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# Mathematical economics

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Undergraduate study in
**Economics, Management, Finance and the Social Sciences**

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# Contents

# Chapter 1
# Introduction

Welcome to **120 Mathematical economics** which is a '300' course offered on the Economics, Management, Finance and Social Sciences (EMFSS) suite of programmes.

In this brief introduction, we describe the nature of this course and advise on how best to approach it. Essential textbooks and Further reading resources for the entire course are listed in this introduction for easy reference. At the end we present relevant examination advice.

## 1.1 The structure of the course

The course consist of two parts which are roughly equal in length but belonging to the two different realms of economics.

- The first part deals with the mathematical apparatus needed to rigoursly formulate the core of **microeconomics**, the consumer choice theory.
- The second part presents a host of techniques used to model intertemporal decision making in macreconomy.

The two parts are also different in style. In the first part it is important to lay down rigourous proofs of the main theorems while paying attention to to assumption details. The second part often dispenses with rigour in favour of slightly informal derivations needed to grasp the essenceof the methods. The formal treatment of the underlying mathematical foundations is too difficult to be within the scope of undergraduate study; still, the methods can be used fruitfully without the technicalties involved: most macroeconomists actively employing them have never taken a formal course in optimal control theory.

If taken as part of a BSc degree, courses which must be passed before this course may be attempted are **66 Microeconomics, 05a Mathematics 1** *and* **05B Mathematics 2** *or* **174 Calculus**, which cover multivariate calculus and integration. As you already have this understanding, we have striven to make the exposition in both parts of the subject completely self-contained. This means that beyond the basic prerequisites you do not need to have an extensive background in fields like functional analysis, topology, or differential equations. Howeve, such a background may allow you to progress faster. For instance, if you have studied the concepts and methods of ordinary differential equations before you may find you can skip parts of Chapter 8.

By design the course has a significant economic component. Therefore we apply the techniques of constrained optimisation to the problems of static consumer and firm choice; the dynamnic programming methods are employed to analyse consumption smoothing, habit formation and allocation of spending on durables and non-durables; the phase plane tools are used to study dynamic fiscal policy analysis and foreign currency reserves dynamics; Pontryagin's maximum principle is utilised to examine firm's investment behaviour and the aggregate saving behaviour in an economy.

## 1.2 Aims

The course is specifically designed to:

- demonstrate to you the importance of the use of mathematical techniques in theoretical economics
- enable you to develop skills in mathematical modelling.

## 1.3 Learning outcomes

At the end of this course, and having completed the Essential reading and activities, you should be able to:

- use and explain the underlying principles, terminology, methods, techniques and conventions used in the subject
- solve economic problems using the mathematical methods described in the subject

## 1.4 Syllabus

**Techniques of constrained optimisation**. This is a rigourous treatment of the mathematical techniques used for solving constrained optimisation problems, which are basic tools of economic modelling. Topics include: Definitions of a feasible set and of a solution, sufficient conditions for the existence of a solution, maximum value function, shadow prices, Lagrangian and Kuhn–Tucker necessity and sufficiency theorems with applications in economics, for example General Equilibrium theory, Arrow-Debreau securities and arbitrage.

**Intertemporal optimsiation**. Bellman approach. Euler equations. Stationary infinite horizon problems. Continuos time dynamic optimisation (optimal control). Applications, such as habit formation, Ramsay-Kass-Coopmans model, Tobin's q, capital taxation in an open economy, are considered.

**Tools for optimal control: ordinary differential equations**. These are studied in detail and include linear 2nd order equation, phase portraits, solving linear systems, steady states and their stability.

## 1.5 Reading advice

While topics covered in this subject are in every economists essential toolbox, their textbook covereage varies. There are a lot of first-rate treatments of static optimisation methods; most textbooks that have 'mathematical economics' or ' mathematics for economists' in the title with have covered theses in various levels of rigour. Therefore students with difffernebt backgrounds will be able to choose a book with the most suitable level of exposition.

### Essential reading

Dixit, A.K. *Optimization in Economics Theory.* (Oxford University Press, 1990) [ISBN 9780198772101]

The textbook by Dixit, is perhap in the felictous middle. However until recently there has been no textbook that covers all the aspects of the dynamic analysis and optimisation used in macroeconomics models.

Sydsaeter, K., P. Hammond, A. Seierstad and A. Strom *Further Mathematics for Economic Analysis.* (Pearson Prentice Hall, 2005) [ISBN 9780273655763]

The book by Sydsaester et al. is a recent attempt to close that gap, and is therefore the Essential reading for the second part of the course, despite the factthatthe exposition is slightly more formal than in this giude. This book covers almost all of the topics in the course, although the empasis falls on the technique and not on the proof. It also provides useful reference for Linear algebra, calculus and basic topology. The style of the text is slightly more formal than the one adopted in this subject guide. For that reason we have included references to Further reading, especially from various macroeconomics text books, which may help develop a more intuitive (non-formal) understanding of the concepts from the application-centred perspective. Note that whatever your choice of further reading reading textbooks is, **textbook reading is essential**. As with lectures, this guide gives structure to your study, while the additional reading supplies a lot of detail to supplement this structure. The are also more learning activities and Sample examination questions, with solutions, to work through in each chapter.

Detailed reading references in this subject guide refer to the editions of the set textbooks listed above. New editions of one or more of these textbooks may have been published by the time you study this course. You can use a more recent edition of any of the books; use the detailed chapter and section headings and the index to identify relevant readings. Also check the virtual learning environment (VLE) regularly for updated guidance on readings.

## Further reading

Please note that as long as you read the Essential reading you are then free to read around the subject area in any text, paper or online resource. You will need to support your learning by reading as widely as possible and by thinking about how these principles apply in the real world. To help you read extensively, you have free access to the VLE and University of London Online Library (see below).

Other useful texts for this course include:

Barro, R. and X. Sala-i-Martin *Economic Growth*. (McGraw-Hill, 1995) [ISBN 9780262025539] The mathematical appendix contains useful reference in condensed form for phase plane analysis and optimal control.

Kamien, M. and N.L. Schwarz *Dynamic optimisation: the calculus of variations and opptimal control in economics and management*. (Elsevier Science, 1991) [ISBN 9780444016096] This book extensively covers optimal control methods.

Lunjqvist, L. and T.J. Sargent *Revcursive macroeconomic theory*. (MIT Press, 2001) [ISBN 9780262122740] This book is a comprehensive (thus huge!) study of macroeconomical applications centred around the dynamic programming technique.

Rangarajan, S. *A first course in optimization theory*. (Cambridge, 1996) [ISBN 9780521497701] Chapters 11 and 12. This book has a chapter on dynamic programming.

Sargent, T.J. *Dynamic macroeconomic theory*. (Harvard University Press, 1987) [ISBN 9780674218772] Chapter 1. This book has a goof introduction into dynamic programming.

Simon, C.P. and L. Blume *Mathematics for economists*. (W.W Norton, 1994) [ISBN 9780393957334] This textbook deals with static optimisation topics in a comprehensive manner. It also covers substantial parts of differential equations theory.

Takayama, A. *Analytical methods in economics*. (University of Michigan Press, 1999) [ISBN 9780472081356] This book extensively covers the optimal control methods.

Varian, H.R. *Intermediate microeconomics: A modern approach*. (W.W. Norton & Co, 2005) [ISBN 9780393927023] Chapters 2–6, or the relevant section of any intermediate microeconomics textbooks.

Varian, H.R. *Microeconomic Analysis*. (W.W Norton & Co, 1992) third edition [ISBN 9780393957358] Chapters 7. For a more sophisticated treatment comparable to Chapter 2 of this guide.

## 1.6 Online study resources

In addition to the subject guide and the reading, it is crucial that you take advantage of the study resources that are available online for this course, including the VLE and the Online Library.

You can access the VLE, the Online Library and your University of London email account via the Student Portal at:
http://my.londoninternational.ac.uk

You should have received your login details for the Student Portal with your official offer, which was emailed to the address that you gave on your application form. You have probably already logged in to the Student Portal in order to register! As soon as you registered, you will automatically have been granted access to the VLE, Online Library and your fully functional University of London email account.

If you forget your login details at any point, please email uolia.support@london.ac.uk quoting your student number.

### The VLE

The VLE, which complements this subject guide, has been designed to enhance your learning experience, providing additional support and a sense of community. It forms an important part of your study experience with the University of London and you should access it regularly.

The VLE provides a range of resources for EMFSS courses:

- Self-testing activities: Doing these allows you to test your own understanding of subject material.
- Electronic study materials: The printed materials that you receive from the University of London are available to download, including updated reading lists and references.
- Past examination papers and *Examiners' commentaries*: These provide advice on how each examination question might best be answered.
- A student discussion forum: This is an open space for you to discuss interests and experiences, seek support from your peers, work collaboratively to solve problems and discuss subject material.
- Videos: There are recorded academic introductions to the subject, interviews and debates and, for some courses, audio-visual tutorials and conclusions.
- Recorded lectures: For some courses, where appropriate, the sessions from previous years' Study Weekends have been recorded and made available.
- Study skills: Expert advice on preparing for examinations and developing your digital literacy skills.
- Feedback forms.

Some of these resources are available for certain courses only, but we are expanding our provision all the time and you should check the VLE regularly for updates.

### Making use of the Online Library

The Online Library contains a huge array of journal articles and other resources to help you read widely and extensively.

To access the majority of resources via the Online Library you will either need to use your University of London Student Portal login details, or you will be required to register and use an Athens login: http://tinyurl.com/ollathens

The easiest way to locate relevant content and journal articles in the Online Library is to use the **Summon** search engine.

If you are having trouble finding an article listed in a reading list, try removing any punctuation from the title, such as single quotation marks, question marks and colons.

For further advice, please see the online help pages: www.external.shl.lon.ac.uk/summon/about.php

## 1.7 Using the subject guide

We have already mentioned that this guide is not a textbook. It is important that you read textbooks in conjunction with the guide and that you try problems from them. The Learning activities, and the sample questions at the end of the chapters, in this guide are a very useful resource. You should try them all once you think you have mastered a particular chapter. Do really try them: don't just simply read the solutions where provided. Make a serious attempt before consulting the solutions. Note that the solutions are often just sketch solutions, to indicate to you how to answer the questions, but in the examination, you must **show all your calculations**. It is vital that you develop and enhance your problem-solving skills and the only way to do this is to try lots of examples.

Finally, we often use the symbol ■ to denote the end of a proof, where we have finished explaining why a particular result is true. This is just to make it clear where the proof ends and the following text begins.

## 1.8 Examination

**Important**: Please note that subject guides may be used for several years. Because of this we strongly advise you to always check both the current *Regulations*, for relevant information about the examination, and the VLE where you should be advised of any forthcoming changes. You should also carefully check the rubric/instructions on the paper you actually sit and follow those instructions.

Two Sample examination papers are given at the end of this giude. Notice that the actual questions may vary, covering the whole range of topics considered in this course syllabus. You are required to answer FOUR of the SIX questions: TWO from Section A and TWO from Section B. All questions carry equal marks and you are advised to divide your time accordingly.

Also note that in the examination you should submit all your derivations and rough work. If you cannot completely solve and examination question you should still submit partial answers as many marks are awarded for using the correct approach or method.

Remember, it is important to check the VLE for:

- up-to-date information on examination and assessment arrangements for this course
- where available, past examination papers and *Examiners' commentaries* for this course which give advice on how each question might best be answered.

# Chapter 2

# Constrained optimisation: tools

## Aim of the chapter

The aim of this chapter is to introduce you to the topic of constrained optimisation in a static context. Special emphasis is given to both the theoretical underpinnings and the application of the tools used in economic literature.

## Learning outcomes

By the end of this chapter, you should be able to:

- formulate a constrain optimisation problem
- discern whether you could use the Lagrange method to solve the problem
- use the Lagrange method to solve the problem, when this is possible
- discern whether you could use the Kuhn-Tucker Theorem to solve the problem
- use the Kuhn-Tucker Theorem to solve the problem, when this is possible
- discuss the economic interpretation of the Lagrange multipliers
- carry out simple comparative statics using the Envelope Theorem.

## Essential reading

This chapter is self-contained and therefore there is no essential reading assigned.

## Further reading

Sydsaeter, Knut, Peter Hammond, Atle Seierstad, Arne Strom *Further Mathematics for Economic Analysis.* Chapters 1 and 3.

Dixit Avinash K. *Optimization in Economic Theory.* Chapters 1–8 and the appendix.

## 2.1 Introduction

The role of optimisation in economic theory is important because we assume that individuals are rational. Why do we look at constrained optimisation? The problem of scarcity.

In this chapter we study the methods to solve and evaluate the constrained optimisation problem. We develop and discuss the intuition behind the Lagrange method and the Kuhn-Tucker theorem. We define and analyse the maximum value function, a construct used to evaluate the solutions to optimisation problems.

## 2.2 The constrained optimisation problem

Consider the following example of an economics application.

**Example 1** *ABC is a perfectly competitive, profit maximizing firm, producing $y$ from input $x$ according to the production function $y = x^{.5}$. The price of output is 2, and of the price of input is 1. Negative levels of $x$ are impossible. Also, the firm cannot buy more than $k$ units of input.*

The firm is interested in two problems. First, the firm would like to know what to do in the short run; given that its capacity, $k$, (e.g., the size of its manufacturing facility) is fixed, it has to decide how much to produce today, or equivalently, how much units of inputs to employ today to maximise its profits. To answer this question, the firm needs to have a method to solve for the optimal level of inputs under the above constraints.

When thinking about the long run operation of the firm, the firm will consider a second problem. Suppose the firm could, at some cost, invest in increasing its capacity, $k$, is it worthwhile for the firm? By how much should it increase/decrease $k$? To answer this the firm will need to be able to evaluate the benefits (in terms of added profits) that would result from an increase in $k$.

Let us now write the problem of the firm formally:

$$\text{(The firm's problem)}\ \max\ \mathbf{g(x)} = 2x^{.5} - x$$
$$\textbf{s.t.}\ \mathbf{h(x) = x} \leq k$$
$$\mathbf{x} \geq 0$$

More generally, we will be interested in similar problems as that outlined above. Another important example of such a problem is that of a consumer maximising his utility trying to choose what to consume and constrained by his budget. We can formulate the general problem denoted by *COP* as:

$$\text{(COP)} \max \ \mathbf{g}(\mathbf{x})$$
$$\textbf{s.t.} \ \mathbf{h}(\mathbf{x}) \leq k$$
$$\mathbf{x} \in x$$

Note that in the general formulation we can accomodate multidimensional variables; In particular $g : x \rightarrow R$, $x$ is a subset of $R^n$, $h : x \rightarrow R^m$ and $k$ is a fixed vector in $R^m$.

