Mathematics Navigator

A Sample of Mathematics Misconceptions and Errors (Grades 2-8)

AMERICA'S CHOICE®

Introduction

Students bring prior knowledge of mathematical concepts to class, and research shows that working with that prior knowledge can lead to deeper understanding and long-term learning (Askew, 2002). Research has shown that teaching becomes more effective when common mistakes and misconceptions are systematically exposed (Swan, 2005). Alan Bell suggests that when students face a challenge to their cognitive structure, they are much more willing to stretch themselves intellectually. Conceptual gains realized in this manner promote “transfer from the immediate topic to wider situations” (Bell, 2006). Alan Bell (2005) argues that without “exposure of pupils’ misconceptions and their resolution through conflict discussion, students may not know why a mistake occurred.” Mathematics interventions should use a subtle process to expose flawed thinking and allow students to confront their own misconceptions and, consequently, discover for themselves the source of their mistakes (Bell, 2005; Bell & Swan 2006; Donovan & Bransford, 2005).

Mathematics Navigator places an emphasis on prior knowledge and *revising misconceptions*. This document contains some of the misconceptions that are the focus of Mathematics Navigator.

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Askew, M. (2002). The changing primary mathematics classroom—the challenge of the National Numeracy Strategy. In L. Haggerty (Ed.), *Aspects of Teaching Secondary Mathematics: Perspectives on Practice.* London: Routledge Falmer.

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Place Value

1. When counting tens and ones (or hundreds, tens, and ones), students misapply the procedure for counting on and treats tens and ones (or hundreds, tens, and ones) as separate numbers.

> Example
>
> When asked to count collections of bundled tens and ones, such as | | | • •, students count 10, 20, 30, 1, 2, instead of 10, 20, 30, 31, 32.

2. Students have an alternative conception of multidigit numbers and see them as numbers independent of place value.

> Example
>
> Students read the number 32 as "thirty-two" and count out 32 objects to demonstrate the value of the number, but when asked to write the number in expanded form, they write "3 + 2."
>
> Students read the number 32 as "thirty-two" and count out 32 objects to demonstrate the value of the number, but when asked the value of the digits in the number, they respond that the values are "3" and "2."

3. Students recognize simple multidigit numbers, such as thirty (30) or 400 (four hundred), but they do not understand that the position of a digit determines its value.

> Example
>
> Students mistake the numeral 306 for thirty-six.
>
> Students write 4008 when asked to record four hundred eight.

> Example
>
> When asked the value of the digit 8 in the number 18,342,092, students respond with "8" or "one million" instead of "eight million."

4. Students misapply the rule for reading numbers from left to right.

Example

Student reads 81 as eighteen.

5. Students order numbers based on the value of the digits, instead of place value.

Example

69 > 102, because 6 and 9 are bigger than 1 and 2.

6. Students undergeneralize results of multiplication by powers of 10 and do not understand that shifting digits to higher place values is like multiplying by powers of 10.

Example

When asked to solve a problem like $\boxed{?} \times 36 = 3600$, students either divide or cannot respond.

7. Students have limited understanding of numbers to one or two representations.

Example

Students may be able to read and write the number 4,302,870 in standard form but the do not link this number to a representation using tally marks in a place value chart or to expanded form.

8. Students apply the alternate conception "Write the numbers you hear" when writing numbers in standard form when given the number in words.

Example

When asked to write the number five hundred eleven thousand in standard form, students writes 500,11,000 with or without commas.

Example

When asked to write the number sixty-two hundredths, students write 62.00 or 6200.

9. Students misapply the rule for "rounding down" and actually lower the value of the digit in the designated place.

> Example
>
> When asked to round to the nearest ten thousand, students round the number 762, 398 to 750,000 or 752,398.

> Example
>
> When asked to round to the nearest tenth, students round the number 62.31 to 62.2 or 62.21.

10. Students misapply the rule for "rounding up" and change the digit in the designated place while leaving digits in smaller places as they are.

> Example
>
> Students round 127,884 to 128,884 (nearest thousand).

> Example
>
> Students round 62.38 to 62.48 (nearest tenth).

11. Students overgeneralize that the comma in a number means say "thousands" or "new number."

> Example
>
> Students read the number 3,450,207 as "three thousand four hundred fifty thousand two hundred seven."

> Example
>
> Students read the number 3,450,207 as "three, four hundred fifty, two hundred seven."

12. Students lack the concept that 10 in any position (place) makes one (group) in the next position and vice versa.

Example

If shown a collection of 12 hundreds, 2 tens, and 13 ones, students write 12213, possibly squeezing the 2 and the 13 together or separating the three numbers with some space.

Example

$$0.72 + 0.72 = 0.144 \quad \text{or} \quad \begin{array}{r} 32{,}871 \\ +\ 9{,}324 \\ \hline 311{,}1195 \end{array} \quad \text{or} \quad 3{,}111{,}195$$

Addition and Subtraction

1. Students have overspecialized their knowledge of addition or subtraction facts and restrict it to "fact tests" or one particular problem format.

