tangible material.

Offering the learner tangible material was progress when drawing and writing tools were too expensive to be handed out lavishly in the classroom, when teachers on a platform watched classes of children sitting on benches, when instruction had to look like military drill and children's initiative and activity were suppressed and nobody even thought about living contexts. Tangible material has a tremendous value in teaching geometry, provided the learner is allowed to structure it himself. If we condemn the learner to the prison of prestuctured blocks -- whether counting ones or logical -- we should not wonder why they are not able to put the mathematics learned to good use. The best palpable material you can give the child is its own body.

At this moment I cannot resist quoting myself from 2.1.2,1:

> The set of guests around the table or of children sitting in a circle on the floor is structured by its cyclic order – "didn't you forget anybody""? Or the children may count themselves, each adding 1 to the number of his neighbour at the right. Or number one, as soon as it is his turn, may continue numbering himself, one more round, two, three, in infinitum – unwinding, as it were, the cyclic order. In a group of seven, which numbers does the first one get? And what kind of? And the seventh, the third? And who gets number 100? Have it marked on the number line – unwinding the circle as it were! This is a marvellous example of prospective learning (2.4.5,3) of multiplication and its tables, as well as of division and, if reflected upon, of vertical mathematising. Strangely enough I have never seen this most natural example of guidance applied.

2.2.4 *Paradigms*

"Paradigm" would have deserved a section or subsection of its own in the first chapter, were it not that there, among the manifold occurrences of the term, it would have looked like a needless repetition. The present chapter, however, offers an opportunity to reconsider paradigms as places where bonds are attached to reality.

"Paradigm" means "example", although not as used in "for example", where it properly means an *after-example.* If such a word existed, I would rather say that a paradigm is a *fore-example.* As far as I am familiar with the history of instruction the term "paradigm" originated in teaching foreign languages (including native language which may have looked or sounded to quite a few of the learners as though it was a foreign one). In (2.1.4) I cited the French *donner* as a teaching paradigm for the first conjugation, that is, for all the verbs ending in *er* (the irregular ones, of course, excluded). To what degree are paradigms really needed? People acquire a great deal of their idiom -- structural paradigms included -- before, without, and even in spite of institutionalised instruction. This certainly holds for their mother language and for languages acquired in daily life. They learn languages by listening and by reproductive reinvention -- guided among other things by corrective measures. and possibly reinforced by increasing awareness -- and this process continues when listening and speaking are supplemented by reading and writing. As far as I am familiar with modem methods, they take advantage of this experience by trying to approach an everyday life style in language teaching. Structures, such as paradigms, rather than being imposed, are given the opportunity to be reinvented by the learner. A striking counter-example is our Dutch spelling. It is one of the easiest of the world. Its notoriously difficult part is the one governed by rules set in advance.

2.2.4.1 *Paradigms imposed*

Let us leave language teaching! The term "paradigm" has meanwhile conquered wider instructional ranges, which is why it was applied frequently to mathematics instruction in the first chapter. Why paradigms? Mathematics is a universal tool, the most flexible instrument to be handled. But universals are no teaching matter, or at least not in the manner of what is called concept attainment. Universals are learned by paradigms, and the most efficient paradigms are those which allow the easiest or the widest transfer. Improperly chosen, however, an intended paradigm can as well block the transfer, by its singularity or by lack of flexibility, or simply because of a wrong view on learning by means of paradigms.

In everyday life, in language instruction, in science teaching, etc., paradigms are as abundant as they are unengaging. Mathematics can do with a relatively small number of paradigms, which engage the learner unambiguously, or so it would seem. Is an algorithm a paradigm? Of course not. But, traditionally, algorithms are taught by paradigms, long division, for instance, by means of a numerical example (or if need be, by a few of them), which promises transfer by inviting imitation. Some children have a knack for algorithms and for learning them in this exemplary way, and some among them will even, at least implicitly, understand why they work, even though they are not interested in knowing why. Others will fail to learn algorithms this way. Does it matter? Long division is now the job of pocket calculators. But long division is only the bitter end. Let us look for the roots of arithmetical algorithms!

Arithmetical algorithms used to be taught by numerical paradigms. I chose the plural "algorithms", although properly understood, for one single arithmetical operation one paradigm should be just enough, or if really more are needed, they should virtually be one, in the sense of being somehow isomorphic. Indeed, non-isomorphic ones would instead endanger the intended uniform algorithmic-paradigmatic character and as a consequence confuse algorithmically less gifted learners. The ever lasting trouble, however, with the strategy of teaching arithmetical algorithms by numerical paradigms, was that, a from column addition onwards, one paradigm was not enough. Neither teachers nor developers were aware of the problems created by a multiplicity of paradigms needed for one and the same operation, a multiplicity that in turn creates just as many sources of failure. I said "from column addition onwards", which would seem a bit exaggerated. But do examples like

24	24
35 +	38+

really suffice as paradigms (at least for additions below 100) or are not sums taught in agreement with this strategy such as

24	24	5	5
5+	8+	32 +	38 +

very likely to suggest to learners that they are just as many fresh paradigms, and wouldn't teachers think they are needed as such in teaching? And what about, say

24	24	24	2	7
39	3	9	24	24
19	12	19	3	19

a list of new-looking paradigms, still far from complete yet long enough to frus trate learners, and for that matter their teachers as well? Long enough, I said, and yet I have restricted myself to additions below 100 of two or three summands. What about more summands, about more empty positions, multiple transfers, and so on?

Although the other operations only involve two term, they certainly do not rank second to addition with regard to the variety of what may be looked upon by the learner as brand new paradigms. Think about all those perfidious zero digits involved in the other arithmetic algorithms, about decimal points, about comparing decimal fractions with one another -- each new difficulty to be conquered by a new rule, introduced by new paradigms!

How real are these difficulties? To tell the truth, I did not invent this story, but took it from the literature; there has been plenty research to prove again and again that something is wrong with teaching column arithmetic, while little attention has been paid to *what* is wrong and *why*, though one thing is clear: that teaching arithmetical algorithms by numerical paradigms heavily clashes with the essence of what arithmetical algorithms are supposed to mean and to perform. Even if teaching column arithmetic is not such a failure as some researchers have concluded, learning arithmetical algorithms by numerical paradigms is a tremendous waste of time.

I am not against numerical paradigms as such, and certainly not where they are meaningful -- see the numerous examples in (1.3.3 and 1.3.5). Numerical paradigms can be the key and stepping stone to general problems and algebraic relations. But as patterns for learning algorithms they are doomed to fail. Of course, arithmetical algorithms are learned *numerically*, rather than by *general rules*, but this principle must not be interpreted by numerically formulating ready-made general rules. In the process of reinventing, algorithms arise by progressive schematising, with the stress on shortening, and the very thing paradigmatical in this process is progression and short-cuts rather than numerical examples.

Word problems are a similar case, or even worse if their context is a textbook page, headed by one item that looks like a paradigm, that shows how to solve all the other items on the same page. To be sure, it works. It does work until the context is extended to the textbook as a whole and the problem arises which page applies under which circumstances. Didacticians of yesteryear liked to embroider all word problem paradigms on frames, intended as algorithms and looking like them. Or at least they tried, albeit unsuccessfully since learning these pseudo-algorithms was even harder than understanding the word problems they were to facilitate.

Algebra looks a bit different, or does it? Cuneiform texts show how Babylonian mathematicians solved linear and quadratic equations, or pairs of them with two unknowns, yet all they left us is numerical examples. Did teachers use them as paradigms, or did they explain to their students in general terms what to do? We don't know anything about Babylonian didactics. Do we know any more about teaching algebra today? We know textbooks -- good ones and bad ones -- but what do we know about teaching? What do we know about how paradigms are used by teachers and interpreted by learners in instructional practice? Are they seen as a form to be filled out or, even worse, as a sequence of acts to be imitated? In [40, p.632] I gave an example of how problem solving is understood by students who might have been taught in this fashion. Let me repeat it!

To prove that $3x^2 + 12x$ has a minimum and to determine the x where it is reached.

(I) $3x^2+12x = 3(x+2)^2+4$ has a minimum, which is assumed for $x= -2$. Reason: $(x+2)^2$ must become 0 --> -2 if this is multiplied by 3 --> still 0 then 4 added and one gets the minimum which is assumed: 4.

(II) The expression $3x^2+ 12x$ is a quadratic equation. If in both parts 3 is put outside brackets the expression does not change. Then inside the brackets is (x^2+4x), that does not equal $(x+2)^2$ for we have added 4. To make it correct, -4 must be subtracted. The -4 must be multiplied by 3. Then we get $3(x+2)^2- 12$ and this is entirely equal to $3x^2+ 12x$. The expression has a minimum of -12 since a square is always positive or 0. It must be multiplied by 3, thus it is at least 0, namely if $x=-2$ one gets $3(-2+2)-12$, and this is -12. The value of the square is always positive, x 3 remains positive thus the minimum is -12.

(Examples by J.van Dormolen, reported by J.S.ten Brinke in *Euclides* 45(1970), 327-336.)

What is wrong here and who is guilty? Too highbrow a paradigm is sheepishly applied in a common sense situation. Let us skip the question of whether the whole problem is worth the trouble. Anyway, one thing is clear: that the students acted out a paradigmatic sequence which had been imposed rather than reinvented, or so they saw it.

2.2.4.2 *Paradigms reinvented*

We don't know how Babylonian mathematics teachers used their numerical paradigms. Let us take a long step. To prove the infinity of the set of prime numbers, Euclid showed nothing more than how, given three prime numbers (indicated by letters), one can find a fourth, but his celebrated proof is what I called *quasi-general*[12], that is, such that everybody understands how to generalis it and then to apply it to an arbitrary finite set of given prime numbers. For centuries general theorems on figurate numbers were proved by quasi-general methods, that is, paradigmatically, and even now numerical paradigms are a stepping stone to theorems and proofs, in number theory in particular. But mind, this function of the paradigm vastly differs from the one I censured in the teaching of arithmetical algorithms and word problems, which was quasi-copying a pattern, while having too feeble an understanding of its essentials.

Let us review a few examples from (1.3.3 and 1.3.5)! Finding the middle of 16 and 72 on the numberline was just a paradigm. So were the three roads joining A and B and the four roads joining B and C, with the question of how many roads there are from A via B to C. When arriving at figurate numbers, I was sick of *numerical* paradigms, or so it would seem, but in Fig. 4 -- 5, as illustration, the dots represent five, rather than n odd numbers, and the fourth and fifth, rather than the $(n - 1)$-th and n-th triangular number, as I would have done in teaching.

In the learner's mind such numerical examples may exert their paradigmatical influence unconsciously; it may be reinforced by consciousness, and even more by verbalising, certainly if supported by labelling variables through the use of letters, which is quite easy with the first and second example -- just say "the middle of a and b", and "m and n roads". The other two cases aren't that easy. Euclid didn't hesitate to denote numbers by letters (as he did with points and lengths in geometry) but he didn't dare to speak about n prime numbers, and so he had to proceed quasi-generally, which in a way is paradigmatically. Archimedes[13] didn't do any better: where he needed an arbitrary number of line segments, he called them *A,B,C,D,E,F,G,H*[14] and even introduced new letters for objects derived from them (for instance by addition) rather than create algorithmically constructed names. This then was the habit adhered to up to Vieta (16th century)[15]. Nevertheless as late as in Hilbert's *Grundlagen der Geometrie* the Archimedean style, though somewhat refined, is still witnessed in such a sentence as

> Satz 6. Sind irgendeine endliche Anzahl von Punkten einer Geraden gegeben, so lassen sich dieselben stets in der Weise mit A,B,C,D,E,. . ., K bezeichnen, daß der mit B bezeichnete Punkt zwischen A einerseits und C,D,E,. . .,K andererseits, ferner C zwischen A und B einerseits und D,E, . . ., K andererseits, sodann D zwischen A,B,C einerseits und E, . . ., K andererseits usw. liegt. . .

The need for variables was felt quite early on in the history of mathematics (both in the sense of indeterminates and of variable objects), and where names for them were required, Babylonian mathematicians used words like "length" and "width". The Greeks did it with letters; yet there are not enough letters in the alphabet to satisfy the need for a potentially infinite number of names for variables. Subscripts ranging through the entire infinity of whole numbers are a historically late invention in mathematics, and letters as subscripts are even of more recent date, as are subscripts of subscripts[15].

Why am I telling this story? I have repeatedly emphasised that history may be a good advisor to teaching us that things are not as simple as they Seem to habitual users. History may be able to warn us against having the learner step in at the present level of ready-made mathematics. In instruction too the use of letters for variables should answer a need, and developers and teachers should create situations where this need is so urgently felt that it stimulates reinvention, and they should this strategy as the use of letters for variables gets more and more refined.

> Let me tell a story about a colleague of mine and his little son – I don't remember his age, but in the mean time the son has become a distinguished mathematician himself. The boy had calculated the number of diagonals of – stop, one cannot tell this in English so let me use the German terms – the number of diagonals of a *Viereck, Fünfeck,* Sechseck – I don't remember how far he had proceeded, when his father asked him: "What about a *n-Eck?"* The boy answered: *"k* times *n* divided by 2."
> 3 is subtracted from *n* by going three steps back in the alphabet. Isn't it logical?

Even historically the trick for indicating the predecessor of n by $n - 1$ (and its successor by $n + 1$) had to wait for its invention, as well as for such things as indicating odd numbers by $2n - 1$, but for a long while the sum of the first n odd numbers had still to be written as

$$1 + 3 + 5 + \ldots + (2n - 3) + (2n - 1).$$

Even the invention of subscripts and sum symbols hardly changed this style. Mathematicians have always been slow to change their notational habits, and they never did so unless they felt an urgent need for change. I am pleading, as I often have, for the same right to be granted to young learners, that is, to reinventors.

"The *fifth* triangular number", if understood according to its essentials, that is, as a paradigm for any triangular number, changes from content to form, only to change again into content in such statements as "any triangular number plus the preceding one equals the square of the same rank". How to formalise and, in particular, to algorithmise such a clumsy statement? The learner has to be guided along a road of alternating contents and forms, where no step may be skipped, and none shall be longer than the learner's pace.

It would take me too long to describe this road in detail but at a later opportunity I will try to make up for this. Combinatorics is of course not the only field (though in my view it is the most excellent one) where subtle mathematical language can be acquired paradigmatically, that is, as though it were one's vernacular. Teaching mathematical language is a problem worthy of our attention. Let me illustrate this by a well-known example[16].

> Let S be the number of students and P the number of professors at some college, where it happens that there are 6 times as many students as there are professors. Express this by a formula!

The percentage of subjects (even among university people) that answered

$6S = P$

was distressingly large. What if S and P are replaced by X and Y or some other pair of letters? I was told this made no difference but I could not check it. But what about the following formulation:

> At some college there are 6 times as many students as there are professors. Put this statement into a formula!

None could tell me whether this has ever been tried. Indeed, it is not the traditional way to formulate problems. But this does not matter. We don't need new investigations to know that *something* is wrong; we do need phenomenology based didactical research to know *what* is wrong and *why.* In the present case choice is between bluntly confronting learners with letters and formulas 16a, and guiding them to experience formulas as new content derived from paradigmatical form.

Let me add a brief note on what phenomenology based didactical research may mean in the present case!
Letters and letter combinations are put to use in mathematical and similar texts in various ways. Let me mention a few of these:
Labelling fixed objects in a given context, for instance, to answer the question of how many orders are possible for three buses to leave a garage.
Labelling, and thereby fixing arbitrary objects in a given context, for instance, points on a map in order to make statements on their particular mutual distance.
Labelling, and thereby singling out arbitrary, but somehow equivalent objects in a given context, for use in general statements; points in the plane, for instance, or line segments, or numbers (as did Euclid).
Labelling geometrical or physical objects that vary within a given context (abscissas, ordinates, angles, and so on in a figure; times, path lengths, and so on in a kinematic process), in order to express their mutual relations by formulas.
Labelling numbers and other mathematical, or mathematised objects, that vary in dependence from one another, in order to express this dependence by formulas.

> Labelling fixed numbers, functions, and so on, by fixed conventional symbols such as π, e, log, $\sqrt{\ }$, Σ, # (number of), $ (mapping numbers on monies), cm (mapping numbers on lengths), kg (mapping numbers on weights), and so on.
> Labelling logical objects, such as propositions, predicates, and so on, as occurs in symbolic logic.
> Labelling both some species and its members by the same letter or letter combination, often chosen as abbreviations, for instance F for females, M for males; A for Americans, B for British, C for Chinese, and so on.
> It is worth noting that the last kind is one of the oldest (or perhaps even the oldest) symbolic use of letters, which in traditional logic, and even in mathematics proper (although now looking old-fashioned) has coexisted up to our days with the modern ones. Even nowadays such formulas as "each A is a B", dating at least as far back as Aris-

totle, and formalising such propositions as "all men are mortal", are not yet out of use. Half a century ago, after such a definition as *A topological space R is a set such that. . .* it was decent mathematical style to casually continue *Let S be an R* and then to map *R on S.* Yet in a few other contexts, computer science, in particular, similar formulations are still not unusual.

This is hardly surprising since it is a quite natural and therefore common use of letter symbols. In my opinion that coexistence amply explains the "6S = P" phenomenon, even in the case of adolescents and adults who should be acquainted with the now prevailing mathematical fashion; in addition to this influence, however, the unsatisfactory formalisation of the "number of. . ." function in school mathematics should not be neglected either. It goes without saying that children, unacquainted with the present use of letter symbols in mathematical formulas, cannot at all be expected to react in the way the interviewer judges to be the only acceptable one, as happened in [16a].

In the preceding analysis I did not include computer science, which would have required a special study.

2.2.4.3 Acts, actions and activities as paradigms

If the fifth triangular number is taken as a paradigm for any triangular number, it is so as a mathematical object. When I spoke in the first chapter of paradigms, the things to be taken paradigmatically looked rather like activities. Let us reconsider paradigms from this viewpoint! Indeed, the mere act of taking the fifth triangular number as a paradigm for any triangular number may again be a paradigm for similar actions, for instance, if numerical squares or cubes are used paradigmatically for squares and cubes of whole numbers in general. Labelling some triangular number as the n-th in a sequence, its predecessor as the $(n-1)$-th and its successor as the $(n+1)$-th may be paradigms along with other sequences for labelling terms together with their predecessors and successors. Getting the n-th triangular number from its predecessor by adding n may be a paradigm for similar procedures such as getting the n-th square from its predecessor by adding the n-th odd number, and eventually for the strategy of viewing sequences as series. Stepping

from $(n-1)^2$ to n^2

as expressions for successive squares, and calculating their difference as the n-th odd number

$2n-1$

may be paradigmatical for all such procedures where a given sequence is interpreted as a series arisen from the sequence of the successive differences, and this procedure may in turn be paradigmatical for forming second, third, or any order difference sequences from a given one.

If the (n -1)-th triangular number has been constructed as the part below the diagonal of the n-th square, that is,

$$\frac{(n^2-n)}{2} = \frac{n(n-1)}{2},$$

finding the n- th triangular either by replacing $n-1$ with n, or otherwise, by adding *n* may be paradigmatical. Anyway the trick

If the *n*- th triangular number is *supposed*

$$\frac{(n+1)n}{2},$$

then by adding $n+1$ the $(n+1)^{th}$ triangular number is proven to be

$$\frac{(n+2)(n+1)}{2}$$

which is the same as the result of replacing *n* with *n* + 1, and so it proves the expression for any triangular numbers so ever

has a wide paradigmatical range. As a matter of fact, it is a paradigm for complete induction.

The preceding exposition is not meant to be a didactical sequence. I was therefore able to abstain from overburdening it with more details. I also renounced references to the many back and forth switches involved between content and form. As a frame for viewing activities as paradigmatic, however, it is still too narrow and deserves to be broadened.

"Looking for essentials" in (1.3.2) may serve very well as a check-list. Situations, problems, procedures, and so on, may be paradigmatical *objects* while the corresponding "across" may aim at a paradigmatical *activity.* Yet let us fix our attention on one special kind of activity. Inventions are edited, prior to communication, and so should reinventions as well. Redundancies are cut out, paths are shortened, concepts and notations are smoothed out, the matter is rearranged, often to the degree of what I have called, putting things upside down. All this, as far as needed, can better be learned paradigmatically than by filling out prescribed forms.

And, last but not least, what a paradigm is and how it works, is learned paradigmatically.

2.2.5 *Applications*

I began the first chapter with a historical explanation of the plural "mathematics". The medieval *quadrivium* has now been superseded by a *dualism: pure* and *applied.* By their very names the two oldest mathematical journals claimed to cultivate both of them, although applied mathematics had still to wait one more century to get a journal of its own. At universities there were (and in some places still are) separate chairs for each of them. The famous mathematician G.H. Hardy wrote a famous textbook "Pure Mathematics". He was proud to assert he had not created anything mathematical that could ever be applied. He was wrong but he didn't live long enough to see this. Indeed, in mathematics one never knows.

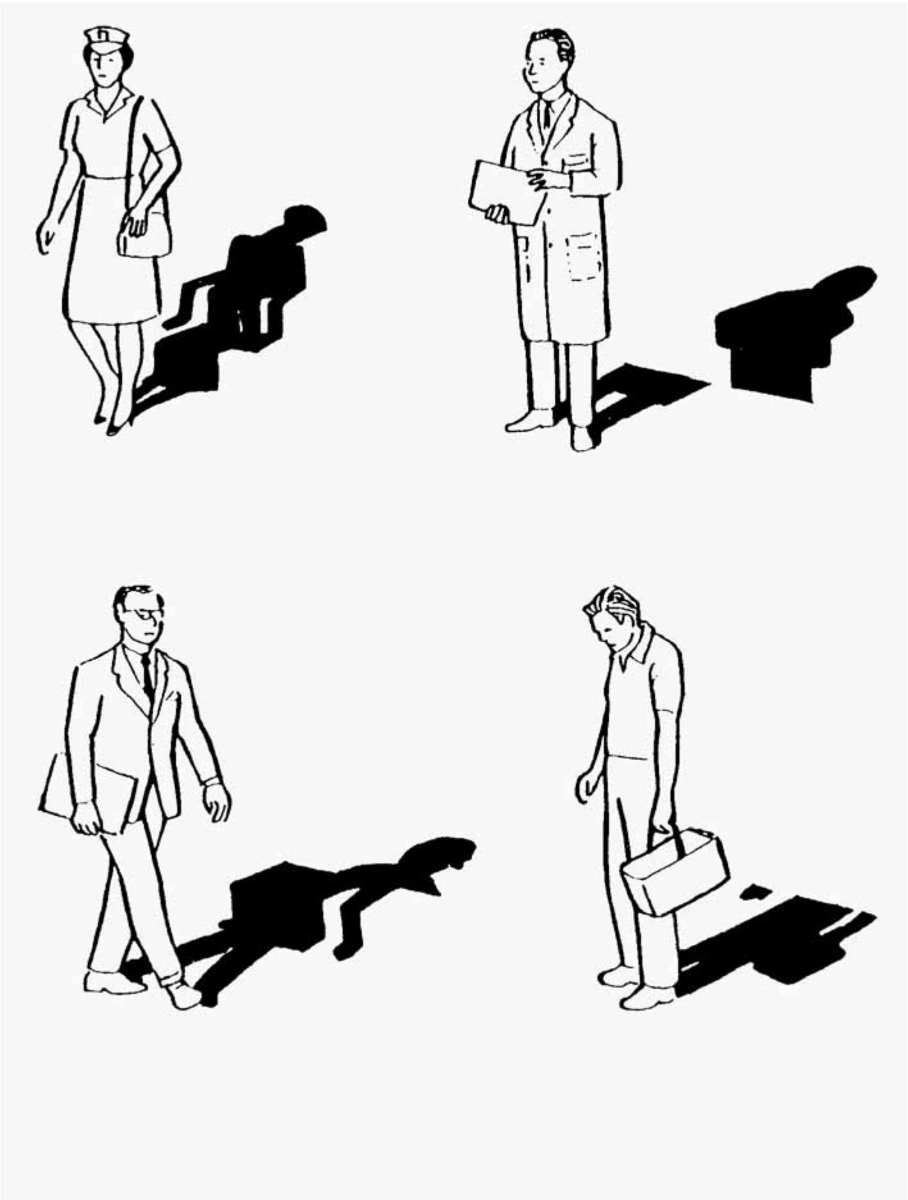
There are good reasons for researchers to specialise and for higher education to be departmentalised. But *pure* and *applied* is an awkward dualism, as are *theory* and *practice,* or (in textbooks) *theory* and *exercises,* or (in mathematics education) *insight* and *drill,* and so on. Reality is a virtual unity, where for practical purposes at any moment some part is focused and others are out of focus. Yet shifting focuses is no less important than focusing itself and it should not be impeded by what I formerly called isolating impermeable membranes. This applies in particular to learning mathematics. So I propagated rich contexts, just to fight impermeability.

The most serious trouble with dualisms like those above is the tendency to favour one-way traffic. In instruction this means: first theory and then exercises, first insight and then drill -- the first for conscience's sake only and very likely to be skipped. After all, the mastery of the first does not imply that of the second whereas the second may bring about the first, or one would hope.

If history means searching for the roots of the present in the past, then history of pure mathematics is a tree deprived of its strongest roots. Teaching is not different. In the past, most mathematics was never applied or meant to be applied by those who learned it at school, but things have changed spectacularly. More and more people use mathematics more often than they are aware of. When I said this I continued: They use mathematics because they cannot do without it. Indeed, the amount they use is astonishing, but have they learned the mathematics they eventually use, only because they were once taught mathematics? Mind, I don't ask whether they use the mathematics they were taught. One can readily deny it, since most of the stuff they learned is of no use at all for most of those who were supposed to learn it. My proper question was whether learning the mathematics they were taught has been conductive to educating mathematical behaviour. Or rather, if there was some influence, could it not as well have been detrimental?

I agree that, in adaptation to social needs, views on teaching mathematics have changed. Henceforth, one now postulates, students shall also be taught to apply the mathematics they are to learn; so reformers and developers call and look for more applied mathematics alongside the pure. As so happens in the course of teaching, "alongside mathematics" is interpreted as "after", and this is true at various curricular levels: The eventual result is a pair of curricula, rather than one: first, pure theory; second, application -- a menu of applications, that is. Or the two curricula are shuffled into one another as packs of cards: parts or chapters alternate with one another -- theory first applications after. Or every bit of theory is followed by another bit of application.

It does not matter which variant is chosen. The view itself is wrong and it is even worse if it is identified with mathematising, which is a learner's rather than a teacher's activity, or at least should be. Applying mathematics is not learned through teaching applications. The so-called applied mathematics lacks mathematics' greatest virtue, its flexibility. Ready-made applications are anti-didactical. Mathematising some problem situation in nature or society should not be demonstrated by the textbook author or the teacher but left to the learner to be reinvented. In this regard applications are no different from mathematics as such, and for that matter the present section could as well not have been written, were it not for the urgent need of using every opportunity to propagate starting and keeping mathematics in reality, while switching back and forth between realities -- natural, social, and mathematical. Pure and applied is a highbrow dualism rather than a learner's concern.



2.3 LEARNING PROCESSES

2.3.1 *"Learning process" as a didactical principle*

What led me to deal with learning processes and their observation, in particular, in a chapter on Didactical Principles? Aren't these rather tools and subjects of *research?* Unfortunately, whatever educational research may mean -- and it may be quite a number of things -- one cannot but agree that little attention has been paid in the past to learning processes and even less to their observation. Even at the present state of knowledge with regard to learning processes, didactics can hardly draw on a substantial capital of educational research. This, then, explains and justifies the prevailing practice of short-cuts in education where, without much ado (let alone any reflection) and trusting one's intuition, more or less conscious experiences on learning processes are applied wherever needed in didactical situations. Let me account for this fact -- and where I consider it a gap try to mend it -- by dealing with learning processes in the present chapter! *Mathematical* learning processes, that is, although most of what I have to say is also true for the didactical ones.

Anyway, the use of and the emphasis on *processes* is a *didactical principle.* Indeed, didactics itself is concerned with processes. Most educational research, however, and almost all of it that is based on or related to empirical evidence, focuses on states (or time sequences of states when education is to be viewed as development). States are *products* of previous *processes.* As a matter of fact, *products* of learning are more easily accessible to observation and analysis than are learning *processes* which, on the one hand, explains why researchers prefer to deal with states (or sequences of states), and on the other hand why much of this educational research is didactically pointless.

Why can -- even dense -- sequences of states if appropriately interpolated, never be equated with processes? The reason is again a didactical principle. In [87, p.165] I formulated and italicised the thesis:

What matters in learning processes are the discontinuities.

Interpolation between states obviously applies only to continuous processes. The process approach is fundamentally different, even if its scope is restricted to *short* processes. Let this be enough for the moment, and let me delay justifying my thesis didactically and elaborating on it!

2.3.2 *Teaching/learning processes*

Since Herbart, if not earlier, views on teaching processes have influenced didactics and, though in instruction teaching and learning processes are strongly interrelated, there are few people left, even among practitioners, who profess the outright identification of learning with teaching. Moreover, one cannot disregard the large part of learning -- even at school -- which is independent of teaching. To my mind, the amount of what is learned by formal instruction is grossly overestimated. But since my concern is didactics, more often than not when I speak of learning processes, this will be shorthand for "teaching/learning processes", or "instructional processes" (which includes both learning and teaching).

1. *Pure and mired learning processes*

Knowledge of *pure* learning processes may greatly help us to understand and organise instructional processes, but whatever pure learning processes may mean, this field still looks like a desert: shrubs of unconscious experience, incidental or intentional observations, and sparse efforts to organise this experience. There is one oasis in this desert: Piaget's early work[17], based on continuous observations of his own children during their first and part of their second year. Of course, active observation, too, is a kind of teaching but to a much lesser degree than the influences of the social environment in general. It is a pity that Piaget's original records have never been published.

> Indeed, Piaget's material, as selected and offered by himself, is processed to fit into the stream of his profound theory. It is chosen in order to describe what he calls the sensory-motor stage. One strange feature should be mentioned: his complete neglect of language. The subjects never say a word (as they never weep or cry). Did Piaget reserve language for another volume that never appeared. or did he underestimate the part language plays in the "construction of reality"? By the quite sophisticated tasks these little children are able to perform they distinguish themselves from animals to such a degree that one is inclined to conclude that this is due to their ability to name things, events and acts, and by this way to imprint them more strongly in their memory. It should be noted that Piaget did not pay adequate attention to the part played by language in cognition in his later work either.

