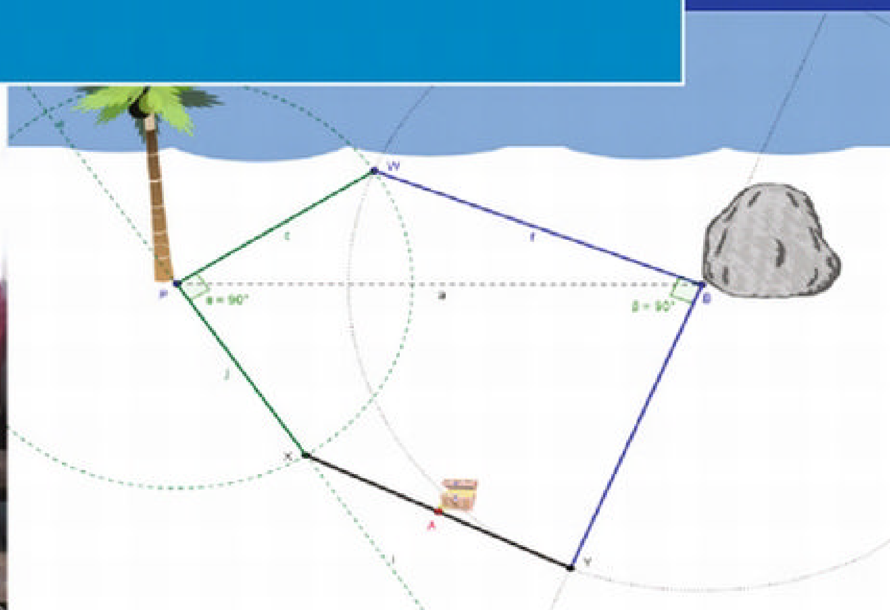
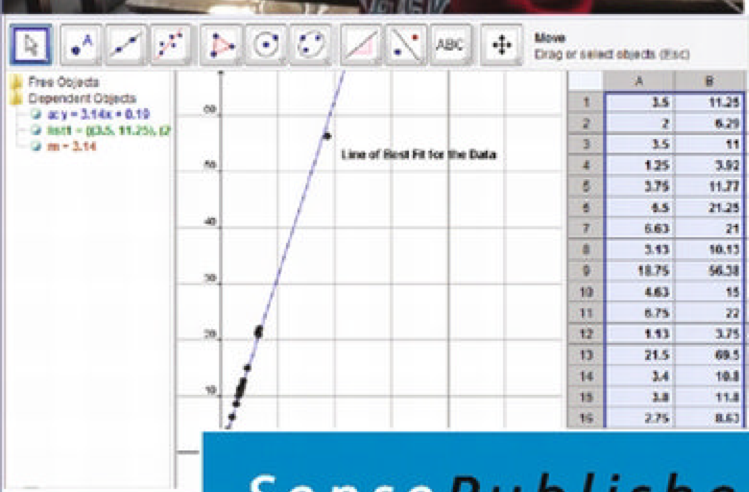
MODELING AND SIMULATIONS FOR LEARNING AND INSTRUCTION

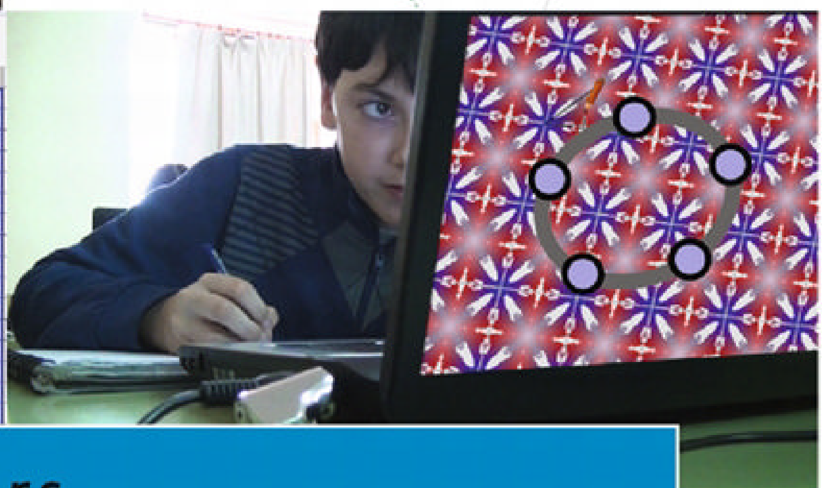
Model-Centered Learning

Pathways to Mathematical Understanding Using GeoGebra

Lingguo Bu and Robert Schoen (Eds.)







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Model-Centered Learning

MODELING AND SIMULATIONS FOR LEARNING AND INSTRUCTION
Volume 6

Scope

Models and simulations have become part and parcel of advanced learning environments, performance technologies and knowledge management systems.This book series will address the nature and types of models and simulations from multiple perspectives and in a variety of contexts in order to provide a foundation for their effective integration into teaching and learning. While much has been written about models and simulations, little has been written about the underlying instructional design principles and the varieties of ways for effective use of models and simulations in learning and instruction. This book series will provide a practical guide for designing and using models and simulations to support learning and to enhance performance and it will provide a comprehensive framework for conducting research on educational uses of models and simulations. A unifying thread of this series is a view of models and simulations as learning and instructional objects. Conceptual and mathematical models and their uses will be described. Examples of different types of simulations, including discrete event and continuous process simulations, will be elaborated in various contexts. A rationale and methodology for the design of interactive models and simulations will be presented, along with a variety of uses ranging from assessment tools to simulation games. The key role of models and simulations in knowledge construction and representation will be described, and a rationale and strategy for their integration into knowledge management and performance support systems will provided.

Audience

The primary audience for this book series will be educators, developers and researchers involved in the design, implementation, use and evaluation of models and simulations to support learning and instruction. Instructors and students in educational technology, instructional research and technology-based learning will benefit from this series.

Model-Centered Learning

Pathways to Mathematical Understanding Using GeoGebra

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SENSE PUBLISHERS
ROTTERDAM/BOSTON/TAIPEI

A C.I.P. record for this book is available from the Library of Congress.

ISBN: 978-94-6091-616-8 (paperback)
ISBN: 978-94-6091-617-5 (hardback)
ISBN: 978-94-6091-618-2 (e-book)

Published by: Sense Publishers,
P.O. Box 21858,
3001 AW Rotterdam,
The Netherlands
www.sensepublishers.com

Printed on acid-free paper

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J. MICHAEL SPECTOR

FOREWORD

Today's students live in a world of ubiquitous technology. However, these technologies have not been adequately incorporated into learning and instruction. Mathematics education has evolved with the times and the available technologies. Calculators eventually made their way into schools. The battle goes on to persuade educators and parents and others that what was important was the ability to solve complex problems – not the ability to perform complex calculations on paper. Architects and engineers and scientists do not perform very many complex calculations on paper. They use sophisticated calculating devices. Within the context of authentic learning, it makes sense to make similar tools available to students.

Graphing calculators have been introduced in courses involving mathematics, engineering, and science. Why is that happening? It seems to be a natural evolution of the use of technology in education. Now that the burden of performing complex calculations has shifted to machines, the new burden of understanding the data that can be quickly calculated is receiving greater attention. Graphing calculators can help in understanding complex functions through a visual and dynamic representation of those functions.

Have calculators and graphing calculators had a significant impact on students' ability to understand relationships among variables and complex sets of data? It is probably the case that the impact has been less than advocates of these tools and technologies would like to believe. Given the lack of significant impact of previous innovative tools in mathematics education, what lessons can be learned that will contribute to future success with new tools?

I believe there are two important lessons to be learned. The first is that the proper preparation and training of teachers is critical to success when introducing new instructional approaches and methods, new learning materials, and innovative tools. The second is that new tools and technologies should be used in ways that support what is known about how people come to know and understand things. It is now widely accepted that people create internal representations to make sense of new experiences and puzzling phenomena. These internal representations or mental models are important for the development of critical reasoning skills required in many professional disciplines, including those involving mathematics. Using appropriate pedagogical methods and tools to support these internal representations is an important consideration for educators.

This volume is about GeoGebra, a new, cost-free, and very innovative technology that can be used to support the progressive development of mental models appropriate for solving complex problems involving mathematical relationships (see http://www.geogebra.org/cms/). GeoGebra is supported with many additional free resources, including lessons, examples, and activities that can be used to support the training of teachers in the integration of GeoGebra into curricula aligned with standards, goals and objectives. The topics herein range widely from using GeoGebra to model real-world problems and support problem solving, to provide visualizations and interactive illustrations, and to improve student motivation and cognitive development.

In short, this is an important book for mathematics educators. It is a must read for all secondary and post-secondary math teachers and teacher educators who are interested in the integration of GeoGebra or similar technologies in mathematics education. In addition, it is a valuable resource for all educators interested in promoting the development of critical reasoning skills.

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ACKNOWLEDGEMENTS

This book project was initially inspired by a series of discussions the editors had with Dr. J. Michael Spector in early 2008 about the roles of modeling and simulations in complex learning, including the potential applications and implications of GeoGebra in mathematics education. The editors are grateful to Dr. Spector for his unfailing support and ongoing encouragement in both theory and practice in relation to the book development. During the initial call for proposals and subsequent review and selection of manuscripts, Dr. Markus Hohenwarter and Dr. Zsolt Lavicza played a helpful role in contacting the international GeoGebra community to invite proposals and manuscripts. We extend our thanks to all the chapters' authors, who not only reviewed and contributed to their colleagues' work, but also made many constructive suggestions to the overall coverage of the book. We also would like to thank our graduate assistants at Southern Illinois University Carbondale for carefully reviewing and commenting on early drafts of the chapter manuscripts: Yazan Alghazo, Ashley Launius, Gilbert Kalonde, and Jia Yu. The Department of Curriculum and Instruction at Southern Illinois University Carbondale provided generous funding in support of travel and consulting related to the book project. We further wish to thank Peter de Liefde, at Sense Publishers, for his support and patience with us during the lengthy reviewing and editing process. Finally, we are indebted to our families, whose understanding and support helped us bring the book to fruition.

Lingguo Bu and Robert Schoen

LINGGUO BU AND ROBERT SCHOEN

GEOGEBRA FOR MODEL-CENTERED LEARNING IN MATHEMATICS EDUCATION

An Introduction

But common as it is, much of education clings too stubbornly to abstraction, without enough models to illustrate and enliven them. The cure for this on the learner's side is to call for more models. Learners need to recognize that they need models and can seek them out.

—Perkins (1986, p. 147)

Mental models serve a twofold epistemological function: They represent and also organize the subject's knowledge in such a way that even complex phenomena become plausible.

—Seel, Al-Diban, & Blumschein (2000, p. 130)

It makes no sense to seek a single best way to represent knowledge—because each particular form of expression also brings its own particular limitations.

—Minsky (2006, p. 296)

GeoGebra (http://www.geogebra.org) is a community-supported open-source mathematics learning environment that integrates multiple dynamic representations, various domains of mathematics, and a rich variety of computational utilities for modeling and simulations. Invented in the early 2000s, GeoGebra seeks to implement in a web-friendly manner the research-based findings related to mathematical understanding and proficiency as well as their implications for mathematics teaching and learning: A mathematically competent person can coordinate various representations of a mathematical idea in a dynamic way and further gain insight into the focal mathematical structure. By virtue of its friendly user interface and its web accessibility, GeoGebra has attracted tens of thousands of visitors across the world, including mathematicians, classroom math teachers, and mathematics educators. Through the online GeoGebra Wiki and global and local professional conferences, an international community of GeoGebra users has taken shape. This growing community is actively addressing traditional problems in mathematics education and developing new pedagogical interventions and theoretical perspectives on mathematics teaching and learning, while taking advantage of both technological and theoretical inventions. Meanwhile, in the fields of learning sciences and instructional design, researchers have highlighted the theoretical and practical implications of *mental models* and *conceptual models* in complex human learning (Milrad, Spector, & Davidsen, 2003; Seel, 2003). A

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model-centered framework on learning and instruction does not only help us understand the cognitive processes of mathematical sense-making and learning difficulties, but also lends itself to instructional design models that facilitates meaningful learning and understanding. Thus, we see in the GeoGebra project a kind of synergy or concerted effort between technology and theory, individual inventions and collective participation, local experiments and global applications. GeoGebra has created a positive ripple effect, centered around technology integration in mathematics teaching and learning, which has reached out from a graduate design project at the University of Salzburg across international borders to all major regions of the world, from university students to children in rural areas. For the most part, GeoGebra and GeoGebra-based curricular activities have been a grassroots phenomenon, motivated distinctively by teachers' professional commitment and their mathematical and didactical curiosity.

This volume stands as an initial endeavor to survey GeoGebra-inspired educational efforts or experiments in both theory and practice in mathematics education across the grade levels. The focus of the book is centered on the international use of GeoGebra in model-centered mathematics teaching and learning, which naturally goes beyond traditional mathematics instruction in content and coverage of concepts. The chapters in this volume address broad questions of mathematics education, citing specific examples along the way, with a clear commitment to mathematical understanding and mathematical applications. In addition to being a computational tool, GeoGebra has been characterized by several authors to be a conceptual tool, a pedagogical tool, a cognitive tool, or a transformative tool in mathematics teaching and learning. This *tool* perspective underlines the versatile roles of GeoGebra in mathematical instruction and mathematics education reforms. In general, the chapters address mathematics teaching and learning as a complex process, which calls for technological tools such as GeoGebra for complexity management, multiple representations, sense-making, and decision-making. In what follows, we briefly introduce the key ideas of each chapter along six themes that run naturally through all the chapters.

History, Philosophy, and Theory

In Chapter 1, Hohenwarter and Lavicza review the history and philosophy behind the initial GeoGebra project and its subsequent and ongoing evolution into an international community project. They further envision a community-based approach to technology integration in mathematics education on an international scale. Chapter 2 features a theoretical paper by Bu, Spector, and Haciomeroglu, who review the literature on mathematical understanding from the psychological, philosophical, and mathematical perspectives, shedding light on the relevancy of mental models in reconceptualizing mathematical meaning and understanding. They put forward a preliminary framework for GeoGebra-integrated instructional design by synthesizing major principles from Model-Facilitated Learning, Realistic Mathematics Education, and Instrumental Genesis. The overarching goal is to identify design principles that foster deep mathematical understanding by means of

GeoGebra-based conceptual models and modeling activities. They also call for increased attention to the mutually defining role of GeoGebra tools and students' instrumented mathematical behavior, especially in complexity management.

Dynamic Modeling and Simulations

In Chapter 3, Pierce and Stacey report on the use of dynamic geometry to support students' investigation of real-world problems in the middle and secondary grades. Dynamic models of real-world scenarios, as they found, help students to make mathematical conjectures and enhance their understanding of the mathematical concepts. Furthermore, the multiple features of dynamic modeling contribute to improving students' general attitudes toward mathematics learning.

Burke and Kennedy (Chapter 4) explore the use of dynamic GeoGebra models and simulations in building a bridge between students' empirical investigations and mathematical formalizations. Their approach to abstract mathematics illustrates the didactical conception of vertical mathematization, a process by which mathematical ideas are reconnected, refined, and validated to higher order formal mathematical structures (e.g., Gravemeijer & van Galen, 2003; Treffers, 1987). They aim to provide model-based conceptual interventions that support students' development of valid mental models for formal mathematics, an important practice that typically receives inadequate treatment in upper-division mathematics courses. In Chapter 5, Novak, Fahlberg-Stojanovska, and Renzo present a holistic learning model for learning mathematics by doing mathematics—building simulators with GeoGebra to seek deep conceptual understanding of a real-world scenario and the underlying mathematics (cf. Alessi, 2000). They illustrate their learning model with a few appealing design examples in a setting that could be called a mathematical lab, where science and mathematics mutually define and support one another in sense-making and mathematical modeling.

GeoGebra Use, Problem Solving, and Attitude Change

Iranzo and Fortuny (Chapter 6) showcase, from the perspective of instrumental genesis, the complex interactions among the mathematical task, GeoGebra tool use, and students' prior mathematical and cognitive background, citing informative cases from their study. GeoGebra-based modeling helped their students diagnose their mathematical conceptions, visualize the problem situations, and overcome algebraic barriers and thus focus on the geometric reasoning behind the learning tasks. Students' problem solving strategies, as the authors observe, are the result of the nature of the instructional tasks, students' background and preferences, and the role of the teacher. In Chapter 7, Mousoulides continues the discussion about the modeling approach to GeoGebra-integrated problem solving in the middle grades, where GeoGebra is employed as a conceptual tool to help students make connections between real-world situations and mathematical ideas. Students in his study constructed sophisticated dynamic models, which broadened their mathematical exploration and visualization skills.

Chapter 8 features an article by Arranz, Losada, Mora, Recio, and Sada who report on their experience in modeling a 3-D linkage cube using GeoGebra. In the process of building a GeoGebra-based flexible cube, one encounters interesting connections between geometry and algebra and develops problem solving skills while resolving intermediate challenges along the way. The cube problem and its educational implications are typical of a wide range of real-world modeling problems in terms of the mathematical connections and the ever expanding learning opportunities that arise, sometimes unexpectedly, in the modeling process (e.g., Bu, 2010).

Haciomeroglu (Chapter 9) reports on his research on secondary prospective teachers' experience with GeoGebra-based dynamic visualizations in instructional lesson planning. His findings highlight the impact of GeoGebra use on participants' attitudes toward mathematic teaching and the importance of collaborative group work in GeoGebra-integrated teacher education courses.

Gómez-Chacón (Chapter 10) adopts a multi-tier, mixed methods research design, which consists of a large-scale survey ($N = 392$), a small focus study group ($N = 17$), and six individual students, to investigate the influences of GeoGebra-integrated mathematics instruction on secondary students' attitudes toward mathematics learning in computer-enhanced environments. While GeoGebra use is found to foster students' perseverance, curiosity, inductive attitudes, and inclination to seek accuracy and rigor in geometric learning tasks, the findings also point to the complex interactions between computer technology, mathematics, and the classroom environment. The author further analyzes the cognitive and emotional pathways underlying students' attitudes and mathematical behaviors in such instructional contexts, calling for further research to find ways to capitalize on the initial positive influences brought about by GeoGebra use and foster the development of students' sustainable positive mathematical attitudes.

GeoGebra as Cognitive and Didactical Tools

Karadag and McDougall (Chapter 11) survey the features of GeoGebra from the cognitive perspective and discuss their pedagogical implications in an effort to initiate both theoretical and practical experimentation in conceptualizing GeoGebra as a cognitive tool for facilitating students' internal and external multiple representations (cf. Jonassen, 2003; Jonassen & Reeves, 1996). Along a similar line of thought, Ronchi (Chapter 12) views GeoGebra as a methodological or didactical resource that supports the teaching and learning of mathematics by helping teachers and their students visualize formal mathematical knowledge and promote their sense of ownership through dynamic constructions in a lab setting.

Curricular Initiatives

In Chapter 13, Little outlines his vision for a GeoGebra-based calculus program at the high school level, showcasing the distinctive features of GeoGebra for facilitating students' and teachers' coordination of algebra and geometry, which is at the very core of learning and teaching calculus. As seen by Little, the simplicity of GeoGebra's user interface and its computational architecture allow students to construct their own mathematical models and, by doing so, reinvent and enhance their ownership of calculus concepts. In Chapter 14, Lingefjärd explores the prospect of revitalizing Euclidean geometry in school mathematics in Sweden and internationally by taking advantage of GeoGebra resources. Perhaps, a variety of school mathematics, including informal geometry and algebra, can be reconsidered and resequenced along Little and Lingefjärd's lines of thought. In response to increased computational resources and the evolving needs of society (exemplified often by applications of number theory, for example), our conception of mathematics has changed significantly over the past several decades. It is likely that the open accessibility and the dynamic nature of GeoGebra may contribute to or initiate a similarly profound evolution of school mathematics and its classroom practice.

Equity and Sustainability

GeoGebra has also inspired research and implementation endeavors in developing countries, where access to advanced computational resources is limited. In Chapter 15, De las Peñas and Bautista bring the reader to the Philippines to observe how children and their mathematics teachers coordinate the construction of physical manipulatives and GeoGebra-based mathematical modeling activities. They also share their approaches to strategic technology deployment when a teacher is faced with limited Internet access or numbers of computers. Jarvis, Hohenwarter, and Lavicza (Chapter 16) reflect on the feedback from international users of GeoGebra and highlight a few key characteristics of the GeoGebra endeavor—its dynamic international community, its sustainability, and its values in providing equitable and democratic access to powerful modeling tools and mathematics curricula to all students and educators across the world. As GeoGebra users join together with mathematicians and mathematics educators, the authors call for further research on the development of GeoGebra-inspired technology integration and the influence and impact of GeoGebra and the GeoGebra community in the field of mathematics education.

It is worth noting that, given the international nature of this first volume on GeoGebra and its applications in mathematical modeling, the editors encountered great challenges in the editing process in terms of languages and styles. With certain manuscripts, extensive editorial changes were made by the editors and further approved by the chapter authors. Meanwhile, the editors tried to maintain the international flavor of the presentations. We invite our readers to consider the context of these contributions, focus on the big ideas of theory and practice, and further join us in the ongoing experimentation of community-based technology integration in mathematics education, taking advantage of GeoGebra and similar technologies.

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MARKUS HOHENWARTER AND ZSOLT LAVICZA

1. THE STRENGTH OF THE COMMUNITY: HOW GEOGEBRA CAN INSPIRE TECHNOLOGY INTEGRATION IN MATHEMATICS

The dynamic mathematics software GeoGebra has grown from a student project into a worldwide community effort. In this chapter, we provide a brief overview of the current state of the GeoGebra software and its development plans for the future. Furthermore, we discuss some aspects of the fast growing international network of GeoGebra Institutes, which seeks to support events and efforts related to open educational materials, teacher education and professional development, as well as research projects concerning the use of dynamic mathematics technology in classrooms all around the world.

INTRODUCTION

During the past decades, it has been demonstrated that a large number of enthusiasts can alter conventional thinking and models of development and innovation. The success of open source projects such as Linux®, Firefox®, Moodle®, and Wikipedia® shows that collaboration and sharing can produce valuable resources in a variety of areas of life. With the increased accessibility of affordable computing technologies in the 1980s and 90s, there was overly enthusiastic sentiment that computers would become rapidly integrated into education, in particular, into mathematics teaching and learning (Kaput, 1992). However, numerous studies showed only a marginal uptake of technology in classrooms after more than two decades (Gonzales, 2004). There were many attempts and projects to promote wider technology integration, but many of these attempts led to only marginal changes in classroom teaching (Cuban, Kirkpatrick & Peck, 2001). While working on the open source project GeoGebra, we are witnessing the emergence of an enthusiastic international community around the software. It will be interesting to see whether or not this community approach could penetrate the difficulties and barriers that hold back technology use in mathematics teaching. Although the community around GeoGebra is growing astonishingly fast, we realize that both members of the community and teachers who are considering the use of GeoGebra in their classrooms need extensive support. To be able to offer such assistance and promote reflective practice, we established the International GeoGebra Institute (IGI) in 2008. In this chapter, we offer a brief outline of the current state of both the GeoGebra software and its community, and we also hope to encourage colleagues to join and contribute to this growing community.

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GEOGEBRA

The software GeoGebra originated in the Master's thesis project of Markus Hohenwarter at the University of Salzburg in 2002. It was designed to combine features of dynamic geometry software (e.g., Cabri Geometry®, Geometer's Sketchpad®) and computer algebra systems (e.g., Derive®, Maple®) in a single, integrated, and easy-to-use system for teaching and learning mathematics (Hohenwarter & Preiner, 2007). During the past years, GeoGebra has developed into an open-source project with a group of 20 developers and over 100 translators across the world. The latest version of GeoGebra offers dynamically linked multiple representations for mathematical objects (Hohenwarter & Jones 2007) through its graphical, algebraic, and spreadsheet views. Under the hood, we are already using a computer algebra system (CAS) that will be made fully accessible for users through a new CAS view in the near future. GeoGebra, which is currently available in 50 languages, has received several educational software awards in Europe and the USA (e.g. EASA 2002, digita 2004, Comenius 2004, eTwinning 2006, AECT 2008, BETT 2009 finalist, Tech Award 2009, NTLC Award 2010).

Apart from the standalone application, GeoGebra also allows the creation of interactive web pages with embedded applets. These targeted learning and demonstration environments are freely shared by mathematics educators on collaborative online platforms like the GeoGebraWiki (www.geogebra.org/wiki). The number of visitors to the GeoGebra website has increased from about 50,000 during 2004 to more than 5 million during 2010 (see Figure 1) coming from over 180 countries.

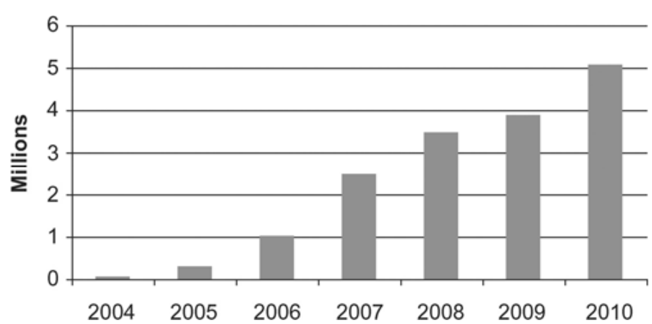

Figure 1. Visitors per year to www.geogebra.org (in millions).

INTERNATIONAL GEOGEBRA INSTITUTE (IGI)

The growing presence of open-source tools in mathematics classrooms on an international scale is calling for in-depth research on the instructional design of GeoGebra-based curricular modules and the corresponding impact of its dynamic mathematics resources on teaching and learning (Hohenwarter & Lavicza, 2007). Thus, we gathered active members of the GeoGebra community from various countries at a conference in Cambridge, UK in May 2008, and founded an international research and professional development network: the International GeoGebra Institute (www.geogebra.org/igi). This not-for-profit organization intends to coordinate international research and professional development efforts around the free software. The main goals of the International GeoGebra Institute are to:

– Establish self-sustaining local GeoGebra user groups;
– Develop and share open educational materials;
– Organize and offer workshops for educators;
– Improve and extend the features of the software GeoGebra;
– Design and implement research projects both on GeoGebra and IGI;
– Deliver presentations at national and international conferences.

FUTURE AND VISION

In order to provide adequate support and training, we are in the process of establishing local groups of teachers, mathematicians, and mathematics educators who work together in developing and adapting the software as well as educational and professional development materials to serve their local needs. For example, through a recent project funded by the National Centre for Excellence in Mathematics Teaching (NCETM), we have been collaborating with nine mathematics teachers in England to embed GeoGebra-based activities into the English curriculum and develop adequate professional development programs (Jones et al., 2009). This project aspired to nurture communities of teachers and researchers in England who are interested in developing and using open source technology in schools and in teacher education.

Since May 2008, more than forty local GeoGebra Institutes have already been established at universities in Africa, Asia, Australia, Europe, North and South America (Figure 2). For example, the Norwegian GeoGebra Institute in Trondheim comprises of more than 50 people in a nation-wide network of GeoGebra trainers, mathematicians, and mathematics educators who provide support for teachers and collaborate on research projects in relation to the use of free educational resources. Since the first international GeoGebra conference in July 2009 in Linz, Austria, more than a dozen local conferences have been held or scheduled in America, Asia, and Europe. These conferences as well as workshops and local meetings are shared and publicized through a public events calendar on GeoGebra's website (Figure 3). For example, several European countries are collaborating in a recently awarded grant to establish a Nordic GeoGebra Network focusing on joint seminars and conferences.

Several local GeoGebra Institutes are also involved in pioneering projects featuring the use of netbook and laptop computers. For example, three million laptops with GeoGebra preinstalled have just been given out to students by the government of Argentina. The GeoGebra Institute in Buenos Aires is actively involved in corresponding teacher training and curricular development activities.

Similar laptop projects are in progress in Australia and Spain. More information on the different GeoGebra Institutes and their activities can be found on http://www.geogebra.org/igi.

Figure 2. Network of local GeoGebra Institutes: www.geogebra.org/community.

Figure 3. GeoGebra events map and calendar: http://www.geogebra.org/events.

DEVELOPMENT OF INSTRUCTIONAL MATERIALS

On the GeoGebraWiki (www.geogebra.org/wiki) website, users have already shared over fifteen thousand free interactive online worksheets that can be remixed and adapted to specific local standards or individual needs. In order to better support the sharing of open educational materials in the future, we are working on a material sharing platform that will also allow users to provide comments and rate the quality of materials. Furthermore, GeoGebra materials will also be useable on mobile devices and phones in the future (e.g., iPhone®, iPad®, Android® phones, Windows® phones).

Concerning the software development of GeoGebra, we are engaging more and more talented Java programmers with creative ideas for new features and extensions through our new developer site (www.geogebra.org/trac). With the recent addition of a spreadsheet view, GeoGebra is ready for more statistical charts, commands, and tools. The forthcoming computer algebra system (CAS) and 3D graphics views will provide even more applications of the software both in schools as well as at the university level. With all these planned new features, it will be crucial to keep the software's user interface simple and easy-to-use. Thus, we are also working on a highly customizable new interface where users can easily change perspectives (e.g., from geometry to statistics) and/or rearrange different parts of the screen using drag and drop.

OUTREACH

As an open source project, GeoGebra is committed to reaching out specifically to users in developing countries who otherwise may not be able to afford to pay for software. Together with colleagues in Costa Rica, Egypt, the Philippines, Uruguay, and South Africa, we are currently investigating the possibilities of setting up local user groups or GeoGebra Institutes, and developing strategies to best support local projects in these regions. For example, we have recently developed a special GeoGebra version for the one-laptop-per-child project in Uruguay. Involving colleagues in our international network could create new opportunities to support countries with limited resources and exchange educational resources and experiences.

SUMMARY

With this introductory chapter, we hope to raise attention to the growing GeoGebra community and encourage our colleagues in all nations to contribute to our global efforts in enhancing mathematics education for students at all levels. It is fascinating and encouraging to read about the various approaches our colleagues have taken to contribute to the GeoGebra project. If you are interested in getting involved in this open source endeavor, please visit the GeoGebra/IGI websites, where we will continue to discuss together which directions the GeoGebra community should take in the future.

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2. TOWARD MODEL-CENTERED MATHEMATICS LEARNING AND INSTRUCTION USING GEOGEBRA

A Theoretical Framework for Learning Mathematics with Understanding

This chapter presents a model-centered theoretical framework for integrating GeoGebra in mathematics teaching and learning to enhance mathematical understanding. In spite of its prominence in the ongoing mathematics education reform, understanding has been an ill-defined construct in the literature. After reviewing multiple perspectives from learning theories and mathematics education, we propose an operational definition of understanding a mathematical idea as having a dynamic mental model that can be used by an individual to mentally simulate the structural relations of the mathematical idea in multiple representations for making inferences and predictions. We further recognize the complexity of mathematical ideas, calling for a model-centered framework for instructional design in dynamic mathematics. Synthesizing theoretical principles of Realistic Mathematics Education, Model-Facilitated Learning, and Instrumental Genesis, we contend that GeoGebra provides a long-awaited technological environment for mathematics educators to reconsider the teaching and learning of school mathematics in terms of the human nature of mathematics, contemporary instructional design theories, and the influences of digital tools in mathematical cognition. We present three design examples to illustrate the relevance of a model-centered theoretical framework.

INTRODUCTION

Mathematics learning and instruction is a highly complex process as has been unveiled by more than three decades of research in mathematics education (Gutiérrez & Boero, 2006; Lesh, 2006; Lesh & Doerr, 2003). Under the surface of symbols and rules lies a rich world of mathematical ideas that permeate a host of contexts and various domains of mathematics. The cognitive complexity of mathematics in general reflects the human nature of mathematics and mathematics learning and instruction that can be characterized in multiple dimensions (Dossey, 1992; Freudenthal, 1973). First, mathematics learning is both an individual and a social process, where diverse ways of individual experiences interact with the normative elements of a field with thousands of years of history. Second, there are virtually no isolated mathematical ideas. From numeration to calculus, each mathematical concept is connected to other

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 13–40.

concepts and vice versa. `It is within such a web of connected concepts that each mathematical idea takes on its initial meaning and further evolves as learners come into closer contact with a variety of related concepts and relations. Third, these interconnections among mathematical ideas are frequently solidified by their multiple representations and the connections among the multiple representations (Goldin, 2003; Sfard, 1991). A parabola, for example, is connected to and further understood in depth by virtue of its relations to lines, points, conics, squares, area, free fall, paper-folding, projectiles, and the like. It is further represented by verbal, numeric, algebraic, and geometric representations, and in particular, their interconnections. Fourth, mathematical representations are ultimately cultural artifacts, indicative of the semiotic, cultural, and technological developments of a certain society (Kaput, 1992; Kaput, Hegedus, & Lesh, 2007; Presmeg, 2002, 2006). For example, although the abacus has been used in some Asian cultures for centuries as a primary calculation device, it now coexists with graphing calculators and computer software. Technology changes, and it further changes what we do and what we can do as well as the way we handle traditional instructional practices (Milrad, Spector, & Davidsen, 2003). With a growing variety of new tools available for mathematics learning and teaching, traditionally valued mathematical operations such as graphing and factoring are becoming trivial mathematical exercises; learners and teachers alike are faced with new choices with regard to the use of tools and the redesign of learning activities (Puntambekar & Hubscher, 2005). All these aspects of mathematics education contribute to its growing complexity, only to be further complicated by the evolving role of mathematics and changing goals of mathematics education in an ever-changing information society (diSessa, 2007; Kaput, Noss, & Hoyles, 2002).

The complexity of mathematics learning and instruction lends itself to a variety of theoretical frameworks and new interactive learning technologies. The theory of Realistic Mathematics Education (RME) (Freudenthal, 1978; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Streefland, 1991; Treffers, 1987) stands out among the contemporary theories of mathematics education because it is grounded in the historical and realistic connections of mathematical ideas. RME conceptualizes mathematics learning as a human activity and a process of guided reinvention through horizontal and vertical mathematizations. In horizontal mathematization, realistic problem situations are represented by mathematical models in a way that retains its essential structural relations; in vertical mathematization, these models are further utilized as entry points to support sense-making within a world of increasingly abstract mathematical ideas in a chain of models. Within RME, models are used primarily as didactical tools for teaching mathematics to situate the origin and the conceptual structure of a mathematical idea (Van den Heuvel-Panhuizen, 2003). However, with natural extensions, such didactical models can be used to generate more advanced ideas and foster problem solving skills, especially in vertical mathematization. The instructional principles of RME are further supported by new interactive mathematics learning technologies, which typically provide multiple representations, dynamic links, and simulation tools. Among the various mathematics learning technologies, GeoGebra (www.geogebra.org) has gained growing international recognition since its official release in 2006 because of its open source status, international developers, and a growing user base of mathematicians, mathematics educators, and classroom teachers (J. Hohenwarter & M. Hohenwarter, 2009; Hohenwarter & Preiner, 2007). As a 21st-century invention, GeoGebra is one of several next-generation mathematics learning technologies that are reshaping the representational infrastructure of mathematics education and providing the world community with easy and free access to powerful mathematical processes and tools (Kaput et al., 2002).

Viewed from the theoretical perspective of RME, GeoGebra affords a variety of digital resources that allow learners to mathematize realistic problem situations, invent and experiment with personally meaningful models using multiple representations and modeling tools, and further proceed to formulate increasingly abstract mathematical ideas. GeoGebra is open source and thus is freely available to the international community; it is also Web-friendly and is thus supportive of both individual reflection and Web-based social interactions. This integration of RME principles and GeoGebra technological features finds a similar theoretical framework developed in the instructional design community—Model-Facilitated Learning (MFL) (de Jong & van Joolingen, 2008; Milrad et al., 2003). As a technology-integrated instructional design framework grounded in Model-Centered Learning and Instruction (MCLI) (Seel, 2003, 2004), MFL tackles complex subject matter through modeling and simulations using systems dynamic methods and emphasizing the use of concrete scenarios, complexity management, and high-order decision-making. The existence of GeoGebra provides an intellectual bridge that connects a domain-specific theory of mathematics education, RME, and a general instructional design framework that is grounded in contemporary learning theories. Indeed, Seel (2003) characterizes RME as one of the exemplary domain-specific theories that operationalizes the basic tenets of MCLI. In our efforts to seek a theoretical framework that facilitates GeoGebra-integrated mathematics learning and instruction, we found it useful to synthesize RME and MFL principles, incorporating recent developments in the use of technology in mathematics education, in particular, the theory of Instrumental Genesis (IG) (Guin, Ruthven, & Trouche, 2005; Trouche, 2004), which sheds light on the mutually defining relationship between technology use and learners' evolving ways of mathematical reasoning. We believe that these three theoretical frameworks, in spite of their different origins and theoretical orientations, are collectively informative with regard to the ongoing use of GeoGebra in mathematics education.

In this chapter, we synthesize the major principles of RME and MFL in an effort to develop a preliminary theoretical framework toward model-centered learning and instruction using GeoGebra. We recognize both the didactical and the mathematical complexity of subject matter and the integral role of technology in mathematics learning and teaching, aiming for deep mathematical understanding and meaningful learning.

UNDERSTANDING AND DYNAMIC REPRESENTATIONS

A recurring and dominant theme in mathematics education reform is *understanding*, which is frequently used in conjunction with sense-making or meaningful learning in such phrases as *teaching for understanding* and *learning with understanding* (Brenner et al., 1997; Darling-Hammond et al., 2008). Understanding has, in effect, become a means and a goal of mathematics education. However, there is no clear definition of mathematical understanding. By contrast, it is relatively easy to identify specific cases where learners show a lack of understanding. For example, some students may automatically resort to subtraction in response to □ + 7 = 21, but cannot explain why they did that or if their answer 14 is correct. Similar examples are abundant in school mathematics.

Johnson-Laird (1983) suggests that the term *understanding* has plenty of criteria but may not have an essence. In his theory of comprehension, he contends that in understanding an utterance, learners first construct propositional representations and further make use of such propositional representations for the construction of a mental model, which preserves the structural relations in a state of affairs and enables the learner to make inferences. Mental models can be recursively revised and dynamically manipulated in support of deeper comprehension and inferences. Our understanding of a certain phenomenon amounts to the construction of a mental model of it; our interpretation depends on *both* the model *and* the processes involved in the construction, extension, and evaluation of the mental model. Indeed, as Johnson-Laird (1983) argues, “all our knowledge of the world depends on our ability to construct models of it” (p. 402).

Johnson (1987) examines understanding from the perspective of embodied cognition and describes understanding as “an event in which one has a world, or, more properly, a series of ongoing related meaning events in which one’s world stands forth” (p. 175). He characterizes meaning as a matter of understanding, which is always about relatedness as a form of intentionality in that “[a]n event becomes meaningful by pointing beyond itself to prior event structures in experience or toward possible future structures” (p. 177). Johnson conceives image schemata as organizing mental structures for human experience and understanding. Image schemata, which are functionally similar to Johnson-Laird’s (1983) mental models, are dynamic in nature because they are conceived to be flexible structures of activities. In summary, according to Johnson (1987), understanding is “an evolving process or activity in which image schemata, as organizing structures, partially order and form our experience and are modified by their embodiment in concrete experiences” (p. 30).

Furthermore, Perkins (1986) approaches understanding from his theory of *knowledge as design*, calling our attention to the metaphorical meaning of the term understanding. To understand means *to stand under* or be an insider of a problem situation. Our knowledge of a situation is accordingly a matter of design that has a purpose, a structure, model cases, and related arguments. Conceptual models, in particular, mediate human understanding, where mental models play a critical role, pervading, enabling or even disabling the cognitive processes.

The brief review above points to a set of common criteria for understanding as conceived in the fields of psychology, philosophy, and learning sciences:

– Understanding of a situation relies on a mental model that preserves the relevant or salient structural relations of a perceived or intuited state of affairs.
– A mental model is dynamic in nature and evolves with experience.
– The human ability to use mental models involves a system of relations that manages complexity and simulates a situation, enabling us to experience meaning and make inferences (cf. Seel, 2003).

Mental models are internal structures that are formulated in one's mind. But where do they come from? Johnson-Laird (1983) suggests that mental models are originally constructed through one's perceptual experience of the world, depending "both on the way the world is and on the way we are" (p. 402). A mental model therefore plays the role of a mental world that connects human imagination and the outside world. Johnson's (1987) image schemata are conceived as "recurring structures of, or in, our perceptual interactions, bodily experience, and cognitive operations" (p. 79). Perkins (1986) also recognizes the central role of mental models in framing our understanding, arguing that both mental and physical models are designs that are necessary components of human knowledge acquisition. Along the same line of thought, Norman (1983) regards mental models as naturally evolving models of a target system, which are not necessarily accurate but are functional in enabling people to make decisions or predications. As such, people's mental models also include their beliefs about themselves as well as the target system.

It follows, accordingly, that our understanding of a mathematical topic is a matter of having a functional mental model for it. Such a mental model does not only represent internally the state of relations of the mathematical topic but also runs dynamically in support of problem solving, including making wrong inferences (Norman, 1983; Seel, Al-Diban, & Blumschein, 2000). As internal entities, mental models cannot be directly assessed or constructed in an instructional setting. To assess one's mental models, it is necessary to have them externalized by means of cultural artifacts such as linguistic resources and mathematical notations. To support learners' construction of mathematically viable mental models, instructional designers need to provide model-eliciting activities, including intellectually appropriate conceptual models. In either direction, this leads to the discussion of multiple representations and conceptual and procedural understanding in mathematics teaching and learning.

Mathematics is a system of ideas developed over centuries as an outcome of the individual and collective endeavor of human experience (Dossey, 1992). The abstract nature of a mathematical idea is much similar to that of a mental model, which is not surprising at all, since one's mathematical ideas are mental models. Just as a mental model has two major components (i.e., a structure that preserves the relevant relations and the corresponding processes that allow the model to run dynamically), a mathematical idea is conceived as an interplay between one's conceptual and procedural knowledge (Hiebert & Carpenter, 1992; Silver, 1986). For historical reasons, mathematics has been taught with too much emphasis on its procedural aspects, resulting in a host of learning problems among students who can perform some procedures correctly, but are little aware of what they have done and why their result may be correct (National Council of Teachers of Mathematics [NCTM], 2000). Reform efforts since the 1980s have explicitly called for the pedagogical coordination of the two aspects of mathematical knowledge, especially in problem solving situations (Silver, 1986). To understand a mathematical idea therefore is to have a mental model that integrates both its conceptual and its procedural aspects.

In light of the complexity of mathematical ideas and the limitations and affordances of mathematical representations, mathematical understanding has accordingly been characterized as a person's ability to navigate through a system of multiple representations such as verbal expressions, diagrams, numeric tables, graphics, and algebraic notations (Goldin, 2003; Hiebert & Carpenter, 1992) and to grasp the relationships among the various representations and their structural similarities and differences (Goldin & Shteingold, 2001). This emphasis on multiple representations in mathematics education is consistent with similar principles involving complex subject matter in the learning sciences (Milrad et al., 2003; Minsky, 2006) since each representation carries its own limitations as well as affordances. From a practical perspective, if a learner can coordinate a variety of representations as a mathematically competent person does, there is solid evidence that he or she understands or, in other words, has a valid mental model for the underlying mathematical idea. Furthermore, each representation, such as a table or a graph, is characterized as the totality of a product and the related processes, which refers to "the act of capturing a mathematical concept or relationship in some form and to the form itself" (NCTM, 2000, p. 67), including both external and internal representations. Thus, each representation ought to be conceived as a mental model on the part of the learner, which is used to recursively transform his or her mental model for the mathematical concept. For example, to understand a linear relation, a learner should be encouraged to seek a comprehensive mental model that synthesizes the underlying structure behind its verbal descriptions, problem situations, numeric tables, graphs, algebraic expressions, and the various connections among them. A graph, as a constituent sub-model, represents the linear nature of the relationship. It also facilitates the corresponding procedures such as graphing, calculating its slope, finding inverses, and making predictions. When a learner's mental model for the graph is enriched through experience, the comprehensive mental model is recursively enhanced. Understanding thus occurs as the learner constructs increasingly mature mental models of the mathematical idea. There is not a clear endpoint in most cases.

When multiple representations are utilized to illustrate various aspects of a mathematical idea, they contribute to the complexity of the learning environment and the cognitive load on the part of learners. Learners who seek deep understanding are expected to grasp not only the dynamic nature of each representation but also the dynamic connections among the multiple representations. Attaining such a level of mathematical understanding, which exists in the mind of mathematically proficient learners, is a daunting endeavor in traditional educational settings since it typically spans a long period of time. The invention of dynamic and interactive technologies, however, has reshaped the representational infrastructure of mathematics, allowing for personally identifiable dynamic representations and, more importantly, automated linking of multiple representations (Hegedus & Moreno-Armella, 2009; Kaput, 1992; Moreno-Armella, Hegedus, & Kaput, 2008). The interactive nature of new technologies further support and constrain the co-actions between learners and the target system (Moreno-Armella & Hegedus, 2009), establishing a kind of partnership of cognition (Salomon, Perkins, & Globerson, 1991). The interplay between dynamic representations and mathematical ideas further enhances the social communication about mathematics, leading to discoveries of pedagogically powerful "synergies between representations and concepts" and "conceptually better-adapted versions of old ones" (diSessa, 2007, p. 250).

In summary, our understanding of a mathematical idea depends on a viable mental model that captures its structural relations and the corresponding processes. Given the complexity of mathematics, it is essential that learners interact with and construct its multiple representations. These multiple representations can be separately constructed and manipulated and also dynamically coordinated using emergent learning technologies, such as GeoGebra, in an environment that supports co-actions between the learner and mathematical representations. In theory, dynamic representations are well aligned with our conception of mental models as the foundation of mathematical understanding and are typical of the behavior of mathematically proficient learners (Nickerson, 1985). In practice, however, dynamic multiple representations pose serious challenges to instructional design. Given the complexity of mathematical ideas, mathematics instruction calls for a starting point that gradually guides learners' development of increasingly powerful and complex mathematical understanding. Thus, teaching mathematics using dynamic technology is an instructional design problem. In the next section, we discuss instructional design principles that may support learners' mathematical development when dynamic mathematics learning technologies are integrated as infrastructural representations (diSessa, 2007).

MODEL-FACILITATED LEARNING FOR DYNAMIC MATHEMATICS

Technology is becoming pervasively influential in mathematics education in that it is playing a "fundamental yet invisible role" (Kaput et al., 2007, p. 190) in much the same way that electricity, mobile phones, and emails are pervasive and influential and mostly taken for granted, especially when readily available and in good working order. Our teaching practices and beliefs about teaching and learning traditional mathematics are facing challenges from new technological tools such as WolframAlpha® (www.wolframalpha.com) and open-source environments like GeoGebra (www.geogebra.org). Indeed, virtually all traditional K-12 mathematical problems can now be readily solved by WolframAlpha®, which accepts natural-language input and also provides a host of related concepts and representations. Predictably, the technologies are getting more intuitive and powerful. Indeed, the very goal of mathematics education is challenged by these technologies. If our primary goal for school mathematics were to enable children to solve those problems, then there would not be much they need to know beyond software navigation skills. By contrast, if our goal is to empower children to understand mathematics in the sense of having a valid, culturally acceptable mental model for making decisions, judgments, and predictions, then there is very little such technical tools can offer to school children. Those tools are powerful and informative resources, but the rest belongs to careful instructional design and classroom implementation.

In recognition of the epistemic complexity of mathematics (Kaput et al., 2007) and the generative power of dynamic learning technologies, we contend that Model-Facilitated Learning (MFL) (Milrad et al., 2003) can be adopted as an overarching framework for reconceptualizing mathematics instruction that takes advantage of emergent dynamic technologies. We further seek domain-specific principles from the theory of Realistic Mathematics Education (RME) (Freudenthal, 1973; Streefland, 1991; Treffers, 1987) and the instrument-related perspectives from the theory of Instrumental Genesis (IG) (Guin et al., 2005).

Model-Facilitated Learning

Decades of research and development in instructional design have identified a few fundamental principles of learning and instruction. Noticeably, understanding is grounded in one's experience; meaning is situated in a context; and learning occurs when changes are made in an integrated system of constituents (Spector, 2004). As a theoretically grounded framework, Model-Facilitated Learning (MFL) (Milrad et al., 2003) draws on such basic principles, well-established learning theories, and methods of system dynamics to manage complexity in technology-enhanced learning environments. MFL seeks to promote meaningful learning and deep understanding, or a systems view of a complex problem situation. The MFL framework consists of modeling tools, multiple representations, and system dynamics methods that allow learners to build models and/or experiment with existing models as part of their effort to understand the structure and the dynamics of a problem situation. MFL recommends that learning be situated in a sequence of activities of graduated complexity, progressing from concrete manipulations to abstract representations while learners are engaged in increasingly complex problem solving tasks. Through the use of multiple representational tools, MFL further maintains the transparency of the underlying models that drives the behavior of a system simulation.

As an emergent theoretical framework for instructional design, MFL represents a well-grounded response to the affordances of new technologies and the needs to engage learners in the exploration of complex problems. As advanced instructional technologies are integrated into the teaching and learning of mathematics, and mathematics instruction is integrated with other disciplines of science, mathematics educators and instructional designers find themselves facing similar issues that MFL promises to address. In particular, the MFL component of policy development holds promise in fostering learner reflection and resolving the validation issue raised by Doerr and Pratt (2008), who found that in a virtual modeling activity, learners tended to validate their emergent understanding solely within the virtual domain without connecting it back to the starting problem scenario.

Didactical Phenomenology and Realistic Mathematics Education

Along with other researchers (e.g., Brown & Campione, 1996; Merrill, 2007) in learning and instruction, we contend that effective instructional design starts with a deep understanding of the content knowledge. In mathematics education, Freudenthal's (1983) didactical phenomenology of mathematical structures serves as a theoretical lens through which we can analyze a mathematical concept, including its historical origin, its realistic connections, its extensions, and its learning-specific characteristics. Such analysis lays the foundation for model-based instructional design and further yields a learning trajectory that starts from "those phenomena that beg to be organized and from that starting point teach[es] the learner to manipulate these means of organizing" (Freudenthal, 1983, p. 32).

Along such a learning trajectory, various representations are necessary for the learners to describe and communicate their experiences of the phenomena. This is where the new technologies come into play and facilitate the realization of learning potentials. The new dynamic technologies provide not only traditional forms of representation but also dynamic links and transformations. The dynamic links and transformations are significant in that it captures the dynamic *process* of representation as well as the static *product* of representation.

Freudenthal's didactical phenomenology lays the foundation for the theory of Realistic Mathematics Education (RME), which has at its core the principle that mathematics is a human activity, in which students make sense of realistic problem situations, re-inventing mathematical ideas under the guidance of competent instructors, and gradually creating increasingly abstract mathematical ideas. In such a process of mathematization, students are engaged in the use of a chain of models, which evolves from models *of* concrete learning tasks to models *for* abstract mathematical structures (Gravemeijer et al., 2000).

Instrumental Genesis

Tool use is an essential component of mathematical learning. As learners make use of tools, including both traditional and digital tools, to facilitate their mathematical activities, such tools and their uses also constitute their mental world. A technical tool, which organizes and facilitates an activity, may eventually be internalized as a psychological tool, or rather, an instrument that mediates a learner's mental processes (Vygotsky, 1978). In other words, a technical tool may become a part of a mental model that enables a learner to make inferences in a problem situation. Mariotti (2002) characterizes such an instrument as an internal construction of an external object, which is "the unity between an object … and the organization of possible actions, the utilization schemes that constitute a structured set of invariants, corresponding to classes of possible operations" (p. 703). A common example is the relationship between students' use of the compass as a circle construction tool and their conception of a circle, which automatically has a center and a radius, but makes it difficult for students to grasp other properties of a circle such as those concerning the diameter being the longest chord or locating the missing center of a given circle.

Within the context of a mathematical activity, the interactions between a tool or artifact and the learner are captured in the notion of *Instrumental Genesis* (IG), in which the learner builds his or her own schemes of action for the tool in a process called *instrumentalization* and the tool also shapes the learner's mental conception of the tool and the activity in a process called *instrumentation* (Guin & Trouche, 1999; Hoyles, Noss, & Kent, 2004; Mariotti, 2002; Trouche, 2005). Instrumental genesis is a long-term process that evolves as a learner internalizes more mathematical and technical artifacts and thus becomes more mathematically proficient. In light of the close relationships among the artifacts, the learner, and the specific mathematical activity, it is reasonable to conceptualize instrumental genesis as a triadic theoretical framework that helps make sense of general human activities, including mathematics learning where new tools have become distinctively instrumental (Fey, 2006). In particular, when new dynamic tools are used, the learning outcome is frequently different than the instructor's intentions (Hollebrands, Laborde, & Sträβer, 2008). For example, while the mid-point tool in GeoGebra was intended as an alternative way to find the mid-point of a line segment, we found, in a professional development project in the US Midwest, that some classroom teachers would choose to use it when they were asked to find the mid-point of a segment whose two endpoints were explicitly given as pairs of coordinates.

In summary, instrumental genesis is a kind of descriptive learning theory that has a solid grounding in the social theory of learning and provides a theoretical lens through which we can make sense of learners' use of technological tools and the potential impact of tool use on their mental processes in the context of mathematical activities. It contributes significant ideas to our understanding of mathematics learning in a model-centered perspective, where tools and artifacts are integral components at all stages.

Model-Facilitated Learning (MFL) for Dynamic Mathematics

The notion of dynamic mathematics dates back to Kaput's (1992) conception of representational plasticity when digital media are used to support various forms of representation and has been further developed by other researchers (Moreno-Armella et al., 2008) from historical and epistemological perspectives as new dynamic technological tools become widely accessible. We tend to think of dynamic mathematics as a systematic correlation between the didactical phenomenology of a mathematical idea and the corresponding technological representations and tools. A mathematical idea is dynamic in that it is connected to a variety of other ideas in realistic and mathematical contexts and is also executable in the sense of a mental model and that of a conceptual digital construction. This notion of dynamic mathematics is well aligned with our characterization of mathematical understanding. In a real sense, all mathematical ideas are dynamic in the mind of a mathematically competent person. That process, which takes a long period of time to develop, can be facilitated by new dynamic technologies in support of learners who are on the way to mathematical proficiency. Using Doerr and Pratt's (2008) notation, dynamic mathematics could be conceptualized as a set of (task, tool) pairs, which serve as the modeling infrastructure. Over such a mathematical and technological infrastructure, we seek to apply the MFL principles, thus establishing a preliminary instructional design framework for integrating dynamic technologies into the teaching and learning of mathematics. Our overarching goal is to promote students' deep understanding of mathematics by focusing on the mathematical processes that are involved in problem solving and tool use.

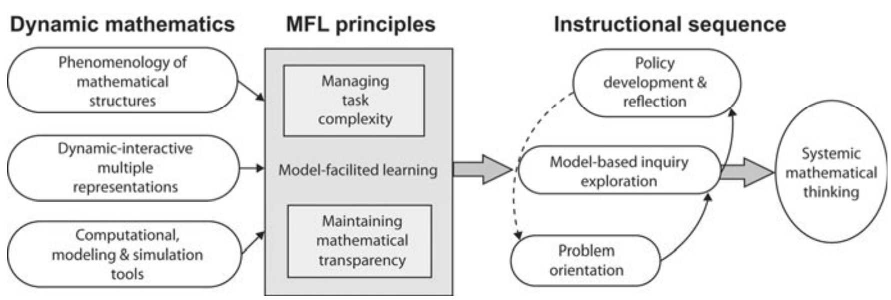

Figure 1. A model-facilitated instructional design framework for dynamic mathematics.

As shown in Figure 1, a lesson design cycle starts with a didactical phenomenological analysis of a mathematical idea, which charts out its structural connections, including its historical, realistic, and formal relations. Next, the technological tools are aligned with the mathematical connections with respect to the global learning objectives. Then, the MFL principles are applied in order to manage the complexity and maintain the mathematical transparency of the mathematical and technological or (task, tool) system. Models of multiple representations and structures play a major role in allowing students to explore and mathematize a starting scenario, develop increasingly abstract understanding, and further make informative decisions about the problem situation. We note that through the stage of policy development or reflection, learners will have an opportunity to examine their modeling activity as a whole, validate or modify their emerging insight into the problem situation. Furthermore, in a modeling environment, the learner development cycle may assume a cyclic form, achieving higher levels of systemic understanding with each cycle. Noticeably, technical tools play an integral role in the modeling process. What initially is a digital representation, a computational utility, or a simulation tool may become integrated into learners' mental world of mathematics as a psychological instrument. Therefore, we should pay special attention to the changing roles of tools in the learning cycle as a way to understand the challenges and opportunities of learners' experience with dynamic mathematics.

Specifically, in app0lying MFL principles to dynamic mathematics, we recommend the following guidelines:

- Conduct a phenomenological analysis of the mathematical idea concerned and identify some historical, realistic, or contemporary problem scenarios to situate the learning process.
- Select a realistic scenario and conduct a thought experiment about the possible stages of learner development and the corresponding scaffolding strategies in what may be called a hypothetical learning trajectory.
- Present problems of increasing complexity and maintain a holistic view of the opening scenario.
- Guide learners in making sense of the problem scenario in mathematical ways such as model building and model use, maintaining awareness of technical tools and their intended functions.
- Challenge learners to examine their modeling process, reflect on the meanings of their tool-enhanced actions, and further develop insight into their actions through decision-making and model-based inquiries.
- Involve learners in group discussions about their learning processes and develop arguments for or against different ways of mathematical thinking and dynamic constructions.

In summary, dynamic mathematics learning technologies, such as GeoGebra, provide an innovative platform to experiment with the basic tenets of Realistic Mathematics Education, in particular, its focus on using realistic contexts as sources of mathematical concepts and guided reinvention as a primary method of mathematization. Guided reinvention involves modeling as a fundamental process of mathematics learning (Gravemeijer & van Galen, 2003). When the complexity of mathematics learning is recognized and further appreciated in the context of emergent digital technological tools, MFL stands as a well-conceived theory-based instructional design framework that addresses the learning of complex subject matter using system dynamics methods and interactive technologies. The theory of instrumental genesis further sheds light on the mutually constitutive relationship between technical tools and learner development. By incorporating the major theoretical principles, we seek to develop a comprehensive design framework to conceptualize the integration of GeoGebra and similar technologies in mathematics teaching and learning. In the next section, we look at three examples that involve the implementation of some of the principles discussed above.

DESIGN EXAMPLES

Quadratic Relations

In this section, we present a model-based learning sequence for the teaching of quadratic relations to algebra students. Quadratic relations exist in various forms in mathematics and are frequently summarized in its algebraic form $f(x) = ax^2 + bx + c$, where a, b, and c are some constants. If $a = 0$, it is reduced to a linear relation. Such a rule, familiar as it is to most algebra students, barely touches on the rich connections and mathematical significance of a quadratic relation. Certainly, we could make use of GeoGebra sliders and animate the effects of *a, b, c* on the shape and location of the parabola (Figure 2). However, this approach does not add much meaning to the mathematical relation.

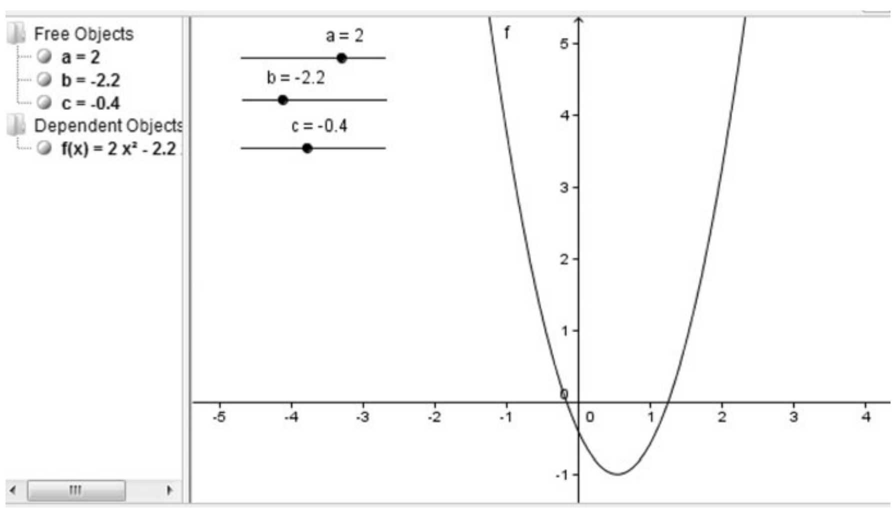

Figure 2. Exploring the effects of a, b, and c on the graph of a quadratic relation.

A preliminary phenomenological analysis of quadratic relations reveals a variety of contexts that may be employed as the foundation for concept formation. First, a quadratic relation can be found in conic sections or similar geometric activities such as paper-folding. Second, it can be found in a context that involves the area of a rectangle with certain width and length. A realistic problem could be stated as: If the width of a rectangle is given as x inches, and the length is two inches longer than the width, how is its area related to the width? Third, a quadratic relation can be found in the real-world construction of a dish antenna, which may use the directrix and focus description of a parabola. Fourth, the quadratic relation can be found in a water fountain or a similar situation involving a projectile. A focal question could be "why does the water stream behave the way it does once it leaves the spout?" Fifth, a quadratic relation exists in the phenomenon of free fall in physics. Other analysis may eventually reveal the fact that a quadratic relation can be developed out of natural or artificial phenomena that involve two dimensions such as width and length for area or time and speed for distance.

Such a phenomenological analysis reveals the complexity of the mathematical idea and justifies the necessity of using multiple models in mathematics instruction. In our work with prospective mathematics teachers, we found the free fall phenomenon interesting since it is naturally familiar to and yet mathematically challenging for most of them. It further provides a context to ground discussion and investigation of multiple related mathematical topics, including constant functions, linear functions, and quadratic functions. Also, the situation could be simplified pedagogically to manage complexity while maintaining the integrity of the whole task. For example, if students find speed change difficult, a sub-problem could be posed for them to model distance changes in the case of an object moving at a fixed speed with no acceleration.

Thus, we can use the free fall phenomenon as a starting point for our discussion of quadratic functions by posing the following problem. The goal is to solve the problem by modeling the scenario and/or derive a formula to solve the problem.

> The Sears/Willis Tower in Chicago is about 442 meters from its roof to the ground. Mark takes a baseball to the roof, and somehow gets it out of the window with no force imposed on it. Now the ball falls freely toward the ground. Assuming that the air has no significant influence on the baseball and the gravitational acceleration in Chicago is approximately 10 m/s^2, Mark wonders, without using calculus: (1) How fast is the ball falling? (2) How does the distance from the roof to the ball change over time?

According to our experience with preservice mathematics teachers, few students had a clear idea of what was going on, although nobody had trouble imagining such a situation. The primary challenge we encountered as instructors was an immediate call for a formula. For various reasons, most students tended to expect a formula to solve a given problem. While a formula does exist in this scenario, it is the least important part of the learning process, at least, under our circumstances. Instead, we could create a dynamic GeoGebra model to make sense of the scenario. The process is rarely sequential, but we need to follow a step-by-step approach in our presentation below.

First, we recognize the fact that there are only three initial parameters involved: the Earth's gravitational acceleration estimated at 10 m/s^2, the height of the tower, and the flow of time. Since we may want to play with these parameters, we choose to use names (or GeoGebra sliders) to represent them. This step is not required, but it leaves room for us to explore the dynamics of the problem. The initial values and the intervals of these sliders can be adjusted according to the real situation.

Second, we want to see how the speed of the baseball changes over time. Some students may choose to graph the function *speed(x) = 10x*, using the *x*-axis for time. That is a reasonable method if they understand the meaning of the function. However, we choose to graph the speed over time point by point, using the fact that at a given time designated by the slider *Time*, the speed of the ball is *Gravity* × *Time*. Therefore, we can plot the point using a command line input: *Speed = (Time, Gravity * Time)*. This allows students to simulate the situation. When slider *Time* is dragged, the point *Speed* changes accordingly, indicating the change of speed over time. By turning on the *Trace* feature for the point *Speed*, they could see how the speed changes over time (Figure 3). Of course, a point such as *(Gravity, Time)* could also be plotted to visualize the (lack of) change in *Gravity* over time. In the light of the multiple relations in the problem scenario, students should be encouraged, at all stages, to explore their own methods or externalize their conceptions, followed by small-group or whole-class justifications and reflections. For instance, the point-wise graph (Figure 3) can be shown to coincide with the continuous graph of *speed(x) = 10x*, which can be taken advantage of to introduce the meaning of an algebraic function *f(x) = mx*.

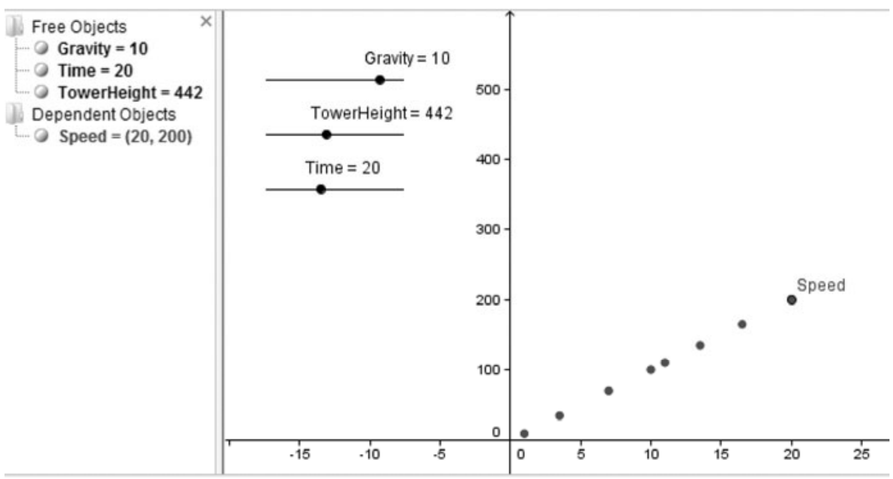

Figure 3. A point-wise plot of the speed-time relation.

