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A Journey in Mathematics Education Research

Insights from the Work of Paul Cobb

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Paul dedicates this book to his mother, to the memory of his father, and to Jenny, his partner in life for over 35 years.

Foreword

Paul Cobb's contributions to research in mathematics education in the past quarter century have been outstanding and exemplary. The progression of his understanding represented in this very valuable and enjoyable volume corresponds in important ways to a progression of concepts and methods that developed in the field of mathematics education research and, more generally, in the learning sciences. Cobb's trajectory is a prototype of the field's progress. He has played, and continues to play, a major role in shaping the problems, methods, and explanations that the field has developed, and, in turn, the problems, methods, and explanations he has developed have been shaped by those that others in the field have developed.

In this foreword, I hope to make two points, both of which relate Cobb's extraordinary contributions to the advancement of the learning sciences, especially in mathematics education research, during the time that he (and I) have been active members of these communities. First, I describe my understanding of some ways in which the field has progressed. Cobb's progression is prototypical and seminal in the field's conceptual trajectory during this period, as I understand it, although I believe the field's progress has been less linear than Cobb's has been, at least as it is presented in the reconstruction he has given us in this book. The second point I try to make here is primarily methodological, relating to recent and current discussions aimed to strengthen the scientific quality of educational research. By reflecting on the progression of methods in Cobb's research program, I believe that we gain in our understanding of what makes research scientific, conceptually productive, and potentially valuable for efforts to improve educational practice.

Cobb's, and the Field's, Learning Trajectory

The view of learning that Cobb developed (explicitly, in the middle four papers of this collection) focuses on progressive changes in the practices of a group and changes in the participation of individuals in those changing practices. A strong methodological claim, stated by Cobb, is that any event in a classroom can be considered productively both by focusing on the practices that are enacted collectively and on the participation of individuals in those practices. I believe that the same can be said about the kind of learning that constitutes scientific progress. The materials in this book focus appropriately on advances in concepts, findings, and methods that were achieved in the contributions of Cobb and his collaborators. Of course, during the time that Cobb has been working, the field of mathematics education research has also advanced, for which Cobb and his colleagues bear a considerable share of the responsibility.

The metaphor of a journey is apt. Cobb and his colleagues have traveled along a conceptual and methodological pathway, and the opportunity this book gives us to travel their route provides valuable insights into important features, structures, and resources of the domain of mathematics education research. Cobb encourages a spatial metaphor when he states, more than once, that he had "modified [his] theoretical position." There are different kinds of journeys, of course, and the journey that Cobb and his colleagues have traveled has been more than a tour. It has had the property of a quest, in which each new location has presented a challenge that had to be overcome. But this modern scientific quest lacks a property of the classical quest narrative. Classically, the challenge of each stage of a quest is met if the protagonist survives, and the quest succeeds because the protagonist and (usually some of) his colleagues are strong, brave, pious, clever, or lucky enough to avoid a catastrophe that would destroy them all. We scientists usually don't risk our lives to solve the problems we work on. In addition, Cobb's, and our, quest has an important property that classical quests lack: the achievements that we accomplish along the way are cumulative. By making progress toward understanding some aspects of phenomena that we study, we develop conceptual and methodological resources that we carry along and that can be utilized when we encounter the next challenge. Indeed, the resources that we develop as we go along are influential in shaping our understanding of what subsequent challenges are.

Challenge #1: Explaining Elementary Understanding of Numbers. Cobb's first paper in this collection presents part of the product of work done by Cobb with his mentors, Steffe and von Glasersfeld, while Cobb was a doctoral student. The challenge, as it was understood then, was to advance scientific understanding of children's early understanding of number beyond the conclusions that Piaget and his associates had provided (e.g., Piaget, 1942). Steffe and Cobb, with von Glasersfeld (1988) provided a stunning analysis that focused on units that children constructed and used in counting operations, which progressed from external objects to motor actions to entirely mental entities as their understanding developed. Methodologically, this study was an exemplary adaptation of a teaching experiment, which is the focus of the reprinted paper (Cobb & Steffe, 1983).

The understanding provided by research conducted at about this time included models of children's understanding of additive and subtractive relations between quantities (Carpenter & Moser, 1983; Nesher, 1983; Riley, Greeno & Heller, 1983), as well as Gelman and Gallistel's (1978) contribution that showed that preschool children's conceptual understanding of number includes implicit cognizance of principles that are represented explicitly in mathematical formulations of the concept. Anderson (1983) constructed a computational model that simulates the information structures involved in solving geometry proof exercises, along with hypotheses about processes of learning. The theoretical and methodological achievements of this early cognitive period were carried forward into later work, including Cobb's detailed designs and analyses of learning trajectories, for which Steffe et al.'s (1988) developmental trajectory provided a kind of prototype.

While contributions in the early 1980s that drew on information processing and cognitive developmental approaches were complementary and convergent, there was another program that provided an antithesis to the cognitive thesis. The spearhead of this counter-program was contributed by Lave, Murtaugh, and de la Rosa (1984), who argued that reasoning with and about quantities should be understood as an achievement jointly produced by individuals and resources in the environment, not just as operations on symbols in mental representations, an argument that Lave (1988) developed extensively. Bauersfeld (1980) also had raised issues that challenged the adequacy of analyzing mathematics learning without taking into account aspects of interaction in the classroom. Lave, Smith, and Butler (1988) argued that problem solving in school mathematics should be understood as an everyday practice, embedded in school activity. More generally, Searle (1980) and Suchman (1985) presented critical arguments against the assumption that cognition can be understood as occurring in a self-contained mental system, and Lave and Wenger (1991) reviewed analyses of apprenticeship learning and proposed a framing of learning as trajectories of participation in communities of practice.

Challenge #2: Explaining Learning in Interaction. The analyses focused on individuals' understandings that Cobb and others developed were evidently incomplete. Cobb's response to this challenge included the papers reprinted in Parts II, III, and IV of this volume. Cobb and his colleagues concluded that students' emotional acts depend (partly) on properties of classroom practices, not just on their individual emotional tendencies. They designed and studied curriculum sequences, considering them as occasions for the students and teacher to progress through a trajectory of practices, corresponding to increasingly sophisticated understanding of mathematical concepts and principles. And they analyzed the progress of mathematical understanding achieved by a class as the group advanced through increasingly sophisticated practices, but also analyzed variations between individual students in their participation in the practices, thereby showing that considering learning at the level of the classroom and at the level of individual students are not only compatible perspectives, but that it is productive to examine relations between findings that result from framing analyses with each of them.

During the years that Cobb and his colleagues wrestled with the challenge of reconciling individual and social levels of analysis, others in the field were similarly engaged. The result was that early in the twenty-first century, the field had developed a strong beginning toward a body of concepts, principles, and methods for understanding, designing, and studying productive classroom environments, especially in mathematics. Examples include Lampert's (1990, 2001) and Ball's (Ball & Bass, 2000) analyses of the interactions they organized in their classrooms, Boaler's (1997/2002) findings that different classroom practices resulted in different "forms of mathematical knowledge," which she documented with assessments and interviews with individual students, Brown and Campione's (1994) design and studies of the curriculum and learning environment called Fostering Communities of Learners, and Engeström's (2001) design and study of learning by two groups of medical professionals, interpreted as participants in two activity systems, through interaction organized to address and solve a problem of practice.

While Cobb's studies documented and interpreted variation in the ways that students participated in classroom practices, this variation was unexplained by the concepts Cobb and his colleagues presented in the papers reprinted in Parts II, III, and IV. Also during the 1990s, though, others in the field began to develop concepts of individual identity in perspectives of participation in activity systems. Holland, Lachicotte, Skinner, and Cain (1998) developed a concept of positional identity, characterizing patterns of participation in which individuals comply with or resist prevalent ways of participation in what they called figured social worlds. Wenger (1998) characterized individual identities as trajectories of participation within and across communities of practice.

Challenge #3. Explaining Personal Continuities Across Situations. The perspective that emphasizes students' activity and learning as participation clarifies many aspects of learning, but it also leaves many questions unanswered. Cobb and his colleagues' interpretive framework for understanding aspects of identity and issues of equity provides one way to begin to account for sources of difference between ways of participating that different students develop. The idea that an individual's identity is co-constructed and emerges in interaction in an activity system was also developed in recent studies by Gresalfi (2004; 2009) and by Nasir and Hand (2008).

The general proposition that people learn by participating in practices applies to teachers as well as to students. Research on teachers' learning has been an active program, and a general finding is that teachers' efforts to develop more effective practices benefit from interacting in a committed professional community in which they reflect on their teaching aims, accomplishments, and challenges (e.g., Little, 1994).

Challenge #4. Explaining Teachers' Progress in Developing Practices. In their discussion of institutional contexts, Cobb and his colleagues contribute observations and thoughtful reflections on two cases in which school leaders differed in supporting teachers opportunities to learn in interaction with each other (and with university-based designer-researchers) to significantly different extents.