Piaget's original approach was time-consuming. The laboratory research carried out by his assistants represents methodologically another extreme: instantaneous exposures, as it were, where learning is considered to be a disturbing nuisance and therefore counteracted and eliminated as much as possible.

> Let us add one remark: Rather than the *population of subjects,* the only constant in the explored time sequence of states is the *complex of tasks* to be tested; yet, since they are propounded at linguistic levels corresponding to the subjects' ages, one can, in each particular case, question what has actually been tested: levels of task performance or of linguistic proficiency.

Factual learning processes are mixtures, depending on the involvement of unintentional and intentional, informal and formal teaching. However important observations on everyday learning may be, didactics, which includes both teaching and learning, is best served by knowledge of more formal teaching/learning processes and the interaction between the guides and those guided.

2. *Participation and reservation*

Another dilemma in learning processes is related to observation: Participation and reservation of the observer. Piaget himself, in his early work, behaved as a participant, although he did so with the utmost self-restraint: the success or failure of his subjects in doing tasks was fed back into his strategy as an observer. For instance, after an initial failure, the same task was offered anew, often ten to twenty times. Piaget's assistants, however, used repetitions to make sure whether a subject's reaction as was by chance or consistent.

The teaching component increases along with the observer's participation in the learning process,. Yet teaching/learning processes, say in the classroom, can be attended and registered just as well by non-participating observers. Personally, I do not like the idea of non-participatory observation. I even feel unable to refrain from intervening, and I still regret some cases in the past where I missed the opportunity or lacked the readiness for intervention, for instance, when I failed to ask the person observed why he did what he did, and why he did think what he thought.

3. *Short-term and long-term learning processes*

As is the case with the interdependence of teaching and learning processes, I emphasise here that my proper aim with regard to learning processes are long term ones, which is in fact the title of (2.4). Short-term learning processes look as if easy subjects, easier than they actually are. They are, indeed, easily accessible to direct observation, which no longer holds true for their long-term effects, whether the short-term processes observed are embedded in long-term ones or not. Of course long-term learning processes are the most difficult thing to observe.

When I objected, on the ground of the discontinuity of learning processes, to interpolating between consecutive states as a procedure to grasp and to guide *processes,* I did not mean to exclude the reconstruction and the pre-construction of long-term learning processes from sequences of short-term ones. The conditions for realising these goals will be the main concern of this section and the following (2.3/4).

2.3.3 *Observation as a didactical principle*

1. *Observing*

Observation is an indispensable tool for any research based on experience. Observing natural phenomena under the microscope is quite different from observing patients, or wildlife, or people in the supermarket, or children in the classroom. Inventive and influential methodologists have developed models for organising classroom observations according to a pattern they themselves consider to be as characteristic of the natural sciences. Assistants, preferably students, armed with check-lists, are charged with observing teachers' and children's classroom behaviour, and according to the minutes spent on certain activities they are to classify the actors in a system of predesigned categories -- a methodology that is a caricature of observation as understood in the natural sciences. Even the naturalist does not trust check-lists to explain unexpected events, which as a matter of fact may prove to be more important than the anticipated ones. His strategy of observation is instead controlled by vast experience and a certain mount of theory. Proceeding according to check-lists is a good thing for starting an engine or any other predesigned and controlled process, but it is an impediment to acquiring new experience or to learning new behaviour.

All the same, educational *research* will be a subject of the next chapter. In the present chapter the observation of learning processes will be dealt with as a *didactical principle*. For many years I have propagated observing learning processes, although in fact it is an art that is practised more often than people would admit -- the observers themselves included. What I meant to ask for was to *report* on such observations, which presupposes *getting conscious* of them. Teaching and didactical experiences result from observing learning processes. Teaching and didactical behaviours are justified by experience, and more explicitly by observations inside and outside the classroom which have consciously been registered.

2. *Observing and recording*

Are conscious observations of learning processes as exceptional as they seem I don't believe so. But if these are so many, then why do so few of them get a chance to be reported? Are non-professional observers afraid that reporting observations would come across as telling classroom anecdotes? If so, they are right to avoid this appearance. Indeed, telling classroom anecdotes will not acquire scientific respectability unless it is organised by means of all-embracing educational or psychological categories. Or, rather, this postulate is the very reason why developmental research as practised by Piagetians is blossoming: classroom or laboratory observations are classified according to categories and hierarchies or at any rate make a pretence of doing that. I admit there are no ready-made tools for organising observations of learning processes but, as long as no substantial material is available, the quest for organising it is meaningless. Nevertheless, there is one valid criterion for estimating whether observations are by chance worth being reported or not: whether they are *paradigmatical.* As a matter of fact, this extends to educational research, where paradigms play a part unrivalled by statistics, but in the present chapter we are concerned with the paradigm as a didactical principle. It was introduced here as an answer to the quest for experiences worth being reported, though this is not at the heart of the matter. As valuable as reports may be for those who are expected to read them, the didactical value of reporting resides first of all in its being an incentive to observation, with the aim of becoming conscious of one's observations and having them analysed.

Learning by paradigmatising would seem to be a vicious circle: on the one hand, the criterion for knowing what observations are worth recording resides in their paradigmatical character while, on the other hand, learning what is paradigmatical, must take place in the course of observations worth recording. It <u>is</u> circular, yet so are learning processes in general. In this particular case we are, indeed, concerned with a specific learning process which, first of all, has to be identified as such: the process *of learning to observe,* as passed by educational practitioners and didacticians; this process is guided by the experience of *observed learning processes* -- those of their subjects -- and moreover, as happens or should happen in teacher training, reinforced by exterior guidance. It is an apparent circularity that is not vicious at all; and in the right perspective the process is a spiral rather than circular.

Later on, we shall look for more objective criteria for what is worth being observed and recorded and at the same time fill in a gap in our argumentation. Meanwhile, we will consider the described process of learning by paradigmatising, albeit lifted up to a higher plane.

3. *Self-observation*

Observing learning processes, whether formally recorded or not, is itself a learning process - a meta-process, as it were. It becomes an opportunity to learn how to observe learning processes as soon and in the measure as the observer is aware of details of this meta-process. What kind of details? The answer is again: details of what is paradigmatical. (In [87, p.205] I described the paradigm in my own learning process as an observer that led me to shaping the idea of paradigm.) But this is too easy an answer. Observing oneself can be a difficult undertaking as everybody knows who has ever tried it; in particular, any mathematician who has tried to observe himself solving a mathematical (or for that matter, any other) problem knows that proving a hypothesis often means finding out the reason why one believed it to be true. Let me therefore deal for a moment with self-observing in a broader context than that of only observing learning processes; this anticipates themes to be dealt with in (2.4.2).

The difficulty of self-observation is that of being actor and spectator simultaneously. It can be learned, to be sure, and where it may mean the observation of *learning processes,* teachers and didacticians should indeed learn it; but self-observation is also worth learning with regard to the acquisition of *subject matter,* and particularly if this subject matter is *mathematics.* Anyway, it is a difficult thing to learn, which may require considerable endurance. There is, however, a short-cut. Let me first describe it in terms of first line mathematics learning and teacher training!

Observing others is definitely easier than observing oneself. In the past I have fought individualised learning (which is fashionable today) even harder than the traditional teacher-centred classroom variant. I will come back to this point when dealing with "Mathematics for All" but let me say in advance that I favour small learning groups of, say, four learners, and that this preference extends to teacher training. Observing the learning processes of others is implicit to learning in small groups, and what happens in such groups should be made explicit by the learners themselves or by their supervisor, although this is hard to do and we know little about how to do it systematically.

The counterpart for teachers and didacticians (and even for researchers) of what the classroom and teacher training learning groups mean for pupils and student teachers is workshops: learning to observe oneself by observing others. Indeed,

observing others and oneself is a source of consciousness for one's own physical and mental actions, and this again is an important aspect: reflecting on what one does and thinks. Of course, we would not be able to act in everyday life if tried to plan everything in advance and think about it afterwards. A large part of our incentives and intuitions are unconscious and will always remain so, but making unconscious things conscious as often as possible is a way to improve the operational value of our unconsciousness.
Was *self-observation* a misleading title? No, its scope has been extended quite naturally from the *individual* to the *collective.*

4. *Discontinuities in learning processes*
I promised more objective criteria than having a paradigmatic character for indicating what is worth observing and recording; at the same time I would like to fill a gap in my argumentation. To do this let me repeat my self-quotation: What matters in learning processes is the discontinuities -- or jumps, as I prefer to call them. In what sense does this matter? First of all, it is the jump in a learning flow that superficially looks continuous -- the learners' overt or covert *aha* -experiences -- that strikes the observer and challenges him to participate by reinforcing their influence.
How to recognise jumps in the learning processes? There may be objective criteria, such as in the cases I reported in (2.1.4,4): for instance, a clumsy procedure for solving linear equations like "first multiplying by the denominator, then dividing by the numerator", may after a long while be spontaneously shortened to multiplying "upside down". Was this a chance event? To make sure whether such an occurrence ***is*** a jump, the observer should offer the learner a sequence of similar problems and after a while and as it happens resume testing once it is clear that the occurrence was not accidental. As a matter of fact, in all the cases I mentioned above (and in many others) there was no relapse whatsoever, which proves that the jumps -- or what I had diagnosed as such -- were real. Clearly the girl did not dare to leave the beaten track until she was sure she would not get lost. I myself was satisfied to reinforce through mere repetition, lest explicitation of rules should harm her. Indeed, as I mentioned earlier, her teacher (by supplying her with explicit rules for operations on fractions) had managed to undo the effects of many weeks of reinventive learning. So I was afraid that, unless she verbalised by herself, rules would prove harmful rather than helpful. It seems that she never had learned to verbalise mathematics on her own, in spite of her otherwise normal mastery of language. Attempts to rectify this shortcoming might seriously have confused her, or so I feared. Notwithstanding the lack of verbalisation, she understood very well what she was doing - later on I will report on how I tested her insight.
Good verbalisers are easier subjects where learning processes are to be observed. My publications contain a large number of references to a boy whom I started observing a few months after his second birthday (more systematic reports are being found in [58, 64, 73,84,98, 109,149]). It was an early habit of his to formulate

his intentions in advance as well as whatever plans he was going to carry out, and to reflect on his own speech and thought, sometimes even while speaking and thinking. On the other hand, great discoveries, or what he felt to be such, were accompanied by heavy emotions and explosive utterances. He was a most appropriate subject for observing learning processes of all kind. Although my thesis about discontinuities being the significant thing in learning processes dates from long before his birth, he has taught me a lot, for which he has been credited in my various publications.

I have just sketched two extremes of learners. From old material and as a "revisitor" I could add a few more reports[18]. All they have in common are *jumps* with paradigmatical consequences - anyway they do not provide enough indications for attempts at categorisation. Still, let me mention what I personally learned from these experiences: all together they have helped me to get rid of the prejudice that something learnable may be self-evident. What is *now* a matter of course was once a matter of learning. It sounds trivial but then it is one of the trivialities that one should be reminded of again and again. For this let me report a striking example!

5. *A group learning process*

The two cases I just reported were learning processes of individual children. For instance, if instead of the one girl, I had been teaching two together, I would have followed an entirely different strategy: The need for communication between the two of them would automatically have stimulated verbalisation, which I would have cautiously guided.

Anyway teachers as well as developers are dealing in the classroom with collective, rather than individual learning processes. My own first experiences in this respect, well over thirty years ago, are related to Dina van Hiele-Geldof's marvellous reports[19] on classroom observations. She took notes while teaching; and I was astonished how many more -- and more important -- things she observed than I was able to do. Since then, modem technical tools have become a great help for observers in the learning environment. On the other hand, small groups are a better environment for observing learning processes than large classrooms. Let me give an example:

> After *measuring* a few angles, it is time to *draw* angles of a given sue. Mariska tries to draw an angle of 20°. She draws at random. She adapts the protractor to one leg, carries the pencil to the other one, reads off it is wrong. She erases the angle and draws another random angle, while taking into account the first measurement. She measures again, and again it is wrong. She erases the second angle, swearing at it. Her neighbour Monique is alarmed by her swearing. She watches the third attempt, and then she explains how to do it at one go. Anja is listening; "that is how I did it" she says: "So I have done it.' Mariska draws the next angle -- 13° -- in the new way. Meanwhile, Cecil has learned how to do it from Anja. Cecil consoles Mariska: "I didn't know it either." The next misfortune is that a point to be marked falls outside the sheet of paper. Anja helps. Anja and Monique always wait for the others before going on to the next task.

When I recommended observing learning processes as a treatment against unwarranted prejudices on self-evidence, the above proceedings were one among many examples I had in mind; I found it a distressing experience, and at the same time a warning. Thirteen years old and behaving like this just to draw an angle of 20°, isn't this sheer stupidity? Or were they really that stupid, these two among the four 13 year-old girls? Would they have acted the same way if the task had been to draw a line segment, say of 5.3 cm? Certainly not! Why weren't they able to use the circular protractor in the same way as they would have used the linear scale of the ruler? Thirteen year-olds who behave like Mariska and Cecil, what have they missed in their most elementary education? In order to answer this question more profoundly, it should be posed in a much broader context. What is behind this distressing experience?

6. *Change of perspective*
It is not merely the switch from measuring to constructing. It is a kind of jump, which I have called *change of perspective* - an important and indispensable activity, characteristic of a mathematical attitude - in this particular case, from *measuring a given thing* to *making a thing with a given measure* and, more generally, from examining a given thing and stating its properties to making a thing with prescribed properties (and of course if need be, the other way around). In traditional geometry teaching this duality was called *analysis and construction.* Geometry, though not necessarily of the traditional kind, is an excellent training field for changing perspective, although its domain of transfer is much wider, both inside and outside mathematics. Change of perspective takes place as soon as the child starts to conquer the so-called egocentrism which, in my experiences happens much earlier than Piaget's laboratory experiments Seem to indicate. Learning processes are marked by a succession of changes of perspective, which should be provoked and reinforced by those who are expected to guide them. My diagnosis with regard to the failure of these two girls is: victims of rigid instruction that never left any room for individual initiatives. Change of perspective develops most naturally in reinventive learning and, if consciously trained, allows for a wide area of transfer, as do all behavioural attitudes. Anyway one should begin exercising it quite early. Is 13 too late to recover what has been missed? Am I too pessimistic if I say it is? What a tremendous gap between the two girls who succeed and the two who fail! Can we do anything to narrow it? Or better, anything to avoid it? I am not at all hopeful as long as teacher trainers fail to teach their students how to observe learning processes. Or shall I call it learning to observe when they send their students into classrooms with prefabricated questionnaires or have them observe classrooms on the monitor? But let us not say too much about "teacher training" beforehand!

7. *Communication*
I almost forgot that the above story was meant as an example of a collective learning process. It is, however, too short and too simple to be a good example. For two of the girls the jump was too easy, if there were any jump at all. The boost

the two others needed and received was too straightforward. I could repeat more sophisticated examples from earlier publications but this would take me too far afield, while high quality research is just now taking place in this domain. The general feature of the observational evidence at my disposal is: four children co-operate more or less efficiently to solve a problem. An accidental remark of one venturing may mean a hint for the other; or the one who, venturing the jump, cannot explain it herself but another can and thus enables other members of the group to follow suit. Linguistic tools arise from the need for communication. Discussions evolve, and participants are compelled to reflect on their own utterances and actions. The teacher, walking between the tables, whether or not requested for by a particular working group, should not intervene actively before having listened to the - perhaps confusing - report on their previous learning process. The mere fact of reporting may mean such great progress in the learning process of the group itself that the teacher's intervention can be restricted to polishing language and adapting *ad hoc* invented terms to the conventional terminology. Or the report may include keys to guide the group in the right direction. Or exchange with more successful groups may be helpful.
This then is observing group learning processes - a scarcely explored field.

8. *Thought-experiment*
What use is the knowledge acquired in observing learning processes for the instruction developer? Certainly, there is some use. As a developer he can profit from these pieces of knowledge, provided he views them in a broad sense as paradigms. As early as 1961, if not even earlier, I propagated the *thought-experiment* as an instrument in educational development:

> Imagine a student, more or less distinctly defined, and have him in his mind reinvent a mathematical idea, observe his actions and analyse them by logical methods. [14, P.32]

> That is the Socratic method, or as I would prefer to say, the method of the thought-experiment. The latter expression comes from Mach, who described it as a method of theoretical physics; it was the central method from Galileo to Einstein (and maybe. still is today). In didactics, I mean by thought-experiment the attitude of a teacher or text-book author of imagining a student or a group of students and teaching them in mentally while reacting in advance to their probable reactions. The imaginary students are active, and their activity allows the teacher to determine his way of proceeding. In a narrower sense I will assume, as Socrates did, that the teaching matter is reinvented or re-discovered in the course of teaching. Rather than being presented dogmatically, the subject matter originates before the students' eyes. Though the students' own activity is a fiction in the Socratic method, the students should be left with the feeling that the teaching matter arose while teaching, that it was born during the lesson, and that the teacher was in effect only a midwife. . .
> In the Socratic method "reinvention" was not understood literally; it was simulated rather than being true reinvention. It could not have been otherwise, could it? The teacher's authority was still dominant. . . The initiative was only on the part of the teacher. Not only did he lead the student, he also showed him how rediscovery works, he rediscovered on behalf of the student. [40, p.100-102].

The variants I proposed were, of course, thought-experiments for real, albeit guided, reinvention. Moreover, I postulated that textbook authors and developers, whether or not they shared my educational philosophy, would reveal the very essentials of their thought-experiments (if there were any); this would, in my view, be the right way to prove, as it were, the soundness of their approach or at least to bring this point up for discussion; and it would eventually lead to having the thought-experiments checked in the classroom - let me stop here and not anticipate too much on "educational development" (3.2/3).

2.3.4 *Levels in the learning process*

What matters in learning processes are the *discontinuities* - quoting myself - or, another word that I have frequently used: thejumps. I owe the conception of the level structure of learning processes to my collaboration with the Van Hieles, a couple who embodied, as it were, the marriage of theory and practice. Let me quote from [40, p.121 sq.], which actually dates back to the spring of 1969, while pieces relevant to the present exposition are mere translations from the French [14, p.32-34], written in 1961, as I afterwards noticed to my own surprise:

> When the Van Hieles started teaching they were just as unprepared as many other young teachers; nobody had told them how to do it. They had, of course, passively undergone teaching, and maybe even observed how their teachers performed, but this was not enough. As time went on, they had the opportunity to discuss their own teaching with each other and with others. They subjected their own actions to reflection. They observed themselves while teaching, recalled what they had done, and analysed it. Thinking is continued acting, indeed, but there are relative levels. The acting at the lower level becomes an object of analysis at the higher level. This is what the Van Hieles recognized as a remarkable feature of a learning process, namely in the learning process in which they learned teaching. They transferred this feature to the learning process that was the goal of their teaching, namely the learning processes of pupils who were learning mathematics. There they discovered similar levels. To me this seems an important discovery.

In my exposition this is followed by a series of examples, which included the levels on the -- historical as well as individual -- road from the intuitive, unreflected and, as it were, incidental *practice* of complete induction (see 1.3.5, 14) to the formulation of Peano's *axiomatic system* of natural number. Then it continues as follows:

> Here the levels of the learning process stand out in bold relief. On the lowest of the levels under consideration complete induction is acted out. On the next level it is made conscious as an organizing principle and can become a subject matter of reflection. On the same or a higher level it is put into a linguistic pattern. From here to the Peano system the path is no longer locally determined. Now the data is no more a bunch of mathematical derivations in which a common principle (like that of complete induction) is hidden; it is rather the organization of an entire field, in which linguistic formulations (that of complete induction) among other mathematical activities become the subject matter of reflection; the complete induction then undergoes a re-interpretation; rather than describing a mathematical principle it is now interpreted as postulating a property

which, along with others, should characterize the natural numbers.

....

How levels stratify this [the mathematical] activity has been shown by an example: the means of organization of the lower level become a subject matter on the higher level. Which phases the learning process passes through from one level to the next is a pedagogical question, which should be answered in specific cases by pedagogical experience. The relation, however, between one level and the next is overwhelmingly logical and accessible to logical analysis. Often the level-raising tools are heavy quantifiers -- over all properties of natural numbers in the case of complete induction.

....

The learning process is structured by levels. The activity of the lower level, that is, the organizing activity by means available at this level, becomes an object of analysis on the higher level; the operational matter of the lower level becomes a subject matter on the next level. The pupil learns to organize by mathematical means, he learns to mathematise his spontaneous activities. Or rather it would be desirable to have him taught in this way.

Let me stop quoting from what I wrote early in 1969 and turn to similar passages, written in the late fifties and published in [13, p.26 sq.]:

> The analysis of mathematics as a ready-made subject presents a deductive system in which all steps are fundamentally equivalent. The analysis of mathematics as an activity, however shows a layered structure. This feature is accounted for by the level theory of the Van Hieles. The stages through which the learning process passes from one level to the next, are of pedagogical concern (which can be disregarded by the present exposition). The relation between one level and the next, however, is preponderantly logical. To discover it, we can use logical analysis. ...
>
> What it is that characterizes the level structure may be expressed in a few words by saying that the operational matter of a lower level may become subject matter on a higher level. If, on a certain level, complete induction has been a significant activity, this activity may become a matter of conscious consideration and, finally, of explicit formulation on the next level. The higher level is clearly indicated here by the heavy quantification over all properties of natural numbers which occurs in the statement of complete induction. ... Operations with integers such that multiples of a certain *m* are considered inessential, and operations with pairs of integers according to the equivalence relation for fractions are unproblematic activities. On a higher level they can be promoted to subject matter and finally described by the device of abstraction through equivalence classes. On a still higher level the equivalence relation and the abstraction by equivalence can become subject matter.
>
> In geometry, space phenomena are studied on the lowest level. The properties discovered on the lowest level are subject matter on the next level, where these properties are related to each other. The relations acquired on this level are then studied on the next one, in order to build up what are usually called theorems. The logical connections between the theorems are subject matter on a yet higher level, which is still below that of logic as a subject matter.
>
> The level theory of learning is closely akin to what can be characterized in logic by the words "theory and metatheory". It is not by chance that the Van Hieles seized upon this idea. To my knowledge they were the first who wrote a textbook in which the learning process is purposefully initiated and kept up as a process of re-inventing. Traditional methods, however, often show the opposite tendency: to descend from higher levels to lower ones instead of climbing from the lower to the higher level.

Let me finally yield the floor to the Van Hieles themselves[20], for dealing with the learning process in geometry at the secondary level:

> . . . Take the example of the rhomb! Before learning geometry children possess the "image" of the rhomb. They are acquainted with rhombs that are special concrete objects. Now they will experience what "rhomb" means in the geometrical context. They discover properties of this figure, which the teacher calls "rhomb". They form the "symbol" rhomb. In this symbol they condense all internal properties of the rhomb they have experienced. . .
>
> Thanks to the possession of a stock of symbols the pupil will be able to organize the subject matter. Mainly as a routine the symbols become signals. From the properties belonging to the symbol rhomb (equal sides, parallel sides, equal opposite angles, diagonals that halve each other under right angles) the rhomb as a signal (equal sides) arises, or rather one property (equal sides) becomes a signal for the symbol rhomb as the complex of all its known properties. The pupil is able to anticipate . . .
>
> The pupil who has started at "level 0" with undifferentiated visual structures, is now at the first level. At level 0 concrete objects and images are the subject matter which is organized by the use of symbols and signals. At the first level the symbols and signals of level 0 will become the subject matter. The system of properties of geometrical figures will be organised by means of relations connecting the geometrical figures to which the symbols and signals of level 0 refer. At the first level relations like congruency, similarity, parallelism appear. At first they have the character of symbols, later on that of signals (e.g. congruency of two figures is a symbol for the fact that the two figures cannot be distinguished in any geometrical aspect; a congruency theorem for triangles may be a signal). However, at this level relations are not yet a subject matter as are the properties of figures.
>
> At the second level the relations which have been devices of organizing at the first level will become a subject matter. Here the organizing devices are relations between relations, mainly of a logical character. The symmetry of a relation between geometrical figures, the interrelatedness between relations by means of implication can be used at this level, but they can become a subject matter at the third only.
>
> At the third level the pupil will be able, e.g. to distinguish formally between a proposition and its converse. Finally at the fourth level (hardly attainable in secondary teaching) logical thinking itself can become a subject matter . . .
>
> Learning is a discontinuous process. The discontinuities are, as it were, the jumps in the learning curve. These jumps reveal the presence of levels. The learning process has stopped. Later on it will continue, as it were, by itself. In the meantime the pupil seems to have "matured". The teacher no longer succeeds in explaining the subject. He (and also the other pupils who have reached the new level) seem to speak a language which cannot be understood by the pupils who have not yet reached the new level. . .

The attentive reader will have noticed a fundamental distinction between the picture unveiled in 1969 and that of a decade earlier. There are common elements: the discontinuity of the learning process; its level structure, discernable by a kind of mathematical analysis; and the levels as determinants of the discontinuities. The pictures *diverge* from one another with respect to the bonds by which successive levels are interrelated.

In the fist version the learner's *operational matter* on the lower level becomes his *subject matter* on the higher level.

In the second version the learner's lower level *activity* becomes an *object of analysis* to him on the higher level, or in other words: on the next level this activity is made conscious and can become a subject matter of *reflection.*

The second version adds to the first the means by which operational matter on the lower level is transformed into subject matter on the higher level:
Being made conscious and subjected to reflection is the means of level-raising, which is applied, to the - still intuitive (operational) - *activities* of the lower level, rather than to the subject matter.
When did this change of view take place and why, or is such a historical query worth the soul-searching? There is some indication in the old exposition of a foreshadowing of the newer view, and some relapse in the newer exposition to the older viewpoint, but I don't believe that this really matters. Was the change of view caused by *observations* of learning processes made in between both events? There are a few of these I might conjecture to have caused the change but I am not sure whether they were really responsible for it. A great many observations of learning processes relevant to the level theory were made *after* the crucial date, although I would hardly have noticed them as such if I had not at that time been in full possession of that theory, or so I believe. So my guess would be that the theory was rationally acquired and only afterwards tested by observation.
What kind of considerations could have guided me? In the first paragraph quoted at the start of the present subsection from [40], I related Pierre van Hiele's own report on how he discovered levels in learning processes, that is, as a matter of fact, first in the process he and his wife went through as young teachers while learning to teach, and how he transferred those levels and their features to their pupils' processes of learning mathematics. The remarkable feature was that of levels related by *reflection:* "At the higher level the acting at the lower level becomes an object of analysis." However, the levels as characterised in my first version, and as substantiated by themselves[20] are of a quite different nature -- I will come back to this point.
I think that the change between the two versions of level theory was influenced by my drawing the full consequences of Pierre van Hiele's report on the discovery of the levels, which were caused by reflecting on their own learning process and then made parallel with the learning processes of their pupils.
But there is more to it. While writing these lines, I discovered to my surprise that the change between the two versions took place as early as 1961. Let me quote from [14, p.33]:

> Ce qui caractérise l'hiérarchie des niveaux en général, c'est que la technique des opérations à un certain niveau devient matière *à* réflexion *à* un niveau supérieur - relation apparentée à celle qui, en logique, est caractérisée par les mots "théorie et metathéorie".

The main example illustrating this statement is what I described above as "the - historical as well as individual - road from the intuitive, unreflected and, as it were, incidental *practice* of complete induction to the formulation of Peano's *axiomatic system* of natural number", and with regard to which I asserted:
Here the levels of the learning process stand out in bold relief. This, indeed, seems to me essential for the reconstruction of the change from the first to the

second version. Obviously I judged the level theory important enough to be extended from geometry -- where it was discovered and in terms of which it was defined by Pierre van Hiele -- to other domains of mathematics teaching and learning. This, indeed, was in accordance with my main interest at that time, the middle and upper grades of secondary education.

For two millennia mathematicians had practised complete induction[12] -- the ancient “side and diagonal numbers” being the most profound example[12] -- when Pascal (and independently, James Bernoulli) couched it for the first time in the now well-known principle; there is not the slightest evidence for there having been any conscious reflection on this practice during this long historical period, although figurate numbers would have offered an excellent opportunity. The following steps on the road to the modem theory of whole number took much less time in the learning process of mankind, but all of them seem to be due to reflecting on previously unreflected activities.

In the first chapter I repeatedly emphasised reflection as being characteristic of mathematical thought, in particular in the interplay of form and contents (1.1.3); this was substantiated by some “examples” (1.1.3.1). I don’t remember when I discovered reflective thought as a forceful motor of mathematical *invention.* It certainly happened by introspection, that is, by tracing my own ways of thinking (which is itself reflective thought), long before I tried to discern it in the thoughts of others and in the learning process of mankind. Let me save the topic of reflective thinking for the next subsection, where it will be dealt with more systematically! For the moment let me confine myself to one example, which will explain why I attach such importance to reflective thought as the motor of mathematical invention -- it is an example I have adduced many times in lectures and papers.

> Why do the diagonals of a parallelogram divide each other in half each other? Of course you can prove they do by applying congruency theorems, but that isn’t the answer I expect to my question, since I am sure you were aware of this property of the parallelogram long before you had any idea of geometry. But *how* did you know it? What was the source of this knowledge? After so many years of mathematical instruction you will not succeed recalling it “except you be converted and become as little children”. But such conversion requires greater effort than most people wish to spend. It is much easier to observe the little ones playing with such toys as sets of mosaic tiles. They know that any parallelogram turned around fits into its own hole, “which proves the theorem”; indeed, by this turn both diagonals are turned around as well while their intersection remains fixed.
>
> Try the experiment! Draw a parallelogram with its diagonals! Have the child discover properties of the figure, and if need be pave the way to the particular property you have in mind the diagonals dividing each other in halves! Ask: “why do you think they do?” “Because I can see it.” “How do you see it?” Insist on an answer! Depending on the age of the child you will eventually get one, which at that age can be considered as a valid proof, obtained by thought-searching, by reflection.

If reflective thought is, indeed, a forceful motor of mathematical invention, it is only natural to put it to good use in such educational design as is based on the principle of learning by reinvention - by guided reinvention, which means that the guide should provoke reflective thinking.

But let us return to the start of this long story, where I asked when and why I switched from the first to the second version of the level theory! My first version is merely a report on the Van Hiele's ideas, although illustrated by examples beyond geometry (to which the Van Hieles had restricted themselves), in particular examples of complete induction. In the Van Hieles' practice I had much earlier identified reinvention as the leading didactical principle, unless my memory deceives me. I certainly knew the part played by reflection in mathematical *invention* and perhaps its level-raising function (at least with regard to my own mathematical activity) and I judged it important enough to transfer it to *reinvention.* This then appears in my second version of the level theory: reflecting on the higher level on one's activities of the lower one.