**Definition 1** *$z^*$ solves $COP$ if $z^* \in Z, h(z^*) \leq k$, and for any other $z \in Z$ satisfying $h(z) \leq k$, we have that $g(z^*) \geq g(z)$.*

**Definition 2** *The feasible set is the set of vectors in $R^n$ satisfying $z \in Z$ and $h(z) \leq k$.*

---

**Learning activity 2.1**

Show that the following constrained maximisation problems have no solution. For each example write what you think is the problem for the existence of a solution.

(a) Maximize $\ln x$ subject to $x > 1$.

(b) Maximize $\ln x$ subject to $x < 1$.

(c) Maximize $\ln x$ subject to $x < 1$ and $x > 2$.

**Solutions to learning activities are found at the end of the chapters.**

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To steer away from the above complications we can use the following theorem:

**Theorem 1** *If the feasible set is non-empty, closed and bounded (compact), and the objective function $g$ is continuous on the feasible set then the $COP$ has a solution.*

Note that for the feasible set to be compact it is enough to assume that $x$ is closed and the constraint function $h(x)$ is continuous. Why?

The conditions are sufficient and not necessary. For example $x^2$ has a minimum on $R^2$ even if it is not bounded.

---

**Learning activity 2.2**

For each of the following sets of constraints either say, without proof, what is the maximum of $x^2$ subject to the constraints, or explain why there is no maximum: (a) $x \leq 0$, $x \geq 1$
(b) $x \leq 2$
(c) $0 \leq x \leq 1$
(d) $x \geq 2$
(e) $-1 \leq x \leq 1$
(f) $1 \leq x < 2$
(g) $1 \leq x \leq 1$
(h) $1 < x \leq 2$.

In what follows we will devote attention to two questions (similar to the short run and long run questions that the firm was interested in the example). First, we will look for methods to solve the COP. Second, having found the optimal solution we would like to understand the relation between the constraint $k$ and the optimal value of the COP given $k$. To do this we will study the concept of the maximum value function.

## 2.3 Maximum value functions

To understand the answer to both questions above it is useful to follow the following route. Consider the example of firm ABC. A first approach to understanding the maximum profit that the firm might gain is to consider the situations that the firm is indeed constrained by $k$. Plot the profit of the firm as a function of $x$ and we see that profits are increasing up to $x = 1$ and then decrease, crossing zero profits when $x = 4$. This implies that the firm is indeed constrained by $k$ when $k \leq 1$; in this case it is optimal for the firm to choose $x^* = k$. But when $k > 1$ the firm is not constrained by $k$; choosing $x^* = 1 < k$ is optimal for the firm.

Since we are interested in the relation between the level of the constraint $k$ and the maximum profit that the firm could guarantee, $v$, it is useful to look at the plane spanned by taking $k$ on the $x - axis$ and $v$ on the $y - axis$. For firm ABC it is easy to see that maximal profits follow exactly the increasing part of the graph of profits we had before (as $x^* = k$). But when $k > 1$, as we saw above, the firm will always choose $x^* = 1 < k$ and so the maximum attainable profit will stay flat at a level of 1.

More generally, how can we think about the maximum attainable value for the COP? Formally, and without knowing if a solution exists or not, we can write this function as

$$s(k) = \sup\{g(x); x \in x, h(x) \leq k\}.$$

We would like to get some insight as to what this function looks like. One way to proceed is to look, in the $(k, v)$ space, at all the possible points, $(k, v)$,that are attainable by the function $g(x)$ and the constraints. Formally, consider the set,

$$B = \{(k, v) : k \geq h(x), v \leq g(x) \text{ for some } x \in x\}.$$

The set $B$ defines the 'possibility set' of all the values that are feasible, given a constraint $k$. To understand what is the maximum attainable value given a particular $k$ is to look at the upper boundary of this set. Formally, one can show that the values $v$ on the upper boundary of $B$ correspond exactly with the function $s(k)$, which brings us closer to the notion of the maximum value function.

It is intuitive that the function $s(k)$ will be monotone in $k$; after all, when $k$ is increased, this cannot lower the maximal attainable value as we have just relaxed the constraints. In the example of firm ABC, abstracting away from the cost of the facility, a larger facility may

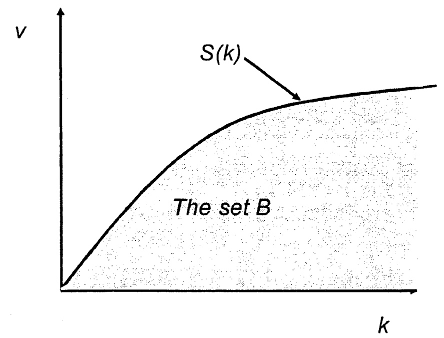


Figure 2.1: The set $B$ and $S(k)$.

never imply that the firm will make less profits! The following Lemma formalises this.

**Lemma 1** *If $k_1 \in K$ and $k_1 \leq k_2$ then $k_2 \in K$ and $s(k_1) \leq s(k_2)$.*

**Proof.** If $k_1 \in K$ there exists a $z \in Z$ such that $h(z) \leq k_1 \leq k_2$ so $k_2 \leq K$. Now consider any $v < s(k_1)$. From the definition there exists a $z \in Z$ such that $v < g(z)$ and $h(z) \leq k_1 \leq k_2$, which implies that $s(k_2) = \sup\{g(z); z \in Z, h(z) \leq k_2\} > v$. Since this is true for all $v < s(k_1)$ it implies that $s(k_2) \geq s(k_1)$. ■

Therefore, the boundary of $B$ defines a non-decreasing function. If the set $B$ is closed, that is, it includes its boundary, this is the maximum value function we are looking for. For each value of $k$ it shows the maximum attainable value of the objective function.

We can confine ourselves to maximum rather than supremum. This is possible if the $COP$ has a solution. To ensure a solution, we can either find it, or show that the objective function is continuous and the feasible set is compact.

**Definition 3 (The maximum value function)** *If $z(k)$ solves the COP with the constraint parameter $k$, the maximum value function is $v(k) = g(z(k))$.*

The maximum value function, if it exists, has all the properties of $s(k)$. In particular, it is non-decreasing. So we have reached the conclusion that $x^*$ is a solution to the $COP$ if and only if $(k, g(x^*))$ lies on the upper boundary of $B$.

**Learning activity 2.3**

$XYZ$ is a profit maximizing firm selling a good in a perfectly competitive market at price 4. It can produce any non-negative quantity of such good $y$ at cost $c(y) = y^2$. However there is a transport bottle-neck which makes it impossible for the firm to sell more than $k$ units of $y$, where $k \geq 0$. Write down $XYZ$'s profit maximisation problem. Show on a graph the set $B$ for this problem. Using the graph write down the solution to the problem for all non-negative values of $k$.

## 2.4 The Lagrange sufficiency theorem

In the last Section we defined what we mean by the maximum value function given that we have a solution. We also introduced the set, $B$, of all feasible outcomes in the $(k, v)$ space. We concluded that $x^*$ is a solution to the $COP$ if and only if $(k, g(x^*))$ lies on the upper boundary of $B$. In this section we proceed to use this result to find a method to solve the $COP$.

Which values of $x$ would give rise to boundary points of $B$? Suppose we can draw a line with a slope $\lambda$ through a point $(k, v) = (h(x^*)), g(x^*))$ which lies entirely on or above the set $B$. The equation for this line is:

$$v - \lambda k = g(x^*) - \lambda h(x^*))$$

**Example 2 (revisiting example 1)** *We can draw such a line through* $(.25, .75)$ *with slope* $q = 1$ *and a line though* $(1, 1)$ *with slope* $0$.

If the slope $\lambda$ is **non-negative** and recalling the definition of $B$ as the set of all possible outcomes, the fact that the line lies entirely on or above the set $B$ can be restated as

$$g(x^*) - \lambda h(x^*) \geq g(x) - \lambda h(x)$$

for all $x \in x$. But note that this implies that $(h(x^*)), g(x^*))$ lies on the upper boundary of the set $B$ implying that if $x^*$ is feasible it is a solution to the $COP$.

**Example 3 (revisiting example 1)** *It is crucial for this argument that the slope of the line is non-negative. For example for the point* $(4, 0)$ *there doesn't exist a line passing through it that is on or above* $B$. *This corresponds to* $x = 4$ *and indeed it is not a solution to our example. Sometimes the line has a slope of* $q = 0$. *Suppose for example that* $k = 4$, *if we take the line with* $q = 0$ *through* $(4, 1)$, *it indeed lies above the set* $B$. *The point* $x^* = 1$ *satisfies:*

$$g(x^*) - 0h(x^*) \geq g(x) - 0h(x)$$

*for all* $x \in X$, *and as* $h(x^*) < k^*$, *then* $x^*$ *solves the optimisation problem for* $k = k^* \geq 1$.

Summarising the argument so far, suppose $k^*, \lambda$ and $x^*$ satisfy the following conditions:

$$\begin{aligned} g(x^*) - \lambda h(x^*) \geq g(x) - \lambda h(x) &\text{ for all } x \in x \\ \lambda \geq 0 & \\ x^* \in x & \\ \text{either } k^* = h(x^*) & \\ \text{or } k^* > h(x^*) \text{ and } \lambda = 0 & \end{aligned}$$

then $x^*$ solves the COP for $k = k^*$.

This is the Lagrange sufficiency theorem. It is convenient to write it slightly differently: adding $\lambda k^*$ to both sides of the first condition we have

$$
\begin{aligned}
g(x^*) + \lambda(k^* - h(x^*)) &\geq g(x) + \lambda(k^* - h(x)) \text{ for all } x \in x \\
\lambda &\geq 0 \\
x^* &\in x \text{ and } k^* \geq h(x^*) \\
\lambda[k^* - h(x^*)] &= 0
\end{aligned}
$$

We refer to $\lambda$ as the **Lagrange multiplier**. We refer to the expression $g(x) + \lambda(k^* - h(x)) \equiv L(x, k^*, \lambda)$ as the **Lagrangian**. The conditions above imply that $x^*$ maximises the Lagrangian given a non-negativity restriction, feasibility, and a **complementary slackness** (CS) condition respectively. Formally:

**Theorem 2** *If for some $q \geq 0, z^*$ maximises $L(z, k^*, q)$ subject to the three conditions, it also solves $COP$.*

**Proof.** From the complementary slackness condition, $q[k^* - h(z^*)] = 0$. Thus, $g(z^*) = g(z^*) + q(k^* - h(z^*))$. By $q \geq 0$ and $k^* - h(z) \geq 0$ for all feasible $z$, then $g(z) + q(k^* - h(z)) \geq g(z)$. By maximisation of $L$ we get $g(z^*) \geq g(z)$, for all feasible $z$ and since $z^*$ itself is feasible, then it solves $COP$. ■

**Example 4 (Example 1 revisited)** *We now solve the example of the firm ABC. The Lagrangian is given by:*

$$L(x, k, \lambda) = 2x^{.5} - x + \lambda[k - z]$$

*Let us use first order conditions, although we have to prove that we can use them and we will do so later in the Chapter. Given $\lambda$, the first order condition of $L(x, k^*, \lambda)$ with respect to $x$ is:*

$$x^{-.5} - 1 - \lambda = 0 \text{ (FOC)}$$

*We need to consider the two cases (using the CS condition), $\lambda > 0$ and the case of $\lambda = 0$. Case 1: If $\lambda = 0$. CS implies that the constraint is not binding and (FOC) implies that $x^* = 1$ is a candidate solution. By Theorem 1, when $x^*$ is feasible, i.e., $k \geq x^* = 1$ this will indeed be a solution to the COP. Case 2: If $\lambda > 0$. In this case the constraint is binding; by CS we have $x^* = k$ as a candidate solution. To check that it is a solution we need to check that it is feasible, i.e., that $k > 0$, that it satisfies the FOC and that it is consistent with a non-negative $\lambda$. From the FOC we have:*

$$k^{-.5} - 1 = \lambda$$

*and for this to be non-negative implies that:*

$$k^{-.5} - 1 \geq 0 \Longleftrightarrow k \leq 1$$

*Therefore when $k \leq 1$, $x^* = k$ solves the $COP$.*

**Remark 1** *Some further notes about what is to come:*

- *The conditions that we have stated are sufficient conditions. This means that some solutions of $COP$ cannot be characterised by the Lagrangian. For example, if the set $B$ is not convex, then solving the Lagrangian is not necessary.*
- *As we will show later, if the objective function is concave and the constraint is convex, then $B$ is convex. Then, the Lagrange conditions are also necessary. That is, if we find all the points that maximise the Lagrangian, these are all the points that solve the $COP$.*

- *With differentiability, we can also solve for these points using first order conditions.*
- *Remember that we are also interested in the maximum value function. What does it mean to relax the constraint? The Lagrange multipliers are going to play a central role in understanding how relaxing the constraints affect the maximum value.*

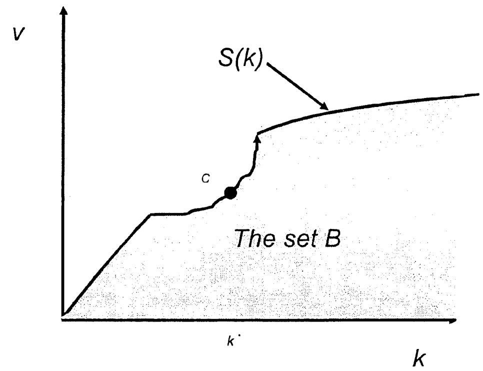


Figure 2.2: Note that the point $C$ represents a solution to the COP when $k = k^*$, but this point will not be characterised by the Lagrange method as the set $B$ is not concave.

## 2.5 Concavity and convexity and the Lagrange Necessity Theorem

In the last section we have found necessary conditions for a solution to the $COP$. This means that some solutions of $COP$ cannot be characterised by the Lagrangian method. In this section we investigate the assumption that would guarantee that the conditions of the Lagrangian are also necessary.