> Example
>
> Students complete addition or subtraction facts assessments satisfactorily but do not apply the knowledge to other arithmetic and problem-solving situations.

2. Students may know the commutative property of addition but fail to apply it to simplify the "work" of addition or misapply it in subtraction situations.

> Example
>
> Students state that $9 + 4 = 13$ with relative ease, but struggle to find the sum of $4 + 9$.
>
> Students write (or say) "12 –50" when they means 50 – 12.

3. Students think that subtraction is commutative.

> Example
>
> $5 - 3 = 3 - 5$

4. Students may know the associative property of addition but fail to apply it to simplify the "work" of addition.

> Example
>
> Students labor to find the sum of three or more numbers, such as $4 + 7 + 6$, using a rote procedure, because they fail to recognize that it is much easier to add the numbers in a different order.

5. Students try to overgeneralize immature addition or subtraction methods, instead of developing more effective methods.

Example

Students may have learned the early childhood method of "recount all" and stopped there. When the numbers get too big to recount, students have nothing else to draw on.

6. Students may be unable to generalize methods that they already know for addition and subtraction to a new situation.

Example

Students may be perfectly comfortable with addition facts, such as 6 + 7, but do not know how to extend this fact knowledge to a problem, such as 16 + 7.

7. Students have overspecialized during the learning process so that they recognize some addition and/or subtraction situations as addition or subtraction but fail to classify other situations appropriately.

Example

Students recognize that if there are 7 birds in a bush and 3 fly away, you can subtract to find out how many are left.

However, students may be unable to solve a problem that involves the comparison of two amounts or the missing part of a whole.

8. Students know *how* to add but do not know *when* to add (other than because they are told to do so, or because the computation was written as an addition problem).

Example

Students cannot explain why they should add or connect addition to actions with materials.

9. Students can solve problems as long as they fit one of the following "formulas."

$a + b = ?$

$$\begin{array}{r} a \\ b \\ \underline{+\ c} \end{array}$$

$b - a = ?$

$$\begin{array}{r} b \\ \underline{-\ a} \end{array}$$

Students have over-restricted the definition of addition and/or subtraction.

Example

Given any other situation, students respond, "You can't do it," or resort to "guess and check."

10. Students see addition and subtraction as discrete and separate operations. Their conception of the operations does not include the fact that they are linked as inverse operations.

Example

Students have difficulty mastering subtraction facts because they do not link them to addition facts. Students may know that 6 + 7 = 13 but fail to realize that this fact also tells them that 13 – 7 = 6.

Students can add 36 + 16 = 52 but cannot use addition to help estimate a difference, such as 52 – 36, or check the difference once it has been computed.

11. When adding or subtracting, students misapply the procedure for regrouping.

Example

$$\begin{array}{r} {}^{1\,1\,1\,1} \\ 63{,}842 \\ +\ 24{,}036 \\ \hline 98{,}888 \end{array}$$

12. When subtracting, students overgeneralize from previous learning and "subtract the smaller number from the larger one" digit by digit.

Example

$$\begin{array}{r} 62{,}483 \\ -\ 58{,}575 \\ \hline 16{,}112 \end{array}$$

Multiplication and Division

1. Students have overspecialized their knowledge of multiplication or division facts and have restricted it to "fact tests" or one particular problem format.

Example

Students complete multiplication or division facts assessments satisfactorily but do not apply the knowledge to other arithmetic and problem-solving situations.

2. Students may know the commutative property of multiplication but fail to apply it to simplify the "work" of multiplication.

Example

Students state that $9 \times 4 = 36$ with relative ease, but struggle to find the product of 4×9.

3. Students may know the associative property of multiplication but fail to apply it to simplify the "work" of multiplication.

Example

Students labor to find the product of three or more numbers, such as $8 \times 13 \times 5$, , because they fail to recognize that it is much easier to multiply the numbers in a different order.

4. Students see multiplication and division as discrete and separate operations. Their conception of the operations does not include the fact that they are linked as inverse operations.

Example

Students have reasonable facility with multiplication facts but cannot master division facts. They may know that $6 \times 7 = 42$ but fail to realize that this fact also tells them that $42 \div 7 = 6$.

Students know procedures for dividing but have no idea how to check the reasonableness of their answers.

5. Students have overspecialized during the learning process so that they recognize some multiplication and/or division situations as multiplication or division yet fail to classify others appropriately.

Example

Students recognize that a problem in which 4 children share 24 grapes is a division situation but state that a problem in which 24 cherries are distributed to children by giving 3 cherries to each child is not.

Example

Students recognize "groups of" problems as multiplication but do not know how to solve scale, rate, or combination problems.

6. Students know *how* to multiply but do not know *when* to multiply (other than because they are told to do so, or because the computation was written as a multiplication problem).

Example

Students cannot explain why they should multiply or connect multiplication to actions with materials.

7. Students know *how* to divide but do not know *when* to divide (other than because they are told to do so, or because the computation was written as a division problem).

Example

Students cannot explain why they should divide or connect division to actions with materials.