I always knew that my levels differed from those of the Van Hieles and at many opportunities and I stressed this at many occasions; my levels were relative rather than absolute ones, I said, although I gave Pierre Van Hiele the full credit for the level idea as such. I should confess that never before have I as consciously considered that difference as I am doing now.

This brings me back to a question I have delayed discussing. If you read my quotation from Van Hieles' text[20], you may notice that the last paragraph does not properly link up with the preceding text. This may be a fault of mine since I edited that text on the basis of their publications and other data, but I think there is more to it than this. Discontinuities that reveal themselves by transitory stops in the individual learning processes are certainly more numerous and diversified than can sufficiently be explained by structuring geometry-learning according to three or four levels; these levels are moreover said to be determined by a structure of geometrical thought which, though significant, is however too global. When I hit on reflection as the thing responsible for the jumps in the learning process, the road was prepared for as many discontinuities and levels in a multitude of learning processes as there are significant occurrences of reflection. These, then, rather than three or four levels, were what I began to observe as levels in the learning process. As I am now looking back on my various reports on learning processes, I cannot but state that almost all of them include examples of learning by reflection. I dealt with reflection in [97,98,137] more systematically, the contents of which will somehow be resumed in the next subsection.

I still have to add another remark. Mathematising and reflecting are closely connected to each other. Indeed, I mentioned reflection among the various aspects of mathematising. Of course, routine mathematising does not require reflection. Similarly, there is routine reflection, that lacks the level-raising jump feature. Moreover, Treffers' distinction between horizontal and vertical mathematising should not be confused with the relative height of level. Horizontal mathematising may just as often mean a jump from reality to fresh mathematics and vertical mathematising a mere routine, as well as vice versa.

When I asked the question of whether this whole story is worth that much soul-searching, I did so in such a way that the reader could have anticipated my answer to be: No! Reporting on one's own learning processes (in the present context, with regard to educational research) is likely to elicit the reproach of overestimating one's own importance; though it requires some courage, this is worth being felt as a challenge. Self-effacement is good style in scientific reports. As I have stressed often enough, keeping silent about the process of inventing may mean a loss to those who are expected to learn by reinvention. This is even more true in the social sciences than it is in the mathematical and natural sciences. The lack of hard and objective evidence can at least partially be made up for by knowledge of the soft and subjective experiences and considerations that were crucial in the development of some theory; thanks only to this frankness can it become accessible to understanding and criticism.

By no means can I assert that I have been as faithful to the policy of disclosing my own learning processes (in which I firmly believe) as I have asked others to be. In the course of the years, however, thanks to my growing insight in the value of reflection, I have mended my ways. I could have shortened this subsection by keeping silent about the fact that only during this historical search did I discover I had introduced reflection as the bond connecting one level to the next as early as 1961. While preparing the next subsection, I met yet another explanation of the divergence between the first and the second version of the learning level theory. It would be unfair to conceal this fact, but let me save it for the next subsection.

Meanwhile let me add another remark, which I could not put at the place where it belonged without interrupting the historical report, that is, after the first quotation in the present subsection: The Van Hieles' discovery is a striking example of a didactical insight obtained through horizontal followed by vertical didactising.

2.3.5 *Reflection*

1. *Reflection discovered*

My memory includes much of what I would now call reflection. The least insignificant discovery I ever made in mathematics -- more than half a century ago -- was the work of probably less than half an hour, including theorems and sketches of proofs. Yet it took me at least a fortnight to fill the enormous gaps between the apparent mile-stones along the hazy road of reasoning -- soul-searching work that consisted in asking myself why I believed in the truth of one or another particular statement. To be sure, at that time I didn't use the word reflection. Von Neumann once said: "A fortnight before you prove something, you have to know it is true, since verifying is easier than proving proper." Maybe it stands for the same feeling as I -- and certainly many other mathematicians -- have experienced many times in the course of their mathematical life.

I don't remember when I first used the verb "to reflect" for such experiences of mine or others. If I may believe a conjecture uttered in [98], I even shed away from it for a long time because it seemed too colourless a term for a mental activity that I could describe more substantially. Indeed, more often than not people, simply use it synonymously with "to think". In [98] I even suggested that the divergence between the first and the second version of the learning level theory was due to my aversion of the term "reflection". The story I dug up in the preceding subsection was different but, as a matter of fact, not until the sixties can I find any instance of "reflection" in my papers and, as far as I see, up to the late seventies it occurs only in the context of the level theory.

My paper [98] starts as follows:

> About a year ago I suddenly realised that I may have overlooked precious evidence in the past when observing children. Did I look consciously enough for symptoms of reflective thinking and of the growth of reflective thinking? – this was my problem. The idea of a possible shortcoming occurred to me while I was preparing a paper for a conference of which the theme was "Proof in Mathematical Instruction".
> I had set out to consider the activity of proving, not as an isolated activity with roots of its own, but as the final stage of activities that develop naturally and might be developed didactically. The stage previous to proving I imagined was constructing which, when it happens mentally, means implicitly proving the truth of existence. But if constructing was a stage previous to proving, there must be an intermediate stage, which, I decided, was reflecting.

The paper mentioned at the end of the first paragraph was [97]. As a matter of fact, the theme of that conference was: *Beweisen im Mathematikunterricht.* Anyway, I postulated that one should teach *proving* rather than *proofs,* as is the habit in traditional instruction; my examples were taken from geometry, which explains why I considered *constructing* as *proving existence* -- as a matter of fact, this still applies outside geometry, provided "constructing" is broadly enough understood. In no other field of reinventive learning does reflection play as conspicuous a part as it does in geometry. That what is in the eye of the beholder is transferred into the mind of the thinker, and this happens by reflection: Insist on asking yourself -- or others, as I did above -- why you see it like this, why you think like that.

My notes include so many observations on reflection that I now believe my fear at the time that I could have overlooked precious evidence on learning by reflection was ill-founded. Indeed, the first note of this kind dates as far back as July 12, 1943:

> It concerns three brothers I will call Yed (7;7), Matthew (5;10), and Tom (3;10). John and Matthew attend different schools. Yed tells his father that, while playing on the street with Matthew, he was greeted by Matthew's teacher "Hi, Yed!" "How did she know my name?", Yed asks his father. "What do you think yourself?", Father answers. "Maybe Matthew once told her about his brothers", Yed explained and immediately continued "How did she know I was Yed (rather than Tom)?" "What do you think yourself?" "She saw that I was older than Matthew, instead of younger, as Tom would be", said Yed answering his second question himself.
> My second relevant entry is a sentence taken from a technical exposition about fishing by aforesaid Matthew (at that time 6;9;15):
> Because then the fish think that the men are mistaken thinking they are dead.

With a view on the evidence now available, I would describe my state of mind at the time I set out to prepare my paper [97] as reflecting more consciously than ever before on the idea of reflecting, and more precisely, on how reflective thinking develops. At that time[21], while wrestling with the problem of how proving arises, I felt for the first time the need for a more compact term for thinking about one's own (and other people's) thinking, and when I hit on "reflection", I realised it was not as colourless as I had thought before. If I now use the term, there is an undertone of physics in it: reflecting as mirrors do, mirroring.

> I don't know whether philosophers or physicists were the first to pick up this term. In any case Latin *speculum* for mirror was not useful since *speculatio* means exploring; both, however, do have the common root specus, which means eye socket.

Indeed, when I use the word "reflection", I mean mirroring oneself in someone else in order to look through his skin, to explore him, to take him in. And, consequently, since somebody else is like oneself --a human -- this is an experience about human behaviour and, finally, knowledge about one's own behaviour. So from mirroring oneself in someone else follows -- as the night the day -- the mirroring of oneself in one's own person, that is, introspection. It becomes reflecting on oneself, on what one did, felt, imagined, thought, on what one is doing, feeling, imagining, thinking. Reflecting, once started, is an activity we perform every moment, in order to determine our course of action, yet, as a mental exercise, it can become an aim itself.

2. *How does reflection arise?*

At the age of three, children start asking "why?". It can be meant causally or purposively even though the adult may interpret it in a logical sense. The *child's* "why?" may ask for arguments. If however, *you* wish the child to argue something, you had better ask, "how do you know?" Perhaps authorities will be cited, parents or friends. Sometimes, in the cognitive field, the answer may be "I just know it", or, if it is a representation ''I see it". May we find fault with the child for such a short answer? As adults, do we always know why we know something or haven't we exercised reflection and introspection for a long time?

> I coupled the question of what is reflective thinking to that of its origins. In fact we know little about them. Lack of means of expression is one reason and the other – more important – is lack of attention to *how* we experience *what* we experience. Or rather than lack of attention shouldn't I say: inability to pay attention? Nevertheless I ventured to assert that reflecting in one's own mind is triggered by reflecting in another's mind.
>
> I feel that this view is fortified by Piaget's research on the origin of imitation[22], which is based on observations of his own rather than of others. Imitation starts reflectively, or as Piaget says, circularly: initially the child imitates only those sounds and motions of the adult that are more or less faithful reproductions of sounds and motions produced by his own, perhaps haphazard, activity. (The importance of this discovery is unfortunately obscured by such arbitrary terminology as "assimilation and accommodation", which has been overextended in its application.) Imitation starts reflectively, that is not as self-imitation but the mirroring of one's own behaviour in someone else's. What about other utterances, intentions, thoughts? For instance, experiencing intention as such and making it conscious for oneself — when does it start, before or after intention as such is ascribed to others? Noticing unexpected acts or unusual be-

> haviour may or may not lead to questions like "why did you do it?', "did you rightly do so?", or "would other people do the same in this situation?" Is it a long step from here to asking oneself the same question? And, even then, what is asked for may be a reason rather than an intention.

There is one argument why reflective behaviour in general should start with mirroring someone else's mind. The argument is language, or more generally, communication. The child learns to say "I feel, I want, I think" from other people. Well, before saying it the child must interpret it. How does it succeed? The analogy with what happens in imitation would indicate that the child succeeds interpreting "I feel, I want, I think" as his own action because he is prepared to, because somehow he "knows" what it means. But even then, the outside stimulus is needed to make it explicit.

Let us shelve these questions! We simply know too little to be able to answer them. Learning how to observe children is a prerequisite to more knowledge. The little I know are (translated) quotations from diaries.

3. *Modes of reflection -- shifting standpoints*

Let us turn to more easily accessible questions! Reflection unfolds itself under many aspects. One of them is what I would call shifting one's standpoint -- shifting mentally though the standpoint itself may be physical or mental, while the shifting may take place in space, time, or any other, say, mental dimension.

With respect to the "points" considered let us distinguish a few possibilities of shifting -- without any pretence of being exhaustive. One is

shifting from A to B in order to look back at A,

which I call

reciprocal shifting.

Another is

shifting from A to B while considering C,

which I call

directed shifting.

And finally

shifting A's environment to B's,

which I call

parallel shifting.

The most concrete realisation of reciprocal shifting is looking into a mirror in order to know how one appears to others. Another example is a reciprocal shift in time, concretised by a shift to an older or younger person: when I grow older, I will be able to climb higher than now; when I was younger I could not yet do what I can now. The mental reservation is also a reciprocal shift of standpoint: after I receive supplementary information I shall judge the present situation better.

An example of directed shifts of standpoint, and then in succession, is describing a path. Well, the easiest way to do this is by pointing.

> We took our walk. Bastiaan (2;8) found the top of a bike-bell. When he dropped it, it made a sound. So he dropped it repeatedly. He threw it, he kicked it. I found it danger-

ous. “Let us go to New Clarenburg!” (a building with a broad terrace), I said and put the bell in my pocket. At New Clarenburg he asked for the bell. I warned him that it could fall down from the (one metre high) terrace, which indeed eventually happened. Bastiaan looked down at the bell and then said “get it, down the stairs” while pointing the way with his hand. This took place at a distance of about 10 metres from the stairs. Not only did he make a plan of how to recover the bell, but he also disclosed it verbally.

Describing a path is an example of a directed shift of standpoint or of a sequence of such shifts in succession. As I said, the easiest way to do this is by showing, as Bastiaan did to recover the lost bell, albeit complemented by the verbal explanation “down the stairs”.

Monica (4;4) wants to define some place by means of the walk leading to that point. Her forefinger raised yet motionless, she exclaims: “and then you go so, and then you go like this, and then you go like this, . . .”. Her wide eyes betray her intuitive vision: she clearly sees the path in her imagination but she both lacks the mimic and verbal resources to describe it. Bastiaan (6;4) in a similar situation shows with his hand which direction to follow “after crossing the sluice gates”.

Describing or explaining a real or imagined situation may require mentally shifting one’s standpoint.

We passed a spot by car where an accident had happened there was a damaged car, and across the road something I could not identify, a tube or a pipe. Bastiaan (4;2) said something like: “It was a streetlamp, and I could see that because I am sitting on this side and you couldn’t because you are sitting on that side.”

Bastiaan shifted as it were to my seat in order to explain why I couldn’t identify the fallen streetlamp. But, in general, one may ask whether shifting one’s standpoint is facilitated by assuming another standpoint or may in fact sometimes be blocked by this. In order to draw a child’s attention to some object, one may point to it or turn the child’s head in the desired direction, or move or lift the child to overcome some obstruction. The child is well acquainted with these procedures -- it is kind of baby language -- which may stall its learning what is relevant in particular cases and developing more sophisticated means.

When the sun had just set, Monica (4;8) asserted she saw Venus near the crescent moon; probably she was right, whereas my eyes’ acuity failed. So she wanted me to sit on my heels in order to see the planet from her – lower – viewpoint.
Her sister Daphne has always resisted having her body or her head moved as a device for showing her an object, and she refuses to be helped in almost all physical activities.

Asking preliminary questions before tackling the main question may also be an example of directed shifting of one’s standpoint. Before interviewing someone on a certain subject, one tries to find out where he is standing, and in order to do so, one changes one’s own standpoint. In the long run, this may be the most striking feature of reflective thinking.
Reflecting oneself in someone else may be facilitated by family relations. The generation shift is an outstanding example of what I called parallel shifting.

In Bastiaan’s case it started very early with the relation between family levels. At a

> very young age he mirrored his relation to his younger sister Monica in the relation between his parents; (this was reinforced by the relation between his grandparents, who lived nearby). When they grew up, he said, both of them would live in a certain now vacant house nearby and would have a little Bastiaan of their own. From the intra-generation relation the stress was soon shifted to the inter-generation relation: when his mother was a child, Grandpa was her father and took walks with her, and when Grandpa was a child, he had also a grandpa. Quite early, Bastiaan structured the past, in particular the technical development, by means of the generation shift: "what did cars look like when you (grandpa) were as old as I am now?", "was there T.V. when Mum was my age?', and many more and more complicated questions.

Of course in general, young children are not in Bastiaan's situation to perform such clear double generation shifts. It is perhaps this situation that explains his early differentiation of the past. As a contrast let me cite a teaching experiment with third-graders, the vast majority of whom were unaware of any differentiation of the past until they were taught it (which also happened by means of generation shifts)!

4.*Conclusion*

I could continue in this way for many more pages, although when rereading what I had written I asked myself repeatedly whether I should not spare the reader these stories. Indeed,. most of what I have recounted here about reflection does not properly belong in a chapter titled "Didactical Principles". Or does it?
I finally decided to include these stories (albeit shortened). I did so because of the concluding paragraph of [98] (from which these examples have been extracted):

Reflective learning at school

> Unfortunately, this section, which should be the most important, is almost empty. Observations of other children (except for Bastiaan, then 9;2) would not fit into the present frame. I never observed Bastiaan at school, nor did he grant me more than a few glances into the growth of his scholastic achievements. At that age he fully understands the meaning of the arithmetical operations, he knows how to perform them, though his performance is not yet flawless. He accepts all he learns without any criticism, and in scholastic matters he never asks the question "why?', though he is still asking it in any other matter. He is not interested in questions like "why does 3 x 4 = 4 x 3?", let alone in reasoning about spelling.
> However, I think this is a transitional stage. School is a phenomenon he has still to learn to cope with.

To what degree he eventually succeeded I am unable to judge, although later on I noted splendid examples of reflection in his diary [124, 137,140,141,142,143, 145, 148, 160, 166, 184, 185], most of them related to mathematical experience though none of them to school mathematics, which, as a matter of fact, he managed to master as indifferently as easily. This was thanks to the character of the curriculum and instruction, which matched his algorithmic talent on the one hand and on the other lacked any bonds with reality, and moreover any challenge to reflect on his own activities.
This is not exceptional. Little attention is paid to reflection in mathematics in-

struction, although this can hardly be blamed on the teachers alone, since not much more attention is paid to it in research, development, and teacher training. Let me characterise traditional mathematics instruction by copying a sentence from a few pages ago:

> In order to draw a child's attention to some object, one may point to it or turn the child's head in the desired direction, or move or lift the child to overcome some obstruction. The child is well acquainted with these procedures – it is kind of baby language – which may stall its learning what is relevant in particular cases and develop ing more sophisticated means.

2.3.6 *Reflection and observation*

The present section started with my demand to *observe* learning processes. *Jumps,* I claimed, are the most striking 'observables' -- blockades are perhaps just as easy to notice but less easy to appreciate and explain --jumps to higher *levels.* Look for the level-raising *reflection,* concealed or manifested in the jumps! The reflecting mirror is the *observer's* most powerful tool, which closes the circle. Let me quote myself:

> So from the mirroring oneself at someone else it follows as on the night the day, mirroring oneself at oneself, the introspection. It becomes reflecting on oneself, on what one did, felt, imagined, thought, on what one is doing, feeling, imagining, thinking. Reflecting, once started, is an activity we perform every moment, in order to determine our course of action, yet, as a mental exercise, it can become an aim itself.

Observation is the converse: mirroring the observed one in oneself. Or are things more intricate? Who is the actor, who the observer in a learning process? Look for the interaction in a learning group, while involved in solving a problem! Whether involved or not, it is easier to observe others than oneself. Does the other person understand you, does he approve or disapprove? Does he feel what you feel, can he help you better than you can help yourself to understand what you mean? What can a third person listening to the exchange of vague arguments learn from and contribute to it? Why do I not understand you, why don't you understand me, why do these persons not understand each other? Is it lack of information, or what information does one of them possess and consciously or unconsciously conceal, which prevents the other from understanding? Or is the other feigning that he does not understand? How can we know what to tell and what is too trivial to be told?

> We are driving by a shop belonging to the Jamin chain. The name "Jamin" appears twice in Neon letters on the front of the building, horizontally and vertically, but separated. In the horizontal version the letter M is missing. I drew Bastiaan's (8;2) attention to it. He said This is only possible with [an] odd [number of letters]. I didn't understand what he meant. He became angry: "You know very well what I mean." I told him, as I had often done in the past, that I never lost my patience if he didn't understand me. (He should do likewise and think about the reason why I didn't understand.) Finally he said "If they cross each other." It was now clear what he meant. The

number of letters must be odd if the words are to be put together to form to form a regular cross. – Let me add that at that period the ideas of even and odd strongly occupied his mind.

2.4 LONG-TERM LEARNING PROCESSES

The present title was announced in (2.3) as being my actual aim. Again *long - term learning processes* means: long term teaching/learning processes. How long? Life long? Extending beyond the limits of school life? These questions are not to the point. The limits, whether reached or not, are set by the teaching goals as viewed by the designer. Well, there is no absolute limit: each goal, as definite as it may look, can be outdone by more ambitious ones, which, however, are meaningful only from the viewpoint of those previously viewed as definite.
Textbook series are examples of *long term planning* of teaching/learning processes. I mean another thing: *planning long term* teaching/learning processes. Later on the difference will become clear.

2.4.1 *Learning to forget*

An ambiguous heading: learning *how* and *what,* or *in order* to forget? Both of them are worth discussion. As every teacher knows, breaks in learning *processes* may cause total losses of learning products. Starting a new school year often amounts to starting with a clean slate. In my country successful final examinations are celebrated by hanging satchels at the flagstaffs. Indeed, after the examination, most of the stuff learned may be forgotten never to be remembered, and this extends even beyond the narrow limits of what was taught with the sole aim of the examination. It is too old a story to be told again: people who, though failures in school mathematics, succeeded in life, or so they claim. Or even worse: people who assure you that, thanks to a mathematics they never used explicitly, they learned a lot of valuable things (especially logical thinking) and, whether asked for or not, substantiate this assertion by examples that are as many proofs to the contrary.
No doubt, in every respect, discarding is no less important an issue than keeping. Beyond a multitude of subjects and activities, one has to learn which ones are worth forgetting and which ones remembering. Much depends on the learner, on individual inclinations and aversions, that foreshadow the learner's future life. This does not absolve teachers and educational developers from the task of steering and designing the process of forgetting as well as that of recalling. Mathematics can again be distinguished from other areas of learning by this fact itself and by the way this task is to be tackled.
We viewed mathematics as an activity, and learning mathematics as guided reinvention. In contrast with traditional instruction, the stress was shifted from instructional *products* to instructional *processes.* This shift cannot but influence the

line to be drawn between potential remembering and forgetting. Anyway, if learning processes are so important that they are specified by guided reinvention, it is quite improbable that their ultimate destination should be obliteration by their products, although this feature does seem characteristic of mathematics. I am going to explain why this is a serious misapprehension, caused by a wrong perspective.

Let us take an example from geometry! A famous theorem, by folklore ascribed to Pythagoras, is learned by the great majority of young people, and the fact that it is famous proves that it is even remembered by a great many adults, or so it seems. Many teachers and even more textbook authors feel obliged to provide learners with some proof, though it is an undeniable fact that the theorem is successfully applied innumerable times by people who never learned proving it or forgot the proof they learned. To be honest, even in the highest regions of mathematics, people often apply results obtained by others without checking the proofs or Caring at all how to prove them.

So why learn proofs if afterwards they may be forgotten? Once one of my colleagues was asked by a freshman: "Excuse me, Professor, isn't it a curious habit of yours that in your course you prove each of your statements? Why do you do so? Are you afraid your students don't trust you?" It is a question that can be answered in manifold ways. Mathematics is different, I claimed. It is a remarkable fact that, in mathematics, each statement can be corroborated by aproof, although it is not the fact that matters here where we are concerned with *teaching and learning* mathematics, rather than with mathematics as a *deductive system.* As we noted several times, the relation in mathematics between *theorems and proofs* is not akin to that between *products and processes* in mathematics instruction. What then can be the didactical value of mathematical proofs? The answer depends on how a particular proof is expected to function in a come as a whole. A proof can be held up as an example to what proving properly means, and let us hope that as such it is paradigmatical enough to convince the learner! Or a proof can be an opportunity to teach a number of isolated facts or activities in an integrated way, that is by *logical* linking, although the course of *reinvention* itself may be as forceful a linkage, or even more so. And finally, by analysing its proof, the teacher may discover how to guide the learner to reinvent the theorem it is to prove. Proofs refashioned for this aim are valuable in themselves as sources of insight. On the other hand, proofs with the sole aim of being forgotten after having produced the theorem they are to confirm, are didactically worthless.

This, then, is the *intrinsic* didactic fallacy of traditional teaching of mathematics, as well as that of its all time high in New Math: the large extent of subject matter learned solely in order to be forgotten, aggravated by the eventual criterion of the examination. In addition to this there is the *extrinsic* problem of what school means for life: How to adapt the relation between remembering and forgetting to the diversity of demands of future lives?

2.4.2 *Remembering learning processes*

"If learning processes are so important that they are specified by guided reinvention," I said, "it is quite improbable that their ultimate destination should be obliteration by their products". Learning processes have a value of their own, which entitles them to be remembered. Not in detail., of course, as there are very few things we remember in detail, if any at all. Rather than details, we record essentials, or what we take to be such. Anyway, that is all we can recall and, if need be, put to good use.

Learning processes, or at least part of them, can be more essential than their products. As far as they fulfil this condition, and in the way they doe this, they should remain accessible to memory. Not in detail, to be sure, but to the extent and in the fashion that they are essential. Few people, if any, remember how they learned arithmetic. This is a good thing if they learned it thanks or in spite of bad teaching. Fortunately there are people who can reconstruct the learning process as it would have taken place if they had been allowed to reinvent arithmetic. It is those people to which the old adage, cited in (2.1.4) applies: first learn, then understand. And the others? They are better served by guided reinventive learning, duly recorded.

Written records, even the most circumstantial ones, are far from being complete, but a good record allows others and oneself to repeat the process recorded, maybe in a more streamlined way than it actually took place. Mental records are no different. Mental records are due to reflection. This, then, is what makes processes of learning by guided reinvention accessible to memory: built-in opportunities to reflect, which can be reinforced by verbalisation and by communicative interaction with the guide and between the guided ones.

Mathematising is to a great degree liable to obliteration by its result, that is, ready-made mathematics. Once schemes and forms have been consolidated, once concepts have been attained, once short-cuts have been performed, the stages of schematising, formalising, constituting mental objects and short-cutting may be dismissed as dispensable nuisances. Algorithms are an utterly extreme case. Once mastered, or believed to have been mastered, they are most likely to disavow their origin. Indeed, it is the great virtue of algorithms that they can be performed mechanically, as it is their drawback that they become useless or even dangerous as goals in themselves: mathematics identified with performing algorithms. It is algorithms that created what looks like the fundamental antinomy within didactics of mathematics: insight versus drill.

2.4.3 *Insight*[23]

A cherished antinomy in teaching and learning mathematics is putting on one side of a deep gorge such noble ideas as

- insight, understanding, thinking,

and on the other side such base things as

- rote, routine, drill, memorising, algorithms.

If I were malicious, I would add another pair of opposites

- theory versus practice,

suggesting that learning by insight is a noble theory while base practice is learning by rote and memorisation. However, it is not that simple, and it has never been so. Even in our computerised age, children memorise tables of addition and multiplication and acquire certain skills by rote, though one might argue that the balance has shifted in favour of the nobler activities due to the rise of the computer.

It is not that simple, firstly because the question is not which side of the gorge to choose but rather to bridge it by the learning process that I called schematising and formalising. Secondly, I do believe that, at any time more mathematics has been *taught* from the viewpoint of insight and more has been *learned* by insight than we are aware of -- indeed, common sense is insight. Everyone agrees and textbook writers have witnessed that elementary arithmetic cannot be *learned* in any other way than by insight, whether it is *taught* that way or not. But it is also true that, as things go on, as teaching proceeds to ever higher grades, to column addition and multiplication, to long division, to fractions (ordinary and decimal), to algebra, to learning mathematical language -- the part played by insight changes. The learner's insight tends to be superseded by the teacher's, the textbook writer's, and finally by that of the adult mathematician. And the same tends to occur on the long winding road which leads from concretely understood word problems to highly formalised and badly understood applied mathematics.

This is why people who advocate learning through insight, disagree about what insight is. New Math's wrong perspective was to replace the learner's insight with the adult mathematician's.

Yet this is not my main point. I have still to explain why we are not aware of how much is learned by insight nevertheless. It is quite natural that, once an idea has been learned, the learner forgets about the learning process, once the goal has been reached, the trail is blotted out. Skills acquired by insight are exercised and perfected by training, intentional and unintentional. This is a good thing. What is bad, is

- sources of insight clogged by acquired routines, never to be reopened,

though that is what usually happens. It explains why upper grade teachers so often complain about teaching habits in lower grades. If it is restricted to the first acquisition of some idea, learning by insight does not deserve this name.

What is crucial, is

- retention of insight,

which is gravely endangered by drill, that is

premature training,
too much training,
training as such.

The problem is

- how to keep the sources of insight open during the training process,
- how to stimulate retention of insight, in particular in the process of schematising and formalising.

How can this goal be pursued? The solution I have proposed is

- having the learner reflect on his learning process.

As I explained earlier, mathematics is to a great degree reflection on one's own and other people's physical, mental and mathematical activity. The origin of proving theorems is arguing things that are apparently obvious. Nobody tries to prove a thing unless he knows it is true. This he knows by intuition, and the way to prove it is, as I claimed, to reflect on one's intuitions. Successful learning processes, if observed, should be made conscious to the learner in order to be reinforced and in order to be recalled if needed. This, however, is not what usually happens. Let me illustrate this by an example I have cited many times.

> Many children and adults can tell you that in order to multiply by 100 "you have to add two zeros" (which is only true for whole numbers) yet most of them cannot explain why. Even worse: most of them don't even understand that the matter can be argued and why this should be done. Did they learn such rules by rote? I don't believe so. I have observed too many children applying such rules intuitively before they were verbalised and formally taught at school. Rather than being taught the rules, they should have been taught to argue their intuitions, to reflect on what appears to be obvious. But this requires more patience than teachers can afford. Indeed, compare this with my story of the girl who was taught the rules on fractions after she had acquired a good working knowledge of fractions. (See 2.1.4,3.)

2.4.3.1 *Testing insight*

When, in 2.1.4,3, I discussed the jumps in the learning process of the girl I had been observing for a few years, I promised to provide an example to show how I tested her insight, that is, to see whether its sources had not been clogged during a period of training.

She had come to the point that she solved equations like

$$\frac{1}{2}-\frac{1}{3}x=\frac{1}{2}x-\frac{1}{3}$$

according to the rules of the art, automatically as it were. I had a strange feeling: the better she performed, the more urgently I asked myself whether she still understood anything. Finally I was terrified by her excellent performance. On the other hand, as explained earlier, I didn't dare to have her verbalise her behaviour. Then I got an inspiration. I drew a horizontal numberline, that is, without numbers, having only an origin and equidistant marks. I took an interval between thumb and forefinger, around the origin yet not symmetrical, and asked her: "If x is here in between, show me where *2x* is?" She did it with her thumb and forefinger. "Where is $\frac{1}{2}x$?" "Where is $x + 2, x - 2$?" "Where is *-x?"* This was the first hitch but it was soon conquered. Then I asked the converses. "If *2x* is here in between, where isx? If $\frac{1}{2}x$ is here in between, where is *x?* After a series of questions

like these I gave her written problems. She already knew the notation $-3 < x < 2$ for *"x* between -3 and 2". Step by step I led her to sums like
if $-5 < \frac{2}{3}x - 1 < 6$

what can you tell about *x?* Or
if $\frac{1}{2} - \frac{1}{3}x < \frac{1}{2}x - \frac{1}{3}$

what can you tell about *x?*
So I became convinced that her insight had been preserved in spite of training.
As an afterthought, the above seems to be the didactically most promising means of formalising the mental object of variable. (Cp. [119a].) It is a pity that I didn't mention it in [146].