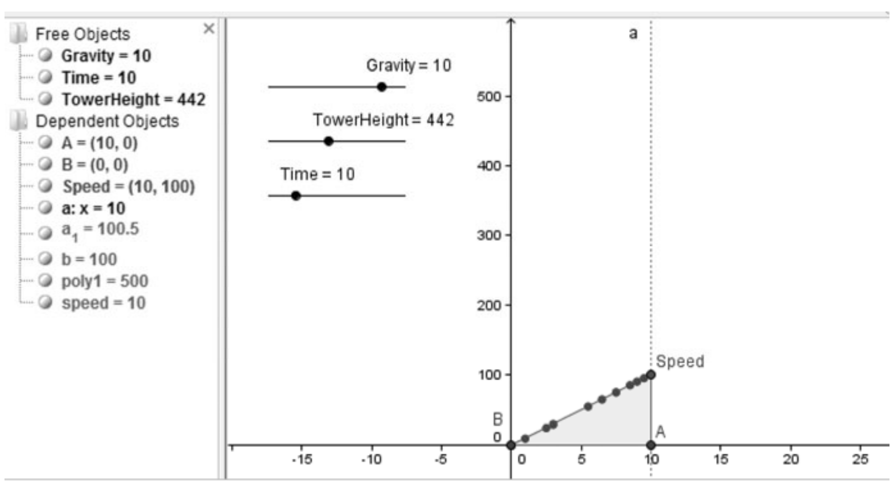

Figure 4. The relationship between distance traveled and area of a triangle.

Third, to find the distance the ball has traveled at a certain time, some cognitive support is necessary, including the use of simpler problems and prompts. For example, the instructor could pose a question about the area of the triangle formed by the points *(0, 0)*, *Speed*, and the corresponding point on the *x*-axis *(x(Speed), 0)*. If necessary, a simpler problem about constant speed and distance could be posed for students to relate distance to area in a geometric way. This critical step represents a cognitive leap and calls for the use of analogical reasoning and, more importantly, social interactions among the students and the instructor, where technology plays a very limited role. Eventually students will come to relate the distance traveled at a certain time to the area of the triangle as shown in Figure 4, and the area tool of GeoGebra thus becomes an instrument for students to find the distance.

Fourth, when students relate the distance traveled to the area of a corresponding triangle, it would be appropriate to ask the question: How is the distance traveled related to time? For that purpose, we can plot a point using the command line input: *Distance = (Time, poly1)*, where *poly1* is the name of the triangle and represents its area. Using the triangle as a whole without calculating its area is one of the features of GeoGebra that supports graduated complexity. At a higher level, it may be very appropriate for students to find an explicit way to calculate the area of the triangle. However, at the current step, the focus is to explore the relationship between distance and time. Using the *Trace* feature for the point *Distance*, students can simulate the free fall process and observe the change of distance over time in addition to the previous speed-time relationship (Figure 5).

Fifth, to find when the ball will hit the ground, we could draw a horizontal line *y = TowerHeight* and simulate the free fall until the *Distance* point goes beyond that line as shown also in Figure 5.

Sixth, since the initial conditions *Gravity* and *TowerHeight* are defined using sliders, students can now change the initial conditions, observe their influences, and ask open-ended questions about the problem scenario. For example, what would happen in a place where gravity is 2.5 m/s^2? What if the gravity is zero? By exploring such questions, students can potentially form a perspective on the problem scenario and identify the structure of the problem (i.e., the constant, linear, and quadratic relations).

Finally, the above GeoGebra sequence could be extended to support higher levels of algebraic thinking. While point-wise graphs represent a snapshot of the underlying structure of the problem scenario, they lack efficiency. As mentioned earlier, some students may find it tempting to graph the speed-time relationship using a function like *speed(x) = Gravity * x*. Along the same line of thinking, they could be guided to find an explicit relation between the distance and the time. In other words, at time *x*, what is the area of the corresponding triangle? Using the base-height rule, at time *x*, the area of the triangle is *(1/2) x * speed(x)*, where *x* is the base and *speed(x)* is the height. Since *speed(x) = Gravity * x*, the distance traveled is (*1/2) Gravity* * x^2. Therefore, we could enter *distance(x) = (1/2) * Gravity * x^2* at the command line (Figure 6). Other dynamic explorations are subsequently possible for students to develop a comprehensive mental model of a quadratic relation and its mathematical connections.

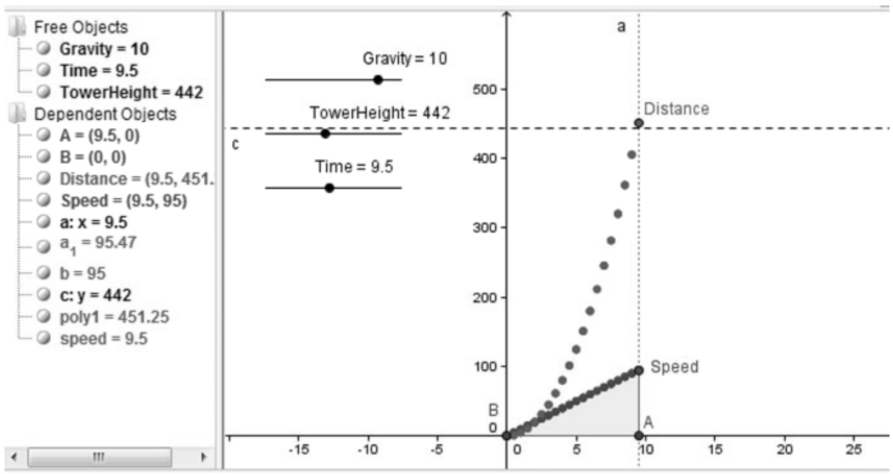

Figure 5. Comparing the relationship between distance and time with the relationship between speed and time.

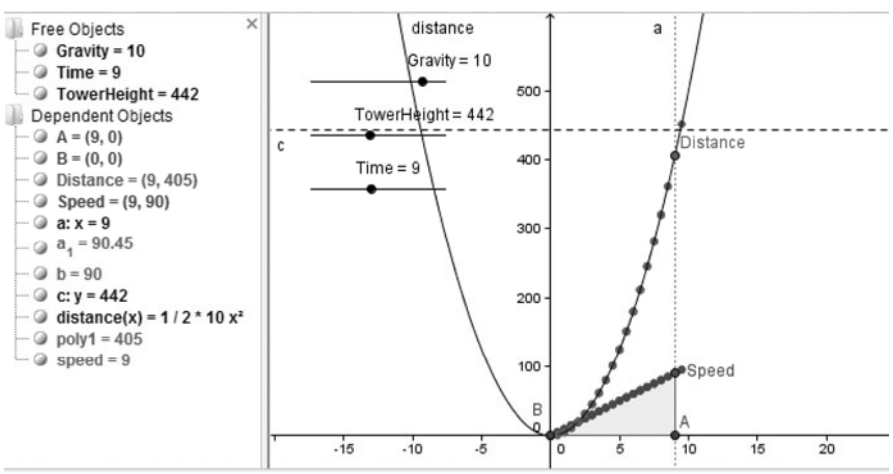

Figure 6. The graph of a quadratic function fits the point-wise simulation of free fall.

To summarize, in the free fall construction we applied the basic principles of RME and MFL in our effort to make sense of not only the problem scenario but, more importantly, the mathematical ideas behind a quadratic relation. The resulting dynamic GeoGebra model serves three main purposes. First, it is the end-product of a problem solving process and can be evaluated to assess a student's understanding of the topic. Second, others can use it as a conceptual model to learn about the problem situation at a different level or as an initial step toward *learning by modeling*. Third, the GeoGebra model can be used as new starting points for higher-order explorations since it can be modified, extended, or incorporated into other instructional units of various mathematical focuses.

Pi

As another design example, we look at the most familiar concept of mathematics, *Pi*, which is defined as the ratio between the circumference of a circle and its diameter. In our experience, few students have trouble recalling the estimated value of *Pi*, but quite a few students cannot describe what it is beyond giving the number *3.14*. There is a long history behind the mathematical idea of *Pi* and many methods for estimating its value (Beckmann, 1976). However, most seem to require advanced mathematical knowledge such as power series or the concept of limit. Our goal is to get students to explore the mathematical idea of *Pi* and build a valid mental model that provides meaning in their future work involving *Pi*.

In light of the ubiquity of circles in the real world and the technological tools provided in GeoGebra, we decided to have students collect data about circles. Specifically, they were asked to find a variety of circles at home, measure them in inches, and record their data in the form of (diameter, circumference) pairs. Except for some measurement errors caused by the ruler, this is a trivial task. It serves as a starting point for further mathematization of the properties of a circle. As a subsequent activity, students were asked to plot these ordered pairs in the GeoGebra environment. Although a single pair of (diameter, circumference) by itself is insignificant, 17 pairs do tend to form a pattern when plotted as shown in Figure 7. This is the first step of *Pi* modeling, where a visual pattern points to the relationship between the circumference and the diameter of a circle.

The next step involves some form of regression analysis, which is beyond the scope of middle-grades mathematics. However, if regression analysis is not the primary objective of instruction, we could take advantage of the technological tools to manage the complexity of the task. Within GeoGebra, we could use the tool *Best Fit Line*, which takes a group of points and generates a line of best fit. With our data, the line of best fit is $y = 3.13x + 0.19$. This best-fit line represents a new type of mathematical model which leads to further discussion about the meaning of the *slope* (3.13) and the interpretation of *y-intercept* (0.19), including the influence of individual points.

A variety of *what-if* and *what-if-not* questions (Brown & Walter, 2005) could be further asked using the dynamic GeoGebra construction. For example, what if someone had made a measurement mistake? What if I drag a point away from the majority of the points? What if we had measured 100 circles of different sizes?

Discussion of these questions and the meaning of slope will eventually help students come to understand *Pi* as a ratio between the circumference and the diameter of a circle and construct a meaningful mental model of a circle, where *Pi* indicates how the circumference changes with the diameter of a circle. Such a dynamic mental model is much more powerful for students to make inferences and predictions about relations involving circles than the narrow conception of *Pi* as a number that is about 3.14 and will in the long run help students appreciate the ideas behind *Pi* and similar linear relations as they move forward to higher levels of mathematics.

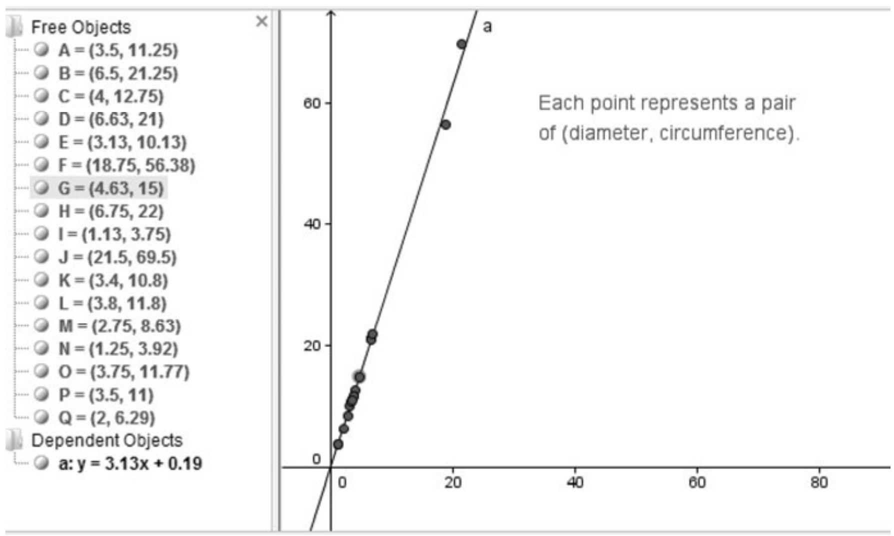

Figure 7. The relationship between the circumference and the diameter of a circle (Point-wise plot and best-fit line).

Similarity

As a third example, we look at the concept of similarity, which is closely related to proportional thinking in the learning processes. Most students have an informal understanding of similarity and can describe it in everyday terms. However, in exploring the concept of similarity in a realistic context, they tend to have difficulty coordinating the multiple quantities involved in a ratio or a proportion. In our work involving GeoGebra, we used the two poles problem as presented below.

> There are two poles erected on the ground (as shown Figure 8). One is six feet tall, and the other is three feet tall. Two ropes are tied from the top of one pole to the bottom of the other, intersecting at point *P*. What is the height of point *P*?

The problem situation should be imaginable to all students. In fact, they could conduct a hands-on or physical experiment and measure the height of Point *P* with respect to the ground. A question that naturally arises with the students is "How far apart are the two poles?" The distance between the two poles is not given in the original problem and this hinders students' attempts to solve the problem. When they are asked to experiment with different distances, however, they would begin to have a tentative finding: Perhaps, it does not matter at all!

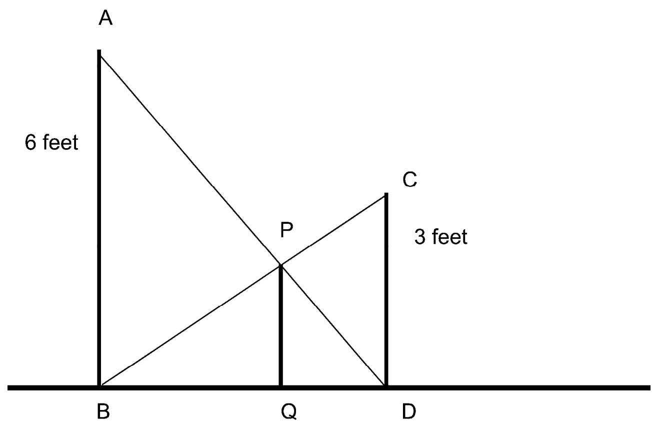

Figure 8. The two-pole problem: What is the height of point P?

While a physical simulation reveals some details of the problem, it has physical limitations: Students can not easily manipulate the problem or extend the problem space. With the physical model as a starting point, students can then move ahead to a GeoGebra simulation, which calls for further mathematization of the problem. In our experience with prospective and in-service teachers, we have repeatedly found that this step is a frustrating and yet motivating stage. When asked to build a GeoGebra model to represent the problem scenario, the vast majority would make a visual model of the original picture using lines and segments without attending to the mathematical aspects and assumptions of the problem. The visual model looks like a GeoGebra-based model of the problem; but when dragged, it collapses to their disappointment. This is a *good* mistake since it reveals the limitations of a visual model and helps students attend to the mathematical aspects of the problem. This shows the diagnostic power of a GeoGebra-based dynamic construction. A brief conversation with the class would quickly lead to the observation that the two poles should be perpendicular to the ground and, indeed, the ground does not have to be horizontal in a mathematical model.

Once students have completed this first step of mathematization from the context to a mathematically valid model, they could move on to the next level and use the GeoGebra model to address the original question about the height of point *P*, which is two feet. Subsequent exploration will unveil the fact that the distance between the two poles is irrelevant. No matter how far apart they are, point *P* is always two feet above the ground, even if the ground is slanted. When the *Trace* feature is turned on for point *P*, we could move the two poles back and forth to collect more evidence in support of the observation, as shown in Figure 9.

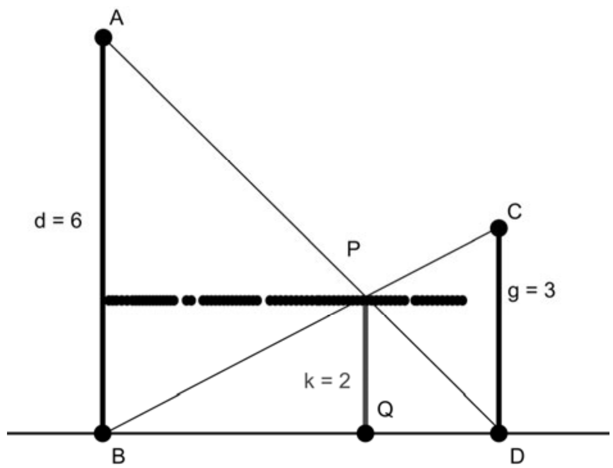

Figure 9. The height of point P stays constant regardless of the distance between B and D.

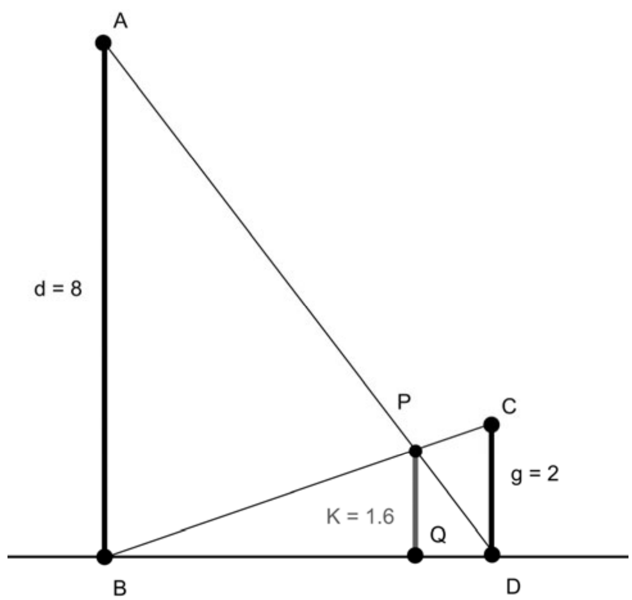

Figure 10. The height of point P is not the ratio between the two poles.

In a dynamic GeoGebra model, the discussion does not end with finding the height of point *P*, which in fact is not the overarching goal of the activity. Students should be prompted for the next level of exploration—how is the height of point *P* related to those of the two poles? Many students might come to a quick hypothesis that the height of point *P* is the length of the longer pole divided by that of the short one, which is indeed true numerically in this specific case. When such a hypothesis is discussed by the whole class, some students would suggest experimenting with the dynamic model, since they could change the initial conditions and generate numerous cases, which would lead to the rejection of their previous hypothesis. Although exploration through GeoGebra-based dynamic modeling does not yield an immediate rule for the relationship between the height of point *P* and the lengths of the two poles, it is a very meaningful and authentic learning process, which may serve as the foundation of a valid mental model for proportional reasoning in that it goes beyond a formula such as $a/b = c/d$ and reveals the structure of the problem scenario.

When students interact with the dynamic GeoGebra model, making and/or rejecting their hypotheses, they may potentially see the invariant relational structure among the numerous cases. There are always two pairs of similar triangles involved, and the two triangles share a common side. Eventually, they may be scaffolded to articulate ratios with the two triangles and derive a rule for the height of point *P*, which is in the form of $(AB \times CD)/(AB+CD)$ and is applicable to all cases. While the rule itself is interesting, it is the holistic experience that will help students appreciate the mathematical ways of reasoning and the rationale behind the rules in mathematics. Along the learning trajectory, GeoGebra plays a variety of cognitive roles. First, it helps students understand the problem and identify gaps in their mathematical knowledge. Second, it helps solve the original problem and open doors for further exploration. Third, it helps students reason with the model, formulating and/or rejecting their hypotheses. Fourth, it serves as a conceptual model for proportional reasoning, which may eventually be incorporated into a student's mental models for future encounters with similar problems. Finally, it shows how tools support and limit our perception of mathematical processes. To summarize, it is the whole experience from physical modeling and GeoGebra modeling, to advanced mathematical reasoning that provides students with a perspective on the complexity of the problem, the human nature of mathematical reasoning, and a paradigmatic case for mathematical problem solving. Indeed, the formula or the recall of such a formula does not illuminate the richness of the relations underlying the problem scenario and its pedagogical values.

In this section, we provided three design examples to showcase the relevance of the theoretical framework discussed previously in our re-conceptualization of school mathematics for the purpose of learning with understanding. There are a variety of mathematical topics that lend themselves to this type of experimentation which strives to provide certain valid mental models for students to make sense of and develop a holistic perspective on the end-products of mathematical investigations—rules and formulas. The scenarios presented above could well be replaced with similar ones, depending on the instructional context, the prior knowledge of the students, and the specific learning objectives. The fundamental principle is that students should be engaged in conceptual modeling in order to develop valid and generative mental models in support of their mathematical learning. They either learn with ready-made models or learn by building or modifying models, using the technological tools available in the learning environment.

CONCLUSIONS

Mathematics is a human activity (Freudenthal, 1973). Mathematics learning and instruction are endowed with all the complexities of such a human endeavor, which range from the multi-dimensionality of the subject matter, the diverse backgrounds of students, to the evolving learning environments, including technological tools and educational goals. Virtually all mathematical ideas, even the very basic ideas in school mathematics, assume their significance with respect to an underlying conceptual system (Lesh, 2006). In this chapter, we first recognized the complexity of mathematics learning and the corresponding call for meaning and understanding in the ongoing mathematics education reforms. While it is relatively easy to identify a lack of understanding such as in the case of rule-based recall of facts, it is challenging to define what understanding is in the context of mathematics learning. A brief literature review further revealed the complexities of understanding in various contexts from learning theories to philosophy. In light of the research and theoretical developments in the past three decades in mathematics education, it seems reasonable to characterize mathematical understanding as a matter of having a world of dynamic mental models that are consistent with the conceptual systems of mathematics and can be called upon in specific situations in support of decision-making and predictions. The complexity of a mathematical idea, especially its connections to an underlying system, further requires the use of multiple representations and their dynamic interconnections. Given the internal nature of mental models, conceptual modeling becomes a necessary mediator to foster changes and developments in a learner's mental world. In exploratory modeling, learners interact with ready-made systems as a way to learn about the underlying structure of the system; in expressive modeling, learners construct or modify models as a way to externalize, reflect on, and modify their mental models (Doerr & Pratt, 2008). At a higher level, as learners solve problems in a model-centered environment, they further construct a mental model of themselves as problem solvers, which includes their beliefs, attitudes, and identity in relation to mathematics learning (Goldin, 2007; Norman, 1983).

The dynamic nature of mathematical understanding and the corresponding needs for multiple representations serve as a theoretical foundation for the integration of technological tools such as GeoGebra, which provides the utilities for learners to construct mathematical models. In a traditional setting, an expert's dynamic understanding of a mathematical idea is usually hidden from the observers. To some extent, GeoGebra models help experts better externalize their mental models of mathematics for the purpose of personal reflection, and more importantly, as conceptual systems to facilitate novices' learning.

From a design perspective, we synthesized the basic principles of Realistic Mathematics Education (RME) and Model-Facilitated Learning (MFL), which, though developed in different fields, share a common theoretical orientation—facilitating the learning of complex subject matter in a meaningful manner using models and modeling as a pedagogical tool to manage complexity and promote increasingly higher levels of understanding. While RME is deeply rooted in the search for meaning in the past three decades of mathematics education research and development, MFL has a solid foundation in learning and instructional design theories, with a strong commitment to the integration of new interactive technologies. We further considered the theory of Instrumental Genesis (IG) as a theoretical lens to examine the use of tools in mathematics learning. As learners make use of new technological resources such as GeoGebra in mathematical problem solving, their mathematical conceptions or mental models may become increasingly instrumented entities. In our work with prospective and in-service teachers, we have collected data in support of this theoretical construct. For example, when asked to find the area of a triangle whose vertices are given in terms of coordinates, many teachers tended to plot the three points, define a polygon, and then read its area from the GeoGebra environment, even if the three vertices were special cases and the area required only simple computations. Once tools become part of their mental resources, they seem to pose challenges to the traditional conceptions of mathematics and especially assessments. In short, RME, MFL, and IG provide a unified theoretical framework for us to examine the design and learning processes in our GeoGebra-integrated mathematics courses and professional development projects. As we gather more empirical data from the teachers and their students on a variety of mathematical topics in a variety of settings, we may need to further refine our theoretical constructs and clarify the relevance and limitations of the basic principles.

As design examples, we presented our preliminary work on the quadratic relations, *Pi*, and similarity in GeoGebra-integrated mathematics courses and professional development, which demonstrate how typical ideas in school mathematics should and could be reconceptualized, recontextualized, and problematized for the purposes of meaningful learning. One common characteristic of the three examples is our effort to engage students in whole-task explorations while providing just-in-time support with regard to component skills (van Merriënboer, Clark, & de Crook, 2002; van Merriënboer & Kirschner, 2007). Basic skills, such as plotting points and constructing perpendicular lines, are meaningful mostly because of their connections to the whole task and they are scaffolded on demand as part of the whole task.

While GeoGebra trivializes a host of traditional mathematical tasks such as graphing functions, solving equations, and finding geometric reflections, it does open the door for much more interesting and motivating scenarios of mathematical explorations and provides a platform for designing and implementing inquiry-based learning (Barron & Darling-Hammond, 2008). Technology changes what mathematics can be investigated with students and how traditional mathematical ideas should be taught (NCTM, 2000). As new technologies enter the lives of students at school and beyond, traditional problem solving could be further considered from a modeling and models perspective, incorporating the evolving needs of students and expectations of society (Lesh & Doerr, 2003). Furthermore, in light of the versatile nature of GeoGebra and its ongoing development, GeoGebra lends itself to a variety of theoretical frameworks for mathematics education. Just as we recognize the dynamic nature of mathematical understanding and the use of GeoGebra, we seek to embrace a dynamic and diverse understanding of instructional and learning theories (Jonassen, 2005) as the world community joins hands in charting out the challenges and opportunities of quality mathematics education for all.

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ROBYN PIERCE AND KAYE STACEY

3. USING DYNAMIC GEOMETRY TO BRING THE REAL WORLD INTO THE CLASSROOM

This chapter reports on the use of dynamic geometry to support the use of real world contexts to enhance the learning of mathematics in the middle secondary years. Dynamic geometry, either linked to real world images or used to create dynamic simulations, not only can provide opportunities for students to collect real or simulated data to make conjectures, but also can improve their understanding of mathematical concepts or relationships through exploration. Tasks, which access these features, can be valuable for both increasing students' engagement and their depth of mathematical thinking. The colour, movement and interaction can create a halo effect valued by teachers for its impact on students' general attitude towards studying mathematics.

INTRODUCTION

Teaching fourteen and fifteen year olds mathematics presents many challenges. Engaging these students can be difficult, especially as the mathematics they study becomes more abstract. Dynamic geometry offers opportunities to bring the real world into the mathematics classroom, to add visualization, colour and animation not possible in a traditional classroom and to deepen the mathematical thinking we expect of the students in various topics of the curriculum.

The four examples of curriculum materials presented in this chapter were developed as part of the RITEMATHS project (HREF1) and they focus on several different affordances of dynamic geometry. The project aimed to investigate the use of real (R) world context problems with the assistance of Information Technology (IT) to enhance (E) middle secondary school students' engagement and achievement in mathematics (MATHS). Teachers from different schools explored how a variety of new technologies could be used, especially in the algebra and functions strand of the curriculum, to assist students to see the links between abstract mathematics and real world situations. Within the project, teachers used graphics calculators, computer function graphing and spreadsheets, computer algebra systems, and digital image and video analysis software, in addition to dynamic geometry from several authors. Dynamic geometry software was amongst the most successful of the technologies that were explored.

GeoGebra offers integrated applications so that dynamic geometry is seamlessly linked to scientific calculator capability and a function grapher. It also allows for some use of text and digital images. The examples given in this paper demonstrate how these facilities allow the teacher to:

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 41–55.

- Give high quality, attractive presentations about real world situations that interest students,
- Provide support for students to engage in problem solving related to real world situations,
- Set tasks that allow students to explore mathematical regularities and variation within one mathematical representation,
- Set tasks that allow students to explore the links between different mathematical representations of mathematical objects.

The value of real world problems for the teaching of mathematics has been well recognised over many years (see for example Burkhardt, 1981). The advantages of providing a curriculum experience for students that is rich in reference to real world problems have been summarized by Blum and Niss (1989) as:

- Fostering general competencies and attitudes (such as a belief that mathematics is useful);
- Preparing citizens who have critical competence (knowing, for example, how to analyse data carefully);
- Equipping students with the skills to utilise mathematics for solving problems;
- Giving students a rich and comprehensive picture of mathematics (including it's applications) and
- Sometimes motivating students to learn mathematics of a traditional type (e.g. by showing how a particular subject is relevant to a student's chosen career)

In addition to these traditionally recognized benefits of real world problems, as a result of our observations in the RITEMATHS project we reported (Pierce and Stacey, 2006) an additional advantage of real world problems. Teachers often use real world problems in order to enhance the image of mathematics by creating a 'halo effect'. The term 'halo effect' dates back to the 1920's work of Edward Thorndike (1920) who showed that assessments of specific traits of a person or thing are markedly influenced by an overall impression, which may itself be based on little evidence. In marketing, the term 'halo effect' is used to describe the phenomenon of extending a positive view of one specific attribute or item to an entire brand. In the RITEMATHS project, many mathematics teachers took advantage of this psychological phenomenon. They aim to capitalise on students' appreciation of colour, fun, and pleasant experiences, some associated with an out-of-school situation, to create a 'halo effect' that extends from the pleasant experience to the immediate mathematical task which uses this real world problem and then beyond to the study of mathematics in general. Some examples of how this operated are presented in this paper.

With dynamic geometry, the real world may be brought into the mathematics classroom in two ways: through the use of digital images and through the use of simulations. Like Arcavi and Hadas (2000), we aim to convey the general spirit and special characteristics of activities as examples of the pedagogical and cognitive potential for the use, in this case, of dynamic geometry. In this chapter the first three examples illustrate the use of digital images while the fourth illustrates the use of simulation. In the sections below, each example will be illustrated and discussed with attention to both the cognitive and affective aspects of student learning which are built into the design. Lesson plans, worksheets and files for these lessons and many others can be downloaded from the RITEMATHS website (HREF1).

EXAMPLE 1: FEDERATION SQUARE

At Federation Square in the centre of the city of Melbourne there is a building façade made of 22000 zinc, glass and sandstone triangles. Small triangles are progressively arranged into larger geometrically similar triangles, using a famous pinwheel tiling. Figure 1a shows a section of the façade with line segments and shading used to highlight the way in which larger triangles are each composed of 5 smaller triangles. The photo has been inserted into a GeoGebra file. All of the small triangles are congruent and all the highlighted triangles are similar. For our local students this is a familiar landmark that many will have visited privately. For many classes of students, it will also evoke pleasant shared memories of looking at this feature as part of a school excursion to the city especially if they have completed the city ‘maths trail’ (Vincent, 2007). The mathematics is linked to the real world because the students have seen the façade. They will learn that the fascinating and aesthetically pleasing attributes of the patterns in the façade are linked to the mathematical properties of the shapes used.

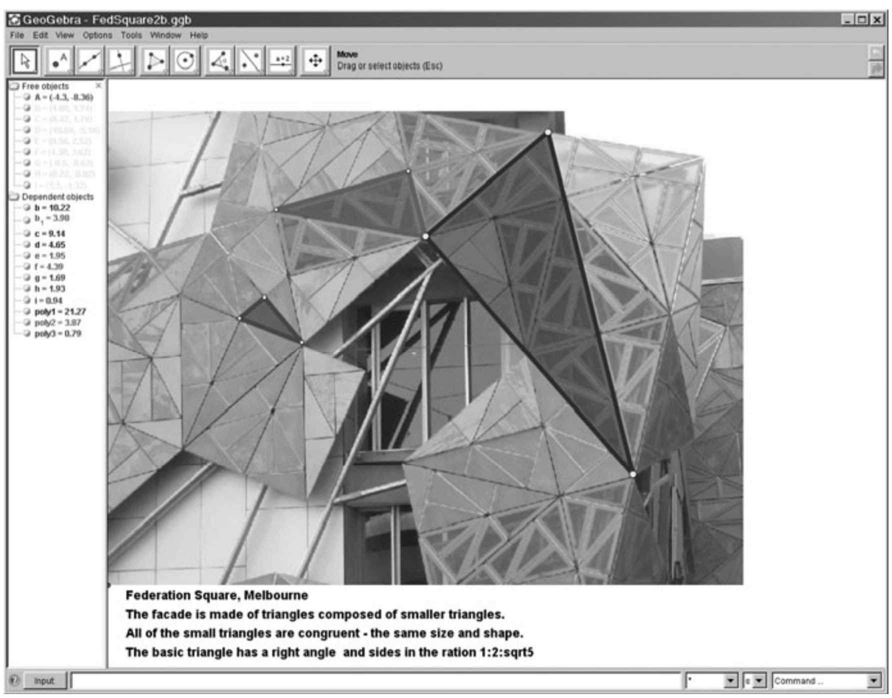

Figure 1a. The façade of federation square, Melbourne, uses a pinwheel tiling: three sizes of triangles are highlighted.

We have prepared a GeoGebra file based on Federation Square to be used when learning about Pythagoras' theorem and dealing with the exact arithmetic of square roots. The lesson takes advantage of the GeoGebra facilities to insert pictures; create points, lines and triangles in front of the image; measure the length of lines; and perform arithmetic calculations such as finding ratios of side lengths. Two sorts of constructions are used in the file: (i) triangles drawn on the photo highlight features as in Figure 1a and (ii) constructed right-angled triangles with side lengths that are multiples of 1, 2, √5 to represent the triangles of the theoretical tiling (see Figure 1b). Investigation of the mathematical properties of the triangles in the façade prompts discussion of properties of triangles, measurement, congruent triangles, similar triangles, tessellations and ratio. If students know that the triangles highlighted in Figure 1a are all similar and right-angled, then they can calculate that the side lengths are in the ratio 1:2:√5 (because the sides of a large triangle are made from 1 and 2 copies of the hypotenuse of the next smaller triangle). For students at this level, it is a challenge to calculate the exact lengths of the sides of the successively larger triangles and to check their work by matching with the measurements from the picture. Teachers who used this approach to working with irrational square roots found that the link with measurements gave a reality to the abstract symbols, and impressed upon these young students that these irrational square roots, written with a funny symbol, were indeed numbers (Stacey & Price, 2005).

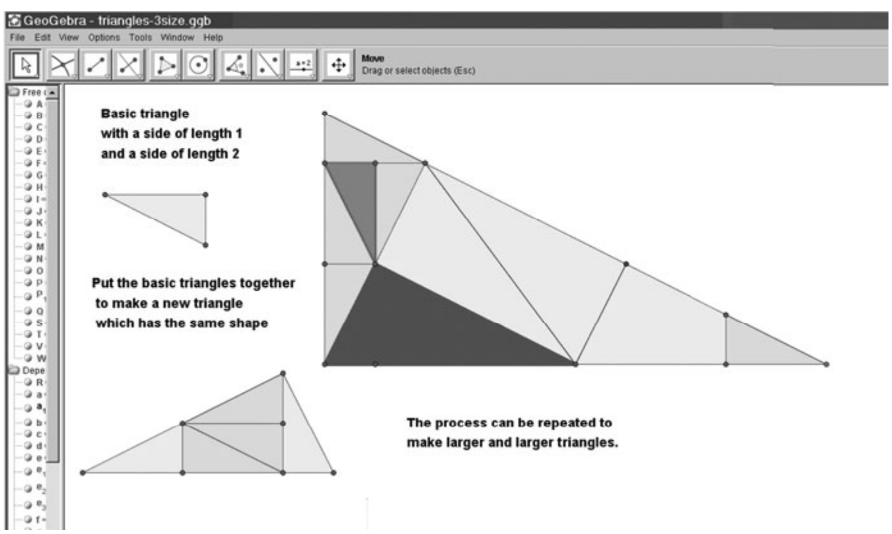

Figure 1b. Illustration of how to make the pinwheel tiling.

None of these images are dynamic. What, then, does the use of dynamic geometry software in this way offer for teaching? It is true, in this case, that the construction, investigation and mathematical operations could be carried out by students using photocopies, pen, paper, ruler, scissors and calculator. Indeed we have found that some manipulation of real paper with concrete measuring is beneficial. However, GeoGebra, used with a data projector, offers the opportunity to use the photo of the real façade and examine it within the orderly confines of whole class discussion. GeoGebra has, in this case, been used primarily as a presentation tool with built-in measuring and calculating.

Olivero and Robutti (2007) point out that measuring in the real world, the digital world and the theoretical world of mathematics have important differences. It is important that students appreciate that using the measuring tools of GeoGebra on the digital image (e.g. to find lengths and angles of triangles) is not exactly the same as measuring in the real world or in theoretical mathematics. The differences between real world measuring of Federation Square and measuring on the digital image are important. Although the triangles are congruent, measures for equivalent sides may slightly vary with the orientation of the triangle (a software phenomenon), the relative position of the camera to the real walls of Federation Square, and the systematic distortion the camera lens may give. Some of this variation requires discussion with the students. If students are to use their mathematics to solve real problems then it is vital that they learn to understand the practical constraints associated with measurement, whether it is using real world objects such as a ruler or on screen measurements such as through dynamic geometry.

The contrast between measuring (finding lengths and angles) in the theoretical mathematical world and in the digital world also provides an opportunity for learning. In ideal (theoretical) mathematics the ratios of length of the sides of all the triangles will be exactly $1:2:\sqrt{5}$. When the lengths are measured (in any way, but in this case using the dynamic geometry software) the ratios can only ever be approximations to this, with accuracy according to the number of decimal places set. Pierce and Stacey (Stacey, 2008) show that exploiting the contrast between ideal and machine mathematics is a common source of pedagogical opportunities provided by teaching with technology.

EXAMPLE 2: RESIZING DIGITAL IMAGES TO EXPLORE SIMILARITY

This example illustrates the use of dynamic geometry to support the development of concepts (in this case, similarity) in a real world context, and to promote exploration and guided discovery (in this case, of the numerical relationships between lengths of similar figures). The world of digital cameras and resizing images is familiar to our students. So too is the variation in aspect ratio (widescreen, letterbox, 16:9, 4:3 etc) which must be considered when showing DVDs on computers, televisions or large screens. We have used these real world experiences to introduce concept-building ideas of similarity using rectangles rather than the classic approach with triangles. For initial explorations, we draw on the pre-mathematical concept of “looks the same” as the fundamental meaning given to similarity. Digital images inserted into a dynamic geometry screen, such as those shown in Figures 2a and 2b, offer links between mathematics and the real world. The image in Figure 2a and 2b is an end-on view of a large circular water tank on the back of a truck. Students are asked to manipulate the image by dragging one corner, changing its size but keeping it ‘looking the same’ (in this case the tank still looks circular). The intention is that students will build up an intuitive sense of similarity and will be able to discover for themselves the relationship between the width and height of similar images.

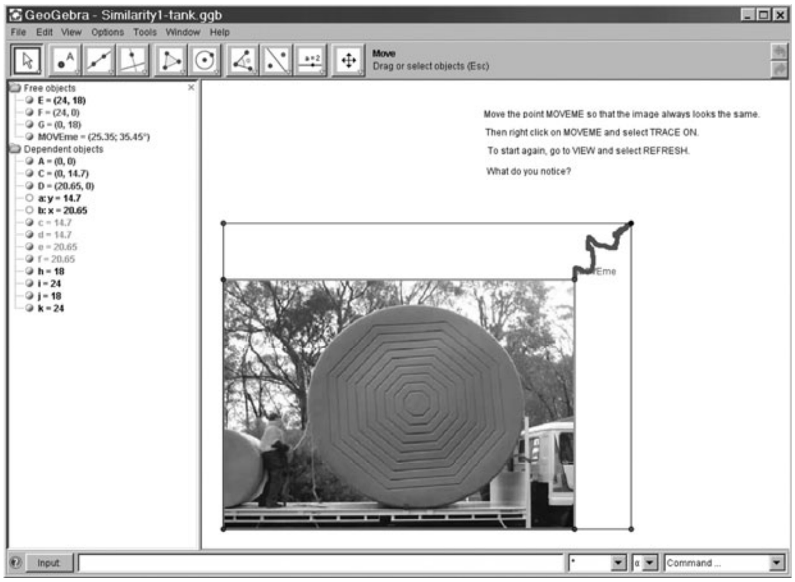

Figure 2a. Explore effects of resizing image-wandering dragging encouraged.

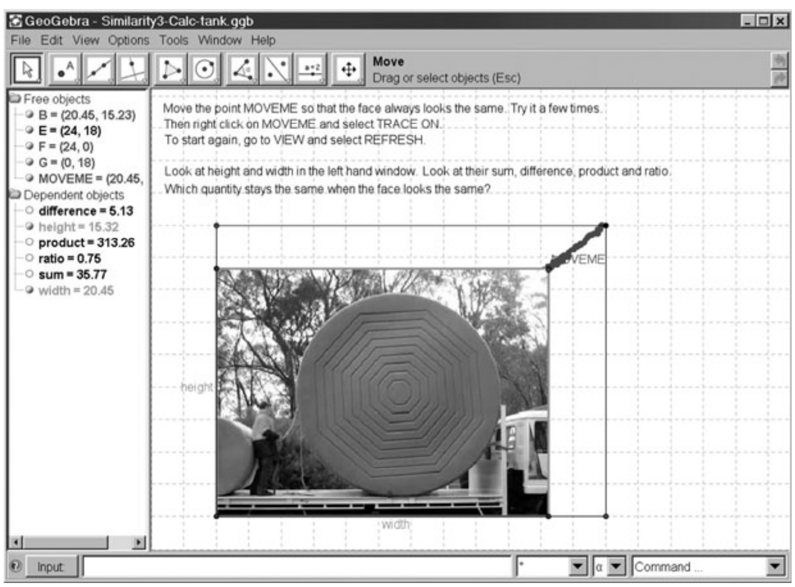

Figure 2b. Explore effects of resizing image-bounded and guided dragging.

Through various modalities of dragging (as discussed below) students watch the changing image and thereby explore the regularity and variation of the length and breadth and also the ratios, sums and differences of these measures. When the image is dragged so that it 'looks the same' the ratio of the length and breadth remains constant, but the actual measurements and other quantities calculated from them alters. These experiences are intended to build concepts about ratio, aspect ratio and similar figures. In this way, dynamic geometry is used to encourage students to develop a sense of discovery as active learners.

In this example, we have chosen the image of the base of a large water tank because the circular base and regular octagons assist students to identify similarity but more positive affective impact, and therefore a halo effect onto mathematics in general, may be gained by choosing other photos. These could be of local places, of the students' faces (used with sensitivity), or students' favourite photos.

In this example, different dragging modalities are used to develop and test theory and these are structured into the design of the dynamic geometry worksheets. Arzarello, Olivero, Paola, and Robutti (2002) describe the ascending and descending modalities of a lesson using dynamic geometry: ascending as the learning activity focus moves from perception of drawings to analysis through theory then descending when the focus moves from theory back to drawings. Descriptive names have been given to the various ways in which the user may use dragging to explore mathematical regularity and variation. Arzarello et al., (2002) name 'wandering dragging', 'bounded dragging', 'guided dragging', 'dummy locus dragging', 'line dragging', 'linked dragging' and 'dragging test'. A number of these dragging modalities are illustrated in the examples in this paper. It is important to note that from classroom research, Arzarello et al., note that the students' solution processes develop through a sequence of different modalities following an evolution from perceptive to theoretical and through ascending and descending modalities. Exploratory tasks require careful didactical design to support students in such a learning process.

In the worksheet illustrated by Figure 2a students are asked to drag the corner of the photo so that the image 'looks the same'. They are encouraged to use what Arzarello et al., (2002) call 'wandering dragging' and observe the effect on the shape of the image and on the lengths, ratios, sums and differences of the side lengths. From this experience they form conjectures about the numerical relationships that preserve aspect ratio. Initially, for example, students find that the images that look the same have the same ratio of length and breadth. Later, they also usually note that the one dragged corner stays on an (extended) diagonal of the image. Putting a trace on the dragged corner (as shown in Figures 2a and 2b) reveals this. This property can be linked then to the invariant ratios of length to breadth, and the equality of the factors that stretch or shrink the length and breadth.

The dynamic geometry task has a series of worksheets. Figure 2a is for initial conjecturing. Figure 2b, which follows, provides both the grid on top of the image and a diagonal line. This may either be used as a guide for students who have not developed a theory ('guided dragging') or to help students test the theory that they have developed. The grid also makes it easier to see other mathematical relationships: for example, when the aspect ratio is constant, the length and breadth have undergone the same dilation. Other examples of the use of dynamic geometry and other image software are given by Pierce et al. (2005).

EXAMPLE 3: EXPLORING VARIATION BY FITTING CURVES TO HIGHLIGHT REAL WORLD IMAGES

This example also illustrates the use of dynamic geometry to promote exploration and guided discovery. In this case, the function "graphing facility" is used. An important goal for the teaching of functions is that students understand how modifications to the symbolic rule transform the graph of the function. Teachers teaching about quadratic functions, for example, very frequently require students to create (using pen-and-paper or a function grapher) graphs for several related rules e.g.: $y = x^2$, $y = x^2 + 2$, $y = x^2 - 3$ and observe the regularities. This activity may be used to encourage discovery learning of the relationships between the graphs, or to practise the application of patterns that have been explained by the teacher. The task illustrated below is in this mode, but with much stronger real-world links. Students are challenged to fit a linear (Figures 3a, 3c) or quadratic (Figures 3b, 3d) curve to features on a digital image by finding the rule for the appropriate mathematical functions. Students typically begin with a guess-and-check approach but quickly become more systematic in their attempts, as they begin to appreciate how to use their knowledge of the effect on the graph of modifications to the symbolic rule.

The choice and positioning of the digital image allows the teacher to vary the difficulty of the task. For example the scissor lift shown in Figures 3a and 3c is deliberately placed symmetrically about the vertical axis. This positioning emphasizes vertical transformations and reflections. The multiple parallel lines of the scissor lift provide repeated opportunities to use (or re-discover) the idea that graphs of $y = mx + c$ with the same value of m will be parallel, and that different values of c move the graph up and down. Another excellent teaching feature is the fact that there are lines with the same gradient rising to the left and the right, so that if $y = mx + c$ traces one support, $y = -mx + c$ traces another. The task illustrated in Figure 3c, has been made more difficult by requiring outlining only the support struts for the scissor lift. This requires the student to restrict the domain of each straight line. This reinforces students' knowledge of how the domain of a function is determined by the x-values, and not the y-values.

Our classroom experiences show that students enjoy the challenge of finding a line that fits accurately over an image. Using these real world images underlines the fact that mathematical situations are all around us, in our daily lives. Again, students' engagement may be increased by making use of digital images which the students have taken themselves. Once again, the halo effect can be harnessed. Teachers in our project felt, for example, that their students would enjoy fitting curves to the famous quadratic McDonald's' arches, because they associate this image with pleasant social occasions, or using pictures of 'exotic' locations that they dream of visiting one day. Care needs to be taken in selecting photos so that the geometric feature of interest (e.g., the shape of an arch) is not masked by inappropriate camera position. The photo of water from a hose (see Figure 3b) links with physics learning that the motion of a projectile is a parabola. Again, some subtleties lurk in these activities. It is important, for example, to note that the functions are only being fitted to an image and not to the real thing. The real world is un-coordinatised.

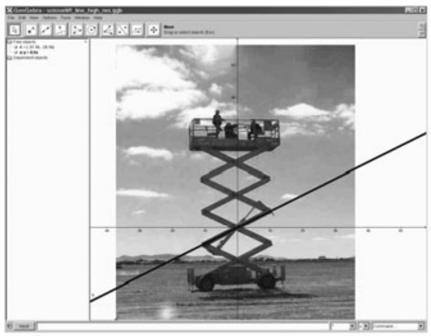

Figure 3a. (left) Linear function 'fitted' to support strut of scissor lift at air show.
Figure 3b (right) Parabolic water spray from a garden hose.

GeoGebra's facility to insert digital images combined with its function graphing capabilities support activities that prompt students to

- See that curves which may be described by function rules are observed in the real world;
- Explore the effects of varying parameters in function rules;
- Practise finding function rules to describe a variety of straight lines or parabolas;
- Appreciate the value of the different 'standard forms' of the function rules: especially the $y = a(x - g)^2 + h$, 'completing the square' form for a quadratic;
- Discuss how well these function rules 'model' the situation, including whether there is a physical reason for this model (as is the case for the water spray shown in Figure 3b);
- Engage in friendly competition with their classmates to find the 'best fit'.

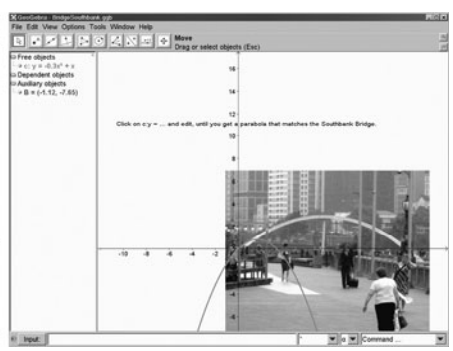

Figure 3c. (left) Scissor lift outlined using linear functions with restricted domains.
Figure 3d. (right) Quadratic to 'fit' curve of bridge over river.

EXAMPLE 4: THE BIGGEST VOLUME FOR AN OPEN BOX

The fourth example requires pre-calculus students to explore an optimisation problem: *what is the maximum volume of an open box that can be made from a standard-sized sheet of paper*? In this case, dynamic geometry is used for simulation and data capture. The diagram in Figure 4b shows how the open box is made by cutting squares from the corners of the paper, then folding it up. Changing the size of the squares removed changes the volume of the box. Our teachers have reported that the impact of this task is greatest, and students remember the possibility of optimization when they learn calculus some years later, if each student makes their own box first. Each student is allocated a size of corner to remove, draws the appropriate net, constructs the box, calculates its volume and plots a point on a single scattergram of class results. Since this is probably the first experience that these students have with optimization, some students think all the boxes will have the same volume because they are made from the same sized piece of paper; others think that the volume increases or decreases steadily, and a few predict there will be a maximum or minimum. This practical and numerical work, culminating in the hypothesis making, precedes the experiences with the dynamic geometry and also precedes the algebraic formulation. Greatest affective impact is gained by using brightly coloured paper, looking at the boxes in order of size of square removed (a teacher can call students to show their boxes in this order), and finally stacking the boxes as in Figure 3a, thereby exhibiting their decreasing bases and increasing height. The difficulty of judging which box has the greatest volume must be resolved using mathematics because the stacking does not help with that but tends to hides it. Like Arcavi and Hadas (2000), we found that the initial visualisation of the problem and personal conjecture led to engagement with experimentation then surprise as they used the dynamic geometry to collect values and graph the results for the volume of the series of boxes. This feedback prompted the need for ‘a proof’ or at least discussion of ‘why is it so?’

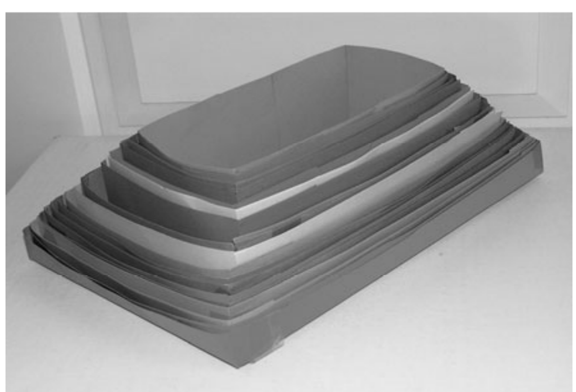

Figure 4a. Coloured open boxes from one class each constructed from A4 paper.

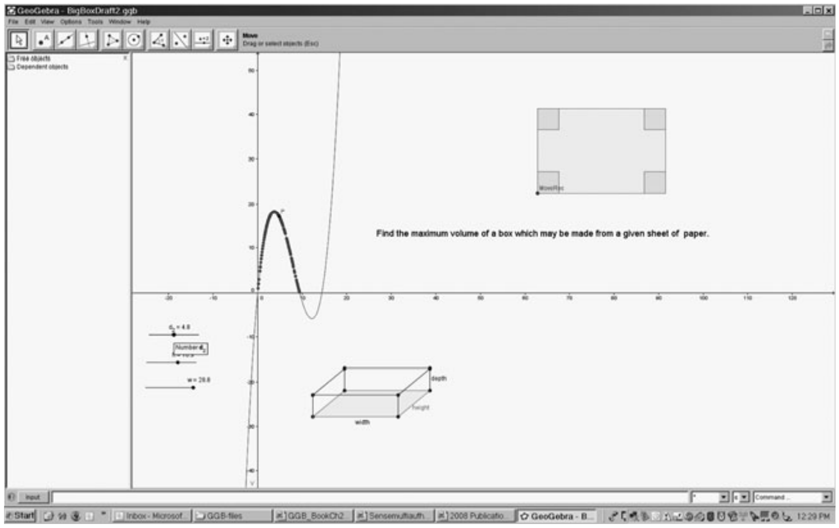

Figure 4b. Dynamic file of open box, its net and with the graph of volume displayed.

When the lesson turns to the pre-prepared dynamic geometry file, students can change the size of squares removed with a slider and see how the shape of the box changes. (In a later lesson, they can also use sliders to change the size of the paper.) They can plot the volume of the box against the length of the square removed by tracing the plotted point. Because dynamic geometry is now linked with data capture, data graphing and list construction, students can:

– Examine and explore a virtual model of the paper cutting, fully mathematised, which corresponds to their paper model yet with a readily adjustable size;
– Obtain first hand virtual experience of how the volume changes when the height of the box is altered (with the kinaesthetic of dragging);
– Capture data to create a table of values of the height and volume (this step highlights empirical aspects of mathematical discovery);
– Display the data graphically as isolated points and as a curve; and
– Test whether the formula for volume, which is later derived, matches the data.

Classic writing on multiple representations for algebra (see, for example, Kaput, 1992) has considered three representations (e.g. the numerical, symbolic and graphical representations of a quadratic function), but with dynamic geometry used to simulate real world problems as in this example, we move between five modes of representation:

– Real world situation of paper boxes;
– Dynamic geometry simulation with corners of variable size removed;
– Numerical representation – tables of values;

– Graphical representations – plotted points from the table, dragged point from simulation or graph of symbolic function);
– Symbolic representation – formula linking volume to size of corner removed.

CLASSROOM PRACTICES

The examples above illustrate the range of successful uses of dynamic geometry that emerged from a design research project. As noted above, teachers could select from many technologies in this project, and dynamic geometry grew in popularity as the project proceeded. The focus of the project was on creating lesson designs that were practical for teachers to implement and which they felt resulted in good learning and which helped them to bring the real world into the classroom. Evidence of success has come from lesson observation and from teacher reports rather than detailed analysis of how students have learned from these uses of dynamic geometry. We did see clear evidence of improvement in affect from using these real world situations within dynamic geometry. We also saw clear evidence that lessons that use technology and aim at engaging students in higher order thinking are susceptible to the same causes of 'decline' to lower order thinking (Henningsen & Stein, 1997) that are observed in lessons without technology or a real world emphasis. However, we were not able to collect evidence for cognitive effects and it is therefore the case that many important research questions remain, including:

– Does understanding the differences between measurements in real situations, dynamic geometry and theoretic mathematics enhance most students' learning or simply complicate it?
– How is the learning with multiple representations best handled to maximise growth in understanding?
– How do students make use of abstractions or cognitive models constructed from virtual activity when they are working in the real world situation or in the mathematical world?

In our project, we were able to address such questions in specific cases when working with graphics calculators (see, for example, Bardini, Pierce & Stacey, 2004; Pierce, 2005), but we have not done so for dynamic geometry.

Our experiences in assisting teachers to adopt the use of dynamic geometry have led us to recommend using pre-prepared files (as above) in almost all instances. When we began our work, we expected that students would make their own files for most lessons. However, lessons that began by making files for subsequent exploration were often unsatisfactory and we now believe that the use of pre-prepared files is usually preferable. These files can be better constructed, more attractive and more robust. Students do not spend precious mathematics learning time on mastering application-specific skills that they are not using regularly (Of course, the argument is different if dynamic geometry will be used regularly). It is much easier to teach students the skills for using a prepared file, such as using a slider or tracing a point being dragged. Moreover, our lesson observations showed that it is often more effective to use the dynamic geometry file with a data projector as a class demonstration and discussion starter, than to have students working on individual computers. It is excellent for students to actively engage in exploring a situation, but without adequate scaffolding and sustained pressure from the teacher for seeking explanation and meaning, free student exploration can often decline into unsystematic behaviour devoid of mathematical content (Henningsen & Stein, 1997).

Sinclair (2003) found value from using carefully prepared dynamic geometry files that encourage discovery without confusion and which can be delivered within a limited time frame. Such files should also be accompanied with carefully designed questions that help students "learn how to use change to explore and how to extend their visual interpretation skills" (p. 312).

During our experiences in the RITEMATHS project, it also became clear that preparing well-constructed dynamic geometry files is not very practical for teachers, at least at this early stage of skill development for the teachers with whom we worked. Creating a robust file (i.e. one which does not 'break' when students perform unexpected actions) for anything other than the simplest of situations requires expertise, and for the foreseeable future will only be undertaken by enthusiasts. This points to the need for excellent mechanisms for sharing good files and associated teaching materials, first amongst the teachers of one school, then across districts and also around the world. This is an important role for internet GeoGebra sites, but sharing resources across platforms and programs is also desirable.

CONCLUSION

The examples above have illustrated four different types of uses of GeoGebra to bring the real world into the classroom. In Example 1 (Federation Square), the dynamic geometry was especially used to enhance the presentation of the material. Students could have used the traditional tools of the pen-and-paper classroom, but the computer added positive associations, colour and clarity. In our research project, we particularly noted the very strong emphasis that teachers placed on these uses of technology, simply to brighten the classroom. At first we were inclined to dismiss such uses as superficial, but we now understand the teachers' strong need to make mathematics lessons more attractive to students. Our analysis of this phenomenon led us to the psychological literature on the halo effect (Pierce & Stacey, 2006). We now understand that an important motivation for teachers to use real world problems, which was previously neglected in the literature (see for example, Blum and Niss, 1989), is to associate mathematics with pleasurable parts of students' present lives. This is different to the way in which previous writers valued the choice of contexts which had substantial mathematical content or important subject matter that was intrinsically interesting to students or essential for their use in the future. Teachers hope (maybe not consciously) that using pleasant contexts, images and objects in mathematics lessons will, by association, make students feel more positively disposed towards learning mathematics. These presentation aspects of dynamic geometry and other software may (quite literally) be superficial, but it does not mean they are not important.

Example 2 (using the water tank image) inherently required technology to manipulate images. The purpose of the manipulation is to develop a concept and to explore the associated regularity and variation. Arzarello's research group had identified the roles of different types of dragging in conducting investigations in their work with Italian schools, and it is interesting that several of these had been independently created for use in these lessons. The concept of aspect ratio has newly entered public consciousness, mainly through visual digital technologies, and it is good to be able to use these same technologies to teach the underlying mathematics.

Example 3 (fitting functions to images of real world objects) provides an illustration of the use of GeoGebra to encourage students to systematically explore links between symbolic function rules and transformations of their graphs. Here the image is static and used only to provide a stimulus and for engagement, but GeoGebra is used for its function graphing capability.

Example 4 (box of maximum volume) shows the expansion of representations in algebra that is supported by dynamic geometry, beyond the classic symbolic-numerical-graphical to include simulations and the associated possibilities for data capture. In this way, dynamic geometry is not just mathematically able software, but also acts to some extent as a real world interface.

In conclusion, GeoGebra has much to offer for teaching mathematics through real world problems. Dynamic geometry is far more than geometry.

ACKNOWLEDGEMENTS

The tasks described in this paper were developed as part of the RITEMATHS project (HREF1) The researchers thank the Australian Research Council, our six partner schools, Texas Instruments for their financial support of this project and most especially the teachers and students who took part in this study.

An early version of this article was published as Pierce, R. & Stacey, K. (2008) Dynamic geometry enriches the design of curriculum materials for middle secondary school mathematics. In J. Vincent, J. Dowsey, & R. Pierce *ConnectEd Maths. Proceedings of the annual conference of the Mathematical Association of Victoria 2008,* 292–303. Melbourne: MAV. The authors also wish to thank the reviewers for their insightful comments and for drawing our attention to other relevant and interesting literature.

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4. GEOGEBRA: FROM SIMULATION TO FORMALIZATION IN TEACHER PREPARATION AND INSERVICE PROGRAMS

Upper division mathematics courses often leave students with the sense that mathematics is essentially an abstract subject whose development is founded on formalized reasoning within formalized contexts. But mathematics is more than this. A dynamic geometry environment like GeoGebra offers professors in proof-oriented geometry courses an alternative to immediate formalization. In this chapter, we describe two classroom situations, one online and one face-to-face, in which students developed and experimented with dynamic geometry models, leading to the formal construction, refinement, and validation of mathematical concepts and theories at the heart of the formal content goals of the courses. In both case studies, students encounter mathematics as a dialectical blending of empirical investigations and formalizations.

INTRODUCTION

University instruction in mathematics at the upper division or graduate levels is usually theory-focused and oriented toward formal proof. Cooney and Wiegel (2003) describe this as a "formalistic and structured approach to teaching", which leaves students with the view that mathematics is essentially an abstract subject whose development is founded on formalized reasoning within formalized contexts. Cooney and Wiegel argue that mathematics is more than this and propose that mathematics, particularly in teacher preparation programs, be developed through instruction as a "pluralistic" subject in contrast to the immediate formalization typical in traditional instruction. Specifically, they propose a "broader view" of mathematics in which its development as a formalized system of logical consequences is mediated by its development as an intuitive and empirical system that is authentic and valuable in its own right. In this respect, doing mathematics is seen as a dialectical blending of empirical investigations and formalizations leading to the construction, refinement, and validation of mathematical concepts and theories.

The challenge in upper division and graduate mathematics courses is to devise and implement the intuitive and empirical investigations that promote the development of broader views of mathematics while supporting the canon of formal mathematics outlined in course descriptions. Technology-rich environments can provide a means for achieving this pluralistic approach (Cooney & Wiegel, 2003). In this chapter, we will illustrate how a dynamic geometry environment like GeoGebra can offer professors in

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 57–72.

proof-oriented geometry courses an alternative to immediate formalization and describe how it can engage their students in empirical investigations, centered on dynamic geometry models, that directly support informal and formal content goals of the course.

THE PROBLEM SETTING

The study of transformational geometry in college geometry and in graduate courses typically involves students in proving many theorems about affined transformations of the Euclidean plane including similarity transformations and isometries. The curriculum includes several major theorems such as the "Classification Theorem for Isometries" and the "Three Reflections Theorem." The Classification Theorem asserts that every distance-preserving transformation (isometry) of the Euclidean plane is a translation, a rotation, a reflection through a line or a glide reflection. The Three Reflections Theorem refines this by asserting that all isometries are either reflections through a line, or the composition of two reflections through lines, or the composition of three reflections through lines. A theorem that follows from the Classification Theorem in a fairly straightforward manner is the Midpoint Connection Theorem (See Usiskin et al., 2003, Theorem 7.44). It asserts that if *A* and *B* are two congruent figures in the plane with corresponding points oppositely oriented on the figures, then the midpoints of all segments connecting corresponding points on *A* and *B* are collinear (See Figure 1).

By generalizing and embedding it in a real world context that can be modeled by GeoGebra, we turn the Midpoint Connection Theorem into a question that can be empirically investigated. We call it "Flippin Squares." Its investigation leads students into the theory of transformations that is at the foundation of the Midpoint Connection Theorem. Such investigations can be designed to shed light on many other areas in the formal study of geometry in teacher preparation programs.

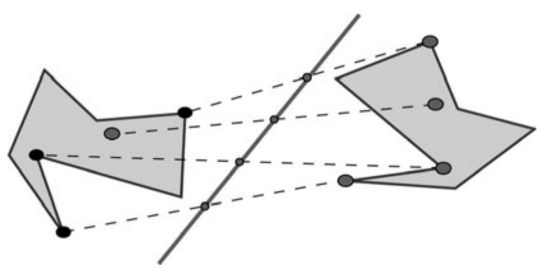

Figure 1. Two congruent figures with oppositely oriented points.

Flippin Squares

Suppose you are given a plastic or cardboard square ■*A'B'C'D'* that is an exact copy of another plastic or cardboard square ■*ABCD*, with corresponding vertices appropriately labeled. Further suppose that the squares are flipped into the air and land as shown in Figure 2. In this case the midpoints of the paths between corresponding vertices fall on a straight line! Find the probability of this alignment randomly happening when the squares are flipped.

With GeoGebra, students can create simulations of the Flippin Squares task and many other such tasks in the transformational geometry domain. Such simulations can be viewed as empirical models that can be manipulated by students to investigate informally the fundamental notions of transformational geometry and discover patterns before being introduced to the major theorems or even the definitions and properties of various transformations. These simulations help students to see the generality of patterns in a phenomenon and provide compelling evidence that the patterns will continue to hold in the infinite space of unexamined cases. They help students to refine conjectures by ruling out cases and even suggest strategic paths to proving conjectures by revealing invariant relationships in the phenomenon. The Flippin Squares task is part of a larger problem space in which students move back and forth between model-centered investigations and synthetic/analytic formalizations thereby contributing to the pluralistic view of mathematics proposed by Cooney and Wiegel (2003).