8. Students do not understand the distributive property and do not know how to apply it to simplify the "work" of multiplication.

Example

Students have reasonable facility with multiplication facts but cannot multiply 12×8 or 23×6.

9. Students apply a procedure that results in remainders that are expressed as "R#" or "remainder #" for all situations, even those for which such a result does not make sense.

> Example
>
> When asked to solve the following problem, students respond with an answer of "10 R2 canoes," even though this makes no sense.
>
> > There are 32 students attending the class canoe trip. They plan to have 3 students in each canoe.
> >
> > How many canoes will they need so that everyone can participate?

10. Students undergeneralize the results of multiplication by powers of 10. To find products like $3 \times 50 = 150$ or $30 \times 50 = 1{,}500$, students must "work the product out" using a long method of computation.

> Example
>
> $$\begin{array}{r} 300 \\ \underline{\times\ 500} \\ 000 \\ 0000 \\ \underline{+\ 150000} \\ 150000 \end{array}$$

11. Students generalize what they have learned about single-digit multiplication and apply it to multidigit multiplication by multiplying each column as a separate single-digit multiplication.

> Example
>
> $$\begin{array}{r} 34 \\ \underline{\times\ 62} \\ 188 \end{array} \qquad \begin{array}{r} {}^{1}128 \\ \underline{\times\ 71} \\ 848 \end{array}$$

12. Students can state and give examples of properties of multiplication but do not apply them to simplify computations.

Example

Students multiply 6×12 with relative ease but struggle to find the product 12×6.

or

Students labor to find the product 12×15 because they do not realize that they could instead perform the equivalent but much easier computation, 6×30.

or

Students have reasonable facility with multiplication facts but cannot multiply 6×23.

13. Students misapply the procedure for multiplying multidigit numbers by ignoring place value.

Example

Students multiply correctly by ones digit but ignore the fact that the 3 in the tens place means 30.

$$\begin{array}{r} 60 \\ \underline{\times\ 38} \\ 480 \\ \underline{+\ 180} \\ 660 \end{array}$$

Students multiply each digit as if it represented a number of "ones." Students ignore place value completely.

$$\begin{array}{r} 47 \\ \underline{\times\ 52} \\ 14 \\ 8 \\ 35 \\ \underline{+\ 20} \\ 77 \end{array}$$

14. Students misapply the procedure for regrouping.

The first step (multiplying by ones) is done correctly, but the same numbers are used for regrouping again when multiplying by 10s whether it is appropriate or not.

Example

$$\begin{array}{r} {}^{3}37 \\ \underline{\times\ 65} \\ 185 \\ \underline{+\ 2120} \\ 2305 \end{array} \qquad\qquad \begin{array}{r} {}^{1\ 4}128 \\ \underline{\times\ 75} \\ 640 \\ \underline{+\ 8860} \\ 9500 \end{array}$$

15. Students overgeneralize the procedure learned for addition and apply it to multidigit multiplication inappropriately.

Original process for addition: When performing addition with regrouping, students first add the amount that is regrouped to the appropriate amount in the topmost addend and then continue by adding the remaining amounts in that place value column.

Inappropriate generalization: When performing multiplication, students first add the amount that is regrouped to the amount in the multiplicand and then multiply (instead of multiplying first and then adding the amount that was regrouped).

Example

Students regroup the 4 tens. They then add 4 to 0 to get 4 and multiply that by 8 (4 • 8) to get 32, instead of multiplying 8 • 0 and then adding the 4, to get 4.

$$\begin{array}{r} {}^{3\ 4}206 \\ \underline{\times\quad 18} \\ 4028 \\ \underline{+\ 2060} \\ 6088 \end{array}$$

16. Students think that division is commutative.

Example

$5 \div 3 = 3 \div 5$

17. Students think that dividing always gives a smaller number.

18. Students think that multiplying always gives a larger number.

19. Students think that they should always divide the larger number into the smaller.

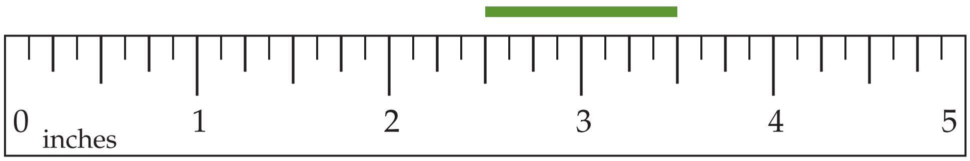
> Example
>
> $4 \div 8 = 2$

20. Students think that the operation that needs to be performed (+, –, ×, ÷) is defined by the numbers in the problem.

Fractions

1. Students have restricted the definition of fractional parts on the ruler so that they think that an inch is the specific distance from 0 to 1 and do not understand that an inch unit of length is an inch, anywhere on the ruler.

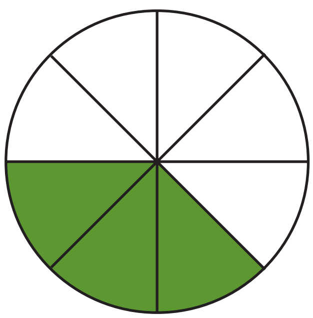
Example



Students say that the line segment is $3\frac{1}{2}$" or that you cannot tell how long the green bar is.