The Scientific Quality of Cobb's Research

In his introduction to Part I of this volume, Cobb writes, ". . . there is no substitute for sustained, first-hand engagement with the phenomena that we seek to understand." I take this to be one of two fundamental commitments of Cobb's research methodology. The other is the importance of constructing examples of phenomena that are worthy of close study and exemplary of learning resources that would strengthen our institutions of education.

I believe that the test of a methodology is whether its use is productive in producing findings and especially (following Toulmin, 1972) in advancing the field's explanatory capabilities. The field of mathematics education research has progressed very significantly in the years that Cobb and his colleagues have been active in it, and Cobb and his colleagues' contributions have played a major role in that progress. Their, and the field's, progress has included advances in empirical findings, explanatory concepts, and methods, and Cobb and his colleagues have been particularly helpful in reflecting on properties of the methods they and others use that contribute to their and others' success. As the scope of scientific research has expanded throughout the history of science, methods that provide empirical information to constrain explanatory theories have been assembled and tailored to characteristics of the domain. The methods shown in the sequence of studies reprinted in this volume illustrate this, as the transition from intensive interviewing of a small group of children, to intensive observation of classroom activity along with interviewing individuals, as Cobb and his colleagues shifted their theoretical focus from understanding mathematics by individuals to including growth of mathematical understanding by classroom groups of students, to mathematical identities of students, to conditions that support teachers in changing their practices.

This progression of substantive issues and methods has been a remarkable achievement in Cobb and his colleagues' research, as it has been for the field of mathematics education research and development. We are fortunate to have had Paul Cobb's participation and leadership during this past quarter century.

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James G. Greeno

Acknowledgments

We wish to acknowledge the assistance of a number of people who contributed significantly to the successful development and production of this book. Special thanks go to Alan Bishop for suggesting the idea of a book that incorporates major aspects of Paul Cobb's work and for inviting us to be the editors.

The challenge we faced in developing this book was to capture the essence of Cobb's extensive body of work to date in a relatively small number of pages. We are especially indebted to Paul Cobb for his part in every aspect of the preparation of the book. He worked closely with us in developing the overall plan for the book and suggested the names of colleagues with whom he had worked closely on various projects. He authored the introduction to each part of the book, incorporating the input of these colleagues. However, we are indebted to Paul most of all for his extraordinary contributions to mathematics education research which provided the occasion for the book.

We also wish to thank Paul's colleagues who so generously and eagerly contributed their recollections and detailed accounts of the research teams' inner workings. It is a tribute to Paul's ability to inspire young researchers that many of them were doctoral students or post-doctoral associates at the time they first worked with him. Many of these colleagues told us they had retained comprehensive accounts of project meetings that they reviewed in preparing their recollections.

We are also grateful to James Greeno for contributing the Foreword to this book. Greeno's extensive experience in educational research in general, and in mathematics education research, in particular, make him uniquely qualified to view Cobb's work from a broader perspective and to situate it within the larger body of educational research. In addition, we thank the staff at Springer who provided invaluable assistance throughout the preparation and publication of the book. Finally, we thank the members of our families for their support throughout this entire project.

The project of developing this book has been an opportunity for us to reflect on our own close working relationships with Paul and recognize the contributions he has made to us individually over the years. Two of us, Erna and Koeno, have collaborated with Paul for many years. Anna has maintained a close professional relationship with Paul for over two decades. Each of us has benefited greatly from the opportunities and challenges presented by working with Paul to wrestle with difficult research questions and pragmatic problems. An added benefit is that we have all become good friends with Paul and with each other. For this we are grateful.

Hammond, Indiana — Erna Yackel
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About Paul Cobb

Paul Cobb completed an undergraduate honors degree in mathematics and a 1-year course for prospective secondary mathematics teachers at Bristol University in the United Kingdom. He then taught high-school mathematics for two years before moving to the United States with his wife Jenny in 1978 to commence graduate studies in mathematics education at the University of Georgia. In the course of these studies, he participated in a 2-year constructivist teaching experiment directed by Leslie Steffe. His dissertation study drew on the data generated in this experiment and focused on young children's development of thinking or derived fact strategies for solving a range of elementary arithmetic problems.

Cobb accepted his first faculty position at Purdue University in Indiana in 1983 and soon established a close collaborative relationship with Erna Yackel and Terry Wood. Together, they conducted a series of classroom design experiments that aimed to support and understand second and third graders' development of central arithmetical concepts and strategies. These classroom experiments in turn constituted the context for 10-year collaboration with Koeno Gravemiejer that commenced in 1990. The work conducted at Purdue University resulted in analyses of classroom social and sociomathematical norms, as well as several instructional sequences and associated domain-specific instructional theories.

Cobb continued to work closely with Yackel and Gravemeijer in conducting classroom design experiments after he moved to Vanderbilt University in Nashville, Tennessee in 1992. These experiments initially focused on arithmetical learning in the early elementary grades and then later on statistical learning in the middle grades. Kay McClain contributed to the initial experiments as a doctoral student, and later became both a faculty colleague at Vanderbilt University and a lead investigator for the statistics experiments. During his first 6 years at Vanderbilt University, Cobb worked with Janet Bowers and Michelle Stephan, two doctoral students, to develop the notion of a classroom mathematical practice. In the context of the statistics design experiments, he worked closely with Lynn Liao Hodge, also a doctoral student, to investigate how classrooms might be organized to ensure that learning opportunities are distributed equitably. This issue has continued to be a focus of his work, first in collaboration with Melissa Gresalfi and currently with Kara Jackson, both post-doctoral research associates.

In 2000, Cobb and McClain began a 5-year design experiment in which they sought to support and investigate the learning of two groups of middle-grades mathematics teachers. Initial analyses of this work have been published in recent years but a number of analyses that are being conducted with Melissa Gresalfi and with three current or former doctoral students, Chrystal Dean, Jana Visnovska, and Qing Zhao, are still ongoing. One of Cobb's primary goals in this teacher development experiment was to begin to understand how the organizational conditions of the schools in which teachers work support and constrain the types of instructional practices that they develop. This issue is central to his current work in which he is collaborating with a policy researcher, Thomas Smith, to investigate what it takes to support the improvement in the quality of mathematics teaching (and thus students' learning) on a large scale. This ongoing work is being conducted in partnership with 120 middle-grades teachers and 80 school and district leaders from four urban school districts that serve a total of 360,000 students.

Cobb is the co-author or co-editor of seven books, and is the author or co-author of 150 journal articles and book chapters. He received the award for the outstanding article published in the *Journal for Research in Mathematics Education* in 1996, and was elected to the US National Academy of Education in 2002. He received the Hans Freudenthal Medal for his cumulative research program over the prior 10 years from the International Commission on Mathematics Instruction (ICMI) in 2005, and became the first incumbent of the Peabody Chair in Teaching and Learning at Vanderbilt University in 2007.

About the Contributors

Janet Bowers is Associate Professor in the Department of Mathematics and Statistics at San Diego State University. She worked with Paul Cobb between 1992 and 1996 while earning her PhD at Vanderbilt University. As her major advisor, Cobb supported her efforts to conduct developmental research focusing on the design and implementation of computer software to enhance students' conceptual understanding of place value. Cobb's focus on the social setting of the classroom provided a fundamental, guiding design principle that enabled the research and development team (Cobb, Yackel, Gravemeijer, McClain, Whitenack, Stephan and Bowers) to anticipate how students' collective reflections on their actions with technology might support rich shifts in their mathematical understandings. At the time, the novel perspective of technology as a social mediator represented a significant shift away from the prevailing view of computers as individualized tutors.

Paul Cobb completed an undergraduate honors degree in mathematics and a 1-year course for prospective secondary mathematics teachers at Bristol University in the United Kingdom. He then taught high-school mathematics for 2 years before moving to the United States with his wife Jenny in 1978 to commence graduate studies in mathematics education at the University of Georgia. In the course of these studies, he participated in a 2-year constructivist teaching experiment directed by Leslie Steffe. His dissertation study drew on the data generated in this experiment and focused on young children's development of thinking or derived fact strategies for solving a range of elementary arithmetic problems.

Chrystal Dean is Assistant Professor of Mathematics Education at Appalachian State University. Her research interests include pre-service and in-service mathematics teacher education. Currently, she studies the emergence and concurrent learning of professional teaching communities in both face-to-face and online professional development contexts. Paul Cobb was Dean's advisor at Vanderbilt University while she pursued her PhD in Mathematics Education, which she received in August 2005. During her graduate work, Dean's collaboration with Cobb focused on the first 2 years of a 5-year professional development design experiment working with a group of middle-school mathematics teachers in the content area of data analysis. This design experiment was often and adequately described as "building a plane while flying it." Naturally, this "opportunity for growth" included frequent plane trips on the best airline (Southwest), prayers while traveling on Interstate 40, and a great deal of "rubbishing."