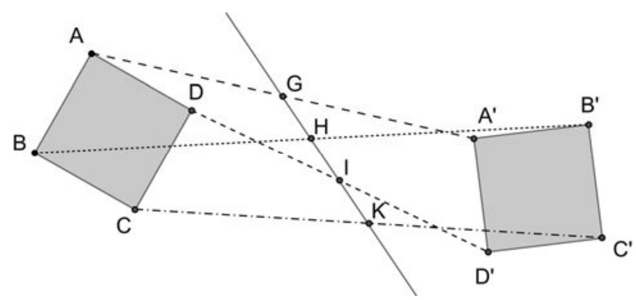

Figure 2. Two congruent squares flipped, with midpoints of preimage/image segments marked.

Classroom Settings

In spring 2008, the Flippin Squares problem was given to 12 secondary teachers in an online geometry class that is part of a mathematics education masters program at Montana State University. The teachers had been using GeoGebra in the study of various challenging problems and were familiar with most of its basic tools. Several of the teachers had experience teaching high school geometry and formal proof. Without forewarning that this problem had anything to do with transformational geometry, the teachers were asked to work in groups and come up with solutions and plausible explanations.

In spring 2009, the problem was given to 27 students in an undergraduate college geometry course at Colorado State University. The course serves pre-service secondary mathematics teachers as well as other mathematics majors. The course has a prescribed and crowded syllabus. It follows a textbook that covers Euclidean geometry from synthetic, analytic, and transformational perspectives, as well as topics in projective, spherical and hyperbolic geometries. Almost by necessity, it was more structured and teacher-directed than the graduate course in which the professor had more freedom to vary topics and coverage. The students in this course were familiar with formal proof and some had prior experience using geometry software.

At the start of the graduate class, teachers were introduced to the tutorials available from the GeoGebra Wiki and quickly became familiar with its main functions. In the undergraduate class, students were introduced to GeoGebra during the first week in the context of reviewing basic ruler and compass constructions. For homework, they were asked to do the same constructions using Geogebra and to become familiar with the software. Without any direct instruction they were able to do all of the Euclidean constructions and quickly became proficient with the menu structure and syntax of GeoGebra.

Preparation for the Problem

Unlike the graduate course, the undergraduates were introduced to transformations before launching into the Flippin Squares task. Before they formally talked about isometries (i.e. transformations of the plane that preserve distance) students explored them by transforming a right triangle using the translation, rotation, and reflection tools in GeoGebra. Without any direct instruction in this "Isometries Activity," students discovered the main defining characteristics of these transformations and, in a class discussion facilitated with GeoGebra, came up with precise formulations of the ideas of "glide reflection" and "isometry."

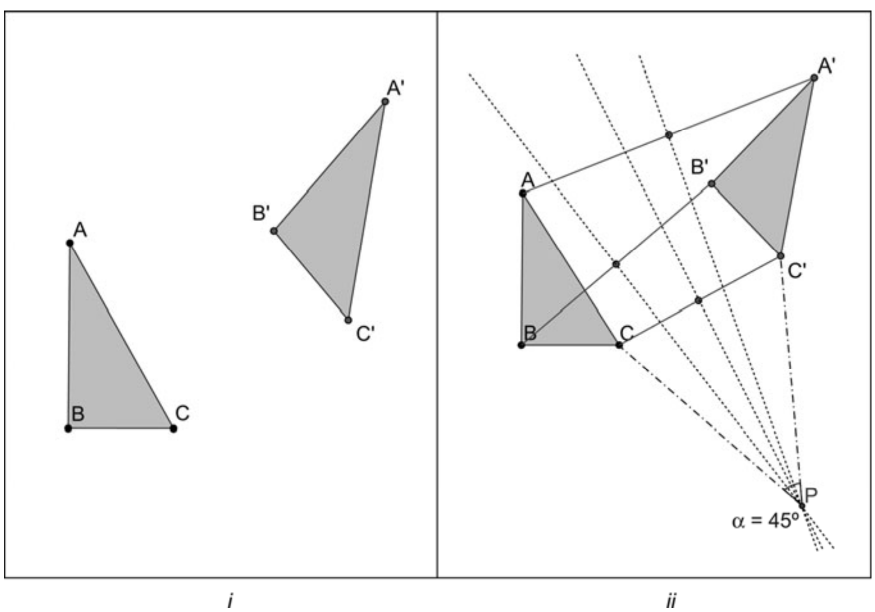

Figure 3. Given an image and preimage of a triangle (i), determine transformation (ii).

Through guided instruction, students were asked to "undo" various transformations by finding the vector/length, center/angle, and lines of reflection for the various transformations. This exercise in reverse engineering was greatly facilitated by the compelling precision and speed of GeoGebra. Figure 3 shows a transformation students believed was a rotation and then validated using GeoGebra. Here, students identified the perpendicular bisectors of preimage-image segments and noticed that they all were concurrent at a point P. Then, using GeoGebra's measuring options, they determined that the angles $\angle APA'$, $\angle BPB'$, and $\angle CPC'$ were all congruent and thus yielded the angle of rotation around P. They verified their work by having GeoGebra rotate ΔABC about P using this angle of rotation.

The following quotations are taken from student journals as they reflected on the Isometries Activity. Significantly, they point to the value of the GeoGebra environment in allowing students to create models and simulate the mathematical components at the heart of an eventual formalization of the Flippin Squares problem.

– From using GeoGebra, I could tell easily what sort of transformations had happened.
– It was nice to learn these things before being told about them. There was an element of discovery.
– GeoGebra really allowed me to see and understand the transformations visually. I don't think if I had been reading this information out of the book that I would have been as successful.

The Flippin Squares Activity - Hands On Exploration

In the graduate class, teachers constructed cardboard squares and labeled them. They flipped the squares and constructed midpoints of the preimage/image segments. In the undergraduate class this was done as a whole-class activity with two colored algebra tiles. In the online graduate class, at least one person in each group discovered that midpoints only aligned when the squares landed in reverse orientation from each other. In other words, the midpoints aligned only when the clockwise ordering of the vertices of the two squares was *A-B-C-D* and *A'-D'-C'-B'*, or *A-D-C-B* and *A'-B'-C'-D'*. The online students worked together asynchronously in the WebCT virtual environment and some spent much time flipping the squares without noticing this pattern. In contrast, once one student in the face-to-face undergraduate class made this observation, all students quickly checked it and, as a class, concluded it appeared to be true.

Both classes concluded that flipping two squares was equivalent to placing ■*ABCD* on the floor and flipping ■*A'B'C'D'* into the air. The relative positions of the two squares after the flip were important and not their actual positions on the floor. Both classes concluded the probability of alignment was 50% since there was a 50% chance that the flipped square would have a different orientation than the fixed one. "It is just like flipping a coin," some observed. However, some skepticism lingered. First, empirical estimates of the probability ranged from 40%–60%. Second, by-hand constructions were imprecise and some teachers in the online class maintained that they had encountered examples where orientations of the squares were reversed but the midpoints did not exactly line up. Finally, the rather lengthy process of verifying even one example led both classes to conclude that simulating the process in GeoGebra would lead to a quicker and more reliable exploration of the range of possibilities.

The Flippin Squares Activity – Simulation using GeoGebra

In the undergraduate class, the instructor guided the students with the following directions for simulating the process of flipping a square and finding midpoints of preimage/image segments. The class discussed the importance of the randomness of the initial point E and the point selected on the circle (Steps 1c and d) and noted the irrelevance of the size of the squares. All of the undergraduates successfully completed the simulation.

1. Use Geogebra to simulate flipping a square in the coordinate plane.

   a. Construct a 2x2 square, ■*ABCD*, centered at the origin.
   b. Pick an arbitrary point, *E*, in the plane.
   c. Construct a circle at *E*, with radius *AB*.
   d. Pick any point *F* on the circle and construct segment *EF*, a segment with length equal to *AB*.
   e. Complete the construction of a square with *EF* as one of its sides. Any GeoGebra tools can be used here, including the regular polygon tool.
   f. Rename the vertices to yield a square, ■*A'B'C'D',* with the same orientation as ■*ABCD* and save. This is Case I.
   g. Reverse the orientation of the labels and save. This is Case II.

2. Complete the following for Case I.

   a. Construct the preimage/image segments and their midpoints.
   b. What do you observe when you move ■*A'B'C'D'* in various random ways?
   c. Verify (b) by using the measurement tool and geometric reasoning.
   d. Determine the type of transformations that yield Case I.

3. Repeat steps 2a-2e only using the construction in Case II.

Figure 4 illustrates the two cases described in the simulation. In Case I, no matter how the students slid or rotated ■*A'B'C'D'*, the midpoints of the preimage/image segments formed a square and in one extreme case, a single point. The extreme case corresponded to a 180° rotation of the plane about this point. In Case II, no matter how the students slid or rotated ■*A'B'C'D'*, the midpoints of the preimage/image segments aligned. At this point most students were satisfied that the probability of alignment had to be 50%.

A much different sequence of insights occurred in the graduate course where the teachers were not told how to simulate the flipping process. A few teachers thought the simulation would involve finding a way for GeoGebra to depict a square flipping over in space and landing randomly on one of its sides. This was quickly dispelled by others in their groups who concluded that they only needed to simulate the final positions of the two squares resting on the floor. Some students did not think about using two separate cases in their simulations corresponding to the two possibilities of the squares' orientations relative to each other. They floundered on the issue of how to label the vertices in their single-case simulation. Once again, group discussions resolved this issue. Nearly all of the teachers discovered that the midpoints coincided at some point *C* when the squares formed a 180° rotation of each other about *C*. This led several teachers to discuss the probability of flipping two squares and getting an *exact* 180° rotation. They concluded this probability was about zero.

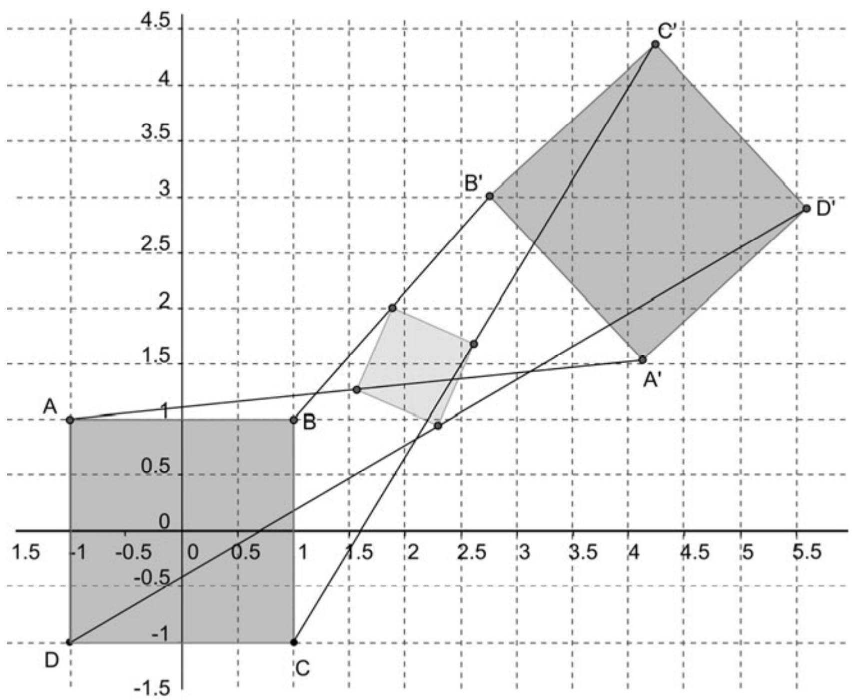

Figure 4a. Midpoints when there is no change in orientation (Case I).

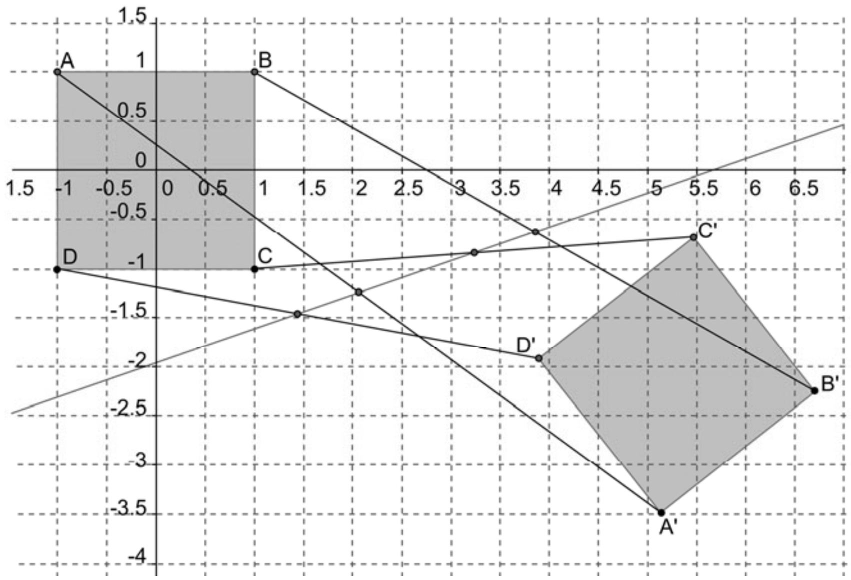

Figure 4b. Midpoints when there is a change in orientation (Case II).

Even after splitting the simulation into two cases, some teachers used the regular polygon tool to place two squares that *appeared* to be the same size on the GeoGebra drawing pad and then studied the midpoints of preimage/image segments. This led to conflicting results. When the squares had the same orientation, the midpoints appeared to form a square as in Figure 4a. However, in the case of reverse orientations, the midpoints did not exactly align (Figure 5).

Sharing GeoGebra files online is easy, so teachers frequently compared their work. The teachers who worked together to construct congruent squares that could be manipulated independently by dragging key points consistently achieved results like those in Figures 4. Several recognized the sufficiency of leaving one square fixed while moving the other in the simulation. Those who had used squares that were not congruent (Figure 5) concluded the task's stipulation that the squares be congruent was an important condition. Interestingly, the teachers who did not use congruent squares had stumbled upon an extension of the theorem underlying Flippin Squares, not found in typical textbooks: When a square is transformed by a dilation (size change) composed with an isometry, the midpoints of the preimage/image segments form a square if the orientations are the same and form a parallelogram (possibly a square) if the orientations are reversed. Because these teachers thought they had made a mistake in their simulation they did not pursue this line of reasoning.

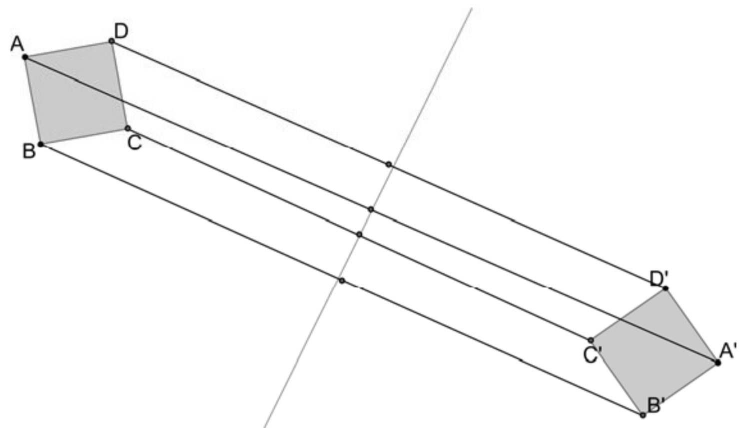

Figure 5. Case of reverse orientation where midpoints appear not to align.

Verification by Synthetic Geometry Proof Techniques

With cases like that in Figure 5 where the conjecture appears not to work, students in both classes were willing to admit that, while the simulation was extremely convincing, they were not absolutely certain that their conjecture about midpoints was true. Furthermore, even though they believed the conjecture was true, they wanted to know why it worked, a reaction often arising in dynamic geometry environments (Hölzl, 2001). Both classes were challenged to further validate their conclusions using geometric arguments.

A few graduate students remembered the Classification Theorem that every isometry of the plane is one of four types: a reflection, a glide reflection, a translation or a rotation. Since this theorem is found in some high school textbooks, it is possible they had encountered it in their own teaching. Their strategy might also have been prompted by their use of the GeoGebra isometry tools when trying to simulate the flipping and manipulations of squares on the Drawing Pad. Nonetheless, they tried to build arguments based upon what would happen to the midpoints in each of the four cases. Alternatively, after the undergraduates had studied the Classification Theorem with the help of GeoGebra, they were given the four cases and asked to prove that the midpoints align for reflections and glide reflections, but do not align for translations and rotations (except a 180° rotation).

All students, graduate and undergraduate, were able to prove that the midpoints aligned when the transformation was a reflection or a glide reflection. Some students needed a hint to focus on a single arbitrary point. The following proofs capture the typical arguments.

- **If *M* is a reflection of the plane through a line *L*, then the midpoints of all preimage/image segments are on the line of reflection.** Figure 6 shows an arbitrary point *P*, not on *L*, and its image *P'* under a reflection through line *L*. By the definition of reflection, *PP'* is perpendicular to *L* at some point *Q* and *PQ* = *P'Q*. Hence *Q* is the midpoint of *PP'*. If *P* is on *L*, then *P* is its own image and hence it is also the midpoint of its preimage/image segment if trivial segments are allowed.

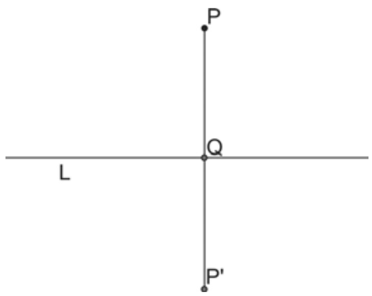

Figure 6. Reflection of P through line L.

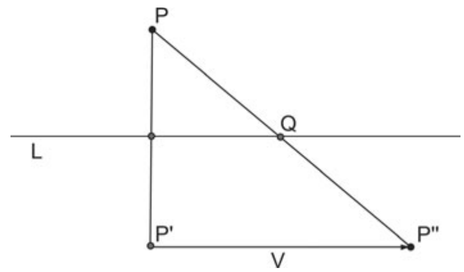

Figure 7. Glide Reflection moving P to P''.

- **If *M* is a reflection through a line *L* followed by a translation by vector $\vec{V}$ parallel to *L*, i.e., *M* is a glide reflection, then the midpoints of the preimage-image segments are on the line of reflection.** Figure 7 shows an arbitrary point *P* not on *L* and its image *P"* under the glide reflection *M*. Since *L* bisects segment *PP'* and is parallel to $\vec{V}$, then by the midpoint theorem for triangles, *L* bisects *PP''*. So, the midpoint *Q* of *PP''* is on *L*. If *P* is on *L* then *P* = *P*' and *P"* is on *L*. Hence, *P*, *P"*, and the midpoint *Q* of *PP''* are on *L*.

In the case of translations, various proofs were offered. They essentially involved the notion that every point in the plane gets moved a fixed distance *d*. Thus, the midpoint Q of an arbitrary preimage/image segment *PP'* is the image of *P* under a translation of distance ½*d* in the same direction. The following argument embodies this kind of reasoning.

- Let *M* be a translation by a vector $\vec{V}$. If *Q* is the midpoint of an arbitrary preimage/image segment *PP'*, then a translation by $-\frac{1}{2}\vec{V}$ maps *Q* back to preimage *P*. So, if the midpoints of a square's preimage/image segments align under a translation M, then the preimage points of these segments also align, since the translation by $-\frac{1}{2}\vec{V}$ preserves linearity. But this would contradict the fact that the preimage points form a square. Therefore, the midpoints of a square's preimage/image segments cannot align.

In the case of rotations, both graduate students and undergraduate students struggled. Some linked the rotation tool in GeoGebra to a slider and demonstrated that the midpoints always formed a square as the slider moved between -180° and

180°. With instructor guidance, students were led to proofs for the case of rotations. One possibility is given below.

- **LEMMA**. Suppose M is a rotation of the plane about point C by an angle $0< \theta<180$ (Figure 8). Suppose P and Q are the midpoints of preimage/image segments BB' and DD', respectively, with F and G the midpoints of the circular arcs $\widehat{BB'}$ and $\widehat{DD'}$. Note that F is on $\overrightarrow{OP}$ and G is on $\overrightarrow{OQ}$ It is easily shown that $\Delta B'PF \sim \Delta D'QG$ and $\Delta CPB' \sim \Delta CQG'$. Using properties of similarity this means that $PF/PC = QG/QC$. (It should be noted that a simple argument using trigonometry also leads to the following conclusion.) ***Thus, for any rotation by an angle θ about a center C, where 0 < θ < 180, there is a constant K such that if P is the midpoint of a preimage-image segment and F the midpoint of the corresponding preimage-image arc, then PF/PC = K.***

- Now, **suppose** P, Q, and R are the midpoints of preimage-image segments $\overline{BB'}$, $\overline{DD'}$, and $\overline{AA'}$, respectively, under a rotation by θ about point C, with $0° < \theta < 180°$, and suppose P, Q, and R align. We will show that A, B, and D must also align. Let G be the midpoint of the preimage-image arc $\widehat{DD'}$. Let L be the line through G parallel to $\overleftrightarrow{OR}$ and intersecting lines $\overleftrightarrow{CP}$ and $\overleftrightarrow{CR}$ in F and E respectively (Figure 9).

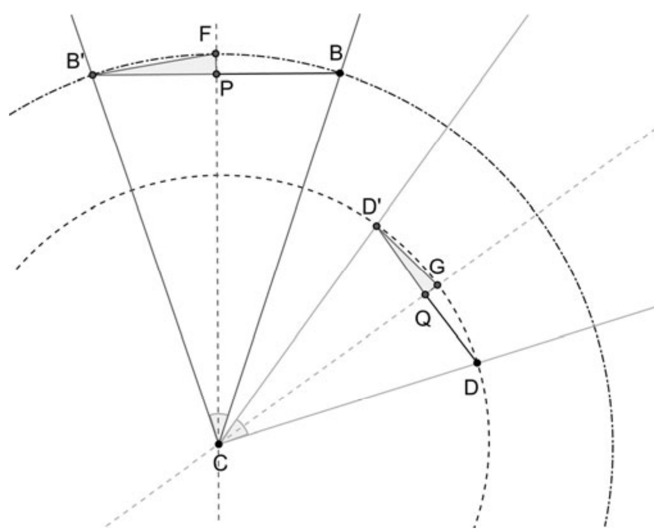

Figure 8. Comparing two preimage/image segments under a rotation about point C.

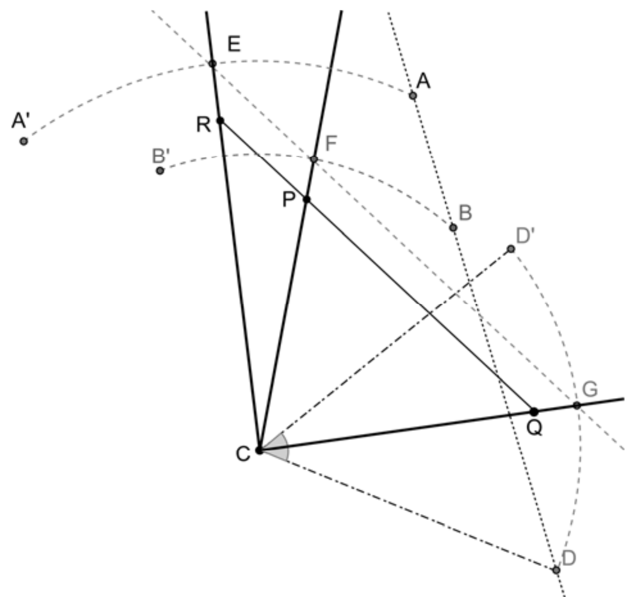

Figure 9. P, Q, and R are collinear midpoints of preimage/image segments.

By the parallel projection theorem for triangles *QG*/*QC* = *PF*/*PC* = *RE*/*RC*. But, by the above Lemma, *QG*/*QC* = *K*, the constant ratio relating the midpoints of the preimage-image segments to the midpoints of the corresponding preimage-image arcs. Thus, since *PF*/*PC* = *K* and *RE*/*RC* = *K*, *F* and *E* must be the midpoints of the preimage-image arcs $\widehat{BB'}$ and $\widehat{AA'}$, respectively. Hence, the midpoints *E*, *F*, and *G* of all three preimage-image arcs are collinear. Furthermore, a rotation of $-\frac{1}{2}\theta$ about *C* maps *E*, *F*, and *G*, respectively, to *A*, *B*, and *D*. Since rotations preserve collinearity, and *E*, *F*, and *G* are collinear, then *A*, *B*, and *D* must also be collinear. Thus, for any rotation $0° < \theta < 180°$ about a point *C*, if three midpoints of preimage-image segments are collinear, then the three preimage points are also collinear. This means that if four collinear points are the midpoints of preimage segments under a rotation $0° < \theta < 180°$, then the preimages cannot be the four vertices of a square. *The precision of GeoGebra graphics in Figure 9 makes a hard-to-visualize argument appear visually compelling.*

Verification by Analytical Techniques

Many graduate students not familiar with the Classification Theorem for isometries chose analytical approaches in their proofs. This seems to have been motivated by their use of GeoGebra's Cartesian coordinate system in the development of their simulations. This approach did not require the teachers to know any of the central theorems of transformational geometry. Several teachers succeeded using an analytic approach and helped others in their groups achieve success as well. In all cases, they fixed ■*ABCD* with center at the origin and vertices on simple lattice points. The most challenging part of the strategy was representing the coordinates of an arbitrary square ■*A'B'C'D'* that is congruent to ■*ABCD*. Several teachers gave appropriate representations, such as that in Figure 10, and went on to prove that *Ma*, *Mb*, *Mc*, and *Md* were collinear or not collinear depending on the orientation and whether the slopes between Ma and the other three midpoints were all equal. A few teachers were frustrated by algebraic or trigonometric errors. Here, the availability of a computer algebra system (CAS), which was not available within GeoGebra at the time, would have been quite helpful. Figure 11 illustrates a computer algebra demonstration that the slope between *Ma* and *Mb* equals the slope between *Ma* and *Md*, thus implying that *Ma*, *Mb*, and *Md* are collinear. A similar argument could be applied to proving *Ma*, *Mb*, and *Mc* are collinear.

In the undergraduate class, students were given a figure similar to Figure 10 and asked to find the formulas for the midpoints of the preimage/image segments and prove that these midpoints were collinear when the orientations were different and not collinear when the orientations were the same. This was an extra assignment and students worked independently. Most were successful comparing the slopes using trigonometry and algebra.

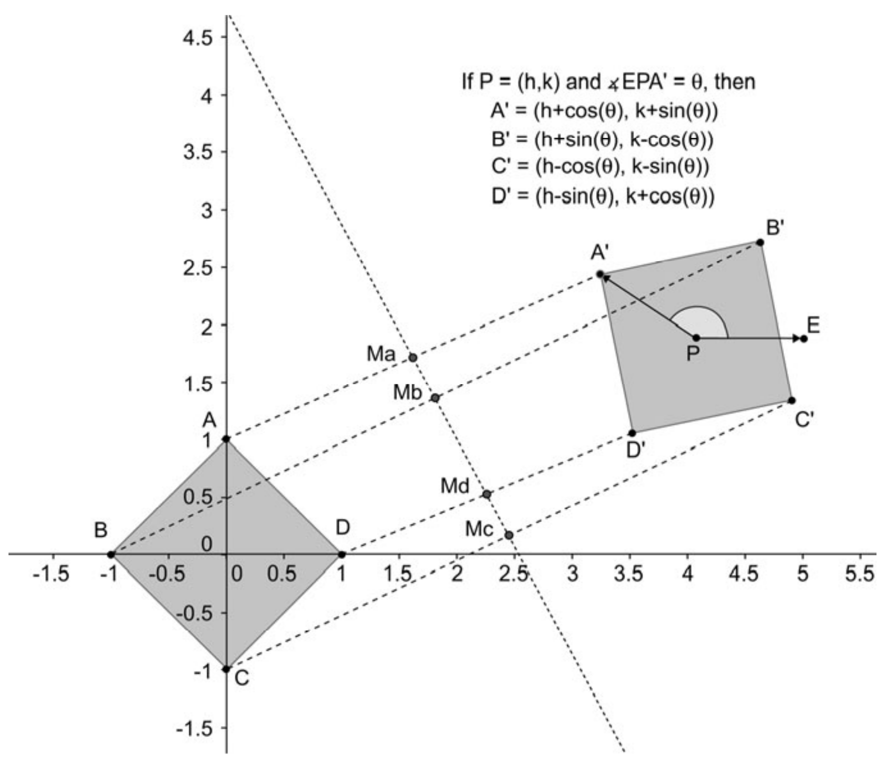

Figure 10. Analytic representation of two squares.

Reflecting on the Results

At the conclusion of the activities, the undergraduate class responded to journal prompts to assess the importance of the dynamic geometry software. Many commented that "discovering" the defining characteristics of the isometries through the use of GeoGebra was a powerful, much better way to learn than if they had just read about isometries or had been directly taught. Some commented on the critical importance of GeoGebra simulations to their success in the Flippin Squares activity. One student said "*I really truly think that my understanding of this problem came from working with GeoGebra. Being able to do this sort of "hands-on" learning is very beneficial for any grade level. A program like this can really make a positive impact on understanding the materials, rather than just regurgitating the information learned in a lecture class because you were told "this is how it works . . . see."*

Many of the students commented that both the Isometry Activity and the Flippin Squares activity deepened their understanding of isometries and made it easier to grasp the significance and proofs of the Three Reflections Theorem and the Classification Theorem for transformational geometry. Students directly attributed their success on tests over the material to the GeoGebra activities. One student said, "[As for] *the test success I can fully attribute [it] to GeoGebra as the more I worked with the program the more comfortable I became with transformations. It was also helpful that I could perform many transformations on the computer rather than only the few I could have done in the same amount of time on paper."*

$\dfrac{\frac{k+\sin()+1}{2} \quad \frac{k-\cos(\theta)}{2}}{\frac{h+\cos(\theta)}{2} \quad \frac{h+\sin(\theta)-1}{2}}$	$\dfrac{\cos(\theta)+\sin(\theta)+1}{\cos(\theta)-\sin(\theta)+1}$
©This shows that the slope of line between Ma and Mb is * $\dfrac{\cos(\theta)+\sin(\theta)+1}{\cos(\theta)-\sin(\theta)+1}$	
$\dfrac{\frac{K+\sin(\theta)+1}{2} - \frac{K+\cos(\theta)}{2}}{\frac{h+\cos(\theta)}{2} - \frac{h-\sin(\theta)+1}{2}}$	$\dfrac{-(\cos(\theta)-\sin(\theta)-1)}{\cos(\theta)+\sin(\theta)-1}$
© This shows that the slope of line between Ma and Mb is * $\dfrac{-(\cos(\theta)-\sin(\theta)-1)}{\cos(\theta)+\sin(\theta)-1}$	
$\text{tCollect}\left(\dfrac{\cos(\theta)+\sin(\theta)+1)}{\cos(\theta)-\sin(\theta)+1} - \dfrac{-(\cos(\theta)-\sin(\theta)-1)}{\cos(\theta)+\sin(\theta)-1}\right)$	0
©This shows that, except possibly for cases where a denominator is 0, the above slopes are equal.	
tCollect(cos(θ)-sin(θ)+1)	$\cos\left(\theta+\dfrac{\pi}{4}\right)\cdot\sqrt{2}+1$
$\text{solve}\left(\cos\left(\theta+\dfrac{\pi}{4}\right)=\dfrac{-\sqrt{2}}{2},\theta\right)$	$\theta=\dfrac{(4\cdot \mathbf{n3}+1)\cdot\pi}{2}$ or $\theta=(2\cdot \mathbf{n3}-1)\cdot\pi$
tCollect(cos(θ)+sin(θ)-1)	$\sin\left(\theta+\dfrac{\pi}{4}\right)\cdot\sqrt{2}-1$
$\text{solve}\left(\sin\left(\theta+\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2},\theta\right)$	$\theta=\dfrac{(4\cdot \mathbf{n4}+1)\cdot\pi}{2}$ or $\theta=2\cdot \mathbf{n4}\cdot\pi$
© The above result show that for $0 \le \theta < 2\pi$, the denominators are 0 only if $\theta = 0$ or $\theta = \pi$ or $\theta = \frac{\pi}{2}$. Substituting these value for θ in the midpoint formulas quickly reveals that the midpoints are collinear for each of thes values.	
	10/99

Figure 11. Computer algebra manipulations supporting the argument that midpoints align.

Interestingly, the work with GeoGebra and with geometric simulations inspired some students to use these ideas in their content methods class. "*I've used GeoGebra for more than your geometry class. I used it to create a model of the parent exponential function with sliders for the transformation parameters. Another group in our education methods class created a GeoGebra model with sliders for students to explore the relationship between particle position, velocity and acceleration . . . Cool lesson!*"

CONCLUSIONS

In the transition from simulation to formal validation, our students needed to conceptualize the elements of the simulation in terms of abstract mathematical forms that could represent all of the infinite possibilities only pointed to by the simulation. This took two directions. On one hand, students conceptualized the flipping process as isometries of the plane and hence as instances of four types of isometries that their mathematical theory indicated exhausted all possibilities. On the other hand, students conceptualized the set of all possibilities of the flipping process as arbitrary pairs of congruent squares, represented as sets of ordered pairs of coordinates in the Cartesian plane, which had the same or different orientations. Significantly, both abstract representations, so critical to the formalization that is needed for proving their conjectures in the Flippin Squares task, were suggested and supported by the student simulation work in the GeoGebra environment.

Comparing the outcomes of the Flippin Squares task under the model-centered, pluralistic approach supported by GeoGebra with our prior experiences using the task in courses where tasks were immediately formalized without the use computer simulations, two conclusions emerge. First and foremost, GeoGebra simulations of geometric phenomenon provide concrete, empirical models that can be publicly shared, manipulated, and scrutinized in face-to-face and in on-line environments. Not only did this make the sharing of ideas far more precise and productive in the group settings where diverse talents and perspectives were brought to the table, but it provided a richer and more "sensible" terrain for doing the empirical work of mathematics advocated by Cooney and Wiegel. This work included the refining of their conjectures by discerning the key aspects of the phenomenon, such as orientation and congruence, aspects that ultimately featured in the formalization of that phenomenon.

Second, the students developed a much stronger grasp of the components of the theory, including the definitions and properties of isometries, used in formalizing the Flippin Squares phenomenon. Indeed, they developed a much better understanding of the main theorems, such as the Classification Theorem, that they ultimately used in their proofs. In addition, the GeoGebra environment helped their proof efforts by providing precise, clear, and dynamic representations of key theoretical ideas used in their reasoning, such as the case of the dynamic diagram used in the somewhat complicated proof of the rotation theorem (Figure 9).

Finally, empirical investigations with GeoGebra models can lead students to discover unexpected patterns. We should have planned for this and should have been prepared to pose extension tasks that would allow students to explore the unknown and unexpected terrain. In the situation depicted in Figure 5, the teachers had inadvertently eased one of the constraints of the Flippin Squares task. An ensuing investigation would have led them to the discovery of a new pattern if they had not assumed they had made a mistake. Designing activities to allow for and encourage such individual or group excursions beyond the paths of the specified curriculum could provide students with deeper experiences of mathematics from the broader view endorsed by Cooney and Wiegel (2003).

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5. BUILDING SIMULATORS WITH GEOGEBRA

This chapter defines a holistic learning model where students and teachers join together in an adventure of learning mathematics by doing the mathematics. The idea is that, to better understand the process or the function of a scenario, one builds a simulator, that is, a model using GeoGebra that simulates the problem. Because GeoGebra dynamically animates and integrates the algebra, the geometry, and the tabular data of a problem, these simulators help both teachers and students comprehend the whole picture. This chapter includes examples of simulators that are presented at various levels, starting from the end user with polished ready-to-use simulators up to basic do-it-yourself construction problems, where students own the design by creating a simulator themselves. In keeping with this holistic model, this chapter also includes examples of good guestions and methods for surveying, rather than testing, understanding that can help students monitor their own progress under the guidance of teachers.

INTRODUCTION

How can math education catch up with and make use of the dizzying innovations occurring in computer technology and digital communications? Too often, by focusing on product rather than process, new math software and websites actually contribute to innumeracy. Genuine math literacy requires holistic thinking and communal dialogue. These two features define GeoGebra, a ground-breaking mathematical tool with global implications.

Learning by Doing and Multiple Representations

As Nicholas of Cusa realized over 500 years ago, nothing exists in isolation. Everything is interconnected, he believed, from the most popular songs to the most rarefied equations. Surveying the city marketplace from a barbershop, the great philosopher and mathematician saw people changing money, weighing wares, and measuring oil and concluded that math informs all human activity. It not only defines art, music, and architecture but the mechanics and dynamics of body and mind. Even the most abstract mathematical concepts are ultimately concrete. Math is the language of reason just as music is the language of sound. If we can use different forms of notation and a wide range of instruments to express music, why not employ many modes and methods to express math?

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 73–89.

Consider the interactive nature of classical math education. Pythagoras' academy at Samos was a conservatory and a gym as well as a math school. Besides learning geometry, his disciples practiced the lyre and the flute and played early versions of basketball and field hockey. Pythagoras based his pedagogy on a simple question: Why do people love to make music and play sports? Because, as participants, they learn by doing! Mathematics, he thought, should be learned the same way.

So one can conclude that in order to be effective, mathematics education must employ many modes and methods to express math and mathematics must be learned by doing.

Teaching in the Real World

On the other hand, the design and implementation of math learning interactivities must be integrated with the realities of today's educational systems. While some institutions do have modern hardware and software and teachers experienced in using this technology to teach mathematics, most do not.

This paper will demonstrate a new experiential way to teach and learn math that focuses on being freely and readily integrable into the daily curriculum of any mathematics program, on learning to create and see multiple expressions of a mathematical situation and then on the doing and exploring of the mathematics by building simulators all within an environment that can be used in the classroom, at home and for both group and individual instruction.

To that end, it contains a link to a web page of models and exercises. By installing the free and open source software GeoGebra and using this web page, readers can test this pedagogical model for themselves and share it with their students.

THE NEW MODEL

This new model operates in the following way:

- An expert teacher creates comprehensive web lessons, which may include class plans, labs, and video presentation.
- A teacher in another place (the same institution or another country) accesses these web lessons via computer.
- Teachers/students select the projects most meaningful to them. There should be a variety from which to choose and students can propose their own projects. As a starter, the teacher might suggest a scenario associated with society or nature.
- Students can collaborate to complete the project.
- Intermediate Self Testing (EST) is based on answering good questions using Surveys (as this paper will show). Unlike a conventional quiz or test, a word survey leaves power and control in the hands of the students. After all, students are motivated to get tested, if only to prove they have learned the material and can graduate from the course. Practice and knowledge lead to self-confidence.

– Final testing is based on *product* and *performance*. The natural parallel here is a music academy, where extensive private practice culminates in a public recital before teachers and peers.

Simulators – What Do They Do and Why GeoGebra Simulators?

Simulators, which are small applications or *applets*, animate a function as it takes on its different values. This process increases comprehension of an occurring process. It sounds elementary, but getting students to truly understand a process or function is one of the most difficult tasks in math education.

This struggle begins at the earliest age – when we ask toddlers to count and tell us the number of objects in front of them They may point at each object and count "one, two, three, four," but to truly declare "There are four objects here!" they must understand that counting is not a form of labeling, such as identifying different colors. Actually, it is an independent process or function – namely addition.

Of course, it is much easier to understand a process with concrete objects than with variables. However, conveying the idea of process or function with variables remains essential – not merely so students can learn mathematics but so they can develop such vital life skills as modeling, connecting, critiquing.

GeoGebra provides access to teachers and students to free, easy-to-use software for building simulators. Together with guided interactivities, these simulators can increase the understanding of process (what we call *function*) at all levels of mathematics.

To help the reader get a "feel" for these simulators and what they can do, we created a web page with the simulators discussed in this paper's "still examples". We suggest readers play with the simulators while reading the paper.
Link to all links: http://www.geogebra.org/en/wiki/index.php/Simulators

We will discuss our ideas by working through examples:

> **Scenario:** Galileo stands on a 50m cliff with a mortar, which shoots round, cast-iron shells at a speed of 30m/s. Assisted by the younger and stronger Torricelli, he wheels and braces the weapon at the edge of the cliff and fires it straight up. When will the shell hit the ground at the base of the cliff?

A Step backward: Graphics Calculator Simulator

To the student, the "goal" is to get 7.49 or "the answer." Notice that the variable, and the unit are not included in "the answer."

Using a graphics calculator (see Figure 1) students may get the false impression that the Ball is actually moving along the curved path and thus understand the problem on a mechanical (not meaningful) level and no real learning has occurred. They just trace over to where the quadratic crosses the x-axis and "get" 7.49. Done. He may not even realize that the answer is ***time*** – that 2.94 is ***seconds***. He can easily think it is distance.

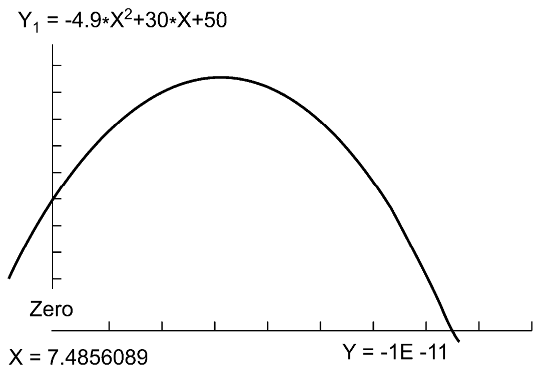


Figure 1. Graphics calculator simulator.

Less understanding is conveyed here than if we had worked entirely theoretically with just the function: $h(t) = -4.9t^2 + 30t + 50$. At least this function correctly labels the independent variable as *t,* time, and the dependent variable as *h*, height. Compare this with a GeoGebra simulator which we developed (see Figure 2).

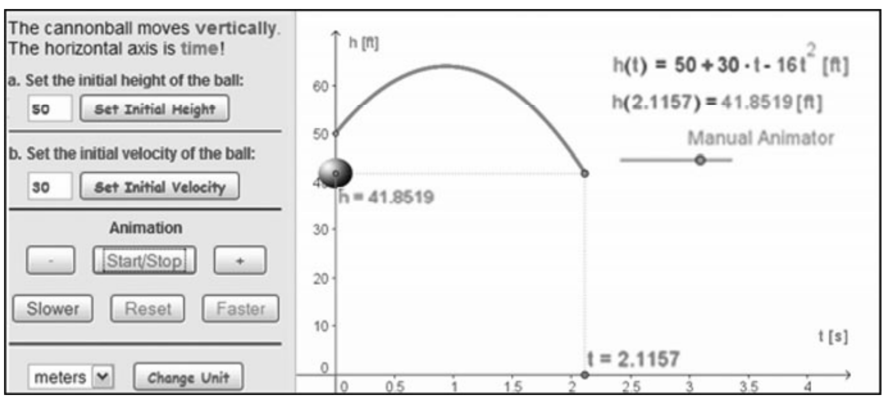


Figure 2. Vertical projectile motion - educator built simulator.

GeoGebra Classroom Simulator

Let's examine the parts of the GeoGebra Classroom Simulator for Vertical Motion. (http://www.mathcasts.org/gg/student/quadratics/motion/motion_v3.html)

Of course this is difficult without animation so click on the link to play with the actual simulator. Two points are vital:

– Most important is that the ball is always on the *y*-axis. That is, the student realizes that the **ball only moves vertically.**

– Secondly, the axes are correctly labeled t and $h(t)$. That is, the student realizes that the **horizontal axis is time** and not horizontal movement.

So, at time $t = 0$, we see only the ball at (0,50) – that is, there is no upside down quadratic until time advances. Upon clicking on the Start/Stop button, the ***ball begins to move vertically*** along the y-axis and at the same time the function $h(t)$ is drawn. The red line for time increases along the horizontal t-axis and the green line (for height) corresponds to the height of the ball. Furthermore, the animation speed can be decreased or increased depending on whether one wants to study the relative speed of the ball or just get to the answer – both of which are important. We suggest that the reader play with it on the live simulator before reading further.

Student built GeoGebra simulator. If students learn best by becoming involved in an actual process, why not get them to build the simulator – to design, create and test the process itself? Below, in a separate section, we will discuss this part of the learning model in detail.

SIMULATORS AND GOOD QUESTIONS

Finally, in this section, we want to emphasize that just playing with simulators is not enough to promote understanding and meaningful learning. We must help both the experienced and inexperienced teacher and the students themselves by providing *Good Questions* –guides to using and building the simulators that promote understanding of the mathematics. We now head in this direction.

Good Questions

The Project *Good Questions* initiated at Cornell University defines the concept as "a pedagogical strategy that aims to raise the visibility of the key concepts and to promote a more active learning environment" ("Good Questions for Calculus," n.d.). Some of the characteristics of Good Questions as defined by the Cornell team are:

– stimulate students' interest and curiosity in mathematics
– help students monitor their understanding
– support instructors efforts to foster an active learning environment.

By asking students meaningful questions they become active learners and develop their reasoning skills. Without meaningful questions students can fall back to the passive learning by rote style where there is little or no understanding.

The art of developing good questions is crucial for effective teaching and is not limited only to simulators but in the case of simulators that are used in the classroom it is important for teachers to learn how to create good questions on their own because the nature of the questions must fit the context and the particular needs of students. Another aspect of good questions is that it helps the teacher assess the progress of the class as a whole as well as individual students.

For example, a colleague of ours who teaches Algebra I used the above "Vertical Motion" simulator in his two classes. In both classes, he asked his students to describe what they thought was happening at different stages in the simulation. The results amazed him. "Poorer" students realized that at the topmost point, the speed of the ball was 0 whereas students in his advanced class did not.

We point this out because as educators we know that students have different kinds of mathematics skills. But we only grade a certain subset of these skills. By combining a good simulator with good questions we can really see what they understand. Once students speak or write real sentences containing mathematics (and not merely numbers and symbols) everybody learns more.

Using Surveys to Ask Good Questions

Surveys, we discovered, are an effective technique to generate questions. Our first attempt was to use the free version of Survey Monkey software (see http://www.surveymonkey.com/) but later we discovered that **Google Forms** is more effective since it is integrated together with Google documents. If you are not using Google documents you may find Survey Monkey better since it allows including pictures but Google Forms is very easy to use. It also takes little time to submit to students. The Surveys give teachers a quick way to assess how students are learning. Either way, surveys take a short time to build. Surveys also allow students to reflect and to think deeper about the material. Here is a quick explanation of using Google docs. (You need a Gmail account in order to do this.)

Step 1: Open Google Documents (sign into and open your Gmail account, click on Documents at top). Then open a New Form (on the Google docs menu, click on command New and then from the drop-down menu, click on Form) – see *Figure 3*.

Figure 3. A new google doc.

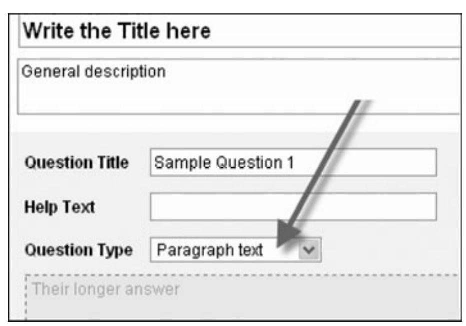


Figure 4. Setting up the doc.

Step 2: Write the questions (we recommend that you use paragraph text for most of the answers unless you want to have quick True False survey). The Paragraph text option allows the students to write in their own way without limitation of space (Figure 4).

Step 3: Click on “Email This Form” Box and paste the email addresses of your students.

Once the students have answered the survey, the result appears as a Google Spreadsheet in your Google documents. In the first row are the actual questions and the rest of the rows are the answers of students who responded (see Figure 5).

The spreadsheet is created automatically once the survey is emailed and is updated automatically as new answers come in. It can be copied to a new spreadsheet and the teacher can delete the Name Column (if there is one) if she wants to share the results with the whole class anonymously (it may be a good idea to use File/Make copy… first if you want to keep the original)

	B	C	D	E	F
	Name	**The temperature of a stone is defined by the function f(x)=x^2 + 1. Where x is changing from 0 to 2 hours. Imagine that the temperatue is tested every ten minutes. Describe how would you model the experiment in a GeoGebra SpreadSheet**	**Explain why there are exactly 13 rows in the spread sheet**	**Right click on the data in the spreadsheet and create a list of points. Turn the grid on and notice the points on the graph**	**Every point on the graph represents an event (or a situation). What event is represented by the point P7 and why.**
2		First I put the number 0 in column A1. Then in column A2, I put = A1+1/6. Then I dragged it down to column A13. Then in B1, I put = A1^2+1. Then I dragged that down to B13. Then I selected both A and B columns and created a list of points. The x-axis was time and the y-axis was temperature. Dani feedback: Good answer A	There are 13 rows because the temperature is checked every 10 minutes. Dani feedback: The answer is not complete. It has to be connected to the two hours and show the computation B	In the graph, the numbers on the spreadsheet are used. Column A, which is time, is the x-axis, and Colimn B, which is temperature, is the y-axis. Good A	It represents that after one hour, at 1am, the temperature was Dani: Not complete but almost… B

Figure 5. Sample spreadsheet of student responses to survey.

For each one of the simulators explained in this paper we will provide at least one example of good questions that can be posed in such a survey. Next we move on to the heart of the chapter.

STUDENTS BUILDING SIMULATORS

The next step, after working with the educator built simulator, is to get the students to build their own simulator. Because GeoGebra is freeware, the amount of teacher involvement in this process depends on available class time, available laboratory time, students’ mathematical understanding, and students’ Information Communication Technology (ICT) and GeoGebra skills. Similarly, the sophistication of the required end product is also flexible. This paper provides examples of simulators that can be built as early as 4th grade.

Example 1. *Spaceball Simulator Kit*

Scenario: *Kirk and Spock decide to play ping-pong in space. The ball bounces back and forth between their paddles, when a Klingon warship approaches. In their haste to return to the Enterprise, they leave their paddles and the ball behind.*

After the battle (in which the Federation and the Klingon Empire both claim victory), Kirk and Spock look out and see the ball still bouncing back and forth between the paddles. ***Make a simulator to show the paddles and ball.***

Spaceball is a simulator kit. That means it comes with a GeoGebra starter file, all the images necessary to make the simulator interesting, a pdf file containing complete step-by-step directions on building the simulator, interesting mathematics/science questions about the simulator and answers for the educator and a finished GeoGebra simulator file.

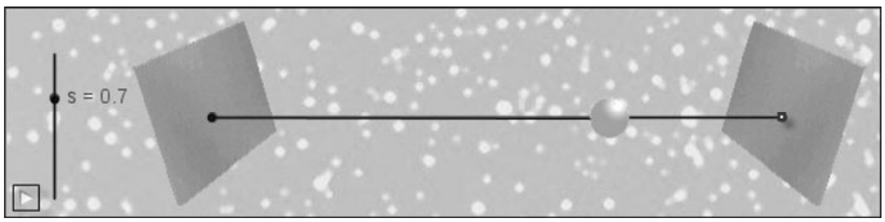


Figure 6. Spaceball simulator.

The simulator has 5 objects. It can be built and tested in minutes. Then the student adds the images to make it fun to use.
URL: http://www.mathcasts.org/mtwiki/InterA/Spaceball

Example 2. *Car Race Simulator Kit*

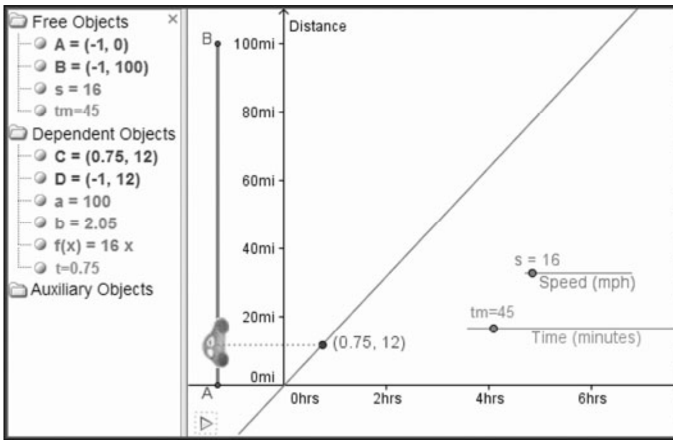
One of the most difficult mathematical concepts to understand are processes that depend on variables because the static picture doesn't "match" the dynamic physical scenario.

Building the Basic One-Car Simulator

Scenario with 1 car: *Every day Rila drives her red car the 100 mile distance across the plains of Flatlandia between the cities AbsolutelyNowhere and BeyondBoring. She drives the whole route at a constant speed using cruise control. A stopwatch displays the number of hours that have passed and a GPS displays the number of miles the car is from AbsolutelyNowhere.*

Today Rila leaves AbsolutelyNowhere at 9:00 AM. She resets the stopwatch to 0 and she sets her cruise control at 16 mph. What two numbers are displayed at 9:45AM? Starting when she resets the stopwatch to 0, every 15 minutes, the car prints out the numbers on the display. What does the printout look like at 9:45AM? What time is it when Rila arrives in BeyondBoring?

We want our students to make a simulator of a car driving from point A to point B. We want this to be both engaging and educational experience. Getting the little car to drive up the road makes it engaging. On the other hand, as with the simulator for vertical motion, it is crucial that the student understands that the car is driving in a straight line and that when we make a 2D graph, the horizontal axis represents time and the vertical axis represents distance. By building a GeoGebra simulator like this themselves (see Figure 7 et al.,), even young students understand the situation clearly. They get a concrete experience of a concept of a mathematical function which is crucial (and so often missing) for understanding more advanced courses like Calculus. They get 4 views of the math – Algebraic, Physical, Graphical and Tabular. By having all a variety of representations students get a better chance to understand the problems and even start liking math if they do not already like it.



	A	B
1	Time	Distance
2	0	0
3	0.25	4
4	0.5	8
5	0.75	12
6	1	16
7	1.25	20
8	1.5	24
9	1.75	28
10	2	32
11	2.25	36
12	2.5	40
13	2.75	44
14	3	48
15	3.25	52
16	3.5	56
17	3.75	60
18	4	64

Figure 7. Car simulator.

Crucial Points When Getting Students to Build Simulators

We point out some crucial points that the educator must emphasize when teaching students to build simulators.

Part 1: Examine the process –a car is driving along a line (road).
Crucial: A vital point in building simulators with GeoGebra is to get the students to think about the process and what we want the simulator to do.
Q: What is the process here? Time increases and the car travels along the road.
Q: Which depends on which? The distance the car travels increases as time increases. So distance depends on (is a function of) the variable time.
Q: What do we want to simulate? We want to simulate time passing and see the position of the car change. We want to animate time, *t*.
Q: What values of *t* are possible? Time starts at $t = 0$ and increases.
Q: Where is the car at time $t = 0$? How does it move as t increases? In the educator simulator, point to the position of the car at $t = 0$ and show how the car moves as *t* increases. What else moves in this simulator?

*Part 2: What is the function? What is **x** and what is **t**?*

Crucial: Another vital point in building simulators with GeoGebra is differentiating between the ***independent variable x of the function*** and the ***slider t***. Remember that when we are explaining the algebra of the problem *t* is the independent variable of the function! However, in a simulator, *x* is the independent variable and *t* is the number we substitute for *x* to see time change.

***f*(*x*) is the whole road and *f*(*t*) is the position of the car on the road at time *t*.** To see this (and before having the students start building a simulator):

Create a slider *t*. At any moment in time, *t* is always a number. Look in the algebra view (left). It always shows *t* as a value. We can change this value either manually by sliding (clicking and dragging) or with animation.

Now define the function $f(x) = 3*x$. Here *x* is the independent variable or argument of this function. In the algebra view, we see the function. In the graphics view (the drawing pad) we see a line. We do not see *x* **anywhere** except inside this function. That is, *x* is a letter and never a number. Because *t* is a number, we can substitute it for *x* into the function *f*(*x*) to get the value of the function for that number. So *f*(*x*) is a function and *f*(*t*) is a number.

Adding a Start Position to the Car Simulator

Scenario with 1 car and a start position: *Today Rila is 10 miles outside of AbsolutelyNowhere when she remembers to reset the stopwatch. Her cruise control is set at 16 mph. What time does the display show one and a half hours later? What time does it show at her arrival in BeyondBoring? What numbers are on the printout?*

After building this simulator, it is easy to add a slider for the starting position of the car and change the function appropriately (see Figure 8).

The student then uses the simulator to get an understanding of the question posed in the scenario. She slides *s* to 16 and *r* to 10, calculates that 1½ hours is 90 minutes and slides *tm* to 90. In the three different windows: Algebra, Graph, and Spreadsheet, she can find the answers to the questions (see Figure 8). Then she can do the mathematics and check her answers against reality and adjust appropriately.

Using the Construction Protocol to Learn How to Create Simulators and Make Adjustments

GeoGebra provides a quick and effective way to help students learn how an existing simulator was created and thus help them gain control and create the simulator themselves. The Race Car Simulator is a good example. The link is: http://www.mathcasts.org/mtwiki/InterA/CarRaceSimulator. First download the simulator. To do this, double-click on either of the simulators. It will open in a GeoGebra window. Select command File->Save. Now the GeoGebra file is saved to your hard disk. Next select the command View -> Construction Protocol.

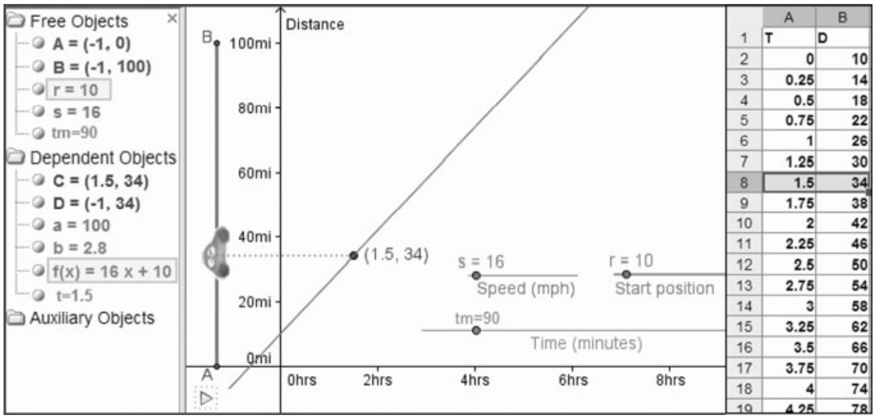


Figure 8. Car Simulator with start position.

You will then see the protocol for the steps that were used by the author to create the simulator. Click on the ⏮ to get back to the first step. Then click on the ⏩ to advance 1 step. As you advance the object created be highlighted in the protocol and shown in the Algebra View, the Graphics or Spreadsheet View (Figure 9).

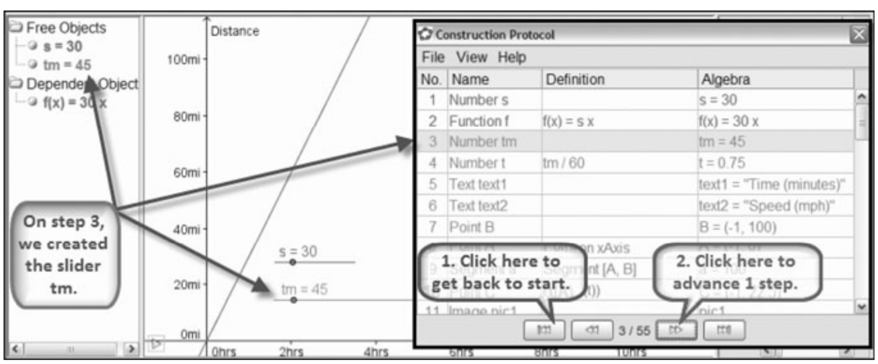


Figure 9. Construction protocol for 1 car: car1.ggb.

Using the Car Simulator

With this simulator, many questions can be posed and the teacher can require both appropriate screenshots from the simulator and algebraic answers. For example:

Q: Set r = 32 miles. How fast must red car go to reach the endpoint in 4 hours?

The student must create a screenshot and write down his steps either in the screenshot (see **Figure 10**) or separately.

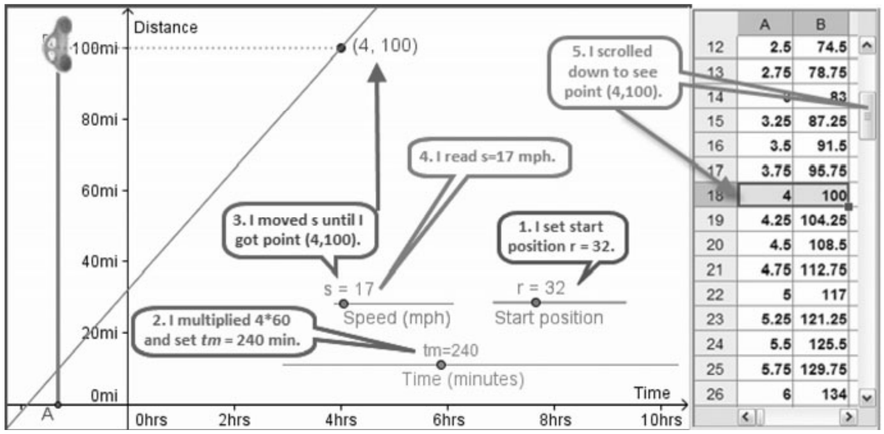


Figure 10. Screenshot of simulator use.

He must also write his answer and his math (Figure 11):
Answer: The car must go 17mph.

start = 32 mi	$d = st$
$t = 4$ hr	$68 = s \,.\, 4$
$d = 100 - start$	$s = \frac{68}{4}$
$d = 100 - 32$	$s = 17$ *mph*
$d = 68$ mi	

Figure 11. Accompanying algebraic solution.

Adding a Second Car – Car Race Simulator

Scenario with 2 cars and start positions: *Bobi sets out in his blue car at 20mph and a while later Rila sets out in her red car at 25 mph. They are talking on their hands-free cell phones when they both decide to stop where they are. Bobi is 30 miles from AbsolutelyNowhere and Rila is 20 miles from AbsolutelyNowhere. At exactly the same moment, they get back in their cars and both reset their stopwatches to 0 and continue driving at their set speeds. Will Rila meet up with Bobi before he gets to BeyondBoring? If so, when and where? What does each of their printouts show?*

Having built the simulator with one car, the student can now add a second car (with sliders for speed and position). To answer the question in the above scenario, the student must create the following screenshots (see Figure 12 and Figure 13).

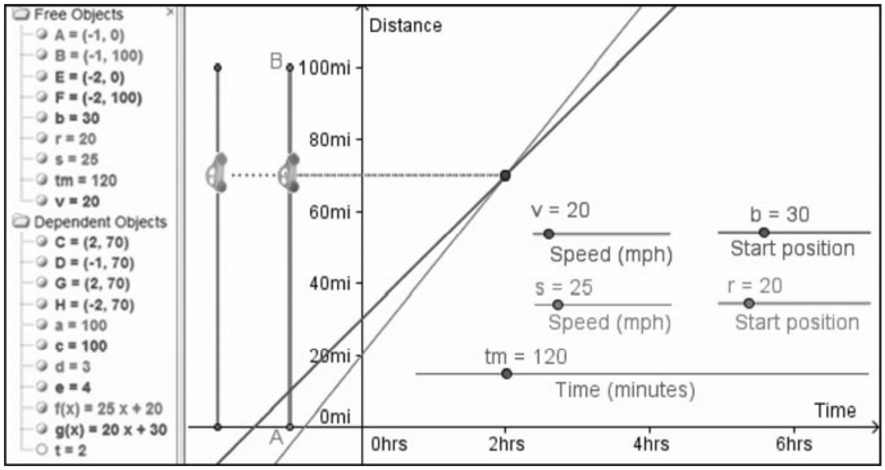


Figure 12. Simulator with 2 cars – solution.

He must also write out his answer and show his math (Figure 14).
Answer: Blue car starts out ahead. The cars meet after 2hrs and 70 miles from A.

	A	B	C	D
1	T-Red	D-Red	T-Blue	D-Blue
2	0	20	0	30
3	0.25	26.25	0.25	35
4	0.5	32.5	0.5	40
5	0.75	38.75	0.75	45
6	1	45	1	50
7	1.25	51.25	1.25	55
8	1.5	57.5	1.5	60
9	1.75	63.75	1.75	65
10	2	70	2	70
11	2.25	76.25	2.25	75

$f(x) = 25x + 20$
$g(x) = 20x + 30$
$f(x) = g(x)$
$25x + 20 = 20x + 30$
$5x = 10$
$x = 2$

$x = 2$
$t = 2$ hr
$d = f(2) = 25.2 + 20$
$d = 70$ mi

Figure 13. Spreadsheet view. *Figure 14. Algebraic solution.*

Complete details including images and step-by-step instructions for building this simulator and educator pages with good questions (and answers) can be found at: http://www.mathcasts.org/mtwiki/InterA/CarRaceSimulator

Example 3*. Related Rates Simulator*

In calculus, some of the most difficult problems for students to solve involve "related rates." In the article GeoGebra: Freedom to Explore and Learn (Fahlberg-Stojanovska & Stojanovski, 2009), the authors give a specific example of a simulator for a question posed on answers.yahoo.com.

Link: http://mathcasts.org/gg/student/calculus/RelatedRates/rates.html

***Example 4**. Numeric Solutions: The Boat Landing Problem Simulator*

There are many problems that students can easily understand and construct the mathematics but which cannot be solved except numerically. Building simulators for these problems is both fun and educational and in fact our first simulator and the idea for building simulators arose in this way.