2. Students write fraction as part/part instead of part/whole.

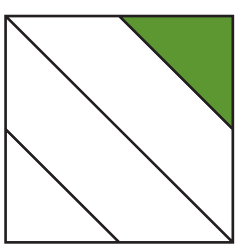
Example



Students say that $\frac{3}{5}$ are shaded.

3. Students do not understand that when finding fractions of amounts, lengths, or areas, the parts need to be equal in size.

Example



Students say that $\frac{1}{4}$ of the square is shaded.

4. Students think that when finding fractions using area models, the equal-sized pieces must look the same.

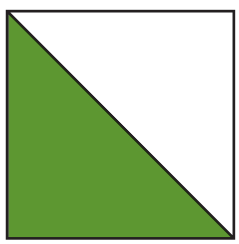
Example

Students say this diagram does not show fourths of the area of the square because the pieces are "not the same (shape)."

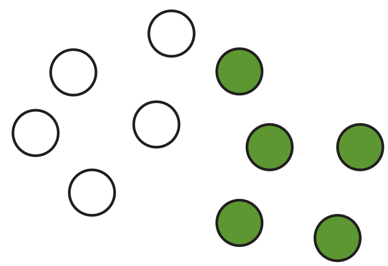
5. Students do not understand that fractions are numbers as well as portions of a whole.

Example

Students recognize $\frac{1}{2}$ in situations like these



one-half of the area is shaded



one-half of the circles are shaded

but cannot locate the number $\frac{1}{2}$ on a number line, or say that "one-half is not a number, it is a part."

6. Students think that mixed numbers are larger than improper fractions because mixed numbers contain a whole number part and whole numbers are larger than fractions.

Example

Students say that $1\frac{4}{5} > \frac{9}{5}$ because whole numbers are larger than fractions.

7. Students are confused about the "whole" in complex situations.

Example

Anna spent $\frac{3}{4}$ of her homework time doing math. She still has $\frac{1}{2}$ hour of homework left to do. What is the total time Anna planned for homework?

When looking at this problem, students can easily become confused about the whole. Is the whole the total time Anna planned for homework, or is the whole one hour?

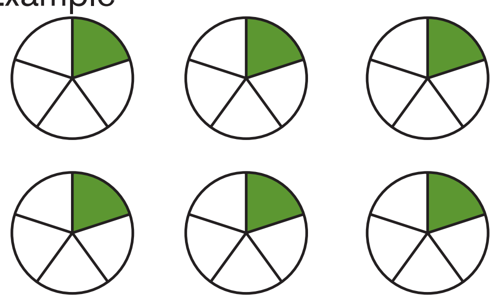
8. Students have restricted their definitions and think fractions have to be less than 1.

Example

When confronted with an improper fraction, students say it is not a fraction because in a fraction the numerator is always less than the denominator.

9. Students count pieces without concern for whole.

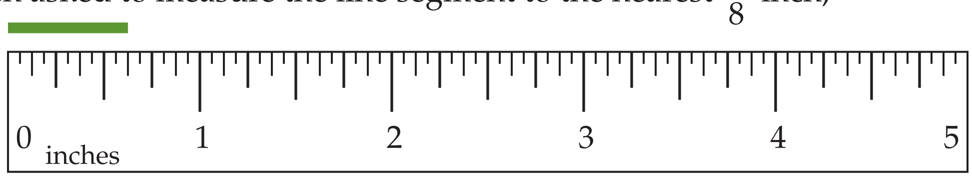
Example



Students say that $\frac{6}{30}$ of a circle is shaded.

Example

When asked to measure the line segment to the nearest $\frac{1}{8}$ inch,



students say that the line segment is $\frac{10}{8}$ inches in length.

10. Students overgeneralize and think that "all $\frac{1}{4}$s (for example) are equal". Students do not understand that the size of the whole determines the size of the fractional part.

Example

Amir and Tamika both went for hikes. Amir hiked 2 miles and Tamika hiked 8 miles.

Students think that when both people had completed $\frac{1}{4}$ of their hikes, they have each walked the same distance because $\frac{1}{4} = \frac{1}{4}$.

11. Students have restricted their definition of fractions to one type of situation or model, such as part/whole with pieces.

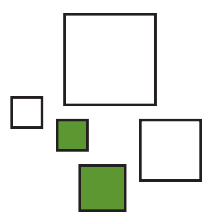
Example

Students do not recognize fractions as points on a number line or as division calculations.

12. Students overgeneralize from experiences with fractions of amounts, lengths, or areas and think that when dealing with a fraction of a set, parts always have to be equal in size.

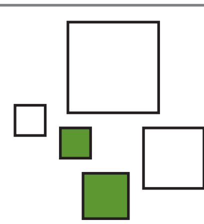
Example

What fraction of the squares is shaded?



Students say, "This is not a fraction because the parts are not equal."