Koeno Gravemeijer is Professor of Science and Technology Education at the Eindhoven University of Technology, The Netherlands. Earlier he held a private chair as Professor of Mathematics Education at Utrecht University. From 1987 through 2008 he was affiliated with the Freudenthal Institute, the cradle of realistic mathematics education (RME). His experience with RME and instructional design was the reason Paul Cobb invited him for a visiting professorship at Purdue University in 1991. This formed the starting point for a productive collaboration of more than a decade in a continuous series of National Science Foundation and Office of Educational Research funded research projects. The RME approach to instructional design and the socio-constructivist perspective on teaching and learning proved to be an inspiring and productive combination that resulted in an elaboration of both RME theory and design research.

Melissa Gresalfi is Assistant Professor in the Learning Sciences at Indiana University. She earned a PhD in Educational Psychology from Stanford University in 2004, and spent the following 2 years doing post-doctoral research with Paul Cobb. During that time, her work focused on issues of mathematical identity with students and teachers. More generally, Gresalfi's research considers cognition and social context by examining student learning as a function of participation in activity settings. With support of the National Science Foundation, the MacArthur Foundation, the Spencer Foundation, and the US Department of Education, she has investigated the development of dispositions toward learning in mathematics classrooms by examining how opportunities to learn are constructed in mathematics classrooms, and how, when, and why different students take up those opportunities. Using this lens, Gresalfi has explored the extent to which classroom practices are equitable and examined categories such as race, gender, and previous mathematical experience as they arise in interaction.

Lynn Liao Hodge is Assistant Professor of Mathematics Education at the University of Tennessee. She is interested in investigating how classroom practices create opportunities for all students to develop both an appreciation and a deep understanding of mathematics. In addition, she is interested in efforts that seek to increase the participation of women and minority students in mathematics and engineering. During her doctoral studies at Vanderbilt, she collaborated with Paul Cobb on two statistics design experiments. Her dissertation, under the direction of Cobb, explored issues of equity and identity in mathematics classrooms.

Kay McClain is currently a math interventionist and instructional specialist with the Madison School District, Phoenix, AZ. She started her career as a classroom teacher in Alabama where she received the Presidential Award for Excellence in Teaching Mathematics. She subsequently attended Vanderbilt University where she had the opportunity to work with Paul Cobb for 18 years on a variety of issues pertaining to mathematics teaching and learning.

Teruni Lamberg is Associate Professor of Elementary Mathematics Education at the University of Nevada, Reno. She received her doctorate from Arizona State University and worked as a research associate/post doctorate fellow with Paul Cobb at Vanderbilt from 2001 to 2004. She worked on Paul Cobb's National Science Foundation funded projects titled "Developing and Sustaining Technology-Intensive Classrooms Where Mathematics is Learned with Understanding" from 2001 to 2002; and she also worked on the project titled "Supporting and Sustaining the Learning of Professional Teaching Communities in the Institutional Setting of the School and School District" from 2003 to 2004. Her work involved collecting and analyzing data on Professional Learning Communities and the institutional context of the school district. Her current research interests include teacher education, institutional context, and children's mathematical thinking, particularly in the area of rational numbers and integrating technology in education.

Anna Sfard is Professor of Mathematics Education at the University of Haifa, Israel and the first holder of the Lappan-Phillips-Fitzgerald endowed chair at Michigan State University. Her research, focusing on the development of mathematical discourses in individual lives and in the course of history, has been recently summarized in the book *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. She is the recipient of the 2007 Freudenthal Medal for research in mathematics education.

Leslie P. Steffe is distinguished research professor in the Department of Mathematics and Science Education at the University of Georgia where he has directed a succession of research projects involving analysis of young children's number concepts and mathematical thinking. Steffe served as Paul Cobb's major professor during his doctoral studies at the University of Georgia.

Michelle Stephan teaches mathematics at Lawton Chiles Middle School in Oviedo, Florida and is a University of Central Florida research scholar. Her current interests involve writing middle-school instructional sequences using Realistic Mathematics Education theory, providing professional development (including cognitive coaching) to support middle-school teachers' incorporation of an inquiry teaching approach, and creating and sustaining mathematical professional learning communities. Stephan worked with Cobb from 1994 to 1998 as a doctoral student and then afterwards to publish work on design research as a *Journal for Research in Mathematics Education* monograph.

Jana Visnovska is Mathematics Education Lecturer at the School of Education of the University of Queensland, Australia. As Paul Cobb's doctoral student and research assistant, she collaborated on longitudinal studies "Supporting and Sustaining the Learning of Professional Teaching Communities in the Institutional

Setting of the School and School District" (2002–2005), and "Designing Schools and Districts For Instructional Improvement In Mathematics" (2006–2007) at Vanderbilt University. Her research interests include classroom instructional practices in which teachers provide all students access to significant mathematical ideas and the design of tools and environments that support teachers in developing such practices.

Terry Wood was professor of Mathematics Education at Purdue University until her untimely death in 2010. She collaborated with Cobb during the early years of his classroom-based research focusing her attention on the teacher's learning. Subsequently her research emphasis was on teacher education.

Erna Yackel is Professor of Emerita of Mathematics Education at Purdue University Calumet. She earned a PhD in Mathematics Education from Purdue University in 1984. In 1986 Paul Cobb invited her, along with Terry Wood, to join him in conducting a year-long classroom teaching experiment in a second-grade classroom. Yackel's collaboration with Cobb continued for more than a decade. Since retirement from Purdue University Calumet, Yackel has devoted her efforts to working with elementary school teachers in Northwest Indiana to foster their development of an inquiry form of instructional practice in mathematics.

Qing Zhao is a graduate student at Vanderbilt University. She is interested in learning about ways to support teachers' learning across the setting of professional development and their classroom. As a graduate student she was fortunate to have had the opportunity to substantially participate in a variety of research projects under the direction of Paul Cobb from year 2000 to 2007. These projects ranged from supporting students' learning of mathematics in a classroom setting to designing teacher professional development programs in two contrasting institutional settings to the most recent one that focuses on investigating support structures in schools and districts that will enhance improvement in teachers' mathematical instructional practices. This unique combination of research experiences has enabled Zhao to bring together perspectives on students' learning, teaching and institutional context while seeking to understand and improve mathematics instruction.

Chapter 1
Introduction

Erna Yackel and Koeno Gravemeijer

[R]esearch is literally that—an unending process of searching. The scientist [or researcher] arrives not at some final answer but a deeper set of questions. There's always a bit further to go, a bit more to learn...

Tom Montgomery-Fate (2009, p. 167)

The last quarter of the twentieth century has seen fundamental shifts in educational theories. These shifts were initiated by the research of Fullan and Pomfret (1977), which revealed fundamental flaws in the curriculum-innovation model of "research, development, and diffusion," known as the RDD model. These flaws showed that one of the key pillars of the RDD model, the presumed fidelity of the implementation of new curricula, could not stand a reality check. They found that the actual implementation process was better described as a mutual adaptation of both the users of the curriculum and the curriculum itself. Also the limitations of corresponding theories about instructional design, such as Gagné's "principles of instructional design" (Gagné & Biggs, 1974) and Bloom's "mastery learning" (Bloom, 1968), both of which focused on learning outcomes, were revealed. Finally, the underlying theories, behaviorism, and later information processing proved ineffective in improving instruction, especially in mathematics. In mathematics education, these shifts coincided with a push for mathematics as a sense-making activity, in which problem solving, understanding, and applying were central tenets. As a consequence, a broad scope of new theories and new approaches in education in general, and in mathematics education especially, had to be developed to replace the discredited ones. We may note, however, that according to Kuhn (1970) scientific theories are not abandoned merely because of the problems they encounter. Instead, theories are abandoned only when there are new theories available that better explain observed phenomena.

In this process, Paul Cobb plays a central role on all fronts, from fundamental background theories to theories about instructional design and research, curriculum

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innovation, and policy issues. The basis for his contributions is invariably grounded in research. Over time this research has stretched from one-on-one teaching experiments, to classroom teaching experiments, to professional development in schools, and to investigations into diversity and equity. In this manner, Paul Cobb has not only made a unique contribution to the field, his work also offers a coherent view on the broad spectrum of all elements that influence and constitute mathematics education reform.

In recognition of his contributions to the field of mathematics education, the International Commission of Mathematics Instruction awarded the Hans Freudenthal Medal to Cobb in 2005. The citation for that award reads in part,

> [Cobb's] work is a rare combination of theoretical developments, empirical research and practical applications. . . .The dynamic character of Paul Cobb's theoretical perspective is a natural outcome of his thoughtful studies. His work shows an acute awareness of the insufficiency of an over-delineated approach, and he has gradually moved the focus of his work from individual learners to teams, to classrooms, and to district-wide infrastructure. Across these settings, he has been systematically examining the consequences of the assumption that human learning is inherently social. . . .Thanks to this systematic foundational contribution, Paul Cobb is today regarded as one of the leading sociocultural theorists in the field of mathematics education and beyond, and his work is currently yielding new insights on issues such as equity and students' identities. (International Commission on Mathematics Instruction, 2005)

The awarding of the Hans Freudenthal Medal to Cobb provided the motivation for this book, which has a twofold purpose. The first purpose is to provide a summary of the school of thought that has evolved through Cobb's work. The book is divided into six parts that represent what might be called stations in Cobb's journey in mathematics education research thus far, while recognizing that he is still an active research scholar. These stations are radical constructivism, social constructivism, symbolizing and instructional design, classroom mathematical practices, diversity and equity, and the institutional setting of mathematics teaching and learning. Each part of this book consists of an introductory chapter written by Cobb himself (sometimes with a number of colleagues), followed by a previously published work, in which Cobb, together with his associates at that time, introduced new theoretical perspectives and methodologies into the literature. In this way, the book provides the reader with a comprehensive view of the major theoretical and methodological contributions of his work up to this time.