Link: http://www.mathcasts.org/mtwiki/GgbActivity/BoatLanding1

***Example 5**. The Box Folding Simulator*

This simulator was also mentioned as example 4 in the chapter by Robyn Pierce and Kaye Stacey (this volume). Try an actual simulator prepared by Linda Fahlberg-Stojanovska and available on mathcasts.org. The direct URL is:

http://www.mathcasts.org/mtwiki/InterA/BoxFolding and some good questions that can be asked even in a non-calculus classroom can also be found here. We give a screen shot of the simulator.

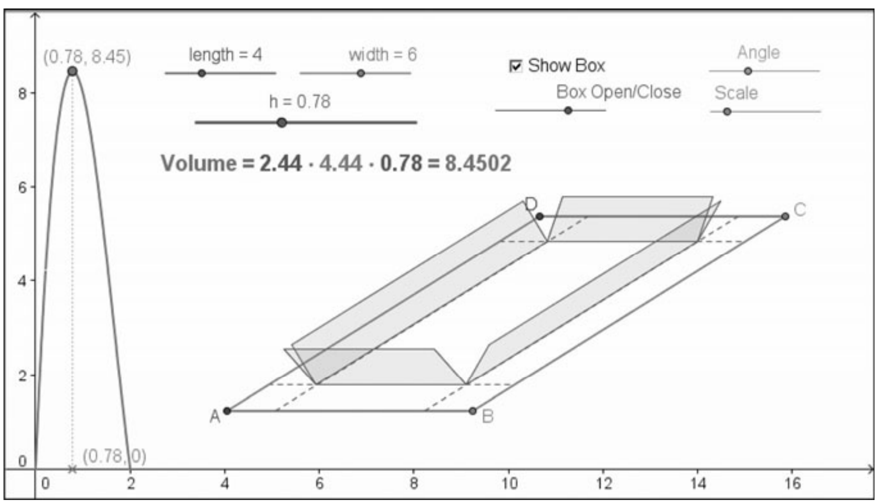


Figure 15. Box folding simulator.

Box Folding Simulator and Good Questions

What is crucial is that the students here become active learners and have time to absorb what is actually happening. That way the concept of a function has a chance to be born in their minds.

The following question was given to the students who had a chance to play with the Folding Box Problem simulator. The answers give a "bird's eye view" of the class understanding as well as individual evaluations of specific students)

> *Explain how to find the volume of a box by cutting 4 squares from the corner of a sheet of paper of dimension 8"×16" sheet. Try to be as clear as you can and imagine you are explaining it to a person that does not know what functions are. Do not just write the final numerical answer.*

The actual *unedited* answers by students are pasted below. The students were given a Google survey and the answer appeared on a spreadsheet for the teacher to check.

- there is a relationship between the length of the two equal cuts one could put into all four corners of a rectangular piece of paper and the volume of space inside the box that would exist if all four of the sides were folded up so that the 2 corners of each corner cut touched at the top. this is assuming that there is a top which would not actually exist if you only had one piece of paper. the length of the cut, which would be cut into each corner twice to create a ninety degree angle, would determine the height of the rectangular prism. as this value increases the length of the 2 short sides or the widths would decrease and as well as the size of the two long sides or lengths. assuming that you did this multiple times each time increasing the cuts in length a very small distance cut would create a small volume and as the cut increased in length the volume would until certain point after which the cut length would continue to increase while the volume would slowly decrease. this relationship as a function would be represented on a Cartesian plane by a parabola or hill shaped line. the function would be written as a function of x where a set starting length and width is determined. if the length of the flat uncut paper started out as 10 units and the width was 5 units the function would be written as f(x) = x(16-2x)(8-2x) because volume equals length (16-2x) times width (8-2x) times height (x) where the value of x decreases the length of the widths and lengths at two times its own value. to make the equation you are simply creating a function that is in the form of a formula for calculating volume. the only difference is that the length and width change proportionally to how much the height changes, which then makes the volume change proportionally to how much the height changes. height is labeled x, width is inside parentheses because a predetermined width must change when cut into and the length changes in the same way.
- You would find the function, or relationship, between the size of the cut and the volume of the box made. If you take a length “x” out of the side on each side, this would result in a decrease of “2x” in the length of the box. The function would be f(x) = (8-2x) * (16-2x) (x) . . . (this is because of L*W*H). This equation gives you the volume of a box made by taking length x from the coroners.
- well, to find the volume of a box, you simply multiply the width by the length by the height. in this case, you have variables that depend on all three of these constants. should the paper be 8"x16", the equation would look something like this: (8-2x)(10-2x)*x.
- I was confused how this plays into the quadratic equation, but i do understand how this is a definite start.

- First plug in each number to the formula, since there are 4 squares cut out make sure you subtract each number by 2x.

  it should look like this (8-2x)(16-2x)
  then don't forget to multiply the whole expression by x for the cut
  the final should look like this x(8- 2x)(16-2x)
  plug into gg to get the max point at (1.7,98.53)
- The volume of this box would be length x width x height. X will represent the 4 squares ripped out of the sheet of paper (8+2x)(16+2x)
- If you cut 4 2" squares from a 8"x16" sheet, 2" is the volume. The width is = (8-2(2))= 4 and the length is = (16-2(2))= 12. In order to find volume you multiply l*w*h. So, 2*4*12= 96.
- The width of the flat sheet of paper is 8 and the length is 16. and if you cut a square from each corner, it will make a box. However many square inches you cut from the paper is going to be the height and you have to adjust the measurements of the width and length accordingly. So, let's say I cut a 2-inch square off the sides. I have to take 2 inches off both sides off the length and width of the paper by subtracting 4. so the width is now 4 and the length is 12. the height is 2. volume is length*width*height. so in this case volume is 12*4*2 = 96 square inches.
- I do not understand this box concept.

The reader can see how diverse the range of understanding of the students is. Regardless of the questions whether the answers were correct you get the sense the many students took the questions seriously since they were treated with respect where many times when they take class tests they cannot really think because they feel afraid.

Student Projects

A wonderful result of teaching a class in simulators to students of computer science this last semester was that a couple of our students built their own simulators. We opened a new page on the GeoGebra wiki to show off their work. This link is: http://www.geogebra.org/en/wiki/index.php/Student_Projects.

SUMMARY

Much work and study remain to be done, but GeoGebra could transform math education. To paraphrase John Louis von Neumann, if students do not believe that mathematics is relevant, it is only because educators too often prevent them from relating mathematics to their lives. Connections abound, however. As Dean Schlicter reminds us, "Go down deep enough into anything and you will find mathematics." With the proper tools and guidance, our students can embrace math rather than endure it.

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6. INFLUENCE OF GEOGEBRA ON PROBLEM SOLVING STRATEGIES

This chapter reports on our research findings about the influence of GeoGebra use on twelve secondary students' problem solving strategies in plane geometry. Using multiple data sources, we analyse the complex interactions among GeoGebra use, students' prior knowledge and learning preferences, and the teacher's role, under the theoretical perspective of instrumental genesis. We identify three levels of instrumentalization and instrumentation and provide specific cases to illustrate students' use of GeoGebra and their evolving mathematical conceptions in relation to GeoGebra tools. In general, the use of GeoGebra helps student enhance their mathematical understanding by enabling alternative problem resolution paths, and, in some case, help diagnose their learning difficulties. We further discuss implications for future GeoGebra use and classroom-based research.

INTRODUCTION

This study is related to research about the integration of computational technologies in mathematics teaching, in particular, the use of dynamic geometry software in the context of students' understanding of analytic geometry through problem solving. In this research we focus on the interpretation of students' behaviour when solving plane analytical geometry problems by analysing the relationships among software use, paper and pencil work, and *geometrical thinking*. Many pedagogical environments have been created such as Cinderella, Geometer's Sketchpad, Cabri géomètre II, and GeoGebra. In this study we focus on the use of GeoGebra because it is free dynamic mathematics software and its focus is not only on geometry, but also on algebra and calculus. Moreover, this may distinguish Geogebra from other geometry software. GeoGebra links synthetic geometric constructions (geometric window) to analytic equations and coordinates (algebraic window). We analyse secondary students' problem solving strategies in both environments: paper and pencil and GeoGebra. As stated by Laborde (1992), a task solved by using dynamic geometry software may require different strategies than those strategies required by the same task solved with paper and pencil, which affects the feedback provided to the student. In this study, we consider the following research questions:

- What is the relationship between students' work in both environments?
- How does the use of GeoGebra interact with the paper-and-pencil skills and the conceptual understanding of 10th grade students when solving plane geometry problems?

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 91–103.

We analyse and compare resolution processes in both environments, taking into account the interactions (student-student and student-GeoGebra), through the instrumental approach (Rabardel, 2001). The primary purpose of our research is to offer didactical knowledge in order to understand the ways in which the use of dynamic geometry software can help promote argumentation abilities in secondary school geometry. Specifically, we will:

- Characterize students' resolution strategies for the proposed problems in both environments.
- Analyse the instrumentation and instrumentalization processes in order to characterize different behaviours among the students.
- Explore the influence of using GeoGebra (conceptual understanding, visualization, resolution strategies) on each type of student.

THEORETICAL FRAMEWORK

We consider different approaches for our theoretical framework. The main theoretical approaches are the instrumental approach (Rabardel, 2001) and the cognitive approach (Cobo, 1998). We also specify some terms in what follows.

According to Kieran and Drijvers (2006), the perspective of instrumentation is a theoretical framework that is fruitful for understanding the difficulties of effective use of technology, GeoGebra in this case. The instrumental approach for using tools has been applied to the study of Computer Algebra System (CAS) in the learning of mathematics and also to dynamic geometry systems. In our research, we apply this framework to the use of GeoGebra, which is a free dynamic geometry software environment that further provides basic features of CAS. The instrumental approach distinguishes between an artefact and an instrument. According to Vérillon and Rabardel (1995), it is important to stress the difference between the artefact and the instrument: An *artefact* is a tool, which could be a physical object, a calculator, or a computer program used by a subject to perform a task. An *instrument* is a mental construction built by the subject from the artefact; so an instrument is a psychological construct.

The process of transforming a tool into a meaningful instrument is called *instrumental genesis*. During the instrumental genesis, the student builds up mental schemes. This process is complex and depends on the characteristics of the artefact, its constraints, and affordances, and also on the knowledge of the user. This process of instrumental genesis has two dimensions: instrumentation and instrumentalization.

Instrumentation refers to the process by which the affordances and constraints of the software influence students' problem solving strategies and their emergent conceptions of the problem situations. According to Vérillon and Rabardel (1995), instrumentation involves forming utilization schemes that provide a predictable and repeatable means of integrating an artefact and the corresponding actions. Instrumentation comprises the rules and heuristics for applying an artefact to a task, through which the task becomes meaningful to the user. We will distinguish, in the analysis of the students' resolutions, three levels of instrumentation in considering the influence of the use of GeoGebra on their resolution strategies and the development of new schemes.

Instrumentalization, by contrast, refers to the process by which an artefact becomes a means of solving the proposed problems. It becomes meaningful to an activity situation and thus is transformed into an instrument. The students' knowledge and their ways of thinking guide the way in which the tool is used. This process depends on the user; the artefact remains the same even if it can be instrumentalized in different ways. We will distinguish, in the analysis of the students' resolutions, three levels of instrumentalization in considering the use of GeoGebra by the students during the resolution process of the problems.

We further specify some terms that will be used in this study of students' GeoGebra-based resolutions such as *figure* and *drawing*. We use these terms with their usual meaning in the context of the dynamic geometry software (Laborde & Capponi, 1994). As stated by Hollebrands (2007), the distinction is useful to describe the way in which students interpret the representations generated with the computer. For example, if a student constructs a rectangle using only measure tools, the figure does not pass the dragging test. This figure is considered a *drawing*. To construct a *figure* with its geometric properties, the student has to know the necessary tools and the geometric properties of the object. It requires technical knowledge about the software as well as geometric knowledge.

Finally, we use the term *basic space* of a problem as defined by Cobo (1998). The *basic space* of a problem is formed by the different paths for solving the problem, which are obtained by a person who solves the problem correctly. We use the term *students' basic space* for the basic space obtained when we consider all the students' resolutions. We adapt these terms to our research in order to carry out the analysis of the problems proposed.

METHODOLOGY

This research is a case study that is analysed from a qualitative-interpretative perspective. We analyse students' mathematical behaviours during the resolution of plane geometry problems (adapted from national exams in the Spanish context). The study was conducted with a heterogeneous tenth-grade group of twelve students from a high school in Catalonia. These students had worked on geometry focusing on a Euclidean approach and problem solving. Their teacher had collaborated with us as a researcher on other occasions. We decided to carry out the study with him because he has extensive experience as a teacher and in teaching problem solving. In choosing the problems for our research, we consulted with the teacher about the prior knowledge of the students, their cognitive characteristics, the units studied at the moment of the data collection and the fact that some of these problems can be solved using GeoGebra. Another important fact is that these students had no previous experience with GeoGebra. For this reason, we planned introductory sessions on the use of GeoGebra. For example, two of the problems considered for this research are the following:

Circle problem: Find the centre and the radius of a circle such that $P = (1,-1)$ and $Q = (3, 5)$ are on the circle and the circle's centre lies on the line r, whose equation is: $x + y + 2 = 0$.

Rhombus problem: A rhombus has two vertices $P = (-2, 1)$ and $Q = (0,-3)$ that form one diagonal of the rhombus. The perimeter is *20* cm. Find the remaining vertices and the area of the rhombus.

Before gathering data from the students, we carried out an analysis of the proposed problems. We considered the *basic space* of the problems in both environments. We analysed the necessary background to solve these problems and considered the different resolution strategies. We identified the conceptual content. In the case of the solutions proposed with GeoGebra, we also analysed the necessary tools to solve the problem following different paths of the *basic space*. All the activities with students were planned to take four sessions of one hour each. In the following sections, we explain the procedures by which these sessions were carried out.

First and Second Sessions (2 hours)

These sessions were carried out in the computer classroom. Every student had access to a computer with GeoGebra installed. The teacher introduced the use of GeoGebra. First, the teacher presented some examples, using an overhead projector, to show students the basic features of GeoGebra. We prepared an introductory booklet with some examples and exercises. The students worked in groups with the help of the teacher, solving some of the examples and construction problems. The purpose of the activities was to familiarize students with the use of the necessary tools for the next sessions. For example, they had to construct a square, given the centre and a vertex, and they had to make some conjectures by exploring the problems situations (Varignon and Vivianni theorems). In these activities, students were only asked to make the construction, state a conjecture, and check it by using the dragging tool; they did not have to prove the conjectures because the aim of the session was to teach them how to use GeoGebra.

Third Session (1 hour)

This session was carried out in the regular classroom. Students had to solve two problems in pairs, as they usually do in a mathematics class but without any help from the teacher. For example, they had to solve the circle problem. We observed the students during this session in order to see what they were attempting to do. We also audio-recorded their interactions.

Fourth Session (1 hour)

During this session, students worked in the computer classroom on their own computer, but they were seated next to the same student as in the previous session. First, they had to solve the circle problem using GeoGebra. Then, they had to solve the rhombus problem using GeoGebra. The students had a worksheet, where they had to explain their constructions and the decisions made while they were working with GeoGebra.

ANALYSIS AND EXAMPLES

Data were collected from the following sources: a) the solution strategies in the written protocols (paper and pencil and GeoGebra resolutions) and the GeoGebra files; b) the audio- and video-taped interactions (student-student and student-GeoGebra); c) construction protocols saved in the computer files; and d) interviews with the teacher.

We analysed all these data sets using an ethnographic approach. Through the analysis of data, we characterized students' learning behaviours and discussed the idea of instrumentation, linking the theoretical perspective and the class experiment. In order to attain these goals, we took into account the following variables: a) heuristic strategies (related to geometric properties, to the use of algebraic and measure tools or to the use of both, their resolution strategies, etc.); b) the influence of GeoGebra (visualization, geometrical concepts, overcoming obstacles); c) students' cognitive characteristics (information provided by the teacher and by the researcher); and d) obstacles encountered in both environments (conceptual, algebraic, visualization, technical obstacles, etc.). In order to illustrate the analysis carried out, we use some students' relevant productions as examples in the following section.

The analysis took place in two phases. During the first phase, we considered individual cases and then conducted a cross-case analysis. For the case analysis, we analysed first the paper and pencil resolutions, considering the written work and the audio-recorded information. We classified the students' resolution strategies to obtain the student *basic space* of the problem and we identified different obstacles encountered by the students. We further analysed how students made use of GeoGebra, their problem solving strategies, and the technical difficulties. For this analysis, we considered the video and audio-recorded information, the GeoGebra files, and the written work (worksheet). In the second phase, we compared the *basic space* of the problems in both environments, and how students made use of the different affordances and constraints of GeoGebra, trying also to identify instrumented techniques such as obtaining the mirrored point at line, reporting lengths, obtaining the area of the rhombus, etc. We analysed the processes of instrumentation and instrumentalization. This analysis allowed us to observe different mathematical behaviours among the students. Finally after conducting a cross-case analysis, we characterized different typologies of the students in the context of the tasks proposed. We will present in next section some examples.

The Case of Sara

Sara is an excellent student and she has quite a global thinking and a mathematically open-minded attitude. She has the ability to see geometrical figures and she does not tend to use algebraic resolutions. She uses graphic representations without the coordinate axes. For example, in the circle problem, Sara considered the perpendicular bisector of the segment *PQ* as the locus of points equidistant from the points *P* and *Q*. She obtained the centre of the circle as the intersection of the perpendicular bisector and the line r, which contains the centre of the circle, whereas the majority of the students would try the strategy that consists of equalising distances and solving the resultant equation. Sara worked individually and she did not interact with other students. She did not have difficulties. She stated that she would be able to visualize the problem without any graphic representation. In the GeoGebra resolution, Sara used geometric properties. She reasoned on the figure and she did not have technical difficulties with the use of GeoGebra. We consider in Sara's case that the levels of instrumentation and instrumentalization are both high.

The Case Marc and Aleix

In the resolution of the circle problem (see Figure 1), Marc and Aleix tried first the algebraic strategy that consists of obtaining the radius of the circle using distances. They wrongly used the formula for the distance from a point to a line (line r which contains the centre of the circle and point *P*) to obtain the radius (R) of the circle. They tried to solve the following system of equations:

$$\begin{cases} R = d(P,r) \\ R = d(Q,r) \end{cases}$$

Finally, they abandoned the strategy because they obtained different values for the radius. We observed that, in the resolution with GeoGebra of the rhombus problem, Marc made the same mistake (misunderstanding of the concept of distance from a point to a line).

In the GeoGebra resolution of the rhombus problem, we observed in the interactions between Marc and Aleix that Marc tried to construct a line at distance 5 from the vertex P instead of using a circle of centre *P* and radius 5. GeoGebra turns out to be a tool, which helps the students and teachers to realise geometrical/mathematical misunderstandings or misconceptions. For example, in the rhombus problem, GeoGebra shows how the concept of the distance from a point to a given line is partially understood by the student (Figures 1 and 2). In this case, the local obstacle, as defined by Drijvers (2002), is not provided by GeoGebra. Rather, it seems that it can be considered as a pre-existing cognitive obstacle that becomes manifest when students work with GeoGebra. In the end, Marc abandoned the GeoGebra distance strategy and followed the same strategy as Aleix. Aleix had no difficulties with the concept of distance from a point to a line. He considered the circle of centre *P* and radius 5 and constructed the rhombus using its geometric properties.

Marc obtained the following figure (Figure 2) and abandoned his strategy. He did not notice that the point obtained is at distance 5 from the vertex *P* (a particular property of this rhombus). We also observed that the tool *segment with given length from point* always produces a segment parallel to *x*-axis. Aleix used the tool *circle with given centre and radius* to obtain the vertices of the rhombus as the intersection points of the two circles centred at *P* and *Q*, respectively, and with a radius of 5 units (Figure 3).

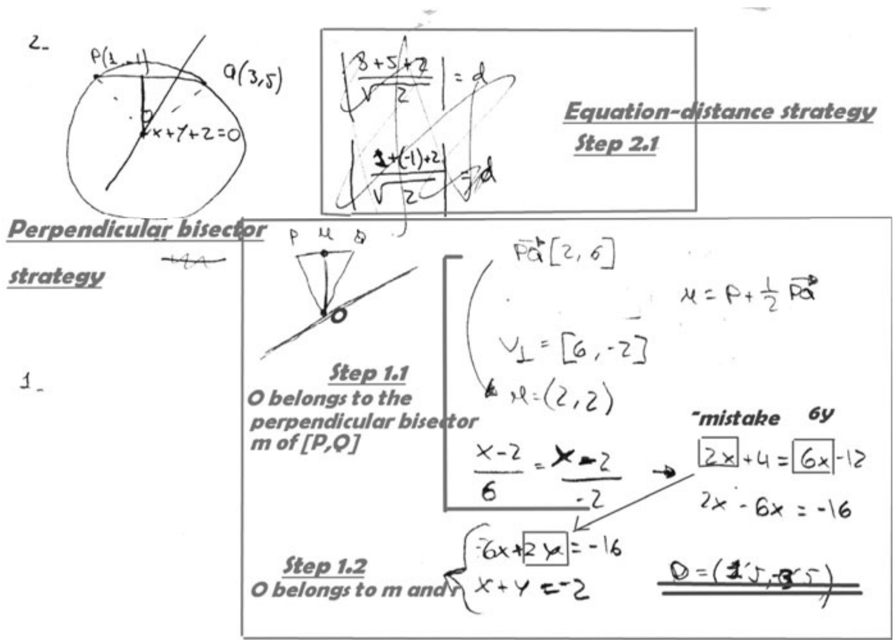

Figure 1. Paper-and-pencil resolution of the circle problem (Marc).

The Case of Julieta and Ofelia

Julieta and Ofelia are weaker students and have difficulties with the basic concepts. In the paper and pencil resolution, they had many difficulties (conceptual and visualization difficulties, algebraic obstacles) and tried algebraic strategies (based on equalising distances). We observed that they also had difficulties in solving the problems with GeoGebra. They had technical and conceptual difficulties and tended to reason on the drawing rather than the figure (their constructions did not pass the dragging test). They used few tools; they had difficulties in finding these tools in the toolbar and then applying these tools. For example, Ofelia tried to apply the tool perpendicular bisector to the points *P*, *Q* instead of considering the segment *PQ*. GeoGebra is flexible enough to help the students apply tools or make geometrical constructions that they partially understand. The level of instrumentalization and instrumentation of these students are low. Nevertheless, we observed that the use of GeoGebra helped these students visualize the problem, even if they tended to reason on the drawing and they constructed the rhombus using measuring tools. They did not use geometric properties of the rhombus for its construction. They obtained the diagonal of the rhombus but they obtained the remaining vertices with the drag tool. They dragged the vertices until they obtained a quadrilateral with sides of length 5.

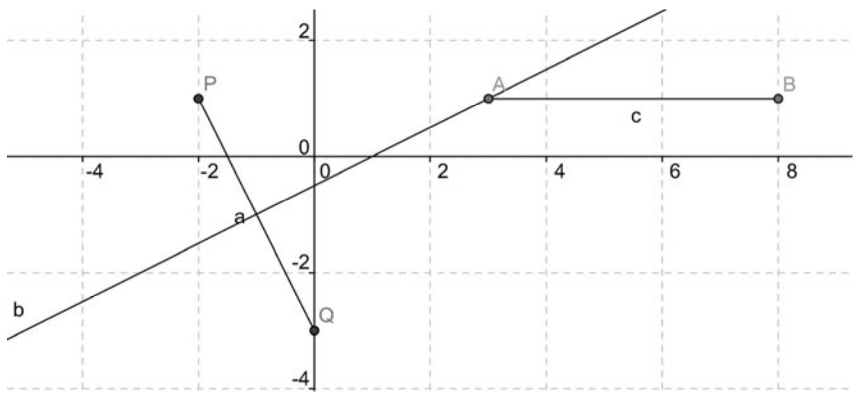

Figure 2. Resolution strategy of rhombus problem based on the wrong concept of distance from a point to a line (Marc).

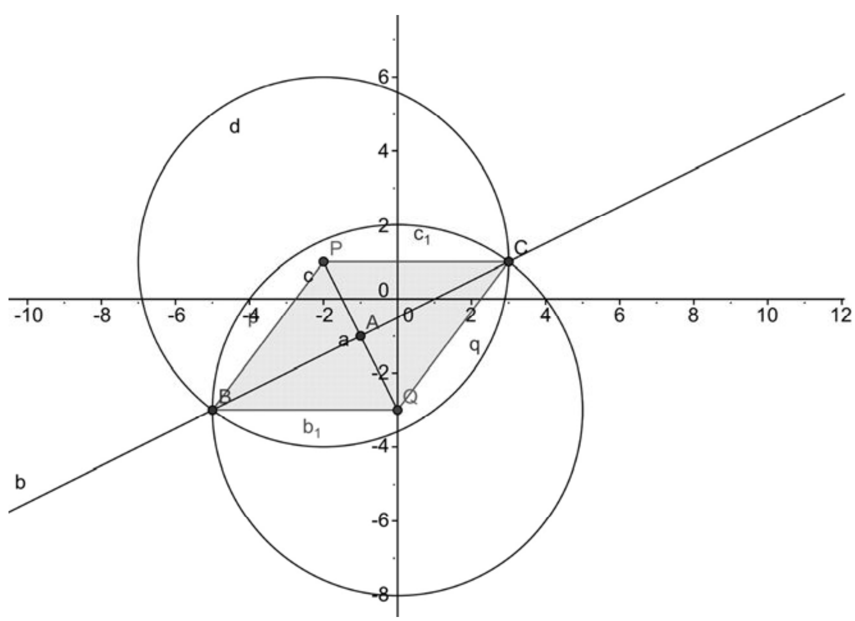

Figure 3. Resolution strategy of the rhombus problem based on the intersection of two circles (Aleix).

RESULTS

We summarize the main results of the study using the instrumental approach. Through detailed analysis of the classroom episodes, we try to characterize the influence of GeoGebra use on students' learning behaviours when solving the given tasks. We have tried to distinguish instrumentalization and instrumentation levels for the different types of students. As stated by White (2007), an artefact may become a very different instrument for one user or task than it is for another. Thus, we try to find behaviour invariants in this particular context. So far, we have obtained the following behaviours.

Autonomous (2 students)

These students (e.g., Sara) are advanced students and they have problem-solving skills. They have to *see* geometrical figures and justify their conjectures. They do not have difficulties in the visualization of this kind of problems or in algebraic calculations. Thus, the use of GeoGebra may help them in the visualization process but it does not help them understand the concepts involved in these problems; it just makes it easier. In this case, the use of GeoGebra facilitates the material aspects of the task but does not change it conceptually (Laborde, 2001). These students tend to reason on the figure and they do not have technical difficulties with GeoGebra. Both instrumentation and instrumentalization levels are high. We conjecture that the use of GeoGebra may provide a support to explore advanced curricular concepts for these students.

Instrumental (4 students)

These students (e.g., Joaquim, see Figure 4) are good students who are more analytical than intuitive in the context of the given tasks. They do not have difficulties with the use of GeoGebra and the use of GeoGebra helps these students visualize the problems. Their paper and pencil strategies are based on algebraic calculations (analytic strategies), whereas the use of GeoGebra fosters more *geometrical thinking* and they tend to reason on the *figure*. *Geometrical thinking* may be exemplified by reasoning that does not rely on a particular approach to geometry, whether it is the algebraic approach to geometry (analytic geometry) or the synthetic approach to geometry. Geometrical thinking would be independent of these approaches. On the other hand, visual thinking is a more general concept than geometrical thinking. Visual thinking may arise in virtually almost all fields of mathematics but obviously it is not understood as a geometrical concept. The instrumentation and instrumentalization levels range from medium to high.

Procedural (4 students)

These students (e.g., Marc) are quite analytical. Although they have some conceptual difficulties (e.g., distance from a point to a line), they usually understand new concepts in class. In the GeoGebra resolution, they reason both on the *figure* and on the *drawing*. The instrumentation level is higher than the instrumentalization level (understanding the affordances of GeoGebra). They (e.g., Marc and Aleix) do not hesitate in applying new GeoGebra tools (without effort), but they do not seem to elaborate their own schemes in their interaction with the software (appropriation).

Naive (2 students)

These students (e.g., Julieta and Ofelia) are weaker students and they have more mathematical difficulties. In the paper and pencil resolution, they encounter algebraic obstacles, visualization difficulties, and conceptual difficulties (basic elements of the triangle, vectors, and distances). The use of GeoGebra helps these students visualize the statement of the problem but they have many conceptual and technical difficulties. They use few tools (measure tools), for example. They have difficulties in finding and applying the tools, and they reason on the *drawing*. The instrumentation and instrumentalization levels are both low. Their constructions do not pass the dragging test.

FURTHER RESEARCH ON THE STUDENTS' BEHAVIOUR

In this section, we discuss some observations that we have made in this study. Some students use different strategies in the two environments. For example, they use the tool *drag* to obtain segments of a given length. The GeoGebra dynamic resolution strategy for the rhombus problem (Figure 4), which is based on obtaining the second diagonal of the rhombus and the remaining vertices A and A' (mirrored point of A at the other diagonal *r*), does not have a clear transfer to a paper and pencil resolution. The student (Joaquim) obtained the rhombus by dragging the vertex A along the diagonal until one of the segments measures 5 units. This construction passes the dragging test.

The dragging option influences the resolution strategy. We have observed different resolution strategies for the rhombus problem. Sara's construction (Autonomous) has a clear transfer to a paper and pencil resolution. Nevertheless, we conjecture that in the paper and pencil resolution, Sara would have used distances instead of circle intersection or she would have followed a vectorial solution by applying the Pythagorean Theorem to obtain the norm of the vector. In this case, the use of GeoGebra fosters a more *geometrical thinking*.

In general, the students do not have difficulties using GeoGebra (except for the naive type). Although autonomous students do not have visualization difficulties, it seems that GeoGebra helps them visualize the problem as in Sara's case. A relevant fact is that all the students used graphic representations without the coordinate axes in the paper and pencil resolutions. This fact is due to the influence of their teacher. We also observe that some obstacles are not due to the software. Rather, it seems that they can be considered as pre-existing cognitive obstacles that become manifest by working in the dynamic geometry environment (Drijvers, 2002). For example, the obstacle encountered by Marc with GeoGebra is a pre-existing cognitive obstacle.

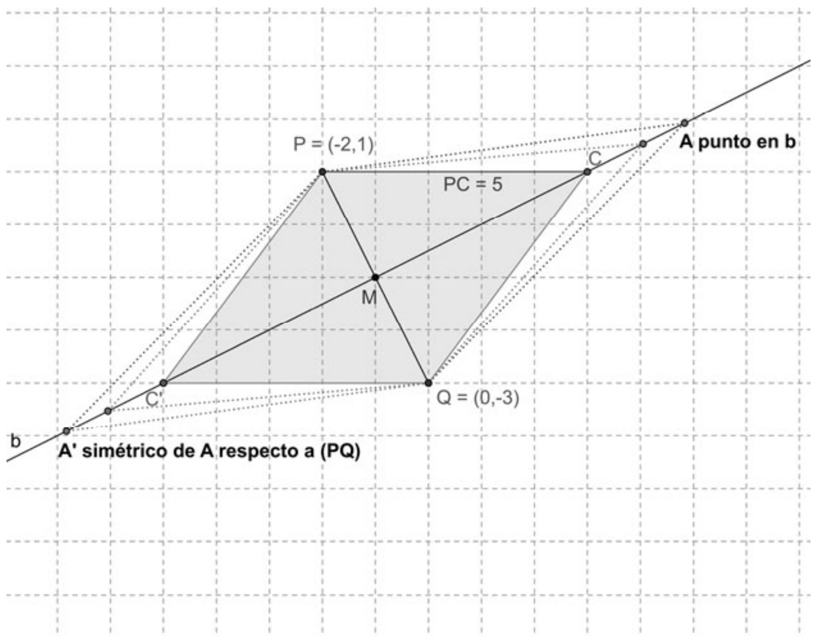

Figure 4. Rhombus construction, dynamic strategy (Joaquim).

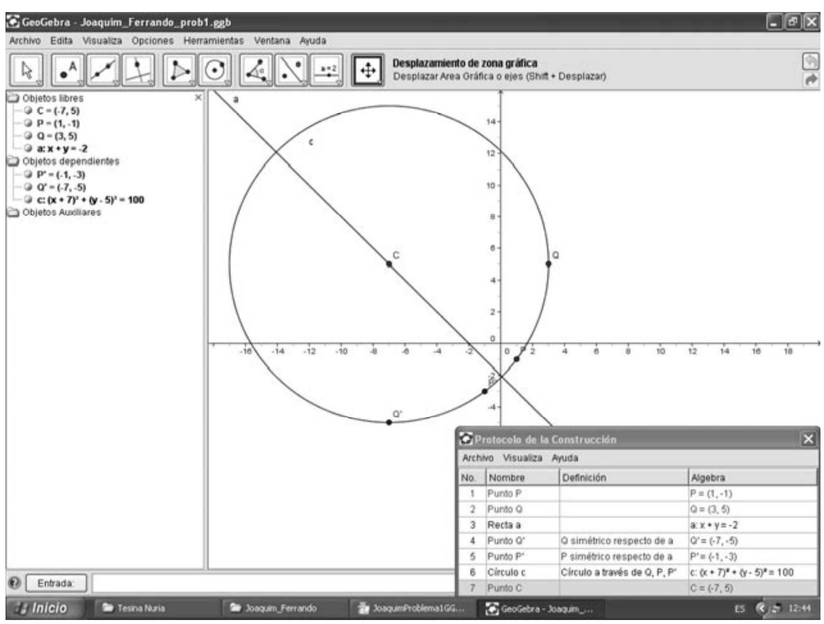

Figure 5. Circle through three points strategy (Joaquim).

As observed before, the transfer of GeoGebra strategies to paper and pencil strategies is not always clear. For example, in the resolution of the circle problem with GeoGebra, one of the students (Joaquim) constructed a third point of the circle *P'* = S_r *(P)* where S_r *(P)*, for a given point P, stands for the reflection of a point with respect a line r. The centre of the circle belongs to the line r; thus P' is a point of the circle if P belongs to the circle. Then, he used the tool *circle through three points* to construct the circle (Figure 5). Finally, he obtained the centre of the circle and the radius by interpreting the circle's equation that appears in the algebraic window in the form of $(x-a)^2 + (y-b)^2 = R^2$. The student observed that he could also obtain the centre of the circle as the intersection point of the perpendicular bisectors of the triangle *QPP'*.

CONCLUSIONS

The majority of students used algebraic and measuring tools in our study and it seems that the use of GeoGebra helped them visualize the problems and avoid algebraic obstacles (such as the equation distance strategy, Figure 1). In general, the students had few difficulties in relation to the use of the software and, as stated before, some obstacles are pre-existing cognitive obstacles that become manifest with the use of GeoGebra. We have also observed that the use of GeoGebra fosters a more geometrical thinking in the context of the given tasks. For example, they considered the intersection of circles instead of equalising distances in the rhombus problem. GeoGebra provided a visual support, algebraic support, and conceptual support for the majority of the students (instrumental, procedural, and naive), as shown in the case of the rhombus problem, where Joaquim realised that there is a symmetry and he obtained the vertices by constructing a point A on the line r and its reflection Sr(A). Then he dragged the vertex A until the length of one side of the rhombus was five units. We found that the use of GeoGebra helped the students construct multiple representations of geometrical concepts and helped them avoid algebraic obstacles to focus on geometrical understanding. Moreover, the use of GeoGebra influenced students' resolution strategies, but this influence depended on the given tasks and the type of students.

It is necessary to carry out further research to better understand the process of appropriation of the software and analyse the co-emergence of machine and paper-and-pencil techniques in order to promote argumentation abilities in secondary school geometry. It is also important to analyse the role of the teacher in future research on the teaching and learning of geometry when using dynamic geometry software.

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7. GEOGEBRA AS A CONCEPTUAL TOOL FOR MODELING REAL WORLD PROBLEMS

This chapter argues for a future-oriented approach to mathematical problem solving, one that draws on the investigation of real world modeling activities with the aid of GeoGebra, a conceptual tool that integrates dynamic geometry, algebra, and spreadsheet features. We give consideration to the Models and Modeling Perspective (MMP) as a means for introducing modeling activities, and we address how GeoGebra enriches students' exploration and understanding when solving such problems. We report on a study in which one class of fourteen-year-olds worked on two modeling activities. Results showed that students developed a number of sophisticated models for adequately solving the problems. Results further revealed that GeoGebra's environment assisted students in broadening their explorations and visualization skills, modeling the real world problems, and making connections between the real world and the mathematical world. Finally, recommendations for implementing technology-based modeling activities in elementary school mathematics are discussed.

INTRODUCTION AND THEORETICAL FRAMEWORK

In the current information age, students face a demanding knowledge-based economy and workplace, in which they need to deal effectively with complex, dynamic and powerful systems of information and be adept with technological tools (Lesh & Zawojewski, 2007). The need to develop students' abilities to successfully use technological tools in dealing with complex problem solving for success beyond school has been emphasized by a number of professional organizations (National Research Council [NRC], 2001; National Council of Teachers of Mathematics [NCTM], 2000).

An appropriate medium for achieving this goal for students is mathematical modeling, a process that describes real-world situations in mathematical terms in order to gain additional understanding or predict the behaviour of these situations (Lesh & Doerr, 2003; Mousoulides & English, 2008). Using the models and modeling perspective, students have opportunities to create, apply, and adopt mathematical and scientific models in interpreting, explaining and predicting the behaviour of real-world based problems.

Mathematical models and modeling have been defined variously in the literature (e.g., Blum & Niss, 1991; Greer, 1997). We adopt the perspective that models are "systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behaviour of some other familiar system"

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 105–118.

(Doerr & English, 2003, p.112). The cyclic process of modeling includes the following steps: a problem situation is interpreted; initial ideas (initial models, designs) for solving the problem are called on; a fruitful idea is selected and expressed in a testable form; the idea is tested and resultant information is analysed and used to revise (or reject) the idea; the revised (or a new) idea is expressed in a testable form; and subsequent extensions. The cyclic process is repeated until the idea (model or design) meets the constraints specified by the problem (Zawojewski, Hjalmarson, Bowman, & Lesh, 2008).

In adopting the models and modeling approach, real-world-based situations are presented to students. In modeling activities, students are presented with complex real-world problems that involve model development and in which students repeatedly express, test, and refine or revise their current ways of thinking as they endeavour to create models that provide significant solutions that comprise core ideas and processes that can be used in structurally similar problems (Lesh & Doerr, 2003). In developing their models, students normally undergo a cyclic process of interpreting the problem information, selecting relevant quantities, identifying operations and variables, and creating meaningful representations (Lesh & Doerr, 2003).

Model Eliciting Activities (MEA) engage students in mathematical thinking that extends beyond the traditional curriculum (Mousoulides & English, 2008). In contrast to typical classroom mathematics problems that present the key mathematical ideas "up front" and students select an appropriate solution strategy to produce a single, usually brief, response, modeling problems embed the important mathematical constructs and relationships within the problem context and students elicit these as they work on the problem. The problems necessitate the use of important, yet underrepresented, mathematical processes such as constructing, describing, explaining, predicting, and representing, together with quantifying, coordinating, and organizing data (Mousoulides, 2007). Furthermore, the problems may allow for various approaches to solution and can be solved at different levels of sophistication, enabling all students to have access to the important mathematical content (Doerr & English, 2003; English, 2006).

In sum, from a models and modeling perspective, these modeling activities are realistically complex problems where the students engage in mathematical and scientific thinking beyond the usual school experience and where the products to be generated often include complex artifacts or conceptual tools that are needed for some purpose or to accomplish some goal (Lesh & Zawojewski, 2007). MEAs present a future-oriented approach to learning, where students are given opportunities to elicit their own mathematical and scientific ideas as they interpret the problem and work towards its solution (Mousoulides, Christou, & Sriraman, 2008).

Recent studies reported that the availability of technological tools can influence students' explorations, model development, and therefore improve students' mathematical understandings in working with modeling activities (Lesh et al., 2007; Mousoulides et al., 2008). Further, these studies showed that the use of appropriate tools can enhance students' work and therefore result in better models and solutions.

In Blomhøj's (1993) research, a group of 14-year-old students were engaged in modeling activities with a specially designed spreadsheet. He reported that students did not find the spreadsheet was a barrier when they were setting up a model. Instead, they often expressed a given relation between variables in the model more easily in spreadsheet notation than in words. In line with previous findings, Christou, Mousoulides, Pittalis and Pitta-Pantazi (2004) reported that students, using a dynamic geometry package, modelled and mathematized a real-world problem, and utilized the dragging features of the software for verifying and documenting their results. Similarly, Mousoulides and colleagues (2007) reported that students' work with a spatial geometry software broadened students' explorations and visualization skills through the process of constructing visual images, and these explorations assisted students in reaching models and solutions that they could not probably do without using the software. In summary, researchers reported that the inclusion of appropriate software in modeling activities could provide a pathway in understanding how students approach a real-world mathematical task and how their conceptual understanding develops.

The present chapter builds on, and extends previous research by examining how GeoGebra, a conceptual dynamic tool for geometry and algebra, can provide a pathway in better understanding how students approach and solve a real-world problem, how software's features and capabilities influence students' explorations and model development, and how students interact with the software in developing technology-based solutions for these problems.

THE PRESENT STUDY

In this chapter we report on a study in which one class of fourteen-year-old students used GeoGebra in creating several different models for solving two engineering modeling problems. The first modeling activity, *Building an Airport*, requires students to develop models for finding the optimal location for building a new airport. Specifically, the problem assigned to students was the following:

> *Problem 1:* The authorities of four towns are planning to build an airport that will serve the needs of their citizens. Identify the optimal place for the airport location so that the needs of the four towns are served in a fair way. Send a letter to the Ministry of Transportation explaining and documenting your solution.

The problem was open-ended and purposefully not well defined. Thus, students had to provide necessary hypotheses in order to clarify the problem situation.

The second modeling activity, *Solar Power Car*, requires students to develop quadratic function models for finding the best selling price for a solar powered car. The second problem that was assigned to students was:

> *Problem 2:* A car making company is launching a new solar powered car. Recent market research showed that one hundred people would buy the car for a selling price of €5000. Further, the market research showed that for every €100 price increase, people's interest in buying the car would decrease

> by one person. Find the best selling price for the car, as to maximize company's sales revenue. Send a letter explaining how you solved the problem to the company's sales manager.

PARTICIPANTS AND PROCEDURES

One class of 21 fourteen-year-olds and their mathematics teacher worked on the two modeling problems as part of a two-year longitudinal study, which focuses on exploring students' development of models and processes and students' interaction with technological tools in working with engineering modeling problems. The students are from a public K-12 middle school in the urban area of a major city in Cyprus. The students had only met such modeling problems before during their participation in the current project, as the mathematics curriculum in Cyprus rarely includes any modeling activities. Students were quite familiar with using GeoGebra, since the software was frequently used in the aforementioned project.

The data reported here are drawn from the problem activities the students completed during the first year of the project. The *Building an Airport* and *Solar Power Car* modeling problems entail: (a) a warm-up task comprising a story or an article, designed to familiarize the students with the context of the modeling activity, (b) "readiness" questions to be answered about the article, and (c) the problem to be solved.

The problems were implemented by the author, two postgraduate students, and the classroom teacher. Working in groups of three to four, students spent four 40-minute sessions on each modeling activity. During the first session the students worked on the newspaper article and the readiness questions. In the next three sessions the students developed their models, and wrote letters to local authorities (for the airport problem) and to the company's sales manager (for the solar power car problem), explaining and documenting their models/solutions, and presented their work to the class for questioning and constructive feedback. A class discussion followed that focused on the key mathematical ideas and the GeoGebra constructions students had generated.

DATA SOURCES AND ANALYSIS

The data were collected through audio- and video-tapes of the students' responses to the modeling activity, together with the GeoGebra files, student worksheets and researchers' field notes. Data were analysed using interpretative techniques (Miles & Huberman, 1994) to identify developments in the model creations with respect to the ways in which the students: (a) interpreted and understood the problem, and (b) used and interacted with the software capabilities and features in solving the engineering problems. In the next section we summarize the model creations of the student groups in solving the *Building an Airport* and *Solar Power Car* activities.

RESULTS

Building an Airport

Three out of the six groups of students who worked on the modeling activity succeeded in developing appropriate models for solving the problem. The three different models are presented in the next section.

Airport Model A. Students in this group spend a considerable amount of time discussing the population of the four cities and the landscape of the region. They decided to simplify the problem by considering that the four cities had the same population. Following this discussion, students decided to consider "fairness" as "equidistance". Moving to the software, students modeled the problem by constructing a rectangle, assuming that the four cities were the vertices of the rectangle. The next extract presents part of their discussion with the researcher:

– Student A: If the vertices of the rectangle represent the four towns, then the best location for building the airport is the centre of the rectangle.
  – Researcher: What do you mean by the "centre" of the rectangle?
  – Student A: The intersection of diagonals.
  – Researcher: Ok. How can you check your hypothesis, that the intersection is the best location?
  – Student B: Since a diagonal is the shortest distance between the two opposite vertices, then the intersection of the diagonals is the shortest distance for all four vertices.
  – Student A: (He points to their construction on the software). We defined a point and then constructed segments from this point to the four vertices (See Figure 1). By dragging the point around the rectangle, we can see that the optimal location of the airport is the intersection of the diagonals.
– Researcher: So, how do you define the optimal location?
– Student B: It is the place where all four distances are equal.

Students considered the best possible location to be the point equidistant to each town. This approach was limited, as students did not consider any other shapes than the rectangle and the square. Students employed the same approach in solving the problem where the four towns were vertices of a square. In that case, students constructed a square in GeoGebra and by constructing its diagonals and a freely moving point; they showed that the intersection of diagonals was also the best possible place for building the airport.

Airport Model B. This group's work was quite similar to the work of Model A group. This group also successfully used software's capabilities for constructing a rectangle and considering the four towns as the vertices of the rectangle. Students then constructed the diagonals and then calculated the distances between the intersection of diagonals and the four vertices. Students documented in their worksheets that the intersection of diagonals was the best possible place for the airport, since this place was equidistance from all four towns.

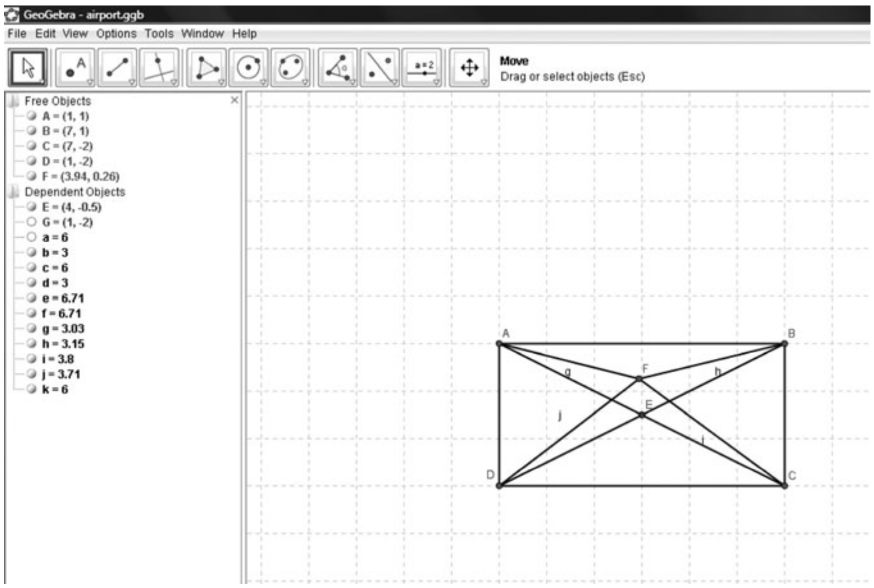


Figure 1. Four towns as vertices of a rectangle.

In addition to the arguments presented by the group who produced Model A, one student in the Model B group encouraged others to work with a circle. In this case, students conjectured that the four points representing the towns should be points on the circumference of a circle. Students moved the construction in GeoGebra, constructing a circle and placing four points on the circumference (see Figure 2a). The next extract shows how GeoGebra assisted students' explorations and how it helped students to refine their reasoning.

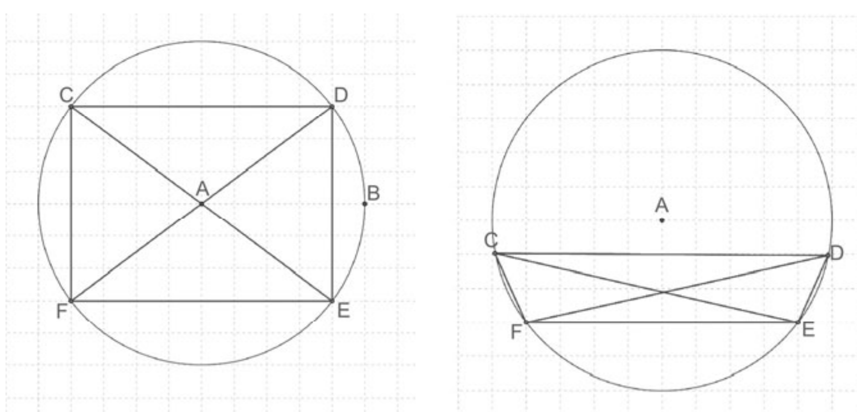


Figure 2. Towns as points on a circle circumference.

– Student A: The best location for the airport should be the centre of the circle, since the centre is equidistant from all four points on the circumference of the circle.
– Student B: Centre is not only the best location in this case, but it is also the best place for any other town, since it is equidistant from all any point on the circumference of the circle.

At this stage, students measured all four distances from centre to the four points and showed that all distances were equal. At this point, the researcher asked them to move some of the "towns" in a way that the centre would not be inside the quadrilateral.

– Researcher: Drag one or two of the vertices of your figure. A little bit more. Well, move the towns in a way that all four are at the bottom of the circle.
– Student B: Oh! The centre of the circle (see Figure 2b) is outside.
– Researcher: Do you think that the centre of the circle is still the best location?
– Student B: Well, the centre is still equidistance from all four towns.
– Student C: No, in this case, equidistance is not good enough, it is just far away from all towns, but it is definitely not the best place.

Students then constructed the diagonals of the quadrilateral and measured the four distances. At this stage, one of the three students reported that the best place would be again (similar to rectangle) the intersection of the diagonals. When the researcher challenged them to somehow prove it, students constructed another point inside the polygon (see Figure 3) and by measuring the distances of that point from the four towns; they concluded that the best possible location (i.e. the minimum total) was the intersection of the diagonals.

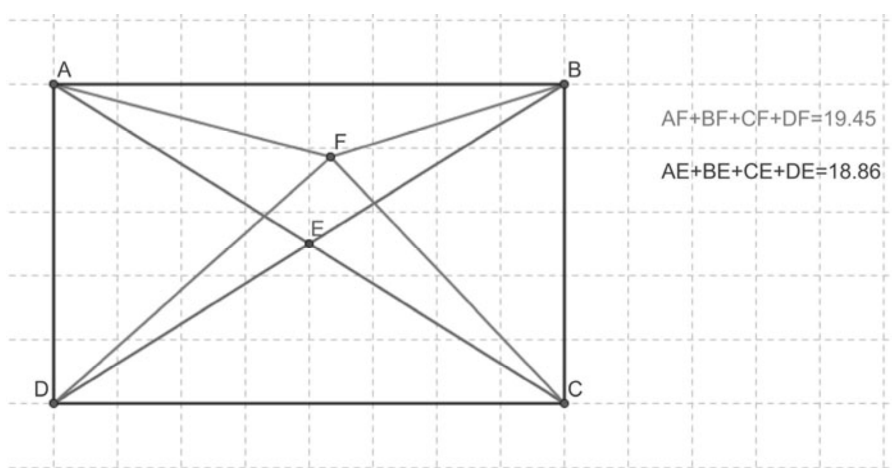


Figure 3. Finding the best place for building the airport.

Airport Model C. Quite different from the other two groups, these students commenced the problem by placing four freely moving points on GeoGebra's screen and by constructing the quadrilateral defined by the four points. Some students constructed the two diagonals of the quadrilateral and their point of intersection and labelled it H. Since students were not sure that the intersection of diagonals was the optimal point, they also constructed a second point G (freely moving) and constructed segments between point G and the vertices of the quadrilateral. Students next moved point G around other possible locations and observed that the point H seemed to be the optimal point. They realized that the diagonals' intersection point H was the point for which the sum of its distances from each of the four vertices was the minimum possible.

During the discussion with the researcher as to whether this example could be generalized for all "types" of quadrilaterals, students considered one example with a triangle (three out of the four points were collinear). Surprisingly to the students, the intersection of diagonals was one of the four points. The next extract shows students reactions and their next steps in solving the problem.

- Researcher: What do you observe?
- Student A: The diagonals do not intersect.
- Student B: They intersect. Only, the point of intersection is point B, one of the four vertices of the shape.
- Researcher: Does it mean that your previous conclusion is not correct? Student B: Why do not we use the same approach? See, our conclusion is still valid. We only need to make explicit that the airport could be in one of the four cities.

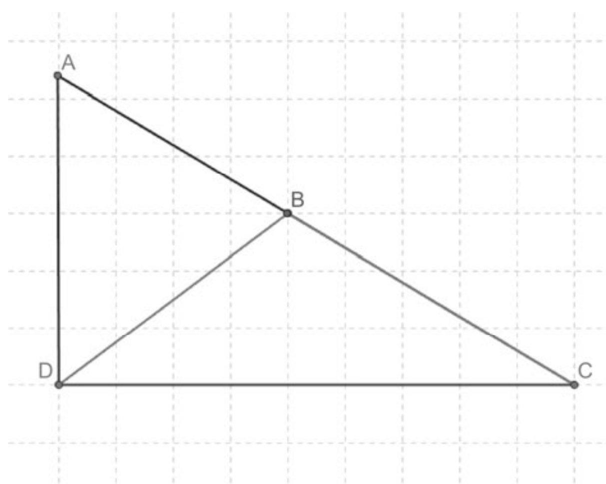


Figure 4. Transforming the quadrilateral into a triangle.

As presented in the above extracts, the environment of GeoGebra provided students with opportunities to investigate the location of airport, by exploring different cases of the original problem; a crucial characteristic in modeling real world problems. The above extract also addressed the conflicts that usually arise from the exploration of the possible extensions in the assigned problems. Surprisingly, students discovered that the optimal point coincided with point B. Students concluded that the location of the optimal point depends on the type of quadrilateral (which in some cases is a triangle).

Solar Power Car

The *Solar Power Car* problem asked students to find the optimal price for selling a solar power car. As presented above, market research showed that one hundred people expressed their interest in buying the car at the price of € 5000. The company could also increase car price and offer more extras. However, any increase in car's price will have an impact on buyers' interest. Specifically, each extra would cost €100 and will decrease interest by one person. Three groups of students successfully developed models using GeoGebra for solving the problems. Their models are presented in the next section.

Car Model A. The first group commenced the question for finding the optimal price for the solar power car by brainstorming different factors that could have an impact on defining a car's price. Students also had long discussions about how they should start working on the problem and how they could use the software. One of the students pointed out that the price of an individual car is not that important, but the total amount of money the company will get is rather more important. This remark assisted students in understanding that they should find a method to calculate the total amount of money the car company would get from selling the solar power car. At this point, students moved into their GeoGebra construction, calculated (using the GeoGebra spreadsheet) the total amount of money and inserted the new data into the available spreadsheet (see Figure 5).

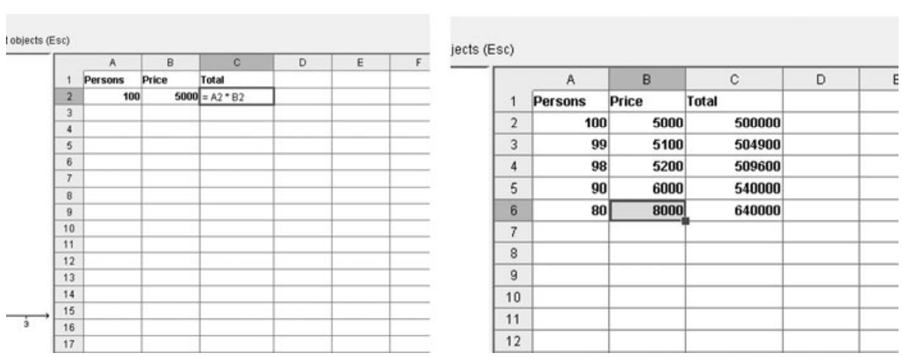


Figure 5. Group A students' spreadsheet.

Students quite easily used the formula *persons*price* (see Figure 5) for calculating the total amount of money. However, students did not use any formula for calculating a new price and the number of persons that would be interested in buying the car. They “manually” enter the new data, instead of subtracting one person for each 100-euro increase, and as a consequence, they faced some difficulties in performing their calculations. For instance, in calculating the new price for the car they made some mistakes, as can be seen in Figure 5b. These difficulties were overcome with researcher’s constructive feedback. Students finally succeeded in making the correct and necessary calculations and reached the optimal price for selling the car. In short, students took advantage of the spreadsheet capabilities of the GeoGebra, but they did not use GeoGebra as a dynamic geometry software for refining or extending their solution.

Car Model B. Similar to the work presented in Car Model A, students in this group reported that the question of the problem could be rephrased into finding the maximum profit the company would get. For the students in this group this decision was not, however, straightforward. On the contrary, students had long debates in understanding the core question of the problem. One student, for instance, reported that the problem was too easy; “the optimal price for selling a car is the starting price, since any change will have a negative impact on buyers’ interest”. As soon as they all agreed that €5000 (starting price) was not the best possible price (by performing some calculations for increased prices), they made use of the software’s spreadsheet capabilities in a similar way to group A; they calculated the total amount of money by multiplying the car price by the number of people. Compared to students’ work in Car Model A, students in this group moved a step further; they calculated new car price and new number of people interested in buying the car, by systematically adding €100 and by subtracting one person respectively, using the appropriate formulas (see Figure 6).

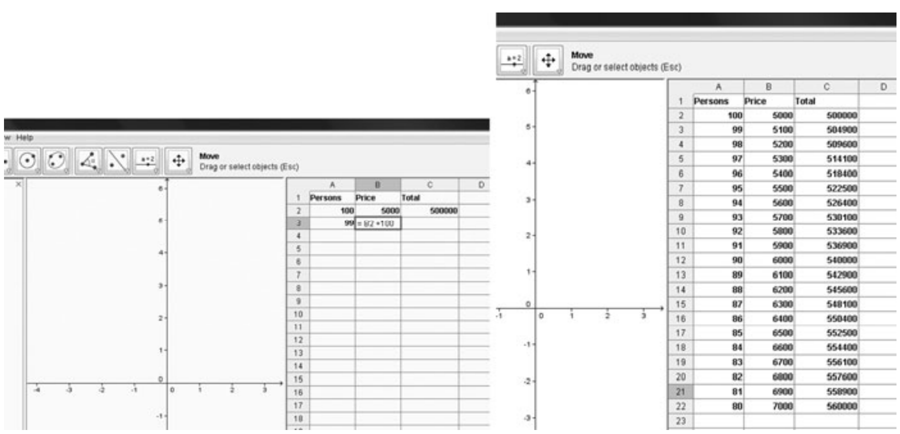
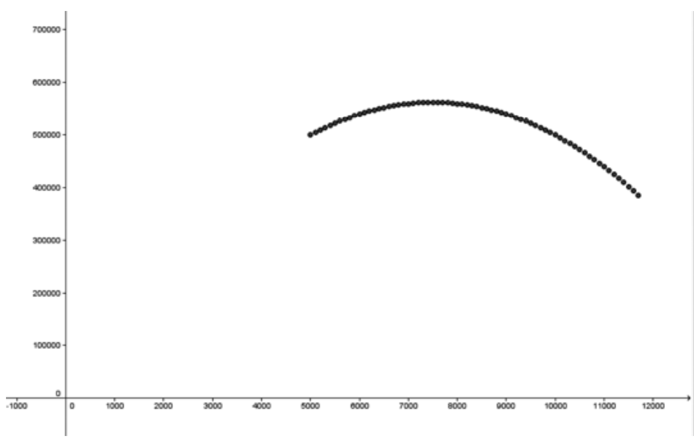


Figure 6. Group B students’ spreadsheet formulae and calculation.

Although students algebraically reached a solution to the problem, by finding that the optimal price was €7500, they also plotted a list of points, in an attempt to generalize their results and to further explore the relation between car price and company's profits.

Students listed the points defined by car price and total amount of money (see Figure 7). Although students did not face any difficulties in creating a list and its associated graph, they failed to explicitly discuss their findings; they enjoyed the fact that the software could handle both an algebraic (spreadsheet) and a graphical (geometry) representation of their solution. Students reported that the graph showed not only the optimal price, but also proved (according to their reports) that no other price could be better, since (7500, 562500) was the maximum point of the curve. In concluding, this group successfully used spreadsheet capabilities for finding the total amount of money, for calculating new price and new number of persons interested in buying the car. However, in plotting the points and in commenting on the shape of the graph, students failed to explicitly identify and discuss the graphs' shape and then to document why the specific car price was the optimal one. They also failed to document how their approach could be used in solving similar problems.



7	95	5500	522500
8	94	5600	526400
9	93	5700	530100
10	92	5800	533600
11	91	5900	536900
12	90	6000	540000
13	89	6100	542900
14	88	6200	545600
15	87	6300	548100
16	86	6400	550400
17	85	6500	552500
18	84	6600	554400
19	83	6700	556100
20	82	6800	557600
21	81	6900	558900
22	80	7000	560000
23	79	7100	560900
24	78	7200	561600
25	77	7300	562100
26	76	7400	562400
27	75	7500	562500
28	74	7600	562400
29	73	7700	562100
30	72	7800	561600
31	71	7900	560900
32	70	8000	560000
33	69	8100	558900

Figure 7. Student calculations and points plotting.

Car Model C. The model presented by the third group was far more sophisticated than Model A and Model B. Students in this group effectively used most of GeoGebra's functionalities and tools. They used spreadsheet capabilities in using formulae for calculating total amounts of money, then dynamic geometry features for plotting points, and constructing various quadratic functions for discovering the relation between the car price and the total amount of money. Since the first two parts of their activity were quite similar to what had been presented earlier as part of Model B work, only the third part of students' activity, testing and validating various different curves for fitting the given points, is presented here.

Given that students rarely encounter similar activities in their textbooks, students had a number of difficulties in finding the relation between the car price and the total amount of money. Initially, students used formulae like $f(\mathrm{x}) = ax^2$, $a > 0$. These first unsuccessful attempts helped them realize that the parameter a in the requested function should be $a < 0$, and that more parameters were needed. Further, since they were used to working with small numbers, they started their next round of explorations using small numbers for parameters b and c. Students ended by formulating a general function $f(\mathrm{x}) = ax^2 + bx + c$. The fact that students could not make even rough estimations of the parameters' values encouraged students to use a simple yet very powerful tool of GeoGebra; sliders. The group C students' final workbook screen is presented in Figure 8.

Students spent a lot of time discussing, exploring and using trial and error to reach the correct function that represents the relation between car price and total amount of money. It was obvious that the software's features and capabilities helped them in performing actions, setting hypotheses, and investigating relations that they could not do with paper and pencil. During the last discussion with the researcher, students asked whether there was an easier way for finding the quadratic function, since they already knew (among other points) the coordinates of the vertex. The researcher encouraged them to search the Web and students easily found that the function could also be of the form $g(\mathrm{x}) = a(x - h)^2 + k$, where h and k the coordinates of the vertex (in this example, $h = 7500$ and $k = 562500$). Students quite easily constructed the graph of the function $g(x)$ and they observed that the function representing the relation between car price and company's profits could be represented in the forms of $f(\mathrm{x}) = ax^2 + bx + c$ and $g(\mathrm{x}) = a(x - h)^2 + k$.

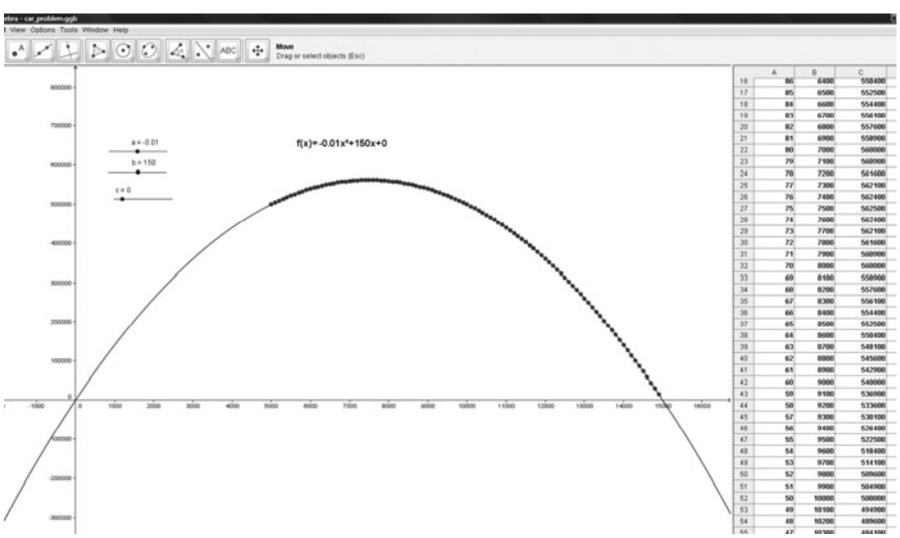


Figure 8. Function plotting and points.