Example



When given this diagram as one of several possible choices for $\frac{2}{5}$, students fail to identify the example as $\frac{2}{5}$.

13. Students overgeneralize fraction notation or decimal notation and confuse the two.

Example

$\frac{1}{4} = 1.4$ or $\frac{1}{4} = 0.4$

14. Students misapply rules for comparing whole numbers in fraction situations.

Example

$\frac{1}{8}$ is bigger than $\frac{1}{6}$ because 8 is bigger than 6.

15. Students overgeneralize the idea that "the bigger the denominator, the smaller the part" by ignoring numerators when they compare fractions.

Example

$\frac{1}{4} > \frac{3}{5}$ because fourths are greater than fifths.

16. Students interpret fractions inappropriately and do not understand that different fractions can name the same amount and are equivalent.

Example

$\frac{2}{3}$ and $\frac{4}{6}$ cannot name the same amount because they are different fractions.

17. Students misapply additive ideas when finding equivalent fractions.

Example

$\frac{3}{8} = \frac{4}{9}$ because $3 + 1 = 4$ and $8 + 1 = 9$

18. When adding two fractions, students add the numerators and multiply the denominators.

Example

$\frac{3}{5} + \frac{1}{2} = \frac{4}{10}$

19. Students overgeneralize results of previous experiences with fractions and associate a specific number with each numerator or denominator when simplifying fractions.

Example

Prime numbers like 2, 3, or 5 always become 1 when you simplify and even numbers are always changed to one-half of their value. Using "rules" like this the student gets correct answers some of the time, like

$$\frac{2}{8} = \frac{1}{4} \text{ and } \frac{4}{6} = \frac{2}{3},$$

but not all the time.

Students ignore the fact that some of the fractions are already in simplest form.

20 When adding fractions, students generalize the procedure for multiplication of fractions by adding the numerators and adding the denominators.

Example

$$\frac{1}{4} + \frac{1}{4} = \frac{2}{8}$$

Note that this error can also be caused by the alternative "conception" that fractions are just two whole numbers that can be treated separately.

21. Students do not use benchmark numbers like 0, $\frac{1}{2}$, and 1 to compare fractions because they have restricted their understanding of fractions to part-whole situations and do not think of the fractions as numbers.

Example

When asked to compare two fractions like $\frac{7}{12}$ and $\frac{5}{13}$ students cannot do so, start cutting fraction pieces, resort to guessing, or perform difficult computations (to find the decimal equivalents or common denominators) instead of comparing both numbers to one-half.

22. Students think that dividing by one-half is the same as dividing in half.

Example

$$4 \div \frac{1}{2} = 2$$

23. When dividing a fraction by a whole number, students divide the denominator by thewhole number.

Example

$$\frac{1}{26} \div 2 = \frac{1}{13}$$

24. Students confuse which number is divided into
or multiplied by which. Students divide the second number by the first.

Example

$$\frac{1}{4} \div \frac{1}{8} = \frac{1}{2}$$

25. When multiplying a fraction by a fraction, students divide both the numerator and the denominator of one fraction by the denominator of the other.

Example

$$\frac{1}{2} \times \frac{4}{6} = \frac{2}{3}$$

26. Students use the numerator and ignore the denominator.

Example

When asked to find $\frac{2}{3}$ of 9 objects, students find 2 objects out of 9.

27. Students think that the denominator is always the number of objects, even if the fraction has been reduced.

Example

Students read $\frac{3}{4}$ of 8 objects as $\frac{3}{8}$.

28. When writing a fraction, students compare two parts to each other rather than comparing one part to the whole.

29. Students think that decimals are bigger than fractions because fractions are really small things.

30. Students think that you cannot convert a fraction to a decimal—that they can not be compared.

31. Students think that doubling the size of the denominator doubles the size of the fraction.

32. Students think that multiplying the numerator and the denominator by the same number increases the value of the fraction.

33. Students think that dividing the numerator and the denominator by the same number reduces the value of the fraction.

34. When subtracting mixed numbers, students always subtract the smaller whole number from the larger whole number or subtract the smaller fraction from the larger fraction.

35. When multiplying fractions, students multiply the numerator of the first fraction by the denominator of the second, and add the product of the denominator of the first and the numerator of the second.

Example

$$\frac{3}{4} \times \frac{5}{7} = (3 \times 7) + (4 \times 5) = 41$$

36. When multiplying fractions, students use the "invert and multiply" procedure by inverting the second fraction and multiplying.

Example

$$\frac{3}{4} \times \frac{5}{7} = \frac{3}{4} \times \frac{7}{5} = \frac{21}{20}$$

37. When dividing fractions, students divide the numerators and divide the denominators.

Example

$$\frac{6}{7} \div \frac{2}{7} = \frac{3}{1}$$

Decimals

1. Students misapply knowledge of whole numbers when reading decimals and ignore the decimal point.

Example

Students read the number 45.7 as, "four fifty-seven" or "four hundred fifty-seven."

2. Students misapply the procedure for rounding whole numbers when rounding decimals. Students round to the nearest ten instead of the nearest tenth, etc.