2.4.4 *Training*

Advocates of insightful learning are often accused of being soft on training. Rather than against training, my objection to drill is that it endangers retention of insight. There is, however, a way of training -- including memorisation -- where every little step adds something to the treasure of insight: training integrated with insightful learning. If I compare our mathematical textbooks -- from the lowest to the highest grades -- I cannot but notice a historical trend toward this kind of integration, In the past there was a clean separation between so-called theory (often nothing more than a model) and its so-called applications (or rather, imitations of the model), which were as numerous as the theory was meagre; and the closer one gets to the present day the more deliberately a more vigorous theory is broken up into insightful steps, each of which may contain both a training and a reinventive aspect. Big problems, posed in contexts, can serve large scale training better than a flood of small applications. In other words, textbook authors have shifted from extensive to intensive training. For teachers who learned mathematics the old way and accuse the new style of being chaotic, it is difficult to use such textbooks adequately. Students, however, adapt to them more easily.
At the primary level there are clear indications[g] that reinventive learning -- learning column arithmetic by progressive schematisation, for instance -- is far less time-consuming than extensive training. This even extends to pure memorising, say of multiplication tables[5], where children are allowed to use their own, so-called informal, methods. At the secondary level, comparisons are not so easily made because of the interweaving of subject matter and teaching style. A more fundamental divergence is, indeed, whether, for instance, exponential functions are introduced as a formal subject or as the result of exponential growth in realistic contexts. The most intimate long-term integration of reinventive learning and intensive training with which I am familiar, is L.Streefland's research[6] on teaching fractions in grades 4 through 6; this research, moreover, includes 13 global learning histories. Streefland's principal instrument was sketched briefly in

(2.1.2, 6) (dividing a number of pizzas among a number of children sitting at a number of tables, where the symbolic recording proceeds from a suggestive pictorial notation towards the standard one by means of formal fractions) as shown in fig. 13-15 from 6, p.203,269, 302. The learning process is an alternation between what Streefland calls the children's own *constructions* and *productions,* which may roughly be distinguished by catchwords like: *open end* and *open start.*

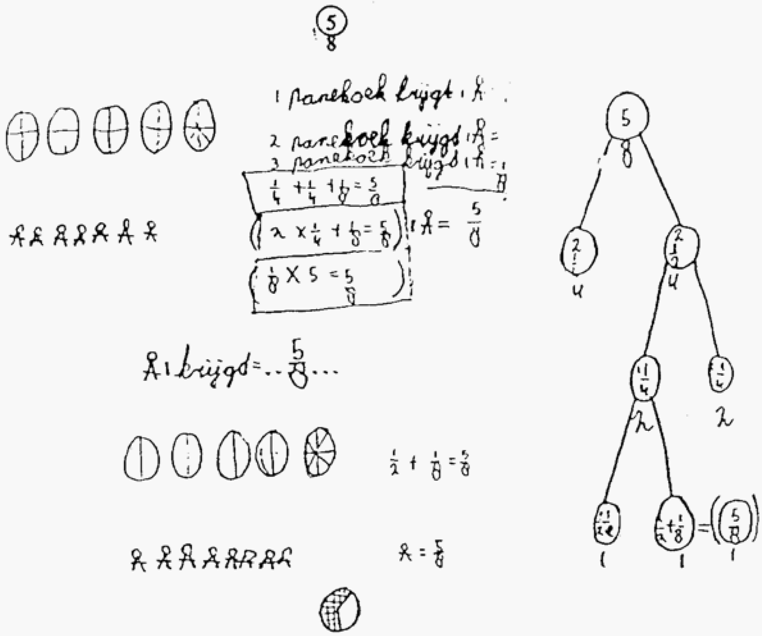
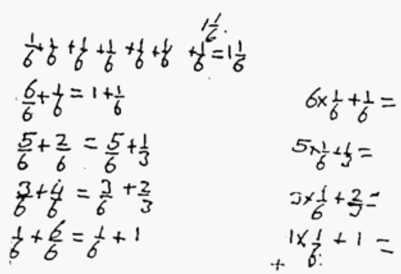


Fig. 13



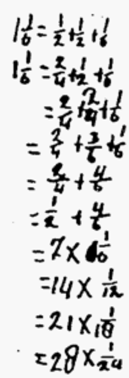


Fig. 14 Fig. 15

2.4.5 *Organising the learning process*

1.*Levels*

Why did we pay so much attention to the level structure in learning processes? Let me quote myself from (1.1.2):

> Sure, some, algorithmically gifted people, learn to apply even imposed algorithms adequately; others – perhaps the majority – fail to identify the new algorithmic procedures with the common-sensical ones from which they should have originated through abridging and streamlining. They fail because some time in the past they were asked to take mental leaps which exceeded their mental powers. Even though they flawlessly learned the algorithm, they will fail to use it in true life situations where common sense counts; they will instead depend on less efficient lower level operations. Researchers have signalled this "relapse" and marvelled on it. Rarely, however, has it been diagnosed as a consequence of instruction, since no alternative instruction was envisaged alongside the imposition of the new algorithm (which was sometimes embroidered with explanations for conscience's sake). The new algorithm, however, never did have the opportunity to reach the state of common sense; afraid of applying a wrong algorithm or the right algorithm wrongly, the learner instead relies on what has remained common sense to his mind.

What I stated there with regard to imposed algorithms applies more widely, I daresay, to almost all imposed mathematics,. Indeed, the fact that researchers signal and marvel at such relapses is shared by a variety of fields. Whatever levels may mean theoretically, it is the duty of the organisers of learning processes to respect them. The fact that a few -- gifted people -- can get along on levels to which they have been lifted inconsiderately, does not justify ignoring levels in a teaching strategy. The question of how jumps in the learning process can be diagnosed, has already been answered. Teachers (or peers in learning groups) need to insist on the justication of what appears to be new knowledge or new procedures, thereby requiring the inventor to reflect on what he -- consciously or unconsciously -- performed. This need not necessarily be done by verbalising, which anyway should not be enforced; a paradigmatical status, witnessed by repetitions, is enough to prove the jump in the learning process.
If levels are to be respected, than learning processes must be organised flexibly enough to be adapted to the needs of learners or learning groups. This demand for flexibility also applies to the goals of the learning processes.

2. *Differentiation*

How fast and how far learning proceeds, depends greatly on the learner and on which way and whether the goals staked out are eventually attained. Reinventive learning cannot possibly be planned rigidly, but lack of planning may be even worse. In 2.1.2 both didactical extremes were presented within one example: the process of memorising the multiplication tables. If pupils are allowed their own methods and stimulated to reflect and, if possible, to verbalise the most appropriate ones, a widely branched network of possible learning roads is then opened to them. It becomes even wider in the case of learning the algorithms of column arithmetic. The standard algorithms are *theoretically* much less compelling than is usually assumed; there are variants, indeed, which don't commit teachers and learners beyond their didactical function as guidelines. Comparative research on such variants, if viewed didactically, has never been more than a hobby, and is unlikely to survive in the computer age. The advantage of reinventive learning, and in the present case of progressive algorithmising, is that individual learners or learning groups may advance at their own pace and eventually reach the best of the goals accessible to them.

In general, this means organising instruction so that, rather than it being differentiated in advance, the learners differentiate it themselves, and do so on levels as high as are accessible to them: spontaneous versus imposed differentiation. In the instructional design, branches are preferred to dead ends and branches that may lead back to the main stream, to branches that lead nowhere. If we may use metaphors I prefer that of the jump, if necessary produced by change of perspective, to the not unusual one of the sharp turn, where no perspective at all is involved.

3. *Prospective and retrospective learning*

It is a well-known experiential fact that column arithmetic may function both as a *motivation* and an *opportunity* to improve one's knowledge of addition and multiplication tables. This experience will be viewed in a broader didactical context, as is indicated by the above heading.

In former times, mental addition, subtraction, multiplication, and division were taught one after the other and separately; arithmetic teaching was divided into stages "up to 10, "up to 100", "up to 1000", and so on; fractions, measures, proportions were dealt with as separate subjects. Things have since changed and will change even more in the future. As everybody can observe, children who have just mastered additive arithmetic up to 10, are proud to tell you that 10 + 10 = 20, or even that 20 + 20 = 40, that 3 x 3 = 9, and to recite the multiplication table of 1, of 10, even perhaps of 5. In their informal arithmetic, doubling and halving often precede systematic learning of the multiplication tables, with as next best, squares which occupy a particularly cherished place in the tables. Rather than suppressing such predilections by premature systematisation, mathematics instruction should take advantage of them, which is actually happening in more and more textbooks[24].

This is what I call *prospective* learning. (Other terms are "anticipatory learning" or "learning by advance organisers".) There are many more examples. Ratio, for instance, has profound visual roots, which can be arithmetised early on by estimate and measurement. There are many informal opportunities in contexts for common sense ratio in everyday language before it is dealt with more systematically and formally. Long before fractioning the traditional "cake", the clock dial is divided according to halves and quarters of an hour, which, unlike the cake segments, have an existence of their own. Functions have visual and numerical roots in graphs and tables, and describe situations and processes, long before they are focused on as such. In general, deep roots should be preferred to virgin soil, and opportunistic to systematic learning.

I chose the term *prospective* for this organisation of learning as a counterpart to what I call *retrospective* learning, which means recalling old learning matter whenever it is apt to do so and worth being reviewed from a higher stance or in a broader context. Retrospective learning serves dual purpose: it roots the new matter in the old one, and it strengthens the old roots. Learning a new idea is often nothing but becoming more conscious of a complex of previously less conscious pieces of knowledge and abilities and of their interrelatedness. If experienced as such, it is retrospective learning. When the laws concerning the arithmetical operations are formulated, their former occurrences should pass the retrospective review in full. This means including -- besides those of an arithmetical character -- also such geometric experiences as: when pacing a circumference to measure it, it does not matter where I start; when measuring areas of rectangles or volumes of beams, the length, width and height are interchangeable data. Combinatorics is a particularly fertile domain in which single paradigms may rule broad fields of isomorphic problems (cp. [87, chap.IV, 9]), antedating for a long stretch of time their general expression and its verbalisation; but as soon as this goal is reached, the old experiences should be renewed and their paradigmatical character should be understood as a general feature of isomorphism.

It has been a habit of adult mathematicians to review old ideas over and over just as it has long been experienced that this activity leads to ever deeper understanding. Young learners should not be treated differently. Prospective learning should not only be allowed but also stimulated, just as retrospective learning should not only be organised by teaching but also activated as a learning habit.

4. *Intertwining learning strands*

Just as prospective and retrospective learning aims at an integration of past and future learning processes, so does *intertwining learning strands*[25] locally, yet with a view on the involved learning processes as a whole. Rather than running on separate tracks which, except for incidental references and loans, are independent of one another, learning should be organised in strands which are mutually intertwined as early, as long and as strongly as possible. When loose ends are inevitable, they are taken up at the first opportunity where they can be connected to other ones in order to be continued. In a sense, examples of prospective and retrospective learning can also serve as such for intertwining learning strands, or at least for points where intertwining can start in order to be continued more consequently. But let us look for other examples!

Ratio and fractions can go together right from the beginning -- and I do mean the very beginning: visual comparison of *separate* objects is paralleled by comparing the parts of *one* object. Double scales on the numberline and proportionality tables are expressions of the same idea as well as tools that connect ratio and fractions with each other. Conventional measures and decimal fractions are inter-

twined in various ways. Functions, graphs, and equations, which in many textbooks form separate chapters, should be intertwined as learning strands. Thanks to the geometrical-algebraical permanence principle (cp.[146], chap.16]), learning negative numbers, vector algebra and geometry, and linear graphs and functions can be so closely knitted that they appear to be one subject[26]. The intertwining of algebra and geometry must not be restricted to traditional analytic geometry; plane and solid geometry must not be dealt with as separate subjects.
Teachers who prefer systematic instruction are likely to accuse this approach of being chaotic. They forget that systematics is an *a posteriori* contraption. What looks like chaos may be well-organised didactically, while, on the other hand the system as such may be a subject of guided reinvention.

5. *Active learning*
In the first chapter I approached mathematics phenomenologically as an activity, and the overruling principle of the second chapter was to do justice to the first, which I tried to do by advocating guided reinvention. Modern textbooks and other recent literature are a wealthy source, providing examples of mathematics as a learner's activity which can be stimulated and realised by guided reinvention. Rather than quoting some of them and keeping silent about others I preferred to look for a paradigm that displays as many features of the reinventing activity as possible. I didn't find it in the literature, although I can hardly believe that it is indeed as new as it looks to me.

Eeny, meany, miney, moe.
Catch a tiger by the toe.
If he hollers let him go.
Eeny, meany, miney, moe.

Isn't this a way of counting? As a matter of fact, in a few languages with which I am familiar, the term for this game of reciting such a rhyme while moving the forefinger in the round is derived from the word for *counting.* It *is* counting; only the words are different. But how much do words matter in counting? After all, the children playing the game give the numbers different names, depending on their mother tongue. In some languages I know counting rhymes that start with an ordinary number sequence; in German one even goes up to 20 and then continues in normal language, where eventually *Danzig* rhymes with *zwanzig.*
So why not all the way count with genuine numerals? How far? One can agree on a limit. For instance, up to 100. This "why" or "why not" may become a point of discussion -- I mean in the classroom, before or after the lesson or sequence of lessons I have in mind.

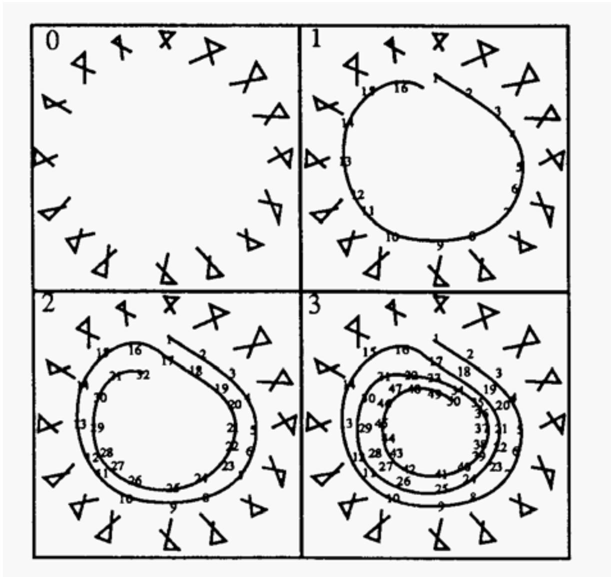


Fig. 16

Let the children be seated in a circle; let them count themselves rather than be counted off by one of them or an outsider but to count themselves. One of them starts with 1, the neighbour says 2, then comes 3, each adding 1 to the number of the predecessor, going on up to. . . No “up to”, because after the last in the circle the starter simply continues counting. One round, a second, a third, . . .
In the drawing in figure 16 I numbered the four stages 0,1,2,3: at the start 0, after the first round 1, after the second 2, and after the third (and a little bit more) 3. (I apologise: I drew it as though I myself had counted off the children, that is, clockwise. I venture a bet that if they counted themselves they would do it the other way round. As in writing things are being done from the left to the right.
Two variants or additional activities: During the counting process a pile of number disks (bingo cards) are passed around, and each child takes one in turn while saying the number out loud. Or each writes his numbers on a sheet of paper.
The sketch shows 16 children. Most classes are larger. But the circle need not include the whole class. On the contrary, it is just to the purpose to start in a small way, preferably in pairs, speaking or writing down in turn the odd and even numbers. After that, groups of 3,4,5, . . .; 10 is particularly nice.
What has this game to do with mathematics? My answer is: all you can make of it. To prove it I will give you a list of questions I would put to individual children or the whole class:

1. If in the present round you have got..., what will you get in the next? And after

two rounds, after three,...? What did you get a round ago, two rounds ago,...? What in the first?
2. What numbers would you have got, if number 1 had started as...? What if number...had started as 1?
3. Who gets number 100, and after how many rounds?
4. What will your first number above 100 be, and after how many rounds? What your last below 100, and after how many rounds?
5. How do your numbers change if counting is inverted?
6. If you did the game with... as many mates, how much faster would they reach 100?

And a more general question:

7. What catches your eye if you look at your set of numbers? Any pattern, any regularity?

It will be clear what I meant when I asserted that this game has as much to do with mathematics as one can make of it. Certainly the four fundamental operations -- the multiplication-tables, and even more striking, the division with remainder stand out in bold relief. What kind of division? *N* numbers divided among *n* children, or *N* numbers arranged in circular groups of *n?* It doesn't matter, and this is one more aspect that does matter.

Remember, the idea was to start with small groups, but in the long run it is didactical wisdom to resume the game again and again, as long as the children enjoy counting out. So it can serve as meaningful *training.* But since even little children can do it with large numbers, as can older children with small ones, it is a marvellous opportunity for *prospective and retrospective learning.* Another striking feature is the tight *intertwining of learning strands.* The learning processes will be *differentiated* according to how the problems are tackled on various *levels,* which can be discovered as such by *reflection* and transmitted by *group cooperation.* Last but not least, it is a *rich context,* the richest one can imagine: a children's game.

As for the guide: an excellent opportunity to *observe learning processes.*

2.4.6 *Acquiring a mathematical attitude*

The original heading of this subsection was: ***Products of learning processes.*** Indeed, learning processes are not aims in themselves. But how to define their products? Hard products, though interminable in high level mathematics and valuable as objectives of mathematical research, are not the motive for teaching and learning mathematics at school, nor is their number significant. The pupils are expected to develop certain abilities. Detached from the learning processes by which they were acquired, these abilities may lead lives of their own as methods and skills, and as such they may be exercised, studied and tested. One can try to list and classify them, but what is the use? It would not answer the question of why they should be products of learning processes. After quite a few false starts I decided to attempt a comprehensive answer under the above heading.

In 1975, in Hungary I happened to take part in a conference *Evolving a Mathematical Attitude in Secondary Education - Age Range 14 -18 Years.* I started my contribution[27] by taking exception to the general theme. I did so for two reasons: First, 14 years is much too late to shape a mathematical attitude, although it is not too late to develop it further, provided one knows on what basis one can continue from 14 to 18 what was achieved at lower age levels. My second, and even more important argument, was that a mathematical attitude is more easily described for lower than higher ages if such descriptions are to be based on observations rather than on logical analysis. Indeed, what matters is the interior rather than the exterior attitude, and the older people grow, the more their interior and exterior lives will diverge from each other; the most extreme instance of this phenomenon is found in publications on mathematical discoveries, where any intellectual or emotional detail of the discovery process is carefully avoided. The only access left to the interior life of adults, is introspection, but this is a hard thing to realise. So, if attitudes are to be described on the basis of observations, one is very likely to find more useful material in the lower age bracket than in the one.

How to describe a mathematical attitude? Such descriptions have often been attempted, though most often in too general terms -- thinking clearly and distinctly, criticism, looking for problems, killing problems. It is the drawback of such descriptions that by the way they are formulated, they do not properly refer to mathematics. The opposite of this procedure can be found in Polya's repeatedly mentioned work, whose tendency, rather than whose details, can be a source of inspiration -- though less of subject matter -- for didacticians of mathematics.

Mathematical attitude can, in my view, be described most efficiently as the mastery of big strategies, and this is the way I will use the term. I will deal with five strategies, while referring to experiences in actual teaching, most of them related to 3-13 year-olds, but viewed as an initial condition for developing attitudes at higher ages.

1. *Developing language* above the ostensive and linguistically relative level, in particular at the level of conventional variables and functional description".

Although linguistic levels are reached by learning processes they should not be confused with the levels in learning processes. Levels in learning processes are there for staying on unless, for some particular reason (for instance as an education developer), one is required to descend. Linguistic levels, on the contrary, are there to for moving on however it works out, that is, at each opportunity on the level that best fits the situation.

Besides the levels, I also distinguish two modes of expression: describing something by means of an action or as a state of affairs. For instance:

To get the mirror image of P with regard to 1, drop the perpendicular from P upon 1 and extend it as much behind 1 as P is in front!

P' is the mirror image of P with regard to 1 if P and P' are at the same distance from 1 at different sides.

I distinguished the following levels:

ostensive language, where pointing with the index finger or pointing mentally may be accompanied by words like “this” and “that”;

relative language, where objects are described by their relation to other objects;

conventional variables, which make relative language function more smoothly;

functional language.

To make things clear let me quote an example from algebra: define the square root!

Ostensive:

$3^2 = 9$ so 3 is the square root of 9,
$5^2 = 25$ so 5 is the square root of 25, and so on.

Relative:

The square root of a number is found by looking for a number that squared reproduces the given number. Or, in another mode:
The square root of a number is the number, the square of which is the given number.
Such clumsy names for variables as are “number” and “given number” are avoided by the use of

Conventional variables:

$x = \sqrt{a}$ if $x^2 = a$.
By means of a new concept the language becomes

Functional:

Taking the square root is the inverse of squaring.

2. *Change of perspective,* a complex field of strategies whose common feature is that the positions of that which is given and that which is sought for (of data and unknowns) in a problem or a field of knowledge are -- partially -- being interchanged; this includes the recognition of wrong changes of perspective”.

3. *Grasping the degree of precision that is adequate to a given problem*[27 28].

4. *Identifying the mathematical structure within a context, if any is allowed, and barring mathematics where it does not apply*[27].
In fact, there are large fields where mathematics is more often illegally than legally applied.

5. *Dealing with one’s own activity as a subject matter of refection in order to reach a higher level.*

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CHAPTER 3

THE LANDSCAPE OF MATHEMATICS EDUCATION

3.1 THEORY OF MATHEMATICS EDUCATION

The heading is wrong but I could contrive no better one. What is wrong is the word "theory" that -- used by so many people on so many occasions with so many different meanings -- needs circumstantial explanations that do not contribute factually much to our subject. I recently read a paper on theories of "theory" where the author claims:

> When a mathematics educator studies the effects of lax and restrictive learning environments on children of different anxiety levels, she presumably has a theory that relates achievement to both anxiety and the structure of the learning environment. Or, when a cognitive psychologist examines classification tasks in the learning of early number concepts, the psychologist most likely has a hunch as to how these tasks are related. Or, when a doctoral candidate designs an experiment in which children are taught several different problem solving heuristics, she presumably has a theory that predicts which of these treatments will be the most effective.

The middle sentence attests a terminology where a "hunch" may have the status of a theory. Indeed, in colloquial language the word "theory" is occasionally used in this way, although under these circumstances most people would prefer the word "hypothesis". However, this is only one extreme of a broad spectrum of meanings. The other extreme (which will be described in what follows) deserves a name as well, and I would not know any name for it if "theory" is used in too liberal a way.

3.1.1 *Theory*

1. *Domains and theories*

People familiar with the terms and with the domains they cover will agree about number theory being a theory, and about distinguishing, if need be, between algebraic and analytic number theory. All the same, they might disagree about some concept or proposition, for instance, whether it falls under analytic number theory or under complex function theory. In fact, depending on the context in which it occurs it may be attributed to both of them. The infinity of the number sequence is neither a theory nor a hypothesis but a mere fact, which nevertheless may be the subject of sophisticated theories. The infinity of even or odd numbers is a fact as well, as is the existence of an infinite number of primes, which -- requiring a non-trivial proof -- has the status of a theorem in number theory.
Neither mathematics nor physics nor chemistry nor biology are theories, al-

though there are a great many -- sometimes competing -- mathematical, physical, chemical, biological theories, that cover various domains and sub-domains of these sciences or deal with their foundations. Mechanics is not a theory but Galilei-Newton mechanics is, as are the theories of elasticity, relativity, and quantum mechanics. Originally, biological evolution was not a theory but a mere tool for organising fossil evidence, which proved successful long before the discovery of evolutionary mechanisms. On the other hand, almost from the onset, astronomic evolution has been a theory.

History is, of course, no theory. Historiography, rather than being a theory, is the technology of historical research. There have been attempts to develop theories on history. "History does not repeat itself" is a theoretical statement on history, as is "History does repeat itself but then as a farce" (Hegel). Although theories on history are more sophisticated than such statements, I doubt whether they have been more successful. Or, in other words, they were topical as long as they did not have the time to outlive themselves.

Language, or any particular language, is of course no theory, but there are a great many theories on various aspects of language and of particular languages, that is, on origins, structures, recording and social functions, interrelatedness, artificiality, and so on.

Economy is no theory, though economics includes a large number of -- competing -- theories. Strangely enough "education" is used in both ways, like "economy" as well as like "economics". Rather than interpreting this twin use as a testimony to integration, I see it as evidence of a shortage of relations between them.

2. *"Theory versus practice"*

The saying "that's mere theory yet practice is different" may mean various things. It may be used if an actual state of affairs does not match the one reported on or desired, or if some tool or recipe does not work the way it was told or expected to. Pure theories are descriptive rather than normative, even though they might have been created under the influence of norms or with a view to establishing norms, and even in spite of a virtual interrelatedness between norm and description. For instance, the all-embracing aristotelian-scholastic theory of *potentia* and actus, which arose from opposing female passivity to male activity, was both the result and the cause of social and sexual norms, although it was formulated to be understood as part of an ontology, that is, as a profound description of the way things are.

The saying may as well be meant to express the more fundamental dissatisfaction about a theory or something like a theory that does not really apply. This can mean that it fails as a description of reality or that there is no reality at all to which it can be tied. In this case, the theoretician is very likely to charge the complaining practitioner with the crime of the so-called confusion of reality and (descriptive) model (see [87]), which is simply a lame excuse. It can also mean that the theory does not provide such *norms* for judging or handling real situations as it seems to promise (or actually promises) under the title of (normative) models (see [87]).

The theoretician might then appeal to the restrictions inherent to the use of models -- an apology that may be accepted provided there is any use at all for the model in question.

3. *Pure versus applied -- descriptive versus normative*

Antinomies can be illuminating. Yet rather than pitting *theory* against *practice,* I would like to pit the desire *for cognition and understanding* against that for *action and change,* pitting *pure* against *applied* research, *descriptive* against *normative* models. To be sure, the interaction between them may be more frequent and fruitful than their contrast. Chapter 1 of the present book was written under the first aspect, chapter 2 under the second, but the result was not a clean separation. Items such as mathematising, mathematics as an activity, mathematics in a context can be introduced as descriptive, pure concepts, even though they were created with application= in mind, in particular in education. Levels are descriptive structural features of learning processes, which are meaningful only from a particular didactical point of view.

In (1.1) I signalled the difference between shades of meaning of the word "arts" in "arts and sciences" and "arts and crafts". From the performer's perspective, "arts" in the two pairs is interrelated, as are "artist" and "artisan". If an integrating rather than differentiating term is desired, one may choose the word *technique,* as performed by technicians, which is derived from the Greek *techne* for art. Then the interaction between what was announced as opposites is known as *technology.*

Education is a technique, and so is mathematics education. A teacher is a technician, as is an engineer, a medical doctor, a lawyer, a clergyman, a social worker. They depend on more or less advanced technologies, which in character extend across the entire range from the mere mastery of a professional language to ap plications of hard science. In [87] I discussed this and many other related ideas in abundant detail, and with as much sophistication as I was able to produce, so I am wary of repeating myself, even in paraphrases, which I am afraid would not be any better or more convincing than the original. I didn't even get the opportunity to defend my ideas since they were, as far as I know, never questioned. Far from believing that any of them was accepted, I even wonder whether they have been noticed at all -- silence does not imply consent[29].

4. *Theorising*

The question that should occupy us here is not whether and under which conditions or circumstances a theory might be true or applicable. We should instead be concerned with the ill-founded belief that mere theorising in itself, if profound enough, produces theory. As valuable as theoretical statements and conceptual tools may be, they do not necessarily, even though assembled to form an extensive collection, constitute what to my mind what is worth being called a theory. As I see it, a theory should pursue understanding in a reasonably well-defined domain, the extent of which does not matter much. It should do so in a reasonably coherent and consistent way, and it should in principle provide its adepts with guidelines on how to tackle questions, relevant to that domain, and how to answer them with unequivocal explanations or by proposing appropriate instruments for action.

This restrictive definition does not intend to underestimate the merits of theorising, which I admitted may be the source of valuable theoretical statements and conceptual tools. But by granting more latitude I would then lack a term for what, in my eyes, is a full-grown theory. Indeed, much of what now rightly claims the status of a theory was for long periods no more than a collection of theoretical statements and conceptual tools, which in the course of time or by some lucky stroke was eventually forged into a theory.

This statement is relevant to our general theme in so far as education, and mathematics education, in particular, is a field of extensive and intensive theorising. The preceding chapters are an example. They display more conceptual tools than would be sensible to enumerate here, such as phenomenological approach, mathematising with all its aspects, context, guided reinvention, levels in learning processes, paradigm, reflection, insight, prospective and retrospective learning, intertwining learning strands, mathematical attitude, and many more. All the same, I would not dare to claim that all together they constitute a theory.

After writing the last paragraph I happened to remember a dozen occurrences -- even in the plural -- of "learning level theory" in 2.3.2. I could plead that these occurred in a context dating from more than a quarter of a century ago, while, in the mean time my way of using "theory" has changed. I cannot exclude the last possibility. Indeed, the lavish use of the term, in particular in education, might have aroused an increasing suspicion with regard to the prevailing pretensions, or at least doubts about its suitability; so I may have become more demanding in granting something the status of a theory.

Yet this does not discharge me from the obligation to check whether learning level theory is, indeed theory in the restricted sense proposed. My answer is: yes, it is. And although this answer may extend to other items mentioned in the same context, which I did not label as theories, but I am wary of checking it. What I actually denied was that they together constitute a theory of mathematics education, and I would now add: not even together with other conceptual tools that I dealt with in the past and did not recapitulate here (see, for instance [87, Chap.IV]).

The fact that, with regard to theorising in education, I focused too sharply on my own results deserves an apology and an explanation: I am not nearly as well-acquainted with other people's work as I am with my own. But it seems to me that I am not the only one who has lost his grasp on the abundant literature, although I guess that on closer inspection (if this is possible) one would find less diversity in results than in terminology. Long lists of references can be misleading. Theoreticians are isolated or restricted to small groups where each understands the other; once or twice a year they meet in larger settings, but continue to speak only their own theoretical idiom.

Let us resume this question later on!

3.1.2 *Theoretic frameworks*

I should confess that I don't feel at all happy with the gulf I created between what I called results of theorising (such as theoretical statements and conceptual tools) and theory proper. Theorising is organising fields of experience, and the result of this organisation, unlike a heap of sand, will be gifted with a structure, thanks to its origin in connected experiences and the processing that took place when they were organised.

The term I propose to bridge the gulf is: *Theoretic framework*. I took the from Treffers, who indeed uncovered and analysed frameworks of various educational theories[7]. I will use the term here in a broader sense, that is, by also applying it to models of educational practice, such as those exhibited by textbook series or by products of experimental educational development, as well as by less formal classroom practice.