DISCUSSION

Computer-based learning environments for mathematical modeling, at the school level, are an appealing notion in mathematics education. Based on the findings of recent research reports, we can claim that the capabilities of dynamic tools, such as GeoGebra, can positively influence students' explorations and understandings in complex problem solving, which is at the core of mathematics education.

There are a number of aspects of the studies presented here that have particular significance for the use of modeling activities and technological tools, like GeoGebra in school mathematics. Results showed that students were able to successfully work with mathematical modeling activities when presented as meaningful, real-world situations and when appropriate technological tools were available. Students progressed through a number of modeling cycles, from focusing on subsets of information to applying mathematical operations in dealing with the data sets, and finally, identifying some trends and relationships. In doing so, students successfully employed a number of Geogebra tools and capabilities. There was evidence that GeoGebra's features and capabilities assisted students in modeling the real problems, and in making connections between the real world and the mathematical world. Further, the software assisted students in familiarizing themselves with the problem and in broadening their explorations and visualization skills through the process of constructing visual images to analyse the problem, taking into account their informal and visual conceptions.

An interesting aspect of this study lies in the students' engagement in self evaluation, through the use of software's features and tools: groups were constantly questioning the validity of their solutions, and wondering about the representativeness of their models. This helped them progress from focusing on partial data to generalizing their solutions and identifying trends and relationships in creating better models. Although only a few students progressed to more advanced models in both problems, they nevertheless displayed surprising sophistication in their mathematical thinking. The students' developments took place in the absence of any formal instruction and without any direct input from the classroom teacher during the working of the problem. We can conclude that the conjunction of GeoGebra's environment and the framework of modeling activities enhanced students' models and solutions, and allowed researchers to gain insights into students' mathematical understandings.

In preparing students for being successful mathematical problem solvers, both for school mathematics as well as beyond school, rich problem solving experiences starting from the elementary school and continuing to secondary school needs to be implemented and appropriate technological tools like GeoGebra need to be effectively used in solving these real-world based problems. Results from research work like the studies presented here can provide both teachers and curriculum designers with details on how technology-based modeling activities can assist students in accessing higher order mathematical understandings and processes. Further research towards the investigation of software's role is needed in promoting students' conceptual understandings and mathematical developments in working with modeling activities.

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8. MODELING THE CUBE USING GEOGEBRA

This chapter presents the context, main concepts, and difficulties involved in the construction of a GeoGebra model for a 3D-linkage representing a flexible cube: a cubic framework made up with bars of length one and spherical joints in the vertices. We intend to show how this seemingly easy task requires the deep coordination of (dynamic) GEOmetry and (computational) alGEBRA, that is, of the specific features of GeoGebra. Finally, the chapter highlights the excellent opportunities to do mathematics when one attempts to solve the many different challenges that arise in the construction process.

We see great value in making physical
models as mathematical experiment . . .
(Bryant, 2008)

INTRODUCTION

This chapter is about the didactical and mathematical values behind the attempts to build a GeoGebra model for a 3D-linkage representing a flexible cube, which is a cubic framework made up with bars of length one and spherical joints in the vertices. Figure 1 displays two models of the cube: one made with GeoGebra and the other with Geomag[1].

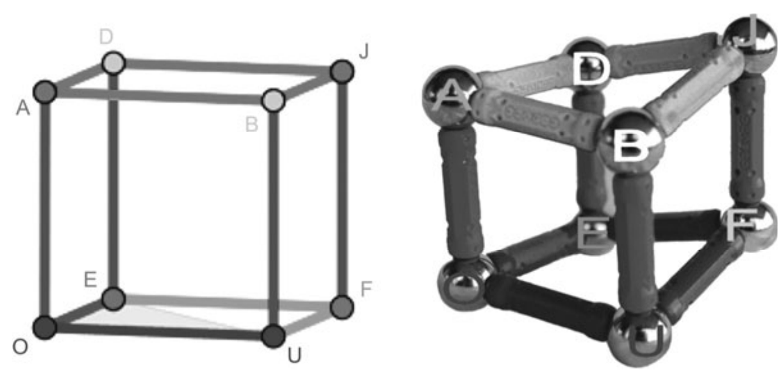

Figure 1. The cube.

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 119–131.

The importance of making physical models of geometric objects has been widely emphasized (Polo-Blanco, 2007); likewise, we would like to highlight the relevant opportunities that modeling with GeoGebra brings for doing and learning mathematics.

The next section provides arguments in this direction and introduces the context, main concepts, and issues involved in our experiment. Then, a detailed description of the modeling process (and its justification) is provided in a new section. We would like to discuss the conjunction of GEOmetry and (computational) alGEBRA that is involved in this process. We end this Chapter by proposing further activities and gathering some conclusions.

LINKAGES, DYNAMIC GEOMETRY, AND GEOMETRY LEARNING

Linkages

Linkages and mathematics have been, for centuries, closely related topics. A lively account of some issues on this historical relation appears in the recent and wonderful book by Bryant and Sangwin (2008). Drawing curves (even simple straight lines) with the help of mechanisms is an intriguing topic in which linkages and mathematics meet since the 18th century. We refer to Kapovich and Millson (2002) for a modern treatment of these problems, including the proof of a statement conjectured by the Fields medalist William Thurston on the universality of linkages: "Let *M* be a smooth compact manifold. Then there is a linkage *L* whose moduli space is diffeomorphic to a disjoint union of a number of copies of *M*". It is perhaps remarkable to notice that some work by the Nobel Prize recipient John Nash, is involved in this proof.

As complementary information, a visit to some web pages, such as those of the *Kinematics Models for Design* Digital Library (KMODDL)[2], at Cornell University or to the *Theatrum Machinarum*[3] of the Universita di Modena, is highly recommended.

Another, but closely related, issue of common interest for mathematicians and engineers is the study of the rigidity (and flexibility) of bar-joint frameworks. As stated in the introduction, in this chapter we will deal with a cube consisting of twelve inextendible, incompressible rods of, say, length one, but freely pivoting at each of the eight vertices. More generally, we could consider other polyhedral frameworks. An important topic is, then, to decide when the given framework has some internal degrees of freedom (i.e., if it has more possible positions than those that are standard for all rigid bodies in R^3, or in R^2 if we are thinking of planar frameworks).

Famous mathematicians, such as Euler or Cauchy, have worked on diverse versions of this problem, and some conjectures in this context have only been settled in recent times such as Robert Connelly's counterexample to the impossibility of constructing flexible polyhedral surfaces with rigid faces. See Roth (1981) for a readable account of this very active field of mathematical research, with applications, for instance, to the design of biomolecules.

Modeling a polyhedral cube as a bar-joint linkage allows us to experiment with these kinds of questions. First of all, if we have in our hands a physical model of a cube framework, it is evident that we can place it around in many different positions, without changing the distances between any pair of its (contiguous or not) vertices. This fact is common to all bodies in three-dimensional space and it is not difficult to verify that there are six parameters governing such displacements, since we can choose an arbitrary position (given by three coordinates in physical space) for one point *O* on the body, and then we can rotate the body as a whole around this point, with such rotation depending on the so-called three (Euler) angles. Thus we say that all bodies, even rigid ones, enjoy six degrees of freedom in R^3.

Since we are mainly interested in the possible "internal" displacements of the cube (those that change the relative position between some vertices, without breaking the linkage), we would like to discount, once and for all, those six "external" degrees of freedom. Thus, let us assume, as a convention, that we have fixed two contiguous vertices (vertices *O* and *U* in Figure 1) and that, moreover, vertex *E* is only allowed to move restricted to a certain plane (for instance, the horizontal plane containing *O* and *U*). In this way we are taking care of six displacement parameters: three for fixing vertex *O*, two for fixing vertex *U* (since it is constrained to be on a sphere of center *O* and radius *1*) and one for restricting *E* to be in the intersection of a sphere of center *O* and radius *1* and in the horizontal plane.

Still, is it possible to move the cube respecting this convention for *O*, *U* and *E*? The answer, obviously, is affirmative (see Figure 1) and, thus, we say the cube is non-rigid or that it is flexible. But, how many parameters now rule, respecting this initial setting, the possible displacements of this framework? In other words, how many internal degrees of freedom does it have? We will see that this question is highly related to the construction process of a GeoGebra model for our cube; its answer should guide the construction and, conversely, a successful construction should allow us to experiment with the existence of the different internal displacement parameters.

Dynamic Geometry

In fact, the above circular statement seems just another example of the need for mathematical insight to produce sound dynamic geometry resources, which, on the other hand, help developing mathematical insight into a geometric problem. Yet we think there are some special circumstances in this context.

As it is well known, when opening a dynamic geometry worksheet for drawing some sketch, we are following the traditional paper and pencil paradigm, replacing physical devices (ruler, compass, etc.) with different software tools. The relevant difference is that, in the dynamic geometry situation, we can benefit from a dragging feature, which is alien to the paper and pencil context.

Now, bar-joint linkages are physical constructions that include the dragging of some of its elements as an intrinsic feature. No one makes a linkage mechanism to let it stand still. In this sense we could think of dynamic geometry programs as especially fit to deal with linkage models. A supporting argument could be a visit to some web pages displaying linkages modeled by dynamic geometry programs;

we cannot refrain from suggesting the collection of GeoGebra and Cabri-Java applets from one of our co-authors[4], exhibiting an interactive collection of about one hundred mechanisms. Wonderful GeoGebra linkages are displayed at some pages by C. Sangwin[5] or by P. van de Veen[6].

Modeling bar-joint frameworks through dynamic geometry software has some advantages, but also presents some difficulties, compared to the classical case of physical models. In fact, both approaches run smoothly when dealing with very simple polygonal or polyhedral figures. But when it comes to more elaborated items, such as the cube, it is not easy to keep the different pieces assembled, or to avoid collisions between the different bars and vertices, which, in physical reality, tend to be thick, far from being intangible lines and points. According to our experience with physical models of cubes, they either have some relatively large dimensions and thus pose construction problems, for instance, with magnetic forces among different elements, or tend to be less flexible than expected. Of course, none of these physical hardships arise with dynamic geometry models.

On the other hand, modeling linkages with dynamic geometry poses other kind of challenges. For instance, it is difficult to model a four-bar planar linkage where all vertices behave similarly, that is, showing in a similar manner the degrees of freedom of the flexible parallelogram when one drags any one of the vertices. Let's fix two contiguous vertices, say, *O* and *U* and consider only the internal degrees of freedom.

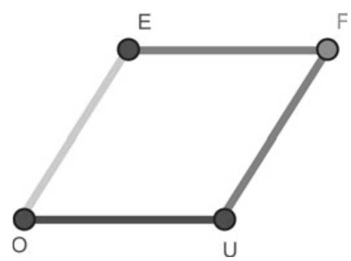

Figure 2. A planar four-bar linkage.

Then the two remaining vertices, *F*, *E*, should have each one degree of freedom, but not simultaneously. Dragging *F*, point *E* should move, and vice versa. But a dynamic geometry construction tends to assign the shared degree of freedom to just one of them, depending on the construction sequence, and not to the other. Typically, if *F* is constructed first, when we can drag it, *E* will move; but we can not drag *E*. To achieve a homogeneous behavior for *E* and *F* we have to use some techniques, such as assigning the degree of freedom to some external parameter and constructing *E* and *F* depending on it, or assign the degree of freedom to, say, one single extra point located in the bar joining the two semi-free vertices. It could seem artificial, but we consider that the reasoning required to explain and to circumvent such difficulties is, by all means, an excellent source of geometric thinking.

Last but not the least, we must consider the 3D issue. Modeling a static 3D object with a dynamic geometry program, which has a 2D display, poses by itself additional problems, not to mention those regarding modeling the movement of the 3D figure. Modeling it, in particular, with GeoGebra, yet without a specific 3D version, is even more challenging. Our experience in this respect is that by using GeoGebra's algebraic features we have been able to simulate, reasonably well, 3D scenes and movements for the cube and that we accomplish it with GeoGebra even better than through some other dynamic geometry programs with specific 3D versions.

Geometry Learning

In the previous two sections we described the mathematical importance of linkages and the potential role of dynamic geometry in modeling such objects. Here we would like to consider the pertinence of introducing linkages as a topic in high school or undergraduate geometry (Recio, 1998).

In many curricula, movements in the plane are introduced with a certain emphasis on their classification such as translations, rotations, symmetries. We can say that movements are considered important for geometry learning, but mostly from a *qualitative* point of view, that is, learning about the different types of rigid movements and their distinctive properties. Now, it is a mathematically challenging task to classify rigid displacements in the plane, very difficult in school mathematics to accomplish it for 3D.

Linkages provide a different approach to work with movements in a *quantitative* and intuitive way: How many parameters determine the positions of a point in the plane? And, how many are needed for a triangle? What about any planar rigid shape? How can we translate this question to the case of a bar-joint framework modeling a triangle, a square, a rectangle, a carpenter rule, etc.? It is easy to reason, at an intuitive level, with such questions, and it is surprising to verify, by direct experimentation with GeoGebra-built linkages, how spatial intuition is, sometimes, wrong. The case of a bar and joint cube framework is one of these models that provide rich learning situations. That is one of the important reasons behind our attempts to construct it with GeoGebra.

Moreover, simple linkages give rise to complicated yet classical high degree curves when we study the traces of some joints. As documented above, tracing curves through linkages is a lively and appealing topic with many historic anecdotes and connections to technology. It also provides lots of classroom activities. Linkages provide, in addition, a good model to understand, through the algebraic translation of the corresponding bar-joint framework construction, systems of algebraic equations with an infinite number of meaningful solutions. This algebra-geometry conversion that linkages naturally provide is, in our opinion, one important source of advanced mathematical thinking. And it is particularly close to GeoGebra's basic design conception of mixing Algebra and Geometry in a single integrated environment.

MODELING A CUBE

This section describes the problems and solutions behind our attempts to build a GeoGeobra model of a joint-and-bar cube.

A Planar Parallelogram

First we analyze the simpler case of a planar joint-and-bar parallelogram with bars of length one (Figure 2). We might consider fixing vertex *O* at the origin of coordinates and vertex *U* at point *(1, 0)* in order to focus only on the *internal* degrees of freedom that add to the 3 degrees of freedom and, at least, have all planar bodies. Then, counterclockwise, vertex *F* and vertex *E* follow. Point *F=(Fx, Fy)* must be on a circle centered at *U* and of radius 1. This means only one coordinate of *F* is free. Finally, point *E* can be constructed as the intersection of two circles of radius 1, which are centered at *F* and *O*, respectively. It will have no free coordinates.

In summary, we obtain the following algebraic system:

```
> R := PolynomialRing([Ex, Ey, Fx, Fy])
> sys := {(Fx - 1) ^ 2 + (Fy - 0) ^ 2 - 1, (Ex - Fx) ^ 2 + (Ey - Fy) ^ 2 - 1, (Ex - 0) ^ 2
+(Ey - 0) ^ 2 - 1}
```

which can be triangularized, using Maple, as

```
> dec := Triangularize (sys, R) : map(Equations, dec, R);
[[Ex - 1, Ey, Fx^2 - 2*Fx + Fy^2],
[Ex*Fx - Fx + Fy^2 - 2*Fx + Fy^2],
[Ex^2 + Ey^2 - 1, Fx, Fy]]
```

We obtain two degenerate solutions (the first and third system in the output above), corresponding to the cases $E = U$ and $F = O$, and one regular solution, in which *Fx* is parameterized by *Fy*; *Ey* is also parameterized by *Fy*; and *Ex* is parameterized by *Fx* and *Fy* (thus, by *Fy* alone). Therefore, algebraically as well as geometrically, we see the parallelogram has just one internal degree of freedom. But this extra degree of freedom can be assigned to anyone of the coordinates of *E* or *F*, depending on the way we order the variables for triangularizing the system or depending on the sequence of the geometric construction.

If we build up a physical joint and bar parallelogram with one fixed side, we observe that we can move any of the two semi-free vertices. Now, no dynamic geometry construction seems to achieve this, since the final vertex that is constructed in order to close the loop, has to be determined by the previously constructed vertices; thus only one of the two free vertices would be *draggable*.

A Spatial Parallelogram

We will now deal with the slightly more complicated case of a 3D joint-and-bar parallelogram (Figure 3). The most evident difficulty for GeoGebra to model this linkage is the lack of 3D facilities. We can circumvent this difficulty by taking advantage of the algebra integrated within GeoGebra. We will associate to each 3-dimensional point *(Px, Py, Pz)* its projection *(Qx, Qy)* on the screen, depending on some user-chosen parameters α and β that represent different user perspectives, as follows:

$$(Qx, Qy) = (Px, Py, Pz)\begin{pmatrix} \sin(\beta) & \sin(\alpha) & \cos(\beta) \\ & \sin(\alpha) & \sin(\beta) \\ & \cos(\alpha) & \end{pmatrix}$$

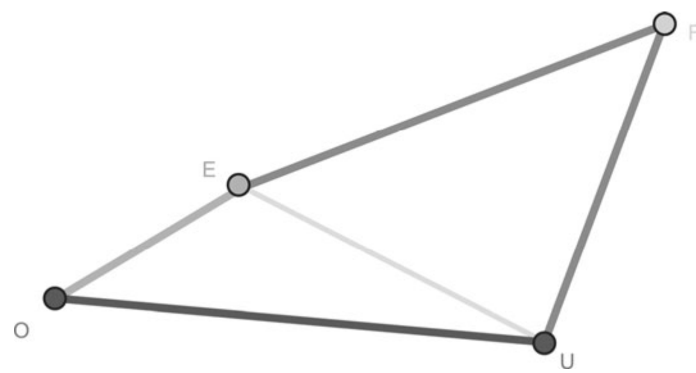

Figure 3. A spatial four-bar linkage.

Once the user introduces, by clicking on some icon such as the two ellipses of Figure 4, the values of α and β, GeoGebra projects on the screen the corresponding values of the different 3-dimensional points that will be introduced through numerical coordinates.

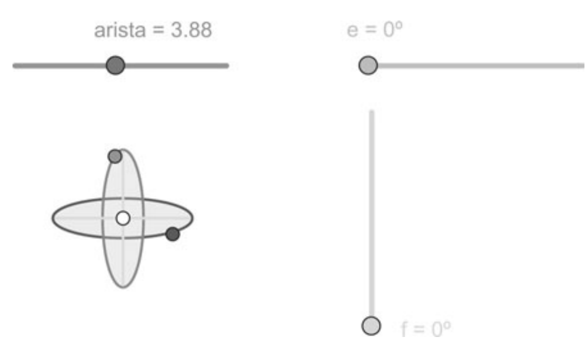

Figure 4. Control icons.

Here we fix two adjacent vertices (say, $O = (0,0,0)$ and $U = (0,1,0)$) and the plane (of equation $z = 0$) where another vertex (say, E) should lie. In this way we take care of the 6 common degrees of freedom for all 3D shapes. Therefore, the coordinates for E are

$$E = (Ex, Ey, 0).$$

Since E must be at distance 1 from O, these coordinates verify:

$$Ex^2 + Ey^2 = 1.$$

That is, introducing a new parameter e:

$$E = (-\cos(e), \sin(e), 0).$$

This parametric representation can be achieved in GeoGebra by constructing a slider (see Figure 4) that will control angle e in order to move point E.

Now, concerning vertex $F = (Fx, Fy, Fz)$, we observe that, being equidistant to E and U, it must be in a plane perpendicular to segment UE through the middle point Q of this segment. But this plane goes also through O, since OE and OU have the same length. Therefore, the coordinates of F verify the following system of equations:

$$\{(Ex - 0)^2 + (Ey - 0)^2 - 1, Ez, (Fx - 0)^2 + (Fy - 1)^2 + (Fz - 0)^2 - 1, Fx\, Ex + Fy(Ey-1) + Fz\, Ez\},$$

and it is not difficult to see that eliminating all variables from this system, except those corresponding to the coordinates of F, one obtains just the sphere

$$(Fx - 0)^2 + (Fy - 1)^2 + (Fz - 0)^2 = 1.$$

A more geometric way of arriving at the same result could be the following. We observe that, for a fixed E, point F describes a circle centered at Q and of radius equal to $k1$ (Figure 5, see below for the value of this parameter).

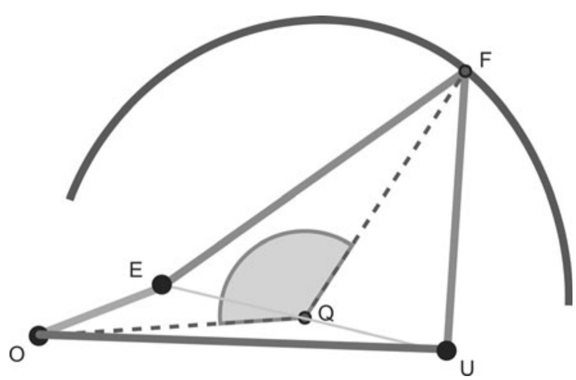

Figure 5. Determining F.

Parametrizing by a new angle *f* the position of *F* in this circle we get:

$$Fx = -cos(e)/2 - sgn(cos(e))\ k_1\ cos(f)\ sin(k_2)$$
$$Fy = (sin(e)+1)/2 + k_1\ cos(f)\ cos(k_2)$$
$$F_z = k1\ sin(f)$$

where k_1 and k_2 are given by:

$$k_1 = sqrt(2 + 2\ sin(e))/2$$
$$k_2 = acos(sin(e))/2.$$

Thus we remark that there are, in total, two internal degrees of freedom (angles *e* and *f*), which are distributed between the two free vertices, one for each vertex, in the following sense: *E* moves on a circle and, for each position of *E*, *F* can be placed at whatever point of another circle (with center and radius depending on *E*'s position). From this description, it is easy to deduce that the locus of all possible placements of *F* is a surface parameterized by circles of variable radius, centered at the different points of the circle displayed by the midpoint of *EU*. After a moment's thought, we check that such a surface is just the sphere centered at *U*, of radius 1, as expected.

The Cube

By considering the case of the spatial parallelogram as a basic building block, we can construct the cube by, first, adding to the parallelogram *OUFE* a new vertex *A* with two degrees of freedom (i.e., lying on a sphere of given radius and centered at the fixed vertex *O*), represented by two parameters *a* and *j*. Parameter *a* allows the rotation of *A* around *O* with *Ax* constant; and the parameter *j* does the same, with *Ay* constant, that is:

$$A = (Ax, Ay, Az) = (sin(j)\ cos(a), sin(a), cos(j)\ cos(a)).$$

Next, from this vertex *A*, two other adjacent vertices *B* and *D* are constructed following the same steps as in the spatial parallelogram case. First, we determine *D* as the fourth vertex of the parallelogram *OAED*. Following the arguments of the previous section, for each position of *E* and *A*, point *D* will be parameterized by an angle d on a circle centered at the middle point *M* of segment *AE,*

$$M = (Mx, My, Mz) = (E+A)/2.$$

Moreover, *D* lies on a plane perpendicular to *AE* and containing *O*. Thus

$$OD = OM + cos(d)\ OM + sin(d)\ |OM|\ n/|n|$$

where *n* is the vector product of *OM* by *EM*,

$$n = (Mz\ Ey,\ -Mz\ Ex,\ Mx(My - Ey) - My(Mx - Ex))$$

which is perpendicular to *OD* and to *EA*.

Likewise, we can determine now (that is, as the fourth vertex of parallelogram *OUBA*, assuming *O*, *U* , and *A* are fixed) vertex *B* depending on a new parameter *b*:

$$N = (Nx,\ Ny,\ Nz) = (U + A)/2.$$

$$m = (Nz,\ 0,\ -Nx)$$

$$OB = ON + cos(b)\ ON + sin(b)\ |ON|\ m/|m|$$

where *N* is the midpoint of *UA* and *m* is the vector product of *ON* by *UN*.

It remains to parameterize vertex *J*. We observe that, for given positions of *O*, *U*, *E*, *F*, *A*, *B*, *D*, this vertex must be on the intersection of three spheres of same radius, centered at *F*, *B,* and *D, respectively*. Therefore, there are, at most, two possible (isomer) positions for *J = (Jx, Jy, Jz)*. We obtain their coordinates by considering that *EJ* (and *UJ*) must be perpendicular to *DF* (to *BF*):

$$(Jx - Ex)(Dx - Fx) + (Jy - Ey)(Dy - Fy) + Jz(Dz - Fz) = 0$$

$$Jx(Bx - Fx) + (Jy - 1)(By - Fy) + Jz(Bz - Fz) = 0$$

The intersection of these two planes (note that only the *J*-coordinates are unknown here) will be a line in the direction determined by the vector product of the normal vectors to these two planes. Finally, we look for the intersection points of this line with the sphere centered at *F* and or radius 1:

$$(Jx - Fx)^2 + (Jy - Fy)^2 + (Jz - Fz)^2 = 1$$

yielding the two possible positions of *J*. The resulting expression is too large to be reproduced here.

Figure 6 displays the cube for some given, through the sliders on the top of the figure, values of the parameters we have introduced in this section. The same values, for another isomer position of *J*, yield the cube at the position displayed in Figure 7.

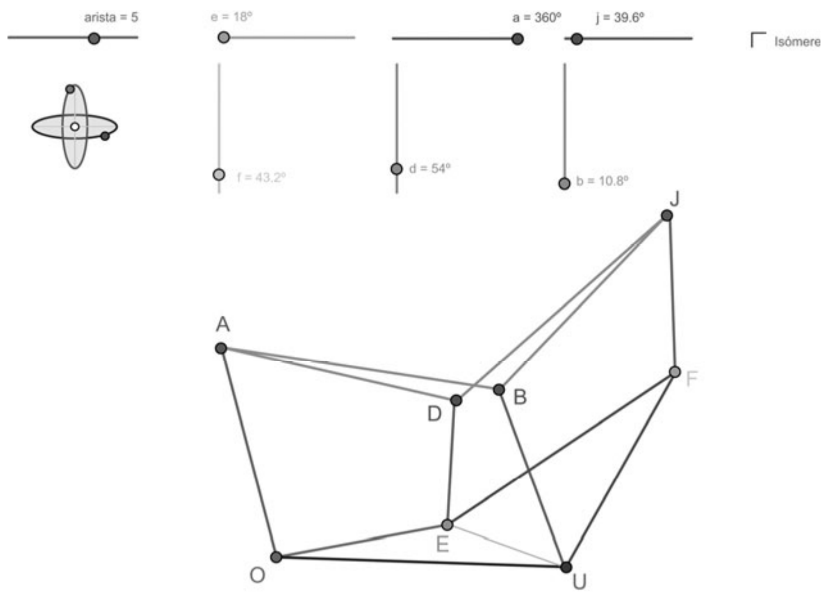

Figure 6. A cube constructed as a result of the analysis.

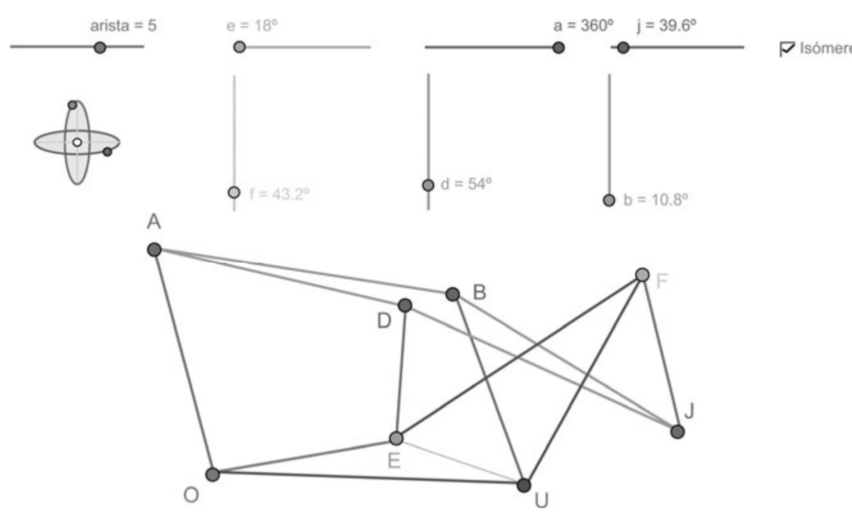

Figure 7. An isomer for the same value of parameters.

OPEN ISSUES AND CONCLUSIONS

The construction of the cube model that we have described in the previous sections behaves quite well in practice. Setting the sliders at different positions, GeoGebra numerically computes the coordinates of the different vertices of the cube, following the corresponding parameterizations and then projects them instantaneously onto the screen at the expected positions by performing some more arithmetical operations. Yet, we have to report that some *jumps* occur between isomer positions, near singular placements. For instance, when $a=270^o$, the parallelogram *AOBU* collapses. In view of the large bibliography on the *continuity problem* for dynamic geometry, it seems a non-trivial task to model a cube avoiding, if possible at all, such behavior.

We remark that the cube we have modeled has six internal degrees of freedom, one for each free parameter we have introduced. But its distribution has not been homogeneous. For instance, the final vertex has been constructed without any degrees of freedom, by imposing some constraints: being simultaneously in a sphere and in two planes perpendicular to some diagonals. This difficulty to make a model where all semi-free vertices behave homogeneously is apparently similar to the planar parallelogram case, but now we cannot conclude that it is impossible to make such a construction, since, after fixing *O* and *U* we still have six vertices and six degrees of freedom. It is probably a consequence of our approach and not an intrinsic characteristic.

In fact, we can think of the dynamic geometry sequential construction process as a kind of triangularization of the system describing a cube. In the planar parallelogram case, the triangularization of the system always yields one semi-free vertex depending on the other one. In principle for a cube, a triangularization should be possible with one new free variable associated to each semi-free vertex, but the triangularization (or Gröbner basis computation) of the algebraic system describing the distance 1 constraints between some pairs of vertices of the cube seems impractical, due to the complexity of the involved computations. If we had succeeded computing automatically this general solution we could have shown automatically that, in fact, the cube has six (internal) degrees of freedom. Right now this important fact can be just proved by considering the specific sequence of solutions presented in our construction, depending on six parameters. In some sense, we see that attempting to build a model of a cube is an example where GeoGebra helps when symbolic computation fails. And, the other way around, it shows how symbolic computation (for 3D coordinates) helps when current GeoGebra features fail.

Building a cube with GeoGebra provides excellent opportunities to learn a lot of mathematics at different levels. Some of them have been summarily introduced in the construction process such as discussing why the intersection of three spheres has at most two points, or why vertex *F* in a spatial parallelogram moves on a sphere. Also of importance is the interaction of algebra (dimension of the algebraic variety defined by the cube's equations, triangular systems, etc.) and geometry that is behind our construction.

Moreover, different classroom exploration situations can be presented to work and play with the GeoGebra cube model, such as:

– Could you fix (say, by pasting some rigid plates) one, two, … facets in the cube and still have some flexibility on the cube? How many internal degrees of freedom will remain?
– For a planar parallelogram, one can feel the one-degree of freedom by checking that once you fix one semi-free vertex, the whole parallelogram gets fixed. The same applies for the spatial parallelogram. You have to fix, one after another, the two semi-free vertices. For the cube, how can you *feel* its six degrees of freedom? Can you fix five semi-free vertices and still move the cube?

The cube, its construction process, and the model itself, seem to us an important source of both algebraic and geometric insight, and, most important, an endless source of fun, thanks, as always, to GeoGebra.

NOTES

1 Geomag is a trademark licensed to Geomag SA.
2 http://kmoddl.library.cornell.edu/
3 http://www.museo.unimo.it/theatrum/
4 http://jmora7.com/Mecan/index.htm
5 http://web.mat.bham.ac.uk/C.J.Sangwin/howroundcom/front.html
6 http://www.vandeveen.nl/Wiskunde/Constructies/Constructies.htm

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9. VISUALIZATION THROUGH DYNAMIC GEOGEBRA ILLUSTRATIONS

The purpose of this chapter is to provide pedagogical strategies and discuss ideas about teaching mathematics using GeoGebra that promote effective use of visualization in a technology-integrated dynamic environment. The author describes his work with prospective secondary mathematics teachers enrolled in a methods course. The results of the study revealed that their perspectives on teaching and learning mathematics with technology were enriched as they worked individually and in small groups to develop and present lessons with GeoGebra, suggesting that creating a collaborative environment for our prospective teachers is as important as incorporating dynamic mathematics software into our teacher education courses.

INTRODUCTION

The rapid advancement of technology has highly influenced the nature of teaching and learning mathematics over the past few decades. In a technology-rich classroom, students can explore, solve, and communicate mathematical concepts in various ways, while at the same time they are exposed to multiple representations, especially graphic representations of mathematical concepts and procedures, and rely on computers to translate among representations. As a result, recent reform movements have given further emphasis on visualization and multiple representations in the teaching and learning of mathematics. For instance, the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000) emphasize the importance of students' abilities to "use visualization, spatial reasoning, and geometric modelling to solve problems" using technological tools. Shortly after the conferences "Toward a Lean and Lively Calculus" (Douglas, 1986) and "Calculus for a New Century: A Pump, not a Filter" (Steen, 1987), addressing the dissatisfaction with traditional calculus, the use of technology and visualization has been integrated into calculus curricula. Moreover, the ongoing reform movements emphasize the use of multiple representations in the presentation of concepts; that is, the concepts should be represented numerically, algebraically, graphically, and verbally wherever possible.

Despite the importance of the use of technology and visualization in mathematics documented in the research literature, in early 1990s, mathematicians and mathematics educators (e.g., Dreyfus, 1991) discussed the merits of technology and visualization in learning mathematics. In describing the role of

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 133–144.

computers in visualization, Zimmermann and Cunningham (1991) put forward significant questions about educational software and mathematical visualization:

> How can the power of computers in general and interactive computer graphics in particular, be used most effectively to promote mathematical insight and understanding? What are the characteristics of good educational software? What are the roles of classroom demonstrations, structured computer laboratory exercises, and free exploration of mathematical ideas? What will be the impact of computers on the mathematics curriculum? (1991, p. 5)

In my view, the questions above are still being discussed in mathematics education and present challenges to mathematics educators. Zimmermann and Cunningham's (1991) questions are consonant with Presmeg's (2006) view that there is a need for research on pedagogy that can enhance the use of mathematical visualization. I agree with Presmeg that research on pedagogy that facilitates students' visualization in a technology-rich classroom should be given high priority because the use of visualization and technology has become an indispensable component of mathematics education.

Recently, innovative and dynamic software environments — for example, GeoGebra — has made possible interactions between students and computers and thereby profoundly impacted mathematics curriculum and classroom dynamics. For instance, we can restructure mathematics courses and re-sequence mathematical skills and concepts; that is, mathematical concepts and applications are more accessible, and students can do and apply mathematics without having to acquire algebraic and computational skills (Heid, 1988). Now the challenge is to design appropriate activities and describe pedagogical strategies for the effective teaching and learning of mathematics with GeoGebra. It is my belief that the technology–integrated dynamic environment not only necessitates changes in mathematics instruction (e.g., the ways teachers and students interact with each other) but also effective pedagogy that facilitates students' ability to form images and visualize through dynamic illustrations. Moreover, there is an increasing importance of dynamic linking of multiple representations in facilitating students' visualization because simply showing pictures or figures is not sufficient to encourage students to form images, visualize, or use various representations (Dorfler, 1991).

In a technology-integrated dynamic environment, as students explore concepts in various representations, they will form and link images to visualize mathematical concepts. Considering various definitions of imagery and visualization in the research literature, the idea of promoting the use of imagery and visualization with technology brings more questions than answers: What is imagery? What is visualization? How can we motivate students to form images or to think visually in a technology-integrated dynamic environment? What are the difficulties associated with the use of visualization in learning mathematics? How can we use mathematics software to facilitate students' visualization?

IMAGERY AND VISUALIZATION

The soul never thinks without an image.
– Aristotle

Visualization and visual representation of mathematical ideas have been an area of interest for researchers for more than one hundred years (Galton, 1907; Suwarsono, 1982; Bishop, 1989). Although visualization has been defined in various ways in the research literature, it appears that forming or transforming images is expressed in definitions of visualization. The word "imagery" is a highly controversial term and has a long history in psychology (Kosslyn, 1980, 1983, 1994, 1999; Pylyshyn, 1973, 1981, 2001). Mental imagery is a phenomenon we experience many times every day without even thinking about it. Images are involved in a variety of mental activities. However, it is not easy to describe what images are (Kosslyn, 1983). Imagery has been debated for a long time to posit a theory explaining how images are stored and transferred in the brain. In particular, the debate over "propositional versus pictorial" was frustrating and unsolvable. As a result, the focus of studies has changed from the nature of imagery to effects of imagery (Anderson, 1978; Clements, 1981).

Having made the preceding claims, it is necessary for me to define the terms that will be used in the remainder of the chapter. It is visual imagery that is relevant to my research, and I define a visual image as a mental construct depicting visual and spatial information, and that visualization is considered as processes involved in constructing and transforming both visual mental images and all of the representations of a spatial nature that may be used in drawing figures or constructing or manipulating them with pencil and paper or computers (Presmeg, 2006). This definition of visualization involves both external and internal representations (or images) because a relationship exists between "external" and "internal" representations created by students. From a constructivist's point of view, mathematical meaning is not inherent in external representations, and mathematical meanings given to these representations are the product of students' interpretative activity. That is, perceiving is an active process and not a passive one because "the representation does not represent by itself. It needs interpreting and, to be interpreted, it needs an interpreter" (von Glasersfeld, 1987, p. 216). The viewer's interpretation resides in his/her activities not in the objects the viewer perceives. The viewer's concepts and mental representations (or images) are dynamic and are not merely replicas of an external world. Internal representations are personal, and students can create different internal representations of the same external representation (Mason, 1987). Lakoff (1987, p.129) said, "Not all of us categorize the same things in the same way. Different people, looking upon a situation, will notice different things. Our experience of seeing may depend very much on what we know about what we are looking at. And what we see is not necessarily what's there."

RELUCTANCE TO USE VISUALIZATION

Visualization and imagery play a crucial role in the work and creativity of scientists and artists. It might be surprising to many people that mathematical reasoning requires creative thinking and imagination. When asked why a certain mathematician became a novelist, "That's completely simple," David Hilbert said, "He did not have enough imagination for mathematics, but he had enough for novels" (Hellman, 2006). In reporting cases of eminent scholars such as James Clerk Maxwell, Michael Faraday, Herman von Helmholtz, Nikola Tesla, James Watt, John Herschel, Francis Galton, Friedrich A. Kekule, and James Watson, Shepard (1978) described how they experienced the appearance of sudden illumination and externalized their mental images in the act of creation. For instance, in developing the theory of relativity, Einstein imagined himself traveling along beside a beam of light. Omar Synder, a scientist, was working on the "canning problem," one of the most difficult problems in the design and construction of the atomic reactor, when he mentally saw the entire process in his mind: "I did not need any drawings; the whole plan was perfectly clear in my head" (p. 145). While engaged in thinking, Francis Galton said:

> A serious drawback to me in writing, and even more in explaining myself, is that I do not so easily think in words as otherwise. It often happens that after being hard at work, and having arrived at results that are perfectly clear and satisfactory to myself, when I try to express them in language, I feel that I must begin by putting myself upon quite another intellectual plane. I have to translate my thoughts into a language that does not run very evenly with them. I therefore waste a vast deal of time in seeking for appropriate words and phrases (Hadamard, 1945, p. 69).

Similarly, for Jacques Hadamard, a noted mathematician, a translation from thought to language or algebraic symbols was cumbersome. He wrote, "I fully agree", as he quotes Schopenhauer as saying that "Thoughts die the moment they are embodied by words." In describing his thinking, Hadamard (1945) said:

> I insist that words are totally absent from my mind when I really think and I shall completely align my case with Galton's in the sense that even after reading or hearing a question, every word disappears at the very moment I am beginning to think it over; words do not reappear in my consciousness. I behave in this way not only about words, but even algebraic signs. I use them when dealing with easy calculations; but whenever the matter looks more difficult, they become too heavy a baggage for me (p. 75).

A review of the mathematics education literature in this field supports the assertion that understanding of mathematics is strongly related to the ability to use visual and analytic thinking. Researchers (e.g., Zazkis, Dubinsky, & Dautermann, 1996) contend that in order for students to construct a rich understanding of mathematical concepts, both visual and analytic reasoning must be present and integrated. Despite the importance of visualization in understanding mathematics, the reluctance of students and teachers to use visualization has been reported in much literature (Lowrie, 2000; Cruz, Febles, & Diaz, 2000). Clements (1984) reported a case of a mathematically gifted student who gained university acceptance at the age of ten. The student had a strong tendency to use analytic thinking despite his well-developed spatial ability. Similarly, Haciomeroglu, Aspinwall, Presmeg, Chicken, and Bu (2009) have shown that students might develop a preference for solutions strategies, which could be different from their ability, for processing mathematical information. Krutetskii (1969, 1976) went so far as to exclude visual thinking from essential components of mathematical abilities. According to Krutetskii, the level of mathematical abilities is determined largely by analytic thinking, and the type is determined largely by visual thinking. Considering that objects, accessible to our senses, are unreliable, and thus visualization or perception could be misleading, Arnheim (1969) tells us why visual thinking is given lower status and how this distinction originated and persisted for centuries. According to Arcavi (2003), difficulties around visualization can be classified into three categories: cognitive, cultural, and sociological. Cognitive difficulties refer primarily to the discussion on whether visual is easier or more difficult. Sociological difficulties include issues of teaching. Cultural difficulties refer to beliefs and values held about what mathematics and doing mathematics means. For instance, the idea that mathematics should be communicated in a non-visual framework is commonly shared in the mathematics community (Dreyfus, 1991; Stylianou, 2002).

Researchers have argued that although an analytic approach was not simpler than a visual approach, students found visual methods more difficult and risky and preferred to think analytically rather than visually (Vinner, 1989; Eisenberg & Dreyfus, 1991). Presmeg (2006), on the other hand, argued that a claim that students are reluctant to visualize is too sweeping and that this claim should not be interpreted to mean that students do not use this mode of mathematical thinking. Further, Stylianou (2002) has shown that advanced students' reluctance to visualize has been changing due to changing curricula and attitudes toward the use of visualization.

INCORPORATING GEOGEBRA INTO MATHEMATICS LESSONS

Studies have shown that enormous diversity exists in individual ability to visualize mathematical objects when different individuals do mathematics (Presmeg 1986, 2006; Haciomeroglu, Aspinwall, & Presmeg, 2009). Haciomeroglu, Aspinwall, and Presmeg (2010) reported the cases of two calculus students for whom visualization was the primary method in their work. For one of the students, visualization produced high levels of mathematical functioning, but without support of analytic thinking, relying on visual thinking alone hindered his understanding of derivative graphs. The other student used analytic thinking to support his visual thinking and avoided difficulties associated with the use of visualization alone. Difficulties experienced as a result of images, or as Bartlett (1967, p. 220) says, are "prices of peculiar excellences." To facilitate the effective use of imagery (or visual thinking) and develop flexibility of thought, it is important to encourage visualization and demonstrate different representations of the same mathematical concept. When typical and similar illustrations or figures are used to introduce mathematical concepts, students may form images that can be a hindrance to their learning. Drawing figures to represent mathematical concepts or problems is beneficial for students, but these figures are static and may not express some aspects of complex mathematical concepts. In short, visualization or visual approach is not a panacea and might be misleading when students have incomplete understanding of static illustrations of mathematical objects or concepts.

Visualization coupled with dynamic illustrations enables students to understand concepts or meanings that might be extracted from algebraic representations and thus plays a significant role for which analytic thinking alone cannot substitute in students' thinking. For instance, Hohenwarter's (2009) work illustrates how students can be encouraged to explore the properties of a geometric figure through guided discovery learning in the dynamic GeoGebra environment. Many teachers, however, are not aware of the potential effectiveness of visual thinking or technology and tend to think that sequential and algebraic approach to school mathematics is more pedagogical and efficient. According to Prensky (2001), our education system fails to address the specific needs of today's students, who are Digital Natives, growing up with new technologies. Prensky further identifies the problem facing education today as the educators themselves, who, as Digital Immigrants, speak an outdated language to their digitally-minded students. Prensky thus suggests that new approaches — such as using games and simulations — to all subjects at all levels be investigated in school environments.

As educators do not have to be skilled in creating computer games for mathematics, but we must incorporate powerful dynamic technologies into our teacher education courses. In three semesters, I conducted studies in which prospective teachers were encouraged to work collaboratively with their classmates to develop lessons with GeoGebra. In their reflections on the use of technology in the learning and teaching of mathematics at the beginning of each semester, many prospective teachers (PT) expressed concerns about their future students' learning and mastery of basic skills and were reluctant to incorporate technology into their teaching although they grew up with technology. Consider the excerpts from their reflections:

PT1: Children become dependent on the technology and they don't learn the basics such as addition, subtraction, multiplication, division, etc.

PT2: I believe that technology can be beneficial in the classroom but it must be paired with strong, traditional classroom instruction and supervision.

PT3: As teachers, we should wait until after the foundation has been set before we start bringing in the tricks and "easier ways of doing things". This way we are not taking away from the actual learning process.

PT4: Most of technology can be very nice tools in the proper situation, but are not essential to learning and teaching in school.

PT5: The only disadvantage to technology is students abusing it and letting technology do all of the work for them.

As seen in the excerpts, prospective teachers were initially cautious or opposed to the use of technology due to their insufficient knowledge of teaching and learning mathematics with technology. Mishra and Koehler (2006), building on the work of Shulman (1986) on Pedagogical Content Knowledge (PCK), established a model for understanding teaching with technology where Technological Content Knowledge (TCK) and Technological Pedagogical Content Knowledge (TPCK) are integrated with PCK. TPCK, in particular, involves deeper understandings of the relationships between content, pedagogy, and technology and the contexts they function within. Since knowing how to use technology does not ensure knowing how to teach using it, Mishra and Koehler proposed learning technology by design as an approach to develop teachers' TPCK. In this approach, teachers actively participate in defining, designing, and refining activities for teaching with technology, focusing on generative mathematical problems and solutions.

At the beginning of each semester, I provided instructions about GeoGebra in two class sessions; the prospective teachers first explored the GeoGebra website, including GeoGebraWiki where worksheets, lessons, and other teaching materials are shared by educators from all over the world. After becoming familiar with the menus and toolbars of the software, they learned how to create basic mathematical objects and figures such as points, line segments, graphs, and polygons. Then, the prospective teachers were asked to explore the GeoGebraWiki website and share their ideas with their classmates about worksheets and lessons of their interest. They were allowed to borrow ideas from the teaching materials posted on this website, but they were required to create their own worksheets and instructions for the worksheets.

The prospective teachers were asked to illustrate a mathematical concept or problem with GeoGebra in any area of secondary mathematics content. Throughout the semester, they were encouraged to work collaboratively with their classmates. Regarding their lessons, they were asked to identify the appropriate national/state standards, goals, and objectives, describe how they would teach their lesson, (e.g., include teacher actions and possible questions to be asked, as well as student actions and possible responses), consider how their future students would explore the problem or concept illustrated with GeoGebra, and explain what and how they would expect their future students to learn through their activities.

In our class discussions, we addressed the importance of creating dynamic illustrations by emphasizing that simply showing static pictures, figures, or diagrams is not sufficient to facilitate students' learning or their use of various representations and visualization. As a result, once the prospective teachers became sufficiently familiar with the software to be able to draw mathematical objects and figures, they collaborated with their classmates to convert static illustrations into GeoGebra worksheets with dynamic constructions. For instance, one of the prospective teachers designed a real-life activity about ranges of cellular phones and forwarding phone signals to switching stations to explore graphs and equations of circles in standard and non–standard forms, and this was a rich and challenging learning task for prospective teachers (see Figure 1). Another prospective teacher created a dynamic activity to draw a graph of a linear or quadratic function and its inverse simultaneously (Figure 2).

Each prospective teacher made a thirty minute presentation of his or her technology supported lesson, and their presentations were critiqued by the class. At the end of each semester, the prospective teachers (PT) wrote open-ended reflections on their experiences of developing technology lesson plans with GeoGebra. They began to see the importance of creating dynamic illustrations in enriching the instruction and overcoming difficulties associated with the use of static illustrations as their future students explore mathematics concepts with GeoGebra. Consider the excerpts from their reflections at the end of the semester:

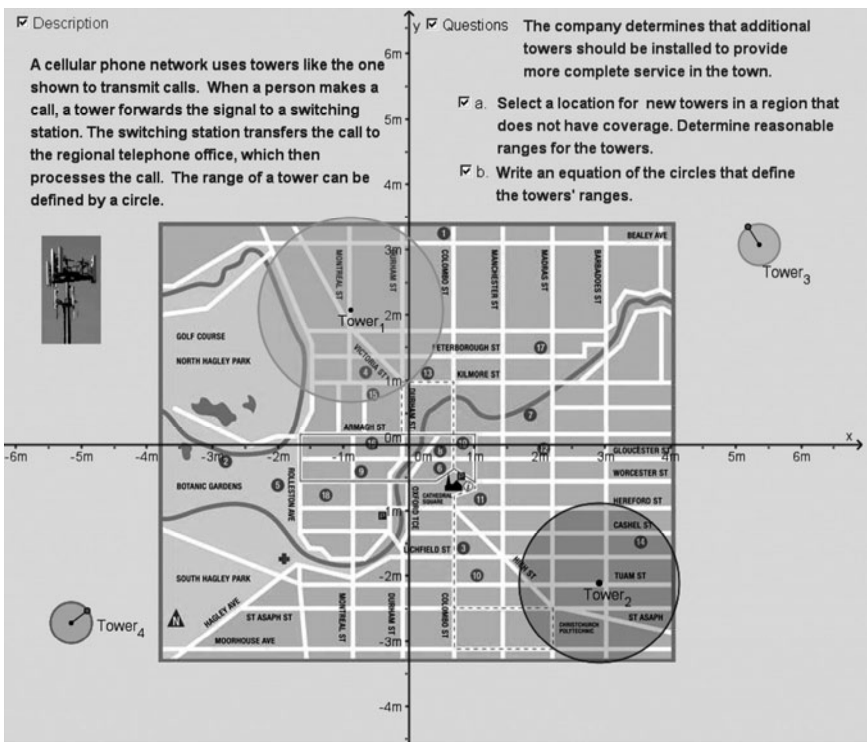

Figure 1. Graphs of circles.

PT1: The graph had sliders and it allowed the students to see what happened to the graph, you change the slope and/or the y-intercept. I then added a picture to the background to make the graph more life-like and applicable.

PT2: I expect them to learn by experience and not just hearing about definitions or seeing it on a black board but to actually create it themselves in GeoGebra to keep them interested and involved in the activity.

PT3: I hope to really use GeoGebra in showing the students a tangible proof of what happens graphically to an equation when you change part of the equation.

PT4: It's a simple function with the use of a slider for Riemann sums to show how an increase in samples leads to a more accurate solution. I expect it to assist a visual when teaching a lesson plan on definite integrals.

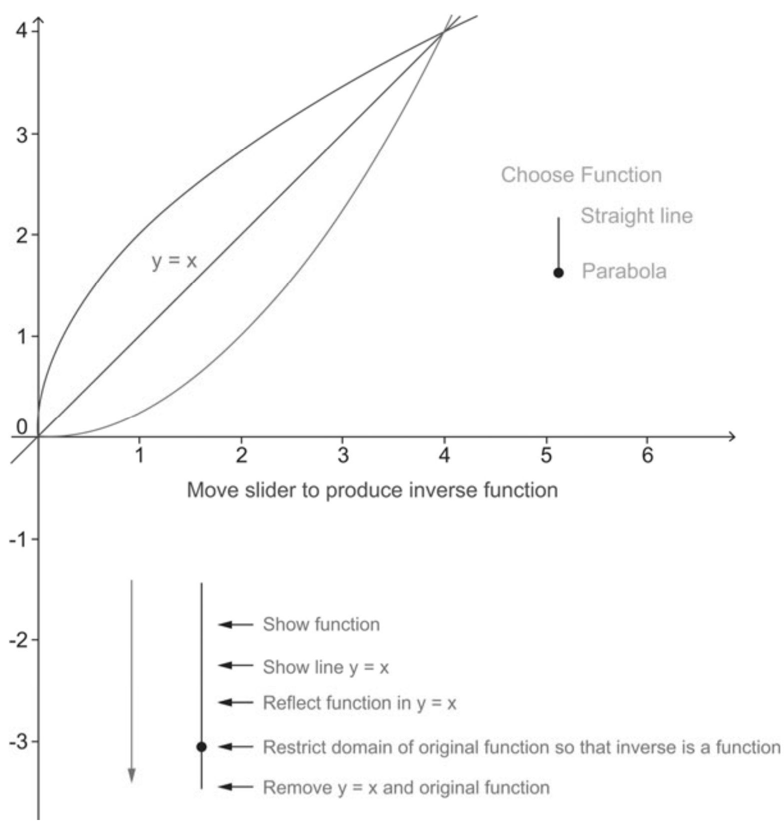

Figure 2. Inverse functions.

Although this is ongoing work, and I conducted the study with the prospective teachers in my methods courses in three semesters, as seen in the excerpts, almost all prospective teachers (PT) expressed positive views about teaching and learning mathematics with GeoGebra. (Swedish teachers' experiences with GeoGebra can be found in Lingefjärd, this volume.) At this point, it is important to note that creating a collaborative environment for our prospective teachers is as important as incorporating dynamic mathematics software into our teacher education courses. Two persons almost never create exactly the same activity. As the teacher and prospective teachers share and discuss their various approaches to create and implement mathematical activities culminating with dynamic GeoGebra illustrations, it is possible that they will enrich their understanding of mathematics and perspectives on teaching and learning mathematics with technology.

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INÉS Mª GÓMEZ-CHACÓN

10. MATHEMATICS ATTITUDES IN COMPUTERIZED ENVIRONMENTS

A Proposal Using GeoGebra

In the last few decades, research on technology-assisted teaching and learning has identified a variety of factors that are involved in the effective integration of such methods. Focusing on one such factor, attitudes, this paper reports on research on attitudes and technology in the context of secondary school mathematics (15- and 16-year-old students). The results reveal significant aspects of attitudes toward mathematics and technology as well as of certain underlying cognitive-emotional processes (inductive attitude, accuracy and rigour, perseverance, critical thinking) that may be useful for teachers integrating software such as GeoGebra into their class curriculum.

INTRODUCTION

Students' learning difficulties and continuous immersion in the world of ICTs (information and communication technologies) lead teachers and researchers to seek more effective use of technology to achieve higher academic performance and favour a change in students' outlook about a subject traditionally regarded to be difficult. In much of this research (Artigue, 2002; Guin & Trouche, 1999; Hohenwarter & Lavicza 2007; Noss, 2002; Lagrange, 2000; Richard, Fortuny, Hohenwarter & Gagnon, 2007), the primary aim is to determine which instrumental and symbolic properties of technology yield the best results in mathematics. Fewer studies, however, have attempted to assess the role of technology in attitudes and the affective aspects of mathematics learning (Gómez-Chacón, 2010).

Studies on learning and affect (McLeod, 1994; DeBellis & Goldin, 1997; Goldin, 2000, 2007; Gómez-Chacón, 2000) tend to refer to the affective reactions that may have a bearing on some of the cognitive and conative processes involved in the development of mathematical thinking (creativity and intuition, attribution, visualization, generalization processes and similar), or to the so-called directive processes (meta-cognitive & meta-affective processes). Consequently, if in the context of technology-enhanced mathematics learning, mathematical tasks are assumed to have affective ties to the above processes, questions such as the following should be posed: How can thinking processes be influenced and the necessary assistance provided to ensure positive interaction between cognition and affect? Does GeoGebra furnish a more suitable environment for this to happen? Can students be

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explicitly taught to acquire mathematical attitudes? What new types of student behaviour can be favoured by an environment such as GeoGebra?

This exploratory study adopted a descriptive line, focused on the aspects of mathematical learning with GeoGebra that may contribute to developing better mathematical attitudes among secondary school students. A number of dynamic geometry applications have been developed in recent years, including Geometric Supposer, Cabri, The Geometer's Sketchpad, Cinderella, GeoGebra and so on. GeoGebra was chosen for the present study because it dynamically integrates geometry and the algebraic expression of graphic objects and because it is freeware geared to secondary school mathematics, making it particularly suitable for the schools where the experiment was conducted.

The paper reviews research literature on "attitudes toward mathematics" and "attitude toward mathematics in technological contexts." It then describes the methodology and results of the study, highlighting the benefits of using GeoGebra for the development of mathematical attitudes and a number of open questions in this regard.

ATTITUDES TOWARD MATHEMATICS AND MATHEMATICAL ATTITUDES IN TECHNOLOGICAL CONTEXTS

While attitudes toward mathematics have long been studied (Di Martino & Zan, 2001; Hannula, 2002; Hernández & Gómez-Chacón, 1997; Kulm, 1980; Leder & Forgasz, 2006; Mcleod, 1994; Ruffell et al., 1998), the study of mathematical attitudes has been less thoroughly developed.

As early as 1969, Aiken and Aiken suggested two classical categories, attitudes toward science (when the object of the attitude is science itself) and scientific attitudes (when the object is scientific processes and activities, i.e., scientific epistemology), which were later adapted by a number of authors (Hart, 1989; NCTM, 1989; Gómez-Chacón, 1997, 2000) to mathematics and denominated attitudes toward mathematics and mathematical attitudes.

The area of technology-mediated learning is most frequently approached from the vantage of attitudes toward mathematics. Research, conducted primarily with undergraduates, has led to the development of instruments to evaluate attitudes in technological contexts (Galbraith & Haines, 2000; Cretchley & Galbraith, 2002; Camacho & Depool, 2002; Gómez-Chacón & Haines, 2008). A more recent model (Forgasz, 2003; Goos et al., 2003; Pierce & Stacey, 2004; Pierce, Stacey & Brakatsas, 2007), developed to study the effectiveness of technology in secondary mathematics teaching, has provided an instrument for evaluating attitudes of secondary students.

Most studies of this nature have used questionnaires (Likert attitude scales) developed from the perspective of a multi-dimensional definition of attitude (cognitive, affective and behavioural), taking mathematics confidence and motivation to be the dimensions with the greatest impact on learning. The findings furnish evidence of the importance of these dimensions of attitude and show that a clearer understanding of student behaviour in the context of technology-based learning calls for an attitude evaluation model that embraces both technical and personal aspects. Moreover, these studies have led to similar conclusions, confirming "the weak relationship between mathematics and computer attitudes (both confidence and motivation) and that students' attitudes toward using technology in the learning of mathematics correlate far more strongly with their computer attitudes than with their mathematics attitudes" (Cretchley & Galbraith, 2002, p. 8).

Although developments in technology and attitudes have focused more on attitudes toward mathematics, both categories (attitudes toward mathematics and mathematical attitudes) are addressed here. Consequently, this discussion will deal with the distinction that should be drawn, in teaching and learning, between attitudes toward mathematics and mathematical attitudes.

Attitudes toward mathematics refer to the valuation of and regard for this discipline, the interest in the subject and the desire to learn it. They stress the affective component–expressed as interest, satisfaction, curiosity, valuation and so on–more so than the cognitive component. Mathematical attitudes, by contrast, are primarily cognitive and refer to the deployment of general mathematical disposition and habits of mind. Disposition refers not simply to attitudes but to a tendency to think and to act in positive ways. Students' mathematical attitudes are manifested in the way they approach tasks such as flexible thinking, mental openness, critical spirit, objectivity and so on, which are important in mathematics (see NCTM Standard 10 (1989), for instance). Due to the predominantly cognitive nature of mathematical attitudes, to be able to be regarded as attitudinal, they must also comprise some affective dimension: i.e., a distinction between what a subject can do (mathematical disposition and habits of mind) and what a subject prefers to do (positive attitude toward mathematics). In 1992 Schoenfeld coined the term *enculturation* to mean that becoming a good mathematical problem solver may be as much a matter of acquiring the habits and dispositions of interpretation and sense-making as of acquiring any particular set of skills, strategies, or knowledge. *Enculturation* is entering and picking up the values of community or culture (Schoenfeld, 1992, p. 340). According to Schoenfeld, students need a socialization process, to be imbued with certain habits of mind and mathematical attitudes such as those mentioned above. Consequently, it is incumbent upon teachers to create environments favouring the inquisitive spirit, pursuit of ideas, research and questioning associated with the practice of mathematics.

Bearing in mind, then, that attitude is defined to be a psychological tendency expressed as the evaluation of an object from the two categories of attitude specified, exploring the cognitive-emotional processes involved in evaluation when doing mathematics on a computer calls for taking two types of evaluation into consideration: students' evaluations when not involved in the task (when answering a questionnaire, for instance) and evaluations of the situation, conducted when the student is involved in a specific mathematical activity. In this case, their circumstance is continuously evaluated with respect to their personal goals and evaluations prompted by the situation (cognitive processes, activity typology and so on).

The present paper describes the attitudes toward technology-mediated mathematics learning observed in the study group, focusing on students' evaluations when not involved in a task, and on the mathematical attitudes and attitudes toward mathematics that are an expression of students' evaluations when they are so involved.

OBJECTIVE AND METHODOLOGY

The present study forms part of a broader project undertaken in 2006 on the relationship between attitudes toward mathematics on the one hand and technology on the other and the design of instruments to evaluate secondary school students' attitudes. The research questions posed in the present study were: What are students' initial attitudes toward technology-mediated mathematics teaching? What levels of students' mathematics confidence, motivation and engagement can be observed when they are doing and learning mathematics in technological environments? What cognitive-emotional processes lead to students' positive or negative evaluation of the use of GeoGebra to learn mathematics? What new types of student behaviour can be favoured by an environment such as GeoGebra? What types of mathematical attitudes are favoured by learning mathematics with GeoGebra and how are they related to performance?

To respond to these questions, the study was divided into two parts. The first part consisted in a survey of 392 15- and 16-year-old students (207 boys and 185 girls) enrolled in secondary schools that had prioritized the integration of new technologies in the mathematics curriculum (School 1, 65 students; School 2, 100 students; School 3, 74 students; School 4, 41 students; and School 5, 112 students). An adapted version of the instruments developed by other researchers (Galbraith & Haines, 2000) to evaluate attitudes toward mathematics and technology was used to measure attitudes in this study (Appendix 1 a sub-scale, can be illustrative of questionnaire items).

In the second part, a qualitative methodology case study was conducted on a group of 17 of the 392 students (class group case study) as well as on six of those 17, who were singled out for more extensive monitoring. Based on the initial survey of 392 students, these six students were representative of the four profiles identified through the survey, with respect to their attitudes toward the interaction between mathematics and computers.

This class group had three weekly 55-minute math classes. The course included traditional academic class work, traditional laboratory classes and problem solving classes using GeoGebra software. The software-mediated problem-solving classes were designed by the researchers (some examples Appendix 2). This suite of activities, called the "instruction workshop", was geared toward introducing students to the educational aspects of GeoGebra.

A number of sources and procedures were deployed to collect the data (questionnaires, class observation, audio-video recordings, interviews, students' class work and so on). All 17 students were asked to complete Likert-style questionnaires before and after the experience. The same questionnaire used with the large group about mathematics and technology (Galbraith & Haines, 2000) and another one about attitude towards solving problems and mathematics confidence that had been validated and used and found to be reliable in previous research (Gómez-Chacón et al., 2001). While the questions on technology were formulated in general terms, in the post-experience survey the students associated this general idea with the specific software used (GeoGebra).

For this group, three key moments in attitude evolution were considered: pre-experience, during the experience and post-experience.

Two types of problems were chosen for the mathematics sessions: problems involving GeoGebra drawings, and proof and modelling problems. These problems called for conceptual and procedural knowledge. The solutions involved a constructive process, a search for alternatives, conceptual knowledge and how to apply it, and the transfer of the information acquired to graphic and symbolic categories.

The questions posed in the problems required students to think in both algebraic and geometric terms. The problems involving proofs and more constructive problems necessitated either the transfer of information from more algebraic to geometric categories, or conjecture, particularization, generalization or modelling. Moreover, each problem was designed to prompt the interaction between mathematics and computers and the development of observable mathematical attitudes (see for example Appendix 2). The exercises chosen for analysis were equally spaced, time-wise, to monitor student evolution.

Mathematical attitudes were evaluated on the basis of the students' own accounts through questionnaires (Appendix 3), recordings of problem-solving sessions and the assessment of their classroom behaviour by teachers and researchers on observation grids (see Appendix 4).

In concluding, we would like to indicate that in order to stabilize the achievement levels in mathematics for each student we used the data provided by the teacher's grades for the whole group, and for the small group we compared the results from pre and post problem solving.

RESULTS AND DISCUSSION

The findings are discussed below. Their description is divided into two parts, one on attitudes toward mathematics and technology and the other on certain cognitive-emotional processes underlying mathematical attitude.

Initially Satisfactory Attitude Toward Computer-Assisted Mathematics

As noted above, the origin of the data and findings discussed in this section is the questionnaire on attitudes toward mathematics and technology completed by 392 secondary school students. The aim was to answer the following questions: What is students' initial attitude toward technology-mediated mathematics teaching? What relationships exit between students' mathematics confidence, motivation and engagement as defined by student responses in a technology-enriched mathematics program?

Before describing the results, definitions of a few of the concepts around attitude are in order. An initially satisfactory attitude is an appropriate disposition from the outset, characterized by dimensions such as confidence, motivation, engagement to learn, and positive beliefs about mathematics and computer-enhanced mathematics learning. While the importance of mathematics confidence, motivation and engagement is generally recognized in the literature cited in this paper, no agreement has been reached about their measurable features. Consequently, definitions are given here for the dimensions measured with the questionnaire: mathematics confidence (mathconf), mathematical motivation (mathmot), mathematics engagement (matheng), computer motivation (compmot) and interaction between mathematics and computers (mathcompint).