The second purpose of this book is to demonstrate with the help of this particular body of work how exemplary educational research is conducted, how it evolves, and why it is useful. While some educational researchers make significant contributions by working within a single paradigm and from a single perspective throughout their entire careers, Cobb's work is characterized by shifts in perspective and in methodology. These shifts were motivated by pragmatic attempts to address and resolve issues and dilemmas that presented themselves in research investigations.

Except for Part I, which presents the field as Cobb found it, the introductory chapters in each part consist of both a detailed discussion of the transition to that station and Cobb's own retrospective comments. These introductions explain the motivation and rationale for developing new perspectives and methodologies and the processes through which they were developed. Each also retrospectively contextualizes, elaborates, and further clarifies the main theoretical constructs of the accompanying chapter in its respective part. In this way, the book provides the reader with heretofore-unpublished material that details the issues and problems Cobb has confronted in his work that, from his viewpoint, have required theoretical and methodological shifts and advances. In addition, the book provides insight into how he has achieved the shifts and advances. The result is a volume that, in addition to explaining Cobb's contributions to the field of mathematics education thus far, also provides the reader with insight into what is involved in developing a powerful and evolving research program.

As the parts in this book reveal, when Cobb confronts problems and issues in his work that cannot be addressed using his existing theories and frameworks, he looks to other fields for theoretical inspiration. A critical feature of Cobb's work is that in doing so, he consciously appropriates and adapts ideas from the other fields for the purpose of supporting processes of learning and teaching mathematics. As a result, Cobb reconceptualizes and reframes issues and concepts so that new ways of investigating, exploring, and explaining phenomena that he encounters in the practical dimensions of his work, including work in classrooms, with teachers, and with school systems, emerge. The effect is that the field of mathematics education is altered. Other researchers have found his "new ways of looking" useful to them, and they, in turn, adapt these ideas for their own use.

The complexity of many of the ideas that Cobb has introduced into the field of mathematics education can lead to a multiplicity of interpretations by practitioners and by other researchers, based on their own experiential backgrounds. Therefore, by detailing the development of Cobb's work, including the tensions involved in recognizing and subsequently reconciling apparently contrasting perspectives, the book sheds additional light on the processes of reconceptualization and thus helps the reader to understand the reasons, mechanisms, and outcomes of a researcher's constant pursuit of new insights.

Cobb sets the stage for the entire book in the introductory chapter in Part I by outlining the initial assumptions and perspectives with which he began his work in the field of mathematics education and indicating which of these assumptions and perspectives he has altered in the intervening years and which commitments remain the same. He concludes the introductory chapter in Part I by explaining the rationale for his shift in epistemological perspective from radical constructivism to pragmatic realism. As he explains, pragmatic realism leads us to acknowledge that we make choices when we adopt theoretical perspectives and that these choices need to be justified. This book demonstrates that a foundational pillar of Cobb's work is that the theoretical choices he makes are driven by the questions he is investigating at any given time and by the purposes of the investigation. As a result, his choices are grounded in the reality of working in classrooms or with teachers in their institutional settings.

Central among Cobb's theoretical and methodological contributions are those that relate to understanding and investigating collective aspects of mathematical learning. These form the major emphases in Parts II–IV. They include the development of constructs and analytic frameworks to account for and document the mathematical learning of a classroom community. Cobb's contributions, while drawing on a variety of sociocultural perspectives, have their roots in symbolic interactionism and ethnomethodology. Consequently, in these three parts, much attention is given to explaining both what is meant by interactive constitution and how it is accomplished and to elaborating analytic means for investigating and documenting collective activity.

In the introductory chapter in Part III, Cobb cites his work in analyzing mathematical learning in collectivist terms as a major shift from his earlier radical constructivist position. He goes on to comment on the extent to which collective constructs are frequently misunderstood and concludes that such misunderstandings indicate that despite increasing emphasis on sociocultural and cultural perspectives, individualistic perspectives remain predominant when it comes to math content.

Part IV provides an in-depth account of both the design experiment methodology and one of the central constructs Cobb has introduced into mathematics education, namely "Classroom Mathematical Practices." This part, more than any other in the book, clarifies what is meant by the coordination of sociological and psychological perspectives. In addition, the notion of emergence of phenomena, which permeates Cobb's work, is elaborated in this part with respect to the emergence of classroom mathematical practices.

Parts V and VI differ from the earlier parts of the book in that they represent shifts to arenas well beyond the classroom. Part V elaborates an interpretive scheme for analyzing the identities that students develop in mathematics classrooms that can inform instructional design and teaching, while Part VI describes an analytic approach for situating teachers' instructional practices within the institutional settings of the schools and school districts in which they work. As the introductory chapters to both parts explain, the issues addressed in the research that formed the basis for the reprinted chapters have their genesis in the earlier research projects. Further, the issues of equity and identity and those related to scaling up are treated in ways that keep students' mathematical learning in the classroom as the central focus.

Cobb is well known for his generosity in collaborating with others. As the authorships of the reprinted chapters in this book and the introductory chapters to the various parts show, his group of collaborators has changed over time. In planning for the book, we worked with Cobb to identify individuals who might make contributions to the part introductions. These individuals were invited to provide their recollections and accounts of the issues that led to the shift in research agenda, the problems that were encountered, and ways the problems were addressed. Cobb incorporated these comments and recollections into each of the introductory chapters. The list of contributors to the introductory chapters reflects those who responded to the invitation. In some cases, one or more of the contributors to the introductory chapter in a part is also a co-author of the reprinted chapter in that part.

As the co-authors, contributors, and collaborators vary from part to part, the groups of individuals represented by the first person plural pronouns that appear throughout the book vary. In some cases, "we" refers to the research team that conducted the relevant research and may include some individuals not listed as co-authors and contributors.

There are several approaches to reading this book, depending on the reader's purpose. Each of the reprinted chapters stands alone in its own right as a scholarly research contribution. The introductory chapters offer unique insights by providing behind-the-scenes views of the motivation for the research that led up to the reprinted chapters. In a sense, the reader has an opportunity to hear first hand about the complexities and the messiness of field-based research. Readers may elect to read individual parts of the book in isolation. However, by reading the book in its entirety, the reader gains a greater appreciation of how and why the research agenda unfolded as it did and gains insights into how an exemplary research program evolves over time.

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Part I
Radical Constructivism

Chapter 2
Introduction

Paul Cobb

My career as a mathematics education researcher began in 1978 when I entered the master's program in mathematics education at the University of Georgia. Prior to moving to Georgia with my wife Jenny, I had completed an undergraduate degree in mathematics at Bristol University in England and was a secondary math teacher for 2 years. Our primary reason for going to Georgia was simply our wish to seize the opportunity to travel. The mathematics education department chaired by James W. Wilson had offered me a graduate assistantship that covered tuition and provided a modest stipend. We had originally planned to spend just 16 months in the US but our plans changed when I became interested in the radical constructivism of Les Steffe and Ernst von Glasersfeld. We eventually spent five happy years in Athens, Georgia, where I finished my master's degree and then completed a doctorate in mathematics education.

Given my secondary background, I initially intended to focus on the teaching and learning of algebra during my graduate studies. However, that changed dramatically at the end of my first quarter in the masters program when Patrick W. Thompson, a fellow graduate student, gave me two theoretical papers on radical constructivism by Ernst von Glasersfeld. I found Ernst's epistemological arguments about the relation between knowledge and reality to be both compelling and jaw dropping. I was also fascinated by the neo-Piagetian model of cognitive development that Ernst had developed, which holds that people reorganize their thinking in order to resolve perturbations or disruptions in the world of their personal experience. At that time, Ernst was collaborating with Leslie P. Steffe on a research project funded by the US National Science Foundation, in which they were investigating the development of young children's arithmetical reasoning. I asked Les to be my major professor so that I could work on this project as a research assistant. Les kindly agreed and so I quickly became immersed in the world of early number learning and left thoughts of focusing on algebra behind.