To this end, we will need to introduce the notions of convexity and concavity. From the example above it is already clear why convexity should play a role: if the set $B$ is not convex, there will be points on the boundary of $B$ (that as we know are solutions to the COP) that will not accommodate a line passing through them and entirely above the set $B$ as we did in the last section. In turn, this means that using the Lagrange method will not lead us to these points.

We start with some formal definitions:

**Definition 4** *A set $U$ is a convex set if for all $x \in U$ and $y \in U$, then for all $t \in [0, 1]$:*

$$t\mathbf{x} + (1 - t)\mathbf{y} \in U$$

**Definition 5** *A real valued function $f$ defined on a convex subset $U$of $R^n$ is concave, if for all $x, y$ in $U$and for all $t \in [0, 1]$:*

$$f(t\mathbf{x} + (1 - t)\mathbf{y}) \geq tf(\mathbf{x}) + (1 - t)f(\mathbf{y})$$

*A real valued function $g$ defined on a convex subset $U$ of $R^n$ is convex, if for all $x, y$ in $U$ and for all $t \in [0,1]$:*

$$g(t\mathbf{x} + (1-t)\mathbf{y}) \leq tg(\mathbf{x}) + (1-t)g(\mathbf{y})$$

**Remark 2** *Some simple implications of the above definition that will be useful later:*

- *$f$ is concave if and only if $-f$ is convex.*
- *Linear functions are convex and concave.*
- *Concave and convex functions need to have convex sets as their domain. Otherwise, we cannot use the conditions above.*

---

**Learning activity 2.4**

$A$ and $B$ are two convex subsets of $\mathbb{R}^n$, which of the following sets are always convex, sometimes convex, or never convex? Provide proofs for the sets which are always convex, draw examples to show why the others are sometimes or never convex. (a) $A \cup B$.
(b) $A + B \equiv \{x | x \in \mathbb{R}^n, x = a + b, a \in A, b \in B\}$.

---

In what follows we will need to have a method of determining whether a function is convex, concave or neither. To this end the following characterisation of concave functions is useful:

**Lemma 2** *Let $f$ be a continuous and differentiable function on a convex subset $U$ of $R^n$. Then $f$ is concave on $U$ if and only if for all $x, y$ in $U$:*

$$\begin{aligned} f(\mathbf{y}) - f(\mathbf{x}) &\leq Df(\mathbf{x})(\mathbf{y} - \mathbf{x}) \\ &= \frac{\partial f(\mathbf{x})}{\partial x_1}(y_1 - x_1) + \ldots \\ &+ \frac{\partial f(\mathbf{x})}{\partial x_n}(y_n - x_n) \end{aligned}$$

**Proof.** Here we prove the result on $R^1$ : since $f$ is concave, then:

$$\begin{aligned} tf(y) + (1-t)f(x) &\leq f(ty + (1-t)x) \Leftrightarrow \\ t(f(y) - f(x)) + f(x) &\leq f(x + t(y-x)) \Leftrightarrow \\ f(y) - f(x) &\leq \frac{f(x + t(y-x)) - f(x)}{t} \Leftrightarrow \\ f(y) - f(x) &\leq \frac{f(x+h) - f(x)}{h}(y - x) \end{aligned}$$

for $h = t(y - x)$. Taking limits when $h \longrightarrow 0$ this becomes:

$$f(y) - f(x) \leq f'(x)(y - x).$$

■

Remember that we introduced the concepts of concavity and convexity as we were interested in finding out under what conditions is the Lagrange method also a necessary condition for solutions of the $COP$.

Consider the following assumptions, denoted by CC:

1. The set $x$ is convex.
2. The function $g$ is concave.
3. The function $h$ is convex.

To see the importance of these assumption, recall the definition of the set $B$:

$$B = \{(k, v) : k \geq h(x), v \leq g(x) \text{ for some } x \in x\}.$$

**Proposition 1** *Under assumptions CC, the set $B$ is convex.*

**Proof.** Suppose that $(k_1, v_1)$ and $(k_2, v_2)$ are in $B$, so there exists $z_1$ and $z_2$ such that:

$$\begin{aligned} k_1 &\geq h(z_1)\ k_2 \geq h(z_2) \\ v_1 &\leq g(z_1)\ v_2 \leq g(z_2) \end{aligned}$$

By convexity of $h$:

$$\theta k_1 + (1-\theta)k_2 \geq \theta h(z_1) + (1-\theta)h(z_2) \geq h(\theta z_1 + (1-\theta)z_2)$$

and by concavity of $g$:

$$\theta v_1 + (1-\theta)v_2 \leq \theta g(z_1) + (1-\theta)g(z_2) \leq g(\theta z_1 + (1-\theta)z_2)$$

thus, $(\theta k_1 + (1-\theta)k_2, \theta v_1 + (1-\theta)v_2) \in B$ for all $\theta \in [0, 1]$, implying that $B$ is convex. ■

Remember that the maximum value is the upper boundary of the set $B$. When $B$ is convex, we can say something about the shape of the maximum value function:

**Proposition 2** *Assume that the maximum value exists, then under CC, the maximum value is a non-decreasing, concave and continuous function of $k$.*

**Proof.** We have already shown, without assuming convexity or concavity, that the maximum value is non-decreasing. We have also shown that if the maximum value function $v(k)$ exists, it is the upper boundary of the set $B$. Above we proved that under CC the set $B$ is convex. The set $B$ can be re-written as:

$$B = \{(k, v) : v \in R, k \in K, v \leq v(k)\}.$$

But a set $B$ is convex iff the function $v$ is concave. Thus, $v$ is concave, and concave functions are continuous, so $v$ is continuous. ■

## 2.6 The Lagrangian necessity theorem

We are now ready to formalise under what conditions the Lagrange method is necessary for a solution to the $COP$.

**Theorem 3** *Assume CC. Assume that the constraint qualification holds, that is, there is a vector $z_0 \in Z$ such that $h(z_0) << k^*$. Finally suppose that $z^*$ solves $COP$. Then:*

*(i) there is a vector $q \in R^m$ such that $z^*$ maximises the Lagrangian $L(q, k^*, z) = g(z) + q[k^* - h(z^*)]$.*

*(ii) the lagrange multiplier $q$ is non negative for all components, $q \geq 0$.*

*(iii) the vector $z^*$ is feasible, that is $z \in Z$ and $h(z^*) \leq k^*$.*

*(iv) the complementary slackness conditions are satisfied, that is, $q[k^* - h(z^*)] = 0$.*

---

## 2.7 First order conditions: when can we use them?

So far we have found a method that allows us to find all the solutions to the $COP$ by solving a modified maximisation problem (i.e., maximising the Lagrangian). As you recall, we have used this method to solve for our example of firm ABC by looking at first order conditions. In this section we ask under what assumptions can we do this and be sure that we have found all the solutions to the problem. For this we need to introduce ourselves to the notions of continuity and differentiability.

### 2.7.1 Necessity of first order conditions

We start with some general definitions.

**Definition 6** *A function $g$: $Z \rightarrow R^m, Z \subset R^n$, is differentiable at a point $z_0$ in the interior of $Z$ if there exists a unique $m{\times}n-$matrix, $Dg(z_0)$, such that given any $\varepsilon > 0$, there exists a $\delta > 0$, such that if $|z - z_0| < \delta$, then $|g(z) - g(z_0) - Dg(z_0)(z - z_0)| < \varepsilon|z - z_0|$.*

There are a few things to note about the above definition. First, when $m = n = 1$, $Dg(x_0)$ is a scalar, that is, the derivative of the function at $x_0$ that we sometimes denote by $g'(x_0)$. Second, one interpretation of the derivative is that it helps approximate the function $g(x)$ for $x's$ that are close to $x_0$ by looking at the line, $g(x_0) + Dg(x_0)(x - x_0)$, that passes through $(x_0, g(x_0))$ with slope $Dg(x_0)$. Indeed this is an implication of Taylor's theorem which states that for any $n > 1$:

$$g(x)-g(x_0) = Dg(x_0)(x-x_0)+\frac{D^2g(x_0)}{2!}(x-x_0)^2+...+\frac{D^ng(x_0)}{n!}(x-x_0)^n+R_n \tag{2.1}$$

where $R_n$ is a term of order of magnitude $(x - x_0)^{n+1}$. The implication is that as we get closer to $x_0$ we can more or less ignore the elements with $(x - x_0)^k$ with the highest powers and use just the first term to approximate the change in the function. We will later return to this when we will ask whether first order conditions are sufficient, but for now we focus on whether they are necessary.

But (2.1) implies that if $x_0$ is a point which maximises $g(x)$ and is interior to the set we are maximising over then if $Dg(x_0)$ exists then it must equal zero. If this is not the case, then there will be a direction along which the function will increase (i.e., the left hand side of (2.1) will be positive); this is easily seen if we consider a function on one variable, but the same intuition generalises. This leads us the following result:

**Theorem 4 (Necessity)** *Suppose that $g : Z \to R$, $Z \subset R^n$, is differentiable in $z_0$ and that $z_0 \in intZ$ maximizes $g$ on $Z$, then $Dg(z_0) = 0$.*

Finally, we introduce the notion of continuity:

**Definition 7** *$f : R^k \to R^m$ is continuous at $x_0 \in R^k$ if for any sequence $\{x_n\}_{n=1}^{\infty}$ in $R^k$ which converges to $x_0$, $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$. The function $f$ is continuous if it is continuous at any point in $R^k$.*

Note that all differentiable functions are continuous but the converse is not true.

### 2.7.2 Sufficiency of first order conditions

Recall that if $f$ is a continuous and differentiable **concave** function on a convex set $U$ then

$$f(\mathbf{y}) - f(\mathbf{x}) \leq Df(\mathbf{x})(\mathbf{y} - \mathbf{x}).$$

Therefore, if we know that for some $\mathbf{x}_0, y \in U$,

$$Df(\mathbf{x}_0)(\mathbf{y} - \mathbf{x}_0) \leq 0$$

we have

$$f(\mathbf{y}) - f(\mathbf{x}_0) \leq Df(\mathbf{x}_0)(\mathbf{y} - \mathbf{x}_0) \leq 0$$

implying that

$$f(\mathbf{y}) \leq f(\mathbf{x}_0).$$

If this holds for all $y \in U$, then $\mathbf{x}_0$ is a global maximiser of $f$. This leads us to the following result:

**Proposition 3 (Sufficiency)** *Let $f$ be a continuous twice differentiable function whose domain is a convex open subset $U$ of $R^n$. If $f$ is a concave function on $U$ and $Df(x_0) = 0$ for some $x_0$, then $x_0$ is a global maximum of $f$ on $U$.*

Another way to see this result is to reconsider the Taylor expansion outlined above. Assume for the moment that a function $g$ is defined on one variable $x$. Remember that for $n = 2$, we have,

$$g(x) - g(x_0) = Dg(x_0)(x - x_0) + \frac{D^2 g(x_0)}{2!}(x - x_0)^2 + R_3 \qquad (2.2)$$

If the first order conditions hold at $x_0$ this implies that $Dg(x_0) = 0$ and the above expression can be rewritten as,

$$g(x) - g(x_0) = \frac{D^2 g(x_0)}{2!}(x - x_0)^2 + R_3 \qquad (2.3)$$

But now we can see that when $D^2 g(x_0) < 0$ and when we are close to $x_0$ the left hand side will be negative and so $x_0$ is a local maximum of $g(x)$, and when $D^2 g(x_0) > 0$ similarly $x_0$ will constitute a local minimum of $g(x)$. As concave functions have $D^2 g(x) < 0$ for any $x$ and convex functions have $D^2 g(x) > 0$ for any $x$ this shows why the above result holds.

The above intuition was provided for the case of a function over one variable $x$. In the next section we extend this intuition to functions on $\mathbb{R}^n$ to discuss how to characterise convexity and concavity in general.

### 2.7.3 Checking for concavity and convexity

In the last few sections we have introduced necessary and sufficient conditions for using first order conditions to solve maximisation problems. We now ask a more practical question. Confronted with a particular function, how can we verify whether it is concave or not? If it is concave, we know from the above results that we can use the first order conditions to characterise all the solutions. If it is not concave, we will have to use other means if we want to characterise all the solutions.

It will be again instructive to look at the Taylor expansion for $n = 2$. Let us now consider a general function defined on $\mathbb{R}^n$ and look at the Taylor expansion around a vector $x^0 \in \mathbb{R}^n$. As $g$ is a function defined over $\mathbb{R}^n$, $Dg(x_0)$ is now an $n-$dimensional vector and $D^2g(x_0)$ is an $n \times n$ matrix. Let $x$ be an $n-$dimensional vector in $\mathbb{R}^n$. The Taylor expansion in this case becomes

$$g(x) - g(x^0) = Dg(x_0)\left(x - x^0\right) + (x - x^0)^T \frac{D^2g(x_0)}{2!}\left(x - x^0\right) + R_3$$

If the first order condition is satisfied, then $Dg(x_0) = \mathbf{0}$, where $\mathbf{0}$ is the $n-$dimensional vector of zeros. This implies that we can write the above as,

$$g(x) - g(x^0) = (x - x^0)^T \frac{D^2g(x_0)}{2!}\left(x - x^0\right) + R_3$$

But now our problem is a bit more complicated. We need to determine the sign of $(x - x^0)^T \frac{D^2g(x_0)}{2!}\left(x - x^0\right)$ for a whole neighbourhood of $x's$ around $x^0$! We need to find what properties of the matrix $D^2g(x_0)$ would guarantee this. For this we analyse the properties of expressions of the form $(x - x^0)^T \frac{D^2g(x_0)}{2!}\left(x - x^0\right)$, i.e., quadratic forms.

Consider functions of the form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ where $x$ is an $n-$dimensional vector and $A$ a symmetric $n \times n$ matrix. If $n = 2$, this becomes

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} a_{11} & \frac{a_{12}}{2} \\ \frac{a_{12}}{2} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and can be rewritten as

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$$

**Definition 8** *(i) A quadratic form on $R^n$ is a real-valued function*

$$Q(x_1, x_2, ..., x_n) = \sum_{i \leq j} a_{ij}x_ix_j$$

*or equivalently,*

*(ii) A quadratic form on $R^n$ is a real-valued function $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ where $A$ is a symmetric $n \times n$ matrix.*

Below we would like to understand what properties of $A$ relate to the quadratic form it generates, taking on only positive values or only negative values.