Indeed, there can be little doubt about the theorising origin of models of mathematical instruction. From the outset they must have been framed, consciously or unconsciously, by cogitation, reflection and argumentation. If need be, such framework can be uncovered by interviewing the authors or by analysing their production. Besides and beyond this, theoretic framework can be designed rationally beforehand in order to be realised by models of instruction, but most often the frameworks will be shaped in an incessant interaction with the matter to be framed. Their relation is comparable to that between *form* and *content* or between theoretically *essential* and *accidental* properties.

Indeed, in any model of instruction we will notice pieces that are less essential than others, and which may be replaced by ones which are somehow equivalent. How far can we go with substituting and rearranging, while at the same time respecting the spirit of the model? What do the eventual *pièces de résistance* look like? The answer to this question is an *a posteriori* definition of what I mean by *theoretic framework*.

If I am asked an *a priori* definition of *theoretic framework* I would say it is a more or less connected set of theoretic statements and conceptual tools, obtained by theorising (and maybe even including mini-theories), and reflecting the essentials of an actual or imagined instructional system in a way that allows the expert reader a reasonably faithful reconstruction. (In order to match the case in which Treffers used it, one has to replace "system" with "theory" in the last sentence.)

"Theoretic framework" is a weaker notion than is "theory". On the other hand, by its mere definition, it is bound more tightly to educational practice, which fact need not be a drawback. As a consequence, by the first token, it applies more broadly and by the second more narrowly than does "theory". Goal description, when extended to all its dimensions (as understood by Treffers[7]) certainly matches my definition, although the expert reader can be satisfied with less detail than the novice. (By an expert reader I mean someone familiar with the theoretic background, for instance, thanks to familiarity with the framework of the underlying theory.)

3.1.3 *Background Philosophy*

The first question I asked in [87] was: What is science? I tried to answer it somehow by means of certain necessary criteria (relevance, consistency, publicity), but otherwise I didn't do much more than to oppose science to -- among others -- technique and faith, without involving any comparative appreciation. I could not, nor would I, restrict "technique" (or "practice" as I called it occasionally) to inanimate nature. Since, together with biotechnique, I had to include medical practice, I saw no reason why the term "technique" should not be applied to the practice of teachers and other practitioners, those of the so-called humanities not excluded.

Until quite recently, almost all technique was practised with no technological support at all, or supported at most by some technology that for its part lacked significant scientific support. This still holds for many techniques, in particular for that of education and instruction -- a statement that again does not mean to suggest any kind of appreciation.

In the past, even up to modern times, much of what is now called science, was part of philosophy -- just think of Newton's *Philosphiae Naturalis Principia Mathematica,* published by the Royal Society, whose fellows called themselves philosophers. Step by step this all-comprehensive science has "de-umbilicated" the present specialised sciences -- to use a suggestive term of O. von Neurath's -- but even today a great deal of what claims the name of science for itself still dwells in the womb of philosophy. I didn't make this statement as a gesture of condescension. On the contrary, what I called background philosophy in [87] (or else ideology) is indispensable as the rationalisation of a faith that in turn comprises the faithful one's picture of world, man and society. It is indispensable, not only to any technique and wherever philosophy decides (implicitly or explicitly) on the choice of technical tools and objectives, but also in "pure" science with its rich choice of goals and methodology.

Did I start my exposition too far from mathematics and mathematics education? Shouldn't I speak rather more specifically about the picture of mathematics and mathematics education? In a sense this is true, since this is my proper aim. Yet more things taught than just mathematics, and in this totality, pictures of subject areas and their instruction are somehow related to each other by their relation to pictures of world, man and society.

For any picture of mathematics and mathematics education one has in mind, it is important where one localises mathematics -- inside or outside the world, bracketed or out of brackets, within brackets that close and open more or less easily. The values attributed to learners as humans determine the ways that they are ex-

pected to acquire their mathematics, how freely or slavishly, under guidance or bridled. They all determine the manner in which one thinks mathematics can contribute to the mutual benefits expected by society and its members from one another, and they depend, at least partially, on one's picture of society and its inherent hierarchy.
Do not misunderstand me! I say "picture of society" rather than merely "society". In primitive societies -- as well as in "primitivising" ones (Freyre, Mellin-Olson) -- "society" and "picture of society" coincide, as they do with the picture of world and man. In our society, a broad spectrum extends between the naive and the utopian picture of society, and even "realistic" ones can differ widely from each other.

3.1.4 *Pictures of mathematics and mathematics education*

Yet the picture of mathematics also influences that of mathematics education. in a direct way. Whoever cherishes a picture of mathematics outside the world -- a deductive system or a catalogue of formula -- is likely to systematise or to interpret mathematics instruction in the Same spirit. On the other hand, whoever experiences mathematics something in the making, vibrating under the impulses of world and society, will be inclined to teach it in the same way -- directly or as an educational developer. There is little need to refer to literature that shows this correlation, between the personal characteristics of teachers; but it is perhaps worthwhile recalling something of a more historic-anecdotal character. It is even great history I am referring to. The colloquium (Melun, 1952)[2], which was crucial on the one hand, for the development of mathematics education in the sixties and seventies, and on the other hand, significant for Piaget's first meeting and confrontation with Bourbaki, personified there by Dieudonné. At his colloquium, E.W.Beth claimed -- with far-reaching consequences for the future of mathematics education -- that

> Le rôle de la formation mathématique dans l'enseignement secondaire consiste presque exclusivement, me paraît-il, à familiariser les élèves avec la méthode déductive.

Although this claim was disputed by no one -- implicitly or explicitly -- it nicely characterises the logician Beth -- the only thing that interested him about mathematics was its deductive system. Dieudonné, on the other hand, confronted the much more encompassing mathematics, exercised and represented by him, with traditional instruction, which, in his eyes, had run aground due to its lack of deductivity (in particular in geometry -- à bas Euclide!). New shipwrecks could be avoided, provided one trusted older and wiser people who have explored the world to the benefit of the young, and used Bourbaki's compass to sail from the poor to the rich structures; many believed this along with Dieudonné.
With the claim quoted above, E.W.Beth only followed his father H.J.E.Beth, who, as a designer of a new mathematics programme, had declared (text[30] trans-

lated):

> It is the main objective of mathematics education to contribute to mental culture and development; transferring useful knowledge is a secondary objective.

In the 19th century (and even before that in The Netherlands) the formal value of mathematics learning was claimed by preference by “modern” schools in their competition with grammar schools where Latin enjoyed a similar reputation. I cite here the formal value of mathematics as an instance of a faith that usurped the state of a theory, while in fact being a hypocrisy. Only a few people dared to unmask it as such -- teachers as well as research mathematicians like D.van Dantzig as is shown by the mere title of one of his articles[31] (translated) ‘The social value of mathematics instruction”, which at that time (1925) was disregarded if not considered an insult.

All my life I have fought this faith in the formal value of mathematics, only -- I confess -- to convert to in the end, although (and this is my excuse) with in mind another kind of mathematics instruction, which has its roots in the most recent past and where deductivity is assigned the place it rightly deserves.

I called it a conversion, and this is no metaphoric language, because it happened in the sphere of faith, where the picture of mathematics is framed by the picture of the world, the picture of the mathematician by the picture of man, and the picture of mathematics education by that of society.

Pictures of mathematics education are made explicit and rationalised by frameworks. Later on (3.1.6), I shall discuss how they are related to background philosophies (3.1.3). Meanwhile, the question arises of whether designing mathematics instruction presupposes making a decision on a background philosophy. The answer is that it does not. Everything can remain implicit: it is a faith that is not being rationalised. On the other hand, the relation between a theoretic framework of mathematics education and the philosophy behind it now looks much closer. But beyond it one can imagine theory (or metatheory) charged with the task of relating theoretic frameworks of mathematics education to background philosophies. An excellent starting point here is Treffers’ classification[7].

3.1.5 *Classifying mathematics education*

I never did like classifying, and I still distrust it for what I consider are sound reasons: I suspect it as both too easy an approach to problems and as too low a bid to buy scientific respectability. In (1.3) I explained my former resistance to Treffers’ distinction of horizontal and vertical mathematisation. Eventually I did accept it as an effective tool to characterise various kinds of mathematics education. The present section shall be concerned with this subject.

Treffers’7 four types of mathematics education are distinguished by a double dichotomy with regard to horizontal and vertical mathematisation: *presence* or *absence* of one or both of these characteristics in the intended learning processes. The unconditional “yes or no” predicts the emergence of *ideal* types, created with

a view to global orientation, yet accessible to refinement, depending on the degree to which the yes-no contrasts are toned down Treffers applied his classification to primary mathematics education, but this is in principle no restriction.
The lack of horizontal mathematising characterises *mechanistic* and *structuralist* instruction: in the first case even the vertical component is lacking, whereas in the second it is in fact cultivated. Horizontal mathematising is fostered in the *empiristic* and the *realistic* approach, in which the former neglects the vertical component and the latter does full justice in the second kind. Seen in a table:

	horizontal	vertical
mechanistic	-	-
empiristic	+	-
structuralist	-	+
realistic	+	+

Fig. 15

Let me illustrate this classification by a somewhat anecdotal report [176,180], a German educational tragedy, as it were:
Lisa Hefendehl-Hebeker[32] tells about a fifth-grader who could not answer the problem
Why is 4 < 9 ? Give three arguments!
and the child's desperate family that, equally unable to do it, made an appeal to her as a mathematics education expert. By looking for the context, that is, the homework in which this problem was one item, she succeeded in suggesting three possible arguments, such as “because in the number sequence 4 precedes 9”, “because on the numberline 4 is to the left of 9”, “because the 4-arrow is shorter than the 9-arrow”. One could also have suggested such answers as “because 9 > 4”, “because 4 < 5 and 5 < 9”, “because 9 = 4 + 5”, “because 9 - 4 > 0”, though it is unlikely that these would have been accepted by the teacher.
Let me remark that, in German, the <-symbol in “4 < 9” is read as what corresponds to “smaller than”. So a smart pupil could have objected that both figures are equally large. I mention this in order to point out that all three symbols mean something, and in order to know what they mean one has to consult the context, which is indeed what Lisa Hefendehl-Hebeker did.
What has this to do with classifying mathematics education?
Well, neither the mechanist nor the empirist would ask such a “why”, whereas the structuralist and the realist possibly would. The teacher who set the problem was obviously a structuralist; the answers as reconstructed match this view. The empirist and the realist would notice that 4 and 9 belong to the vernacular as do, say, 4 marbles and 9 marbles, 4 florins and 9 florins, 4 kg and 9 kg, 4 hours and

9 hours, 4 floors and 9 floors, 4 metres and 9 metres, and then, depending on the context, "4 < 9" can mean that 4 marbles are less than 9,4 florins is cheaper than 9 florins, 4 kg lighter than 9 kg, 4 hours shorter than 9,4 years younger than 9,4 floors lower than 9 floors, 4 metres shorter, smaller, shallower than 9 metres. Mathematics taught and exercised in rich contexts would provide for all of these, and many more, comparatives, and realistic mathematics instruction, as opposed to the empiristic version, would account for their isomorphism by vertical mathematising. Then a question like "Why is 4 < 9?" may validly be answered by "because 4 marbles are less than 9", 4 florins is cheaper than 9 florins, "4 kg is lighter than 9 kg", which would already be three arguments out of many more possible ones.

Due to its rough schematism Treffers' classification applies to theoretic frameworks rather than to theories. The latter should be uncovered by more profound search in actual instructional systems if they are not presented in detail by the designers themselves. Moreover, this classification describes ideal types. In textbooks of the mechanistic trend, which boast a long tradition in teaching arithmetic, concepts and operations may in some way be related to reality but this usually happens in a way that neither teacher nor learner are likely to take seriously. It is often an intellectual alibi for the developer's or the teacher's conscience, a misty cloud veiling an ideology, which in spite of its venerable age can best be described by a modem metaphor. This I will do in the next sub-section[32].

Meanwhile, let me add an indispensable remark. In Treffers' double dichotomy it pays to replace "mathematising" with "didactising" in order to characterise styles of teaching mathematical didactics (rather than mathematics), which happens, among other places, in teacher training and retraining. Philosophies of mathematics education should be judged according to both interpretations. This is not at all a triviality. More than once I have read and heard proponents of realistic *mathematics* trying to implement their ideas in a way that, *didactically* viewed, appeared to be mechanistic-structuralist rather than realistic. The didactical attitude was not matched by its mathematical counterpart.

3.1.6 *Philosophies of mathematics education*

1. *Mechanistic*

According to the mechanistic philosophy man is a computer-like instrument, that can be programmed by drill to perform, on the lowest level, arithmetic and algebraic, maybe even geometric operations, and to solve applied problems, distinguished by recognisable patterns and processed by repeatable ones. This, then, is the lowest level, where man is placed within a hierarchy of ever more skilful computers, which are related to one another as are programmers and programmed subjects. Skinner has forcefully propagated this kind of human society. New prospects have now been opened for this ideology by computer controlled instruction. There are, however, good reasons to ask its proponents why people should be educated to perform tasks on a level where, by many orders of magnitude, computers are faster, cheaper, and more reliable than humans.

2. *Structuralist*

The structuralist view is also historically rooted, in particular in the traditional teaching of geometry. A well-structured system of mathematics or a mathematical domain shall be taught. It is a human right and dignity to learn by insight and understanding and as a rational being he is judged able to perform deductions more efficiently the more systematically the subject matter is structured. In the ideal case, however, of the socratic method he needs the midwife who delivers him of his mathematical ideas -- Socrates says explicitly about the slave, “you see he has expressed nothing but his own opinion”. This, then, was mere “theory” since, in (Socrates’ and) the classroom’s practice, the learner was expected to obediently repeat the master’s deductions. In order to check the quality of the repetition -- whether it was mere parroting or full of insight -- problems were set, which in turn were deactivated by drill. In the nineteen sixties and seventies of our century, under the name of New Math, the structuralist view was advertised and propagated. Yet it became soon clear that this wrong perspective -- from the poorer to the richer structures -- was an obstacle to any kind of genuine mathematising. On behalf of the prestructured mathematics to be taught, a correspondingly structured world was invented of Venn diagrams, arrow schemes, “games” and so on, to be mathematised by the learner. This was, indeed, a kind of horizontally mathematising activity, yet it started from an *ad hoc* created world, which had nothing in common with the learner’s living world. It was mathematics taught in the ivory tower of the rational individual, far from world and society.

3. *Empiristic*

To the empirist the world is a reality, where man can acquire useful experiences -- a respectable point of view provided reality and usefulness are broad-mindedly interpreted. Empirism is deeply rooted in English utilitarian education. Provided with material from their living world, learners get the opportunity to acquire useful experiences, but they are not prompted to systematise and rationalise these experiences in order to break the barriers of the environment and to expand the reality they are familiar with. This matches the picture of a society stratified into layers that correspond to separate realities.

4. *Realistic*

In realistic instruction the learner is given tasks that proceed from reality, that is, from within the learner’s ever expanding living world, which in the first instance require horizontal mathematising. The individual’s and the group’s progress in the learning process -- how far and how fast -- determine the spectrum of differentiation of learning output and the position of the individual learner therein. One example: If, in the course of progressive algorithmising, column multiplication is first carried out as successive addition, in order to gradually be shortened (by using the tables of multiplication and the positional system), individual learners-- in the social context perhaps the whole group --- will eventually acquire the standard algorithm. In the case of long division, the learning output of progressive algorithmising may be more differentiated -- an unavoidable differentiation, which nonetheless deserves a positive qualification.

The realistic picture of mathematics fits without brackets into the world picture. The picture of man is that of the reinventor, who is stimulated to put his abilities to good use. The corresponding society is differentiated continuously, rather than by artificial layers.

3.1.7 *Use of the classification*

Two criteria have been singled out for the design and the analysis of mathematics education: horizontal and vertical mathematisation. There are many more. The choice of these two criteria may be justified by their relevance to background philosophies. It may be useful both for creators and reviewers of instruction to localise instructional creations within a non-trivial scheme. This may help creators to better understand themselves, and reviewers to more easily examine explicit or implicit pretensions of the creators. Whatever the objections may be to comparing instructional creations that start from different didactical -- and, basically, philosophical -- principles, they do not affect the submission of each of them to the test of their own principles. Wherever in the past the mechanistic view prevailed, textbook authors often pleased themselves by embellishing their production -- implicitly or explicitly -- with realistic or structuralist touches, and the same is likely to happen in the future as a defence of mechanism against a realistic trend that is gaining momentum. The classification can prevent educational developers from misleading themselves and others.

3.1.8 *A matter of faith*

We put forth a theory that relates kinds of mathematics education to philosophies. It is determined by the theoretical tool of (horizontal and vertical) mathematising and thus presupposes faith in (guided) reinvention. It is open to anybody to brand this faith as "irrealistic": "Where in the world are those reinventors and those who are able and willing to guide them?" "How well-founded is this faith?" Answering such questions is not easy. In matters of faith tolerance is becoming. If somebody believes in the hierarchical society of programmable human computers, that is his business, although of course tolerance does not extend to incorrect computer programmes.

Detecting errors in computer programs is no great feat, but it is quite another thing to evaluate instruction without any regard for its own basic principles. How to test whether one kind of instruction is better than another? Instruction serves goals. But what to do if the goals are formulated in terms of a philosophy, thereby being a matter of faith? Isn't it a vicious circle? No, it is not if the testing criteria are given together with the -- implicit or explicit --'objectives of the instruction. Sometimes they are, that is, with respect to the *short-term* objectives, where, in

the most favourable case, the term may extend up to graduation.
Is there any way to at least compare the quality of various instructional systems under these favourable circumstances? Given an instructional system, collect all those exam items which are judged representative for it, each with its weight within the system; in order to compare two of them, form their intersection and put it to the test! For the higher levels and with regard to the mechanistic and realistic approach, one can safely predict that this intersection will be almost empty or at least negligible if compared with the remainder, and one can do so with even more certainty if the main goal of the realistic instruction is learning to mathematise. At lower levels, in particular if one restricts oneself to pure and applied elementary arithmetic, the intersection might still be considerable and accessible to empirical comparison.
Although some of the research carried out in my country seems to strongly suggest the superiority of the realistic view, I am still cautious of generalising its conclusions. In one case[9] a small sample of classes and schools adhering to the realistic style were compared with those of other styles (or with the average of that half of schools which were taking a nation-wide test). Meanwhile, a large scale investigation[33] took place, which also suggested the superiority of the realistic style. Unfortunately, because of flaws in the provisional report I must postpone my final conclusion. Yet this is not what matters.
Realistic textbook series for primary education, patterned on the Wiskobas model[10 11], are a relatively recent feature in my country; they are supported by a majority of educationalists. Realistic textbooks are gradually beginning to supplant older ones; and it is not far-fetched to assume that this process started at "better" schools and gradually began to affect more schools in an order corresponding to their quality. At present, somewhat more than 50% of the schools use realistic textbooks. If, in the course of this process of substitution, low-quality schools get involved, the trend that now seems to prove the superiority of the realistic style may become reversed -- a similar phenomenon has been observed in Hungary[34].
In the last paragraph I spoke about good and bad schools. This I did for simplicity's sake. I was not referring to their overall quality but to the measure in which schools that switch textbooks -- in the present case most often from mechanistic to realistic -- are prepared to correspondingly switch philosophies and teaching habits. Indeed, one does not automatically imply the other. Working with realistic textbooks in the mechanistic instructional style to, which one is accustomed may even make things worse; in fact, this may explain the failures as mentioned above.
We shall deal with implementation in due course: rather than selling textbooks, one has to sell a faith.

3.1.9 *Learning theories and philosophies*

Let me first remind the reader that *learning theory* is meant as shorthand for *teaching/learning theory* or *instruction theory,* which, for the moment, will include theoretic frameworks.

There are a great many *general* learning theories, where, according to the authors, ‘‘general’’ means subject-area-independent; only as means or domain of exemplification and concretisation do subject areas -- by preference, mathematics, or what generalists consider to be mathematics - come into play. Usually the view is even held that the general learning theory leaves no room left for learning theories specific to any subject areas, other than those derived from the general one. I strongly distrust general learning theories, even if their validity is restricted to the cognitive domain. Mathematics is different -- as I emphasised before -- and one of the consequences is that there is no didactical analogue to guided reinvention in other fields. I don’t know about learning theories for other areas. I can imagine some general learning theory derived from learning theories specific to a variety of subject areas. This inductive process is primarily the one by which new theoretical knowledge arises, rather than the other way around in a pseudo-deductive way.

As I have emphasised repeatedly, I am allergic to armchair hierarchies of phases, stages, types, levels, and so on, often illustrated by examples from mathematics instruction. Very few of them have ever been properly and successfully applied or empirically corroborated. In [87] I analysed the only one I knew at that time to have been put to the test by its author (or perhaps by his ignorant assistants). It was a most amusing job. Unfortunately, more an more textbook series and models of instruction are allegedly being based on such theories, which are arbitrarily interpreted in order to eventually be degraded to a kind of billboards. Treffers[7] more patiently analysed such general learning theories and convincingly showed how to justify any kind of mathematics instruction -- I mean each of the four styles -- by each of those general learning theories. Maybe one can also do the converse: justifying any general learning theory by any kind of mathematical instruction.

A particularly striking example is *Piagetian instruction.* In his early work[17] [22] “Piaget developed an empirically based descriptive learning theory, which has hardly received any notice[17] -- it seems that he himself even forgot about it. A great many normative theories and practices claim the label “piagetian”, although it Seems to me that they have borrowed words rather than ideas from Piaget, by preference terms like *operation*[35]

As promised in the heading, I will focus on the philosophical background of a few learning theories.

3.1.9.1 *Gal’perin, and materialism*

1. Gal’perin’s levels

The levels distinguished by Gal’perin[36] in any learning process are

- familiarisation with the task and its conditions,

- an act based on material objects, or their material representations or signs,
- an act based on audible speech without direct support from objects,
- an act involving external speech to oneself (with output only of the result of each operation),
- an act using internal speech.

"These levels indicate the basic transformation of an act as it becomes mental." The terms require some explanations, which are easily extracted from Gal'perin's exposition (a few ambiguities may be ascribed to the translator). Western instruction designers who base their work on Gal'perin usually describe his hierarchy as follows:

- orientation basis,
- material act,
- verbal act,
- mental act,

which in no way does justice to Gal'perin's intention.

The philosophical materialism behind Gal'perin's level structure of the learning process is unmistakable, or so it seems to me. But is it as operational as it looks? "Material act" -- as it is called by Gal'perin's followers -- evokes associations with manipulability as a way (particularly in arithmetic) to start with concrete, that is solid, material, which today is available in a great variety, indeed. If Gal'perin was aware of any of this material at all, he must have considered it irrelevant. It is clear from his description that *material* or -- as he later says -- *materialised* has a much broader -- or even quite different -- meaning from palpability. As emphasised once more later on[36], p. 253, he also admits -- or even prefers -- *material representations or signs,* which he equates straightforwardly with materials. To state this in the extreme, a tree the child can touch, a picture of a tree recognisable as such to the child, the printed word "tree" as soon as the child can read, a printed definition of "tree" as soon as the child can understand it, all belong to the same learning *level* (albeit in different learning *processes),* but as soon as they are voiced *audibly* or *inaudibly* or *internally,* they belong to the next levels, respectively, in the corresponding learning processes.

In the same way is "verbal act" a misleading reproduction of Gal'perin's intention. According to Gal'perin, written or printed material is as "materialised" as are pictures. What matters at the third and fourth level is speech (to others or to oneself), and at the fifth its elimination. (This interpretation is confirmed by[38], 1977, p. 33-38.)

Although this sounds mad, it is absolutely rational and intelligible, as I will promptly explain. In order to understand it, one has only to forget about one's own ideas on learning. The impression of madness is due to the extreme example of "tree", which doesn't fit in this context. The context that fits is that of *teaching arithmetic,* in particular column arithmetic, which becomes clear from the one example that has been elaborated in detail[36], p. 270-273. Or, more precisely, *Soviet instruction of arithmetic* as it was familiar to Gal'perin at the time he applied his psychological ideas and wrote the article we are concerned with. Middle-aged readers in more Western countries will perhaps be able to relate column arithmetic as they learned it themselves, to this frame -- it is a kind of instruction that is likely to have become obsolete much earlier in Western countries than in the Soviet Union, where it is now equally outdated, or so I think. The wine of gradually renewed instruction had better not be bottled in a dated learning theory, unless the "bottle" is adapted to the new contents, which implies a preposterous use of the original label if it should happen.

"Dated" is not meant to blame Gal'perin. As a psychologist, he registered and analysed the then prevailing instruction, and did so carefully and not bent on educational additions and ambitions. Maybe he emphasised the level structure that resulted from his analysis too strongly, and extended it too widely, but if somebody is to be blamed for this, then it is rather those who believe that such level structure is suitable to frame instruction of a quite different character.

2. *Gal'perin, and initial arithmetic*

Let us consider initial arithmetic as framed by Gal'perin's levels! From the very start "material" means quite another thing than what is available as such in our classrooms. The "orientation base for action" in "abstract counting" (without numbers) is an *instruction card* with stairs picturing piles of an increasing number of objects, and the instruction:

> "Next number" means "one more object than on the pile shown"!
> "Preceding number" means "one less object than on the pile shown"!

(Notice that the number of objects on the pile "far exceeds the arithmetical knowledge" of the pupil!)

At any assignment the pupil first consults the instruction card in order to fulfil the task, which, when this is done, is *materialised action.* Then the card is turned over and eventually removed, in order to have the pupil repeat the task interpretation orally while fulfilling it, which is the *vocalised form of action.* Then the pupil is asked to "solve the problem silently, reporting only the result", which is *action at the intellectual level.*

This is to say that, after the instruction "name the next (preceding) number", the pupil acting at the materialised level reads the card and points to the next higher (lower) pile (or does the whole class do so in unison?); at the level of audible speech, he does so without consulting the card; next he acts the same way while whispering before pointing, and finally he points immediately to the "number" asked for.

It becomes clear from the above that, rather than referring to piles of objects, Gal'perin's "material" refers to instruction cards. This is confirmed by the next example.

3. *Gal'perin, and column arithmetic*

Here the procedure looks somewhat different. One might expect that the way of using an instruction card would be again articulated according to the four stages from "materialised" to "mentalised", but things have now become complicated by a gradual transformation of the card itself. After a start restricted to two-digit numbers, the pupil gets a sheet of paper divided into columns to learn numbering, addition and

Billions			Millions			Thousands			Single Units		
100	10	1	100	10	1	100	10	1	100	10	1

Fig. 16

subtraction. He also has small ready-made numerals at his disposal, which, when placed in the columns, create numbers to be read or formed, added and subtracted by the pupil. After the *orientation basis* has been laid he enters the *stage of the materialised action.*

Next, the verbal descriptions are removed from the triple columns while the pupils still have to pronounce them. Then the thick lines are replaced by thin ones.In the next step, the figures above the columns disappear, while still being pronounced by the pupils. One step further, the horizontal lines are also removed, and eventually the entire grid disappears; the ready-made numbers are laid out on the table or else corresponding digits are recorded directly in the notebook but vocalisation of numbers and operations is still maintained. Here the *audible stage* ends, and that of *external speech to oneself* begins, from which, finally, that of mental action takes over.

4. *Gal'perin, and geometry*

Gal'perin's interpretation of "material" also extends to teaching geometry[37]. It is the most traditional and the barest form of concept formation: reading and learning to apply definitions, such as those of straight line, angle, bisector. The GDR's Lompscher[38] has described this kind of instruction which is aiming at concept attainment. Although the example set forth in[37] reminds one of a catechism rather than of geometry instruction, our point is how it fits into Gal'perin's hierarchy, rather than its intrinsic quality.

In this type of geometry instruction, geometrical figures don't play any significant part as a *material* element. The materialised level is represented by written or printed orientation cards that disclose the various definitions of geometrical objects. With these instructions before their eyes the pupils have to decide whether something is a straight line, an angle, a bisector, and so on. At the next level they must answer such questions without using the cards, while reciting aloud the defining criteria. Via the stage of whispering the pupils arrive at that of directly announcing the result; only if need be -- for instance in testing - -are they to reproduce the definitions.

5. *Gal'perin: conclusion*

Gal'perin's learning theory distinguishes itself by the refined elaboration of all details by means of the given examples and by the straightforwardness of the intended learning process. Properly stated, it is a pure teaching theory, inspired by a kind of instruction that allows the learner no initiative. If Gal'perin starts his article, saying

> Our basic hypothesis is that the formation of mental acts passes through a series of stages. At each stage a given activity is carried out in a new form and undergoes changes in several directions.

and subsequently specifies this, among others, by his stages or levels, he undoubtedly has in his mind a certain instructional system in which his basic hypothesis can possibly be tested. Within the limits of this system such a model can be considered to be descriptive, but in a broader frame it is a quite restrictive normative model. Could it have been more liberal than suggested by the above examples? Probably not! To corroborate this judgment let me quote Gal'perin[39] himself:

> If, however, we reproduce the material activity systematically step by step, first in its verbal and thanks to it in its ideal form, we can keep supervising it, whereas in the case of "spontaneous" transition we don't know what has happened nor how it did. As soon as we lose the command of the process, we can no longer understand it.

Gal'perin's main concern is the checkability of the learning process; which is considered to be mere interiorisation. What is the use of it and what can one expect from it? The answer depends on one's picture of mathematics and mathematics education, that is, on one's philosophy. If it fits philosophical materialism at all, then it does so according to a strange interpretation of "material", or so it seems to me. In Treffers' classification, Gal'perin's learning theory is an extreme example of mechanism.

For justice's sake, let me add that more recently Gal'perin's level theory -- in particular his strong insistence on the material and materialised level in the teaching process -- has experimentally and with great success been challenged, which in fact has happened in his own school[40].

3.1.9.2 *Constructivism, and "Kant"*

-The plenary sessions of the PME-11 meeting[41], Montreal 1987, were dedicated to constructivism. Through personal talks during the meeting and by afterwards studying the literature I tried to find out whether and in what respect "constructivism" were more than a new slogan. To no avail -- I must confess. What follows is the result of my analysis.

1. *Language*

There are words that can mean everything and its opposite. "Construction" is one of them. This doesn't matter as long as it has a clear meaning to whoever pronounces, writes, hears or reads it -- clear not in analytic isolation but in a clear context. For topical words one creates a fitting context for oneself, which is not bad as long as one is aware of this fact. What is bad, is being unaware of the need for a meaningful context, at least for the user, who risks being deemed a misuser. I don't aspire to the office of linguistic judge, and even less of legislator. On the contrary, flexibility is a great virtue of language, and can hardly be overestimated. pictorial and metaphorical language can enrich both language and thought. But flexibility does not mean weakness; pictures should be focused.