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The research project that Les directed was called Interdisciplinary Research on Number (IRON) and also involved John Richards,[1] a philosopher of mathematics, and Pat Thompson, a senior doctoral student. My most vivid recollection of my first few months as a member of the project team is of periodic project meetings that lasted between 2 and 4 days. Initially, there was little that I could contribute and I found the high level of discussions in these meetings intimidating. Les typically showed (black and white) video-clips of interviews or one-on-one teaching episodes that he or Pat Thompson had conducted with children. His goal in doing so was to try to illustrate his recent insights into the development of arithmetical reasoning. Ernst and John would push Les and Pat for greater clarity and would frequently challenge their interpretation of the children's problem-solving activity. The discussion of a single video-clip often lasted an hour or more and frequently became very heated. I was struck both by the passion with which they engaged in these exchanges and by the fact that the arguments were never personal. The object of the exercise was not to win the argument as an end in itself, but to develop and refine theoretical constructs that gave insight into the development of arithmetical reasoning. As a consequence, what counted was the substance of an argument rather than the formal position or role of the person making it. As my grounding in the issues under debate improved, I gradually began to make initial, tentative contributions. Les, Ernst, Pat, and John showed me great kindness in treating these initial proposals with more seriousness than they probably merited as they scaffolded my induction into their research enterprise.

Within hindsight, I regard the IRON project as a near ideal learning environment for a beginning researcher and consider myself to have been remarkably fortunate. In addition to participating in project meetings, I was involved in an investigation in which we taught six children one-on-one twice each week in order to bring about and thus study what Les termed the critical moments when cognitive restructuring occurs.[2] Commencing in my third year as a graduate student, I also became a member of an interdisciplinary faculty reading group at Ernst's invitation. This group met once a month to discuss a paper selected by one of the members, whose fields included biology, philosophy and history of science, and literary criticism. The papers shared by this group proved to be a rich source of analogies and ideas for someone attempting to understand young children's arithmetical learning.

In many respects, my experience as a graduate student at the University of Georgia was that of an apprentice researcher whose induction into academia in general and mathematics education research in particular was supported at every step of the way. In this process, I learned much that is intangible from my two primary mentors, Les Steffe and Ernst von Glasersfeld, two scholars of great intellectual

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[1] John later became senior education advisor to CNN, the US cable news network.

[2] The findings of this teaching experiment were eventually published in Steffe and Cobb (1988). In my doctoral dissertation, I analyzed the six children's development of thinking or derived fact strategies.

and personal integrity. A few years ago, Kenneth Tobin[3] asked me to write a short reflection on Ernst's contributions to science education for a paper that he was writing in honor of Ernst's 90th birthday. The paragraph I wrote is reproduced below. The sentiments that I expressed apply equally to Les.

> I had the great privilege of studying with Ernst as a graduate student. As has been the case for many others, learning about epistemology and cognition with him proved to be a life-changing experience. Most importantly, Ernst has provided us with a model of what it means to be a scholar. It is a form of scholarship that permeates the very essence of how one understands oneself, others, and the world. He has taught us many things. One of the simplest and yet most profound is that there is no substitute for sustained, first-hand engagement with the phenomena that we seek to understand. Ernst's influence on research in science (and mathematics) education is far reaching and ranges from basic issues of epistemology, to the methodologies we use, to how we think about cognition and communication. In a very real sense, he has precipitated a change in the discourse of science education. For this, we are very much in his debt. (Tobin, 2007, p. 529)

The chapter reprinted in this part of the book was written while I was a doctoral student at the University of Georgia. It reflects a strong commitment to the constructivist view of mathematical learning developed by Les and Ernst. For example, we portrayed individual students as active constructors of increasingly sophisticated forms of mathematical reasoning. Further, we characterized the process of mathematical learning as one in which students simultaneously reorganize both their mathematical activity and the worlds in which they act. One of our goals when analyzing students' mathematical activity from this perspective was therefore to identify qualitative changes in students' mathematical reasoning, in the process delineating how the worlds of meaning and significance in which students act change in the course of development.

It is now over a quarter of a century since I completed my studies with Les and Ernst. As the chapters reprinted in the successive parts of this book document, my basic assumptions about the process of mathematical learning have changed significantly over the years. At the same time, however, several recurrent themes run throughout my work and are evident in the chapter reprinted in this part of the book.

First and foremost, I continue to believe in the importance of sustained, direct engagement with the phenomena under investigation, be it young children's understanding of measurement, the collective mathematical learning of the teacher and students in a classroom, or the learning of a professional teaching community. A number of years ago, I read Bruno Latour's (1987) book *Science in Action* in order to learn something about the role of representations in scientific practice. I was struck by Latour's account of how the typical career trajectory of scientists move away from what he termed bench science and toward research administration in which the head of a lab is the public face of a number of large projects but is removed from the concrete practice of inquiry. Latour's description of the work of senior scientists

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[3] Ken moved from Australia to begin his doctoral studies in science education at the University of Georgia at the same time that I entered the masters program in mathematics education.

served as an early warning for me and, in recent years, I have sought to remain true to Les' and Ernst's dictum about the importance of staying close to the processes into which we seek to gain insight.

Second, I remain committed to the view that people's practices are reasonable from their perspective, be they mathematics students, teachers, or school administrators. The challenge for researchers as I see is to attempt to understand the rationality of the people whose practices we study. As a consequence, I consider an analysis to be lacking in explanatory power if it portrays the people as less than rational given their personal histories and current circumstance. This basic rule of thumb has served me well over the years. As an illustration, the students with whom my colleagues and I have worked when conducting classroom design experiments have occasionally developed some very strange forms of mathematical reasoning. In these situations, we always assumed that we unknowingly taught the students to think in these peculiar ways. On each of these occasions, when we have reviewed video-recordings of prior instructional sessions, we have found that this was indeed the case: it was a failure of our instructional design for supporting the students' learning, rather than of the students per se. And once we have identified the source of the problem in this manner, we are immediately in a position to adjust our instructional design accordingly. More generally, I continue to believe that a solid understanding of why people's current practices are reasonable from their perspective is a prerequisite for developing effective designs for supporting their learning. An approach of this type helps us move beyond an exclusive focus on what people currently do not know or cannot do and orients us to view their current practices as resources on which to build when supporting their learning.

Third, I continue to see considerable value in the types of distinctions that Les, Ernst, Pat Thompson, and other constructivist researchers have identified in the quality of students' mathematical reasoning in particular mathematical domains. To be sure, I now view the form of reasoning attributed to a student as a relation between the student and the situation rather than as an inherent property of the student (see the chapter reprinted in the fourth part, Classroom Mathematical Practices). Even with this modification, I find that the qualitative distinctions in students' mathematical reasoning made by constructivist analyses continue to give us insight into the world of students' mathematical experience (with the understanding that those experiences are socially and culturally situated). As a consequence, analyses of this type help us understand why students' mathematical activity is rational from their perspective.

Fourth, I continue to subscribe to core tenets of the constructivist teaching experiment methodology as outline in the reprinted chapter and regard it as a direct antecedent of the design experiment methodology. As we make clear in the reprinted chapter, the primary goal when conducting a constructivist teaching experiment is to gain insight into the development of students' mathematical reasoning. For their part, researchers conducting a design experiment frame the development of instructional designs for supporting students' mathematical learning as an explicit goal. There are a number of additional points of contact between the two methodologies. For example, we clarify in the reprinted chapter that our intent when conducting a constructivist teaching experiment was to test and revise conjectures about students' mathematical reasoning. This conjecture-testing process foreshadowed the tightly integrated cycles of design and analysis that characterize design research. In addition, we indicated in the reprinted chapter that the primary products of a constructivist teaching experiment are conceptual models composed of theoretical constructs that have some generality in that they can be used to account for the learning not merely of the students who participated in the experiment, but for the learning of other students as well (Thompson & Saldanha, 2000). Design experiments also have a theoretical orientation in that the goal is not merely to refine a particular instructional design, but also to develop domain-specific instructional theories by systematically studying students' development of particular forms of learning and the means by which their learning was supported (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Thus, researchers conducting constructivist teaching experiments and those conducting design experiments both strive for generality by means of explanatory frameworks that provide process explanations of causality (Steffe & Thompson, 2000).

Finally, in the chapter reprinted in this part of the book, we focused explicitly on what we termed "the context within which the child constructs mathematical knowledge" (Cobb & Steffe, 1983, p. 84). To be sure, we were referring to what might be termed the individual psychological context within which a particular child approaches and attempts to solve mathematical tasks. However, we also considered that this personal context was influenced by interactions with the teacher and argued that the teacher could (and should) help students reconstruct the contexts within which they learn mathematics. Our concern for students' personal mathematical contexts drew directly on Fish's (1980) discussion of the influence of context on the interpretation of literary text, as well as on a number of other ideas that were in wide currency at the time (e.g., Kuhn's, 1962, notion of a scientific paradigm). With hindsight, it would be fair to say that we psychologized ideas from a number of fields by bracketing out their social and cultural dimensions. In many ways, the chapters reprinted in this book document an ongoing attempt to come to grips with the notion of context, be it the social context of the classroom, the cultural context of students' out-of-school lives, or the institutional context within which teachers develop and refine their instructional practices.