### 2.7.4 Definiteness of quadratic forms

We now examine quadratic forms, $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. It is apparent that whether $\mathbf{x} = \mathbf{0}$ this expression is equal zero. In this section we ask under what conditions $Q(\mathbf{x})$ takes on a particular sign for any $\mathbf{x} \neq \mathbf{0}$ (Strictly negative or positive, non-negative or non-positive).

For example, in one dimension, when

$$y = ax^2$$

then if $a > 0$, $ax^2$ is non negative and equals 0 only when $x = 0$. This is positive definite. If $a < 0$, then the function is negative definite. In two dimensions,

$$x_1^2 + x_2^2$$

is **positive definite**, whereas

$$-x_1^2 - x_2^2$$

is **negative definite**, whereas

$$x_1^2 - x_2^2$$

is **indefinite**, since it can take both positive and negative values, depending on $\mathbf{x}$.

There could be two intermediate cases: if the quadratic form is always non negative but also equals 0 for non zero $\mathbf{x}'s$, then we say it is **positive semidefinite**. This is the case, for example, for

$$(x_1 + x_2)^2$$

which can be 0 for points such that $x_1 = -x_2$. A quadratic form which is never positive but can be zero at points other than the origin is called **negative semidefinite**.

We apply the same terminology for the symmetric matrix $A$, that is, the matrix $A$ is positive semi definite if $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is positive semi definite, and so on.

**Definition 9** *Let $A$ be an $n \times n$ symmetric matrix. Then $A$ is:*

- *positive definite if $x^T Ax > 0$ for all $x \neq 0$ in $R^n$,*
- *positive semi definite if $x^T Ax \geq 0$ for all $x \neq 0$ in $R^n$.*
- *negative definite if $x^T Ax < 0$ for all $x \neq 0$ in $R^n$,*
- *negative semi definite if $x^T Ax \leq 0$ for all $x \neq 0$ in $R^n$,*
- *indefinite $x^T Ax > 0$ for some $x \neq 0$ in $R^n$ and $x^T Ax < 0$ for some $x \neq 0$ in $R^n$.*

### 2.7.5 Testing the definiteness of a matrix

In this section, we try to examine what properties of the matrix, $A$, of the quadratic form $Q(\mathbf{x})$ will determine its definiteness.

We start by introducing the notion of a **determinant** of a matrix. The determinant of a matrix is a unique scalar associated with the matrix.

Computing the determinant of a matrix proceeds recursively:

For a 2x2 matrix, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ the determinant, det(A) or |A|, is

$$a_{11}a_{22} - a_{12}a_{21}$$

For a 3x3 matrix, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ the determinant is:

$$a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$
$$+ a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

Generally, for a $n \times n$ matrix, $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$, the determinant will be given by

$$\det(A) = \sum_{i=1}^{n}(-1)^{i-1}a_{1i}\det(A_{1i})$$

where $A_{1i}$ is the matrix that is left when we take out the first row and $i'th$ column of the matrix $A$.

**Definition 10** *Let $A$ be an $n \times n$ matrix. A $k \times k$ submatrix of $A$ formed by deleting $n-k$ columns, say columns $i_1, i_2, ..., i_{n-k}$ and* **the same** *$n-k$ rows from $A$, $i_1, i_2, ..., i_{n-k}$ , is called a $k-$th order principal submatrix of $A$. The determinant of a $k \times k$ principal submatrix is called a $k-$th order principal minor of $A$.*

**Example 5** *For a general $3 \times 3$ matrix $A$, there is one third order principal minor, which is det(A). There are three second order principal minors and three first order principal minors. What are they?*

**Definition 11** *Let $A$ be an $n \times n$ matrix. The $k-$th order principal submatrix of $A$ obtained by deleting* **the last** *$n-k$ rows and columns from $A$ is called the $k-$th order leading principal submatrix of $A$, denoted by $A_k$. Its determinant is called the $k-$th order leading principal minor of $A$, denoted by $|A_k|$.*

We are now ready to relate the above elements of the matrix $A$ to the definiteness of the matrix:

**Proposition 4** *Let $A$ be an $n \times n$ symmetric matrix. Then*

*(a) $A$ is positive definite if and only if all its $n$ leading principal minors are strictly positive.*

*(b) $A$ is negative definite if and only if all its $n$ leading principal minors alternate in sign as follows:*

$$|A_1| < 0, |A_2| > 0, |A_3| < 0 \text{ etc.}$$

*The $k-$th order leading principal minor should have the same sign as $(-1)^k$.*

*(c) $A$ is positive semidefinite if and only if every principal minor of $A$ is non negative.*

*(d) $A$ is negative semidefinite if and only if every principal minor of odd order is non positive and very principal minor of even order is non negative.*

**Example 6** *Consider diagonal matrices* $\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$. *These correspond to the simplest quadratic forms:*

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2.$$

*This quadratic form will be positive (negative) definite if and only if all the $a_i's$ are positive (negative). It will be positive semidefinite if and only if all the $a_i;s$ are non negative and negative semidefinite if and only if all the $a_i's$ are non positive. If there are two $a_i's$ of opposite signs, it will be indefinite. How do these conditions relate to what you get from the proposition above?*

**Example 7** *To see how the conditions of the above Proposition relate to the definiteness of a matrix consider a $2 \times 2$ matrix, and in particular its quadratic form*

$$Q(x_1, x_2) = (x_1, x_2)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= ax_1^2 + 2bx_1x_2 + cx_2^2$$

*If $a = 0$, then $Q$ cannot be negative or positive definite since $Q(1, 0) = 0$. So assume that $a \neq 0$ and add and subtract $b^2x_2^2/a$ to get:*

$$Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 + \frac{b^2}{a}x_2^2 - \frac{b^2}{a}x_2^2 =$$
$$a(x_1^2 + \frac{2bx_1x_2}{a} + \frac{b^2}{a^2}x_2^2) - \frac{b^2}{a}x_2^2 + cx_2^2$$
$$= a(x_1 + \frac{b}{a}x_2)^2 + \frac{(ac - b^2)}{a}x_2^2$$

*If both coefficients above, $a$ and $(ac - b^2)/a$ are positive, then $Q$ will never be negative. It will equal 0 only when $x_1 + \frac{b}{a}x_2$ and $x_2 = 0$ in other words, when $x_1 = 0$ and $x_2 = 0$. Therefore, if*

$$|a| > 0 \text{ and } \det A = \begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0$$

*then $Q$ is positive definite. Conversely, in order for $Q$ to be positive definite, we need both $a$ and $\det A = ac - b^2$ to be positive. Similarly, $Q$ will be negative definite if and only if both coefficient are negative, which occurs if and only if $a < 0$ and $ac - b^2 > 0$, that is, when the leading principal minors alternative in sign. If $ac - b^2 < 0$. then the two coefficients will have opposite signs and $Q$ will be indefinite.*

**Example 8 Numerical examples**. *Consider* $A = \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}$. *Since $|A_1| = 2$ and $|A_2| = 5$, $A$ is positive definite. Consider* $B = \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}$. *Since $|B_1| = 2$ and $|B_2| = -2$, $B$ is indefinite.*

## 2.7.6 Back to concavity and convexity

Finally we can put all the ingredients together. A continuous twice differentiable function $f$ on an open convex subset $U$ of $R^n$ is concave on $U$ if and only if the Hessian $D^2 f(\mathbf{x})$ is negative semidefinite for all $\mathbf{x}$ in $U$. The function $f$ is a convex function if and only if $D^2 f(\mathbf{x})$ is positive semidefinite for all $\mathbf{x}$ in $U$.

Therefore, we have the following result:

**Proposition 5 Second order sufficient conditions for global maximum (minimum) in** $R^n$. *Suppose that* $x^*$ *is a critical point of a function* $f(x)$ *with continuous first and second order partial derivatives on* $R^n$. *Then* $x^*$ *is:*

- *a global maximiser for* $f(x)$ *if* $D^2 f(x)$ *is negative (positive) semidefinite on* $R^n$.
- *a strict global maximiser for* $f(x)$ *if* $D^2 f(x)$ *is negative (positive) definite on* $R^n$.

The property that critical points of concave functions are global maximisers is an important one in economic theory. For example, many economic principals, such as marginal rate of substitution equals the price ratio, or marginal revenue equals marginal cost are simply the first order necessary conditions of the corresponding maximisation problem as we will see. Ideally, as economist would like such a rule also to be a sufficient condition guaranteeing that utility or profit is being maximised, so it can provide a guideline for economic behaviour. This situation does indeed occur when the objective function is concave.

## 2.8 The Kuhn-Tucker Theorem

We are now in a position to formalise the necessary and sufficient first order conditions for solutions to the COP. Consider once again the COP:

$$\max_{\substack{\text{s.t. } h(x) \leq k^* \\ x \in x}} g(x)$$

where $g : x \to R$, $x$ is a subset of $R^n$, $h : x \to R^m$, and $k^*$ is a fixed $m$–dimensional vector.

We impose a set of the following assumptions, $(CC')$ :

1 The set $x$ is convex,

2 The function $g$ is concave,

3 The function $h$ is convex,

(these are assumptions $(CC)$ from before), and

4 The functions $g$ and $h$ are differentiable

Consider now the following conditions, which we term the Kuhn-Tucker conditions:

1. There is a vector $\lambda \in R^m$ such that the partial derivative of the Lagrangian
$$L(k^*, \lambda, x) = g(x) + \lambda[k^* - h(x)]$$
evaluated at $x^*$ is zero, in other words:
$$\frac{\partial L(k^*, \lambda, x)}{\partial x} = Dg(x^*) - \lambda Dh(x^*) = 0$$

2. The Lagrange multiplier vector is non negative:
$$\lambda \geq 0$$
3. The vector $x^*$ is feasible, that is, $x \in x$ and $h(x^*) \leq k^*$.
4. The complementary slackness conditions are satisfies, that is,
$$\lambda[k^* - h(x^*)] = 0$$

The following Theorem is known as the Kuhn-Tucker Theorem.

**Theorem 5** *Assume (CC').*

*(i) If $z^*$ is in the interior of $Z$ and satisfies the K-T conditions, then $z^*$ solves the $COP$.*

*(ii) If the constraint qualification holds (there exists a vector $z_0 \in Z$ such that $h(z_0) << k^*$), $z^*$is in the interior of $Z$ and solves $COP$, then there is a vector of Lagrange multipliers $q$ such that $z^*$ and $q$ satisfy the K-T conditions.*

**Proof.** We first demonstrate that under CC and for non-negative values of Lagrange multipliers, the Lagrangian is concave. The Lagrangian is:
$$L(k^*, q, z) = g(z) + q[k^* - h(z)]$$
Take $z$ and $z'$. Then:
$$\begin{aligned} tg(z) + (1-t)g(z') &\leq g(tz + (1-t)z') \\ th(z) + (1-t)h(z') &\geq h(tz + (1-t)z') \end{aligned}$$
and thus with $q \geq 0$, we have:
$$\begin{aligned} & g(tz + (1-t)z') + qk^* - qh(tz + (1-t)z') \\ \geq \; & tg(z) + (1-t)g(z') + qk^* - q(th(z) + (1-t)h(z')), \end{aligned}$$
It follows that:
$$\begin{aligned} & L(k^*, q, tz + (1-t)z) \\ = \; & g(tz + (1-t)z') + q[k^* - h(tz + (1-t)z')] \\ \geq \; & t[g(z) + q(k^* - h(z))] + (1-t)[g(z') + q(k^* - h(z'))] \\ = \; & tL(k^*, q, z) + (1-t)L(k^*, q, z'). \end{aligned}$$
This proves that the Lagrangian is concave in $z$. In addition, we know that $g$ and $h$ are differentiable, therefore also $L$ is a differentiable function of $z$. Thus, we know that if the partial derivative of $L$ with respect to $z$ is zero at $z^*$, then $z^*$ maximises $L$ on $Z$. Indeed the partial of $L$ at $z^*$ is zero, and hence, we know that if $g$ is concave and differentiable, $h$ is convex and differentiable, the Lagrange multipliers $q$ are non-negative, and
$$Dg(z^*) - qDh(z^*) = 0$$
then $z^*$ maximizes the Lagrangian on $Z$. But then the conditions of the Lagrange sufficiency theorem are satisfied, so that $z^*$ indeed solves $COP$. We have to prove the converse result now. Suppose that the constraint qualification is satisfied. The $COP$ now satisfies all the conditions of the Lagrange necessity theorem. This theorem says that if $z^*$ solves the $COP$, then it also maximises $L$ on $Z$, and satisfies the complementary slackness conditions, with non negative Lagrange multipliers, as well as being feasible. But since partial derivatives of a differentiable function are zero at the maximum, then the partial derivatives of $L$ with respect to $z$ at $z^*$ are zero and therefore all the Kuhn-Tucker conditions are satisfied. ■

**Remark 3 (Geometrical intuition)** *Think of the following example in* $R^2$. *Suppose that the constraints are: 1)* $h_1(z) = -z_1 \leq 0$. *2)* $h_2(z) = -z_2 \leq 0$. *3)* $h_3(z) \leq k^*$. *Consider first the case in which the point* $z_0$ *solves the problem at a tangency of* $h_3(z) = k^*$ *and the objective function* $g(z) = g(z_0)$.*The constraint set is convex, and by the concavity of* $g$ *it is also the case that*

$$\{z : z \in R^2, g(z) \geq g(z_0)\}$$

*is a convex set. The Lagrangian for the problem is:*

$$\begin{aligned} L(k^*, q, z) &= g(z) + q[k^* - h(z)] \\ &= g(z) + q_1 z_1 + q_2 z_2 + q_3(k^* - h_3(z)) \end{aligned}$$

*In the first case of* $z_0$, *the non negativity constraints do not bind. By the complementary slackness then, it is the case that* $q_1 = q_2 = 0$ *so that the first order condition is simply:*

$$Dg(z_0) = q_3 Dh_3(z_0)$$

*Recall that* $q_3 \geq 0$. *If* $q_3 = 0$, *then it implies that* $Dg(z_0) = 0$ *so* $z_0$ *is the unconstrained maximiser but this is not the case here. Then* $q_3 > 0$, *which implies that the vectors* $Dg(z_0)$ *and* $Dh_3(z_0)$ *point in the same direction. These are the gradients: they describe the direction in which the function increases most rapidly. In fact, they must point in the same direction, otherwise, this is not a solution to the optimisation problem.*

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## 2.9 The Lagrange multipliers and the Envelope Theorem

### 2.9.1 Maximum value functions

In this Section we return to our initial interest in Maximum (minimum) Value functions. Profit functions and indirect utility functions are notable examples of maximum value functions, whereas cost functions and expenditure functions are minimum value functions. Formally, a maximum value function is defined by:

**Definition 12** *If* $x(b)$ *solves the problem of maximising* $f(x)$ *subject to* $g(x) \leq b$, *the maximum value function is* $v(b) = f(x(b))$.