2. Language: Construction.

"Construction" was soon extended from the building trade to other manual work and to mental work as well. It even came to signify poor work: artificial, a mere construction.

"Construction" presupposes a maker. Who has made it? The designer, the foreman, the bricklayer slave, the drawing board slave, the assistant to the architect or the architect himself?

"Constructing" rings like "creating", and if this is no more than a ring, then reconstruction is mere re-creation. But reconstructions can also be of greater value than constructions. For instance, efforts to reconstruct a dilapidated or destroyed building, or an obscure course of affairs can by far surpass the efforts that once were needed for the original construction.

"Construction" does not reveal anything about the constructor's own contribution, nor does the finished product do so. The "Do it yourself" package contains every little part as well as directions on how to form the desired object. Only by reflecting on the own activity can the re-creator become a re-creator. If the directions are missing, the reconstruction of a decomposed jigsaw puzzle can require more inventiveness than many constructions.

3. *Constructivism -- philosophically*

Is our world picture a picture of the world, and if so, how faithful is it to the original? Since olden times this question has been asked again and again. How many tricks does not Nature play on us, and is not Reason the means to unmask Illusion? But how reliable is Reason? How often has the lie not been given to Reason? Yet there is one thing that cannot be denied: that in the course of history our world picture has become widely extended -- in the macro-world of the universe, in the sub-atomic micro-world, and towards ever and ever greater wealth in the intermediate dimensions of our direct sensation.

Yet we have to put up with people who object that this whole picture is a mere reconstruction of what, albeit unconsciously and unintentionally, was constructed by ourselves beforehand -- reconstruction by the very means by which it was first constructed. Well, we need not bother about a question and an answer that are equally meaningless. But what about poor Kant, to whom the problem and the answer are ascribed by people, who probably never read a single word of his? My answer to this question was to surround his name in the title of the present subsection with quotation marks. This, of course, does not refute what *philosophically* some people call constructivism. *Instructionally* it refutes itself by its irrelevance.

4. *Construction and constructivism -- mathematically*

In Euclid, besides theorems *(quod erat demonstrandum)* we find constructions *(quod erat faciendum)* -- constructions by means of the ruler and the pair of compasses. Although, in the past, definitions and theorems arose by construction as well, they are presented here as ready-made articles; yet this does not prevent constructive activities, such as drawing and producing lines and assuming points, from sneaking into the proofs.

To be sure, in mathematics (and not only in mathematics), inventing and reinventing must be distinguished from reporting. As emphasised repeatedly, a variety of styles are possible. Reports can be restricted to the ready-made matter, but they can also do more or less justice to the process of making. Less, if it is mere phraseology; more, if the reporter tries to play the part of inventor, and thus smooths the road for the reinventor.

If, from a given infinite set, Cantor forms the set of its subsets, is this a construction or idle verbosity? Language on behalf of itself or transmission of something that can be transmitted in no other way?

Constructivism in the foundations of mathematics means restricting oneself to activities that take place in time, or that are understood as such: In an unrestricted time, in a restricted time, arbitrarily accelerated, or reeling off with theoretically or practically constant speed.

The set of subsets of a set is formed in one breath, provided these words are pronounced fast enough. With the natural numbers one can progress towards eternity, or towards the endpoint of an interval halved again and again; with ever faster computers one gets a grip on ever larger numbers, whereas number language conjures up an exhaustible number sequence.

5. *Constructivism in developmental psychology*

Construction starts as reconstruction, to wit by imitating oneself or others, yet the required physical-mental tools must first be shaped and improved by ever more purposeful efforts. At the same time, and entangled in this process, a powerful cognitive tool is developing and maturing: interpretation. With a profundity that can hardly be matched, let alone surpassed, Piaget, in his unnoticed early work[17] [22], uncovered the roots of interpretation. Unfortunately, and for incomprehensible reasons, he didn't even touch here on the role played by language, which is equally entangled in the process -- the children he observed never babble nor weep. Is this indifference to vocal utterances somehow related to his later habit of taking linguistic failures of his subjects for cognitive ones, as though this were obvious? It can be a neutral reconstruction of the play-pen if the child within it scans the sequence of the bars with eyes or fingers to close the cycle, but as a structure the play-pen is so obvious that it provokes this cyclic interpretation. Invisibly structured playthings, in particular mechanical ones, are different: most, if not all of them, are black boxes, where the reaction on the stimulus is a datum, inaccessible to analytic interpretation.

We have already discussed construction in the sense of slavish obedience to assembly directions. The development of language is to be viewed quite differently: it is a reconstruction rather than imitation, which is extrapolated by the creation of words and clauses never before heard, the novelty of which is witnessed by sins against morphology, grammar, and syntax.

Reconstruction is again most productive where it is guided by global structures. An example we mentioned already is that of the first mathematics (or is it the first?) to emerge in development. When, by listening and speaking, the child acquires the number sequence (that is, the sequence of spoken numbers), counting *starts* with a verbal imitation of a fragment or of fragments, separated by gap. This is then superseded by imitation of the structure, introduced by interpretation into the linguistic system, which allows the learner himself to close gaps and playfully continue the sequence. Playfully, that is, beyond need and necessity. Without limit? New milestones are needed; million, billion, trillion, etc. suggest something fading away in limbo. More profound analysis is needed to transform ". . .illion" into a recursion exhausting the number sequence. (As a matter of fact, even with regard to the decimal positional system nobody bothers about the feature that the size of numbers runs far ahead of the size of the means by which they are defined -- on a higher level, a source of paradoxes.)

The preceding is cited as an example of where re-creation stands out in bold relief against the kind of construction and reconstruction -- manual or mental -- that is mere imitation, e.g.copying a drawing, blindly obeying assembly directions. What does the self-styled constructivist mean here by "construction"? "Construct the perpendicular bisector!", whether the one charged with the task, already knows how to do it, or not?

6. *Constructivism -- didactically*

What matters here is teaching and learning, pedagogy and didactics. Only in order to show their irrelevance did I regard the constructivism of an empty philosophy and of a poor developmental psychology.

For shaping teaching and learning processes there is as little need to discuss whether the world is a mere construction and the world picture a reconstruction of what was constructed by oneself as there is for the claim that the individual in his development constructs or reconstructs his world, that is, as long as this is a mere claim, and as long as this claim is not substantiated by any explanation of how the individual manages to do that -- indeed, only this knowledge could help one to better understand and to exploit the constructivist thesis.

If "constructivism" is to mean anything didactical, it must indicate the one who is expected to "construct". There are people who call themselves constructivists because they allow the teacher (rather than the textbook author or some other authority) to construct education. Constructivism also means an artistic, say Bauhaus, style. The teacher who adheres to this necessarily creates structuralist mathematics. If I were to accept the term "constructivism", I would mean a programme having a philosophy that grants learners the freedom of their own activity. Then it doesn't matter whether this activity is called construction or reconstruction or whatever, as long as such words do not prejudice anything that concern the learners' freedom. Whatever is imposed upon the learner remains imposed, whether this is called construction or reconstruction or something else. There is no use pleading here that learning in freedom is more dignified than under coercion, since the hope that the first is more efficient than the second is being strengthened daily by an increasing number of examples. Lacking a convincing context, such terms as construction, reconstruction, and constructivism are doomed to remain slogans. The only context that counts didactically is instruction itself, that is, instruction developed from the direction of the design onwards towards its realisation. If a term is, indeed, needed I prefer guided reinvention, which I discussed in (2.1).

7. *Von Glasersfeld*

One would expect Von Glasersfeld's name to appear in this exposition. I didn't mention him originally because I didn't know where to place him, but a recent publication of his has made things easier. His paper starts as follows[41a]:

> During the last three decades faith in objective scientific knowledge, a faith that formerly served as the unquestioned basis for most of the teaching in schools and academia, has been disrupted by unsettling movements in the very discipline of philosophy of science.

Indeed, and this is a most striking symptom of the ever broadening and, to my mind, most deplorable gulf between the philosophy of scientists and the philosophy of philosophers of sciences, at least as preached by the noisiest among them. Would any scientist who lacks faith in objective knowledge have ever had Voyager 2 launched for a visit of the big planets -- as far as Neptune -- to gather knowledge about them? Would he have had big accelerators built that are to discover missing elementary particles? Would he analyse long DNA strings in order to manipulate them? Would he excavate ancient cities and decipher ancient inscriptions? These are just a few questions out of an interminable variety, and yet I did not even mention the mathematician searching for new knowledge.

Rather than trying to uncover the scientist's implicit philosophy and methodology, philosophers and methodologists of science, if unacquainted with real science or frustrated by it, dream up their own fancies. To be sure, Von Glasersfeld is in good company, but it is a company of spectators rather than of actors in science. Yet, in his particular case, it is a third kind of philosophy that should count: philosophy of education. These three may be mutually related provided they are not confused with each other. At any rate, I cannot see any bond between mathematics instruction on the one hand and an alleged or assumed lack of faith in objective mathematical knowledge on the other hand, whether it is called constructivism or anything else.

3.2 RESEARCH IN MATHEMATICS EDUCATION

3.2.1 *Research*

Research is being carried out in ever more fields, by ever more people, albeit for a rather limited and invariable number of reasons. The question "what is the use of it?" can be answered differently as soon as one adds "for whom?". Landau said: "Number theory is good, thanks to it one can get a Ph.D." This is a good argument, which can be elaborated on by stating that, thanks to number theory, one can get papers published, make a living as a professor, become famous, earn prizes. Landau did not live to see number theory applied in cryptography. Well, ready mathematics as applied in cryptography was far below Landau's highbrow level. But, once applied, it produced fresh flowers, reasonably comparable to number theory's classics.

99% (or even more) of contemporary mathematics research will never be applied, except to create new mathematics. This statement even extends (perhaps by a smaller percentage) to that brand of mathematics that is intentionally created to be applied outside mathematics. Anyway, my statement admits that some 1% (or is it 1%?) will eventually be applied in a near or distant future. The big problem, however, is to know which 1%. But what about the 99%? Much of it has proved useful enough to be recycled, or, when it is rejected, can be used to close off dead ends.

What is the use of it? The question is answered differently according to whether we spend our money to buy the thing as a taxpayer or as a consumer. Art is useless unless the artist is able to sell. Why, for heaven's sake, do people produce things they are unable to sell? Obviously, because they like to. In the sense of Arts and Sciences, mathematics is an art as well as a science, and for many mathematicians even more an art than a science. A longing for abstract beauty has been a forceful motor and a trustworthy guide in mathematics and in the so-called exact sciences. Yet, whatever beauty may mean for its creator, it is aimless unless there are more people to enjoy it, and somehow they will have to pay for it, either as a buyer or as a taxpayer. Is beauty measured by its return? Few people are ready to appreciate the beauty in low-level mathematics, and fewer, including even experts, mathematics at the research level. Therefore, as far as mathematics research is concerned, beauty is not a convincing answer to the question "What is the use of it?".

There is, however, another aspect: mathematics is true. *Wis en zeker* -- sure and certain. Research results can be checked, errors can be tracked down, and, if experts disagree, they may do so about good or bad, rather than about true or false. In the so-called exact sciences, checking becomes more troublesome the farther away one moves from mathematics. Anyway the ultimate judge is Nature, who rewards good questions with good answers.

Man and Society are much less co-operative and reliable than is Nature, and this is the very reason why, in the social sciences research is a tremendously bigger problem than in mathematics and the natural sciences. It is a problem that, unfortunately, bothers the assiduous researcher less than it does the not so compliant user -- as long as it is useful to the former, in one way or another.

As early as the beginning of the 17th century Thomas Mun and Edward Misselden discovered and explained the fluctuations of exchange rates by the balance of foreign trade[42]. It was true and useful; as a discovery it was beautiful and historically it is still so since it happened in a period where money meant coined metal and paper currency was at most embodied by bills of exchange. In the 20th century this is no longer true, let alone useful or beautiful.

Research on Man and Society badly needs and distressingly lacks criteria of truth, which are present in all kinds of research on Nature. Just when the following lines were being written, it so happened that both professionals and the public at large were surprised by the announcement of a new and almost trivially simple method of nuclear fusion. Lo and behold, within a few days researchers all over the world undertook to imitate the experiments as described and to check the alleged effects. Not even for a second did I doubt that, long before these same lines would be printed, it will have been settled for ever what was and was not true about it. To be sure, most often it is not as simple as that: it can take more time, greater trouble, and bigger machinery to test an alleged effect. Where Man and Society are concerned, however, each experimental falsification can be met by the objection that it is a fact life that two situations can never be the same; moreover, once an experiment has been terminated, later discovered mistakes in design or execution (if admitted at all), cannot be undone, because the costs of repetition would be prohibitive.

Can the lack of criteria of truth be reduced by trust in the capability and honesty of the researcher? The question is too rhetorical to be answered. Man and Society are developing at a much faster pace than Nature (if something at all like the social version of evolution applies to Nature). Nature is more uniform, or so it Seems to the investigator, who can profit from centuries of experience provided by a stable Nature. This, to my mind, justifies or even requires a more utilitarian attitude towards social and societal research than one is inclined to grant the natural sciences if the question "what is the use of it?" is to be answered.

Keeping the problems we are concerned with here in mind, let me disregard what are called humanities, on the one hand, are as much an art as are their subjects, and, on the other hand, are on no score that matters here inferior to the so-called exact sciences. There are, however, examples of psychological research I like because of their beauty. For instance - I forgot the author's name - the experiment where subjects were given a weight, a string, and a nail to be fixed on the wall in order to construct a pendulum: whether the subjects did or did not succeed to solve the problem depended on whether the string was attached to the weight beforehand.

There is a kind of inquiry that characterises itself by such attributes as listing, stock-taking, registering, recording and fact-finding. If carried on at secretary level these can be enormously useful. They can also be useful at research level, provided the attributes attest to modesty or are as critically handled as they deserve. They can be useful as rough material according to the degree to which they are still rough material. The question "what is the use of it?", however, becomes more urgent the more theorising attitudes dominate or reshape what once was (or never was) rough material.
Henceforth, where research is concerned, my crucial question will be "what is the use of it?" I mean here its use for the consumer rather than for the producer, albeit in a broad-minded interpretation; so broad-minded that the mere intention of being useful is counted as high as the result.
But let us now approach the actual aim of our discussion more closely!

3.2.2 *Educational research*

1. *What is the use of it?*
By education I mean a practice, though often enough this word is used as a synonym for "research on education"[43]. For the present moment, when I speak about educational research in *general* I intend it to include research on *mathematics education* as long as, for the researcher, mathematics is no more than an easily available and easily handled subject matter, chosen to test and apply general ideas and methods, with no regard for the specific nature of mathematics and mathematics instruction.
The enormously rich and useful educational literature, which addresses itself most directly to its presumed users - parents, teachers, trainers, counsellors - is to such a degree and so straightforwardly the result and the expression of everyday experience, craft and philosophy that however they might have been influenced by research, these influences can hardly, if at all, be retrieved nor their impact be evaluated in this vast and wealthy field. This is the reason why I shall restrict myself to the more professional literature on education. Let me summarise my negative feelings beforehand: as a general trend, the greater the pretention with which something is presented as research, the less satisfactorily it comes across as an answer to the question "what is the use of it?" Mind, I do not expect all research (or even a substantial part of it) to be somehow useful. I would simply like the researcher, whatever his undertaking, to ask himself the question "what is the use of it?", and then in a way as though he were one of the intended users himself.
To be sure, a great deal is demanded here. It is a hard thing to place oneself in another person's environment, let alone in his mind; and, on the other hand it is a long way from the researcher's armchair or laboratory to the user's classroom, even though both of them may be located in the same building. Indeed, their mutual distance is properly determined by the length of the chain of mediators leading from the one to the other, provided there are any mediators at all; or other-

wise, by the depth of a gulf that nobody is able or cares to bridge. In both cases, mediation means interpretation, either of *facts,* which is the most fortunate case, or of *words,* which can be preposterous -- producing a situation not unlike the one I depicted in (3.1.9).

My statement about the long way from the researcher's workshop to that of the user may be a source of misunderstanding. Rather than "from" I should have said "between". It is a symmetric relation. Or rather, wherever fresh trails are to be laid, the trail-blazing should start in the classroom; it makes a difference here whether it is the researcher himself who starts the trail-blazing, or some delegates who are expected to carry out his prefabricated instructions, as well as whether the teacher is considered to be a fellow traveller or is expected to stay home.

I failed to mention curriculum developers among the users of educational research; this gap will be closed in 3.2.3. Nor did I mention decision makers, policy agents and legislators. In what respect, if at all, can they be counted among the users of educational research? I do not know about other countries but at ours two kinds of government-advising bodies are to be distinguished: Fact-finding and opinion-shaping commissions on the one hand, and, on the other, councils of -- presumably scientific -- advice. When asked, they will commission groups of researchers to write reports, which, in the case of education, are no more than sources of vocabulary; this helps them to pretend that their final advice, whatever it may be, is not merely a product of common sense. Policy agents behave in a similar way. Whoever their advisors are and whatever their advice may be, they are not used for anything else than rationalising a politically based decision.

2. *Methodology*

Too much metaphorical language? Yes, because as far as I have been informed, trail-blazing is not what the future researcher is told and taught is his business in education. His business is rather: collecting data and processing them according to ever refined standard methods, which are neither argued nor questioned, but simply taught with the aim of being obediently applied to a variety of themes, depending on the prevailing fashion or the whims of the researcher, his director, or his client. I don't remember *when* it happened but I do remember, as though it were yesterday, the bewilderment that struck me when I first heard that the training of future educationalists includes a course on "methodology". This is at any rate the custom in our country but, judging from the literature in general, this brain-washing policy is an international feature. Please imagine a student of mathematics, of physics, of -- let me be cautious, as I am not sure how far this list extends -- impregnated, in any other way than implicitly, with the methodology of the science he sets out to study; in any other way than by having him act out the methodology he has to learn! In no way do I object to methodology as such -- I have even stimulated the cultivation of it, but it should be the result of *a posteriori* reflecting on one's methods, rather than as an *a priori* doctrine that has been imposed on the learner.

I readily admit that the principle of "learn first, apply later" works in educational methodology no better than it ever did in mathematics, that is, where it works it does so to the benefit of a small *minority* of learners only -- the future specialists of methodology. Yet, fortunately, the intimidated *majority* can count on the precious assistance of this authoritative guild, the pure methodologists, whose strength consists in knowing all about research and nothing about education. They gladly leave the educational researcher a responsibility of his own to fill empty methodological vessels with educational contents, and they are unconcerned with the question of whether these fit or not. Mathematics can afford a vivid interplay between form and contents because in playing this game they can be neatly separated from each other. In educational research, however, separating form and content is not feasible, and, wherever this is attempted the consequence is estrangement. This is covered up, if need be, by distinctions like those between *validity,* which is the researcher's business, and *reliability,* which is the methodologist's responsibility and which by mere convention is measured by a formula that, in fact., does not measure anything that deserves this name. Please imagine a biologist who has defined a measure for immunity against some plant disease that can be determined with a high degree of precision while it is most doubtful whether it has anything to do with resistance against that disease! Who would accept such a measure, reliable up to the third decimal if nothing is known about its validity? This may be a rhetorical question in biology but not in educational research, where it can only be answered historically: formalised testing was first concerned with intelligence; when the query what, if anything at all, the I.Q. did measure, had to be shelved, one became satisfied with measuring this same thing as reliably as possible. From the I.Q. this easy answer spread to psychometric measures in general, as did the habit of measuring first and trying to explain later. This is a bad habit, since it might be a short step from bad to worse: as soon as scientific scruple and honesty are degraded to mere formalities, it is almost unavoidable that precision and accuracy will count only as far as they can be expressed in numbers of decimal positions.

This may sound exaggerated, and in a sense it is. Let me quote myself from [87, p.145]:

> According to a well-worn joke there are three kind of lies: lies, damned lies, and statistics. When the joke was invented, mathematical statistics had not yet come into being. Otherwise one would have added three kinds of surreptitious credibility: by word of honour, by oath, and by mathematical statistics. Or, three methods to assume a learned air: footnotes, bibliography, and correlations tables.

This is a witticism, to be sure. But even so it is the gist of many years of -- sometimes disgusting -- experiences, which I gained as a reader, reviewer, referee, and editor, and of which only a small but paradigmatical part has been published. At no point have my detailed scrutinies of small and large pieces of educational research ever been refuted, nor, to the best of my knowledge, has anybody ever seriously tried to challenge them on minor or major points, whether published [99, 102, 116] or not.

Matters are even worse. The collection of poor research I criticised was nothing more than a sample chosen from what I came across. My actual target was the underlying research method, imposed by means of the authority of a misleading and misled methodology, which is still taught today, although I feel its authority is waning. Anyway, I don't know of any attempt to defend this methodology or parts of it against my criticism. Or is this a policy of killing by silence? Yet I received no positive signals either. As far as I know, up to now nobody has ever tried to explain the mathematical meaning and the cognitive function of correlations within the various contexts -- where educational researchers have computed them by tens of thousands and arranged the min tables -- and their use other than for impressing laymen and sponsors. Nobody has ever tried to vindicate the normality of distributions that, by definition, are not normal, or to justify linear regression as such, in particular, if applied to connections which are obviously non-linear, as in the case of stable equilibrium. I have never seen any explanation of how mathematical methods from biotechnics -- where their applicability is warranted by models from physics, chemistry, and biology -- can be transferred to education, which lacks any fundamental insights for building intermediate models.

I cannot but recognise the fact that some things have changed in the mean time. Taxonomy, for instance, has fallen into disgrace (though not because of its inherent weaknesses but because it is too strongly a contents-dependent tool) and is now successfully being shipped to developing countries. Test theory and technology, on the other hand, have boasted ever new successes in fighting sound education. Qualitative research is gaining ground, but we are still far from the point where mathematical methods can add a finishing touch to qualitative knowledge, and many researchers are even farther from the insight that mathematics is not able to do more than just this. But it is hard to fight the prevailing superstition that ready made mathematics can solve all problems.

Reluctantly I reiterated my old complaints on educational research. My intention was actually to review them while asking the question "what is the use of it?", other than cultivating professional and public relations.

There is a hard struggle for life and grants going on in the community of educational (and some social) research. One cannot do without advertising and the headlines in newspapers, or so people think. Is this a necessary evil, and if so, why can it not be used for good things? Or do people who are wary of this policy suffer from allergic reactions to it?

Are good things not expensive enough to be supported? When the first IEA cycle started it was the most pretentious project ever undertaken in educational research, and the most costly -- some 50 million dollars if corrected for inflation, I would estimate - and when it was finished it turned out to be a mess: a heap of spurious data, processed by means of nonsense mathematics, and - what is worse -- useless to such a degree that after a few years even in the most extensive bibliographies ignored it. What was the use of it? Was the road to New Math paved by the vast publicity that IEA achieved in the USA with their first report, which proved the inferiority of American mathematics education? I don't believe this

was really a major factor, although years after New Math's eventual failure, T.Husén, proud of the hundreds of headlines it had earned him in the American newspapers, still thought it was.
In my country, IEA received even Dutch TV coverage because our Netherlands pupils had scored so badly in a subject area they had never been taught -- a fact unknown to the inexpert researchers -- and because they had performed so excellently in other areas, thanks to international test instruments, debased by translators, who had not been informed about the aim of the translation.
As far as I know, no international report on the second - much less pretentious - EA Mathematics Study has been published. In our country it made the headlines because we finished second or third in the international field, albeit thanks to the fact that among the various types of schools the possibly lowest scoring one had been left out of the tested population: this was "special" education, which, according to our (by international standards unusual) definition includes about 10% of the school population under consideration[43a].
A similar mistake has marred an otherwise excellent national study of ours on the quality of primary mathematics instruction. For our daily papers girls *score much lower than boys* was the most important news it revealed. As a *fact,* this contradicted worldwide experience at the primary level, but rather than being a fact it was an *artifact,* due to the same omission as mentioned above, and easily explained: only behind the desk do researchers forget that, in our "special" education, boys form the overwhelming majority. The blunder could easily have been avoided, but the report was not even made available under embargo to experts, who had to wait one more day than the press to get information and to discover the blunder. This in fact happened at an educational meeting the day after the press conference. Too late, of course, since *girls equal to boys or even better* is not worth the trouble of a correction.
Who does not remember the Coleman report -- the most extensive, pretentious, expensive and impressive report, ever carried out in the USA? Or rather, who would at present still remember it for any use whatsoever, even if it had not been wrecked by sophisticated analysis [68] [44].

3. *Comparative research*

A large part of educational research is comparative. Cognitive or affective results of education are compared with one another according to such variables as nationality, instructional system, educational philosophy, schools and school system, gender, ethnicity, race, socio-economic background (of pupils or teachers), the teaching and evaluation methods applied, the aids, instruments, textbook series used -- a list that is all but complete. I do not object to comparisons. Whoever judges a given kind of education, will relate it somehow to the experiential or ideal models he has in mind; the more consciously and systematically this is done, the more valuable the result may be, in particular, for enriching experience and improving models. But "systematically" should not mean: according to a preconceived system, in the mechanically thoughtless way, as taught by methodologists.

In education, comparing is a most precarious business. What are comparable situations and how to get them? We just gave examples of rather simple and transparent situations where researchers overlooked obvious differences. How to avoid this? One way is to hold strictly controlled instructional experiments; this is, in fact, both the most efficient and the most useless way, since rigid instruction is the worst instruction one can imagine. One can try to eliminate disturbing factors, and if this is done by common sense, I would not object either. But common sense is suspect to methodologists bent on employing refined mathematical methods such as factor, variance, and regression analysis. To the best of my knowledge they have never been rationally justified, either in general or in some particular case, and, depending on the way they are applied, can often be used to prove anything and its opposite. Moreover, little if anything at all is known about influential factors, for want of a fundamental theory on the subjects under consideration. Some people try to make up for this by the sheer number of factors, chosen *ad lib,* a policy that increases the impressiveness and decreases the credibility of the applied mathematical technique. However it is done, this is merely an attempt to apply mathematics to bunches of numerical data, irrespective of the way they were gathered, with no intervention of any model based on theory. People who know that this is wrong excuse themselves by bestowing the dignity of model on any linear regression equation they write down; they do not bother to justify the hypothesis of linearity, even where common sense suggests that it cannot be trusted, as in case of equilibrium.

4. *Tests*

Rationality requires statements to be tested for their truth instructional methods, for the benefit of education, and individuals, for their own benefit and that of society. Tests should be trustworthy, and what is more trustworthy than numbers, obtained by measurement, and moreover amenable to mathematical processing? I already emphasised that, in the study of Nature, measurement is rather a finishing touch, applied to models of physical reality. These models are there to explain what has to be measured by what yard-sticks -- the more rigid the better -- and for what aim. Man and Society are much less accessible to such models, and often enough, in particular in education, they are not accessible at all; yet in no way has this hindered the tremendous growth of psychometry in education from its first roots onwards, from early in our century to its present state. Let me quote B.S.Bloom & A.W.Foshay[45] as witnesses for the high esteem in which tests were (and by many still are) held.

> There is one field in which a considerable sophistication has developed since 1920: the field of achievement testing. It is possible now to study the degree and nature of a student's understanding of school subjects with a subtlety not previously available. Modern objective achievement tests, when properly developed and interpreted, offer one of the most powerful tools available for educational research. Findings have been made through their use that rise far above common sense.

In all fairness I must say that less high-spirited appreciations are voiced by others who co-operated in the same IEA-study from which this quotation stems; yet, as far as tests are concerned neither Bloom's celebrated Taxonomy[46], nor work accomplished under his guidance [cp.99] can be looked upon as paragons of sophistication. And what about the last statement of the quoted passage? Where should one look for such "findings"?

It cannot be denied that, bit by bit, a highly sophisticated testing technique has been developed, and measuring sticks have been constructed, which, from the viewpoint of rigidity, can compete fairly well with those of the natural sciences, although this might probe to be a disadvantage rather than an advantage. In the narrow sense of this technique, the word "test" now belongs to the common vocabulary of most languages. The attribute "objective", which is only added when strictly necessary, nowadays means that the formerly expensive job of testing can be left to the much cheaper computer. (In my own country, *objective examiner is* a governmental-hypocritical euphemism for cheapest examiner.)

Objectivity, if opposed to subjectivity, is indeed praiseworthy, if people are to be judged. Yet, rather than atpeople to be judged, testing technique aims at objects defined through tests or test batteries and allegedly independent of human caprice, although they are given names, which suggest connections to human life and learning. It is a process of rectification by which new objects are created, with the sole aim of being studied numerically, and where no regard is paid to their effectual state nor their usefulness for understanding and explaining phenomena. This process is started, sustained and steered by a quite natural tendency to detach form from content, which, although it has proved enormously fruitful in mathematics, thanks to its, as it were, self-controlling character, is condemned to sterility wherever it escapes control.

Although the present exposition started with a quotation involving achievement tests, it applies to any kind of tests in the technical sense, including opinion polls, such as preposterously used for research in the affective domain.

The above criticism of formal testing technique with regard to research in mathematics education may appear to be outdated: There is an unmistakable trend away from it among educationalists with a strong mathematical background, who have good reasons to do so and are courageous enough to no longer be afraid of being reprimanded by methodologists for neglect of "objective" methods. Open testing, interview, and observation of learning processes are gaining ground, and well-designed intersubjectivity has is given a chance to supersede ill-conceived objectivity. Little attention, however, has been paid to the methodology behind the new approach -- J. deLange's work[47] is here quite exceptional. Anyway, there are promising perspectives on fresh approaches. Will they be successful in the long run?

Technique and techniques are indispensable and beyond praise in our society, at least as long as they deserve praise. If they are blamed, it is for what nowadays is called pollution. How far has the pollution by testing techniques spread in education? The negative influence examinations can exert on education by their mere existence is a well-worn truth, and nobody can tell how much testing technique has added to it. How much has diagnosis and how much have diagnostic abilities suffered under the influence of badly understood diagnostic tests? And how much profound qualitative research has never been undertaken because it could not compete with shallow quantitative pseudo-research?

3.2.3 *Developmental research*

1.*Change*

Formally, the present subsection looks like it is one more part of the preceding one. I could have formally separated them by simply adding the adjective "traditional" to the title of the former subsection, but I preferred to avoid this because of its negative connotations. Moreover, what we are now accustomed to define as developmental research is by no means a new thing. On the contrary, it is perhaps the oldest kind of educational research, though it has not been recognised as such.