Referring back to Galbraith and Haines (1998, pp. 278–280), mathematics confidence is defined to be the dimension found in students "who believe they obtain value for effort, do not worry about learning hard topics, expect to get good results, and feel good about mathematics as a subject." These authors regard computer confidence to be self-assurance in using computers, a belief that computer procedures can be mastered, confidence in answers when found on a computer and, in the event of error when working with a computer, confidence in one's own ability to solve the problem. Computer motivation, in turn, is finding that computers make learning more enjoyable, liking the freedom to experiment provided by computers, spending long hours at a computer to complete a task, and enjoying the possibility to test out new ideas on a computer. Lastly, for those authors, computer engagement is behaviour-related. It consists in a preference to work through examples rather than learn given material, to test understanding through exercises and problems, to link new knowledge to existing knowledge, to elaborate material with notes, and to review work regularly.

To study attitudes towards the use of technology for learning mathematics, Galbraith and Haines defined a construct they called "computer and mathematics interaction" and claimed that "students indicating high computer and mathematics interaction believe that computers enhance mathematical learning by the provision of many examples, find note-making helpful to augment screen based information, undertake a review soon after each computer session, and find computers helpful in linking algebraic and geometric ideas" (Galbraith & Haines, 2000, p. 22).

Table 1 shows the means, standard deviation and internal consistency (Cronbach's α) for each of these sub-scales of the survey (The survey is on a 5-point Likert scale, of 1 to 5). The data revealed that student mathematics confidence, motivation and engagement were acceptable (mean = 3.23; mean 3.08 and mean = 3.33, respectively). When asked, students agreed that if they made an effort, their performance improved, although they stressed that they were more concerned about mathematics than other subjects and admitted their uneasiness in this regard. They found mathematics to be abstract and hence less interesting, and so devoted little time to the subject.

Confidence in computer use and student interaction with mathematics and computers were high (mean = 3.28; mean = 3.30). Students reported that computers helped them establish mathematical connections, such as between a graph and its equation. The scores for items on how to work with computers were lower, however (they tended not to take note of processes or results nor did they generally check mathematical exercises done on a computer).

Table 1. Mean, standard deviation and internal consistency (Cronbach's α) for each sub-scale

Scale	Mean	Std deviation	(Cronbach's α)
Mathematics confidence	3.234	.651	.756
Mathematical motivation	3.080	.642	.799
Mathematics engagement	3.335	.498	.634
Computer motivation	3.286	.473	.487
Student interaction with mathematics and computers	3.304	.474	.665

In the initial survey, the results confirmed conclusions obtained in prior studies (Galbraith & Haines, 2000 and Galbraith, 2006) to the effect that the interaction between attitudes toward mathematics and toward computer use was low. The data obtained in that preliminary study showed that the correlation was higher among students who were using technology to learn mathematics than among those who were not, particularly when the parameters measured were mathematics confidence and mathematics and computer motivation. For instance, a correlation was found between mathematics confidence and motivation. The correlation between mathematics and computer confidence was weak. Mathematics engagement was strongly associated with mathematical motivation and the interaction between computers and mathematics was associated with computer confidence and mathematics engagement (Tables 2 and 3).

Table 2. Correlation between attitude scales

	mconfid	mathmot	matheng	compmot	mathcompint
mathconfid	1.000	.502	.288	.017	.178
mathmot		1.000	.554	.104	.291
engmat			1.000	.223	.390
compmot				1.000	.437
mathcompint					1.000

These figures suggest that the data for the five sub-scales cluster around two axes (factors), which explain 69 % of the variance (see Table 3).

Table 3. Factor pattern matrix (data on attitudes toward mathematics)

Sub-scale	Components	
	1	2
Mathematics confidence (mathconfid)	**.819**	– .177
Mathematical motivation (mathmot)	**.861**	.047
Mathematics engagement (matheng)	**.636**	.340
Computer motivation (compmot)	– .150	**.884**
Student interaction with mathematics and computers (mathcompint)	.195	**.763**

Moreover, cluster analysis was used to identify attitude four profiles among students.

– *Profile 1* (116 respondents = 30%): students scoring low on the interaction between computers and mathematics and on mathematics confidence, motivation and engagement, and exhibiting lower than group computer motivation.
– *Profile 2* (111 respondents = 28%): students exhibiting high mathematics - confidence, motivation and engagement, but scant computer motivation or low interaction between computers and mathematics.
– *Profile 3* (105 respondents = 27%): students with high computer confidence and motivation and high scores for interaction between computers and mathematics, but lower mathematics performance.
– *Profile 4* (60 respondents = 15%): students with high values for all variables: mathematics confidence, motivation, engagement and interaction between computers and mathematics.

In a nutshell, the figures showed that learning mathematics on a computer was more strongly correlated to attitudes toward computers than to attitudes toward mathematics. In the group studied, high scores in the sub-scale were obtained when attitudes toward mathematics were measured in terms of confidence, motivation and engagement. Nonetheless, subjects' attitudes toward the use of computers for learning mathematics varied and were often negative. Moreover, an overall cluster analysis of the findings revealed different student profiles. Based on these profiles, questions can be posed about the cognitive-emotional processes that lead students to positive or negative evaluations, as well as about other variables relating to students' computer work habits (styles), technical command, emotional issues, and the impact on attitude of using a given type of software. The following section discusses how these issues and questions were approached in a qualitative case study in learning situations with GeoGebra.

Mathematical Attitude or Cognitive and Emotional Attitudes Toward Learning with GeoGebra

The questions posed in the qualitative study were: What cognitive-emotional processes lead to students' positive or negative evaluation of the use of GeoGebra to learn mathematics? What new types of student behaviour can be favoured by an environment such as GeoGebra? What types of mathematical attitudes are favoured by learning mathematics with GeoGebra, and how are they related to performance?

In addition to mathematics confidence, motivation, engagement and interaction between mathematics and technology, the following mathematical attitudes were studied in the 17 students chosen for closer monitoring: flexible thinking, critical spirit, visual thinking, inductive attitude, curiosity, perseverance, creativity, independence, systematization, cooperation and teamwork.

The findings showed that their mark (mathematics performance) improved significantly, from a mean of 5.50 to a mean of 8.18 (on a high store of 10). Of the 17 students, 15 earned higher marks, one the same and one a lower mark after than before the three-month intervention. While this finding suggests that the use of GeoGebra contributed to improving academic performance in mathematics, it cannot be regarded to be conclusive, because many other factors came into play, such as the subject studied – geometry – and the novelty of using a new tool. Any such conclusion would call for more case studies and a larger sample, and the definition of the variables specific to GeoGebra and not found in other dynamic geometry software.

The data also indicated that fifteen students' motivation to solve problems rose, while for the other two it declined. This rise may be interpreted to mean that in such situations, in which computers play an active role, students were more motivated to tackle mathematical problems. Nonetheless, the questionnaire data showed no sharp rise in their confidence in their computer-assisted problem-solving skills. Furthermore, students were less prone to check problems after than before the trial period. Class observations and interview data were examined in an attempt to seek an explanation for these results. Some of the reasons given were:

> "I'm more motivated in the class with the computer, it's more relaxing" (Ruben' interview)
>
> "More entertaining, time goes faster" (Javier' interview)
>
> "I think this class is a good idea because it gets you out of the rut of listening to the teacher, taking notes; in the laboratory you do assignments as if they were experiments." (Alejandro' interview).

These data showed that students' motivation to learn mathematics on a computer was closely related to the widely varying approach and teaching methods deployed by classroom teachers. The class with or without computer has different didactical contracts. The students are aware of this institutional dimension and they generate different expectations. Another aspect that data showed was the students' confidence in their attitudes depended heavily on their technical command of the software.

The data on the long-term (3-month) use of GeoGebra, in turn, showed that this tool favoured the development of several mathematical attitudes, including:

– *Perseverance* or the tendency not to give up easily when solving problems. Students were almost always observed to become deeply involved, making many attempts to find the solution, despite infrequent success. GeoGebra favoured a return to and confrontation with the problem, enabling students to acquire a clearer understanding (transfer of responsibility) and making them increasingly aware of what impels them to act. Returning to the problem is useful when the answers are not a mere true or false, but call for deeper involvement.
– *Mathematical curiosity*, expressed not only as a simple desire to learn or know more about mathematics, but also the desire to explore mathematical ideas by formulating new problems both during and after an exercise.
– *Inductive attitude*, in which students progressed from collecting specific information (early enquiries, trial and error, simple calculations, short enumeration and so on) to discovering general patterns through observation, regularity and consistency.
– *Accuracy, rigour and surmounting visual obstacles*. The data provided insight on the relationship between drawings and geometric objects in learning geometry. Geometric drawing is not necessarily interpreted by students as something that brings a geometric object to mind. GeoGebra software helps them to view drawings within a wholly geometric interpretation, in which drawings should bring to mind objects defined theoretically. Any given geometric drawing can in fact be interpreted in many ways, and perception intervenes in interpretation building when students lack sound theoretical geometric knowledge that enables them to transcend an initial perceptive reading. The perceptive aspects of drawing were found to favour or hinder secondary school students' geometric reading. The latter was observed, for instance, when attention was focused on elements irrelevant to that reading (a finding that confirms the results of research conducted on Cabri software for instance (see Mariotti, 2002). In the classroom situations observed, no mention was made of the relationship between the mathematical object and the drawing, i.e., either the difference was silenced or the existence of a natural bond between them was taken for granted. In the present pursuit of cognitive-affective processes underlying attitude, the relationship between mathematical object and drawing was found to be a key factor, both in the analysis of individuals (cluster profiles) and the comparison of students with a positive attitude toward learning mathematics on computers, to students with low scores in this regard. The students for whom working with GeoGebra afforded greatest satisfaction and enjoyment were the ones with more highly developed visual skills (or who adopted a visual approach to problem-solving) and who had a sufficient command of mathematics and technique to handle objects and dynamic constructs. The students least satisfied or indifferent tended to transfer drawings as reflected on paper (statically) and to fail to differentiate between the drawing and the geometric object.

Affective-Cognitive Pathway: With Their Graphics and Numerical Calculation, Computers (GeoGebra) Help me Learn Mathematics

Some of the foregoing assertions are illustrated in the following example. For reasons of space, this article describes only one of the affective-cognitive pathways and two of the case studies.

The problem that students were to solve individually was: In the following drawing, which segment is longer, AD or EG? (Figure 1) (You can do the assignment with pencil and paper or use GeoGebra).

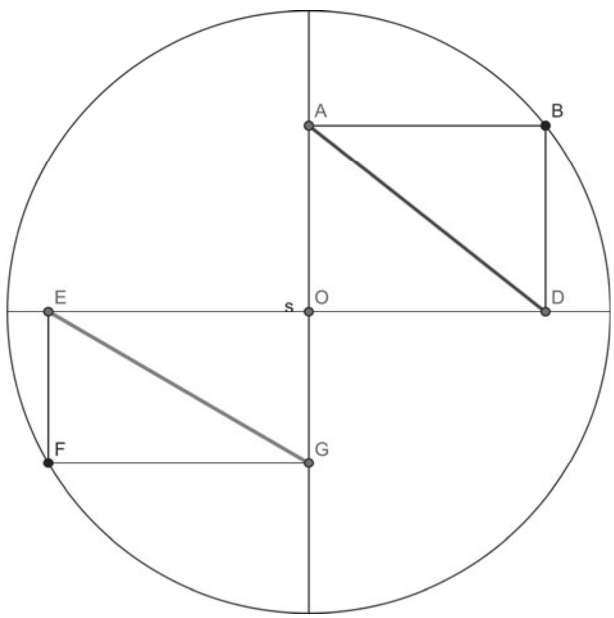

Figure 1. Length problem.

The exercise was worded as an open problem to oblige students to think through the steps they needed to take to build the object. It was designed, moreover, to focus on two aspects of GeoGebra problem-solving functions: construction-related aspects on the one hand and the reasoning leading to the realization and proof that the segments were equal on the other. The mathematical attitudes the exercise was intended to foster included the development of critical thinking and the realization that perception may be deceiving. As the wording of the problem infers, the two segments, AD and EG, appear to be different.

Table 4 summarizes the class-group information on getting stuck, emotional reactions, cognitive aspects of representing and visualizing the problem and a few dimensions relating to its solution. Note the impact of working with GeoGebra on positive emotions and getting stuck and that a high proportion of students (over

50%) reported that they could not visualize or think through the problem without GeoGebra.

To better understand the effects of software on students' attitudes, several who scored high on attitude (motivation, confidence and engagement) (cluster 2), but with different attitudes about learning mathematics on a computer, were chosen for further study. The question posed was what cognitive-emotional processes take place in these subjects to predispose them favourably or unfavourably toward technology-enhanced learning (GeoGebra). Table 5 shows their attitude scores as deduced from the questionnaire (mathematics confidence, motivation, engagement, mathematics-technology interaction).

The problem as solved by two students of six in-depth case studies, Rubén and Alberto, is described below.

Rubén's problem (Figure 2). Rubén began working with pencil and paper, and jotted down the following:

$\alpha = 45^{o}$

Perpendicular line through the midpoint of segment OC.

Perpendicular line through the intersection between the circumference and point F

Use angle α to draw a square

Deduction that the length is equal in β and ω.

That makes me see that if points G and F are moved along the perimeter of the circumference, the length of the diagonals in the two figures is the same.

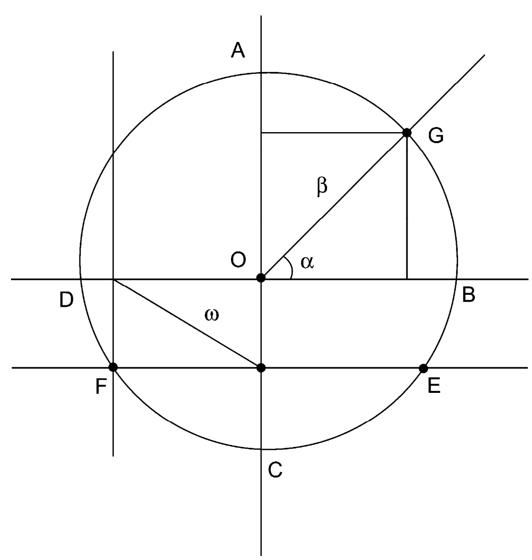

Figure 2. Rubén's problem.

Table 4. Synthesis of cognitive-emotional elements in the length problem (17 student's solutions)

Use of drawings in problem solving	Problem visualization and reasoning	Approach to solution	Emotional reactions in P&P solutions	Emotional reactions with GeoGebra	Getting stuck with P&P	Getting stuck with GeoGebra
No 7s 42 %	No 9s 53%	App 1, 7s 42 %	Positive 3s, 18 % Joy	Positive 13s, 76% Joy, self-assurance, relief, happiness, calm, pride, fun, satisfaction	Getting stuck 7s 42%	Getting stuck 2s 12 %
Yes 5s 29%	Yes 5s 29 %	App 2, 5s 29 %	Negative 10s, 58 % Rejection, despair, frustration, anger	Negative 2s, 12 %	Not stuck no answers 1s 6%	Not stuck 15s 88 %
Some-times 5s 29 %	More or less 1s, 6 %	App 3 2s 12 %	Indifference 4s 24 %	Indifference 2s 12 %	More difficult 9s 53 %	Easier 12s 71%

Table 5. Case study, class group and total group scores on mathematics and technology questionnaire scales

Student	Math. conf	Math. mot	Math. engag.	Comp. mot.	Comp.-math. interact.
Ruben	4.35	3.63	3.70	2.71	3..75
Alberto	3.50	3.13	3.25	4.00	3.38
Alejandro	3.63	3.50	3.88	3.86	4.63
Class-group (17 students)	3.49	3.11	3.35	3.38	3.46
Total group (392 students)	3.23	3.08	3.33	3.28	3.30

After thinking through and nearly solving the problem with pencil and paper, he moved on to use GeoGebra. At first he tried to build figures with angles, but his command of the software was too weak to parameterise the problem and build the figure with sliders. He therefore used the software as a tool to verify and validate his conjectures (t = 00.12’). He used the software to draw the figure on the screen (Figure 3).

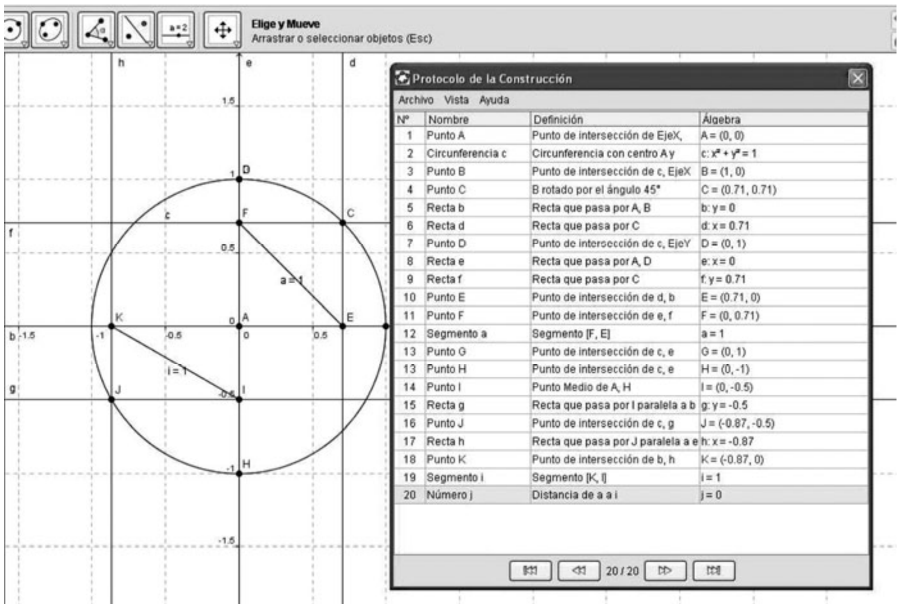

Figure 3. Rubén’s GeoGebra solution.

But since he didn’t know how to make a dynamic drawing with GeoGebra, he was unable to move the point as he had conjectured on paper. His comment was “UGH!! All that work and it still won’t move.”

He could draw parallel and perpendicular lines and circumferences but was unable to drag the point to a new position (in this case, on the circumference). He did not work on mathematical objects. And yet GeoGebra enabled him, with the distance command, to verify that the diagonals were equal (validate) in this specific case, so he needed to go no further (approach 1 in Table 4). His solution consisted in conjecture and validation in a specific case. Sharing his emotional reaction, his comments shared in the interview were “Well, this should move… Well, I guess so, but this is no different from drawing it on paper… Well, at least I can check the measurements automatically.” He added that he felt no satisfaction when he solved the problem because unlike his classmates, he was unable to visualize a dynamic image.

Alberto’s problem. When he received the problem, Alberto went straight to GeoGebra to set up a system of coordinates. He said that working with pencil and paper only he couldn’t come up with anything. On the paper drawing he tried to build a larger rectangle, extending AG, BC, FG and FE. He thought that would give him the answer, but “it got [him] nowhere. Rectangle and no measurements… Relationship between GF and BC, equal? Not equal…” (See Figure 4).

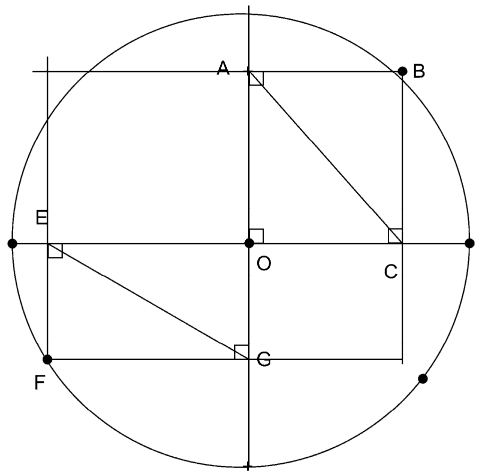

Figure 4. Alberto's problem.

Then he drew the figure with GeoGebra. To do so he began with a circle and the two perpendicular diameters. Then he positioned points B and F on the circle and continued the drawing according to the instructions set out in the problem. He said: "I realized that without GeoGebra I wouldn't have noticed the rectangles formed by each triangle and then I saw that since I knew that the diagonals of a rectangle are equal, I'd be able to solve the problem. GeoGebra helped, because since I had to draw it, that's when the idea came to me and I saw the tack to take" (Notes from Alberto's recording).

He represented the drawing with GeoGebra as shown below (Figure 5):

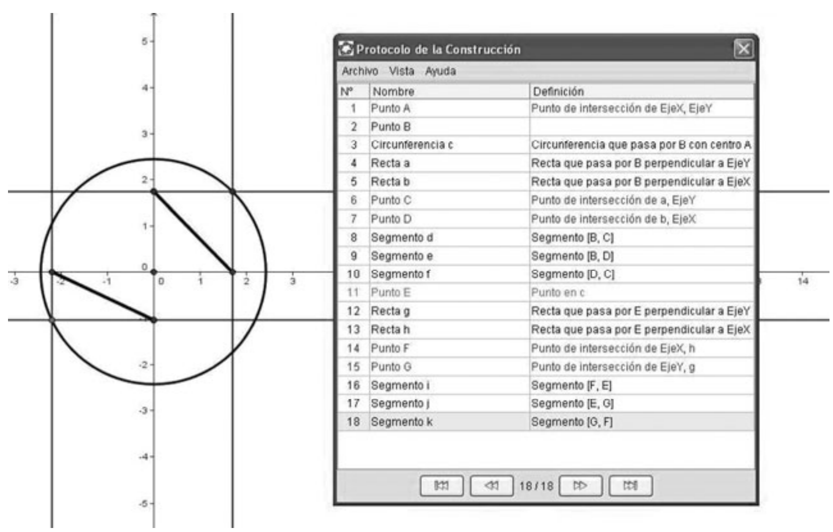

Figure 5. Alberto's GeoGebra solution.

He returned to the idea that he had plotted on paper, but using a figure drawn with GeoGebra. "Relationship between GF and BC, equal? I can see that it's a perfect square, I can see the measurements. With Pythagoras you figure the measurements and that's that."

Alberto intuitively perceived that the diagonals were equal and confirmed his intuition with his knowledge of the properties of rectangles. He tried to progress further and answer the question: How can you use GeoGebra to prove that the diagonals are equal?

"I was on the wrong track when I was working with pencil and paper; I'm trying to do it by drawing the rest of the rectangle ($t = 00.15'$)". He came back to the idea of a large rectangle. "All I need to do is prove that the diagonals of the two rectangles are equal, if I join the diagonals of the two rectangles OB and OF, I get the diameter of the circumference. That's it. Each diagonal is the radius of the circle". He focused on the properties of rectangles, but discovered that proof of equality depended on the presence of the circle and hence the importance of placing the points on the circle. The segments to be compared in the problem are of the same length as the radius of the circumference, which is constant. This is clear because when point C is moved (and with the software, it moves when point B is moved), rectangles are always formed in the quadrant where C is located. Therefore, if the opposite vertices are joined by segments FG and AC, they must be equal and AC is a radius of the circumference (approach 2 in Table 4).

By illustrating the cognitive-affective processes that underlie subjects' mathematical attitude toward technology-aided learning (in this case GeoGebra), this emotional episode helps characterize the underlying dimensions of such attitudes.

Two case studies. Attitudinal features Episodes such as described above, recorded over three months of class-group observations, served as a basis for identifying certain key components in subjects' attitudes toward mathematics and working with mathematics in computer environments, and enhanced the initial cluster analysis. Rubén's and Alberto's experiences are compared below.

– *Affective component*: Rubén claimed to like to solve mathematical problems and explore new ideas. When not doing an assignment (e.g., when answering a questionnaire), he said that working on a computer for his personal use and for mathematics class was fun. But this evaluation changed when he was actually doing an assignment: then he said he preferred to solve problems with pencil and paper and rejected the use of a computer for learning mathematics. Alberto, in turn, asserted that mathematics was one of the subjects he had the most fun with and spent time on, and that he liked computers. He enjoyed using a computer to do mathematics. He realized that the Internet could be useful for mathematics assignments.
– *Cognitive component of belief*: Both Rubén and Alberto regarded mathematics to consist in problem-solving and exploring new ideas, although Alberto attributed a significant role to memorizing concepts and procedures. On the

attitude scales both said that using a computer helped them and made it more fun to learn mathematics, although in class Rubén clearly tended to solve problems with pencil and paper. Alberto claimed that GeoGebra helped him study mathematics and establish mathematical connections.

- *Cognitive technical competence*: Rubén and Alberto were both very self-assured about their own conceptual and instrumental knowledge of mathematics, but Rubén was much less so about his computer skills. Rubén lacked a mastery of certain software commands, while Alberto used the GeoGebra command sequences and menus reasonably well, although the class recordings showed that he received support from a classmate, RB. Both students sometimes encountered difficulties with formula syntax and the use of certain graphics. Alberto could readily move from working with pencil and paper to working on the computer (and vice-versa).
- *Value and use of computing*: Rubén devoted little time to computers in his personal and academic activities (according to a second interview and classroom observation). In the first interview he claimed to use computers often. His use of computers for academic purposes was limited to writing up papers and finding information. Alberto on the contrary had fully integrated computers as a tool for school work, as a way of putting what he had learned into practice.
- *Behaviour or disposition to use computers* and particularly specific software such as GeoGebra. While he had developed a strategic method for working with pencil and paper, Rubén failed to apply it when working on the screen, where he adopted a trial and error (random) approach. His modus operandi and approach were approximately 20% geometric and 80% algebraic. Alberto, by contrast, adopted a much more strategic method, laying and following a plan. His modus operandi and approach were approximately 80% geometric and 20% algebraic. Alberto's method was both theoretical and experimental; he interpreted, compared and persevered as long as necessary to do assignments on the computer. He proved to be very communicative with his classmates, self-confident, and confident of his own curiosity and creativity.
- *Difficulty*: the two students viewed their difficulties differently. Rubén found learning mathematics on the computer to be difficult, although he didn't say so on the attitude scales; this was a classroom observation. Albert on the contrary found it easy to learn mathematics on a computer.
- *Expectations of success*: Rubén's expectations of success were low and he felt unable to surmount the technical difficulties involved in transferring his skills to computer screen. Conversely, Alberto had high expectations of success and believed that he would surmount any technical difficulties he might encounter.
- *Evolution in a three-month period*: Rubén's motivation to use computers and solve problems grew and his performance improved, even though his confidence in his ability to use technology for mathematics declined. Alberto, in turn, acquired a more realistic attitude toward the use of computers and technology-mediated learning with respect to recognizing and working with mathematical objects. His command of screen

information improved, while his problem-solving flexibility in geometric and algebraic working environments grew.

CONCLUSIONS

In this study group, the interaction between attitudes to mathematics and to technology fit different profiles, and attitudes toward computers and mathematics were scantly related. The data clearly showed that attitudes toward computer-aided learning of mathematics were more closely related to attitudes toward computers than to attitudes toward mathematics, particularly in terms of mathematics confidence and motivation. (These findings concur with the results of previous studies, as described in section 1).

The preceding conclusion is based on data from the questionnaires. A look at the qualitative results (affective-cognitive pathway in case studies such as Rubén and Alberto, described above), however, reveals the existence of cognitive-emotional processes that generate a positive or negative evaluation on the part of students using technology to learn mathematics. These episodes show that evaluations underlying attitudes are closely related to cognitive, instrumental or technical competence. When subjects are not engaged in a specific task and the stimulus to define their attitude is a questionnaire, the emotions summoned have to do with associations and prior experience in mathematics class and with computers. In this case – as in all six in-depth cases studies conducted – computers are associated with leisure, networking with friends, and laboratory (computer) classes whose more flexible structures (allowing for exchange with classmates) are regarded to be more "fun." When they find they have to perform specific tasks, students perceive that learning is more demanding. The processes leading to the mastery of the device (hardware or software) prove to be instrumental in the formation of usage schemes and their attitudes toward use.

Therefore, bearing these results and the context of this research in mind, different constructs have been identified on which the evaluation of attitudes to mathematics learned with technology should be based: affects (emotions, feelings toward the computer); cognition (evaluation and perception of and information on computers); conation or behaviour (intended behaviour and action with respect to the computer); and perceived behaviour (mastery or otherwise of computer skills) and perception of the utility of technology against the backdrop of general goals (the extent to which the individual feels such know-how will be useful in future).

The present study highlights some of the positive results of using GeoGebra in connection with the development of mathematical attitudes such as perseverance, critical thinking, curiosity, accuracy, rigour and an inductive attitude. Other findings contribute to advancing in the understanding of the handling of cognitive processes underlying attitude. The data show that the role of cognitive processes such as visualization, reasoning and the identification and interpretation of key mathematical objects in the development of mathematical attitudes merits more attention. The use of software helps to overcome certain obstacles to examining and interpreting diagrams and graphics, despite the fact that the teaching scenarios studied here do not address the relationship between drawing and geometric (mathematical) object. The experimental potential (adding elements to the drawing not mentioned in the problem or by the teacher) is not perceived, and much less exploited, by students. This draws attention to the fact that these mathematical attitudes or habits of mind are not the result of decisions made spontaneously by students, but must be explicitly taught in the classroom with the aid of instrumental schemes.

Lastly, further to an issue mentioned in the results section, attitudes toward technology are scantly related to the performance of mathematical tasks. This defines new areas of research and, for technology-aided teaching and learning (in particular with GeoGebra), poses questions around how to capitalize on the initial enthusiasm for and positive attitudes toward computers to build sound, positive mathematics attitudes. Further studies are therefore needed to enlarge the range of attitudes or attitudinal dimensions from the cognitive and emotional standpoint, as well as teacher training, in light of teachers' key role in technological contexts.

ACKNOWLEDGEMENTS

The author wishes to express her gratitude to all the students and teachers who participated in this project from 2006 to 2009. In 2006 and 2007 the author received funding to conduct studies on technology and mathematics at Paris 7 University (Complutense University of Madrid Research Committee grant 23/6/2005 and Erasmus Staff Mobility Action grant 2007) and also in 2007 from ASTROCAM project (S-0505/ESP/000237) funded by Directorate General of Universities and Research, Madrid Regional Department of Education to stay as a research visitor at the City University London. Finally, I would like to thank as well Maurice Burke from Department of Mathematical Sciences at Montana State University and Michèle Artigue from Diderot-Paris 7 University for the helpful comments to this paper.

APPENDIX 1: EXAMPLES OF THE ITEMS OF THE SUB- SCALES OF THE QUESTIONAIRE COMPUTER CAND MATHEMATICS INTERACTION

1. Computers help me learn by providing many examples to work on (+)
2. When I read a computer screen, I tend to gloss over the details of the mathematics (+)
3. I rarely review the material soon after a computer session is finished. (–)
4. With their graphics and numerical calculations, computers help me learn mathematics (+)
5. Computers help me to link knowledge, e.g. the shapes of graphs and their equations (+)
6. I find it difficult to grasp what I see on a computer screen. (–)
7. I can't retain the steps involved in solving a mathematical problem on the computer (–)
8. When I work with a computer I get distracted by the keyboard instructions (–)

APPENDIX 2: SAMPLE PROBLEMS

Problem	Problem's type	Mathematical attitude	Cognitive processes
Trapezoid: Draw any trapezoid ABCD such as the one in the Figure 6. Find the mid-points of the non-parallel sides and join them with a segment EF. What is the relationship between the dimensions of EF and bases AB and CD? Explain your answer. Put the result (theorem) obtained in your own words and give it a name.	Type 1 Problems involving GeoGebra drawings,	Self-confidence Accuracy	Conjecture and justification Visualization and representation
Lengths: In the following drawing (Figure 7), which is longer, segment AD or EG?	Type 2 Proof and modeling problems.	Critical spirit and thinking	Conjecture and justification Particularization and generalization

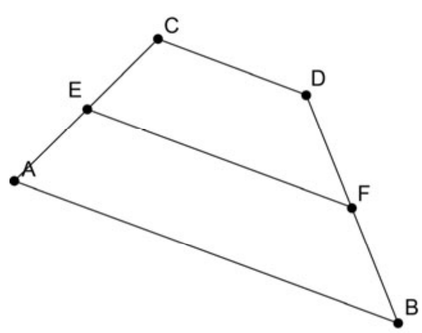

Figure 6.

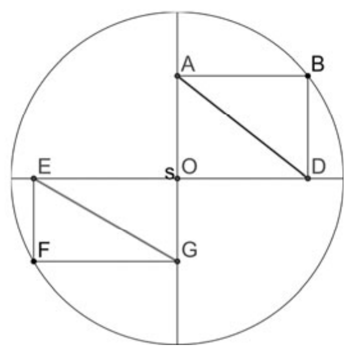

Figure 7.

APPENDIX 3: STUDENT QUESTIONNAIRE'S EXAMPLE

Now that you've finished, answer the following questions:

1. Was this problem easy or hard? Why?
2. What was hardest for you?
3. Do you usually use drawings when you solve problems? When?

4. Were you able to visualize the problem without drawing a figure?
5. What mathematical concepts and notions did you learn?
6. What kind of information would you give classmates about GeoGebra so they could solve this problem?
7. Describe your emotional reactions, your feelings and say whether you got stuck when doing the problem with pencil and paper or with a computer.

APPENDIX 4: STUDENT EVALUATION BY TEACHERS AND RESEARCHERS. MATHEMATICAL ATTITUDE EVALUATION SCALE

Below you'll find a series of questions about mathematics classes and MATHEMATICAL ATTITUDES that may be favoured by GeoGebra. Answer each question on a scale of 1 to 5, where 5 is the highest and 1 the lowest valuation. The possible meanings of each score are shown below, by way of indication:

Never or hardly ever, not at all or rarely, poor, very scant, I hardly agree at all
Infrequently, scant, poor, I don't agree very much
Sometimes, average, normal, so-so
Often, pretty much, pretty good, I agree pretty much
Always or nearly always, a lot or totally, very good, I wholly agree

Using the above scale, reply to the following questions:

FLEXIBLE THINKING

1. Lets you solve problems in more than one way.
2. Shows an interest in classmates' approach if different from his/hers.
3. Changes his/her mind if given good reasons

VISUAL THINKING

4. Prefers visual to analytical methods.
5. Prefers visual interpretations even when the problem is couched in analytical terms
6. Prefers verbal and algebraic strings
7. Pays no attention to the right relationships or graphic elements
8. Prefers to use drawings, even if they contain errors

INDUCTIVE ATTITUDE

9. Collects specific information (early enquiries, trial and error, simple calculations, short enumeration and so on)
10. Discovers through observation, regularity and consistency
11. Perceives general patterns

CRITICAL SPIRIT

12. Analyzes the solution found and reflects on whether it's right.
13. When unable to find the right answer, goes over the steps taken to check for possible errors.

14. Realizes that he/she can't find the solution or the solution found is wrong, but doesn't bother to try to find out why or to keep on trying.

PERSEVERANCE

15. Gives up easily before finding the answer to a problem.
16. When failing to find the answer to a problem, doesn't try again, but settles for the wrong answer.
17. Doesn't give up until he/she finds the answer to the problem.

ACCURACY

18. Dislikes making mistakes and is careful with arithmetic.
19. Thinks that arithmetic errors are unimportant.
20. Settles for approximate solutions. Is not rigorous.

SELF-CONFIDENCE

21. Doesn't think on his/her own or feels no need to find the solution him/herself.
22. Needs encouragement to participate in activities and constantly refers to the teacher for an opinion.
23. Needs encouragement when unsuccessful, a stimulus when stopping in the middle of the process.
24. Becomes readily involved in activities and persists when unsuccessful.

CURIOSITY

25. Takes no interest in anything: this doesn't mean a lack of curiosity, simply that it goes unexpressed under these circumstances.
26. Simply skims over the application, touches the buttons, gets bored, moves erratically from one thing to the next.
27. When is surprised by a result, begins to reorganize his/her thoughts and pose questions about facts and events, focusing on his/her subjective vision.
28. When in doubt about a situation or fact, goes back over the preceding steps. Poses specific questions that arouse class interest and lead to subsequent exploration. Makes specific observations: hi/her curiosity leads to constructive intellectual activity.

CREATIVITY

29. Simply repeats ideas by rote.
30. Likes to invent new strategies or problems.
31. Doesn't explore new or different strategies.

INDEPENDENCE

32. Prefers not to think for him/herself and asks the teacher or classmates what to do.
33. Works independently.

SYSTEMATIZATION

34. When working, knows where he/she is headed.
35. Acts by rote: he/she is unaware of the purpose of the activity.
36. Is able to synthesize calculations or results.

COOPERATION-TEAMWORK

37. Prefers to work alone.
38. Sometimes works alone and sometimes with classmates.
39. Interacts with his/her partner, they like to work together.

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11. GEOGEBRA AS A COGNITIVE TOOL

Where Cognitive Theories and Technology Meet

In this theoretical chapter, we explore the features of GeoGebra under the perspective of cognitive theories and further discuss its pedagogical implications. As a dynamic and interactive learning environment, GeoGebra is relatively new to many of us and needs to be explored, discussed, and understood under various perspectives. Our goal is to initiate this cognitive discussion and motivate researchers and practitioners to join the ongoing conceptual and practical experimentation. This chapter has two main sections. The first section reviews the literature by exploring cognitive processes and the role of GeoGebra as a cognitive tool, while the second section focuses on how to use GeoGebra to explore, explain, and model mathematical concepts and processes. In the discussions, we reflect on the examples presented and conclude the chapter by suggesting future directions.

INTRODUCTION

Much of the mystery of learning and doing mathematics stems from the fact that mathematical activities are associated with mathematical cognition and that there is a dynamic and two-way interaction between activities and cognition (Kaput, 1992; LeFevre, DeStefano, Coleman, & Shanahan, 2005). In order to identify this phenomenon, Kaput (1992) posits two worlds of mathematical activities and mathematical cognition, “a world of physical operations and a world of mental operations” (p. 522) and a continuous interaction between these worlds. The physical world mentioned here could be any medium, from pencil-and-paper environment to computer environment, to perform mathematical operations. In contrast, the world of mental operations is assumed to be located somewhere in the brain. However, despite the theories matching particular areas of brain to certain cognitive activities, brain imaging experiments suggest that networks of neural areas are reserved for these activities, even for simple tasks, rather than one specific area for each (Matlin, 2005; Posner & Rothbart, 2007).

The goal of this paper is to explore the interaction between these worlds and the use of cognitive tools to support this interaction. The challenge in learning mathematical concepts and studying these concepts demand a strong desire for cognitive tools forming a wide spectrum from paper-and-pencil to sophisticated computer software. The use of cognitive tools in mathematics education could be to create mathematical objects, to manipulate them, and to interact with them.

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 169–181.

Teachers can use cognitive tools, particularly GeoGebra, in their instructional activities (1) to explain, (2) to explore, and (3) to model mathematical concepts and the connections between these concepts. The examples we discuss in this chapter are developed to devote the readers' attention to the possible use of GeoGebra to perform these tasks and to encourage them to reflect and brainstorm on the potential hidden in these visuo-dynamic learning environments as well as their contribution to promoting concept learning (Hoyles & Noss, 2008; Kaput, 1992; Moreno-Armella, Hegedus, & Kaput, 2008).

Prior to exploring possible uses of GeoGebra to promote conceptual learning, we review the theoretical framework related to cognitive theories and cognitive tools in the following section. Then, the bulk of the chapter will be devoted to analysing GeoGebra examples and cognitive processes associated with these examples.

THEORETICAL FRAMEWORK

Cognitive theories help us to explore and understand how mathematical cognition occurs in mind, how one learns mathematics, and what learning and remembering mathematics mean. A brief review of literature suggests that we receive data as stimulus, process stimulus as information, and construct information as knowledge (Matlin, 2005; Shunk, 2008). In other words, data always resides in the physical world and becomes information when we devote our attention to them. The data perceived by sensory registers (Atkinson & Shiffrin, 1968) evolves to information while being transferred from the physical world to our mental world, working memory.

In the process of doing or learning of mathematics, data refers to mathematical objects, illustrated mainly in the physical world such as on paper or on computer screen. Those mathematical objects become information when we attempt to manipulate them or apply some domain specific operations on them. What we mean by domain specific operations could be arithmetic and algebraic operations if we talk about algebra or calculus. For example, one may want to find the points lying on a function by substituting certain values if one knows the algebraic definition of the function. Similarly, if we have a visual representation of a function, we can manipulate a GeoGebra slider to create various points lying on the function. By manipulating the slider, we actually stimulate the data, which has been already on the GeoGebra display window, launch the evolution of that specific data into information, and make it ready to be processed in the working memory. Cognitive tools, as discussed below, are used to support this evolution.

Working Memory

Working memory is a hypothetical place in brain that is responsible to process information, perceived from the physical world, to construct knowledge (Baddeley, 2007; Matlin, 2005). Working memory (WM), in contrast to long term memory (LTM), has a limited capacity and can hold only 7±2 chunks of information (Miller, 1956). LTM has relatively more capacity to store "knowledge being represented as locations or nodes in networks, with networks connected (associated) with one another" (Shunk, 2008, p. 157).

Baddeley (2007) identifies a multi-component working model to explain where mental operations occur: (1) the phonological loop, (2) the visuospatial sketchpad, (3) the central executive, and (4) the episodic buffer. The phonological loop and the visuospatial sketchpad are basically the storage systems with limited capacity whereas the episodic buffer is a more integrated storage system having connection between long-term memory and the other two storage systems. Baddeley (2007) describes episodic buffer as a cognitive workspace with flexible access to long-term memory and a significant unit in thinking. The information coming from the other parts of the working memory, from long-term memory as well as from physical environment are integrated here. The capacity of buffer is limited so the information needs to be coded by chunking.

The central executive is a unit that controls and coordinates cognitive activities, processes information to transform knowledge, and plans strategies (Baddeley, 2007; Matlin, 2005). In other words, the central executive is a unit where mathematical thinking occurs. LeFevre et al. (2005) provides a detailed review of the literature related to mathematical cognition and working memory and conclude that the literature exploring the features of working memory is useful to understand and explain mathematical cognition.

Cognitive Tools

Given that human ability to process and store information is limited (Baddeley, 2007; Matlin, 2005; Shunk, 2008), cognitive tools are used to support cognitive activities in order to overcome this limitation (Heid, 2003). The type of support could be to reduce one's cognitive load by reorganizing tasks (Sweller, 2003), to reorganize one's thinking by offering new representational systems such as visualization (Borba & Villarreal, 2005; Kaput, 1992; Pea, 1985), and to reduce one's cognitive load by storing some information at a place accessible as needed (Karadag & McDougall, 2009). In short, cognitive tools play a flexible role on a spectrum from perceiving data to constructing knowledge.

For example, the basic idea behind using computer algebra systems (CAS) is to share the computing task of mathematical activities and to free students' working memories for higher-order thinking activities (Drijvers, 2003; Heid, 2003). Heid (2003) suggests using CAS for either routine symbolic manipulations or for repeated calculations in exploring a mathematical subject or in solving a problem. Similarly, Drijvers (2003) reflects on the process-concept duality of abstraction and mathematical entities (Tall, 1991) and concludes that CAS can provoke the development of concept aspect of mathematical objects by performing procedural tasks.

GeoGebra serves for a similar goal by demonstrating multiple representations of mathematical objects. It allows users to create certain mathematical objects by using one representation system (i.e. algebraic) and to observe their demonstrations in other representations (i.e. algebraic, visual, and verbal) simultaneously. Moreover, it is possible to manipulate objects and observe the timely change in their attributes. That is, one can create a slider to control and observe consecutive positions of a point moving on a function. Furthermore, GeoGebra extends our understanding by allowing us to add more elements to devote our attention at a specific attribute. For example, in order to make the ordinate of a point more visible, one can import an extra object, a picture of an animal, to the GeoGebra dynamic worksheet (see Figure 1).

In summary, GeoGebra supports users' cognitive activities by making mathematical objects, including concepts and relationships, ready to perceive from the physical environment and to deliver to working memory to be processed for knowledge. The effectiveness of cognitive tools depends on both their own features and the skills of users. While developers will continue to improve features of GeoGebra, we discuss here the possible uses of GeoGebra, such as explaining, exploring, and modelling, in mathematics education and the cognitive processes associated with these uses.

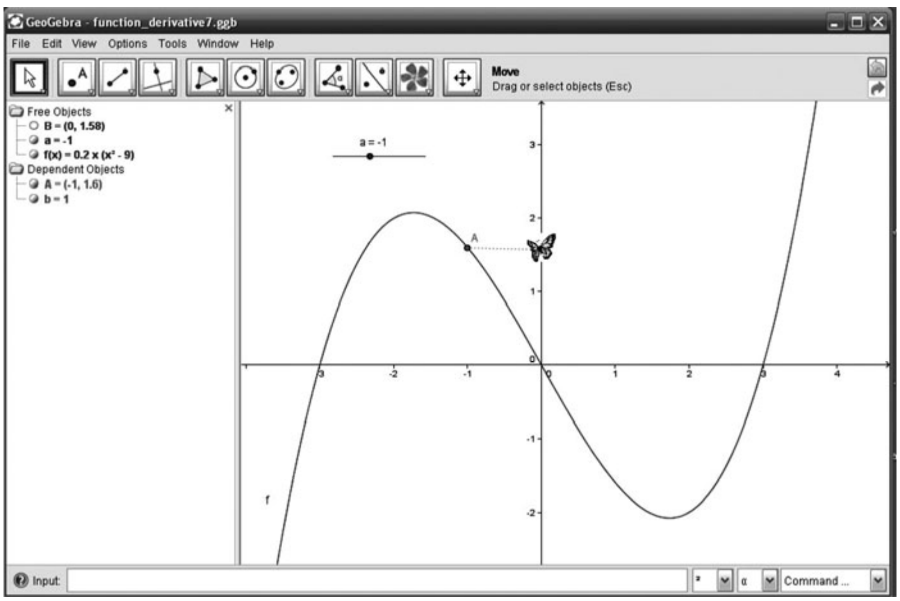

Figure 1. A butterfly moving along the ordinate of point A.

GEOGEBRA EXAMPLES

Referring to Kaput's (1992) *two-world* metaphor, we have identified computers as the physical world, working memory as the mental world, and GeoGebra as the cognitive tool supporting the interaction between these worlds. We have also described how GeoGebra could support this interaction. This section moves beyond that description to incorporate Kaput's metaphor into GeoGebra examples in the context of function-derivative relationship.

In order to incorporate the dynamic feature of the software into the context, let us start by creating a slider (see Figure 2). Although GeoGebra allows dragging mathematical objects to observe their change in time, the slider makes this change more explicit, and therefore, the interaction between objects becomes more visible and perceivable. Teachers can define the boundary values as well as increments to be performed at each step so that students can observe all of the reactions. Besides algebraic and visual representations illustrated in this paper, one can employ other representations (i.e. numeric and verbal) to make math objects more perceivable.

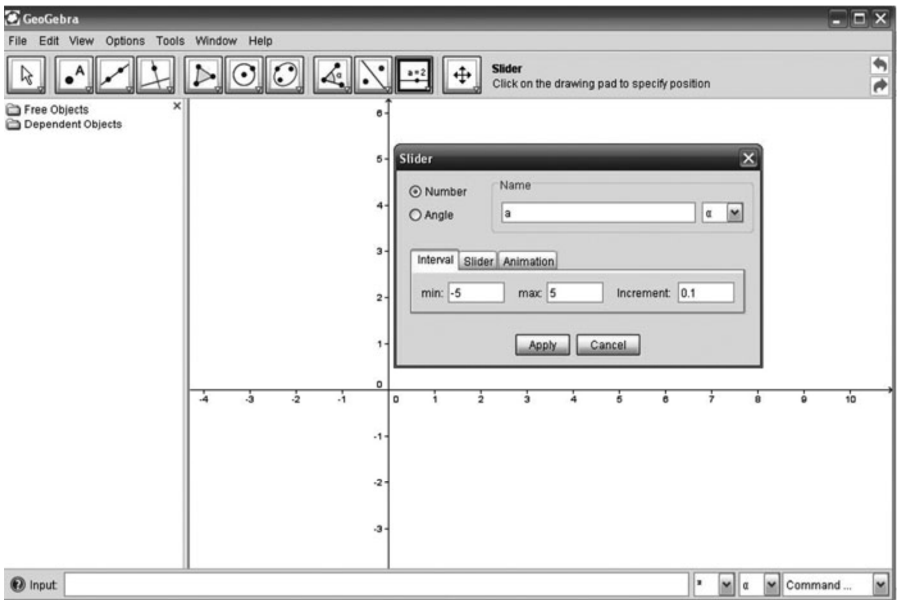

Figure 2. Creating a slider by identifying name, extreme values, and increments.

After creating the slider, teachers can define a function, $f(x) = 0.2x(x^2 - 9)$, and a point on the function, *A (a, f(a))*, where *a* stands for the name and value of the slider. Therefore, each change in the value of the slider is reflected as a movement of the point on the function, because the slider calculates the ordinate of the point by considering the function *f(x)*. Before moving further and analyzing the ways of using GeoGebra in the learning of function-derivative relationship, we need to create a tangent line from the point A to the function *f(x)*. Since tangents are objects used frequently in mathematics, GeoGebra has a specific tool located in the upper menu to create it simply by clicking on the icon and choosing the related element. Now, the stage has been set to move further and to talk about the use of GeoGebra in educational settings.

Explaining

Teachers can use GeoGebra to explain mathematical concepts and relations between these concepts. This section suggests a framework on how teachers can use GeoGebra to explain the function-derivative relationship while the function increases, decreases, and reaches to its extreme points by building on the scenario mentioned in the previous section. The scenario consists of a dynamic worksheet having a function *f(x)*, a slider *a*, and a point A (see Figure 3). This dynamic worksheet could be considered as an example of physical medium (Kaput, 1992) to do and learn mathematics. In fact, GeoGebra acts both as a physical medium and as a cognitive tool in this scenario, because, on one hand, it accommodates mathematical objects, and on the other hand, it encodes data to provide support for cognitive activities.

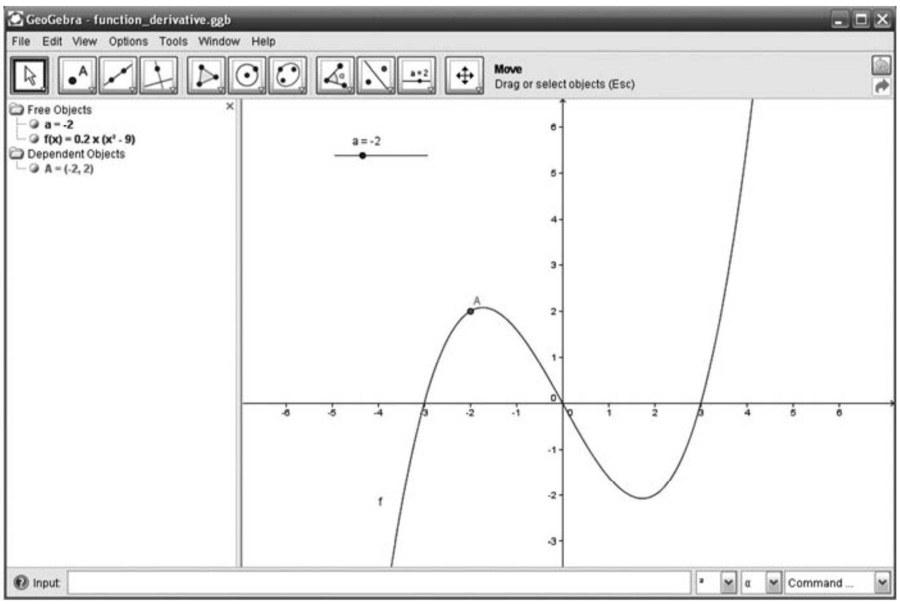

Figure 3. A dynamic worksheet illustrating a slider, a function, and a point moving on the function both on display and algebra windows.

Teachers can start their explanations by evoking students' previous knowledge of functions. Given that students are ready to study the function-derivative relationship, they should have already taken the algebra courses related to the basic features of functions. By evoking their previous knowledge prior to explaining the function-derivative relationship, students become ready to learn new content. Technically speaking, they become ready to create connections between the new content and the old knowledge. This evocative action could be to demonstrate the movement of a point on the function and the relation between its abscissa and ordinate.

There are at least two opportunities to demonstrate the movement of point A: (1) visual representation on the display window of GeoGebra as illustrated on the right part and (2) algebraic representation on the left (see Figure 3). Each value assigned to the slider moves the point A to a different position on the function. While demonstrating this movement dynamically, teachers can lead students' attention to the algebraic representation to verify the relation between abscissa and the ordinate. They can ask students to verify those numeric values through other means such as paper-and-pencil or calculators. The link between those representations, provided by GeoGebra, could help students transfer their visual perception to a formal algebraic notation and consolidate the abstract relation between the point A and the function *f(x)*.

Given that the procept (Tall, 1991) of point-function relationship is evoked, teachers can introduce the concepts of increasing, decreasing, and local extreme values of the function. In a regular calculus course book, the left and right parts of the function (see Figure 3) are identified as "increasing." In fact, the term "increasing" points out a dynamic process, which is quite difficult to understand in a static media. "What does it mean?" and "What is increasing?" are some of the questions that could be raised by novice students. The dynamic feature of GeoGebra provides teachers with opportunities to demonstrate *what is increasing* visually and, then, to formalize this visual information algebraically. Teachers can direct students' attention to the change in the ordinate of the point A while altering the slider value. Each increment in the slider value moves the point A to a new position on the function. Therefore, students are able to observe the increase in the ordinate while abscissa of the point increases.

Each representation created by manipulating the slider can be seen on their own window. For example, the visual information illustrating the movement of the point A resides on the display window. Because the slider also resides on the display window, we can assume, so can students, that visual actions, which are performed on the display window, result in visual representations, which are illustrated on the same window and similarly that algebraic representations, which are associated with visual representations, are represented on the algebra window. In reality, GeoGebra links both representations and allows users to access and control them through one control element, the slider in this scenario.

The demonstration of the dynamic illustration in one representation and the link between two different representations could help students to better process the information and develop a better understanding. "Perception (pattern recognition) refers to attaching meaning to environmental inputs received through senses. For an input to be received, it must be held in one or more of the sensory registers and compared to knowledge in LTM" (Shunk, 2008, p. 141). Once students start comparing this new information to their knowledge already stored in their LTM, teachers can move beyond the current discussion.

Referring to the goal of the illustration stated at the beginning, teachers can demonstrate the layout of the tangent line controlled by the slider, or connect it to the various images of point A visually and algebraically and explain the theory behind this demonstration (see Figure 4). They can explain how tangents behave when the function increases or decreases. Similarly, they can explain why the derivative of a function, the slope of its tangent, becomes zero when the function reaches to its local extreme values. Again, students have the chance to compare various representations and develop an understanding of this relationship.

The second phase, processing information to build knowledge, is also an issue that teachers should deal with and that educational researchers as well as cognitive scientists could explore. Some of these issues are curricular issues because students' previously constructed knowledge guides them to identify the information to be perceived and to be processed. However, teachers play a significant role while students are processing information because they can engage students to create conjectures and to share and discuss the validity of the conjectures with their peers.

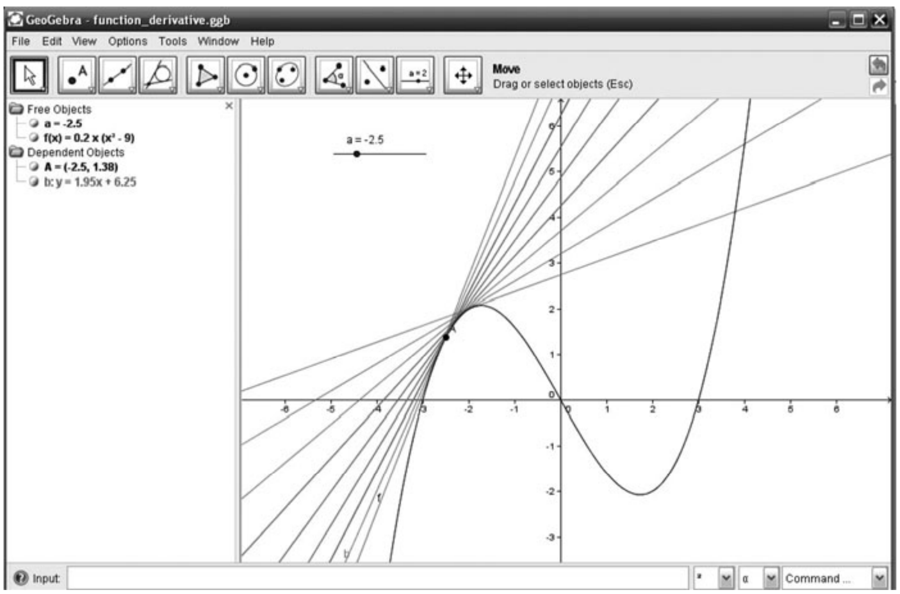

Figure 4. Tangents drawn to the increasing part of a function.

In sum, teachers can develop many scenarios, one being addressed here, while using GeoGebra to explain mathematical concepts. Similarly, educational researchers can develop many research questions to explore the mechanism of understanding and information processing in this dynamic learning environment. GeoGebra supports this understanding process by (1) providing various opportunities to represent mathematical objects, (2) allowing to create dynamic illustrations controlled by sliders, and (3) linking multiple representations. The reason we used the expression "multiple" rather "two representations" here is because one can also create other representations in addition to and linked to visual and algebraic representations. For example, the dynamic text residing above the slider on the display window (see Figure 5) provides a verbal-numeric representation of the slope of the tangent.

Exploring

Learning by exploring demands more cognitive work performed by students compared to explaining because they are expected to develop an understanding of the curricular material by themselves. However, if students are misguided, or not guided at all, during the exploration, they may not accomplish understanding of the content. Teachers need to develop relevant materials or scenarios for students to explore. Furthermore, they should provide formative feedback on a timely basis to guide students.

As discussed in the previous section, teachers start exploration by providing the dynamic worksheet with a slider and a function with a point on it and asking questions in order to encourage students to activate their prior knowledge, stored in their LTM. For example, they can ask students to create two points corresponding to certain values of the slider, say -2 and +3, and calculate the distance between them. Indeed, this is a very good problem to start exploration with because it also engages students to connect their analytic geometry knowledge to their function knowledge. Cognitive science literature provides theoretical principles that support this type of connections.

According to the neural network theory of cognitive science, the human brain consists of a network of nodes and connections between the nodes (Martindale, 1991; Massaro & Cowan, 1993; Matlin, 2005). Moreover, the theory states that connections between nodes have different strengths and that the degree of this strength determines how well knowledge is learned. Therefore, if students are exposed to different representations of knowledge, they are likely to create stronger connections among the conceptual nodes. They recall knowledge related to analytic geometry and functions from their LTM to the episodic buffer of working memory, where they process information regarding two points on the function illustrated on the GeoGebra dynamic worksheet, and process this new information under the control of their previously constructed knowledge. The central executive unit controls these processes, which occur in episodic buffer (Baddeley, 2007).

Similar to the explaining scenario, students could continue exploring the behaviour of the function by focusing on the questions "What does increasing, or decreasing, mean?" and "What is increasing or decreasing, while the abscissa of the function is increasing?" Questions addressing these specific areas of the content could help students develop an understanding of the concrete meanings of the terminology rather than memorizing them as mathematical facts. While exploring in the GeoGebra environment, they have at least two different representations, visual and algebraic, and two cognitive tasks associated to these representations. In this particular example, students are expected to perceive visual and algebraic information and to work on the scenario by considering both representations. The theories addressing information processing in human brain claim that human brain differs from computers because a human is able to process information in parallel rather than in serial whereas most computers process in serial (Shunk, 2008).

Furthermore, students could be asked to observe the change in the slope of the tangent lines around the local maximum point and describe the relation between the function and the slope of the tangent, namely its derivative, at various points (see Figure 5). They can make this observation by exploring the behaviour of the tangent line and the slope of tangent as well as observing the written message as a dynamic text (see Figure 5). Similarly, they could make this observation by examining the change in the equation illustrated explicitly on the algebra window. The effectiveness of these windows and how students pick up and process various forms of information provided on these windows would be interesting for further research.

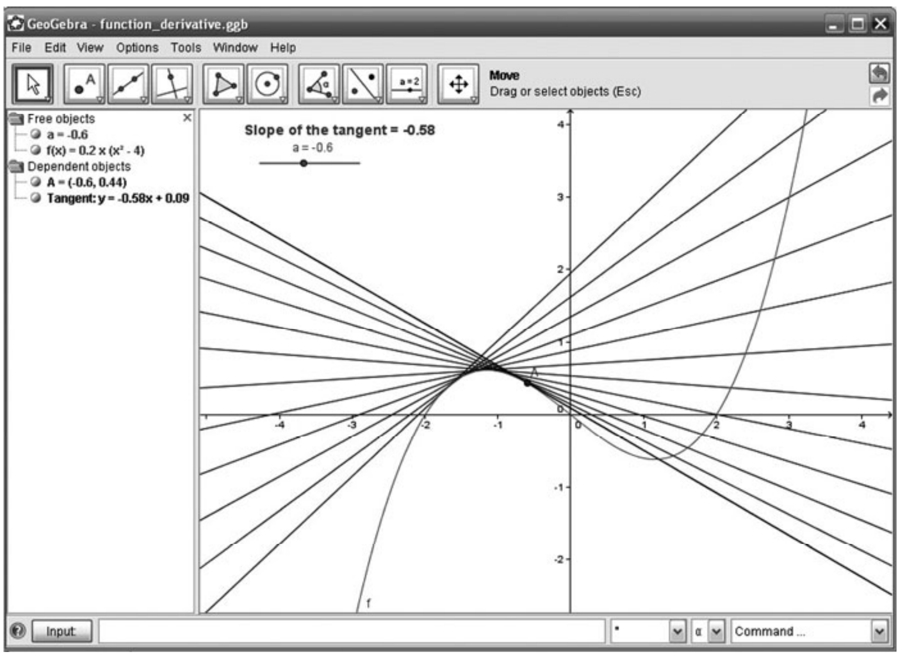

Figure 5. Tangents around the maximum.

Modelling

The ability to model mathematical concepts requires an ability to understand these concepts and their features. Therefore, mathematical modelling demands much more cognitive work in contrast to explaining and exploring. It is because modelling demands a substantial amount of content knowledge and an advanced understanding of the content. The theory supporting modelling in learning (Milrad, Spector, & Davidsen, 2005) advocates that the learning experience better start with explanations and/or explorations and that continue with "the opportunity and challenge to become model builders, to exchange and discuss models with peers, and to experiment with models to test hypotheses and explore alternative explanations for various phenomena" (p. 30). This learning sequence helps students to strengthen the connection between their knowledge nodes (Martindale, 1991; Massaro & Cowan, 1993; Matlin, 2005). In this section, we illustrate an example that shows how models can be used to represent a topic from physics: harmonic motion.

The algebraic representation of a harmonic motion could be either a sine function, *f(x) = A sin(Bx +C)*, or a cosine function, *f(x) = A cos(Bx +C)*, depending on its initial position. For example, if the particle starts moving from its reference point, then *f(x) =A sin(Bx +C)* is used. One description of sine function is vertical projection of the radius of a circle.

Teachers can ask students to create a model representing this relationship. In order to perform this representation, students need to remember what they have already constructed and stored in their long-term memory regarding circle, radius of circle, and its projection on the axes. They could create a circle with the radius r, controlled by a slider, to illustrate the sine function (figure 6).

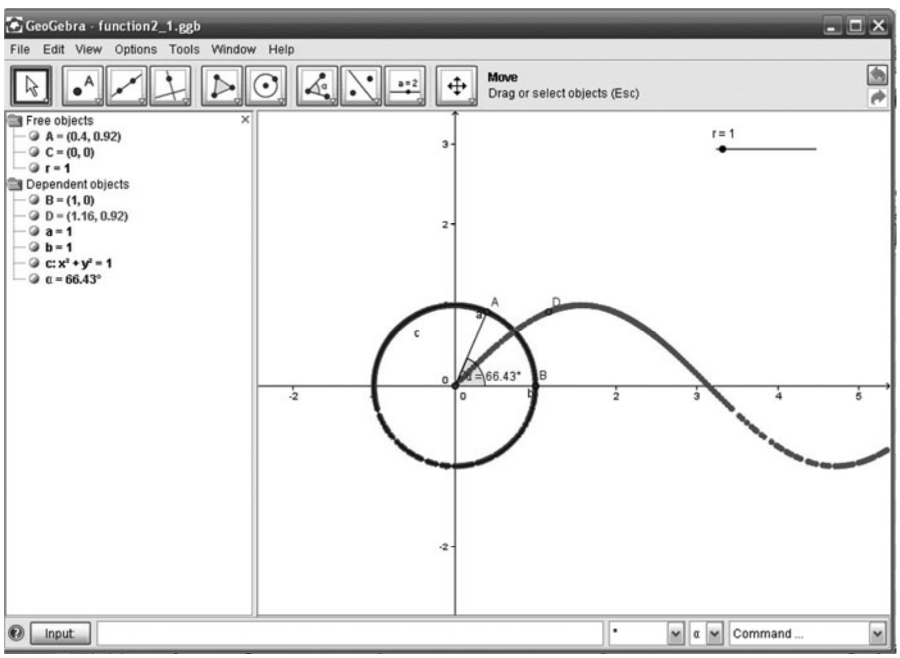

Figure 6. The sine graph as vertical projection of the point A on the circle.