You will remember that such a maximum value function is non-decreasing.

Let us now examine these functions more carefully. Consider the problem of maximising $f(x_1, x_2, ..., x_n)$ subject to the $k$ inequality constraints

$$g(x_1, x_2, ..., x_n) \leq b_1^*, ..., g(x_1, x_2, ..., x_n) \leq b_k^*$$

where $\mathbf{b}^* = (b_1^*, ..., b_k^*)$. Let $x_1^*(\mathbf{b}^*), ..., x_n^*(\mathbf{b}^*)$ denote the optimal solution and let $\lambda_1(\mathbf{b}^*), ..., \lambda_k(\mathbf{b}^*)$ be the corresponding Lagrange multipliers. Suppose that as $\mathbf{b}$ varies near $b^*$, then $x_1^*(\mathbf{b}^*), ..., x_n^*(\mathbf{b}^*)$ and $\lambda_1(\mathbf{b}^*), ..., \lambda_k(\mathbf{b}^*)$ are differentiable functions and that $x^*(\mathbf{b}^*)$ satisfies the constraint qualification. Then for each $j = 1, 2, ..., k$ :

$$\lambda_j(\mathbf{b}^*) = \frac{\partial}{\partial b_j} f(x^*(\mathbf{b}^*))$$

**Proof.** (We consider the case of a binding constraint, and for simplicity, assume there is only one constraint, and that $f$ and $g$ are functions of two variables) The Lagrangian is

$$L(x, y, \lambda; b) = f(x, y) - \lambda(g(x, y) - b)$$

The solution satisfies:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x}(x^*(b), y^*(b), \lambda^*(b); b) \\
&= \frac{\partial f}{\partial x}(x^*(b), y^*(b), \lambda^*(b)) \\
&\quad -\lambda^*(b)\frac{\partial h}{\partial x}(x^*(b), y^*(b), \lambda^*(b)), \\
0 &= \frac{\partial L}{\partial y}(x^*(b), y^*(b), \lambda^*(b); b) \\
&= \frac{\partial f}{\partial y}(x^*(b), y^*(b), \lambda^*(b)) \\
&\quad -\lambda^*(b)\frac{\partial h}{\partial y}(x^*(b), y^*(b), \lambda^*(b)),
\end{aligned}$$

for all $b$. Furthermore, since $h(x^*(b), y^*(b)) = b$ for all $b$,

$$\frac{\partial h}{\partial x}(x^*, y^*)\frac{\partial x^*(b)}{\partial b} + \frac{\partial h}{\partial y}(x^*, y^*)\frac{\partial y^*(b)}{\partial b} = 1$$

for every $b$. Therefore, using the chain rule, we have:

$$\begin{aligned}
&\frac{df(x^*(b), y^*(b))}{db} \\
&= \frac{\partial f}{\partial x}(x^*, y^*)\frac{\partial x^*(b)}{\partial b} + \frac{\partial f}{\partial y}(x^*, y^*)\frac{\partial y^*(b)}{\partial b} \\
&= \lambda^*(b)[\frac{\partial h}{\partial x}(x^*, y^*)\frac{\partial x^*(b)}{\partial b} + \frac{\partial h}{\partial y}(x^*, y^*)\frac{\partial y^*(b)}{\partial b}] \\
&= \lambda^*(b).
\end{aligned}$$

■

The economic interpretation of the multiplier is as a 'shadow price': For example, in the application for a firm maximising profits, it tells us how valuable another unit of input would be to the firm's profits, or how much the maximum value changes for the firm when the constraint is relaxed. In other words, it is the maximum amount the firm would be willing to pay to acquire another unit of input.

## 2.10 The Envelope Theorem

Recall that

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - b),$$

so that

$$\frac{d}{db}f(x(b), y(b); b) = \lambda(b) = \frac{\partial}{\partial b}L(x(b), y(b), \lambda(b); b).$$

Hence, what we have found above is simply a particular case of the envelope theorem, which says that

$$\frac{d}{db}f(x(b), y(b); b) = \frac{\partial}{\partial b}L(x(b), y(b), \lambda(b); b).$$

Consider the problem of maximising $f(x_1, x_2, ..., x_n)$ subject to the $k$ inequality constraints

$$h_1(x_1, x_2, ..., x_n, c) = 0, ..., h_k(x_1, x_2, ..., x_n, c) = 0$$

Let $x_1^*(c), ..., x_n^*(c)$ denote the optimal solution and let $\mu_1(c), ..., \mu_k(c)$ be the corresponding Lagrange multipliers. Suppose that $x_1^*(c), ..., x_n^*(c)$ and $\mu_1(c), ..., \mu_k(c)$ are differentiable functions and that $x^*(c)$ satisfies the constraint qualification. Then for each $j = 1, 2, ..., k$ :

$$\frac{d}{dc} f(x^*(c); c) = \frac{\partial}{\partial c} L(x^*(c), \mu(c); c)$$

Note: if $h_i(x_1, x_2, ..., x_n, c) = 0$ can be expressed as some $h_i'(x_1, x_2, ..., x_n) - c = 0$, then we are back at the previous case, in which we have found that

$$\frac{d}{dc} f(x^*(c), c) = \frac{\partial}{\partial c} L(x^*(c), \mu(c); c) = \lambda_j(c)$$

But the statement is more general.

We will prove this for the simple case of an unconstrained problem. Let $f(x; a)$ be a continuous function of $x \in R^n$ and the scalar $a$. For any $a$, consider the problem of finding $\max f(x; a)$. Let $x^*(a)$ be the maximiser which we assume a differentiable of $a$. We will show that

$$\frac{d}{da} f(x^*(a); a) = \frac{\partial}{\partial a} f(x^*(a); a)$$

Apply the chain rule:

$$\begin{aligned}
&\frac{d}{da} f(x^*(a); a) \\
&= \sum_i \frac{\partial f}{\partial x_i}(x^*(a); a) \frac{\partial x_i^*}{\partial a}(a) + \frac{\partial f}{\partial a}(x^*(a); a) \\
&= \frac{\partial f}{\partial a}(x^*(a); a)
\end{aligned}$$

since $\frac{\partial f}{\partial x_i}(x^*(a); a) = 0$ for all $i$ by the first order conditions. Intuitively, when we are already at a maximum, changing slightly the parameters of the problem or the constraints, does not affect the optimal solution (but it does affect the value at the optimal solution).

# 11 Solutions to learning activities

### Solution to learning activity 2.1

(a) For any possible solution, $x^* > 1$, we have that $x^* + 1 > 1$ and $\ln(x^* + 1) > \ln(x^*)$ and therefore $x^*$ cannot be a solution. As $\ln x$ is a strictly increasing function and the feasible set is **unbounded** there is no solution to this problem.

(b) For any possible solution, $x^* < 1$, we can choose an $\varepsilon$, where $\varepsilon < 1 - x^*$, such that have that $x^* + \varepsilon < 1$. But then we have $\ln(x^* + \varepsilon) > \ln(x^*)$ and therefore $x^*$ cannot be a solution. As $\ln x$ is a strictly increasing function and the feasible set is not **closed** there is no solution to this problem.

(c) As the feasible set is **empty** there is no solution to this problem.

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### Solution to learning activity 2.2

(a) In this case there is no maximum as the feasible set is empty.
(b) In this case there is no maximum as the feasible set is unbounded.
(c) The maximum is achieved at $x = 1$.
(d) In this case there is no maximum as the feasible set is unbounded.
(e) The maximum is achieved at either $x = 1$ or at $x = -1$.
(f) In this case there is no maximum as the feasible set is not closed around 2.
(g) The only feasible $x$ is $x = 1$ and therefore $x = 1$ yields the maximum.
(h) The maximum is achieved at $x = 2$. Note that this is true even though the feasible set is not closed.

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### Solution to learning activity 2.3

In what follows refer to the following figure. The profit function for this problem is $4y - c(y)$ and the problem calls for maximising this function with respect to $y$ subject to the constraint that $y \leq k$. To graph the set $B$ and to find the solution we look at the $(k, v)$ space. We first plot the function $v(k) = 4k - c(k)$. As can be seen in the figure this function has an inverted $U$ shape which peaks at $k = 2$ with a maximum value of 4. Remember that the constraint is $y \leq k$ and therefore, whenever $k > 2$ we can always achieve the value 4 by choosing $y = 2 < k$. This is why the boundary of set $B$ does not follow the function $v(k) = 4k - c(k)$ as $k$ is larger than 2 but rather flattens out above $k = 2$. From the graph it is clear that for all $k \geq 2$ the solution is $y = 2$ and for all $0 \leq k < 2$ the solution is $y = k$.

---

### Solution to learning activity 2.4

(a) $A \cup B$ is sometimes convex and sometimes not convex. For an example in which it is not convex consider the following sets in $\mathbb{R}$ : $A = [0, 1]$ and $B = [2, 3]$. Consider $x = 1 \in A \subset A \cup B$ and $y = 2 \in B \subset A \cup B$ and $t = 0.5$. We now have $tx + (1 - t)y = 1.5 \notin [0, 1] \cup [2, 3]$.
(b) This set is always convex. To see this, take any two elements $x, y \in A + B$. By definition $x = x_A + x_B$ where $x_A \in A$ and $x_B \in B$. Similarly, $y = y_A + y_B$ where $y_A \in A$ and $y_B \in B$. Observe that for any $t \in [0, 1]$, $tx + (1-t)y = tx_A + (1-t)y_A + tx_B + (1-t)y_B$. As $A$ and $B$ are convex, $tx_A + (1 - t)y_A \in A$ and $tx_B + (1 - t)y_B \in B$. But now we are done, as by the definition of $A + B$, $tx + (1 - t)y \in A + B$.

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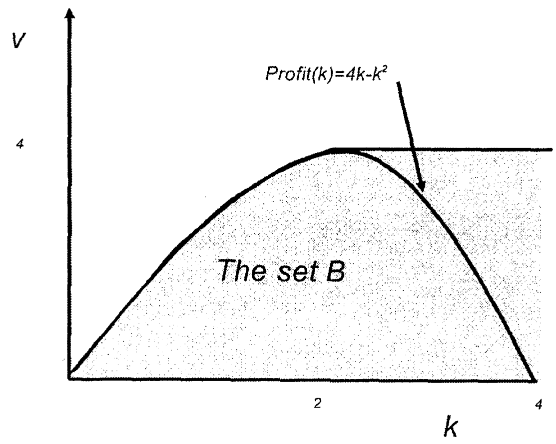


Figure 2.3: Concave profit function in learning activity 2.3

## A reminder of your learning outcomes

At the end of this chapter, you should be able to:

- formulate a constrain optimisation problem
- discern whether you could use the Lagrange method to solve the problem
- use the Lagrange method to solve the problem, when this is possible
- discern whether you could use the Kuhn-Tucker Theorem to solve the problem
- use the Kuhn-Tucker Theorem to solve the problem, when this is possible
- discuss the economic interpretation of the Lagrange multipliers
- carry out simple comparative statics using the Envelope Theorem.

## 2.12 Sample examination questions

**Question 2.1** *A household has a utility* $u(x_1, x_2) = x_1^a x_2^b$ *where* $a, b > 0$ *and* $a + b = 1$. *It cannot consume negative quantities of either good. It has strictly positive income* $y$ *and faces prices* $p_1, p_2 > 0$. *What is its optimal consumption bundle?*

**Question 2.2** *A household has a utility* $u(x_1, x_2) = (x_1 + 1)^a (x_2 + 1)^b$ *where* $a, b > 0$ *and* $a + b = 1$. *It cannot consume negative quantities of either good. It faces a budget with strictly positive income* $y$ *and prices* $p_1, p_2 > 0$. *What is its optimal consumption bundle?*

## 2.13 Comments on the sample examination questions

**Solution to question 2.1** *The utility function, $u(x_1, x_2)$, is real-valued if $x_1, x_2 \geq 0$. therefore, we have $Z = \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \geq 0\}$. Our problem is to maximise $u(x_1, x_2)$ on $Z$ and subject to the budget constraint: $p_1x_1 + p_2x_2 \leq y$.*

*The constraint function is differentiable and convex. The problem is that while the objective function, $u(x_1, x_2)$, is concave (see below), it is not differentiable at $(0, 0)$ as $Du(x_1, x_2) = \left(ax_1^{a-1}x_2^b, bx_1^a x_2^{b-1}\right)$ is not well-defined at that point.*

*Note, however, that in a valid solution both $x_1$ or $x_2$ must be non-zero, for suppose that without loss of generality $x_1 = 0$. The marginal utility for good 1 is infinite, which means that trading a small amount of good 2 for good 1 would increase the utility. As a result the constraint $(x_1, x_2) \in Z$ should be slack at the maximum.*

*We next verify that the objective function is concave. For this we look at the Hessian:*

$$D^2u(x_1, x_2) = \begin{pmatrix} -abx_1^{a-2}x_2^b & abx_1^{a-1}x_2^{b-1} \\ abx_1^{a-1}x_2^{b-1} & -abx_1^a x_2^{b-2} \end{pmatrix}$$

*The principal minors of the first order are negative, $-abx_1^{a-2}x_2^b, -abx_1^a x_2^{b-2} < 0$; the second order minor is non-negative, $|A_2| = (a^2b^2 - a^2b^2)x_1^{2(a-1)}x_2^{2(b-1)} = 0$. Therefore, the Hessian is negative semi-definite and so $u\,(x_1, x_2)$ is concave. As for the constraint, it is a linear function and hence convex.*