Example

Round 3045.26 to the nearest tenth. Students respond, "3050" or "3050.26"

3. Students misapply rules for comparing whole numbers in decimal situations.

Example

0.058 > 0.21 because 58 > 21

2.04 > 2.5 because it has more digits

4. Students read the marks on the ruler as whole numbers.

5. Students mix decimals and fractions when reading decimal numbers.

6. Students think that a decimal is just two ordinary numbers separated by a dot.

Example

The decimal point in money separates the dollars from the cents.

100 cents is $0.100.

Example

The decimal point is used to separate units of measure. 6 meters is 1 meter 6 centimeters.

7. When adding a sequence, students add the decimal part separately from the whole number part.

Example

0.50, 0.75, 0.100 rather than 0.50, 0.75, 1.00

8. Students verbalize decimals as whole numbers without the place value designated.

Example

"point ten" instead of "point one zero"

"point twenty-five" instead of "twenty-five hundredths" or "point two five"

9. Students add or subtract without considering place value, or start at the right as with whole numbers.

Example

4.15 + 0.1 = 4.16 or 12 – 0.1 = 11

10. Students misunderstand the use of zero as a placeholder.

Example

1.5 is the same as 1.05.

11. Students think that decimals with more digits are smaller because tenths are bigger than hundredths and thousandths.

Example

.845 is smaller than .5

12. Students think that decimals with more digits are larger because they have more numbers.

Example

1,234 is larger than 34 so 0.1234 is larger than 0.34

13. Students mistakenly apply what they know about fractions.

Example

$\frac{1}{204} > \frac{1}{240}$, so 0.204 > 0.240

14. Students mistakenly apply what they know about whole numbers.

> Example
>
> $600 > 6$, so $0.600 > 0.6$

15. Students believe that zeros placed to the right of the decimal number changes the value of the number.

> Example
>
> 0.4 is smaller than 0.400 because 4 is smaller than 400, or 0.81 is closer to 0.85 than 0.81 is to 0.8.

16. Students believe that a number that has only tenths is larger than a number that has thousandths.

> Example
>
> $0.5 > 0.936$ because 0.936 has thousandths and 0.5 has only tenths.

17. When multiplying by a power of ten, students multiply both sides of the decimal point by the power of ten. When dividing by a power of ten, students divide both sides of the decimal point by the power of ten.

> Example
>
> $6.9 \times 10 = 60.90$
>
> $70.5 \div 10 = 7\frac{1}{2}$

18. Students do not use zero as a placeholder when ordering numbers or finding numbers between given decimals that have different numbers of significant digits.

> Example
>
> There are no numbers between 2.2 and 2.18.

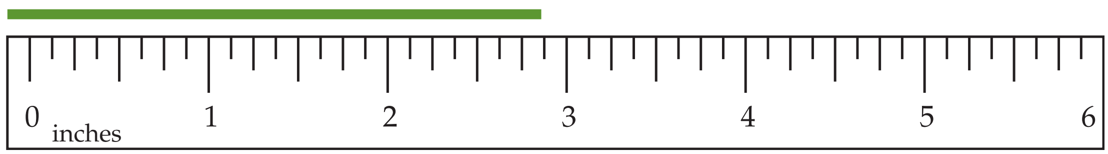
19. Students do not recognize the denseness of decimals.

> Example
>
> There are no numbers between 3.41 and 3.42.
>
> There is a finite number of expressions that will add or subtract to get a given decimal number.

Measurement

1. Students begin measuring at the end of the ruler instead of at zero.

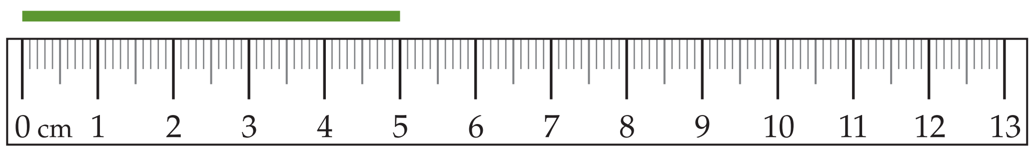
Example



Student response: line segment is $2\frac{7}{8}$ inches in length.

2. When measuring with a ruler, students count the lines instead of the spaces.

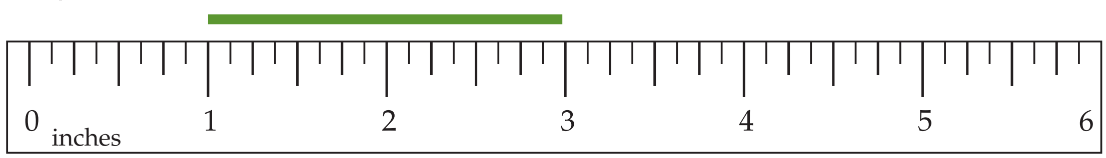
Example



Student response: line segment is 6 centimeters in length.