The five recurrent themes that I have described indicate the influence that Les and Ernst have had on my thinking even as some of my basic theoretical commitments have evolved over the last 25 years or so. In the introductions to the chapters reprinted in the next three parts, I outline some of the major changes in my general perspective on mathematical learning. I would, however, be remised if I failed to acknowledge that radical constructivism as developed by Ernst von Glasersfeld is an epistemology as well as a theory of learning. The radical constructivist epistemology is grounded on the assertion that it is impossible to check whether our ideas and concepts correspond to external reality (von Glasersfeld, 1984). It is important to emphasize that von Glasersfeld did not deny the existence of a pre-given external reality. His central epistemological claim was instead that this reality is unknowable precisely because it exists independently of human thought and action. He therefore argued that knowledge that has proven viable fits with, rather than matches, external reality in that it satisfies the constraints of that reality much as a key satisfies the constraints of a particular type of lock (von Glasersfeld & Cobb, 1984). In developing this position, von Glasersfeld challenged the traditional project of epistemology, that of identifying a universal method for determining whether a particular theory or conceptual scheme matches or corresponds with external reality.

This epistemology proved to be controversial within the mathematics education research community and provoked considerable debate, much of it emotional in nature. In retrospect, I do not believe that these exchanges were particularly productive, especially since none of the leading participants changed their basic positions. At the time that I completed my doctoral studies, I was a strong adherent to radical constructivism as an epistemology. Although I have modified my epistemological stance over the years, I did not find the arguments of critics of radical constructivism compelling. Instead, I came to see great value in the pragmatic philosophy of Dewey (1929/1958) and Peirce (1958), as well as the work of contemporary pragmatist such as Bernstein (1983), Rorty (1979), and particularly Putnam (1987, 1988). In the remainder of this introduction, I outline Putnam's pragmatic realist perspective, in the process indicating why I have changed my basic epistemological commitments.

Like radical constructivism, pragmatic realism seeks to end the traditional epistemology's quest to achieve absolute certainty in our knowledge claims by demonstrating a match or correspondence with external reality. However, whereas radical constructivism contends that it is impossible to bridge the gulf between knowledge and reality, pragmatic realism questions the very notion of reality on which traditional epistemology and radical constructivism seem to agree. This reality is not populated with tables and chairs, students and teachers, or differential equations and geometry proofs. It is instead an imagined, or perhaps better, an imaginary realm that has been the center of philosophical debate since the time of Plato. Putnam (1987) referred to it as Reality with a capital "R" to distinguish it from the world in which our lives take on significance and meaning. For all their differences, realist and radical constructivist epistemologies both take as fundamental the basic image of people as knowers separated from Reality (with a capital "R"). In contrast, Putnam and other pragmatists follow Dewey and Peirce by challenging this dichotomy between a putative external Reality on the one hand, and the concepts and ideas that people use when they think and talk about it on the other. Thus whereas radical constructivism seeks to offer a new solution to traditional epistemological problems, pragmatism questions the way in which epistemological problems have traditionally been framed.

For Putnam and Dewey, the experience is not screened off from reality but is instead a path into it. As Dewey (1929/1958) put it, "experience is *of* as well as *in* nature. It is not experience that is experienced, but nature" (p. 12, italics in the original). In speaking of people being in nature, Dewey eschewed the traditional preoccupation with Reality in favor of a focus on people's activities in the realities in which they actually live their lives. With this change in orientation, the quest for certainty is given up in favor of understanding how people are able to produce fallible truths and achieve relative security in their knowledge claims. The primary concern is then with processes of inquiry as they are enacted by flesh-and-blood people.

As Putnam observed, contemporary science as it is actually enacted by researchers has taken

> away [ahistorical] foundations without providing a replacement. Whether we want to be there or not, science has put us in a position of having to live without foundations ... That there are ways of describing what are (in some way) the 'same facts' which are (in some way) 'equivalent' but also (in some way) 'incompatible' is a strikingly non-classical phenomenon. (1987, p. 29)

Putnam went on to note that scientists in many fields switch flexibly from one perspective to another and treat each set of facts as real when they do so. On this basis, he concluded that these scientists *act as* conceptual relativists who treat what count as relevant facts and as legitimate ways of describing them as relative to the background theoretical perspective that they adopt (cf. Sfard, 1998). This observation led Putnam to reject as irrelevant the traditional epistemological project of determining which of the resulting constellations of phenomena correspond to an imaginary Reality. He termed his position *pragmatic realism* to emphasize it takes at face value the realities that scientists investigate. In his view, questions concerning the existence of abstract mathematical and scientific entities should be addressed not by making claims about Reality (with a capital "R") but by examining disciplinary practices. In this regard, he noted approvingly that Quine (1953)

> urges us to accept the existence of abstract entities on the ground that these are indispensable in mathematics, and of microparticles and space–time points on the ground that these are indispensable in physics; and what better justification is there for accepting an ontology than its indispensability in our scientific practice? (Putnam, 1987, p. 21)

Putnam made it clear that the conceptual relativism evident in scientific inquiry is also apparent in everyday common sense. In addition, he was careful to differentiate conceptual relativism, the notion that the phenomena treated as real differ from one perspective to another, from the view that every perspective is as good as every other. "Conceptual relativity sounds like 'relativism,' but it has none of the 'there is no truth to be found ... 'true' is just a name for what a bunch of people can agree on' implications of relativism" (Putnam, 1987, pp. 17–18). His goal in making this differentiation was to rehabilitate the notion of truth that is dispensed with by radical constructivism while simultaneously rejecting the view that Truth should be ascertained in terms of correspondence with Reality. His primary reason for taking this approach was that the notion of truth is pragmatically real in both scientific practice and everyday life. In the view that he proposed, truths are characterized as fallible, historically contingent, human productions that are subject to correction.

In my view, the grounding of pragmatic realism in analyses of both everyday practice and scientific practice constitutes one of its primary strengths. My shift from radical constructivism to pragmatic realism has clearly involved a significant change in fundamental epistemological commitments. Radical constructivism views the traditional epistemological debate as legitimate and engages in it by denying that Reality is knowable. In contrast, philosophical pragmatism in general, and pragmatic realism in particular, contend that the notion of Reality (with a capital "R") is a discursive construct in a philosophical language game that fails to make contact with the worlds in which people actually live their lives. Pragmatists therefore consider that this debate is irrelevant to both everyday and scientific concerns. As a consequence, they refuse to engage in this traditional debate and instead change the conversation by taking people's concrete practices rather than Reality as their point of reference (cf. Rorty, 1979). In doing so, they treat the concept of truth (with a small "t") as entirely legitimate and focus on the processes by which people individually and collectively establish and revise knowledge claims that attain the status of truths.

As I have indicated, pragmatic realism is primarily descriptive rather than prescriptive in that it does not attempt to tell scientists how they should go about the business of developing insights into the phenomena under investigation. I nonetheless find pragmatic realism valuable because it, in effect, holds up a mirror to scientific practice and clarifies that it is quite reasonable for scientists to act as conceptual relativists. This insight provides a useful orientation as we attempt to come to terms with the plethora of distinct theoretical perspective on learning and teaching that characterize mathematics education as a field of inquiry. On the one hand, it leads us to question claims made by adherents to a particular perspective that their viewpoint gets the world of teaching and learning right – that the phenomena that they take as real correspond to Reality. Pragmatic realism instead leads us to acknowledge that we make choices when we adopt a particular theoretical perspective, and that these choices reflect particular interests and concerns. On the other hand, it alerts us to the danger of inferring from the conclusion that there is no neutral algorithm of theory choice that any theoretical perspective is as good as any other, and that the decision to subscribe to a particular "ism" is a matter of personal preference or taste. Pragmatic realism instead underscores that such choices need to be justified, and that the justification should be pragmatic in nature. The relevant question is then for whom and for what purposes are the forms of knowledge produced within different perspectives useful in enabling them to contribute more effectively to the improvement of classroom teaching and learning.

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Chapter 3
The Constructivist Researcher as Teacher and Model Builder

Paul Cobb and Leslie P. Steffe

The constructivist teaching experiment is used in formulating explanations of children's mathematical behavior. Essentially, a teaching experiment consists of a series of teaching episodes and individual interviews that covers an extended period of time—anywhere from 6 weeks to 2 years. The explanations we formulate consist of models—constellations of theoretical constructs—that represent our understanding of children's mathematical realities. However, the models must be distinguished from what might go on in children's heads. They are formulated in the context of intensive interactions with children. Our emphasis on the researcher as teacher stems from our view that children's construction of mathematical knowledge is greatly influenced by the experience they gain through interaction with their teacher. Although some of the researchers might not teach, all must act as model builders to ensure that the models reflect the teacher's understanding of the children.

Our methodology for exploring the limits and subtleties of children's construction of mathematical concepts and operations is the primary object of attention in this paper. We argue that, in such an exploration, there is no substitute for experiencing the intimate interaction involved in teaching children. We then discuss the constructivist view of teaching and stress the importance of modeling children's mathematical realities. Next, the similarities and differences between constructivist and nonconstructivist teaching experiments are highlighted. In the remainder of the paper, we focus on models and model building.