*As prices and income are strictly positive, we can find strictly positive $x_1$ and $x_2$ that would satisfy the budget constraint with inequality. Therefore, we can use the Kuhn-Tucker Theorem to solve this problem. The Lagrangian is:*

$$\mathcal{L} = x_1^a x_2^b + \lambda[y - p_1x_1 - p_2x_2].$$

*The first order conditions are:*

$$ax_1^{a-1}x_2^b = \lambda p_1$$
$$bx_1^a x_2^{b-1} = \lambda p_2$$

*If $\lambda > 0$, dividing the two equations we have,*

$$\frac{ax_2}{bx_1} = \frac{p_1}{p_2}\{\textit{MRS=price ratio}\}$$

*Plugging this into the constraint (that holds with equality when $\lambda > 0$),*

$$x_1 = \frac{ay}{p_1}, x_1 = \frac{by}{p_2}$$

*As all the conditions are satisfied, we are assured that this is a solution. To get that maximum value function, we plug the solution into the objective function to get,*

$$v(p_1, p_2, y) = a^a b^a p_1^{-a} p_2^{-b}.$$

**Solution to question 2.2** *The utility function, $u(x_1, x_2)$, is real valued if $x_1, x_2 \geq -1$, but the consumer cannot consume negative amounts. Therefore, we have $Z = \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \geq 0\}$. Our problem is to maximise $u(x_1, x_2)$ on $Z$ and subject to the budget constraint: $p_1x_1 + p_2x_2 \leq y$.*

*The constraint function is differentiable and convex. The objective function, $u(x_1, x_2)$, is concave (see below) and differentiable (see below).*

*We first make sure that the objective function is concave. For this we look at the Hessian:*

$$D^2u(x_1, x_2) = \begin{pmatrix} -ab(x_1+1)^{a-2}(x_2+1)^b & ab(x_1+1)^{a-1}(x_2+1)^{b-1} \\ ab(x_1+1)^{a-1}(x_2+1)^{b-1} & -ab(x_1+1)^a(x_2+1)^{b-2} \end{pmatrix}$$

*The principal minors of the first order are negative, $-ab(x_1+1)^{a-2}(x_2+1)^b, -ab(x_1+1)^a(x_2+1)^{b-2} < 0$ and of the second order is non negative, $|A_2| = (a^2b^2 - a^2b^2)(x_1+1)^{2(a-1)}(x_2+1)^{2(b-1)} = 0$. Therefore, the Hessian is negative semi-definite and so $u(x_1, x_2)$ is concave. As for the constraint, it is a linear function and hence convex.*

*As prices and income are strictly positive, we can find strictly positive $x_1$ and $x_2$ that would satisfy the budget constraint with inequality. Therefore, we can use the Kuhn-Tucker Theorem to solve this problem. As in this problem $x_i = 0$ could be part of a solution, we can add the non-negativity constraints. The Lagrangian is:*

$$\mathcal{L} = x_1^a x_2^b + \lambda_0[y - p_1x_1 - p_2x_2] + \lambda_1x_1 + \lambda_2x_2.$$

*The first order conditions are:*

$$a(x_1+1)^{a-1}(x_2+1)^b = \lambda p_1 + \lambda_1$$
$$b(x_1+1)^a(x_2+1)^{b-1} = \lambda p_2 + \lambda_2$$

*There may be three types of solutions (depending on the multipliers), one interior and two corner solutions. Assume first that $\lambda_1 = \lambda_1 = 0$ but that $\lambda_0 > 0$, dividing the two equations we have,*

$$\frac{a(x_2+1)}{b(x_1+1)} = \frac{p_1}{p_2} \{\textit{MRS=price ratio}\}$$

*Plugging this into the budget constraint (that holds with equality when $\lambda_0 > 0$),*

$$x_1 = \frac{ay + ap_2 - bp_1}{p_1}, x_1 = \frac{by - ap_2 + bp_1}{p_2}$$
$$\lambda_0 = a^a b^a p_1^{-a} p_2^{-b}$$

*This is a solution only if both $x_1$ and $x_2$ are non-negative. This happens if and only if,*

$$y > \max\left(\frac{-ap_2 + bp_1}{a}, \frac{ap_2 - bp_1}{b}\right)$$

*Consider now the case of $\lambda_1 > 0$ implying that $x_1 = 0$. In this case we must have that $x_2 = \frac{y}{p_2}$ implying that $\lambda_2 = 0$, as the consumer might as well spend all his income on good 2. From the first order conditions we have,*

$$\lambda_0 = \left(\frac{b}{p_2}\right)\left(\frac{y}{p_2} + 1\right)^{-a}$$
$$\lambda_1 = \left(\frac{bp_1 - ap_2 - ay}{p_2}\right)\left(\frac{y}{p_2} + 1\right)^{-a}$$

*This is part of a solution only if* $\lambda_1 \geq 0$, *i.e., if and only if* $y \leq \frac{bp_1 - ap_2}{a}$ *and in that case the solution is* $x_1 = 0$ *and* $x_2 = \frac{y}{p_2}$. *Similarly, if* $y \leq \frac{-bp_1 + ap_2}{a}$, *the solution is* $x_1 = \frac{y}{p_1}$ *and* $x_2 = 0$.

# Chapter 3

# The consumer's utility maximisation problem

## Aim of the chapter

The aim of this chapter is to review the standard assumptions of consumer theory on preferences, budget sets and utility functions, treating them in a more sophisticated way than in intermediate microeconomics courses, and to understand the consequences of these assumptions through the application of constrained optimisation techniques.

## Learning outcomes

By the end of this chapter, you should be able to:

- outline the assumptions made on consumer preferences
- describe the relationship between preferences and the utility function
- formulate the consumer's problem in terms of preferences and in terms of utility maximisation
- define uncompensated demand, and to interpret it as the solution to a constrained optimisation problem
- define the indirect utility function, and to interpret it as a maximum value function.

## Essential reading

This chapter is self-contained and therefore there is no essential reading assigned. But you may find further reading very helpful.

## Further reading

For a reminder of the intermediate microeconomics treatment of consumer theory read

Varian, H.R. *Intermediate Microeconomics.* Chapters 2–6, or the relevant in section of any intermediate microeconomics textbook.

For a more sophisticated treatment comparable to this chapter read Varian, H.R. *Microeconomic Analysis.* Chapter 7 or the relevant section of any mathematical treatment of consumer theory.

# 3.1 Preferences

## 3.1.1 Preferences

Consumer theory starts with the idea of preferences. Consumers are modeled as having preferences over all points in the consumption set, which is a subset of $\mathbb{R}^n$. Points in $\mathbb{R}^n$ are vectors of the form $\mathbf{x} = (x_1, x_2...x_n)$ where $x_i$ is the consumption of good $i$. Most treatments of consumer theory, including this chapter, assume that the consumption set is $\mathbb{R}^{n+}$ that is the subset of $\mathbb{R}^n$ with $x_i \geq 0$ for $i = 1...n$.

If $\mathbf{x}$ and $\mathbf{y}$ are points in $\mathbb{R}^{n+}$ that a consumer is choosing between:

- $\mathbf{x} \succ \mathbf{y}$ means that the consumer **strictly prefers** $\mathbf{x}$ to $\mathbf{y}$ so given a choice between $\mathbf{x}$ and $\mathbf{y}$ the consumer would definitely choose $\mathbf{y}$
- $\mathbf{x} \sim \mathbf{y}$ means that the consumer is **indifferent** between $\mathbf{x}$ and $\mathbf{y}$ so the consumer would be equally satisfied by either $\mathbf{x}$ or $\mathbf{y}$
- $\mathbf{x} \succsim \mathbf{y}$ means that the consumer **weakly prefers** $\mathbf{x}$ to $\mathbf{y}$ that is either $\mathbf{x} \succ \mathbf{y}$ or $\mathbf{x} \sim \mathbf{y}$.

## 3.1.2 Assumptions on preference

The most important assumptions on preferences are:

- Preferences are **complete** if for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n+}$ either $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$.
- Preferences are **reflexive** that is for any $\mathbf{x}$ in $\mathbb{R}^{n+}$ $\mathbf{x} \succsim \mathbf{x}$.
- Preferences are **transitive**, that is for $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in $\mathbb{R}^{n+}$ $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{z}$ implies that $\mathbf{x} \succsim \mathbf{z}$.

A technical assumption on preferences is:

- Preferences are **continuous** if for all $\mathbf{y}$ in $\mathbb{R}^{n+}$ the sets $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{n+}, \mathbf{x} \succsim \mathbf{y}\}$ and $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{n+}, \mathbf{y} \succsim \mathbf{x}\}$ are closed.

Closed sets can be defined in a number of different ways.

In addition we often assume:

- Preferences satisfy **nonsatiation** if for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n+}$, $\mathbf{x} \gg \mathbf{y}$, that is $x_i > y_i$ for $i = 1, 2...n$, implies that $\mathbf{x} \succ \mathbf{y}$.

Note that nonsatiation is not always plausible. Varian argues that if the two goods are chocolate cake and ice cream you could very plausibly be satiated. Some books refer to nonsatiation as 'more is better', others use the mathematical term 'monotonicity'.

- Preferences satisfy **convexity**, that is for any $\mathbf{y}$ in $\mathbb{R}^{n+}$ the set $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{n+}, \mathbf{x} \succsim \mathbf{y}\}$ of points preferred to $\mathbf{y}$ is **convex**.

Convex sets are defined in the previous chapter and on pages 70 and 71 of Chapter 6 of Dixit and Section 2.2 of Chapter 2 of Sydsæter et al. The convexity assumption is very important for the application of

Lagrangian methods to consumer theory and we will come back to it. Some textbooks discuss convexity in terms of diminishing rates of marginal substitution, others describe indifference curves as being convex to the origin.

Different textbooks work with slightly different forms of these assumptions; it does not matter, the resulting theory is the same.

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**Learning activity 3.1**

Assume there are two goods and that all the assumptions given above hold. Pick a point $\mathbf{y}$ in $\mathbb{R}^{2+}$.

(a) Show the set $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{n+},\ \mathbf{x} \sim \mathbf{y}\}$. What is this set called?

(b) What is the marginal rate of substitution?

(c) Show in a diagram the set of points weakly preferred to $\mathbf{y}$, that is $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{n+},\ \mathbf{x} \succsim \mathbf{y}\}$. Make sure that it is convex. What does this imply for the marginal rate of substitution?

(d) Show the set of points $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{n+},\ \mathbf{x} \geq \mathbf{y}\}$, where $\mathbf{x} \geq \mathbf{y}$ means that $x_i \geq y_i$ for $i = 1, 2...n$. Explain why nonsatiation implies that this set is a subset of the set of points weakly preferred to $\mathbf{y}$, that is$\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{n+},\ \mathbf{x} \succsim \mathbf{y}\}$.

(e) Explain why the nonsatiation assumption implies that the indifference curve cannot slope upwards.

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## 3.2 The consumer's budget

### 3.2.1 Definitions

**Assumption.** The consumer's choices are constrained by the conditions that

$$p_1x_1 + p_2x_2.... + p_nx_n \leq m$$
$$\mathbf{x} \in \mathbb{R}^{n+}$$

where $p_i$ is the price of good $i$, and $m$ is income. We assume that prices and income are strictly positive, that is $p_i > 0$ for $i = 1...n$ and $m > 0$.

**Definition 13** *The* **budget constraint** *is the inequality*

$$p_1x_1 + p_2x_2.... + p_nx_n \leq m.$$

The budget constraint is sometimes written as $\mathbf{px} \leq m$ where $\mathbf{p}$ is the vector of prices $(p_1...p_n)$ and $\mathbf{px}$ is notation for the sum $p_1x_1 + p_2x_2.... + p_nx_n$.

**Definition 14** *The* **budget line** *is the set of points with* $p_1x_1 + p_2x_2 + ....p_nx_n = m$ *and* $x_i \geq 0$ *for* $i = 1...n$.

**Definition 15** *The* **budget set** *is the subset of points in* $\mathbb{R}^{n+}$ *satisfying the budget constraint. In formal notation it is*

$$\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{n+}, p_1x_1 + p_2x_2.... + p_nx_n \leq m\ \}$$

**Learning activity 3.2**

Assume that $n = 2$ and draw a diagram showing

(a) the budget line and its gradient

(b) the points where the budget line meets the axes

(c) the budget set.

**Learning activity 3.3**

Draw diagrams showing the effects on the budget line and budget set of the following changes

(a) an increase in $p_1$

(b) a decrease in $p_2$

(c) an increase in $m$.

# 3.3 Preferences and the utility function

## 3.3.1 The consumer's problem in terms of preferences

**Definition 16** *The consumer's problem stated in terms of preferences is to find a point* $\mathbf{x}^*$ *in* $\mathbb{R}^{n+}$ *with the property that* $\mathbf{x}^*$ *satisfies the budget constraint, so*

$$p_1x_1^* + p_2x_2^* + ....p_nx_n^* \leq m$$

*and*

$$\mathbf{x}^* \succsim \mathbf{x}$$

*for all* $\mathbf{x}$ *in* $\mathbb{R}^{n+}$ *that satisfy the budget constraint.*

## 3.3.2 Preferences and utility

The difficulty with expressing the consumer's problem in terms of preferences is that we have no techniques for solving the problem. However preferences can be represented by utility functions, which makes it possible to define the consumer's problem in terms of constrained optimisation and solve it using the tools developed in the previous chapter.

**Definition 17** *A utility function* $u(\mathbf{x})$ *which takes* $\mathbb{R}^{n+}$ *into* $\mathbb{R}$ **represents preferences** $\succsim$ *if*

- $u(\mathbf{x}) > u(\mathbf{y})$ *implies that* $\mathbf{x} \succ \mathbf{y}$
- $u(\mathbf{x}) = u(\mathbf{y})$ *implies that* $\mathbf{x} \sim \mathbf{y}$
- $u(\mathbf{x}) \geq u(\mathbf{y})$ *implies that* $\mathbf{x} \succsim \mathbf{y}$.

The following result and its proof are beyond the scope of this course, but the result is worth knowing

**Theorem 6** *If preferences satisfy the completeness, transitivity and continuity assumptions they can be represented by a continuous utility function.*

Varian's *Microeconomic Analysis* proves a weaker result on the existence of a utility function in Chapter 7.1. Again this is beyond the scope of this unit.