DISCUSSION

Today's students are more familiar with visual learning because they learn many new technologies such as computers, Internet, and cell phones visually. Therefore, it might be very challenging for them to learn symbolic algebra first. Rather, they may better understand algebraic notations after they developed a visual understanding of mathematical concepts. Based on our analysis of students' learning preferences, we suggest that instructors develop better ways to integrate visual and algebraic representations into their learning activities.

Thus, the next step for us could be to explore the effects of visual learning, which is comparatively new for many educators. Also, we may need to understand how students transfer the outcomes of this dynamic and visual learning to the algebraic world of mathematics. Given that algebraic notations have a significant role in the use of mathematics in other disciplines, such as engineering and economics, how does visual learning affect students' experiences in advanced mathematics and its applications?

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12. GEOGEBRA AS A METHODOLOGICAL RESOURCE

Guiding Teachers to Use GeoGebra for the Construction of Mathematical Knowledge

The aim of this chapter is to stress the role of GeoGebra as a methodological resource. In particular, we contend that teachers need to become aware that an appropriate integration of GeoGebra within the classroom activities could foster the construction of mathematical knowledge. As a consequence, educators need to guide teachers to perceive GeoGebra as a methodological resource so that they would be able to effectively use it. As a theoretical framework we mainly refer to the "instrumental approach" and to the idea of "mathematics laboratory as a Renaissance workshop". We also suggest that, when teachers are "immersed" in appropriate non-standard learning situations, they could experience by themselves that GeoGebra can be very useful for creating a meaningful mathematics learning environment.

INTRODUCTION

Many researchers are now offering greater educational opportunities using tools in the teaching and learning of mathematics (e.g., Lagrange et al., 2003). The ideas presented in this chapter are connected to an ongoing research study developed within a local institution for pre-service teacher training (SSIS) in Puglia in the South of Italy. This study focuses on the use of technological tools as methodological resources to support mathematics teaching and learning activities. In this paper, we aim to present this point of view; and according to the early results of our research, we will try to make some didactical suggestions.

Our main research assumption is that technological tools such as GeoGebra can assume a crucial role in supporting the teaching and learning processes because they allow teachers to create suitable learning environments with the goal to promote the construction of meanings for mathematical objects. In this sense, GeoGebra can be considered as a methodological resource.

However, as underlined by Mously, Lambidin, and Koc (2003), it cannot be taken for granted that technological advances alone can change essential aspects of teaching and learning simply because they can bring about opportunities for change in pedagogical practice.

The Italian Committee for Mathematics Education, for example, highlights in an official document (UMI-CIIM, 2004) that:

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 183–189.

> The meaning cannot be only in the tool per se, nor can it be uniquely in the interaction of student and tool. It lies in the aims for which a tool is used, in the schemes of use of the tool itself. (p. 32)

Innovative learning environments can result from the integration among educational and cognitive theories, technological opportunities, and teaching and learning needs (Bottino, 2000). We believe that teachers should recognise the need for an effective integration of technologies in classroom activities and will see new technologies as cultural tools that radically transform teaching and learning only if they become aware of the potential usefulness and effectiveness of a technological tool such as GeoGebra as a methodological resource, which enables them to foster the construction of a meaningful learning environment (Faggiano, 2009).

The basic use of GeoGebra, for example, can be easily and quickly learned; but, even though GeoGebra allows an interaction between the visual and theoretical aspects of geometry, it is often used to facilitate visualisation more than act in the solution process.

According to Borba (2005), there continues to be resistance to the use of technology in educational environments. Many scholars do not consider the potential role of computers in the reorganisation of thinking and the changes in contents or teaching strategies, and many teachers, too, consider computers only as a tool that expands human memory, increases the turnaround of feedback, and enhances the possibility of generating images.

If GeoGebra is used only as an auxiliary to show a graph or a dynamical geometrical construction, students would have no opportunities to learn mathematics with the computer because that is not part of mathematical contents.

THEORETICAL FRAMEWORK

A very important idea on which this study is based is the idea of “mathematics laboratory,” which comes from both empirical and theoretical studies and is summarised, for example, in the UMI-CIIM (2004) document as follows:

> A mathematics laboratory is not intended as opposed to a classroom, but rather as a methodology, based on various and structured activities, aimed to [promote] the construction of meanings of mathematical objects. (p. 32)

In this sense, a laboratory environment can be seen as a Renaissance workshop, in which the apprentices learn by doing and communicating with each other about their practices. In particular, in the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools; and on the other, to the interactions between people working together, without distinguishing between the teacher and the students.

According to this approach, in the creation of suitable learning environments aiming to construct mathematical knowledge, technological tools assume a crucial role in supporting the teaching and learning processes. However, as claimed by Laborde (2002):

> . . . Whereas the expression “integration of technology” is used extensively in recommendations, curricula and reports of experimental teaching, the characterisation of this integration is left unelaborated. (p.285)

In particular, as already stressed in Faggiano (2009), Laborde underlines the idea that the introduction of technology in the complex teaching system produces perturbation and, hence, for a teacher to ensure a new equilibrium, he/she needs to make adequate, non-trivial choices. Integrating technology into teaching takes teachers time, because first they need to understand why and how learning might occur in a technology-rich situation and then become able to create appropriate learning situations.

This point of view is based on the idea that a computational learning environment could promote the learners’ construction of situated abstractions (Noss & Hoyles, 1996; Hölzl, 2001) and on the “instrumental approach” as developed by Vérillon and Rabardel (1995).

Within the instrumental approach, the expression “instrumental genesis” has been coined to indicate the time-consuming process during which a learner constructs an instrument from an artefact. It is a complex process, both individual and social, linked to the constraints and potentials of the artefact and the characteristics of the learner.

According to the instrumental approach learners need to acquire non-obvious knowledge and awareness to benefit from an instrument’s potentials. However, we also contend that teachers need to take into account the student’s instrumental genesis (Trouche, 2000), which helps the teacher to integrate artefacts into their teaching practices. Pre-service and in-service teacher training need to take into account three different levels of complexity of this integration (Trouche, 2003): first, a mathematical one in that new environments require a new set of mathematical problems; second, a technological one, which seeks to understand the constraints and the potentials of artefacts; third, a psychological one, which seeks to understand and manage the instrumentation process and their variability.

TECHNOLOGICAL TOOLS AS A METHODOLOGICAL RESOURCE

Early Results of an Ongoing Investigation into Teachers’ Perceptions

In the last two years, an anonymous questionnaire was submitted to both pre-service and in-service teachers within the local institution for the pre-service teacher training in Puglia and in many regional high schools, with the aim of understanding if teachers see technologies as learning resources, and to see if they were willing to stay up to date in order to properly design and manage technology-rich classroom activities (Faggiano, 2009).

Findings revealed that most teachers perceived that technology can support their teaching, but only as far as it is a motivating tool that, just by using it, can enable students to understand. Some teachers recognise that, if nothing else, knowledge of how the instrument works is probably not enough to allow a teacher to use it in an effective way.

Being aware of the opportunity to create a new learning environment and change the “economy” of the problem solving process was, however, extremely far from their perception of the use of technology in mathematics teaching/learning activities, both for in-service and for pre-service teachers.

They mostly did not feel the need to be skilled in using technology for their teaching and did not usually consider that their lack of skills caused them any difficulties. Although many teachers recognised the need to have some didactical competence in order to use the new technology, what they asked to know about was, in most cases, simply how to use the software, neither potentials nor constraints. Only some of the pre-service teachers asked to know how to effectively integrate technology use into the teaching practice.

Guiding Teachers to Perceive and Use GeoGebra as a Methodological Resource

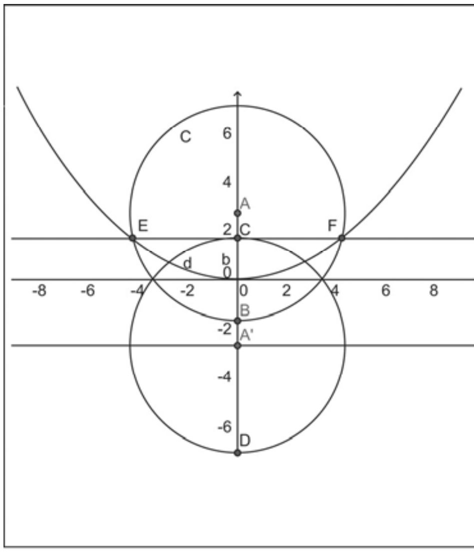
At the present stage of our ongoing research, it can be claimed that most teachers have difficulty in acquiring the awareness of the potentials of technology as a methodological resource. In this sense, GeoGebra is no different. As a matter of fact, we believe it is extremely important to guide teachers to develop a more suitable and effective awareness of the usage of GeoGebra. According to Faggiano (2009), the difficulty teachers have in acquiring this awareness could be overcome by giving them the opportunity to be the subject of a *mise en situation*. That is, we should allow teachers to be an active part of a learning situation, engaging them to solve some unusual problems which require non-standard strategies. In this way, teachers can experience by themselves the difficulties students may encounter, the cognitive processes that they can put into action, and the attainments they can achieve. They also have the opportunity to understand and manage students' instrumental genesis and to become more skilful and self-confident when deciding to exploit the potentials of the software in mathematics education.

In particular, the experiences of *mise en situation* should be promoted to allow teachers to experience the important relationship between the specific knowledge to be acquired by the students and the knowledge the teacher possesses, and that between the specific knowledge to be acquired by the students and their prior knowledge.

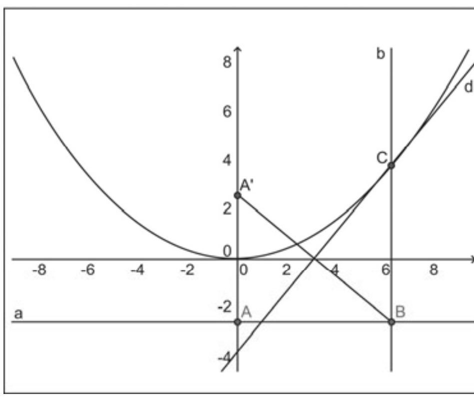
GEOGEBRA EXAMPLES

When a problem of construction is tackled with paper and pencil, attention is directly focused on the drawing itself by observation or by measurement. With dynamic geometry software like GeoGebra, the focus is shifted from the drawing to the thinking processes carried out to realise a *correct* construction.

To draw a parallelogram correctly with GeoGebra, for instance, you need to know and apply the properties of the shape in relation to the tools you have. This can be done in more than one way and, in each case, the shape you have will react in different ways to the dragging test. The interesting thing is not making the different constructions but to foresee and to analyse their behaviour under dragging. This kind of activity allows the learner (both students and the teacher) to understand the logic of the software and to focus on the geometric properties. Moreover, as research has already shown, the construction and the observation of dynamic geometry drawings can foster the construction of some important meanings for mathematical processes such as exploring, defining, verifying, conjecturing, validating.



Construction protocol A
1 Point A - Point on y-Axis
2 Point A'- A mirrored at x-Axis
3 Line a - Line through A' parallel to x-Axis
4 Point B - Point on y-Axis
5 Circle c - Circle through B with center A
6 Segment b - Segment [A, B]
7 Circle d -Circle with center A' and radius b
8 Point C -Intersection point of d, y-Axis
8 Point D -Intersection point of d y-Axis
9 Line e - Line through C parallel to a
10 Point E- Intersection point of c
11 Locus loc1-Locus[F, B]
12 Locus loc2 - Locus[E, B]



Construction protocol B
1 Point A - Point on y-Axis
2 Line a - Line through A parallel to x-Axis
3 Point A' - A mirrored at x-Axis
4 Point B -Point on a
5 Line b - Line through B perpendicular to a
6 Segment c - Segment [A', B]
7 Line d - Bisector c
8 Point C Intersection point of b, d
9 Locus loc1 - Locus[C, B]

Figure 1. Two different approaches to the construction of a parabola.

When using GeoGebra, teachers and students need to be aware of the limitations and potentials of the software. This can be done by doing appropriate activities such as the comparison between constructions carried out by different sequences of steps. Observations and reflections, furthermore, can be usefully promoted by asking students to fill in sentences like the following: If I drag . . . , then . . . moves directly or indirectly because . . . . As shown in Figure 1, for example, a parabola can be obtained in two different ways.

What can be very interesting is to ask learners to do these two different constructions, to verify the robustness of the two constructions to dragging, and to further explain the reasons why both of them can give a parabola.

CONCLUSION

According to the early results of our research study, we contend that teachers need to be guided in recognizing GeoGebra as a methodological resource. In particular, we believe that teachers should be given opportunities to tackle the obstacles that students might encounter. The main idea of this chapter, indeed, is that if "immersed" in appropriate learning situations, teachers can experience by themselves the processes that come into play when bringing GeoGebra into teaching/learning situations: They can understand the cognitive and meta-cognitive processes involved and the results that can be achieved. In this way, teachers can cope with the changes that may be encountered in a GeoGebra learning situation. Moreover, reflecting on their experience, they can also learn how to manage with the students' instrumental genesis. Finally, through the *mise en situation*, teachers can understand how to make better use of Geogebra as a methodological resource in order to create an effective and meaningful learning environment that fosters students' construction of mathematical knowledge.

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CHRIS LITTLE

13. APPROACHES TO CALCULUS USING GEOGEBRA

This chapter outlines how GeoGebra can be used to facilitate a high school treatment of calculus, including differentiation of polynomial, trigonometric and exponential and logarithmic functions, and Riemann integration. While many of the functions of GeoGebra can be performed using graph plotters or other interactive geometry packages, the simplicity of GeoGebra's architecture, with its separate coordinate geometry and algebra windows, is particularly suited to calculus, whose pedagogic challenge is to coordinate algebraic and geometric concepts. Rather than following over-prescribed, step-by-step instructions or using pre-packaged files, this simplicity enables teachers, and indeed students, to focus on the mathematics and construct their own dynamic drawings from scratch and, in doing so, 'reinvent' concepts and results for themselves.

INTRODUCTION

The teaching of calculus has often been regarded as problematic. Its pedagogic challenges are perhaps not surprising, given its long period of gestation in the history of mathematics. In about 450 BC, Zeno proposed his well-known paradox of the arrow which highlighted the problem of modelling motion and infinity:

> If a body moves from A to B, then before it reaches B it passes through the mid-point, say B_1 of AB. Now to move to B_1 it must first reach the mid-point B_2 of AB_1. Continue this argument to see that A must move through an infinite number of distances and so cannot move. (O'Conner & Robertson, 1996, para. 3)

By 225 BC, Archimedes, in the first known example of the summation of an infinite series, had succeeded in calculating the area of a segment of a parabola by constructing an infinite sequence of triangles; he also used a method of exhaustion to find an approximation for the area of a circle and had succeeded in 'integrating', amongst others, the volume and surface area of a sphere, the volume and area of a cone and the surface area of an ellipse (Boyer, 1985, p. 143).

However, no further progress in calculus was made until the 16^{th} and 17^{th} centuries, culminating in the work of Newton and Leibnitz, who are commonly regarded as the founders of modern calculus. Moreover, early attempts to interpret 'infinitesimals' or 'fluxions' were heavily criticised by figures such as Bishop Berkeley for their logical flaws and it was not until the 19^{th} century that a fully rigorous analytical treatment of infinite limiting processes had been developed by Gauss and Cauchy (Boyer, 1985).

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 191–204.

A modern student is required to assimilate ideas, which took the world's greatest mathematicians two millennia to develop, and overcome natural intuitions about the nature of infinity, which are often faulty. For example, it seems clear that an unbounded area between curve and *x*-axis such as that shown in Figure 1 must be infinite. It is counter-intuitive to learn that when the curve is $y = 1/x$ it is indeed infinite, whereas when the curve is $y = 1/x^2$, it is not.

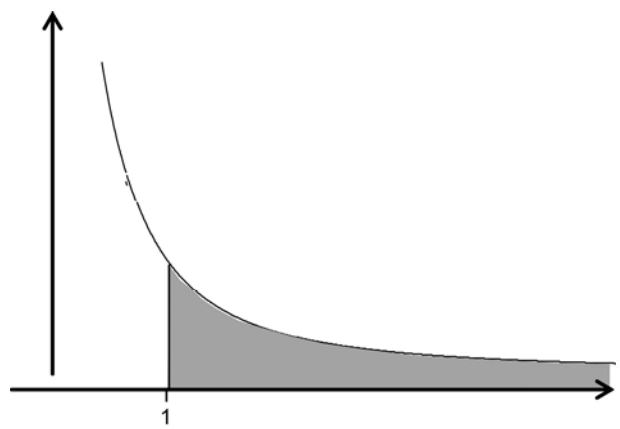


Figure 1. The unbounded area between a curve and the x-axis.

What are the pedagogic challenges to teaching calculus? As well as overcoming misconceptions about infinity, students of calculus are required to visualise geometrical ideas such as gradient, tangent at a point, continuity, area, and graphs of functions, and coordinate these with algebraic concepts and skills such as rates of change, polynomial arithmetic, binomial expansions, algebraic fractions, and the algebraic properties of trigonometric, exponential and logarithmic functions. They need to become fluent with certain notations which are, initially at least, entirely foreign, such as that used for derivatives,

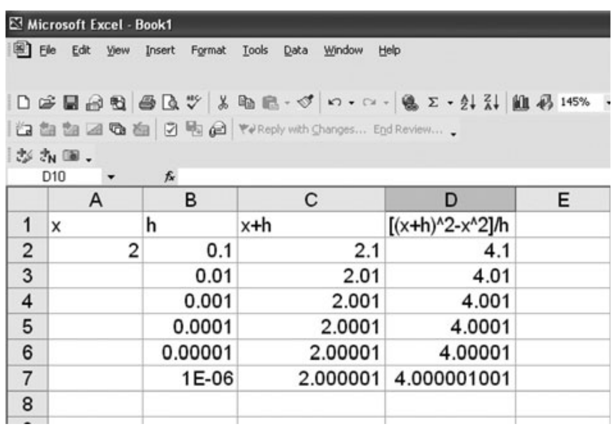
$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Before embarking on a calculus unit, the teacher has to make a number of decisions. What is an appropriate order of presentation? For example, should the Riemann integral and approximate methods of integration precede anti-differentiation? How much rigour should the ideas be presented with? Which notation should be introduced, and when? To what extent should results be established through some level of proof, for example differentiation from first principles, or should fundamental results be stated without attempts at proof, in order to allow students to develop the skills and techniques to apply results to problems?

Many of these issues will be subject to the curricular constraints placed upon the classroom, such as examination syllabi, the timing and organisation of courses, and the abilities, prior knowledge and skills of students. However, technological advances over the last forty years have provided the teacher with powerful tools which enable calculus to be made more accessible to students. These may be divided roughly into tools which have enhanced arithmetic and algebraic computation and those which have enhanced our ability to visualise functions and geometrical ideas. Pocket calculators have superseded logarithmic tables and slide rules, enabling the student to investigate limits such as $\lim_{h\to 0}\frac{(2+h)^2-2^2}{h}$ numerically by quickly performing calculations such as $\frac{2.1^2-2^2}{0.1}$, $\frac{2.01^2-2^2}{0.01}$, and so on. In the 1970s, the first PCs were running short Basic programs to automate the calculation of limiting processes, for example:

```
10    Input X, h
20    LET Y = ((X+h)^2 – X^2)/h
30    PRINT Y
```

Spreadsheets were then developed and routinely used to perform limit calculations more easily and efficiently (see Figure 2).



	A	B	C	D	E
1	x	h	x+h	[(x+h)^2-x^2]/h	
2	2	0.1	2.1	4.1	
3		0.01	2.01	4.01	
4		0.001	2.001	4.001	
5		0.0001	2.0001	4.0001	
6		0.00001	2.00001	4.00001	
7		1E-06	2.000001	4.000001001	
8					

Figure 2. Estimating the limit of an expression using a spreadsheet program.

The development of Computer Algebra Systems (CAS) such as Derive has enabled the derivatives and integrals of functions to be automated. The visualisation of functions and their graphs has benefited from the development of graph-plotting software, and graphic calculators. In the UK, a commonly used graph plotting program is Autograph (Butler & Hatsell, 2003), which allows the user some dynamic control over images. For example, a point can be placed on a graph; a tangent can then be constructed, with its equation displayed in a status box; the point can then be dragged round the curve to investigate the gradient of the moving tangent. Similarly, using this software, the gradient of the tangent as the limit of that of a chord PQ as Q approaches P can be illustrated. Customised buttons for constructing derivatives and area integrals, numerical methods such as fixed point and Newton-Raphson iteration, differential equations, etc., are also available.

Autograph has developed over a period of 20 years to include tools that cover a wide range of mathematical tasks as mandated by the post-16 curriculum in UK schools. A statistics window enables data processing and probability distributions to be illustrated, and three-dimensional vector geometry can now be explored. However, this multi-tasking requires a sophisticated user interface, with over 50 'buttons' to operate different functions. This level of functionality places considerable demands on the professional development of mathematics teachers to exploit it to the full. Moreover, my experience suggests that unless the software is used on a regular basis, it is easy to forget which button to use at a vital point in the lesson!

Just as graphing software has expanded to include some elements of the moving image, dynamic geometry software such as Cabri Geometrie and Geometers' Sketchpad (GSP), developed in the 1980s in France and the US, has extended its original functions to include coordinate geometry tools, thus allowing them to be used in developing calculus ideas. Cabri was originally conceived for Euclidean constructions, and only later developed measuring and transformation geometry tools. Both these packages can be used to enhance the teaching of calculus–a notable example of software-led curriculum development is the work of Clements, Pantozzi, and Steketee (2002), which provides a comprehensive course in calculus, together with bespoke GSP files accompanied by detailed worksheets.

From a UK perspective, however, this worksheet-led approach may be seen to have a number of drawbacks. On the one hand, as is the case with Autograph, the GSP user interface requires considerable knowledge of the software to allow teachers to develop files from scratch, and this limits the flexibility of the materials for classroom instruction purposes. If used by students with the worksheet material, the attention of the student may be distracted away from the mathematical ideas by the details of operating the software. Moreover, the depth of treatment intended by Clements et al., (2002) requires a substantial investment of time by teachers or students.

Despite the exciting opportunities offered by Cabri and GSP for enhancing the teaching of calculus, recent surveys suggest that these programs are less well utilised than graph plotters in UK classrooms (Fischer Trust, 2004). The reasons are explored in some detail in Little (2008) and more generally in Cuban (2001). They may be summarised as follows. It is possible that the technical challenges of using dynamic geometry software for calculus in the classroom have been underestimated. In order to feel confident in their use of technology, teachers need to develop, and maintain, facility with the software; the amount of software-specific know-how required has therefore to be kept to a manageable level, with readily available and understandable on-screen help at hand in case of difficulties; and the design of the software needs to be fit for purpose, rather than adapted to the purpose by extending the original range of functions by adding a plethora of buttons.

By contrast, GeoGebra seems to offer some advantages to teachers in overcoming these barriers to calculus. First, the default presentation of the screen provides the teacher with a geometry window adapted to the requirements of coordinate geometry and an algebra window that enables easy input of algebraic variables and functions. Thus both the geometric and algebraic facets of calculus are readily represented. Second, the menu structure, with its on-screen help, is easy to learn and retain, and its logic supports the choice of the correct function. Third, this ease of use encourages a belief that teachers, and indeed students, may be able to learn the mechanics of the software without the need for extensive professional development, and, in turn, concentrate their attention on the mathematics.

The aim of the rest of this chapter is to sketch how a course in calculus might be supported by GeoGebra. A number of lessons are outlined to illustrate how learning sequences based on GeoGebra files developed from scratch require relatively little prescription. They are therefore adaptable to both the needs of teachers using them to develop and discuss ideas for groups of students, and the needs of students to develop and 're-invent' the concepts of calculus for themselves. The belief – yet to be turned into a reality - is that GeoGebra, through its organic presentation and design, might become a tool as natural as driving a car, or, more accurately, driving a spreadsheet or word processor.

GEOGEBRA TOOLS FOR CALCULUS

The GeoGebra toolbar, with its associated on-screen help, takes little time to assimilate, especially if the options are restricted by customising the tools available. For calculus purposes, we shall not need any circle or symmetry tools, so these can be removed, leaving the following buttons (Figure 3):

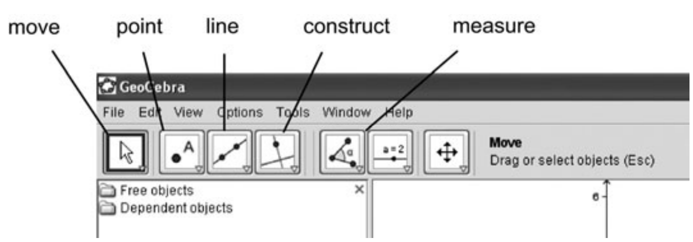


Figure 3. GeoGebra tool buttons can be customised for calculus.

The essential commands required for constructing dynamic drawings illustrating calculus are shown in Figure 4.

These commands are indicated in the lesson outlines below using bold type. In addition, we shall occasionally need sliders to control the values of algebraic variables (right click on the variable and check 'show object'). It is of course possible to 'prettify' the drawings by introducing colour, line styles, changing font sizes, showing (or not showing) labels, all from with the 'properties' menu (right click object, then 'properties').

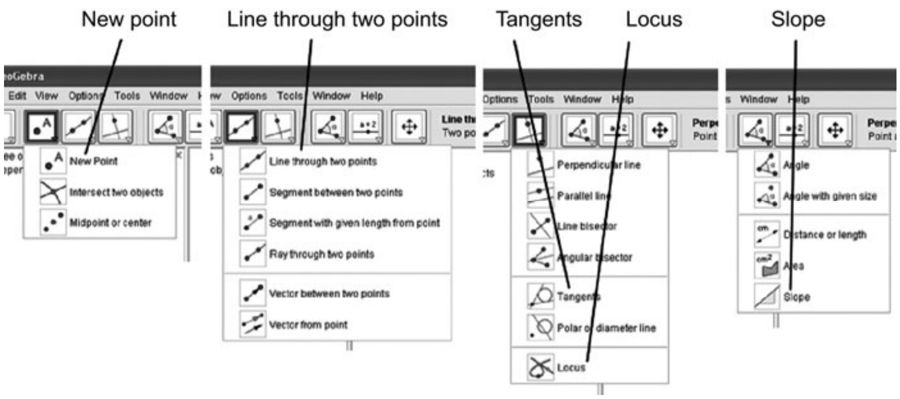


Figure 4. GeoGebra tools that are needed for calculus.

LESSONS IN CALCULUS USING GEOGEBRA

In this section, I provide an outline of an approach to calculus using GeoGebra through describing a sequence of lessons in differentiation. The section concludes with a discussion of integration through area functions. My approach is not to prescribe these lessons too tightly in the hope that teachers will develop for themselves, and thereby 'own', their own more detailed lesson plans. They would need some competence in the software; however, my experience of working with teachers suggests that this can be acquired quite quickly.

Lesson 1: The Gradient of a Line

By constructing two **new points** A and B, we can then construct the **Line through two points** AB, **measure** the slope of AB, and see how this behaves for different positions of A and B. However, rather than utilise the ready-made tool, it might be beneficial to input *(y(B) – y(A))/(x(B) – x(A))* (taking care with brackets), and create a **new tool** (tools menu) called **gradient** which inputs A and B and outputs the gradient *m*. In doing so, the algebraic formulation is reinforced.

Lesson 2: The Gradient of a Curve

What is the difference between a straight line and a curve? What do we mean by the gradient of a curve? Can we find a connection between the position on a curve and the gradient?

Investigating these questions in GeoGebra could not be more natural. We need to know how to create a function $f(x) = x^2$ by **inputting** *f(x) = x^2*. We need to place a **point** on the curve, **construct** a tangent and **measure** its gradient (i.e., **slope**). By **moving** the point round the curve (helpfully snapping on points with integer *x*-coordinates), we observe that the gradient is twice the *x*-coordinate. By changing *f(x)* to, for example, *2 x^2, k x^2, x^3, x^n,* etc., we can observe the rules connecting the gradient with the *x*-coordinate, and 're-invent' the formula nx^{n-1}. This of course is similar to how we might use a graph plotter like Autograph, but the software-specific knowledge required is arguably less and more intuitive.

Lesson 3: The Gradient Function or Derivative

By seeing how the gradient varies with *x*-coordinate, we have constructed the gradient function, which we call the *derivative*. Can we show this function by means of a graph? Well, we have the *x*-coordinate *x(A*) and the gradient m, so let's define a point B with coordinates (*x(A)*, *m*) by inputting *B = (x(A), m).*

Point B appears, and by constructing the trace of B (right click, **trace on**), as we **move** A, we see the gradient function appearing. The **undo** button (top right) erases this, and we could now **construct** the **locus** to display this function more permanently – change *f* (double-click on it in **move**) and the gradient function changes accordingly.

At this point, we can investigate properties of the gradient function. How does it change if we double *f* or add a constant to *f*? We may also wish to introduce some formal notation for the gradient function, say *f'(x)* or *dy/dx*.

We further need to transfer the ideas we have developed from the computer screen to paper, and develop skills in differentiating polynomial functions with some conventional exercise work. We could use GeoGebra to check answers by inputting *f(x)* and *f'(x)* and then observing if they match our locus. However, the merit of this in my opinion is debatable: students need to gain experience writing the algebra themselves.

Making marks on paper is, to my mind, just as important as it was in Newton's day. CAS, which removes the requirement of the learner to perform algebraic manipulations, also denies the learner the psycho-motor experiences required to reify (Sfard, 1991) or encapsulate (Dubinsky, 1991) the mathematical concepts for themselves.

Lesson 4: Differentiation from First Principles

So far, we have used a GeoGebra modelling environment to conduct a pseudo-scientific enquiry. In order to convert our 'discoveries' into mathematics, we need proof. While students may be content, after an hour or so of screen-watching, to take GeoGebra's word, or rather its image, that the derivative of x^n is nx^{n-1}, teachers need to go beyond the pictorial representation of objects on the computer screen, and give students insight into the algebraic processes that establish this result.

For this, a pencil and paper, and facility with certain algebraic manipulation skills, such as expanding brackets, are necessary, and GeoGebra will not help the learner to acquire these. However, the ability to input algebraic expressions in recognisable form may enhance the transferability of ideas from paper to computer screen. Moreover, having a dynamic picture of a chord approaching a tangent, which is readily constructed from scratch, appears to enhance our ability to visualise this limit process.

In order to construct this, it is helpful to be able to control points by keyboard rather than by mouse. Thus, we start, by **inputting** *f(x) = x^2*, *a = 1*, and A = (*a, f(a)*). Now we have three means of controlling the position of A on the curve, either by slider (right click on *a=1*, then *Show object*) or by selecting *a* and using arrow keys, or by double-clicking on *a* and changing its value. These methods are less hit-and-miss than dragging. We then input *h = 1*, show object and, in properties, adjust the slider to go from 0 to 1 in steps of 0.01. This '*h*' will be our small increment in *x*, or Δx, but *h* is more user-friendly at this stage. We then input *B = (a + h, f(a + h))*, create a line through AB, and construct a tangent to the curve at A. We can now see the chord AB approach the tangent at A as *h* gets to zero (Figure 5).

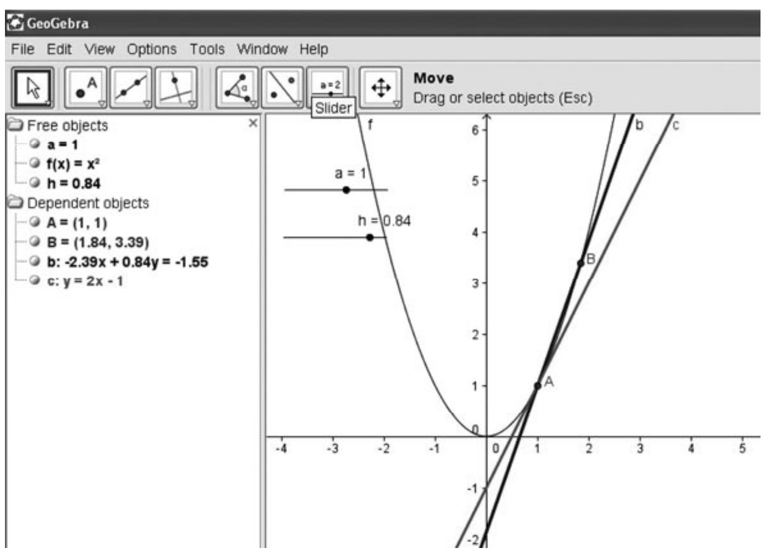


Figure 5. Chord AB approaches the tangent at A when h gets to zero.

With *a = 1*, we can **measure** the gradient (**slope**) of the tangent line, which is of course 2. We could also use **slope** to measure the gradient of PQ, but we know how to do this for ourselves by inputting *m = (f(a + h) – f(a))/h*.

We can see *m* approaching 2 as *h* approaches zero. It even disappears when *h* = 0!. At this point, the student will need pencil and paper to do the algebraic expansion of $\frac{(2+h)^2-2^2}{h}$, followed by $\frac{(a+h)^2-a^2}{h}$. The curve can of course be changed, whilst preserving the structure of the diagram, by editing the function *f*. There is nothing here that cannot be done using Autograph or other dynamic geometry packages. However, it is the logical clarity of the GeoGebra screen and the synergy between geometry and algebra windows that helps, to my mind, to reinforce the understanding of the algebraic and geometric formulations of the limit process. With these four outline lessons, we have laid the foundations of differential calculus, as applied to polynomial functions. These will of course need reinforcing through exercise work.

Lesson 5: Derivatives of Trigonometric Functions

GeoGebra can readily construct, or re-invent, the graphs of *sin x* and *cos x* as functions of the angle made by the radius OP of a unit circle with the *x*-axis and the *y*-coordinate and *x*-coordinate of P (see Figure 6).

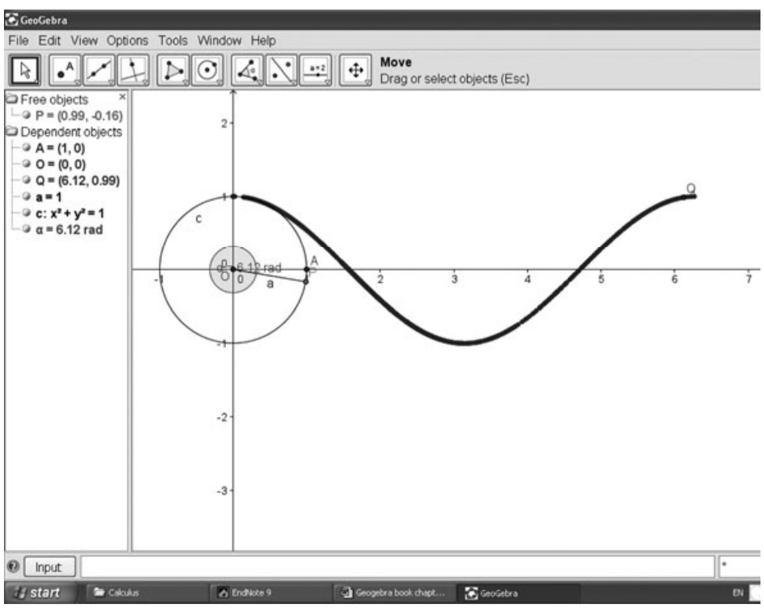


Figure 6. Constructing the graph of cos(x) using a unit circle.

First, we augment our GeoGebra tools with the circle tool. We construct a **circle** with centre (0,0) and radius 1 unit, place a **point** labelled P on the circle and a **point** labelled A at (1, 0). Then **measure** angle AOP (by default named α), first changing the angle unit to radians (**options**). Define a point *Q* = *(α, x(P))*, and trace Q as P **moves** round the circle. To trace the sine, rename Q as *(α, y(P))*.

Unfortunately, we cannot easily extend this trace to angles outside the domain 0 to 2π, but this is perhaps a useful discussion point.

What are the derivatives of the function? Well, we can sneak a look by recreating the figure from lesson 2 for the function *f(x) = sin (x)*. The similarities of the two traces in Figures 6 and 7 are somewhat surprising, and invite the question of what the derivative of *cos(x)* might be!

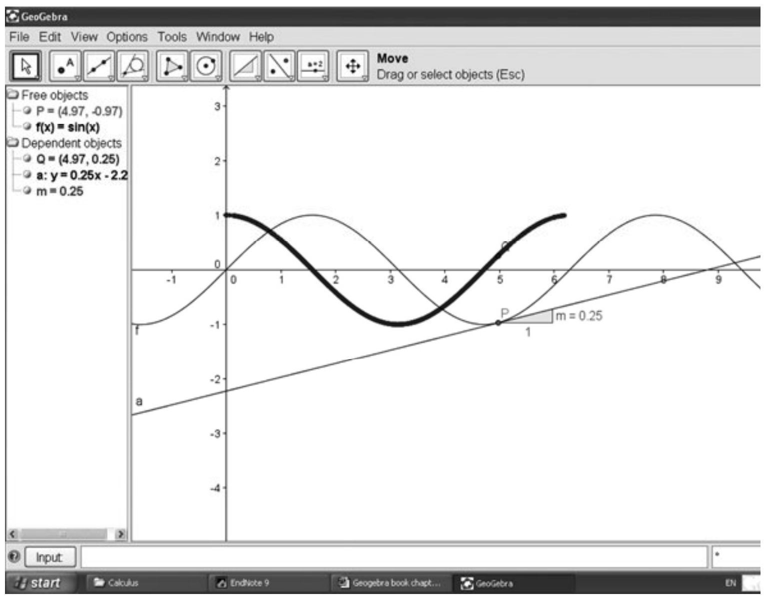


Figure 7. Trace of the derivative of sin x.

We could now, of course, revisit the work of lessons 3 and 4 using trigonometric instead of polynomial functions. A quicker approach here might be to define *f* and *h* (with slider from 0 to 1 in steps of 0.01 as before), and then **Input** *g(x) = (f(x + h) – f(x))/h*. Observe then what the limit of the function *g* is as *h* approaches zero. This would work as a way of investigating derivatives of any function visually.

What about a proof? This requires the use of compound angles or factor formulae, together with the limits of *sin(h)* and *cos(h)* as *h* approaches zero, to establish, but establishing the derivative as the limit of *g* as *h* approaches zero may help to motivate the expansion of *sin(A + B*) and *cos(A + B)*.

Lesson 6: Exponential Functions

A very similar approach may be adopted to establish the derivative of exponential functions and establish the value of *e*. First, we need to define a variable base *a*.

The trace of the gradient function suggests that when *a* is close to 3, the derivative is close to the function *a^x* (see Figure 8).

By plotting the locus of Q (instead of its trace), students can investigate this value of *a* with greater accuracy. The algebra of the limit in this case is beyond an elementary analytical treatment, however. Nevertheless, it would appear that re-constructing a value for the elusive *e* in this way is both instructive and motivating.

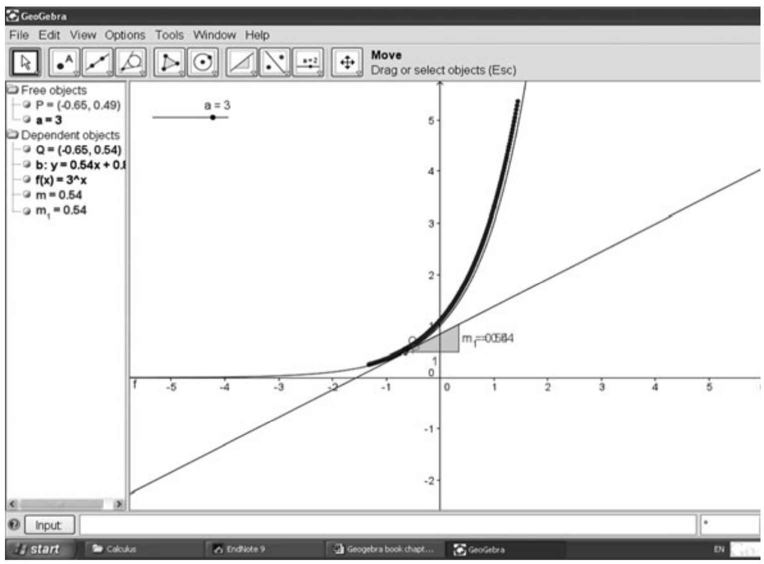


Figure 8. Finding the derivative of a^x and estimating the value of e.

Lesson 7: Integration

Establishing the area function as the inverse of the derivative using GeoGebra relies upon using the *Uppersum* and *Lowersum* functions. However, we could initially investigate the area under *f(x) = x* by tracing the area of the polygon (or triangle, in fact) under the curve (see Figure 9).

For this figure, we **input** *f(x) = x*, create **points** O and A on the *x*-axis, and **input** *P = (x(A), f(x(A)))*. Construct **polygon** OAP (the default area calculated is denoted by poly1), and then **input** a point *Q = (x(A), poly1)*. The trace of Q as A moves on the *x*-axis can be seen to be the function $g(x) = ½\ x^2$. This is easily proved to be the area of triangle OAP.

As polygons are straight-edged, we need to establish the idea of approximating the area under a curve using rectangles. In Figure 10, the function *f(x) = x*, and limits a and b are **inputs** with initial values 1 and 2, the number of rectangles is **input** as n, with values from 0 to 200 (say). Then input *U = Uppersum [f ,a ,b , n]* and *L = Lowersum [f, a, b, n]* and observe how *U* and *L* approach each other as *n* increases.

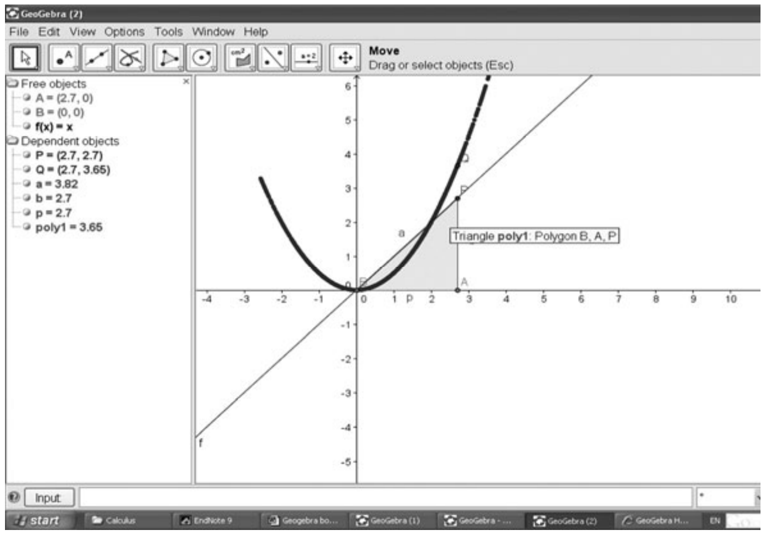


Figure 9. Find the integral by tracing the area of a polygon.

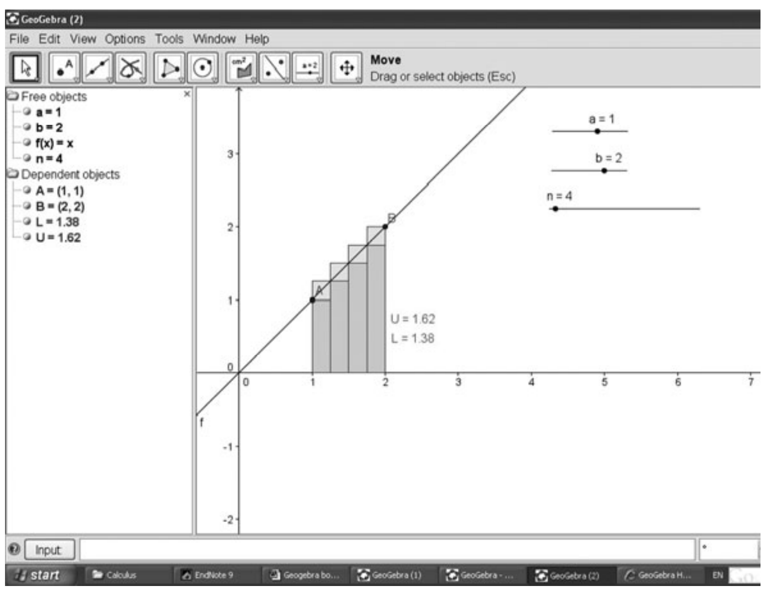


Figure 10. The uppersum and lowersum approach each other as n gets large.

By re-setting the value of *n* as 1000, for example, we have created an approximate area calculator, and tracing, or finding the locus, of a point *Q = (x(B),U)* will create the 'area so far' function. This in turn could be used with *f(x) = 1/x* to create the *ln x* function (see Figure 11).

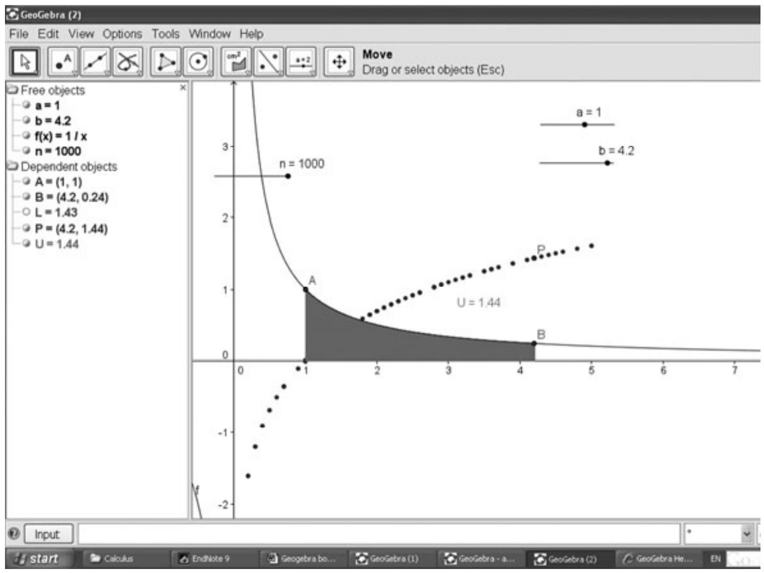


Figure 11. Estimating the ln(x) using the uppersum when n=1000.

SUMMARY

This chapter outlined a model-based approach to calculus using GeoGebra. Differentiation is introduced by investigating the gradient of a moving tangent line, constructing the gradient function as a trace or a locus, and illustrating the derivative as the limit of the gradient of a chord. These stages can be followed initially for polynomial functions and extended to develop results for trigonometric and exponential functions.

An approach to the Riemann integral is also proposed, albeit briefly, using the polygon function to investigate areas under *f(x) = x*, using the *Uppersum* and *Lowersum* functions to develop an approximate area function. Although it is possible to develop calculus in a similar way through graph plotters or other interactive geometry packages, it is argued that there are the following advantages in using GeoGebra:

– the ease and accessibility of the software;
– the ease of use of the program: menus, options and on-screen help obviate the need for teachers to spend time in learning how to use the software;
– the steps in creating the figures can be described simply using key words such as **point**, **input**, **construct**, **measure**; this avoids the need for over-prescriptive worksheets;

– the input of algebraic expressions is close to that used in pen and paper work, thus reinforcing the connection between the geometrical and algebraic aspects of calculus.

Although interactive geometry can be a powerful means of overcoming barriers to learning calculus, it is important to keep a model-based approach in perspective. It will not, and indeed cannot, replace the need for students to master certain algebraic processing skills. There are also aspects of teaching and learning calculus, such as differentiation techniques (the product, quotient and chain rules), which are naturally algebraic in nature, and which are less likely to be enhanced by interactive geometry.

For the author, the most interesting aspect of modelling calculus with GeoGebra is that it appears to provide the student with a powerful tool for researching, and re-inventing, the results for themselves and, in doing so, enhancing the depth of their understanding beyond that achieved by traditional approaches, where the concepts and results are presented as facts and rules to be learned. The gap between intuition and theory is particularly wide in calculus, and in GeoGebra the student possesses a tool that may well help to bridge this gap.

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14. REBIRTH OF EUCLIDEAN GEOMETRY?

For hundreds of years, the ability to construct geometrical objects has been the very essence of mathematical knowledge. During the second half of the 20th century, appreciation of competencies in Euclidean geometry constructions vanished from the curriculum in many countries, including Sweden. It was replaced with a strong emphasis on algebra and functions. GeoGebra, which is open source and free, translated into Swedish (as well as into 50 other languages), and independent of computing platforms, has changed the educational circumstances. With a growing interest in GeoGebra among mathematics teachers in Sweden, there is, at the present time, a possibility that geometrical constructions will come back into the mathematical classroom. GeoGebra can also be used in many other domains of mathematics, which makes it more conducive to geometrical constructions. Perhaps we will see a rebirth of Euclidean geometry?

INTRODUCTION

One characteristic of human beings is that we continuously are confronted with new tasks and problems to solve. How we manage to solve the tasks that we face is to a large extent dependent on the context and the resources that are available. Therefore mankind has always invented tools to mediate the process of problem solving. How we manage to solve tasks is to a large extent also a question of how we make use of the tools we have available. Learning in relation to tools is not a new concern; various types of tools and technologies have always been important for human living, learning, and surviving. While many tools have been mainly physical in the past and thereby increasing our strength, digital tools are mainly cognitive and thereby amplifying our mental power.

There are many research questions within mathematics education that seem to have become persistent during the last thirty years or so. One common question has been “How will the access of (modern) technology affect the teaching and learning of mathematics?” The word technology or even technical aid for doing and learning mathematics is, however, quite ambiguous. Long before the use of even the clumsiest calculator the use of abacus for simple calculations was the modern technology back then. What we consider modern technology is naturally just a reflection of where technology stands at present time, when we are looking.

No technical device for teaching and learning mathematics is, however, created or invented from a total vacuity. Most, if not all, technological inventions are created within and related to economical, social, political and cultural frameworks. The variety of advanced calculators and equally advanced software that are

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 205–215.

available today are all linked back to their historical predecessors in a long chain of developmental processes. At the same time technology seems to have been local to some extent along the course of history. Some of the questions regarding the use of technology for learning and doing mathematics are surprisingly constant over time. Richard Delamain and William Oughtred argued about the use of the slide ruler in the early fifteenth century (Turner, 1981, p. 102), just as researchers and teachers argue about the correct way to use technology in the teaching of mathematics today. But some theoretical constructs regarding the use of modern technology are new and linked to the technical progress of our time. Instrumentalization and precepts are modern constructs.

GEOMETRY

Geometry is a fascinating field of mathematics, which we humans meet from the first time we open up our eyes. From the beginning of time we saw straight lines and circular shapes. In the early days maybe even the first geometry experts were land surveyors in Ancient Egypt. The ancient Greeks gave them the name Arpedonapti, which means "those who knot ropes," and that name was related to the fact that Egyptian experts used to tighten ropes on the land to mark lines and circles. Traces of this ancient usage are still present in several modern languages as, for example, in the saying "to draw a line." According to most historians, the theoretical and mathematical field of geometry was developed later by the Greeks, and it culminated in the classical work the *Elements of Euclid*. The construction of geometric idealities or the establishment of the first proofs were truly improbable events. As the Romans came into closer contact with the Greek culture, the Romans also encountered its repertoire of learning, of which geometry was a significant part. Several Roman authors managed to cultivate and spread the Greek accomplishments around the Roman Empire, even up to the far north of Europe.

At that time and for many years to come, geometry and mathematics in general were recognized and appreciated as integral elements of civilization and sophisticated human culture. To study geometry and be involved with geometry in general were seen as a pursuit that concerned humans on a universal level. The Greeks and the Romans were convinced that the use of mathematics and geometry would develop the brain itself, with benefit to all human life:

> With this argument is connected a reason of more practical nature, which is attested in both Cicero and Vitruvius: the study of the artes, to which geometry and mathematics are reckoned, shapes the mind formally, that is to say, sharpens the wit, promotes readiness at learning, etc. (Bohlin, 2009, p. 230)

And how could geometry have been any different than a challenge for the mind? The Greeks' philosophical view of geometry compelled any scholar of that time into abstract and deep thinking. Some of the geometrical definitions and descriptions were almost religious in the way the Romans recognized and accepted them and spread them over Europe and further:

> punctum esse quod magnitudinem nullam habeat, extremitatem, et, quasi libramentum in quo nulla amnino, crassitudo sit, <sic> lineamentum sine ulla latitudene [carentem].
>
> A point is that which has no dimension, a boundary, and, as is a surface that which has no thickness at all, so is a line destitute of any breadth (Bohlin, 2009, p. 113)

For many years geometry and Euclid were, if not religion, at least very close. As a result of the Roman impact on Europe through the church, the Geometry in Euclid's *Elements* was studied in Latin for thousands of years. That was true for Sweden as well as for most countries. It was not until 1744 that the Swedish mathematician Mårten Strömer published the first Swedish translation of Euclid's *Elements*. It was certainly not the first textbook in geometry in Swedish, but it was the first translation of Euclid.

Geometry was so imbedded in the nature of mathematics that the national curriculum in Sweden labeled the school subject as Arithmetic and Geometry even until the 1950s. The textbooks were mainly based on arithmetic and Euclidean geometry. Many of the ideas (and even most of the geometrical examples) presented were some two thousand years old. Nothing of significance had been added to the curriculum since integral calculus some two hundred years before. So for several hundreds of years, not much was altered in the curriculum.

Then, in the 1960s a two-thousand-year tradition was changed. Euclidean geometry, which had been the historical basis of not only much of our culture but also of deductive work in mathematics, was examined and found to be insufficiently theoretical and rigorous. Alternate geometries without Euclid's "parallel lines never meet" postulate had been developed and found useful. Unfortunately, the famous statement "Euclid must go" by Jean Dieudonné was transformed into a decision that "Geometry must go." That in turn indicates how often "Euclid"' was considered to be synonymous with "geometry" at that time, even by many teachers of mathematics.

In Sweden and in many countries, Geometry in the Euclidean sense was washed away during the 1960s and replaced by more algebra in the curriculum. The concept of function was much more vivid than ancient geometrical theorems, slowly the curriculum became more and more practical, and the construction of geometrical objects was forgotten and disappeared also from teacher preparation programs.

SWEDISH TEACHERS OF MATHEMATICS

It is a fact that Swedish teachers of mathematics are in general quite reluctant to use computer packages in their teaching. The reason is at least two-fold. Students from grade seven have calculators, and students in upper secondary school have graphing calculators that can provide most of the necessary technology aid that is required. Besides that, different software packages cost money, are not translated to Swedish and are difficult to share with students for use on their home computers.

Now it seems that such reasons are no longer warranted, since GeoGebra is free under the regulations of Open Source (GPL) and is also available in Swedish.

During the year 2008 when I was involved in the translation of GeoGebra, I was also invited to many different seminars and asked to give talks about the usefulness of GeoGebra. When I look back at the 15 years I had been using and demonstrating first Cabri Géomètre and then the Geometer's Sketchpad, there was always a huge difference in teachers' appreciation of what was possible to do with these tools, compared to their plans for buying and using these dynamic geometry tools themselves.

I have been able to advise teachers to download and try out GeoGebra for themselves the very same evening after my talk and I have also encouraged the teachers, if still interested, to send me an e-mail for materials. The response has been astonishing, and I dare say that it seems as if the prospect that teachers will change their way of teaching mathematics is much more realistic than ever before. It might be worth mentioning that geometric constructions in the Euclidean sense are not equal to the work with dynamical geometry system in the sense of points with no dimension, lines destitute of any breadth, and so forth.

There are several dynamic geometry systems in practice in 2009 such as Cabri, GeoGebra, Geometer's Sketchpad, Cinderella, GEONExT, GeoPlan/Space, and WIRIS just to mention a few. The fact that dynamic geometry systems all have a Euclidean working method in common does not imply that they are more or less the same. Many of these systems offer much more than just Euclidean dynamic geometry. They are sophisticated, complex instruments that offer quite new challenges and possibilities to their users. Consequently, we think of technology in new and more theoretical ways.

THEORETICAL CONSIDERATIONS

Euclid laid the foundations for axiomatic geometry, where the definitions were abstract. However, the use of geometry, including the construction of geometrical objects was concrete in the past. As technological instruments get more sophisticated, the relationship gets more advanced and complex between the user and the instruments. That relationship can be seen as framed by theories of affordance, constraints, and instrumental genesis. Gibson (1977) defined affordances as referring to all the "action possibilities" latent in the environment, objectively measurable, and independent of the individual's ability to recognize those possibilities.

Obviously, those action possibilities are dependent on the capabilities of the actor and should always be measured in relation to the relevant actors. Norman (1988) talked about perceived affordance. This distinction makes the concept dependent not only on the physical (body related) capabilities of the actors but also on their goals, plans, values, beliefs, and past experiences. Effectively, Norman's conception of affordance "suggests" how an object can be interacted with. But when we start to interact with an instrument or tool, we also change the way we look at the instrument, we adjust the way we understand the instrument, and we accordingly adjust the way we use the instrument. The concept of instrumental genesis is based on the distinction between artifact and instrument with the latter having a psychological component (sometimes called scheme), indicating a dialectic relationship between activity and implicit mathematical knowledge. The activity that employs and is shaped by the use of instruments (instrumented activity) is directed towards the artifact, eventually transforming it for specific aims (instrumentalization):

> The subject has to develop the instrumental genesis and efficient procedures in order to manipulate the artefact. During this interaction process, he or she acquires knowledge, which may lead to a different use of it. Similarly, the specific features of instrumented activity are specified: firstly, the constraints inherent to artefacts; secondly, the resources artefacts afford for action; and finally, the procedures linked to the use of artefacts. The subject is faced with constraints imposed by the artefact to identify, understand and manage in the course of this action: some constraints are relative to the transformations this action allows and to the way they are produced. Others imply, more or less explicitly, a prestructuration of the user's action (Guin & Trouche, 1999, p. 201).

When working with a highly complex and sophisticated instrument as GeoGebra, the instrument itself is a significant part in a complex learning process. We therefore need to bridge the theory of affordances with the perspective of the complex construction of instrumental genesis (Verillon & Rabardel, 1995), further developed by Drijvers (2000) and Guin & Trouche (1999). In this framework, it is important to distinguish between the utilization of instrumentation—how the tool influence and shape the thinking of the user—and the mental instrumentalization—where the tool is shaped by the user. A user working with GeoGebra will be affected and behave accordingly in both these ways.

Instrumentation is an evolutionary theoretical construct, and can be characterized as the concept of mental schemes that emerge when users execute a task such as constructing a circle and dragging it around in GeoGebra. During that process the user will create some mental schemata. But GeoGebra is also an instrument created with specific utilities that allow the user to engage in activities within the constraints of the artifact. Without much reflective effort, the user would drag the circle around since he or she already knows that GeoGebra allows such flexible manipulations.

Instrumentalization is a psychological process that leads to an internalization of the uses and roles of an artifact; it can be viewed as an organization of the mental schemes, but also includes a personalization and perhaps transformation of the tool, and a differentiation between the complex processes that constitute instrumental genesis and those that are critical for teachers to master (Guin & Trouche, 2002). An example of this is when a teacher uses the option to create a new tool in GeoGebra or to save a construction in GeoGebra for later use in her or his teaching of geometry.

The competence *Instrumental Genesis* occurs when a user with specific knowledge and methods acts on an instrument with specific affordances and constraints. The instrument brings instrumentation to the user, while the user brings instrumentalization to the instrument. Instrumental genesis can be viewed as occurring in the combination of these two processes. It seems to the author that the presence of dynamic geometry systems has opened up for almost unlimited instrumental genesis to be developed.

BEYOND EUCLIDEAN GEOMETRY

Obviously many of the ideas and constructions that are used with dynamical geometry systems in classrooms around the world are relatively old. Some of them maybe even thousands of years old. One such example is the Thales theorem (named after Thales of Miletus who lived around 620 BC – around 550 BC), which is an excellent example of the construction of a right triangle.

Nevertheless, some problems within Euclidean geometry can be investigated in a way that was impossible before the invention of dynamic geometry tools. The possibility to check conjectures by moving geometrical objects is a desired feature that was impossible to implement with just paper and pencil. This possibility creates an extra dimension to the instrumental genesis. Some problems also lead into investigations beyond Euclidean Geometry in the sense of possible constructions, explorations, and conclusions. Consider the following problem that was given to a group of Swedish pre-service mathematics teachers in 2002.

1. Construct a triangle ΔABC in the Geometer's Sketchpad.
2. Trisect (in congruent segments) each side of the triangle.
3. Choose for each side the first trisection point (clockwise relative to the vertices A, B and C) and connect each such point with the opposite vertex.
4. Investigate the relation between the area of the central triangle that results from this construction, and the area of the triangle ΔABC.

Within this activity, it is easy to imagine that a complex and impressive instrument genesis is created, in a flow of instrumentation and instrumentalization that is occurring between the instrument and the user of the instrument. In order to construct different parts of the described situation, the user will need to learn more about the instrument as well as make the instrument do what is desired. The user is naturally also learning mathematics when investigating this problem through use of the instrument. The task can be further extended as:

5. Given an arbitrary triangle ΔABC. Trisect (in congruent segments) each side of the triangle. Connect the trisection points to the opposite vertices. Investigate the relation between the central figure's area and the area of the original triangle ΔABC.
6. Given an arbitrary triangle ΔABC. Five-sect (in congruent segments) each side of the triangle. Connect the two central points of the five-section points to the opposite vertices. Investigate the relation between the central figure's area and the area of the original triangle ΔABC.
7. Use the same construction as in 6), but divide each side of the triangle into n parts where n is an odd number larger than 5. Investigate the relation between the central figure's area and the area of the original triangle ΔABC for each such selected value of n.
8. Repeat the procedures of task 6) and 7) but connect instead the two outermost portioned points for each side with the opposite vertex respectively.

It is difficult to even imagine that such an investigation would have been possible to propose before the time of dynamic geometry systems. But with the instrument of GeoGebra or other similar dynamic geometry systems, it seems as if the common user is challenged, and maybe even eager in pursuit of an answer to her or his conjectures. Another characteristic of this problem is that the visualization the dynamic geometry system brings to the user of the instrument, which sharpens the instrumental genesis and also guides further development of conjectures and hypotheses. See Lingefjärd and Holmquist (2003) for a deeper and more extensive analysis of the history of this problem and how the pre-service students worked with this problem and what they accomplished.

TEACHER'S OPINIONS ABOUT GEOEBRA

To be honest, the theories of affordance, constraints, instrumentation, and instrumentalization are not what I promote most in teaching practice, when teachers first come to use GeoGebra. Normally I try to find something that most lower- and upper secondary teachers of mathematics know that they need to bring up with their students and where I know that the dynamic behavior is important. I usually start with the fact that the sum of angles in a triangle is 180°. This is an excellent example of where the dynamical behavior makes a difference (Figure 1).

One interesting detail in this construction is that the left part is very easy to organize in GeoGebra and relatively easy for most teachers to imitate right away, even with just a short introduction to GeoGebra. The right part is however somewhat more complicated and need you to become much more skilled with GeoGebra. Even so, most teachers seem to be rather attracted to this example.

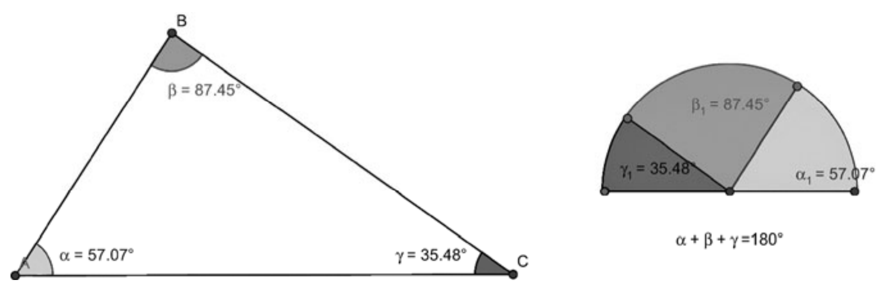

Figure 1. The fact that the sum of angles in a triangle is 180°, illustrated with GeoGebra.

Based on my experience in introducing GeoGebra to more than five hundred teachers, I have noticed that there are some major benefits that teachers appreciate. Teacher's opinions naturally depend on what level of mathematics they teach and with what groups of students. Teachers in primary school have different objectives for their mathematics teaching than secondary teachers and lower secondary teachers have different objectives than upper secondary teachers. Nevertheless, I have observed some mutual appreciation across all levels of school mathematics, which is beyond the fact that GeoGebra is free and translated to Swedish.

1. The generality issue. When teachers see that GeoGebra can not only be used for introducing basic arithmetic on a number line, but can also be used to demonstrate more advanced geometry, algebra, calculus, number theory or statistics, they become more confident that they will have many other teachers to discuss the use of GeoGebra with (Figure 2).
2. The portability issue. The fact that GeoGebra is installed over Java has two major benefits. First, it means that GeoGebra looks and acts the same whether you are running it on a Windows machine, on a Macintosh, or on a machine with Linux. Since I am encouraging them to ask their students to download and run GeoGebra at home, it is important for the teachers to know that it makes no difference what kind of operating system the students have at home. Second, many schools are under a municipality hierarchy that prohibits them from installing software on the school computers. The regulation states that any program installation should be done by the local authority computer technician. Since Java is already installed on the computers at school, the installation of GeoGebra does not require any special administrative privileges. In the worst scenario, GeoGebra can also be launched on the Internet and not installed at the computers at all.
3. The connectivity issue. The fact that GeoGebra allows you to put two points in the Graphics view and immediately see them represented in the Algebraic view and also in the Spreadsheet view is amazing and probably the most beneficial feature from a cognitive point of view (Figure 3).