The question on educational research I started with was, "what is the use of it?" I didn't ask the question "what is the use of education?", which is almost rhetorical as long as it aims at education as such rather than at a special kind, say, actual or topical education. No doubt, education is meant to prepare one for future life, yet neither the future of individuals nor of society are predictable well enough to provide a solid basis for any choice of education, which, conversely, is influential as a condition of this future. Once, when asked by an interviewer whether I thought that attempts at innovation have improved education, I hesitated for a short while, only to eventually stamp this as a wrong question. Pictures of education, taken at different moments in history, cannot be compared. Each society at a given period got the education it wanted, it needed, it could afford, it deserved and it was able to provide. Innovation can effect no more than to adapt education to a changing society, or at best it can try to anticipate the change. This alone is difficult enough. If any proof was needed, it can be seen in the efforts, vaccillating between childish and spasmodic, to prepare for what is commonly called the computer age, though in fact the computer is only one of the features of this age. Or, to take an example from the past, a past that started after World War II: Even now innovation has not yet succeeded in coping with the fast growing demand for education, which is too readily answered by "more of the same", rather than by diversification of the supply. So the gist of this paragraph is to give a the first answer to the question "what is the use of it?": *Change.*

2.*Errors*

If I am not mistaken the first subject formally studied in mathematics education - which means, as early as the turn of the century - was errors, that is to say, *learners' mathematical* rather than *teachers', developers'* and *researchers' didactical* errors, and according to research published up to a few years ago this is still the most cherished subject - the first kind of errors, of course, while the second has hardly been noticed. Even nowfailure rather than success is the main discriminator in many research papers. Even though, the variety of investigated errors has increased in the course of the century. It is remarkable how much attention has been dedicated to the same kinds of errors over and over, albeit more and more related to an increasing number of other parameters, such as age, gender, peer groups, ethnicity, race and social-economic status of parents and teachers, performance in other subjects, intelligence, affective features such as mathematics love or anxiety, and so on.

The attention to errors is as indispensable as to physical or mental diseases, provided it is guided by the wish to cure or to prevent and is undertaken in a context in which this wish has a chance to be satisfied. I recently attended a meeting where a researcher presented a new diagnostic instrument for mental arithmetic in the domain 20 to 100. He modestly added that instruments of remediation were not the researcher's but the expert teacher's concern, and he didn't change his view even after a teacher in the audience urgently asked him about the use of diagnosis.

Mere diagnosis of diseases is a most valuable statistical tool of epidemiology on behalf of medical policy but in education the epidemiological approach to failure testifies to a trust in educational policy that is hardly justified. Diagnosis may be important for employers and insurers, and it cannot be ignored where the product of education is what matters, but even then processual diagnosis should be given a fair chance. The trend in medicine to focus on health rather than on illness is almost too obvious to warrant being mentioned. There are no vaccines against mathematical errors. On the contrary, and in particular if mathematics is to be reinvented under guidance, errors are unavoidable or even welcome provided they are used to stimulate resistance against their own consolidation and recurrence.

Ideas on the need and possibility of aetiology - to stick to the medical terminology - are as rare in research on errors as is the attempt to recover the past of the learning process and insight into the relevance of long-term learning processes.Errors as 62 - 45 = 23 have been signalled hundreds of times, but the strong suggestion that they are due to a premature separation of form and content - positional system and subtraction context, in the present case - has not even been a subject of research as interpreted by the majority of researchers. In education, it does not yet go without saying that aetiology is a condition for effective remediation and prevention.

The inclination to study error is quite understandable. Normal things need no explanation, but what really strikes the beholder are deviations from the norm, that is, from what is felt or agreed on to be normal. Education, as looked upon in its present state, is taken for granted, and against this background of certainty its actual or potential diversity can be a matter of study that handles liability to success or failure as its criterion - comparative research, which is questionable as such - rather than ætiological.

Are educational researchers indeed as conservative as that? Or wasn't New Math a historical counter-example? If at all, this example was research on how to put new wine into old bottles -- is it *Enseignement des mathématiques modernes ou enseignement moderne des mathématiques?* I asked in 1961 [14].

3. *The cradle of developmental research*

Most research is on factual education. Developmental research is different, and, as I already claimed with some reservation, historically it even preceded formal educational research in mathematics (and other fields). In the past, mathematics instruction progressed by the activity of educational developers, in general documented by textbooks, which are still to date the main agents of change. When I claimed that educational research started developmentally, I admitted as well that it was not recognised as such, and I meant -- not even by those who carried it out. I didn't actually say it was the oldest kind; I said "perhaps", which means that the statement depends somehow on what is understood by "research".

In (2.3.1) I recalled that, as early as 1961, I propagated thought- experiments as instruments of educational development. I should rather have spoken about drawing attention to their function in this activity. Indeed, it is clear from the context that I took it for granted that thought-experiments had always functioned this way and that the only thing lacking was consciousness of it. Observing their thought-experiments and reporting on them would have transformed educational developers into educational researchers, who present their findings together with the arguments, rather than dogmatically.

Although factual experiments would be more convincing, thoughtexperiments will remain indispensable (at least in order to prepare factual ones) as long as educational development is being undertaken, and this will happen as long as the need for change is felt. Tradition and change, and their mutual relation. are fascinating subjects. In [40] I dealt with their role in mathematics and in education. A good teacher, who does not blindly follow traditions dictated by his own past or by prescribed and prescribing textbooks, acts as an educational developer, who most often is unaware of this role. In fact, awareness of the motives of one's own activities can be either useful or harmful, depending on the situation, the action and the actor. A teacher can be struck after school by the idea that "today it was different". Was it really different, and if so, what is the "it" that was different and why was it so? Soul-searching isn't everybody's business, but change through accident can be a source of search, just as search can lead to change.

Is this simply a bunch of trivialities? I wished to argue that even nowadays research can start at the grass-root level, that is, under no other conditions than that of becoming aware of one's new experiences, as well as the wish to understand them in the perspective of a change for better, and that at any time it can restart under the same conditions. This is not to belittle high -level research but to prevent it from clogging its most abundant sources.

4. *R&D*

The title of the present chapter promised to deal with Research *in,* rather than *on* Mathematics Education, not in order to exclude the latter but to emphasise the

former. This, then, characterises developmental research: it takes place within the educational environment, which is expected to undergo and to activate change.
"Development" made its entrance as a technical term in the nineteen-sixties, when educationalists were trying to take advantage of the then noisy call for change. It appeared in the combination R&D -research and development - which was soon amplified by another D, that of dissemination. The terminology itself was borrowed from the natural sciences and their technology, though, as I have often emphasised [e.g.80], this happened with little regard for its meaning in that field, and without asking whether it adequately applies in education. In the natural sciences it alludes to the road leading from theoretical work to technical applications: the test tube replaced by retorts of more and more industrial dimensions, the laboratory by ever bigger workshops, the researcher's desk by the drawing board and the computer. This is not the way it takes place in the social sciences -- except perhaps in economics. Theory may be a source of inspiration but seldom, if ever, the place to look for germs of development towards applications. There is research for creating means of organisation and processing information, such as terminology, methodology, test theory, statistics, but all this is applied in a descriptive and inventorising, rather than in a developmental way. For a discovery in chemistry or biology to be developed industrially, buildings get designed in one way or the other, though what really matters is not the structure but what it contains and what is intended to happen there. In education, however, the R of R&D is concerned with the formal element of the building rather than with its contents.

The problem with the educational R&D terminology in everyday practice are the strange consequences, as nicely illustrated in the following example: creating new courses of instruction is filed under "development", even though it requires -- at least implicit -- fundamental research, whereas the mere routine of testing or adapting a given course on behalf of, say, level differentiation, mastery learning, and so on, falls under research, which is likely to be held in higher esteem and correspondingly be subsidised more easily and more generously than development.

The R&D mania stimulated productions like curriculum theories, general goal description theories, taxonomies, hierarchies, model designs. In [87] I still judged these worth punished, while nowadays it is hard to find anybody who even cares about them; so I can only hope that the same will happen to system theory, "cognitive psychology", and other things, which are imposed on or proposed for mathematics education by general educationalists. Indeed, even though the R&D terminology has been dropped, the R&D mentality still subsists, by which I mean the mentality of separating research and development.

5. *Cyclicity*

It still subsists, that is to say, in stale theory. Practice, at least in education, requires a cyclic alternation of research and development, which can be more efficient the shorter the cycle. What was developed behind the desk is put to the test in the classroom in order to be analysed, and development is resumed with the results of this analysis.

This seems to be a trivial statement, and indeed the alternation between thinking and doing, planning and testing is nothing more than the everyday way of life. *Developmental research* means that this alternation is understood as a macro feature, rather than a micro feature of the domain where it takes place and of the flow of events. By this broad field and long-term feature it distinguishes itself fundamentally from what is sometimes called *constructive research.*

How essential is the fall-out of development to deserve the name of theory, how essentially fertile is the development stimulated by such theory? The feature that gives developmental research the right to the name of research is its bringing to consciousness that what is fundamentally new and essentially fertile in research that is arisen fro educational development.

But I should not leave it at this. Educational research as fall-out of educational development - this again erects a dividing wall, which even if only a curtain, is not flexible enough to do justice to what developmental research means. "Development ensuing from research" and "research as fall-out of development" is too weak a synthesis. How can we achieve a stronger and more efficient synthesis in order to answer the question "what is the use of it?"

6. *Proof by process*

Rather than R&D, I borrowed another feature of research in the natural sciences on behalf of education: the thought-experiment. Design, even if undertaken behind the desk, is a better start than stale theory, but it also better than a long series of thoughtless experiments. Physics, as we know it at present, originated neither theoretically nor experimentally, but in a rational way nevertheless: from ideas, suggested by the analysis of nature and tested later -- often even much later. The double helix was a splendid idea for the mental reconstruction of DNA, and it became the model for numerous successful reconstructions before it was actually proved, that is, a quarter of a century after it had been proposed the first time.

The high tide of attempts at educational change after the Sputnik shock flooded mathematics education with new programmes, new teaching matter, new courses, all of them contrived behind the desk, discussed at symposia and, as a matter of urgency, pushed into classrooms. Little was ever put to the test, and still less survived even the most benevolent trial. In fact, innovation is an arduous process, and gives the lie to the most cautious predictions. My criticism at that time did not aim at the "behind the desk" origin of the innovation projects -- how else should one proceed? -- but at the lack of any report on the underlying thought-experiments, if any. Researchers publish *products* of their activity, rather than the *processes* by which they were created; the knowledge of these processes is considered to be their private domain; and so did it happen in this case too.

This policy may work in mathematics, where a proof, if sound, justifies the theorem and itself, although didactically viewed it is less efficient even there, It may still work in experimental sciences, where experiments, if satisfactorily described, can be reproduced by others more or less easily. In the social sciences, however, such a situation is quite exceptional. How, in any case, can one reproduce thought-experiments that have been kept secret?

Knowledge can successfully be presented as a product if the process of its acquisition is reproducible - a characteristic of "hard" science. Wherever this condition is not fulfilled, knowledge presented without any indication of the process that brought it about, lacks all characteristics of rationality that distinguish genuine knowledge from dogma. It is unproven, unargued, unfounded, or so it seems. How to transfer such a knowledge, how to have others reinvent it if one keeps silent about that which is the most essential?

This question is particularly pressing in the case of educational development. Taking notice of the product, which allows for many interpretations, is not enough. In order to apply the product, one must know how it came into being. Not in detail, of course; the point is not to get stuck in blind alleys but to be warned against them. Reproducing does not mean parroting.

7. *Dissemination as an agent of synthesis*

Above I claimed that "development ensuing from research" and "research as fall-out of development" is too weak a synthesis. For reasons I just expounded, the idea of dissemination, which in the RDD conception is a separate component, should be kept in mind the entire time that the developmental process lasts. In short,

developmental research means:

experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience.

In contrast, *constructive research* is a more modest conception than *developmental research* of which *transmission* is an essential feature. To be useful, constructive research trusts its results to *implementation.* In education, or at least in mathematics education, little that can be transmitted with impunity as mere implement. Developmental research asks for more by including the view on *dissemination.*

Anyway, dissemination should not be separated from development. This requires a continuous awareness of what is happening in the ongoing process, a state of permanent reflection, recorded as much as possible[48]. Dissemination is one of the aspects under which hard and soft knowledge differ from each other. Hard, that is wrought iron - it does not matter how it was wrought. Products of educational development can also feel hard - too hard to be applied. Hard science shall meet hard criteria. Where should one look for criteria for soft products? The answer is: among the hard facts of the developmental process. They are needed to bear witness for the product in order to make it plausible and transferable, which demands an attitude of self-examination on the part of the developmental researcher: a state of permanent reflection.

8. *The Landscape*

Dissemination is a misleading term if it suggests sowing after harvesting. In fact, from the outset it should be closely connected to the educational development itself, while the seeded territory is expanding in step with it. Both teachers and pupils are partners on equal standing with the professional developers in the rapid alternation of activities behind the desk and in the classroom; they are simultaneously sowers and harvesters and educational developers, like the professionals. (Let me add that I never did like experimenting with human subjects other than while giving them a chance to profit from the experiments themselves.)

Developmental research includes
- development on behalf of
  - first line instruction
  - teacher training, guidance and retraining
  - guidance and retraining of teacher trainers
  - guidance and retraining of counsellors
  - testing
  - media
- advising and supporting
  - educational development by other agents (schools, working groups, text book authors) in a broad sense, with the emphasis on co-operation
- creating cadre for the increasing territory of dissemination research
  - within the development
  - as fall-out.

Developmental research covers
- longitudinal developments with a view on
  - long-term learning processes

and is itself a
- long-term learning process.

9. *A bit of history*

In order to do justice to a repeatedly professed philosophy by me, I ought to tell in a few words how developmental research, born by necessity and owned by reflection, was baptised.

In 1961 the Dutch government installed a Commission on Modernising Mathematics Education (CMLW), which as late as the nineteen seventies was followed by similar commissions for other subject areas. The way CMLW interpreted its task foreshadowed later developments [24]. Soon the need was felt for a centre that would combine and co-ordinate their numerous activities. Early in 1971 the CMLW was finally given an Institute on Developing Mathematics Education (IOWO), which was loosely connected to Utrecht University. The mere name of the institute implied a programme: its focus would be on educational rather than on curriculum development. From the start onwards, the task of IOWO was interpreted as one of integrated engineering within the total "Landscape", as described above, with the vague perspective on possible theoretic "fall-out" as a look into the future [37,44,46].

General educationalists found fault with this heterodox unprofessional approach. Indeed, IOWO people could not answer questions about the curriculum or learning theories they adhered to, nor could they produce catalogues of learning objectives, simply because they did not have any. The only things they had to show were a philosophy on mathematics and mathematics education -- this had never been heard of -- and a grand experimental design, which, to make matters worse, failed somehow to mention control experiments. To be honest, I admit that IOWO was taken seriously in the long run -- or even dreaded -- by general educationalists, even by those who had other ideas on educational development and research.

Before IOWO's arrival the educational "Landscape" had neatly been divided into sectors of Training, Counselling, Retraining, Development, Innovation, Test Production, and Research, each represented by a single institute or a group of institutions, or this was the theory. At IOWO all this was integrated. So IOWO was a notorious case of unfair competition. As such it was particularly feared in the politically most powerful among these sectors, which up to the present day has unsuccessfully struggled for a genuine educational identity of its own.

Anyway, one day the government decided to put an end to this chaos and, as behooves bureaucracy, to neatly divide the landscape (as far as subsidised) into sectors that were to respect each other's borders -- a ruling that afterwards was even written into law. Although this law strictly forbids any integration, it allows cooperation between workers or working units of two or more institutions of different sectors, which in any particular case requires a tedious procedure of approval by managing directors, boards and councils of the respective institutions. Yet bureaucracy is not almighty. Meaningful work is still being done and subsidy obtained on behalf of this work by people who know the loop-holes, though it is not an easy way of getting work done.

For the IOWO the clearing of the "Landscape" meant its death sentence. Only a short "research" leg should be spared, which was implanted (together with the educational computer centre) as a department into the Faculty of Mathematics of Utrecht University; since that time this leg has grown quite a bit stronger. Formally stripped of educational development, it is now known as Institute for Research on Mathematics Education and Computer Centre (OW&OC).

IOWO, of course, had fought back against its own death sentence, and this struggle gave birth to the term "Developmental Research", which is the reason why I told the story. Under pressure of public opinion and parliament the death sentence was commuted from decapitation into mutilation; as mentioned before, a research leg should be spared so a broad definition of research was in the best interest of IOWO. Our integrated approach had been successful, and none of our people could imagine or trust nor feel able to try another approach as promising as the proven one. What's in a name if only it covers a good thing? If it was a lie, it had short wings. Though it was not, it did not help. Even though the name looked new, Developmental Research was not an *ad hoc* invention, or so I hope to have convinced the reader. It simply was the approach that had proven successful in the past and promised success in the future. Through its struggle for life IOWO, understandably became conscious of its true identity, as well as of the need for a name to distinguish it from others.

10. *Developmental research - a conclusion*
Did I emphasise developmental research unbecomingly and to the detriment of other views in this section on Research in Mathematics Education? My proper criterion for judging educational research was formulated as early as (3.2.2, 1): *What is the use of it?,* interpreted in a broad sense. My answer, in general, was: Change. How much can research contribute to this? Or rather, how far should research participate in change? My answer was: As *Developmental Research.* (Notice the switch from "can" to "should", and from "contribute" to "participate"!) "Developmental research" is meant as an ideal yardstick and a critical touchstone, and it will be used as such in the next section, where developmental research implicit to practice will be made explicit.

3.3 PRACTICE OF MATHEMATICS EDUCATION

3.3.1 *Practice*

In (3.2.3,8) I described in detail the *Educational Landscape,* which along with *first line education* includes what in my country is called *Service for Education.* In the Netherlands this is quite an extensive, and for the most part government-subsidised, bureaucratic structure, strictly compartmentalised by law (WOV), comparable to the guild system of crafts of former centuries (3.2.3,9).
In the preceding section I dealt with the *Research* compartment of the Landscape (that is, with the emphasis on Mathematics Education). As a name for the remainder I could find nothing better than that of *Practice,* which on the one hand is or should be the customer of *Research,* and on the other hand looks like the counterpart of *Theory,* which was dealt with in the first section of the present chapter.
"Practice" has both a descriptive and a normative meaning. If it is understood in the descriptive sense, its qualitative evaluation covers such a broad range that it defies any attempt at description, even when restricted to one single country and one single type of institution or part of it. So the normative element cannot be dispensed with in organising a description, Moreover, individual practitioners or groups of them act according to explicit or implicit norms, which compete with or are derived from more general ones. These norms are subjected to change either pursued by the practitioners, or demanded by the public, the authorities and the theoreticians; and, after all, if there is any use for descriptions, it is in general to serve in general as -- positive or negative -- norms. I shall not try to disentangle these aspects from one another, and as it happens, norm will dominate descrip-

tion.

3.3.2 *A background of competence*

How competent may I feel to deal with practice in its descriptive or normative aspects or in a synthesis of both? Not until 1970 did I shift my interest fully to education, and even though I participated for a decade in the everyday activities of a number of schools and training institutions of various kinds, I should confess that I have never taught school nor even teacher training classes. Although I tried to look inside as deeply as I could and as is needed for being a participant, I remained an outsider, one who casts glances into the educational system and acts within it, without factually being a part of it.

To make things worse, I know almost nothing about other educational systems when compared with my knowledge of the Dutch one. Of course, I have read quite a bit about some of them, but what does this mean? If I were to judge our Dutch educational system on the strength of written and printed documents, I would get a severely distorted picture. Not because people who write about education are liars, but for the simple reason that the bulk of what is written is wishful thinking rather than description. As far as it is descriptive, it is most often written from a distance where details, among which perhaps the most important, are blurred, clouded and obscured. So I must warn you with regard to all that I am going to say -- about what in practice is right or wrong, what can be improved and how it should be done, what should be changed and what is better kept as it is -- that I am my own spokesman and the interpreter of my own experiences, which were acquired in my own country at the places I just happened to visit. This, then, may involve ideas that apply nowhere else or, the other way round, it may mean labouring the obvious.

To be honest, I am not as pessimistic as all that. If I were, I would not dare to tackle practice. Even if various educational systems appear more different than in fact they are, I firmly believe that these are deceptive appearances. In particular, if our gaze is focused on nations that share similar social conditions, it is quite improbable that their educational systems diverge as much from each other as they make it appear. Even in one and the same country, with a narrow or broad scale of diversity, attempts at producing change are defined to a higher degree by social determinants than by individual or group efforts, which of course, cannot at all be dispensed with. To repeat a joke I am proud of: I believe that in my life-time nothing has had as much impact on education as the ball-point, the overhead projector, and xerox -- inventions that have not primarily been made on behalf of education. Or am I exaggerating? In a sense I am. Besides the ball-point, overhead projector and xerox, one needs people who use them, and preferably, people who use them in a creative way.

This restores my confidence: we are living in one world, most of us under similar social conditions, and even though learning one's own language and learning to teach it may mean different things in countries where people speak other languages, mathematics is the same all over the world, and so learning mathematics and learning to teach mathematics and to develop mathematics education are the same as well.

3.3.3 *Taught and learned -- the subject matter*

Let me reconsider my last sentence! Is that what I just claimed to be the same, really the same all over the world? Or is, if anything, the teaching subject the same? If one compares mathematics as taught in various countries, certainly at the secondary level, one would be inclined to say it is not. The Bourbaki caricature taught in France strongly differs from the Kolmogorov version in the Soviet Union (or are they really taught as such?), and both of them differ from what is taught at British O and A levels, and still more from what pupils are expected to learn in highschool in the USA.
Well, there are broad chasms between what is being taught in various countries. But does it matter? I don't think so. As I see it, if things are to be compared, the proper question to be asked is not what is taught here and there but what is learned, what is really learned, what lastingly affects the minds of the learners. I cannot believe that it differs that much in different countries. I don't believe it because I think it cannot possibly be so. How can people in various countries differ intellectually so much that whatever they are taught they can learn such different things as those different kinds of mathematics? I would prefer to say that people on the whole, and in particular young people at a certain age, are all the same under similar social conditions -- even beyond comparing averages. We need not be haunted by feelings of inferiority when confronted with the awfully abstract mathematics that, say, sixteen years olds are supposed to understand and to master according to, say, French, German, or Soviet textbooks. Tests have made it clear beyond any doubt that the vast majority of pupils have not the slightest idea of what is meant by this highly sophisticated mathematics that they are supposed to have learned. Every year at the national examinations of 16 years olds in my own country I am struck by the wide gap (extending over many other subjects areas, indeed) between the boastful pretensions of the tests and the poor performances -- a gap possible thanks to an evaluation of the results that belies the expectations of the test producers.
I like to call it the big lie of our educational system, and I am afraid that this characterisation applies to quite a few others: there is a wide gap between the inflated demands and the allowable passing levels. Exams serve selection: they single out those who can satisfy the most exacting demands. Formal tests are constructed in order to discriminate, as test producers call it, which means a 50% failure rate. But why should we ask the great majority to aim infinitely higher than they are able to reach? Not only infinitely higher, I should say, but also in the wrong directions. Learning to forget was the title for (2.4.1). What is the use of it? Learn-

ing simple mathematics at a reasonable level is a more dignified pursuit than learning complex mathematics at no level of understanding at all.

It sounds almost trivial; this focusing on what learners *did* learn, rather than on what they *did not.* It is the most natural thing during the learning process, so why should it not be natural at its end? Why not adapt the demands qualitatively and quantitatively, and with regard to substance and depth, to the learners' ability, as proven by their accomplishments?

I used the plural "learners" intentionally, and if this is taken into account, my statement is less trivial than it sounds. We teach *classes,* so why shouldn't we expect *classes* to learn? Life is cooperation. Why do examinations focus on individual performances and entirely disregard collective ones? But let us delay answering this question!

The wide gap between what is claimed to be taught and accepted as being learned -- the big lie -- is inherent to any system that strictly relies on exterior control. The gap is reduced or absent at institutions that are trusted to set standards of their own. This is true in any case for my own country, where uniformity of teaching, learning, and examining stops at the doors of Institutions of higher learning and the prospect of the examination is the only measure that puts restrictions on the freedom of teaching. I don't know enough about other countries, but I still remember tests at FRG universities a few decades ago, where the majority of freshmen who had learned a vast amount of Calculus at school, didn't master the most elementary algebra. Have things changed in the mean time? I am still puzzled by the contrast when I look at the kind and level of mathematics that future primary school teachers are supposed to master in my own country and in the FRG. Are their students so much more mathematically minded than ours and is teaching 6-10 year-olds elementary arithmetic so much more demanding in Germany than it is in Holland to justify requirements in mathematics as high as theirs? Nowadays, students who are able to fulfil such requirements are not very likely to aspire to primary school jobs -- this is the case in my country, and I doubt whether they are more likely to elsewhere.

3.3.4 *Taught and learned -- the agents*

Learning and teaching -- aren't these not outmoded terms? In Education the fashion is now to call it change and transformation from *novice* into *expert,* although this is more than a merely linguistic fashion. Two well-defined states are assumed with respect to a certain knowledge or ability: the clean slate, and mastery. The agent, responsible for this transformation, is supposed to be an expert himself, which means a *product* expert. The intended *process* that transforms novices into experts is likened to that which takes place on an assembly line. It has been well-designed on the strength of *cognitive or instrumental analysis* -- a kind of thought-experiment, where the subject to be transformed from novice into expert is supposed to be acted on, rather than to be an agent.

To be sure, one can imagine more refined versions of the expert-novice model, which may grant the learner more freedom: choosing among several assembly lines, for instance, or even within a predesigned network -- the more complex the more costly, and nobody can say where the break-even point is. Can the computer be of great help? I doubt whether enthusiasm alone justifies optimistic expectations[49].

The expert-novice model is a way to organise learning processes. Learning is time-consuming. To be efficient it must be organised, but organising is also something that has to be learned. Machines in which a lot of learning is firmly and flawlessly invested are different: they need not learn themselves, or if they are of the kind that do need to learn, then the required learning process is invested just as firmly and flawlessly. The technological progress is accompanied by an ever increasing estrangement. Everybody can make some tool like a hammer as soon as he needs it. A lot of people can repair electrical switches, radio sets and cars, but few people know how computers work, and still fewer dare repair them. Yet, even in this context, one can try to account for learning as a human activity. Von Neumann once claimed that, in principle, any human activity can be taken over by computers, provided it has been analysed in detail. No doubt this applies to physical activities, taken over by robots. But brain and mind are different. Although the first computers were copies of human reckoners, albeit adapted to the understanding of the computer, our contemporaries in this field apply strategies of their own in order to more efficiently use those facilities by which computers distinguish themselves from humans. Yet chess computers, when first designed, owed nothing to any analysis of human chess: as a matter of fact, such an analysis had not even been attempted, and even now there is a difference between the computers' strategies and the more complex ones of humans, which are less accessible to analysis.

Computers are human-made and some humans know how they work. There is no fundamental knowledge about the working of the brain, and hardly any phenomenological knowledge about that of the mind. Nevertheless, we are able to predict and influence our own and other people's mental activities, without being able to say how we acquired this ability. The computer metaphor of brain and mind is therefore mistaken in both directions. All in all, computers are much too streamlined, much too prestructured and too reliable to be useful as paradigms for learning to mathematise little worlds.

3.3.5 *Taught and learned-- interlaced*

Let us recapitulate: At the assembly line of the expert-novice model, the learner is supposed to be acted on, rather than to be an agent. On the other hand, the computer model lacks descriptive validity as required for its being normative; and as a tool in the expert-novice relation, it still fails to even approach the measure of interactivity achieved in traditional learning. Anyway, strengthening the feedback between teaching and learning is still a most promising way to improve instruction.

Indeed, through personal experience we know about individual learning processes, through generalisation about general ones, and through history about those of mankind. It depends on one's viewpoint whether one qualifies this statement by adding "a little" or "a lot". Anyway, the greater part of this knowledge cannot be verbalised, let alone formalised, and this is the very reason why programmed instruction -- whether by computer or not -- has made little, if any, progress. Fortunately, just as modem travellers need not follow to the trails of old time voyagers, so learners need not repeat all the learning processes that were once required. Most of these processes can be shortened or even reduced to merely accepting the bare product, and teaching is a way of organising this reorganisation The big problem is to distinguish the products which admit the strategy of transfer from those for which the strategy of what I called guided reinvention is indispensable. To what degree may estrangement be either useful or harmful?

All one needs to know in order to use a computer is its exterior reactions to one's acts. This is astonishingly simple, although the style -- either formal or childish -- of the manuals conjures up an image of complexity, which only betrays bad didactics. Even oral explanations may be counterproductive if supplied by experts who lack the ability to place themselves in the shoes of the novice. Until further notice the fastest manner to *learn* to use computers is *to* use them and to ask for help only if some problem seems to demand too much of one's solving ability. Although this is an unsatisfactory and certainly not paradigmatic example of guided reinvention, it should not be surprising that teaching how to use computers has not kept pace with their rapid evolution.

Other teaching subjects can boast a longer history. But history is not yet done; it continues as times goes on. The learning paths leading from here to there form an intricate and never completed labyrinth and each fresh learning process, be it individual or collective, asks for new teaching processes. Even assembly lines, as useful as they may be, require modernisation, which can be a costly venture. Teaching/learning processes, observed and reinforced, entangled and isolated, necessarily require and fortunately allow more flexibility. To be sure, as a whole they must be organised carefully, but the care needed does include caring for flexibility. Interactivity means that both teachers and learners are agents as well as being acted on. Taught and learned should coincide, or, even better, compete with each other on equal terms. These are high demands, and even higher if learning is to mean long-term learning.