### 3.3.3 Cardinal and ordinal utility

One of the standard points made about consumer theory is that the same set of consumer preferences can be represented by different utility functions so long as the order of the numbers on indifference curves is not changed. For example if three indifference curves have utility levels 1, 2 and 3, replacing 1, 2 and 3 by 2, 62 and 600 does not change the preferences being represented because $1 < 2 < 3$ and $2 < 62 < 600$. However replacing 1, 2 and 3 by 62, 2 and 600 does change the preferences being represented because it is not true that $62 < 2 < 600$. The language used to describe this is that the utility function is ordinal rather than cardinal, so saying that one bundle of goods gives higher utility than another means something, but saying that one bundle gives twice as much utility as the other is meaningless.

This can be stated more formally.

**Theorem 7** *If the utility function $u(\mathrm{x})$ and the function $b(\mathrm{x})$ are related by $u(\mathrm{x}) = a(b(\mathrm{x}))$where $a$ is a strictly increasing function, then $b(\mathrm{x})$ is also a utility function representing the same preferences as $u(\mathrm{x})$.*

As there are many different strictly increasing functions this result implies that any set of preferences can be represented by many different utility functions.

**Proof.** As $a$ is strictly increasing and $u$ represents preferences
$b(\mathrm{x}) > b(\mathrm{y})$ implies that $u(\mathrm{x}) > u(\mathrm{y})$ which implies $\mathrm{x} \succ \mathrm{y}$
$b(\mathrm{x}) = b(\mathrm{y})$ implies that $u(\mathrm{x}) = u(\mathrm{y})$ which implies that $\mathrm{x} \sim \mathrm{y}$
$b(\mathrm{x}) \geq b(\mathrm{y})$ implies that $u(\mathrm{x}) \geq u(\mathrm{y})$ which implies that $\mathrm{x} \succsim \mathrm{y}$ so
$b(\mathrm{x})$ is a utility function representing the same preferences as $u(\mathrm{x})$. ■

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**Learning activity 3.4**

Suppose that $f$ is a function taking $\mathbb{R}^{n+}$ into $\mathbb{R}$, and the term $\succsim_f$ is defined by $f(\mathrm{x}) \geq f(\mathrm{y})$ implies that $\mathrm{x} \succsim_f \mathrm{y}$. Explain why the relationship given by $\succsim_f$ is complete, reflexive and transitive.

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**Learning activity 3.5**

Suppose that the utility function is differentiable, and that

$$\frac{\partial u(\mathrm{x})}{\partial x_i} > 0 \text{ for } i = 1...n.$$

Explain why this implies that the preferences satisfy nonsatiation.

## 3.4 The consumer's problem in terms of utility

### 3.4.1 Uncompensated demand and the indirect utility function

**Definition 18** *The* **consumer's utility maximisation problem** *is*

$$\max u(\mathbf{x})$$
$$\text{s.t. } p_1x_1 + p_2x_2 + ....p_nx_n \leq m$$
$$\mathbf{x} \in \mathbb{R}^{n+}.$$

Note that this has the same form as the constrained optimisation problem in the previous chapter. The previous chapter showed you how to solve the consumer's problem with the Cobb-Douglas utility function $u(x_1, x_2) = x_1^a x_2^b$ and the utility function $u(x_1, x_2) = (x_1 + 1)^a (x_2 + 1)^b$.

**Definition 19** *The solution to the consumer's utility maximisation problem is* **uncompensated demand***. It depends upon prices $p_1, p_2..p_n$ and income $m$. The uncompensated demand for good $i$ is written as*

$$x_i(p_1, p_2...p_n, m)$$

*or using vector notation as $x_i(\mathbf{p}, m)$ where $\mathbf{p}$ is the vector $(p_1, p_2..p_n)$. Notation $\mathbf{x}(\mathbf{p}, m)$ is used for the vector of uncompensated demand $(x_1(\mathbf{p}, m), x_2(\mathbf{p}, m)...x_n(\mathbf{p}, m))$.*

The notation $\mathbf{x}(\mathbf{p}, m)$ suggests that uncompensated demand is a function of $(\mathbf{p}, m)$, which requires that for each value of $(\mathbf{p}, m)$ there is only one solution to the utility maximising problem. This is usually so in the examples economists work with, but it does not have to be, and we will look at the case with a linear utility function where the consumer's problem has multiple solutions for some values of $\mathbf{p}$.

Recall from Chapter 1 the definition of the maximum value function as the value of the function being maximised at the solution to the problem.

**Definition 20** *The* **indirect utility function** *is the maximum value function for the consumer's utility maximisation problem. It is written as*

$$v(p_1, p_2...p_n, m)$$

*or in vector notation $v(\mathbf{p}, m)$ where*

$$v(\mathbf{p}, m) = u(\mathbf{x}(\mathbf{p}, m)).$$

There are two important results linking preferences and utility functions.

**Theorem 8** *If preferences are represented by a utility function $u(\mathbf{x})$ then the solutions to the consumer's utility maximising function in terms of preferences are the same as the solutions to the consumer's problem in terms in utility.*

**Theorem 9** *If two utility functions represent the same preferences the solutions to the consumer's utility maximisation problem are the same with the two utility functions but the indirect utility functions are different.*

These results are not difficult to prove, the next learning activity asks you to do this.

### 3.4.2 Nonsatiation and uncompensated demand

The nonsatiation assumption on preferences is that for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n+}$, if $\mathbf{x} > \mathbf{y}$, that is $x_i \geq y_i$ for all $i$, and $x_i > y_i$ for some $i$, then $\mathbf{x} \succ \mathbf{y}$. If the preferences are represented by a utility function $u$, this requires that if $\mathbf{x} > \mathbf{y}$ then $u(\mathbf{x}) > u(\mathbf{y})$, so the utility function is strictly increasing in the consumption of at least one good, and non-decreasing in consumption of any good. In intuitive terms, the assumption says that more is better, so a consumer will spend all available income.

**Proposition 6** *If the nonsatiation condition is satisfied any solution to the consumer's utility maximising problem satisfies the budget constraint as an equality.*

**Proof.** To see this, suppose that the budget constraint is not satisfied as an equality so

$$p_1x_1 + p_2x_2... + p_nx_n < m$$

and increasing consumption of good $i$ increases utility so if $\varepsilon > 0$

$$u(x_1, x_2...x_i + \varepsilon, ...x_n) > u(x_1, x_2...x_i, ...x_n).$$

As $p_1x_1 + p_2x_2... + p_nx_n < m$ if $\varepsilon$ is small enough the point $(x_1, x_2...x_i + \varepsilon, ...x_n)$ satisfies the budget constraint and gives higher utility that $(x_1, x_2...x_i, ...x_n)$ so $(x_1, x_2...x_i, ...x_n)$ cannot solve the consumer's problem. ■

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**Learning activity 3.6**

Explain why Theorem 8 holds.

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**Learning activity 3.7**

Explain why Theorem 9 holds

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**Learning activity 3.8**

Find the indirect utility function for the Cobb-Douglas utility function $u(x_1, x_2) = x_1^a x_2^b$.

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**Learning activity 3.9**

The objective of this activity is to solve the consumer's utility maximising function with a linear utility function $2x_1 + x_2$ subject to the constraints $p_1x_1 + p_2x_2 \leq m$, $x_1 \geq 0$ and $x_2 \geq 0$. Assume that $p_1 > 0$, $p_2 > 0$ and $m > 0$.

(a) Draw indifference curves for the utility function $u(x_1, x_2) = 2x_1 + x_2$. What is the marginal rate of substitution?

(b) Assume that $p_1 \backslash p_2 < 2$. Use your graph to guess the values of $x_1$ and $x_2$ that solve the maximising problem. Which constraints bind, that is have $h_i(x_1, x_2) = k_i$ where $h_i$ is constraint function $i$. Which constraints do not bind, that is have $h_i(x_1, x_2) < k_i$? Does the problem have more than one solution?

(c) Assume that $p_1 \backslash p_2 = 2$. Use your graph to guess the values of $x_1$ and $x_2$ that solve the maximising problem. Which constraints bind? Which constraints do not bind? Is there more than one solution?

(d) Assume that $p_1 \backslash p_2 > 2$. Use your graph to guess the values of $x_1$ and $x_2$ that solve the maximising problem. Which constraints bind? Which constraints do not bind? Does the problem have more than one solution?

(e) Explain why the Kuhn-Tucker conditions are necessary and sufficient for a solution to this problem.

(f) Write down the Lagrangian for the theorem. Confirm that your guesses are correct by finding the Lagrange multipliers associated with the constraints for activities 3.6–3.8. Does the problem have more than one solution?

**Learning activity 3.10**

Find the indirect utility function for the utility function

$$u(x_1, x_2) = 2x_1 + x_2$$

## 3.5 Solution to learning activities

**Solution to learning activity 3.1**

Assume that the consumer's preferences satisfy all the assumptions of Section 3.1. Refer to Figure 3.1.

(a) The set $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{n+}, \ \mathbf{x} \sim \mathbf{y}\}$ is the indifference curve.

(b) The marginal rate of substitution is the gradient of the indifference curve.

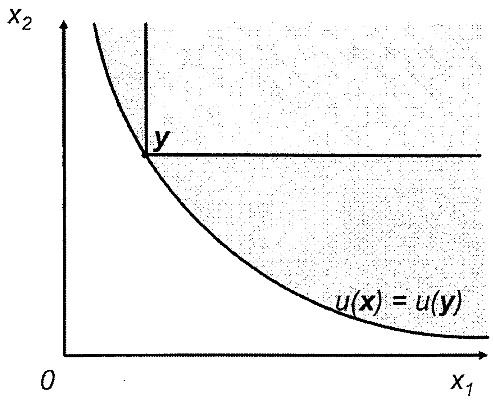


Figure 3.1: Indifference curves and preference relationships

(c) The set $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{n+}, \mathbf{x} \succsim \mathbf{y}\}$ is the entire shaded area in Figure 3.1. The marginal rate of substitution decreases as $x_1$ increases because this set is convex.

(d) The set $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{n+}, \mathbf{x} \geq \mathbf{y}\}$, where $\mathbf{x} \geq \mathbf{y}$ means that $x_i \geq y_i$ for $i = 1, 2...n$ is the lightly shaded area in Figure 3.1. The nonsatiation assumption states that all points in this set are weakly preferred to $\mathbf{y}$, that is $\mathbf{x} \succsim \mathbf{y}$.

(e) If the indifference curve slopes upwards there are points on the indifference curve $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ with $y_1 > x_1$ and $y_2 > x_2$ so $\mathbf{x} \gg \mathbf{y}$. Nonsatiation then implies that $\mathbf{x} \succ \mathbf{y}$ which is impossible if $\mathbf{x}$ and $\mathbf{y}$ are on the same indifference curve.

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**Solution to learning activity 3.2**

The budget line in Figure 3.2 has gradient $-p_1/p_2$ and meets the axes at $m/p_1$ and $m/p_2$.

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**Solution to learning activity 3.3**

Refer to Figure 3.3:

- An increase in $p_1$ to $p_1'$ shifts the budget line from $CE$ to $CD$. The budget line becomes steeper.
- A decrease in $p_2$ to $p_2'$ shifts the budget line from $CE$ to $BE$. The budget line becomes steeper.
- An increase in $m$ to $m'$ shifts the budget line out from $CE$ to $AF$. The gradient of the budget line does not change.

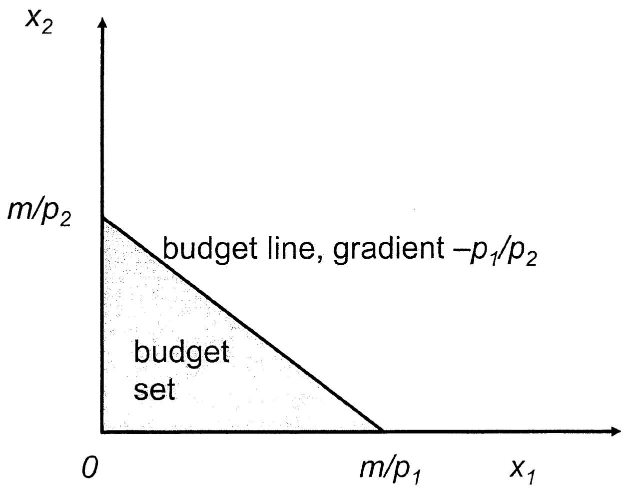


Figure 3.2: The budget set and budget line.

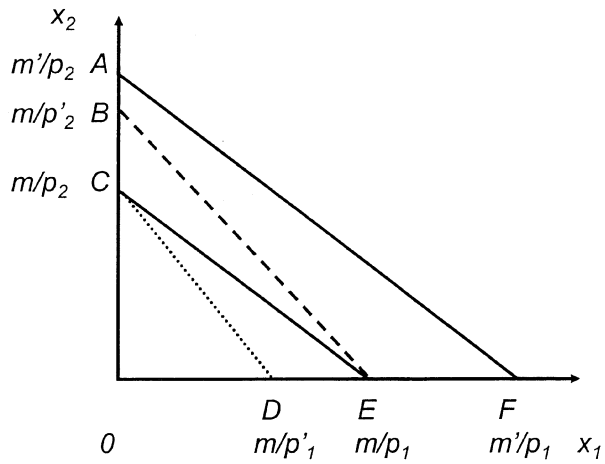


Figure 3.3: The effect of changes in prices and income on the budget line.

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**Solution to learning activity 3.4**

This learning activity tells you conditions on the utility function that ensure that preferences have various properties, and asks you to explain why.

Suppose that $f$ is a function taking $\mathbb{R}^{n+}$ into $\mathbb{R}$, and the term $\succsim_f$ is defined by $f(\mathbf{x}) \geq f(\mathbf{y})$ implies that $\mathbf{x} \succsim_f \mathbf{y}$.

The values of the function $f(\mathbf{x})$ are real numbers. The relationship $\geq$

for real numbers is complete, so for any $\mathbf{x}$ and $\mathbf{y}$ either $f(\mathbf{x}) \geq f(\mathbf{y})$ or $f(\mathbf{y}) \geq f(\mathbf{x})$.

The relationship $\geq$ for real numbers is reflexive so for any $\mathbf{x}$, $f(\mathbf{x}) \geq f(\mathbf{x})$.