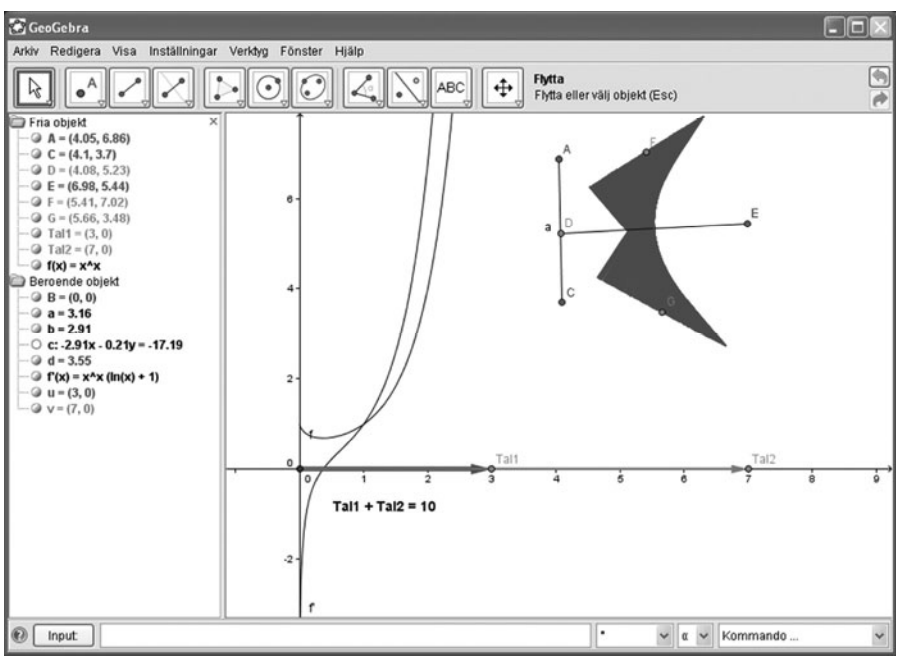

Figure 2. Numbers are added, $f(x) = x^x$ and $f'(x)$ are sketched, a parabola is constructed.

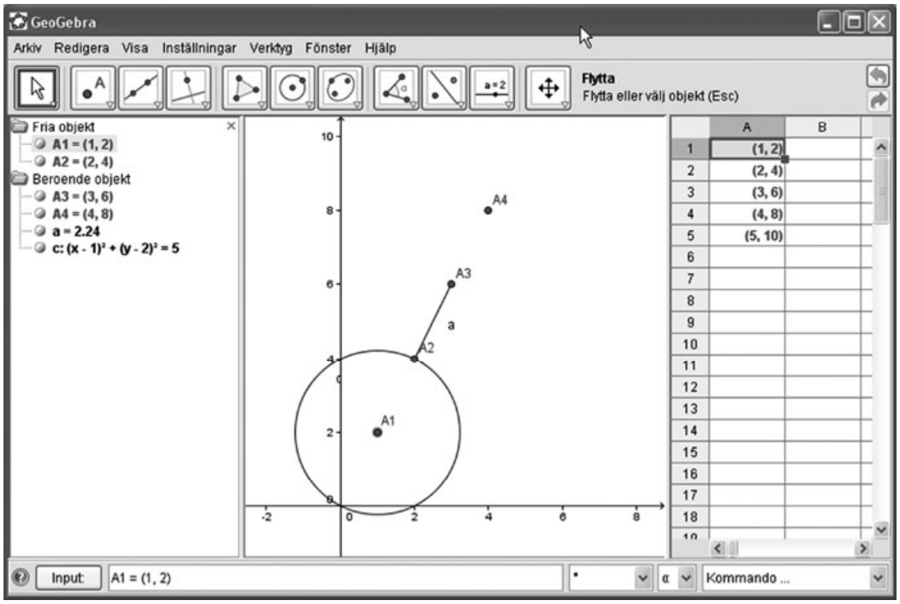

Figure 3. Two points labeled A1 and A2 become objects in three different views.

Swedish teachers also seem genuinely interested in how other Swedish teachers use GeoGebra. I have often made references to the work of Johan Johansson, one upper secondary teacher of mathematics in Sweden who uses GeoGebra as a tool for student's homework. This teacher has asked his students to download and install GeoGebra on their home computers and regularly distribute construction files through e-mail so that his students could be prepared for the next lesson by exploring the GeoGebra file he sent them.

During one of my presentations of GeoGebra for a group of teachers on March 17, 2009, I had the opportunity to connect to the article by Fahlberg-Stojanovska & Stojanovski (2009). This short article is a wonderful example of how the world around us is rapidly changing in the light of technology. Even when you think you have a good hold of what is going on in terms of technology, you might easily get surprised:

> Where do our young people go for help with their mathematics? Check out the mathematics questions at answers.yahoo.com. The questions come in every minute and range from the absolute most simple to complex questions from calculus and beyond. Answers are given by anyone who wants to contribute and again the range in quality and quantity is immense. Finally, the askers and readers rate the answers. This is a breathtaking view into what is happening with our young people and their mathematics education throughout the world. It is also an incredible opportunity for all kinds of mathematical exploration and we will show this through detailed consideration of two examples. (Fahlberg-Stojanovska & Stojanovski, 2009, p. 2)

In this article, the authors showed in a very detailed manner how GeoGebra can act as an almost instant tool to try out and quickly reject or verify conjectures. The teachers to whom I demonstrated the neat GeoGebra constructions in this article were both surprised and challenged. The fact that GeoGebra can be used also for complicated simulations seemed most encouraging to the teachers I talked to. It definitively helped that the article by Fahlberg-Stojanovska & Stojanovski is written in a discussion stylish way. I cited the dialogue and the teachers were both impressed with how GeoGebra could help in the problem solving process, but also by the insight into the world of some students on the Internet:

> This is almost as looking into a quite different world, you know… The students asking questions out on the Internet and then they are receiving very rapid responses that sometimes are wrong… Amazing! And the example you showed us with the ships A and B. I was almost stunned by that example and how you can demonstrate very difficult mathematics with free software. (Anonymous Swedish mathematics teacher, March 2009)

The possibility to build a simulation, the possibility to visualize the fundamental theorem of Calculus, and the tools for measuring areas and lengths are the opportunities most teachers appreciate much more than the construction part of dynamic geometry. So, the rebirth of Euclidean geometry is both here and not here. The pure constructions of Euclid might not be what the teachers seek for first in GeoGebra, but it might very well be something useful when they start exploring geometrical concepts that now are so much easier to investigate. Nevertheless, GeoGebra will convince teachers about changing their teaching through its excellent flexibility and multifarious possibilities.

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15. THE EMERGING ROLE OF GEOGEBRA IN THE PHILIPPINES

In the past few years, researchers in mathematics education have highlighted the use and benefits of integrating technology and mathematics. However, one critical issue in developing countries is the inaccessibility of technology for the study and learning of mathematics. The first part of our chapter pertains to how GeoGebra, as free software with well-planned functionalities, plays a particularly important role in the Philippine setting. We also discuss how GeoGebra is used across classrooms with varying levels of computer and Internet access. In the second half of the chapter, we demonstrate how GeoGebra supports student-centered learning through the creation of interactive websites and exploratory activities.

INTRODUCTION

There is a growing field of research indicating that technology enhances mathematics learning (Balanskat, Blamire, & Kefala, 2006; Gutierrez & Boero, 2006). However, most of the studies on the affordances of technology have been conducted in developed countries where technology is widely available. In developing countries, such as the Philippines, one critical issue is the inaccessibility of technology, which tends to widen the gap between the ‘haves’ and ‘have-nots’ (Dunham & Hennessy, 2008; Hennessy & Dunham, 2002).

GEOGEBRA IN THE PHILIPPINES

In the Philippines, the integration of technology in teaching mathematics remains marginal due to several reasons. The first and primary factor is the lack of infrastructure: insufficient number of computers, difficult access to computer laboratories, and high costs of hardware and software. Even if computers have been introduced in schools, these are limited in number, and are often exclusively used for computer classes. The mathematics syllabus, instructional materials, and the values and norms of mathematics teaching and learning are still defined with respect to the mathematical practices associated with the traditional pen and paper environment. Another reason is the lack of experience, inertia, or even resistance of some Filipino educators.

In this chapter, we will discuss the role of GeoGebra as an accessible technological tool for mathematics education in the Philippines. We will highlight

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra,217–229.

the affordances of GeoGebra and its functionalities and how it offers the benefits of teaching mathematics with technology in the Philippines.

Availability as Open-source Software

A problem faced by teachers in Philippine schools is the lack of access to commercial software packages. Even with the desire to teach mathematics with technology and with a handful of computers at their disposal, mathematics teachers are in a helpless situation if they have no access to mathematical software. Further, computers are used mainly for obtaining data, writing reports, constructing presentations, or sending email. GeoGebra, being an open-source package, thus plays an important role in making the benefits of technology more accessible for Filipino teachers and their students. It is very easy not only for teachers but also for students to access GeoGebra and download the software. The installation file can also be copied directly from one computer to another. Thus, even computers without Internet access could have a copy of the software.

Because Filipino teachers can easily access GeoGebra, we presented the software to inservice teachers from elementary, secondary, and post-secondary institutions who were doing their graduate studies or undergoing teacher training at our university. In addition, we were invited to give a plenary talk during the annual convention of the Mathematical Society of the Philippines in 2007, and to provide hands-on training during the 2008 Campmath for College Teachers. We have also been invited to provide training seminars for teachers from other colleges and universities. During our presentations, we provided examples of GeoGebra sketches and how these may be used to dynamically illustrate mathematical concepts. We also highlighted how the sketches may be used to provide students with the opportunity to investigate mathematical ideas on their own.

The teachers in our audience, many of whom have seen dynamic geometry software for the first time, saw the possibilities of teaching mathematics through the software. The fact that the software is freely available allows teachers to use the software in their classes or in their planning. They, in turn, have reported to us how they used GeoGebra in their own schools.

Functionalities of GeoGebra

User interface. Many Filipino teachers do not have the confidence and experience in teaching with technology. They have limited exposure to computers and mathematical software. In our experience of introducing GeoGebra during seminars, teachers responded positively to the well-planned interface of GeoGebra. Even without the aid of training materials and lengthy workshops, a teacher can manage to learn how to use GeoGebra through its toolbar and menus. One can also shift between the algebra and geometry windows with ease. The interface is user-friendly, which is a big factor in making teachers feel confident to use technology in their teaching.

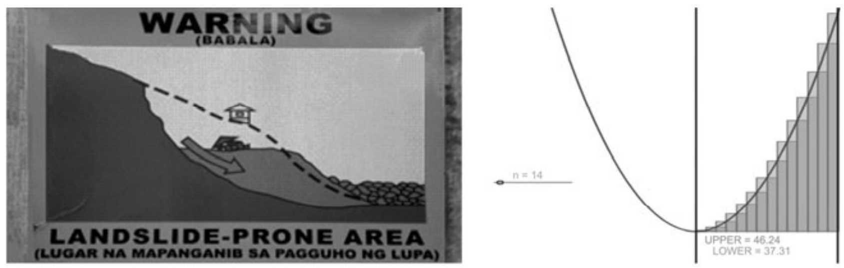
Interactive pre-constructed web pages and applets. The pre-constructed activities are especially helpful for teachers who have limited experience in using technology and manipulating dynamic geometry software. They can either take advantage of the applets written by colleagues who are more experienced in using GeoGebra or those posted in the GeoGebraWiki or in other websites. Because pre-constructed applets are easily accessible, teachers and students do not need to be trained extensively in the use of the software to be able to benefit from the software.

Construction protocol and navigation bar for construction steps. Filipino teachers carry a heavy workload—most teach an average of eight hours daily. They also have additional responsibilities in school, which include committee work, homeroom advising, and mentoring of students. It is very difficult to study new technologies and engage in the creation of instructional materials such as writing activities that incorporate technology in the classroom. There are also time constraints and expenses involved in organizing workshops and training seminars for teachers. With minimal opportunities for GeoGebra training, a teacher may study the construction protocol and navigation bar of pre-constructed applets to replay the construction steps to learn how the activity was created. They can do so in their own free time and at their own pace.

A Concrete Example

We now describe how GeoGebra's functionalities make it possible for a teacher to use GeoGebra in his class. Edward Macagne of Cordillera Regional Science High School in Benguet, in the northern part of the Philippines first learned of GeoGebra from a talk we gave during a mathematics seminar series offered to teachers in the Benguet region in October 2007. The fact that GeoGebra was freely available and was easy to learn encouraged and allowed Macagne to write activities using GeoGebra. Even without having received extensive training, he was able to apply the software and modify the usual instructional approach in the classroom. In particular, he was able to link mathematical ideas to real-life situations. His work has not gone unnoticed. In *the Philippine First National Science-Mathematics Summit for Regional Science High Schools held in* February 2009, with the theme "Building, Enhancing, and Sustaining Meaningful Learnings in Science and Mathematics," Macagne won first place for his lesson in the *Micro-Teaching: Lesson Exemplar (Mathematics)* category. His lesson begins with the following problem posed to his senior students, based on a signage (Figure 1a) from a subdivision in Baguio City, a city close to his high school in Benguet: *How will you compute the area of the dark chocolate region if all you have is a ruler*? The students were then guided with a dynamic GeoGebra worksheet entitled

Riemann sums approximating the area under the curve for them to learn that as more rectangles of constant widths are used, the closer the sum becomes to the actual area (Figure 1b).



(a) (b)

Figure 1.(a) Finding the area of an irregularly-shaped region; (b) Riemann sums as an approximation of the area under a curve.

For this lesson, Macagne further elaborated: “GeoGebra was able to help me in two ways. First, I was able to put across the difficult concept of inserting several rectangles, which is of course impossible to do with the chalk. … Second, I was able to cater to the needs of the visual learners. The auditory learners were catered [to] by my explanations, by our interaction during the discussion. The tactile learners were catered [to] by the worksheets.” He said that students initially thought that the area under the curve is exactly the same as the sum of the areas of the rectangles when these rectangles were made very thin. However, through GeoGebra and its zoom function, the students realised that the rectangle areas were just approximations of the actual area under the curve.

The Use of GeoGebra in Classrooms

Guided by Alejandre’s (2005) framework, we categorised situations in the majority of Philippine schools with respect to technology use. The first is having Internet access. The second is having computers without Internet access. Third, there are scenarios where technology is available, but in very limited numbers. Fourth, there are situations where computers are not available at all. In the discussion that follows, we explain how GeoGebra, through its capability to create web pages and applets, makes technology accessible in each of these scenarios.

When Internet is available, students may view pre-constructed web pages that are accessible online. When Internet is not available, teachers may download worksheets online or create their own worksheets, and load these on each computer. With an additional constraint of having limited numbers of computers, Alejandre (2005) recommends that the computers be used as one of the stations within a number of stations. This is a technique where several areas are set up in the classroom, with each area having its own set of tasks or instructions to be completed by a group of students. For instance, if the topic is about trigonometric functions, and there are only 5 computers, the teacher loads GeoGebra applets on each computer beforehand. Each computer station would include different tasks related to trigonometric functions; one station could show an applet on sine graphs, another on cosine graphs, and so on. This setup is also ideal for group work. Doerr and Zangor (2000) report that group work on a shared computer initiates more mathematical interaction and discussion among students.

GeoGebra and Technology-Based Manipulatives

The final situation in Alejandre's (2005) framework is where there are no computers in the classroom. This scenario is quite common in the Philippines. For example, there are schools with a minimal number of computers to be shared only among teachers; students do not have direct access to the computers in a classroom or laboratory setting. In this environment, a way for a teacher to bring the benefits of technology to the students is to construct technology-based manipulatives that provide opportunities for experimentation and discovery in the classroom. The idea behind a technology-based manipulative is to use concrete mathematical models that simulate a computer-aided activity. These models can be touched and moved around by the students to help them explore and understand mathematical concepts and skills. The models are based on readily available materials and objects present in the students' domain.

Dynamic geometry software provides an environment to explore geometric relationships and lends itself naturally to the creation of manipulatives. Through GeoGebra, a teacher uses dynamic geometry to write exploratory activities and create applets that can be readily accessed by colleagues in her/his school. The activities form a basis for building the manipulatives.

We have used the example given in Figure 2a during workshops and local symposia for teachers. The example shows an applet built using GeoGebra on investigating midpoint quadrilaterals. Using this exploratory activity as a starting point, a simple manipulative was then created as an alternative, using simple cardboard strips and pins to demonstrate the same concepts (Figure 2b).

It is based on this idea that mathematics teacher Teresita Arlante and her students, Catherine Imperial and Charmaine Manalang of Naga City Science High School, created the *CenTheorem gadget*. Before being introduced to dynamic geometry via GeoGebra, she and her students experienced geometry mainly through textbooks with static diagrams. However, after having been introduced to GeoGebra through a course on technology, and after having seen how simple manipulatives may be created to simulate the dynamic geometry environment, Arlante led her students to create their own gadget. Their device was inspired by GeoGebra, the only geometry software the school could access. Geometric explorations were carried out on a GeoGebra applet on the inscribed angle, angle-chord, and angle-secant theorems under the guidance of Arlante (See Figure 3).

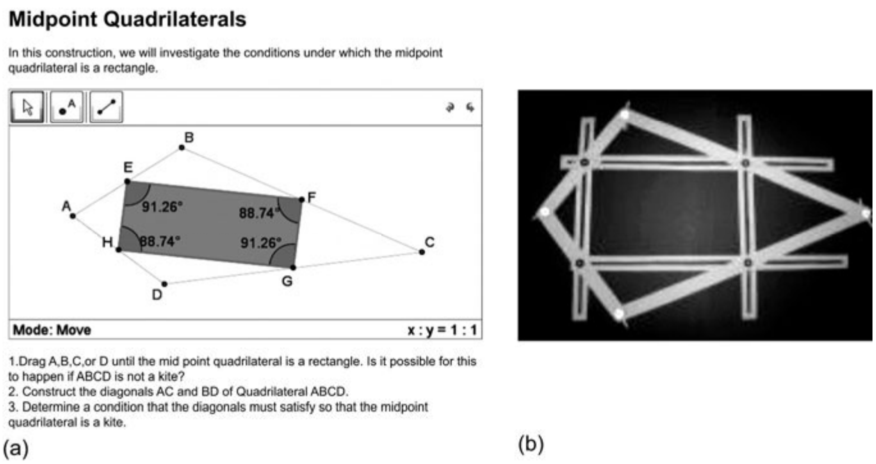


Figure 2. (a) An applet created in GeoGebra on midpoint quadrilaterals; (b) A manipulative using cardboard and pins.

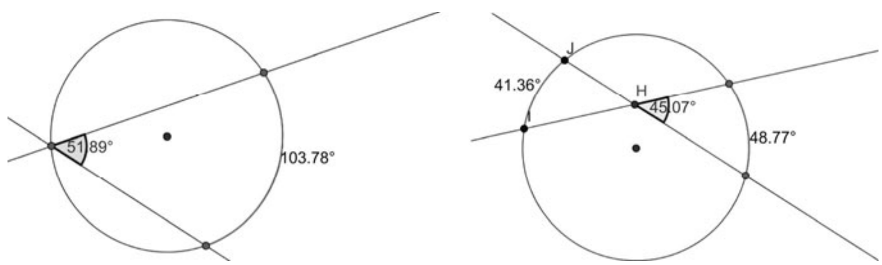


Figure 3. Inscribed angle and angle-chord theorems.

To simulate the dynamic nature of these constructions, a manipulative device was created by the students using illustration boards, used paper folders, washers, nuts and screws (Figure 4). Through a protractor and calibrations marked along the circumference of the manipulative, the concepts shown by the dynamic geometry software were captured, illustrating the relationship of the measures of the inscribed angle with the measures of the arcs they intercept. The manipulative could also demonstrate the relationship of the measure of the angle formed by secants intersecting inside or outside the circle with the measure of the arc the angles intercept.

The manipulative, which Imperial and Manalang refer to as the CenTheorem gadget to stand for "center of all theorems" on angles formed by radii, chords and secants, won for them the *Division of Naga City Schools Most Outstanding Science Innovation*. Further, the manipulative facilitated an understanding of the concept and was excellent in terms of ease of use, presentability, and degree of helpfulness (Arlante, 2008). Arlante's example of helping bring technology to the classroom served as an inspiration to teachers in her division, who continue to find new ways and means to improve students' mathematical skills and develop their passion for mathematics.

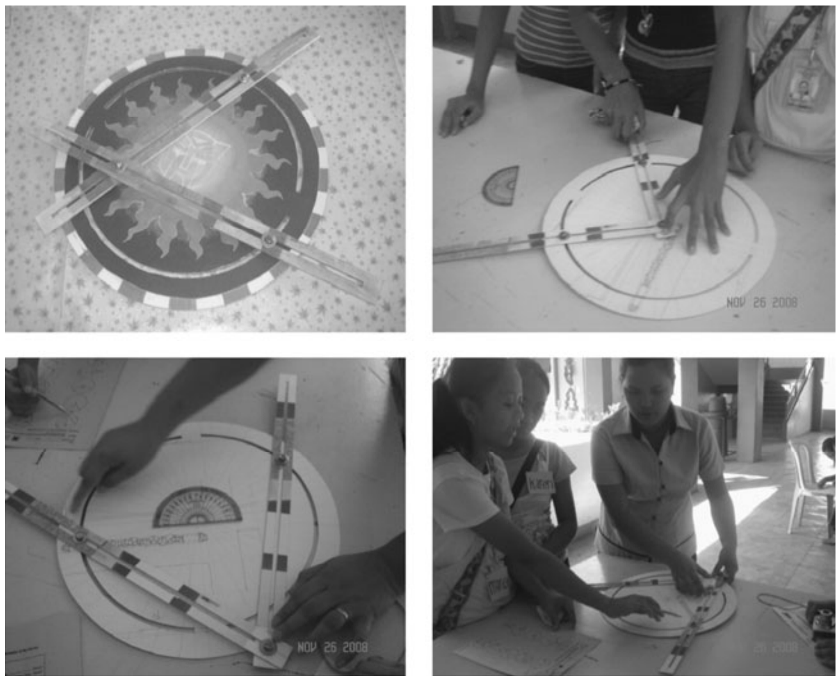


Figure 4. The CenTheorem gadget.

GEOGEBRA AND STUDENT-CENTRED LEARNING

Our university places a strong emphasis on student-centred learning (SCL) and has recently made concrete steps to bring this ideal to reality. The university released an SCL primer where student-centered learning was defined as:

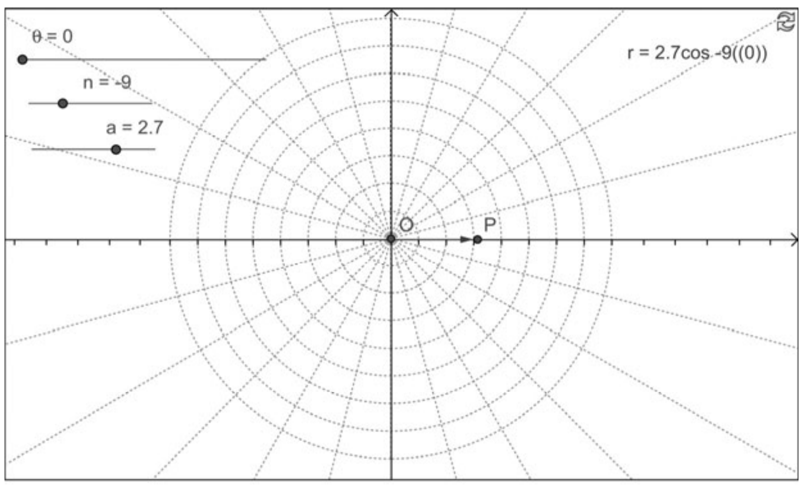
> A system of instruction that places the student in its heart. It is teaching that facilitates active participation and independent inquiry, and seeks to instill among students the joy of learning inside and outside the classroom. (Ang, Gonzales, Liwag, Santos, & Vistro-Yu, 2001, p. 2)

Faculty members undergo extensive and continuous training in developing teaching methods that promote active participation, independent inquiry, and a shared responsibility in the learning process (Vistro-Yu, 2006). As part of our efforts to provide student-centered learning activities in pre-calculus, we used GeoGebra to create interactive applets, which students may access online inside or outside the classroom. These sketches were either uploaded to the course website (i.e., http://sose.ateneo.edu/system.php?LS=staticpages& id=1219457084122) or emailed to students. We found that the GeoGebra sketches provided the students with some flexibility over the time and place of study and allowed them to work at their own pace, on the basis of their prior knowledge (Sparrow, Sparrow, & Swan, 2000). The use of GeoGebra provides opportunities for student-centered learning in several ways: by providing a basis for classroom discussion and assessment, and by facilitating proofs. We illustrate the possibilities below.

Figure 5 shows an example of a GeoGebra applet in our pre-calculus web page. Students were assigned to work on the applet independently or in small groups. Guide questions were provided to enable students to determine the relationships between parameters and the graph. When used in class, discussions followed these explorations, which provided opportunities for students to share ideas and validate each other's responses.

Graphs of Roses (1)

1. Move the θ slider to obtain the graph of r = a cos (n(θ)).
2. Vary the values of n, a and observe the corresponding graph. what does n represent? How about a?



Ma. Louise De Las Penas, September 14, 2008, Created with GeoGebra

Figure 5. Pre-constructed GeoGebra sketch.

Like the example shown in Figure 5, each web-based applet includes pre-constructed GeoGebra sketches and exploratory questions. The applet drew students' attention to various features of the sketch. The questions are important because when students work at home or with their small groups without their teachers' presence, the questions guided them towards the applets' learning objectives. Together, the pre-constructed sketches and questions facilitate the exploration process by helping students observe, pose their own questions, and investigate relationships (Sinclair, 2003).

The GeoGebra applets on our course web page were thus designed with the student as an independent learner in mind. We ensured that the applets' objectives were clear, and that applets were supported by guiding questions to allow students to navigate through the lessons on their own. Teachers may simply assign students to work on particular applets on the course web page, depending on the current classroom lesson. The applets are also appropriate for students from other universities. Inservice teachers who were trained in our university reported that they used the applets on our course web page in their own classes. Some also used the applets as a guide to create their own GeoGebra sketches.

Because the computer laboratory was not always available for hands-on investigations, the web-based applets have also been used for lectures in whole-class settings. When done properly, lectures can become avenues for SCL. GeoGebra sketches have been used in whole classes to promote active thinking. Its dynamic nature enables the teacher to "guide" rather than to "tell." For example, in a lesson on the inverse of a function, teachers may present a pre-constructed GeoGebra sketch, where point A can be dragged (see Figure 6). As point A is moved, students see that point B moves as well. Teachers may then ask students how point B relates to point A. Next, teachers may use the Trace feature of GeoGebra to help student make sense of the inverse of a function (Figure 7). Instead of simply providing the definition of a function's inverse, teachers may use the GeoGebra applet to help students visualize or construct the definition themselves.

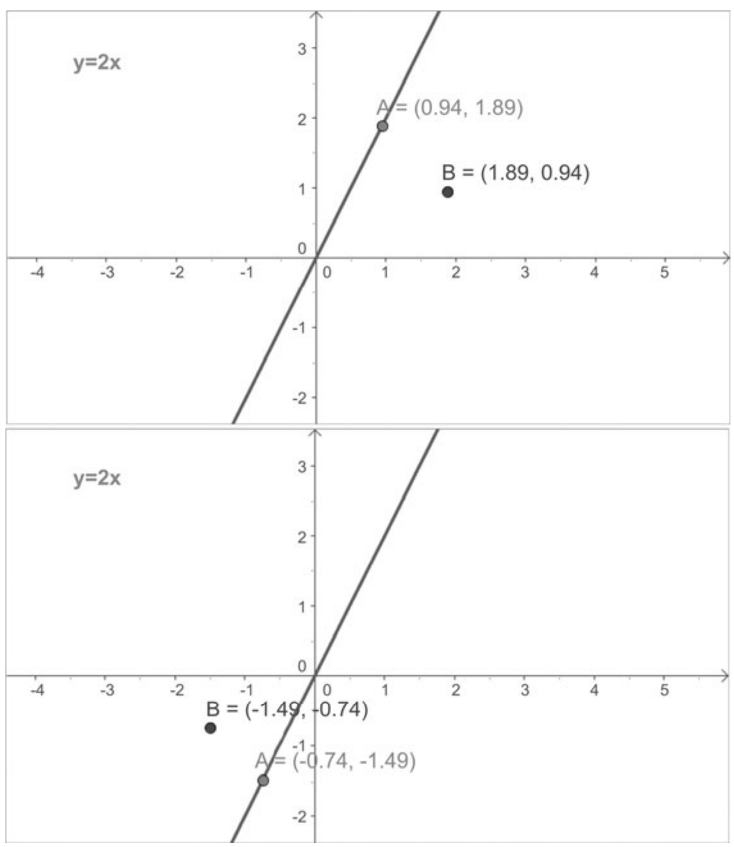


Figure 6. Sketch of the inverse of a function.

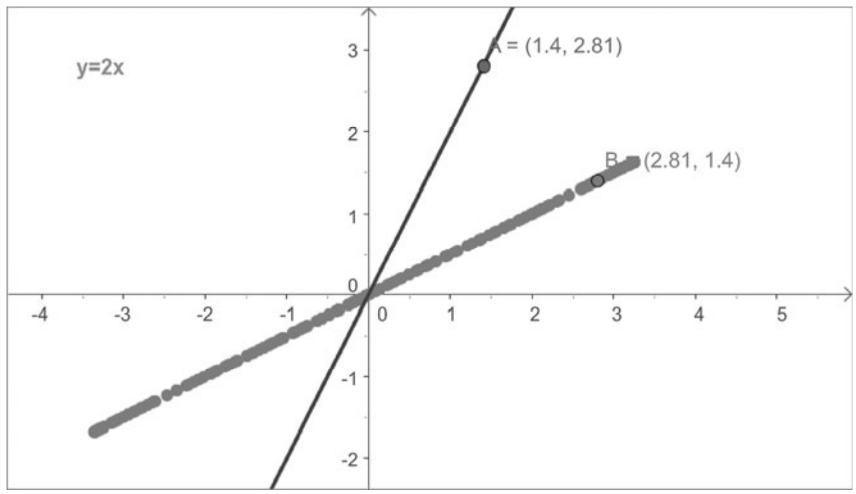


Figure 7. Sketch of the inverse of a function.

Aside from facilitating classroom investigations, GeoGebra may also be used for assessments. We show in Figure 8 an example of a GeoGebra applet, created by Dr. Jumela Sarmiento of Ateneo de Manila University, where students are asked to apply the concepts learned during class. In this example, students have to use their knowledge of trigonometric functions to investigate the word problem. They may inquire independently, and their answers may be verified using the web page.

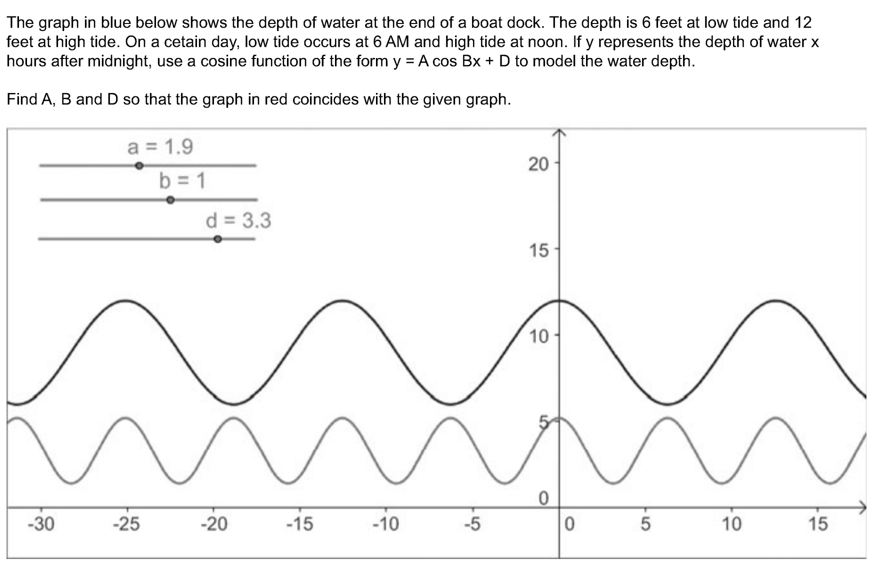


Figure 8. GeoGebra as an assessment tool.

GeoGebra has further been used to facilitate proofs (De las Peñas & Bautista, 2008). Traditionally, teachers read a theorem or a problem from a textbook, and prove the statements on the board. After teachers have modeled the process of proving, they provide exercises for students to work on independently or in groups. GeoGebra provides an alternative to this teaching strategy. A particular feature of GeoGebra that assists students in forming their own proofs is its construction protocol. Consider the example shown in Figure 9, where point A may be dragged along the circle, and the locus of point P may be traced. It appears that the P traces an ellipse, but a proof is necessary to justify this claim. To form a proof, students may view the construction protocol and follow along the construction process as it is shown step by step (Figure 10).

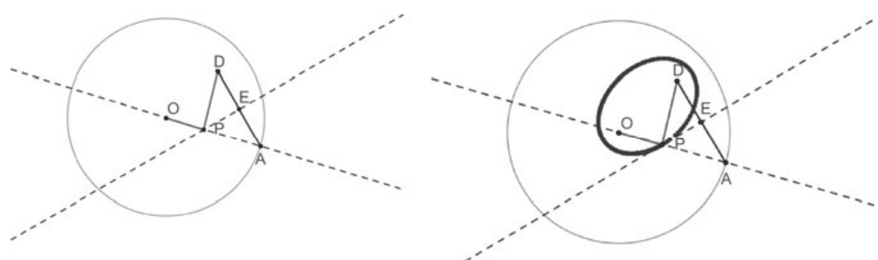
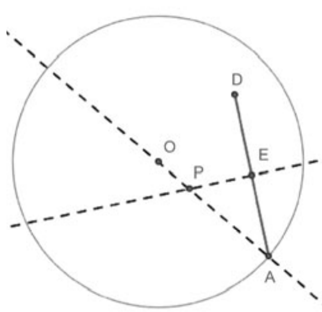


Figure 9. Creating an ellipse.



Construction Protocol

File View Help

...	Name	Definition	Command
1	Point O		
2	Point B		
3	Circle c	Circle through B with	Circle[O, B]
4	Point A	Point on c	Point[c]
5	Point D		
6	Segment a	Segment [A, D]	Segment[A, D]
7	Point E	Midpoint of A, D	Midpoint[A, D]
8	Line b	Line through E	PerpendicularLine[E, a]
9	Line d	Line through O, A	Line[O, A]
10	Point P	Intersection point of d,	Intersect[d, b]

Figure 10. The construction protocol of an ellipse.

Through the construction protocol, students understand how different objects in the applet are related to each other. For example, the construction protocol shows that the line through E is the perpendicular bisector of segment DA. From this, students may conclude from a well-known geometric theorem that the lengths of segments DP and AP are equal, regardless of the position of P. Because the length of OA is fixed, students may conclude that the sum of the lengths of segments OP and DP is invariant, implying that point P traces an ellipse with foci O and D.

With the above example, we have shown how GeoGebra's dynamic nature and its construction protocol have been utilized to facilitate proofs. Other examples of how we have used GeoGebra to facilitate proofs are reported elsewhere (De las Peñas & Bautista, 2008).

CONCLUSION

This chapter described some ways GeoGebra is being utilized in the Philippines. We used GeoGebra to introduce mathematics teachers to dynamic representations of mathematical concepts. Because of the limited access to technology in the Philippines, GeoGebra, being open-source software, becomes particularly appropriate and cost-effective for the scaling up of technology integration in mathematics education. When teachers learn about GeoGebra through seminars, they can apply what they have learned because they can access GeoGebra for free. They are also inclined to integrate technology in their mathematics classes.

However, being free does not mean that GeoGebra is deficient. We provided examples of how versatile GeoGebra's functionalities are. GeoGebra may be used to illustrate concepts in whole-class settings or in small-group stations. The sketches may also be shared online for use by a wider Philippine audience. These teachers may also take advantage of the dynamic nature of GeoGebra to create their own manipulatives. The sketches and technology-based manipulatives provide students with opportunities to investigate mathematical concepts, formulate proofs, and become independent learners.

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16. GEOGEBRA, DEMOCRATIC ACCESS, AND SUSTAINABILITY

Realizing the 21st-Century Potential of Dynamic Mathematics for All

The aim of this chapter is to highlight the international trends in GeoGebra usage since its emergence as a powerful, open-source mathematics software in 2002. GeoGebra Institutes are being formed around the globe under the auspices of the International GeoGebra Institute (IGI), and the GeoGebra Wiki and User Forum are being well used by an increasing number of academics and enthusiasts. Within this chapter, comments made by a variety of these international users will be highlighted to address the questions of democratic access and software sustainability vis-à-vis the open source context, and to offer insight into the meaningful and expanding role that the software now plays in many regions of the world.

INTRODUCTION

The National Council of Teachers of Mathematics (NCTM), in *Principles and Standards for School Mathematics* (2000), presented *equity* as the first of six principles that should guide mathematics education in the new millennium. Specifically, they proposed that "excellence in mathematics education requires equity—high expectations and strong support for all students" (p. 12). Furthermore,

> Well-documented examples demonstrate that all children, including those who have been traditionally underserved, can learn mathematics when they have access to high-quality instructional programs that support their learning. . . . These examples should become the norm rather than the exception in school mathematics education. Achieving equity requires a significant allocation of human and material resources in schools and classrooms. Instructional tools, curriculum materials, special supplemental programs, and the skillful use of community resources undoubtedly play important roles. An even more important component is the professional development of teachers. (p. 14)

Access to such high-quality instructional programs, resources, and development opportunities has often been limited, or even non-existent because of financial realities. The notion of "democratic access to fundamental ideas of mathematics" was popularized by the late Jim Kaput (Lesh & Roschelle, 2006). As plenary

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speakers at the 3rd International Conference on Mathematics Education and Society, Danish researchers, Skovsmose and Valero (2002), shared their related concern regarding democratic access: "Mathematics education presupposes resources, and we believe that it is necessary to ask how these resources—human and material—create opportunities and, more essentially, how resources and opportunities are distributed around the world" (p. 8). In their *Handbook of International Research in Mathematics Education* chapter entitled *Democratic Access to Powerful Mathematical Ideas*, the same researchers describe the complexities of equal access to mathematics education:

> All students, everywhere in the world, have the right to education. We can go further and say that all students in the world should have the chance to learn mathematics. Democratic access, in this sense, refers to the actual possibility of providing mathematics for all. However, the idea of democratic access, understood as the right to participate in mathematics education, is more complex. . . . [A] critical view on the connection between mathematics education and democracy situates the notion of democracy in the sphere of everyday social interactions and redefines it as purposeful, open political action undertaken by a group of people. This action is collective, has the purpose of transforming the living conditions of those involved, allows people to engage in a deliberative communication process. (2008, p. 427)

It was Freire who, in his work *Pedagogy of the Oppressed* (1970/2000), described a revolutionary leadership in which common *reflection and action* would characterize the liberation of a population, in light of democratic access to ideas and resources; a context in which leaders would continue to learn as equal co-participants:

> A revolutionary leadership must accordingly practice co-intentional education. Teachers and students (leadership and people), co-intent on reality, are both subjects, not only in the task of unveiling that reality, and thereby coming to know it critically, but in the task of re-creating that knowledge. As they attain this knowledge of reality through common reflection and action, they discover themselves as its permanent re-creators. In this way, the presence of the oppressed in the struggle for their liberation will be what it should be: not pseudo-participation, but committed involvement. (p. 69)

It is these very issues of democratic access, collective questioning and action, and sustainable growth through distributed leadership and shared learning that form the basis of this chapter. Positioned at the intersection of these complex issues, and presented as "Dynamic Mathematics for Everyone," the software *GeoGebra* in its rapid rise in global popularity (Hohenwarter & Lavicza, 2009) has drawn both praise and criticism from various stakeholder groups. Some may consider the arrival of *GeoGebra* as a potentially defining moment in Kaput's grand vision of democratic access; others maintain that the software still presupposes technological resources which are not always available in financially weak regions; and still others feel that the open-source status of the software is unsustainable in terms of development, growth, and quality control. However, we contend that these latter objections are likely unfounded, particularly in light of other longstanding and successful open-source initiatives (e.g., Linux), and the fact that several large software companies are now developing products that involve open-source technology in order to meet the challenges of sustainability and relevancy within a rapidly changing marketplace.

The permanence and potential, yet ongoing challenges, of technology use in mathematics education were highlighted by several keynote speakers (e.g., Artigue, Kilpatrick, Hoyles) at the 11th International Congress on Mathematics Education (ICME-11) held in Monterrey, Mexico in July 2008. Repeated reference was also made in various working group and workshop sessions to the growing demand for widely-accessible and quality mathematics software resources that could meet the expanding needs of educators and institutions worldwide, particularly in regions wherein existing commercial software is simply not affordable. It was following this ICME-11 event that the creator of *GeoGebra*, Dr. Markus Hohenwarter, invited a number of international colleagues to comment, via email, on their uses of, and opinions relating to *GeoGebra* in light of the above issues. Responses to this email were received from a number of different countries (e.g., Brazil, Costa Rica, Egypt, Spain) and it is our purpose in the remainder of this chapter to highlight some of these thoughts based on the email feedback and on other subsequent discussions with colleagues in the field.

GEOGEBRA SOFTWARE IN TEACHING AND LEARNING MATHEMATICS

The emergence of *GeoGebra* as an increasingly popular mathematics education resource is likely due to a number of key characteristics: versatility (i.e., algebraic, geometric, and now numeric [spreadsheet] representations), open-source accessibility in multiple language translations, and interactivity via an on-line wiki and user forum. Respondents to the above-mentioned email invitation often mentioned these and other benefits. For example, a German university professor noted with enthusiasm both the usefulness of the software in teaching as well as what he perceives to be the positive effects of the software on the learning of mathematics:

> *GeoGebra* is for me and my colleagues the most ingenious mathematical tool that we have known and learned so far. I cannot imagine preparing my professional work without *GeoGebra* for my teaching materials. Several times a week I use *GeoGebra* for demonstration or illustration purposes. Application areas for me include algebra, plane geometry, trigonometry, analysis, vector geometry, imaging geometry, etc. . . . Through the use of *GeoGebra*, many connections are made for the learner, and it is much easier and faster for them to understand. *GeoGebra* is now a very important tool for teaching. It increases the attractiveness of teaching mathematics and

> increases the motivation of the students. . . . Once again, I would like to express my sincere thanks to you all for your great work. I am in my professional life infinitely glad to have *GeoGebra*. I hope that you have, for a long time, the energy, the resources, and the desire to keep working on it. (personal communication[1], September 2008)

Likewise, a Brazilian mathematics professor explained how he uses the software for two main purposes in his teaching, and how he too has noticed student improvement:

> I have used *GeoGebra* with two main purposes: (1) to produce mathematically correct illustrations for the texts that I write, and, (2) to produce interactive applets to illustrate mathematical concepts in my classes. The purpose (1) is not trivial: It's very easy to find incorrect graphs printed in our secondary school books (for instance, the graph of cosine being graphed with semicircles). . . . Indeed, I'll give a workshop showing how to use *GeoGebra* to produce beautiful and mathematically correct illustrations. In the last two semesters, I've combined (1) and (2) to teach calculus: I've used slides that were illustrated with pictures generated by *GeoGebra*. Key concepts were presented with animations and simulations built with your software. It is worth noting that weaker students had a considerable improvement with the use of this approach. (personal communication, July 2008)

Both of the above-quoted respondents also mentioned how the software is easily *accessible* by students and teachers, both in a strictly logistical (i.e., access to the software in their school, region, country), and a more metaphorical sense (i.e., in the sense of an easy-to-use tool). The democratic accessibility sense, in terms of its global, open-source availability now becomes our main focus in what follows.

DEMOCRATIC ACCESS

As we explore the notion of democratic access to fundamental mathematical ideas and, in this particular case, technological resources, we are cognizant of the fact that not all regions of the world presently have the luxury of computers in classrooms. However, with initiatives such as the Massachusetts Institute of Technology Media Lab's "One-Laptop-Per-Child"[2] project (Bullis, 2005), and with the falling costs for small laptop ("netbooks") and tablet computers (e.g., India's $35 machine announced in 2010), we are also hopeful that democratic access to computers, and hence to open-source software, will potentially come within reach of the global community within the near future. This conviction is also foregrounded within the comments made by a number of international colleagues.

A retired professor and a secondary school math/science teacher from Finland describes the ease of use of *GeoGebra* for their students, as well as the significant benefit of being able to teach and learn with the software in one's own mother tongue:

> We like *GeoGebra* because it is free, it is versatile, we can use it in Finnish and, last but not least, it loads automatically, so the students can use it for homework and voluntary studies without any special arrangements. For us it is not insignificant that we can cooperate in an international community. It is the best way to get new and miscellaneous ideas. In order to widen the scope of *GeoGebra* users in Finland we need to operate in Finnish. Therefore, we are proud of having Finnish GeoGebraWiki pages, even though tiny yet. Interest in using *GeoGebra* is increasing rapidly in Finland. . . . In addition to translating *GeoGebra* environment and handbook into Finnish, our main aims are to find suitable ways to use *GeoGebra* in mathematics instruction in keeping with the Finnish curricula, to make up dynamic worksheets, and to develop styles of instruction. . . . Schools have small economic resources so to have a brilliant tool free of charge is a prodigious advantage. For fourteen to sixteen year-old students, the opportunity to work and learn in their mother tongue is essential. (personal communication, August 2008)

Pragmatic issues of computer lab availability and commercial software licensing were described by a veteran Spanish secondary school mathematics teacher:

> Commercial software has a number of inconveniences from which *GeoGebra* does not suffer: Facilities and relocations must be "authorized" on each computer, one must watch for updates and permits, not having the application on any computer that has just an Internet connection, etc. Perhaps for someone accustomed to developing their classes in a computing environment this is only a small inconvenience. However, the vast majority of Spanish math teachers greatly appreciate any simplification of work in an environment that they consider little friendly. Moreover, pupils working outside of school can benefit from the immediate reach of the software—one doesn't need CDs, DVDs, or download keys. Suffice it to say two words: *Google* and *GeoGebra* (I now refer to these as the "four Gs"!). (personal communication, August 2008)

A Brazilian doctoral student and first-year mathematics teacher of engineering courses explained her own increased comfort with *GeoGebra* over time, and also underscored the importance of democratic access and instructor responsibility:

> Unfortunately, during my professional formation, in the 90's, I did not have the opportunity of dealing with computers, so I confess that I was a bit frightened of them. Fortunately, after knowing programs like *GeoGebra* which is easy to manipulate, free, and extremely interesting [for students], I was encouraged to bring those rich tools into my classes. . . . In 2006, I used *GeoGebra* to teach the process of Riemann integration to my students. It was very good to see how they worked seriously and felt stimulated to make experiences related to the subject of the class, using this program. In 2008, I

> repeated the experience introducing new exercises and modifying the old ones. Again, the students "took in their hands" the learning of the integration process mediated by *GeoGebra*. . . . Finally, I'd like to emphasize that *GeoGebra*, besides being an excellent program for the development of mathematics teaching/learning, as a free program it becomes democratically accessible to the teachers. I believe that making *GeoGebra* reach the classrooms is also the duty of the mathematical education professionals, so I hope this community will feel the necessity of [sharing] it. (personal communication, August 2008)

Two Iranian educators discussed how they use *GeoGebra* in their geometry teaching, as well as how collaboration was facilitated for them in Iran:

> [W]e are doing research for interactive education solutions and we have found *GeoGebra* to be very useful in teaching and learning geometry, algebra and calculus. Since *GeoGebra* is free, dynamic mathematics software and supports the Persian language, all Persian students and teachers can use it. We sent an email to Dr. Markus Hohenwarter, the developer and project leader of *GeoGebra*, to get some advice for creating interactive, web-based calculus lessons in Persian. He wrote us that it would be great if we could also provide our materials on [the *GeoGebra* website]. He also forwarded our email to the Persian translator of *GeoGebra* in Iran. Thanks to Markus, we have joined together to organize our activities and develop our goals, since November 2007. On the one hand, our mission is to promote student achievement in learning of mathematics, both in a conceptual and activity-based approach based on *GeoGebra* by developing web-based mathematics education. On the other hand, we support mathematics teachers who would like to effectively integrate dynamic mathematics software into their own teaching practices. (personal communication, August 2008)

A professor of mathematics from Brazil explained his thoughts regarding present and future software accessibility, and shared a prediction about teacher use:

> The federal Brazilian government wants to equip every public school with computers, projectors, and broadband connection by 2010. Currently, it is funding projects for the production of free activities with free software to be used by secondary students. Many of these activities are being done with *GeoGebra*. *GeoGebra* is gaining great popularity in Brazil. Several reasons are contributing to this. The software is user-friendly, it has a lot of features, the author gives support, its community is growing fast, and it is free! I sincerely hope that *GeoGebra* remains free! . . . If *GeoGebra* becomes commercial, it certainly will lose many users, especially here in Brazil, where teachers and students cannot afford to buy educational software. (personal communication, July 2008)

It is, of course, one thing to make universally available, via the Internet, a piece of mathematical software, and yet quite another to ensure that this software will continue to expand and grow in terms of both its functionality and collaborative use. It is towards this latter, more complex, challenge that we now turn our attention.

SUSTAINABILITY

Fullan (2005), an internationally-acclaimed leader in educational reform, noted that sustainable system initiatives require the following context and features:

> From a system perspective, the single answer to the question of how to increase the chances for greater sustainability is to build a critical mass of developmental leaders who can mix and match, and who can surround themselves with other leaders across the system as they spread the new leadership capacities to others. Adaptive challenges such as sustainability, moral purpose for all, deep learning, fine-tuning intelligent accountability, productive lateral capacity building, and getting results never before attained can be tremendously enticing once you start to get good at doing them. People find meaning by connecting to others; and they find well-being by making progress on problems important to their peers and of benefit beyond themselves. (p. 104)

This type of positive synergy was commonly described within the responses received by school teachers and by post-secondary instructors alike. The sharing of *GeoGebra* resources, particularly among colleagues who speak one's own language, via the on-line Wiki and User Forum is noted as a contributing factor, or impetus, for future development. For example, an American mathematics professor, having created and shared many *GeoGebra* activities via his own instructor website, explained how that he feels that technological reform in mathematics education is essentially incremental:

> Part of the rationale I have for working with applets is that I am convinced the most effective way to engage teachers in pedagogical reform is not to propose a whole scale change but to instead offer discrete activities that can be incorporated piecemeal into a curriculum. I now have a substantial portion of my faculty who use some applets in their teaching. My applet website is currently drawing about 300 visitors a day. (personal communication, September 2008)

Again quoting the Spanish secondary school mathematics teacher, we read how he feels that commercial developers actually have more to fear than open-source counterparts with regard to sustainability and future relevance within an era of rapid technological change:

> I find it outrageous—but commercially understandable, unfortunately—the attitude of commercial software developers. Especially considering that the experience I've had in recent years shows exactly the opposite: It is

commercial software which surely will, increasingly, [experience] problems of sustainability. (personal communication, August 2008)

In Macedonia, a visionary enthusiast combines *GeoGebra* with new media for teaching and learning mathematics. She describes how she has developed digital "compass and straightedge" constructions, mathcasts, animations, YouTube™ videos, and even a *GeoGebra* group within the virtual world of Second Life™. On the issue of sustainability, she further shared the following response, "I would like to say that I have heard those kind of remarks before—about freeware not being user-sustainable. That is [ludicrous]—period. You have designed the most remarkable product and the word is getting out" (personal communication, August 2008).

LARGE-SCALE, SYSTEM-WIDE INITIATIVES

During the interim following the email communications described in this paper (2008–09), other *GeoGebra* initiatives have begun which serve to further underscore the importance of *democratic access* and the viability of *sustained software development* by an international user group. For example, the *Technology for Improved Learning Outcomes* (TILO) program has been designed as a joint venture of the U.S. Agency for International Development, the Egyptian Ministries of Education and Communication/Information Technology, the private sector, and communities located across seven governorates as this team seeks to develop a holistic integrated model for introducing technology into school-based reform activities. The goal is to benefit the larger community through increased student learning outcomes. TILO will focus activities in Alexandria, Cairo, Fayoum, Beni-Suef, Minya, Qena and Aswan to reach more than 200 primary schools undergoing school-based reform and 85 public experimental schools to be transformed into technology-enabled schools called "Smart Schools." Two TILO directors wrote to Hohenwarter to express their views:

> The TILO project and the teachers and students of Egypt thank you for creating *GeoGebra* and making it openly available for users globally. We have included *GeoGebra* as a recommended digital resource for use in these schools as we believe that it supports our project objectives to improve teaching and learning in Egypt. We are recommending that other educators in Egypt consider it as well. (personal communication, April 2009)

In her August 26, 2009 article in *ComputerWorld*, reporter Dahna McConnachie described a similar type of situation occurring in New South Wales, Australia in which 267, 000 laptops will come bundled with the *GeoGebra* software.

> The 267,000 Windows 7 based netbooks that the NSW Government has started rolling out to high schools will come pre-installed with open source software. . . . Over the next four years, each Year 9 student will receive one of the devices as a gift, which they can keep once they have left school. . . . A

> spokesperson from the NSW department of Education and Training (DET) today confirmed that *GeoGebra*, *Dia*, *Audacity*, *Freemind* and *MuseScore* would all be included on the devices. (paragraphs 1 and 3)

As clearly indicated within the above-mentioned communication excerpts, the fact that *GeoGebra* has remained a freely-available and open-source (i.e., visible and modifiable code for programmers) software option has been highly significant for educators and Ministries of Education, world-wide. The evidence of past, present, and future development of activities, resources, teacher training opportunities, software adaptations, and international connections via the *GeoGebraWiki* and User Forum speaks to the realistic expectation that the software will continually undergo expansion and refinement through focused and sustained international use.

To provide coherence and guidance to the growing international user population, the International GeoGebra Institute (IGI) was formally introduced in 2007 (Hohenwarter & Lavicza, 2007; Hohenwarter, Jarvis, & Lavicza, 2009). Since that time, national/regional GeoGebra Institutes have begun to be formally recognized by the IGI and presently exist in locations such as Austria, Brazil, Canada, Denmark, Hungary, Norway, Poland, Portugal, Spain, United Kingdom, United States, and Turkey, with the list of countries increasing continuously. A large number of projects and initiatives involving *GeoGebra* are also currently underway in many different countries, a list of which includes: Finland, France, Germany, Iceland, Hungary, Luxembourg, Philippines, South Africa, United Kingdom, and the USA. Development of the software itself also continues under the guidance of Hohenwarter and his core programming team, with version 3.2, which includes the "spreadsheet view," now available on-line and pre-releases continually being updated and made public for user feedback (e.g., CAS functionality). Language translations of *GeoGebra* menus and help files continue to increase in number as international volunteers offer their time and expertise. As a result, the software is currently available in 55 languages.

CONCLUDING THOUGHTS

The communication taking place among and between *GeoGebra* users from different regions, countries, and world hemispheres underscores the power of high-speed Internet within an age of globalization, "brick and click universities" (Gürüz, 2008, p. 85), and ICT-driven economies. Our interconnected, 21st-century context is that which makes even more possible the speaking of, what Freire referred to nearly 40 years ago as, "true words" involving action and reflection:

> As we attempt to analyze dialogue as a human phenomenon, we discover something which is the essence of dialogue itself: the word. But the word is more than just an instrument which makes dialogue possible; accordingly, we must seek its constitutive elements. Within the word we find two dimensions,

> reflection and action, in such radical interaction that if one is sacrificed—even in part—the other immediately suffers. There is no true word that is not at the same time a praxis. Thus, to speak a true word is to transform the world. (1970/2000, p. 87)

As will no doubt be further demonstrated within the covers of this first *GeoGebra*-focused compilation dealing with theory and practice in mathematics education, when enthusiastic people from around the world are intrigued, connected, and supported over space and time, they can achieve incredible goals together. Democratic access to, and sustained development of, the software known as *GeoGebra* are significant issues that will both require further research and investigation in order to more fully understand the ramifications of this particular approach to sharing mathematics resources and ideas. In the interim, it is with excitement and anticipation that we will watch this software continue to grow in educational functionality, and the people associated with its dramatic progress continue to act, reflect, and collaborate in the 21st century.

NOTES

1 "Personal communication" indicates actual responses to the above-mentioned email message. Names of participants remain confidential due to ethical considerations.

2 More information regarding the One-Laptop-Per-Child initiative organized by Chairman Dr. Nicolas Negroponte can be viewed here: http://laptop.org/en/

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REFLECTIONS AND CONCLUSIONS

This book represents an international sample of GeoGebra-inspired mathematics education endeavors during the first decade of its development alongside the open source movement in computing and education (Hohenwarter & Preiner, 2007). More than thirty mathematics educators and mathematicians from more than ten countries reflected on their educational experiments with both theories and practices involving GeoGebra and reported on their experiences, findings, and future directions. This volume covers a wide variety of topics in mathematics education and technology integration—real world simulations, geometry, calculus, cognitive tools, problem solving, mathematical modeling, teacher preparation, model-centered learning and instruction, mathematical attitudes, democratic access, and sustainability.

Throughout the chapters, there is one strong theme that is characteristic of the first decade of GeoGebra use in mathematics education: The GeoGebra community is moving beyond basic facts and skills toward a holistic and comprehensive model-centered approach to mathematics learning and instruction with a clear commitment to *teaching mathematics for understanding*. There are three possible reasons. First, the GeoGebra community is, in general, well informed of the research literature on the use of new technologies in mathematics education and especially the fact that mathematics education has been recognized as a complex enterprise involving multiple dimensions of human experience and sense-making (Richard Lesh, 2006; Richard Lesh & Doerr, 2003). Second, the GeoGebra community is in the process of synthesizing and applying the fundamental principles of contemporary learning and instructional design theories (Merrill, 2002; Milrad, Spector, & Davidsen, 2003; van Merriënboer, Clark, & de Crook, 2002; van Merriënboer & Kirschner, 2007), especially model-centered and/or inquiry-based perspectives on mathematics learning and the impacts of digital technologies. Third, GeoGebra, as a community-sustained open source web-friendly learning environment, is by design conducive to the development of whole-task learning activities, benefiting from its interactive and dynamic environment and world-wide collaboration and feedback community for both software development and its educational uses. Indeed, GeoGebra itself has witnessed significant evolutions since the book project was initiated, with improved user interfaces, functionalities, mathematical tools, and web features. Accordingly, the contributors to this volume have continued to update and enrich their chapters during the editing process. The dynamic nature of the software and that of the emerging communities of mathematical teaching practices are well

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aligned with our present characterization of genuine mathematical understanding on the basis of mental models.

Since this volume gives prominence to a model-centered approach to GeoGebra-integrated mathematics teaching and learning, it is important that we clarify the various uses of the term *model* and its context in order to avoid potential confusion within the GeoGebra community. The term *model* finds itself in numerous fields of research literature ranging from philosophy, psychology, to sciences and mathematics (Seel, 2003). In cognitive psychology, *mental models* are a key theoretical construct, referring to the internal representation of the structure of perceived reality, which functions as the cognitive basis for sense-making and decision-making (Johnson-Laird, 1983; Seel, 2003). In a specific domain of science, *conceptual models* are usually external representations of a certain target system that are established and approved by a scientific community for its validity, accuracy, consistence, and completeness. In instructional design theories, *learning and design models* are theoretical frameworks about the learner, the learning environment, assessment, and the scientific conceptualization of learner development (Spector, 2000; van Merriënboer & Kirschner, 2007). In general, models and modeling represent our ever evolving efforts to understand the world and manage its complexities at various levels. In the learning process, models can be recursively updated (Johnson-Laird, 1983) and/or encapsulated as elements of high-order models, creating of a chain of models as footprints of one's learning progress (e.g., Presmeg, 2006).

With regard to the use of GeoGebra in mathematics education, there are at least three different yet closely connected uses of the term *model*. First, in light of the maturity of mathematics in school settings, the majority of modeling activities can be classified as *didactical modeling,* by which real-world or realistic scenarios are utilized in a classroom in support of students' development of valid conceptual models and subsequently mental models in a progression of model-based mathematization (Freudenthal, 1973; Streefland, 1991; Treffers, 1987; Van den Heuvel-Panhuizen, 2003). Furthermore, didactical modeling can be successfully employed to situate, connect, or reconceptualize the teaching of a variety of curricular mathematical ideas (Bu, 2010; Zbiek & Conner, 2006). Second, *mathematical modeling* by itself is increasingly recognized as one of the primary goals of mathematics education (Mooney & Swift, 1999; Pollak, 2003), especially when traditional problem solving has been reconsidered under a *models and modeling* perspective (Richard Lesh & Doerr, 2003). Third, GeoGebra allows the *simulation* of traditional mathematical ideas. Similar to virtual manipulatives, these mathematical simulations are increasingly used to capture and demonstrate the dynamic nature of mathematical ideas for the purpose of teaching and learning. For example, the construction of a dynamic equilateral triangle by itself can be a worthwhile leaning task, which allows a student to present, diagnose, reflect on, and deepen her conception of the mathematical idea of an arbitrary equilateral triangle. Using GeoGebra, an equilateral triangle can be constructed using the regular polygon tool, an angle-based method, or a side-based method. While modeling/simulating curricular mathematics, students' GeoGebra models tend to change along with the progression of teaching and their understanding of the mathematical idea, and this change is pedagogically informative (cf. Van den Heuvel-Panhuizen, 2003). A dynamic equilateral triangle, for example, further showcases what invariants are essential to the mathematical concept.

As we conclude this first book on Geogebra and its initial uses by in the international mathematics education community, GeoGebra continues to grow. As of October 2010, GeoGebra is available in 55 languages, serving more than 700,000 visitors from 190 countries and regions every month. There are 45 GeoGebra Institutes dedicated to the local implementation and research around the world. As we look into the future of GeoGebra and its growing user and research communities, we venture to make the following recommendations for the purpose of generating more discussions:

– By virtue of the open source nature of GeoGebra and its growing international communities, it is highly feasible and necessary to build an international online database of GeoGebra-integrated mathematical activities/problems together with empirical data such as student-generated artifacts and assessment tools. The Geogebra communities need to develop and agree on a set of standards to facilitate the archiving and reuse of such activities for classroom adaptations and research syntheses.
– In terms of GeoGebra-oriented research, we suggest that GeoGebra communities conduct multi-tier teaching experiments with children, prospective teachers, inservice teachers, and even mathematicians and mathematics education researchers, bringing forth the voices of each group in support of day-to-day classroom teaching and theory development (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Richard Lesh, Hamilton, & Kaput, 2007).
– GeoGebra does not only provide new approaches to traditional mathematical ideas, it may indeed change the very nature of the mathematics being taught, the way students think, the way teachers teach, and many other intermediate elements, such as student attitudes and expectations and public policies, in the complex system of mathematics education, especially when tens of millions of laptops loaded with GeoGebra are made available to school children. Therefore, the GeoGebra community needs to collect and synthesize empirical data about its local or global impact on mathematics education, including curricular development and reforms.
– The open source nature of GeoGebra makes it especially appropriate for technology integration in mathematics education in the rural areas and other technologically under-represented world communities. It will be highly informative to collect data on the influence of dynamic mathematics on students and prospective and inservice teachers in such regions as the world joins hands in understanding and improving mathematics education for all populations.
– GeoGebra-integrated learning materials are interactive and dynamic in nature, presenting great challenges to traditional publication media. For example, the construction protocols of GeoGebra designs capture the solution process of a problem. By playing back such construction protocols, teachers and researchers can look into the problem solution in the making, obtaining insight into a

student's thinking processes. In the long run, an international channel or online peer-reviewed journal would be appropriate for sharing dynamic curricular materials and research findings within the international GeoGebra community.

As GeoGebra becomes mature, incorporating the feedback and the specific needs of diverse users, we anticipate that more practical and theoretical issues will become relevant, creating both challenges and opportunities for the GeoGebra and mathematics education community. Furthermore, we recognize the fact that most of us, the GeoGebra enthusiasts, have already acquired a decent understanding of the mathematics we are (re-)investigating using GeoGebra. Although many of us have developed new insights into the traditional mathematics we are teaching, students' experiences may be very different than ours, pointing to new directions in theory and practice. Indeed, students' conceptions of mathematics and technologies are also ongoing and dynamic, which reminds us of the important principle in learning and instruction theories—the *uncertainty principle* from both mathematics education and instructional design research. We must have an open mind for educational creativity and continued improvement in design and assessment (Streefland & van den Heuvel-Panhuizen, 1998), and we must acknowledge the fact that "people generally know less than they are inclined to believe" (Spector, 2004, p. 274). The uncertainty principle informs us that GeoGebra may eventually bring about positive changes in mathematics education on an international scale and yet we need to continue to critique our own work to open the door for new ideas and new designs. Looking ahead into the future of GeoGebra in mathematics education, we think that Model-Centered Learning and Instruction (MCLI) with GeoGebra stands as a theoretical tool for thought and reflection as well as a call for actions and experimentation, as we seek to promote understanding as the primary goal of mathematics teaching and learning (cf. Shulman, 2002). As GeoGebra becomes integrated in the mathematical lives of millions of children and adults, it may well change the educational landscape of mathematics, creating new opportunities and challenges for theory and practice.

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