3. Students begin measuring at the number 1 instead of at zero and do not compensate.

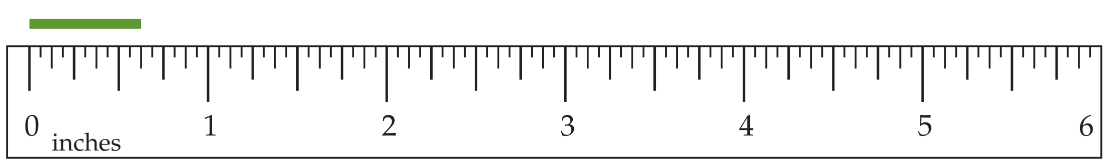
Example



Student response: line segment is 3 inches in length.

4. Students count intervals shown on the ruler as the desired interval regardless of their actual value.

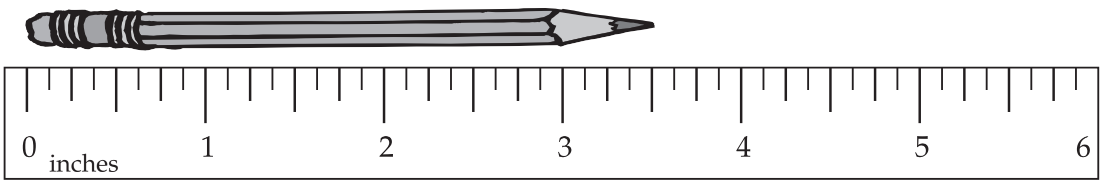
Example



Student response: line segment is $\frac{10}{8}$ inches in length.

5. Students fail to interpret interval marks appropriately.

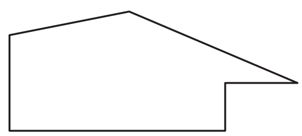
Example



When asked to measure the pencil to the nearest $\frac{1}{8}$ inch, the student responds with $3\frac{3}{8}$ inches or $3\frac{5}{8}$ inches because he fails to interpret the $\frac{1}{2}$ inch mark as a $\frac{1}{8}$ inch mark.

6. Students try to use the formula for finding the perimeter of rectangular shapes on nonrectangular shapes.

Example



Students measure "length" (horizontal distance across) and "width" (vertical distance), then calculates perimeter as 2 × length + 2 × width.

7. Students confuse area and perimeter.

Example

When asked to find the area of a rectangle with dimensions of 12 cm × 4 cm, students add 12 + 4 + 12 + 4 = 32 cm.

Example

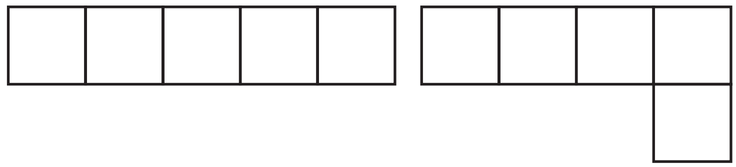
When asked to find the perimeter of a rectangle with dimensions of 8 inches × 7 inches, students multiply 8 × 7 = 56 inches (or square inches).

Example

Students think that perimeter is the sum of the length and the width because area is length times width.

8. Students think that all shapes with a given perimeter have the same area or that all shapes with a given area have the same perimeter.

Example

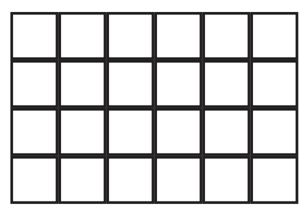


Since both of these shapes have an area of 5 square units and a perimeter of 12 units, students conclude that all shapes with an area of 5 square units have a perimeter of 12 units or that all shapes with a perimeter of 12 units have an area of 5 square units.

9. When counting perimeter of dimensions of shapes drawn on a grid, students count the number of squares in the border instead of the edges of the squares.

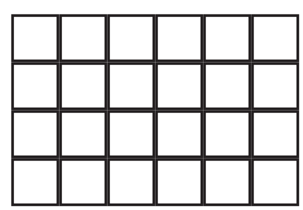
Example

When asked to find the perimeter of this rectangle, students responds with 16 or 16 units or 16 squares.



Example

When asked to sketch a 4 cm × 5 cm rectangle, students sketch a 4 cm × 6 cm rectangle.



10. Students overgeneralize base-10 and apply it to measurements inappropriately.

Example

When asked to change 1 hour 15 minutes to minutes, students respond with 115 minutes or with 25 minutes.

Example

When asked to change 1 hour 15 minutes to hours, students respond with 1.15 hours.

11. Students believe that the size of a picture determines the size of the object in real life.

12. Students have a limited number of units of measure that they know and understand and use those units inappropriately.

Example

Students use wrong notation or labels.

Example

Students choose inappropriate unit of measure or inappropriate measuring tool for task.

Example

When faced with a unit he does not know, students ignore the unit, guess, or do nothing.

13. Students do not understand elapsed time.

Example

Students can read a clock or a calendar but do not apply this knowledge to elapsed time problems.

Example

When faced with an elapsed time problem, students guess or do nothing.

14. Students lack "benchmarks" that allow them to estimate measures.

Example

When faced with a problem that asks students to estimate a measurement, students guess or do nothing.