The relationship $\geq$ is transitive, so for any $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, $f(\mathbf{x}) \geq f(\mathbf{y})$ and $f(\mathbf{y}) \geq f(\mathbf{z})$ implies that $f(\mathbf{x}) \geq f(\mathbf{z})$.

Thus, as $f(\mathbf{x}) \geq f(\mathbf{y})$ implies that $\mathbf{x} \succsim_f \mathbf{y}$, the relationship $\succsim_f$is complete, reflexive and transitive.

---

**Solution to learning activity 3.5**

As

$$\frac{\partial u(\mathbf{x})}{\partial x_i} > 0 \text{ for } i = 1...n$$

if $y_i > x_i$ for $i = 1...n$ then $u(\mathbf{y}) > u(\mathbf{x})$, so $\mathbf{y} \succ \mathbf{x}$, and nonsatiation is satisfied.

---

**Solution to learning activity 3.6**

Theorem 8 states that if preferences are represented by a utility function $u(\mathbf{x})$ then the solutions to the consumer's utility maximising function in terms of preferences are the same as the solutions to the consumer's problem in terms in utility.

To see why suppose that a point $\mathbf{x}^*$ solves the consumers problem in terms of preferences so $\mathbf{x}^* \succcurlyeq \mathbf{x}$ for all points in $\mathbb{R}^{n+}$ that satisfy the budget constraint. A point $\mathbf{x}^*$ solves the consumer's utility maximisation problem if $u(\mathbf{x}^*) \geq u(\mathbf{x})$ for all points in $\mathbb{R}^{n+}$ that satisfy the budget constraint. If the utility function represents the preferences then the set of points for which $\mathbf{x}^* \succcurlyeq \mathbf{x}$ is the same as the set of points for which $u(\mathbf{x}^*) \geq u(\mathbf{x})$, so the two problems have the same solutions.

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**Solution to learning activity 3.7**

Theorem 9 states that if two utility functions represent the same preferences the solutions to the consumer's utility maximisation problem are the same with the two utility functions but the indirect utility functions are different.

This follows directly from Theorem 8, because if $u(\mathbf{x})$ and $u(\mathbf{x}^*)$ represent the same preferences the solutions to the consumer's utility maximisation with the two utility functions are the same as the solution to the consumer's problem in terms of the preferences and so are the same as each other.

## Solution to learning activity 3.8

The examination question for Chapter 2 gave the solution to the consumer's Cobb-Douglas utility function $u(x_1, x_2) = x_1^a x_2^b$ where $a > 0$, $b > 0$ and $a + b = 1$. The solutions is $x_1 = a(m/p_1)$, $x_2 = b(m/p_2)$ so uncompensated demand is

$$x_1(p_1, p_2, m) = a\frac{m}{p_1}.$$
$$x_2(p_1, p_2, m) = b\frac{m}{p_2}$$

so the indirect utility function

$$\begin{aligned} v(p_1, p_2, m) &= \left(a\frac{m}{p_1}\right)^a \left(b\frac{m}{p_2}\right)^b \\ &= a^a b^b \frac{m^{a+b}}{p_1^a p_2^b}. \\ &= \left(\frac{a^a b^b}{p_1^a p_2^b}\right) m. \end{aligned}$$

The third line here follows from the second line because $a + b = 1$.

## Solution to learning activity 3.9

The objective of this activity is to solve the consumer's utility maximising function with a linear utility function $2x_1 + x_2$ subject to the constraints $p_1x_1 + p_2x_2 \leq m$, $x_1 \geq 0$ and $x_2 \geq 0$. Assume that $p_1 > 0$, $p_2 > 0$ and $m > 0$.

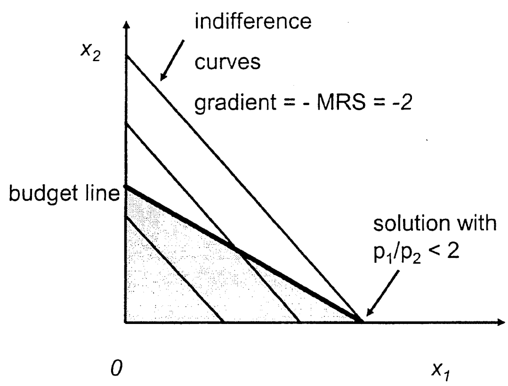


Figure 3.4: Utility maximisation with $u(x_1, x_2) = 2x_1 + x_2$ and $p_1/p_2 = 2$.

(a) The indifference curves for the utility function $u(x_1, x_2) = 2x_1 + x_2$ are parallel straight lines with gradient $-2$ a shown in Figure 3.4. The marginal rate of substitution is 2.

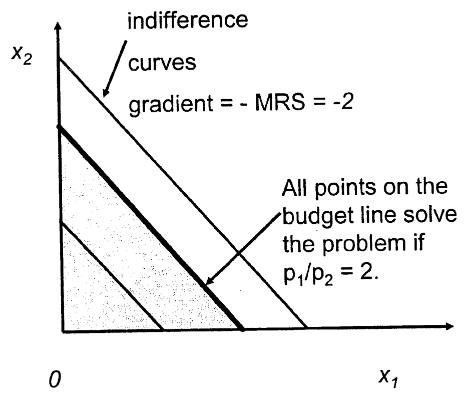


Figure 3.5: Utility maximization with $u(x_1, x_2) = 2x_1 + x_2$ and $p_1/p_2 = 2$.

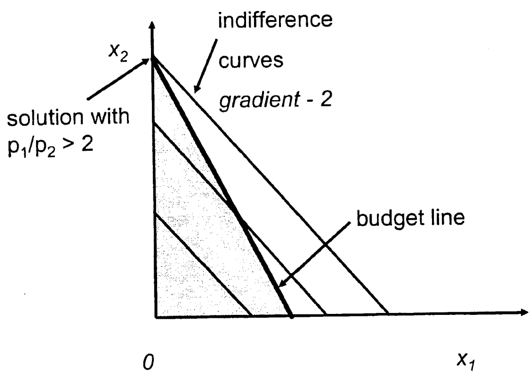


Figure 3.6: Utility maximisation with $u(x_1, x_2) = 2x_1 + x_2$ and $p_1/p_2 > 2$.

(b) From Figure 3.4 if $p_1/p_2 < 2$ the budget line is less steep than the indifference curves. The solution is at $x_1 = m/p_1$, $x_2 = 0$. The constraints $p_1x_1 + p_2x_2 \leq m$ and $x_2 \geq 0$ bind. The constrain $x_1 \geq 0$ does not bind. There is only one solution.

(c) From Figure 3.5 if $p_1/p_2 = 2$ the budget line is parallel to all the indifference curves, and one indifference curve coincides with the budget line. Any $(x_1, x_2)$ with $x_1 \geq 0$, $x_2 \geq 0$ and $p_1x_1 + p_2x_2 = m$ solves the problem. If $x_1 = 0$ the constraints $x_1 \geq 0$ and $p_1x_1 + p_2x_2 \leq m$ bind and the constraint $x_2 \geq 0$ does not bind. If $x_2 = 0$ the constraints $x_2 \geq 0$ and $p_1x_1 + p_2x_2 = m$ bind and the constraint $x_1 \geq 0$ does not bind. If $x_1 > 0$ and $x_2 > 0$ the constraint $p_1x_1 + p_2x_2 \leq m$ binds and the constraints $x_1 \geq 0$ and $x_2 \geq 0$ do not bind. There are many solutions.

(d) From Figure 3.6 if $p_1/p_2 > 2$ the budget line is steeper than the indifference curves. The solution is at $x_1 = 0$, $x_2 = m/p_2$. The constraints $p_1x_1 + p_2x_2 \leq m$ and $x_1 \geq 0$ bind. The constraint $x_2 \geq 0$ does not bind. There is only one solution.

(e) The objective $2x_1 + x_2$ is linear so concave, the constraint functions $p_1x_1 + p_2x_2$, $x_1$ and $x_2$ are linear so convex, the constraint qualification is satisfied because there are points that satisfy the constraints as strict inequalities, for example $x_1 = m/2p_1$, $x_2 = m/2p_2$. Thus the Kuhn-Tucker conditions are necessary and sufficient for a solution to this problem.

(f) The Lagrangian is:

$$\begin{aligned}\mathcal{L} &= 2x_1 + x_2 + \lambda_0 (m - p_1x_1 - p_2x_2) + \lambda_1 x_1 + \lambda_2 x_2 \\ &= x_1 (2 - \lambda_0 p_1 + \lambda_1) + x_2 (1 - \lambda_0 p_2 + \lambda_2).\end{aligned}$$

The first order conditions require that:

$$\begin{aligned}2 &= \lambda_0 p_1 - \lambda_1 \\ 1 &= \lambda_0 p_2 - \lambda_2.\end{aligned}$$

Checking for solutions with $x_1 > 0$ and $x_2 = 0$, complementary slackness forces $\lambda_1 = 0$, so $\lambda_0 = 2/p_1 > 0$ and $\lambda_2 = \lambda_0 p_2 - 1 = 2p_2/p_1 - 1$. Thus non-negativity of multipliers forces $p_1/p_2 \leq 2$. Complementary slackness forces $x_1 = m/p_1$. The point $x_1 = m/p_1$, $x_2 = 0$ is feasible. Thus if $p_1/p_2 \leq 2$ the point $x_1 = m/p_1$, $x_2 = 0$ solves the problem. If $p_1/p_2 < 0$ this is the unique solution.

Checking for solutions with $x_1 = 0$ and $x_2 > 0$, complementary slackness forces $\lambda_2 = 0$, so $\lambda_0 = 1/p_2$ and $\lambda_1 = \lambda_0 p_1 - 2 = p_1/p_2 - 2$. Thus non-negativity of multipliers forces $p_1/p_2 \geq 2$ . Complementary slackness forces $x_2 = m/p_2$. The point $x_1 = 0$, $x_2 = m/p_2$ is feasible. Thus if $p_2/p_2 \leq$ the point $x_1 = 0$, $x_2 = m/p_2$ solves the problem. If $p_2 < p_1/2$ this is the only solution.

If $p_1/p_2 = 2$, and $\lambda_0 = 2/p_2 = 1/p_2$, then $\lambda_1 = \lambda_2 = 0$, so solutions with $x_1 > 0$ and $x_2 > 0$ are possible. As $\lambda_0 > 0$ complementary slackness forces $p_1x_1 + p_2x_2 = m$. In this case there are multiple solutions.

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**Solution to learning activity 3.10**

Finding the indirect utility function for the utility function $u(x_1, x_2) = 2x_1 + x_2$,

if $p_1/p_2 < 2$, then $x_1(p_1, p_2, m) = m/p_1$ and $x_2(p_1, p_2, m) = 0$ so $v(p_1, p_2, m) = 2x_1 + x_2 = 2m/p_1$.

If $p_1/p_2 = 2$, any $x_1$, $x_2$ with $x_1 \geq 0$, $x_2 \geq 0$, and $x_2 = (m - p_1x_1)/p_2$, solves the problem, so:

$$v(p_1, p_2, m) = 2x_1 + \frac{(m - p_1x_1)}{p_2} = \frac{m}{p_2} = \frac{2m}{p_1}.$$

If $p_1/p_2 > 2$, then $x_1(p_1, p_2, m) = 0$ and $x_2(p_1, p_2, m) = m/p_2$ so $v(p_1, p_2, m) = 2x_1 + x_2 = m/p_2$.

This is perfectly acceptable answer to the question, but the indirect utility function can also be written more concisely as

$$v(p_1, p_2, m) = \max\left(\frac{2m}{p_1}, \frac{m}{p_2}\right).$$

## A reminder of your learning outcomes

By the end of this chapter, you should be able to:

- outline the assumptions made on consumer preferences
- describe the relationship between preferences and the utility function
- formulate the consumer's problem in terms of preferences and in terms of utility maximisation
- define uncompensated demand, and to interpret it as the solution to a constrained optimisation problem
- define the indirect utility function, and to interpret it as a maximum value function.

## 3.6 Sample examination questions

When answering these and any other examination questions be sure to explain why your answers are true, as well as giving the answer. For example here when you use Lagrangian techniques in question 2 check that the problem you are solving satisfies the conditions of the theorem you are using, and explain your reasoning.

**Question 3.1** *(a) Explain the meaning of the nonsatiation and transitivity assumptions in consumer theory.*

*(b) Suppose that preferences can be represented by a utility function $u(x_1, x_2)$. Do these preferences satisfy the transitivity assumption?*

*(c) Assume that the derivatives of the utility function $u(x_1, x_2)$ are strictly positive. Do the preferences represented by this utility function satisfy the nonsatiation assumption?*

*(d) Continue to assume that the derivatives of the utility function $u(x_1, x_2)$ are strictly positive. Is it possible that the solution to the consumers utility maximisation problem does not satisfy the budget constraint as an equality?*

**Question 3.2** *(a) Define uncompensated demand.*

*(b) Define the indirect utility function.*

*(c) A consumer has a utility function $u(x_1, x_2) = x_1^\rho + x_2^\rho$ where $0 < \rho < 1$. Solve the consumer's utility maximisation problem.*

*(d) What is the consumer's uncompensated demand function for this utility function?*

*(e) What is the consumer's indirect utility function for this utility function?*

## 3.7 Comments on sample examination questions

As with any examination question you need to explain what you are doing and why.

**Solution to question 3.2** *(d) The uncompensated demand functions for consumer's utility maximisation problem with* $u(x_1, x_2) = x_1^\rho + x_2^\rho$ *are*

$$x_1(p_1, p_2, m) = \left( \frac{p_1^{-\frac{1}{1-\rho}}}{p_1^{-\frac{\rho}{1-\rho}} + p_2^{-\frac{\rho}{1-\rho}}} \right) m$$

$$x_2(p_1, p_2, m) = \left( \frac{p_2^{-\frac{1}{1-\rho}}}{p_1^{-\frac{\rho}{1-\rho}} + p_2^{-\frac{\rho}{1-\rho}}} \right) m.$$

*(e) The indirect utility function is*

$$v(p_1, p_2, m) = \left( p_1^{-\frac{\rho}{1-\rho}} + p_2^{-\frac{\rho}{1-\rho}} \right)^{1-\rho} m^\rho.$$