3.3.6 *Change*

"Taught and learned" is just one facet of the familiar antonym "Theory versus Practice" as I recalled in (3.1.1, 2). Let me now turn to Research as a source of Practice! My answer in (3.2.3, 1) to the question "what is the use of it?" was:

Change.
Change, in particular under the name of "innovation", is tacitly understood to involve improvement. Rapid changes are known as revolutions, yet revolution need not imply innovation. Anyway the innovatory speed depends on the circumstances. This is even true in industry. There it takes time to sell innovation to their directors, to develop new tools, to buy new machinery, to instruct workers how to use them, and to advertise the new products. Even if the time needed is considerable it is nothing when compared to that needed in education. (An estimate I just heard was 1 to 10.) The reason why, is the greater *amount* and *complexity* of feedback required in education, and the lack of past experience.
Innovation, when thoroughly planned, affects each level in the Educational Landscape, a big hierarchy extending from first line instruction to research, where the taught-learned relation is the glue that connects subsequent levels to one another -- a large *number* of feedback spots here, each of which is of a high *complexity.* Curriculum theory as developed in the sixties tried to account for this structure, but in formal educational innovation projects the amount and complexity of required feedback are easily underestimated, which has happened in the past and which still happens. This, then, is the reason why most of them failed in the past. The few that somehow succeeded, did so because all levels of what I called the Landscape were involved in the process of change; they underestimated, however, the time required for teaching/learning at each particular stage of dissemination. In fact, innovation itself is a learning process for the Landscape as a whole and for its particular agents, some of whom may be subject matter experts who nevertheless lack the expertise to recall their own learning processes and to analyse them in order to know what has to be transmitted to learners and how it should be done. If innovation is a learning process, it is quite a singular one, rather than one of guided reinvention. As far as it may be guided, it is so internally, which requires a high measure of flexibility.

3.3.7 *The agents of change*

3.3.7.1 The educational developer as an organiser

Learning **is** time-consuming, indeed. It depends on the learner and the learning matter how much time is required, that is, how much estrangement can be allowed if learning processes are to be shortened. Strangely enough, as history has shown us, this is a bigger problem in the ***service*** regions of the educational landscape than in the first ***line.*** Disregard for this fact is, in my opinion, the actual reason why time estimates at the various stages of the process of change have been so far off the mark.
Though under another aspect, it was already discussed in (3.2.3,6) how the "expert" can use the knowledge of his own learning processes as a means to guide those of the "novice": reporting on thought-experiments in order -- as behooves

the researcher -- to "prove" the validity of proposed instruction to the satisfaction of those who are expected to get acquainted with the proposal. In projects of developmental research the researcher need not rely on mere thought-experiments. As such a project proceeds, thoughtexperiments are gradually superseded by series of ever broader factual ones, only to cede their indispensable role to farther reaching and more profound new ones. At any stage, the results of the preceding stage, rather than being imposed as products on the participants, need to be transmitted as inventions, in order to enable participants to reinvent them.

After (3.2.3), where gradual dissemination was to serve the progress of developmental research, the tables were turned: research now serves dissemination. Even so, dissemination is not understood here as something that happens once, after research and development have been completed. It is a gradual and, in a way, open-ended process. From the start onwards, dissemination is part of any developmental research: at fist, this takes place within the restricted company of the initial actualisers who perform thought-experiments behind the desk. In the developmental process they compete with the professional developers in the activity of adapting and modifying preconceived ideas. As dissemination begins to affect ever extending circles of participants, developmental research includes keeping a grip on what is occurring at an ever more distant periphery. Yet it not a one-way influence. Signals from the broadening circle are fed back into the developmental process -- this aspect of dissemination has already been brought to the fore in the preceding subsection.

With regard to practice I made a distinction between descriptive and normative aspects. Since educational development as such is not a widespread conception, it is no wonder that norm by far outstripped description in the preceding exposition. At closer glance and regarding details quite a few realistic aspects may be recovered as a description of what at present is subconsciously and unintentionally being carried out as educational development. Yet let us now turn to other agents in the process of change!

3.3.7.2 *The teacher*

In the Educational Landscape the teacher is the least specialised actor (not counting, of course, the fist line learner), and the less specialised the teacher the lower the age bracket of the instruction in question. Less specialised is likely to imply less self-conscious of one's role, and for that matter, maybe less conceited. Was the apparent lack of specialisation the reason why curriculum developers, in particular those of New Math, trusted the teacher to be able to implement brand new subject matter in the classroom? On a *tabula rasa* one can write all one wants, and this is what they did in the retraining courses with the subject matter to be taught in the classroom -- somewhat broadened or seen from a higher viewpoint. When this strategy failed, some developers conceived the idea of "teacher-proof curriculum". My dictionaries are not up-to-date enough to include items like

"teacher-proof" but, looking for related items, I feel inclined to believe that it means stuff that even a teacher cannot spoil. Other developers realised that a teacher was no more a *tabula rasa* than anybody else and that the deeper the imprints, the more likely they were to withstand attempts to change or erase them. In the cognitive domain, background philosophy (as I called it in (3.1.3)) is the totality of the deepest imprints. As far as mathematics is concerned, there can be little doubt that such aspects as mathematising are lacking in pictures of mathematics which traditional instruction created in most people's and even in most teachers' minds. The mechanistic style (3.1.5-3.1.6) of the mathematics they were taught pervades the mathematics they learned. If they chose to become teachers, they will continue teaching the mathematics they were taught, in spite of new subjects unless emphasis in retraining is shifted from subjects to attitudes. Most attitudes are acquired unconsciously. The easiest and most efficient way to change them is actively participating in change this need not remain a privilege of that group participating in projects of educational development which I characterised as the initial actualisers of thought-experiments. Participation can be transmitted. At every stage of dissemination teachers are guided to reinvent primordial changes, albeit modified by the experiences at intermediate stages. Although unconsciously acquired attitudes are reinforced by reflection and knowledge of them may be helpful for transmission, the true thing to be transmitted in the course of dissemination -- if it is to be efficient -- is the behaviour itself rather than knowledge of it.
Guided dissemination aims as far as the ultimate frontiers. At every stage it is endangered by its own success. If stopped too early, the campaign can be lost and the instructional system be worse off than before. What does "too early" mean? In medical terms: if the mechanistic view on mathematics is a disease, then it matters to judge how many and how large areas of reinfection may be left with impunity. But what about prevention?

3.3.7.3 *The teacher student*

Mathematics teacher training is a newcomer in the Landscape of Mathematics Education. At any rate, this is true in my country. If there are other countries which can boast a longer tradition of mathematics teacher training, I am not sure what its impact has been on first line mathematics education. In most countries, teachers were educated either as *educational generalists* in order to function in primary education, or as *subject matter specialists* with a professional future in secondary education. The first were educated at teacher training institutions, the second at universities, although in my country extra-universitary examinations kept a continuous stream flowing from the first to the second pool of teachers. As a consequence, in secondary level mathematics, the majority, rather than being university-educated, used to have a professional past as primary school teachers.

However, the number of those who, prior to the extra-universitary examinations, had not had any training as a teacher was steadily increasing. Not until a few years after World War II was some teacher training made compulsory for future secondary school teachers, but, even so, for years this was thwarted and kept down by universitary people in charge of the various disciplines, who jealously guarded the scientific status of their students -- indeed, in this respect didacticians had still to qualify themselves. At the present state of teacher training for primary education, mathematics is integrated with its didactics; teacher training for lower secondary education has mathematics and its didactics taught simultaneously and by the same persons, while, for the upper secondary level, students are trained by didacticians *after* the accomplishment of their study of mathematics, which is still an unsatisfactory state of affairs.

I have re-counted the above in order was to draw attention to two basic issues regarding the relation between the two components of teacher training -- the scientific content and the didactical form. These issues are

equality of status,

the measure of integration that respects the level of both components.

Only where the mutual respect of scientific and didactic trainers and their cooperation can be taken for granted, can students be expected to take both components equally seriously and to integrate them by themselves.

In [115], when dealing with "Major Problems of Mathematics Education", I made the parallelism between mathematising and didactising explicit by drawing a parallel between each of the problems in first line mathematics instruction and its counterpart in teacher training, between each learning problem and a teaching problem, which in turn shows up as a learning problem for the teacher student. So I started with drawing a parallel between

- Why can Jennifer not do arithmetic? and
- Why can Jennifer not teach arithmetic?, only to continue via the pair
- reflecting on one's own learning with its counterpart
  reflecting on one's own teaching and, among many others,
  developing a mathematical attitude with
  developing a didactical attitude to
  How can calculators and computers be used to arouse and increase mathematical understanding? paired with
  How can calculators and computers be used to arouse and increase didactical understanding?

To explain why I changed the paradigmatical "Johnny" into Jennifer, let me quote myself from [115]:

> Why can Johnny not do arithmetic?
> Does this sound sexist? I will not change it into
> Why can Mary not do arithmetic?,
> lest it may sound even more sexist, suggesting that girls are less able than boys. As a matter of fact both are wrong. My problem is not John Roe and Mary Doe. The prob-

> lem is, indeed, why many children do not learn arithmetic as they should, and it is a major one because, more than anything else, failure in arithmetic may mean failure at school and in life. My concern, however, is not, or not primarily, what is wrong in classrooms and textbooks today that creates a host of underachievers.

Let me change the question! I now ask:

- Why can Jennifer not do arithmetic?

> Rather than an abstraction like John and Mary, Jennifer is a living child (though I have changed her name) whom I can describe in every detail. The two details that matter here are that she was eight years old and could not do arithmetic. Meanwhile the question

- Why can Jennifer not do arithmetic?

> is not a question any more, because today Jennifer is eleven and excels in arithmetic. Yet when she was eight, somebody who was observing her stumble with numbers succeeded in answering the question and after ten minutes of remedial teaching, Jennifer's problem had ceased to exist. Was it a miracle? Not at all. It was just an easy case. But there are so many such cases. Noticed and unnoticed, cared for and uncared for. But what about the less easy cases? They have grown out of those easy ones that remained unnoticed and uncared for.

Indeed, the parallelism between learning mathematics and learning to teach mathematics is the leading principle of what I have to say about mathematics teacher training, which in practice is acted out by the parallel between mathematising and didactising. Still, in accordance with the *tabula rasa* metaphor, the second kind of learning is more difficult than the first. Teacher students, in general, belong to the large group of adults where the sources of what they once learned by insight -- be it much or little -- are likely to have been clogged by the knowledge and skills they acquired in the meantime. To say it more concretely: they neither care why multiplication by 100 is carried out by "adding two zeroes" nor about the fact that you can argue about such a piece of knowledge, or why you should do so. So they have to undergo remediation: thy must relearn such facts while teaching children and observing their learning processes. The higher the level of learning, the more paradoxical this conclusion may sound. I have often argued that knowing apiece of mathematics too well may seriously impede one's ability to teach it decently; as a teacher one should be so conscious of the learning process that produced one's own excellence that one can reinvent it. So the teacher student needs to relearn by observing the learning processes of less skilled people -- in fact, of children. But if this is so important, we are faced with one of the big problems in teacher training. Although one can easily arrange for observing *short-term* learning processes in the school environment, it is impractable and hence impossible to do the like for *long-term* learning processes. Thought-experiments, as undertaken by textbook authors, cannot fill the gap if undertaken by unskilled people. Lack of experience in long-term learning processes is the actual cause for the depending of young and even older teachers on textbooks as their only sources of knowledge on long-term learning processes. Inservice guidance and cooperation would seem indispensable means for refreshing these teachers and getting them to re-create.

Anyway, there is good reason to shift the centre of gravity of teacher training from theory to practice, from the training institution to the classroom. From personal experience I know about such efforts in my own country. All I know about them elsewhere stems from written sources. As far as such sources are theory, I am not satisfied. I am afraid that the so-called classroom-based teacher training is too theory-laden to be helpful. Yet, is it still a topic?

I believe that teaching should be learned in the classroom. This is more than a belief; I think it is a fact that almost all of what practising teachers have learned in their lives about teaching and about how to teach was learned in everyday classroom practice. Teaching is learned by teaching -- one could claim -- like walking is learned by walking. But this simple truth is an outright falsehood. It cannot be denied that walking is learned by walking, but as soon as "walking" is replaced by "swimming", it is clear that things are not that simple. To learn to swim one needs aids, material and personal. Learning to speak presupposes listening, and learning to speak better requires personal and material interventions to improve speech; for that matter, even "walking learned by walking" does not apply to handicapped persons, who, without help, would never learn to walk as we would like them to do. But when I claimed teaching to be learned by teaching I meant -- and people who agree with me likewise mean -- that actual teaching is an indispensable component in the process of learning to teach.

Not so long ago teaching, certainly at the secondary level, was learned by imitation, and by trial and error. Imitation was here a crude form of observation, that is, one repeats one's own learning processes, which were recorded from one's own school-life, and supported by the textbooks used in the classroom. The trial and error provided as much feedback as the guinea pigs in the classroom could offer and the teacher was able to process.

No doubt passive observation and active trial and error are still the essential features of learning to teach, just as they are of any kind of learning not sufficiently supported by general theory. The one thing that has changed and should change even more in order to further improvement is the organisation of both observation and trial and error.

A forceful means of doing this is to observe learning processes, which I dealt with in a larger context (including teacher training) in such detail in (2.3.1) that I need not repeat it here.

I think that all would agree that a trainee should get in touch with the practice that is to become his future practice, at the earliest opportunity. Training institutions diverge in answering the question: how early is "early". After three years, four years? Well, this may be correct if it is required by the quantity of the new subject matter the student has to learn in order to teach. Primary teacher training is certainly not such a case.

On the other hand, being confronted with the practice (someone else's or one's own) without any preparation may be counterproductive. Even trial and error presuppose some knowledge about what is to be tried. But if the aim is to observe learning processes, and mathematical ones, in particular, one has the opportunity to start within the walls of the training institution, or even at home -- I mean by observing one's own learning processes: Take a little mathematical problem, solve it, tell me how you did it, and think about what you have learned while doing it! Isn't this a marvellous idea?

As early as (2.3.1) I answered this question. When applied to teacher training, we may conclude that, even before being introduced to practice, the trainee has and should have opportunities in the classroom of the training institution to observe and to analyse learning situations. By this, however, I do not mean what in my country is called institution practice: one of the students playing the teacher and the others simulating being the pupils who are being taught a certain subject in a similar way to what would happen in the genuine classroom. This is too artificial an approach to be successful, and anyway it is not what I mean.

To make clear what I do mean, let me tell a little anecdote, a story from a lesson in a teacher training institution I was observing. While solving a mathematical problem at the blackboard, a student had committed a grave error but neither the trainer nor the other students had intervened to correct it. The student went on and on, and the trainer let her proceed as she did up to the unhappy end when she ran into a flat contradiction, which proved that all she had done was wrong and that the entire work would have to be done over again. Finally, when all this was finished, I intervened to pose the question of whether somebody had noticed any remarkable feature. Nobody had. My guess is that even the trainer hadn't noticed. Anyway, he hadn't asked this question -- I had. Even after I had specified my question a bit and added hints, nobody could answer it. Finally, I myself had to disclose the answer, that is, to draw their attention, and maybe even the trainer's, to the fact that the trainer had not intervened to correct the student's mistake. Hadn't they noticed that mistake as soon as it was made? Did they believe that the trainer hadn't noticed it? For what reason could he have passed over that mistake? Was it a good reason? How should one act in such a situation? Nobody had asked themselves such questions. So I simply had to explain the trainer's didactical trick and to reveal his intention of letting the student proceed as she did until she ran into trouble.

What does this story mean? Observing learning processes is indispensable to learning to teach, and it should start at the earliest opportunity: observing one's own learning processes and those of one's learning group. But in order to be efficient, observing as horizontal didactising has to be complemented vertically.

3.3.7.4 *The textbook author*

When I discussed change and its agents, I did so from the high viewpoint of Developmental Research as presented in (3.2.3). How far can it be approached by Practice? Formal projects is the closest approach; the question of how close, is decided by striking a subtle balance between the desirable and the practically possible. Projects that are designed too narrowly, in particular with respect to the size of the instructional landscape involved (as most of them were in the past) may be useless if not harmful.

So far, in practice, the most efficient agents of change were textbooks, and by the look of it, they will maintain this status for the time being. So I may narrow my former question by asking: "How far can textbooks approach the view on change taken by developmental research?"

Mathematical textbooks -- including similar teaching aids -- have changed in the last few decades. To conclude from reviews and the ways of reviewing, however, they have changed less fundamentally than one would have liked in the wake of their remarkable predecessors dating from at least half a century ago. I admit that my first-hand knowledge of the present textbook literature is quite restricted, but among the few textbooks I do know, I could signal some that have done the utmost to take the desirable view on change, or so I would guess. This is indeed guess work. I never reviewed textbooks and with all respect to the host of reviewers, I wonder how much teaching experience is required to undertake the task of reviewing textbooks before or without having used them. My reluctance stems from confrontations at a semi-teaching level with a few of them: when asked for help by children who had to work with these books, I suddenly saw, as it were, through the learners' eyes, how bad these, at first sight good-looking textbooks were: they were shaky from questions onwards I myself was not able to understand, to didactical tricks that backfired, up to an ill-conceived build-up of the teaching matter as a whole.

I agree that what I called my "semi-teaching" experience is hardly representative. Whether one likes it or not, textbooks are merchandise, and in the marketplace good quality is what appeals to the needs and tastes of prospective customers. In the present case, said customers are teachers who need these books as teaching aids and who prefer to use what they like. This would seem to be a gloomy perspective for change and for the textbook author as an agent of change, were it not that needs can be stimulated and tastes can be educated.

The traditional way to try this is teacher manuals. As far as they supply additional information and disclose richer sources, I do not object to them, but as didactical aids I do not trust them much, because of the serious doubts I have about their factual use. In [87, p.119-121] I cited an investigation into the use of teacher manuals, which included four mathematical textbooks. Only one quarter of the teachers questioned claimed to have used them. This does not look like much, but even it is an over-estimate, since none of the four textbooks could boast a teacher's manual. Not that this prevented the 25% self-styled users from answering the most detailed questions about their ways of using them.

I prefer a textbook style where both learning and teaching at all levels of the teaching/learning process, as well as local and global reflection on it, are implicit in the text itself. For instance, the task of posing meta-questions should be shouldered by the textbook author, rather than leaving it to the teacher. This is at the same time the most effective way to implicitly influence the teacher's attitude, which includes asking more of these questions wherever they are needed. By meta-questions I mean, in the present didactical context, questions that start like

why do you think

and continue like

did I claim that. . .
did I call this . . .
did I supply you with this. . .
did I ask you.. .
did I include the condition. . .
did I not add this conclusion . . .
did I define this by. . . rather than by. . .
did I deal with this in this order rather than. . .,
did I prescribe this task for a group of four

or start like

what do you think

and continue like

did you learn in . . .
did you unlearn in . . .
is the use of this.. .
should be added
could be missed in this context?

For short, I mean didactising questions as the textbook author has asked himself as an educational developer and implicitly answered by writing the textbook in the way he eventually did. Framing these questions to fit the layout of the textbook is the way to have users reinvent the educational development that produced the textbook as though they had participated themselves in its production process; and by users I mean both teachers and learners, each on the level that is suitable to that group.

I started discussing the textbook author as an agent of change with a reference to (3.2.3). From the above it has become clear that I would like him to bear witness to his activity as developmental researcher (whether he did so explicitly or not) in the implicit way of imprinting the process into its product, thereby leaving the users as much freedom as they deserve, among other things by means of meta-questions that suggest alternatives.

Would this strategy be understood and its tactics put to good use in the classroom? It all depends on the existing infra-structure of teacher training and retraining, which should be accounted for as boundary conditions in any educational development whatever, and certainly in textbook production -- this is all part of the Landscape I have been hying to deal with.

3.3.8 *Mathematics for all*

The title of this subsection is a challenge even though it sounds like a slogan. It has been the theme (or one of the themes) of innumerable meetings and conferences -- national and international -- and it has not ceased to be the subject of research and development. In discussions on general education, *numeracy* has become the companion of *literacy.* Indeed, as badly as literacy is needed in a world full of written and printed matter, so indispensable does numeracy seem to be in a world where number emerges at every turn to be used and -- maybe even more often -- to be misused; a world which looks like it is ruled by number. But let me add that, just as literacy is no more than one aspect of language, so is numeracy one aspect of Mathematics for All -- which certainly does not cover all of it. Yet as much as I have refrained from discussing subject matter in the present books so little will I deal with the contents of Mathematics for All. Anyway, it will not be the same thing for each particular learner; there is a great deal of diversity not only of contents, but also or even more so, of breadth and depth of understanding. As a matter of fact, this may already have been asserted with respect to numeracy, and if you wish to add it, for literacy, as well.

Literacy and numeracy are spoken of as companions, but is this really a company of equals? There is a habit of distinguishing what is called *technical* reading from reading *comprehension.* If we do so with *technical numeracy* and *number comprehension,* and consider how closely reading comprehension is knitted to life and reality, shouldn't we ask for numeracy to have an equally strong grip on number in the real world? Or are literacy and numeracy different in principle?

Literacy is a relation between two expressions of language -- spoken and written (or printed) -- or so it would seem, whereas numeracy is having the competence to relate a manifold of manifestations of number to each other, which is harder to learn; and this might be an argument to be satisfied with less numeracy than what is required as literacy.

In fact, neither literacy nor numeracy are uniquely defined. Being illiterate can mean many things, depending on the environment and circumstances; so what is meant to be understood by literacy heavily depends on individuals and groups, communicating with each other in all kind of situations. Numeracy is hardly any different.

But what about mathematics for all? The answer to this question, even if it depends on the group in question, is not given by the mere intersection of all the particular mathematics that concern the individual members of the group, whether as subject matter or as competence; and this again is a property it shares with literacy.

A most striking feature of language is the abyss between passive and active mastery. Each of us, whether illiterate or literately gifted, understands tremendously more language than he is speaking or writing, and than he would ever speak or write, or be able to speak or write, and this difference extends from mere vocabulary to grammar and style, and certainly to the mastery of foreign languages. Nowhere in our mental and instrumental structure does passive mastery exceed its active counterpart to such a degree as in language, and mathematics is no exception to this rule. No wonder -- one would say -- for language is the vehicle of communication, and communicating vessels, whether wide or narrow, are in some respect each other's equals.

Under the aspect of communication I shall look at Mathematics for All. Mathematics can boast a language, so specific that it looks like mathematics itself is a language (which is often, though wrongly, identified with mathematics as such). Language is mainly learned in the course of strongly interactive communication; even reading, thanks to contents, is more interactive than it looks at first sight.

This is the reason (or one of the reasons) why I have pleaded for learning in small groups of say, four learners and, for that matter, in groups of mixed ability [87, p.60-63]. Relieve me of the task of repeating my arguments! In fact, these arguments are the direct consequences of my view on learning processes: they are steered by reflection on one's own mental activities, which is stimulated by observing oneself by means of observing others, and reinforced by the distant levels of the participants in the learning process.

Cooperative learning foreshadows cooperation in adult life and professions. In this perspective Mathematics for All means: as much and as good active mathematics as is needed to participate in even more and even better passive mathematics. I asked:

> We teach *classes,* so why shouldn't we expect *classes* to learn? Life is cooperation. Why do examinations focus on individual performances and entirely disregard collective ones? But let us delay answering this question!

The answer is, as you may expect: They should not. This is a verbal answer, indeed. It will take much time and trouble before these words become deeds. A first step on this long path is to formulate, besides individual objectives, cooperative ones of Mathematics for All, and to describe specific tasks to be performed in cooperation with others.

EPILOGUE

Did I do justice to what I promised and explained in "Apology and Explication"? I myself was astonished how little revisitation could add to the yield of former visits except reorganisation of the contents.
Publishing "Lectures", which are no lectures at all is an old and venerable habit, but I did not realise early enough that nowadays "Lectures" are even less suited to reflect lectures. Indeed, lectures on subjects like those dealt with in this book, are centred around a rich variety of overhead sheets, and elaborated on by informal talk that provides for the indispensable concreteness. Sometimes I have tried surrogates but I could not continue this way indefinitely. In particular, I had to refrain from dealing extensively with teaching matter.
The reader might be disappointed that I did not tackle computer education, including informatics as it is called on the continent. In the perspective of the present book this does not distinguish itself from other mathematical subject matter. And the computer as a teacher is still a far cry away, or so it seems to me.

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Solution of the problem 1.3.3 , 15 (fig. 3)

1. The first step to deal with such problems is distinguishing states, and operations moving states into each other.
2. Under the conditions of the problem some states may be indistinguishable. This is the case for states that arise from each other by rotations of the barrel and by interchanging white and black. This remark also holds for the operations.
3. As a consequence one considers *classes* of states and *classes* of operations. Two states (operations) belong to the same class if they arise from each other by rotations or colour change.
4. This leaves us with four classes of states and three classes of operations, each defined by a representative:

States	Members in the class
O= ○ ○ ○ ○	2
S= ● ○ ○ ○	8
P= ● ● ○ ○	4
D= ● ○ ○ ●	2

Operations	Members in the class
s = - + + +	
p = - - + +	
d = - + + -	

\- means inverting the disk,
\+ means leaving it as it is

5. Although it is not strictly needed, one can draw up a table recording the possible results of operations applied to states:

	D	P	S
d	O	P	S
p	P	D, O	S
s	S	S	P, D, O

6. The system shall be moved into state 0. If it is state D, the operation d does the job. So we first apply d. If it happens to be in state P, it then moves into state D or 0. So if is not yet in state 0, we can move it into state D or 0, and if need be, by another d into 0. Anyway if the system was in state D or P, it moves into 0 by means of dpd (or part of it).
7. If, however, it is in state S, then s moves it into state P, D, or 0. So if we now apply s, and if need be, another dpd (or part of it), we are finished.
8. The solution is

dpdsdpd (or part of it).

NOTES

1. Piaget & Bärbel Inhelder, *La représentation de l'espace chez l'enfant'*. Paris 1948. (English:) *The Child's Conception of Space,* London 1960.
   Piaget, B. Inhelder & A. Szeminska, *'La géométrie spontanée de l'enfant'*. Paris 1948. *(English:) 'The Child's Conception of Geometry'*. London 1960.
   The English translations of the titles are wrong as are those of many details.
2. J. Piaget et al, *L'enseignement mathématique,* Neuchâtel *1955*.
   Notwithstanding the title this symposium contains nothing relevant to mathematics instruction.
3. A.Treffers, *Wiskobas doelgericht* , IOWO 1978. -- English adaptation: A.Treffers, *Three dimensions -- a model of theory and goal description in Mathematics Instruction - The Wiskobas Project* . Reidel, Dordrecht 1986.
4. Anyway before 1962 (see [13]); the idea, realised and witnessed as early as in Dina van Hiele-Geldof's mathematics lessons, is much older (see[4;7]).
5. See, for instance: H.ter Heege, The acquisition of basic multiplication skills, *Educ. Stud. Math.* 16(1985), 375-388.
6. L.Streefland, *Realistisch breukenonderwijs.* Vakgroep OW&OC. Utrecht, 1988. -- An English translation is under press.
7. A.Treffers[3]. In particular, Chap.VII, 3.2.
   L.Streefland & A.Treffers: Produktiver Rechen-Mathematik-Unterricht. Manuscript ,1988.
8. A seminal paper: Hans ter Heege, 'Testing the maturity for learning the algorithm of multiplication', *Educ. Stud. Math.* 9 (1978), 75-83. See also [7]: A.Treffers, Chap.VI.
9. J.Rengerink, *'De staartdeling - een geïntegreerde aanpak volgens het princ-ipe van progressieve schematisering* ', Vakgroep OW & OC, Utrecht 1983.

9a. A paragon of poor context is the world reduced to fingers and matches, as presented by Dagmar Neumann in her Ph.thesis 'The origin of arithmetical skills', *Göteborg Stud.* in Educ. Sci. 62 (1987), in which work I am many more times quoted than understood.

10. *Wiskobas Bull. 5* (1975) #2-3.
11. Most of the materials, mentioned here and in the next few paragraphs, was edited in Dutch; some have been translated into English or French. They are distributed by Vakgroep OW & OC, Tiberdreef 4, 3561 GG Utrecht. Some abstracts were published in *Educ Stud. Math.* **7** (1976), #3.
12. H.Freudenthal, 'Zur Geschichte der vollständigen Induktion'. *Arch.I nt.-d'Hist.des Sci.* 22 (1953), 17-37. --The side- and diagonal numbers are inductively obtained approximations $a_n + d_n$ of the ratio a:d of side and diagonal of the square: $a_{n+1} = a_n + d_n$, $d_{n+1} = 2a_n + d_n$, starting with $a_1 = d_1 = 1$.
13. Peri Helikon X.
14. Of course, with Greek letters.
15. H.Freudenthal, 'Einige Züge aus der Entwicklung des mathematischen Formalismus'.I. *N.Archief Wisk.(3)7(1959),* 1-19.

16. There is no authentic formulation. See J.L.Kaput & J.Clement, 'Letter to the Editor of JCMB' in *The Journal of Children's Math.Behav.* 2(1979),#2,208. The discussion in the same journal, 3,#1, is hardly satisfactory.
16a. Confronting 10-11 years olds bluntly with letters and formulas _ this is the gist of an experiment reported by R.Fischer, 'Didactics, mathematics, and communication'. *For the Learning of Mathematics* 8 (1988)#2,20-30 in particular 20.
17. J.Piaget, *'La naissance de l'intelligence chez l'enfant',* Neuchâtel 1936 J.Piaget, *'La construction du réel chez l'enfant'* Neuchâtel 1937. (English) *'The origin of the intelligence in the child'*, New York,1952. *'The construction of reality in the child',* New York, 1955. See also 22.
18. See [58], [64], [72], [73], [84]. [85], [97], [98], [109], which is an abridged German translation of my address "Learning Processes" to the Presession, Boston, 18 April 1979, of a NCTM meeting, [124], [137], [141], [143], [148], [149], [153], [154], [160], [184], [185].
19. Dina van Hiele-Geldof, *'De didactiek van de meetkunde in de eerste klas van het VHMO'* (The didactics of geometry in the lowest class of the secondary school), Ph.D.Thesis Utrecht 1957.
20. P.M. van Hiele & D.van Hiele-Geldof, A method of initiation into geometry at secondary schools. In [9], 67-80, in particular 74-75. See also Note 19 as well as: P.M.van Hiele, *'De problematiek van het inzicht*, Ph.D. Thesis, Utrecht 1957.
21. The following is largely taken from [98].
22. J.Piaget, *'La formation du symbole chez l'enfant',* Neuchâtel, 1945. (English:) *'Play, dreams and imitation in childhood',* London, 1951, 1967.
23. A large part of this subsection has been taken over from [115].
24. To my surprise I noticed that in France teaching arithmetic in the first grade is restricted to addition, albeit up to 80.
25. An idea emphasised and a term coined by L.Streefland. Cp.6.
26. The idea of learning negative numbers in this intertwined way seems to be so exotic that, after Van Hiele and my own experiments [146,154], it has not found any followers. It was even overlooked by otherwise careful reviewers of [146].
27. Parts of this lecture were edited in [87], which can also be consulted for details that are here omitted.
28. See also [166].
29. Unfortunately I lost the evidence.
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