Percents

1. Students do not understand that percents are a number out of one hundred or that percents refer to hundredths.

2. Students think that percents cannot be greater than 100.

Example

Students write 1.45 as .145%.

3. Students do not realize that one whole equals 100%.

4. Students do not know which operation to use when working with percents.

5. Students have difficulty identifying the "whole" that the percent refers to.

6. Students lack the understanding that percent increase has a multiplicative structure.

7. Students have difficulty using zero as a placeholder when writing a percent as a decimal.

Example

Students write 6% as 0.6

8. Students find the increase or decrease instead of the final amount.

9. Students treat percents as though they are just quantities that may be added like ordinary discount amounts.

10. Students do not recognize the "whole" the percent refers to, and that a second percent change refers to a different "whole" than the first.

11. Students think an increase of n% followed by a decrease of n% restores the amount to its original value.

Functions and Graphs

1. Students confuse the two axes of a graph.

2. Students do not understand the meaning of points in the same position relative to one of the axes.

3. Students think that the points on the graph stay in the same position even if the axes change.

4. Students think that graphs are "pictures" of situations, rather than abstract representations.

> **Example**
>
> Students think that a speed graph of a bicycle coasting downhill and then uphill resembles the hill: first going down and then up.
> A graph with negative slope means the object is falling.
> If the graph is rising, the object is moving upward.
> If the graph changes direction, the object changes direction.
> If two lines on a graph cross, the paths of the objects cross.

5. Students think that graphs always go through (or begin at) the origin.

6. Students think that graphs always cross both axes.

7. Students focus on some attributes of a situation and ignore others.

> **Example**
>
> Students note the existence of local minima but ignoring relative positions or values.

8. Students read the y-axis as speed even when it represents a different parameter.

9. Students think that the greatest numbers labeled on the axes represent the greatest values reached.

> **Example**
>
> If a graph of a race has the distance axis labeled up to 120 meters, the race is for 120 meters (even if it is a 100-meter race).

10. Students think that all sequences are linear or increasing and linear.

11. Students do not discriminate between linear and non-linear sequences.

12. Students do not discriminate between increasing and decreasing sequences.

13. Students do not discriminate between linear and proportional sequences.

14. Students think that linear problems are always proportional.

Example

Students believe that a stack of ten nested cups will be twice the height of five nested cups.
Students believe that if $y = ax + b$ then doubling x will double y.

15. Students think that $n \bullet n = 2n$.

16. Students confuse a table showing an actual situation (for example, 4 teams in a tournament) with a table that represents all of the games in the tournament.

17. Students are unable to generalize the nth case. Students only work with real numbers.

18. Students are always trying to work a problem from the visual to the equation.

19. Students think that the results are random. There is no pattern.

20. Students try to substitute numbers rather than write a formula for each sequence.

Expressions and Equations

1. Students solve problems from left to right no matter what the operations are.

2. Students divide the whole expression by the denominator rather than just the part that is the fraction.

3. Students disregard exponents when calculating expressions.

Example

$2 \bullet 4^3 = 8$

4. Students multiply by an exponent rather than multiply the expression by itself.

Example

$6^2 = 6 \bullet 2$, reading "6 squared" as "6 doubled"

5. Students do not use standard algebraic conventions. Students write expressions as in arithmetic.

Example

$x \bullet 5 = 1$ rather than $5x = 1$

6. Students do not use parentheses when they are necessary to interpret the expression.

Example

$5 + x \bullet 5$ rather than $5(x + 5)$

7. Students do not use standard algebraic conventions for exponents. Students write expressions as in arithmetic.

Example

$x \bullet x$ rather than x^2

8. When simplifying expressions, students write like terms next to each other but do not add them.

Example

$4 + 2x + 6 + x = 10 + 2x + x$

9. Students add unlike terms.

Example

$10 + 2x = 12x$
$10x^2 + 2x = 12x^2$

10. Students do not distribute multiplication to all terms in the parentheses (misuse of the distributive property).

Example

$2(x + 6) = 2x + 6$

11. Students distribute multiplication by a negative term (or subtraction) to only the first term in an expression.

Example

$x - 2(x + 6) = x - 2x + 12 = -x + 12$

12. Students read the equality sign as "makes" without considering what is on the other side of the equation.

Example

$9 + 10 = x + 9$ as $9 + 10 = 19$

13. Students confuse negative signs when adding and subtracting terms.

Example

$2x + 12 = x$ as $x = 12$ rather than $x = -12$

14. Students think that a variable can only stand for one particular number.

15. Students think that different variables must stand for different numbers.

Example

$x + 5 \neq y + 5$ because x and y cannot be the same number.

16. Students think that a variable represents an object rather than a number.

Example

"If there are d days in w weeks, then $w = 7d$ because a week equals seven days." This interpretation is incorrect because w and d are identified as the "objects" week and day, rather than as the numbers of weeks and days. A correct equation would be $d = 7w$, because we would have to multiply the number of weeks by seven to get the number of days.