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Why We, as Researchers, Act as Teachers

We believe that the activity of exploring children's construction of mathematical knowledge must involve teaching. Theoretical analysis by the researcher does play an important role in understanding the significance of children's mathematical behavior. But knowledge gained through theoretical analysis can at best intersect only part of the knowledge gained through experiencing the dynamics of a child doing mathematics. These experiences give the researcher the opportunity to test and, if necessary, revise his or her understanding of the child. The continual tension created by inexplicable or seemingly contradictory observations leads ultimately to a knowledge of the child that supersedes the initial theoretical analysis. The insufficiency of relying solely on a theoretical analysis serves as one reason for our belief that researchers must act as teachers. Some of our most humbling experiences have occurred when knowledge gained through theoretical analysis has failed to be of value in understanding children's mathematical realities. On the other hand, totally unexpected solutions by children have constituted some of our most exhilarating experiences.

A second reason we believe researchers must act as teachers is that the experiences children gain through interactions with adults greatly influence their construction of mathematical knowledge. The technique of the clinical interview is ideally suited to the psychological objective of investigating a sequence of steps children take when constructing a mathematical concept. In an interview, mathematical knowledge can be traced back to less abstract concepts and operations. Further, using the clinical interview, a researcher can specify structural patterns children may abstract from the experience gained through interaction with *their* milieu. However, the researcher conducting a clinical interview does not intend to focus on those critical moments when cognitive restructuring takes place. Consequently, the resulting accounts of abstraction are devoid of the experiential content that a teacher needs when planning specific interventions. More importantly, some children might take a different sequence of steps and construct different concepts in particular instructional settings.

A third reason for acting as teachers stems from the importance we attribute to the context within which the child constructs mathematical knowledge. The crucial role played by context can best be illustrated by example. Erlwanger (1973) found that Benny, a sixth grader, had developed a coherent rationale that accounted for his experiences in using Individually Prescribed Instruction (IPI) materials. Erlwanger concluded that, for Benny, the activity of learning and using mathematics involved formulating unrelated rules that yielded the correct answers to particular sets of problems. Benny frequently searched for patterns in the numerals of related problems. Erlwanger explains that Benny

> thought these rules were invented "by a man or someone who was very smart." This was an enormous task because, "It must have took this guy a long time.. . about 50 years ... because to get the rules he had to work all of the problems out like that." (p. 17)

Benny had arrived at his conception of mathematics by reflecting on, and attempting to make sense of, his past experiences of doing mathematics. This conception served as an encompassing framework within which he formulated anticipations of what sorts of events might happen and how he should respond when he worked with IPI materials (cf. Skemp, 1979, for a discussion of meta-learning).

Confrey (1982) also found it necessary to consider the context within which students do mathematics when she investigated ninth graders' mathematical abilities. She concluded that "in order to understand a student's mathematical performances and to judge their abilities, one must consider the influence which the context of classroom instruction has on those performances" (p. 27). By acting as teachers, and by forming close personal relationships with children, we help them reconstruct the contexts within which they learn mathematics. In particular, we help them differentiate between the contexts of doing mathematics in class and doing mathematics with us. This is essential given our objective of exploring the limits and subtleties of children's creative possibilities in mathematics.

In summary, we believe that the failure to observe children's constructive processes firsthand denies a researcher the experiential base so crucial in formulating an explanation of those processes. Researchers who do not engage in intensive and extensive teaching of children run the risk that their models will be distorted to reflect their own mathematical knowledge.

The Constructivist View of Teaching

The actions of all teachers are guided, at least implicitly, by their understanding of their students' mathematical realities as well as by their own mathematical knowledge. The teachers' mathematical knowledge plays a crucial role in their decisions concerning what knowledge could be constructed by the students in the immediate future. Through reflecting on their interactions with students, they formulate, at least implicitly, models of their students' mathematical knowledge. Constructivist and nonconstructivist views of teaching differ in the emphasis they place on the activity of modeling children's realities. In the constructivist view, teachers should continually make a *conscious* attempt to "see" both their own and the children's actions from the children's points of view. This emphasis stems from an analysis of teaching as primarily the activity of communicating with students. As Schubert and Lopez Schubert (1981) point out, teaching in this sense is rooted in action:

> It is in the subtly powerful interaction of some teachers . . . with their students. It is in the daily striving of teachers who try to understand their students' sources of meaning, their out-of-school curricula, their personal "theories" or sense-making constructs. It exists in attempts made by teachers to determine how their experience and knowledge can bolster their students' quest for meaning. (p. 243)

In the course of an interaction, both the teacher and the children attempt to make sense of each others' verbal and nonverbal activity. The children, for example, interpret and give meaning to the teacher's actions in terms of their current conceptual structures. In some cases, these interpretations are influenced by the children's intuitions about the teacher's motives and intentions. In any event, the teacher acts with an intended meaning, and the children interpret the actions within their mathematical realities, creating actual meanings (MacKay, 1969; von Glasersfeld, 1978).

Obviously, to communicate successfully with children, there must be some fit between the intended and the actual meanings. The likelihood that a teaching communication will be successful is increased whenever the teacher's actions are guided by explicit models of the children's mathematical realities. From this perspective, the activity of teaching involves a dialectic between modeling and practice. The teacher's actions are formulated within the framework of his or her current models. The plausibility of these models is in question when the teacher attempts to make sense of observations of the children's behavior in subsequent encounters.

Teaching Episodes and Clinical Interviews

In our work, we use teaching episodes as well as occasional clinical interviews as an observational technique. The interviews are used when we want to update our models of the children's current mathematical knowledge, usually after a vacation or when the child has been absent for some time. However, the main emphasis is on the teaching episodes, as these give us a better opportunity to investigate children's mathematical constructions. Our primary objective is to give the children opportunities to abstract patterns or regularities from their own sensory-motor and conceptual activities. Guided by our current model, we hypothesize certain patterns or regularities that it is possible for a child to abstract. Activities are then initiated in the hope that the child will reflect on and abstract those patterns or regularities from his or her experiences. For the constructivist teacher, the key is to help children hold their own mathematical activity at a distance and take it as its own object. This is the crucial aspect of reflection (von Glasersfeld, 1991).

The teaching episodes (and the occasional clinical interviews) are routinely videotaped. These tapes serve as a record of the episodes and permit a longitudinal analysis of a child's mathematical development. Members of the research team can discuss interpretations of the child's behavior when viewing the tape. Although the researcher responsible for making the video-recording of the episode is free to intervene during or after an episode, the teacher has the right to ignore interventions made during an episode. The research team try to help the teacher in two ways. First, they help the teacher explicate his or her intentions and interpretations by asking appropriate questions. Second, they suggest alternative interpretations and propose activities that the teacher might wish to initiate. During these discussions, attention is given to the child's conception of the activity of doing mathematics as well as to his or her mathematical knowledge.

Our emphasis on formulating and revising explicit models of children's mathematical realities in the context of acting as teachers is in harmony with Vygotsky's research as modeling rather than empirically studying mathematical processes (El'konin, 1967, p. 36). Unfortunately, this emphasis has been submerged in the literature dealing with the methodology of the teaching experiment (Kantowski, 1978; Kieran, 1985; Menchinskaya, 1969a). In the following sections, we first consider the characteristics shared by all teaching experiments and then differentiate between the constructivist teaching experiment and other variants of the method.

Teaching Experiments

One general characteristic shared by constructivist and nonconstructivist teaching experiments is the “long-term” interaction between the experimenters and a group of children. A second is that the processes of a dynamic passage from one state of knowledge to another are studied. What students do is of concern, but of greater concern is how they do it. A third characteristic is that the data are generally qualitative rather than quantitative. The qualitative data emanate from two possible sources. The first source is teaching episodes with the children. For example, Davydov (1975) reports anecdotal data obtained from his observations of teaching in classes for which he had designed learning material. The data took the form of verbatim exchanges between the teacher and her students as well as descriptions of the instructional contexts and the students’ responses in those contexts. The second source is clinical interviews conducted at selected points in the teaching experiment.

Macroschemes and Microschemes

Menchinskaya (1969a) identifies two types of teaching experiments reported in the Soviet literature. She calls the first type a *macroscheme*: “Changes are studied in a pupil’s school activity and development as he makes the transition from one age level to another, from one level of instruction to another” (p. 5). This type is exemplified by Davydov’s (1975) teaching experiment. He constructed *teaching material* that reflected his view of quantity, a view derived from his previous work with children. Children were expected to compare objects on various attributes, reverse the sense of an equality, reverse the sides of an equality, reason transitively, add a quantity to equalize inequalities, and so on. Because Davydov was interested in processes expressible in terms of an ontological notion of quantity, his experiment involved teaching children to behave in certain ways so that they would experience these processes. Consequently, he used intact classes for his teaching experiment.

The second type of teaching experiment identified by Menchinskaya is the *microscheme*, where “in a single pupil the transition is observed from ignorance to knowledge, from a less perfect mode of school work to a more perfect one” (p. 6). Kantowski (1977) conducted a teaching experiment of this type in the United States. She went outside the realm of mathematics for processes. Goal-oriented heuristics, analysis, synthesis, persistence, and looking-back strategies were among the processes of interest in her study. These processes, being “thought” oriented rather than “content” oriented, led her to investigate individual students as they attempted to acquire the processes in the context of her instruction in geometry.

These examples illustrate the considerable variation in how the passage from one state of knowledge to another is investigated. In general, a macroscheme such as Davydov's has a curriculum orientation, and a microscheme such as Kantowski's has a psychological orientation. Given our focus on children's constructive activity, it is clear that constructivist teaching experiments are microschemes.

Constructivist and Nonconstructivist Microschemes

Kantowski's outstanding investigation exemplifies a characteristic shared by non-constructivist teaching experiments. The processes studied are determined a priori to be the ones of interest. Alternative processes are of secondary importance. This characteristic reflects beliefs about how learning and teaching are related. Menchinskaya (1969b) states the Soviet position succinctly when she says that "neither scientific nor everyday concepts spring forth spontaneously; both are formed under the influence of adult teaching," (p. 79). One can see, then, that she believes that children form scientific concepts as a result of receiving instruction in specific school subjects and that the processes of mastery can be studied only in the context of these subjects.

We, too, believe that adults can help children as they attempt to learn mathematics. However, it is not the adult's interventions per se that influence children's constructions, but the children's experiences of these interventions as interpreted in terms of their own conceptual structures. In other words, the adult cannot cause the child to have experience *qua* experience. Further, as the construction of knowledge is based on experience, the adult cannot cause the child to construct knowledge. In a very real sense, children determine not only *how* but also *what* mathematics they construct. Consequently, we do not attempt to study children's construction of certain preselected processes in instructional contexts. Instead, we attempt to understand the constructions children make while interacting with us.

Model Building in a Teaching Experiment

We contend that children's mathematical knowledge can be modeled in terms of coordinated schemes of actions and operations (von Glasersfeld, 1980). Our goal is to specify these schemes and to intervene in an attempt to help the children as they build more sophisticated and powerful schemes.

We offer a brief glimpse of the steps that Jason, a six-year-old child in 1980, took when constructing his counting scheme. We also mention some of our concomitant intentions when teaching Jason. However, this brief illustration focuses on Jason's counting behavior and our interpretation of it rather than on our *teaching episodes* (cf. Steffe & Cobb, 1982, for an elaboration of a teaching episode). We make this choice to emphasize that our models are of children's understanding. It will be seen that we conduct an experiential analysis of Jason's progress within the framework of a theoretical model of children's counting types.

Development of Jason's Counting Schemes

We distinguish between the activities of counting and of reciting a sequence of number words. When we say a child counts, we mean the child coordinates the production of a sequence of number words with the production of a sequence of unit items (items that are equivalent for the child in some way). This activity of counting follows the establishment of a collection of countable items. The result of the counting activity is a collection of counted items.

Our investigations indicate that the quality of the items that children *create* while counting undergoes a developmental change (Steffe, von Glasersfeld, Richards, & Cobb, 1983). In October 1980, Jason was what we call a *counter of motor unit items*. His most sophisticated counting activity involved counting his movements as substitutes for visual items screened from view. For example, in a clinical interview, we asked Jason to find out how many marbles there were in all when four were hidden beneath the interviewer's hand and seven were visible. Jason counted the visible marbles, pointing to each in turn. He continued his count of the entire collection by pointing rhythmically over the interviewer's hand while synchronously uttering the number words "8-9-10-11." Jason was successful because his counting acts completed a temporal pattern. He also counted a partially screened collection of seven squares, where three were visible, by first counting the three visible ones and then continuing over the screen, stopping when his counting acts completed a square pattern. When five squares were screened, however, he did not know when to stop counting.

Our immediate intention was to help Jason develop spatial and temporal patterns in counting activity involving more than four items. He might then continue until the pointing acts in the continuation completed, say, a domino five pattern. This decision was based on our belief that the coordination of spatial and temporal patterns with the counting scheme can play a crucial role in the process of constructing more sophisticated types of unit items. Our technique was to hide a collection of felt squares beneath a cloth, lift the cloth briefly, and then ask Jason how many squares he thought were hidden. When Jason did not recognize the pattern, the teacher would ask Jason to count "what you saw." Jason counted by pointing over the cloth where his points of contact completed a spatial pattern analogous to what he had seen. The squares were arranged both randomly and in regular patterns, such as in linear and square four patterns (.... and ::). Within three sessions (in December 1980) Jason had constructed patterns for the number words up to "seven." Further, these patterns eventually were coordinated with his counting scheme. For example, on 8 April 1981, to find out how many checkers were under two cloths after being told that nine were under one and five under the other, Jason pointed over the first cloth while synchronously uttering "1-2-...-9" and then continued by pointing over the second cloth while uttering "10-11-12-13-14." He stopped when his points over the second cloth completed a domino five pattern.

We hypothesized that Jason always counted starting from 1 because what was significant for him while counting was the perceptual and motor activity. We thought that Jason would not be able to find out how many items there were in all by counting-on over the second cloth until the verbal aspect of counting became significant for him (Steffe et al., 1983). Starting at 1 was not merely a habit. From his perspective, he had to count all the items—he had no alternative. His failure to count-on was ultimately related to the types of unit items he could create while counting.

The difficulty Jason had in curtailing his counting activity over the first cloth was exemplified in the 8 April teaching episode. Our goal was to help Jason take his utterances as substitutable items (i.e., to count verbal unit items) and thus curtail his motor activity in counting. We presented a sequence of ten tasks in which Jason was asked to count all the items hidden under two screens. Jason folded his hands while counting to perform the last two tasks, indicating that he was counting verbal unit items. He finally showed some awareness of the superfluousness of counting over the first cloth when performing the last task. After uttering "1-2-. . .-12," he said "wait," pointed to the cloth, and said "12" before continuing. Immediately afterwards, the interviewer presented three addition sentences ($9 + 3 = \_\_$, $17 + 3 = \_\_$, and $25 + 3 = \_\_$). Jason counted to solve them starting with 1, gesturing in the air in synchrony with uttering number words. In the teaching episode, Jason made progress toward becoming a counter of verbal items (i.e., he could substitute vocal utterances as well as movements for visual items) in the context of counting the hidden items. But his solutions to the number sentences indicate that his progress in this episode was limited to particular local contexts.

The limitations of being a counter of motor items can be illustrated by Jason's almost total lack of knowledge of the basic addition facts. Although we do not want to give the impression that counters of motor unit items cannot learn the basic facts, their spontaneous methods and strategies for doing so are very limited. For example, in a teaching episode held on 23 March 1981, Jason found the sum of 3 and 4 by simultaneously extending three fingers, simultaneously extending four fingers, and then counting them all. He clearly did not know the sum of 3 and 4. To find the sum of 7 and 4, Jason sequentially put up seven fingers in synchrony with uttering "1-2-. . .7," and continued sequentially putting up fingers while uttering "8-9-10-11." He reused one finger he had already put up. These primitive methods inhibit children from using addition facts they already know to help them find a given sum. There were indications that Jason knew the sums $3 + 3$ and $5 + 4$, but he did not spontaneously use this knowledge to find the sums $3 + 4$ and $7 + 4$. He found each sum independently as a new task. This was a recurrent feature of his mathematical behavior.

Throughout April and most of May 1981, Jason's counting scheme remained sensory-motor. However, he made rapid progress during this period and could soon count verbal unit items in a variety of contexts. Further, by the end of May 1981, he had a flexible, adaptive counting scheme that he could use to solve a variety of problems. He could take a sensory-motor unit itself as a unit to be counted (or create abstract units). He solved problems by intentionally finding out how many times he counted. Counting was an expression of numerical structure. For example, in an interview in November 1981, he counted 16 units beyond 14 to find the sum of 14 and 16 by sequentially putting up fingers in synchrony with uttering "16-17-. . .-31" (he stopped at 31 because of an executive error). He intentionally kept track of his utterances. Moreover, he counted backwards “16-15-14---13” to find how many marbles were left in a tube containing 16 after 3 had been removed, which had been completely beyond him in April. This great flexibility in his use of the counting scheme was consistent with the adaptiveness he displayed in the November interview. Initially, he could not solve a missing addend task like 11 + __ = 16 by counting. However, with only minor suggestions from the interviewer, he solved not only that particular task, but others as well. In addition to using his counting scheme creatively in finding sums, missing addends, and differences, Jason displayed powerful numerical strategies. For example, after counting to solve 5 + __ = 12, he knew that 9 was the answer to 5 + __ = 14, because 9 is 2 greater than 7!

We have discussed some of Jason’s typical mathematical behavior within the framework of our model of counting types. In particular, we accounted for Jason’s use of increasingly sophisticated solution procedures in terms of changes in his counting scheme. This experiential analysis allows us to further elaborate the counting types model.

Model Building—The Quest for Generality and Specificity

Thus far, the discussion of model building has focused on interactions between children and a teacher. The reader might well have inferred that our sole objective is to account for the mathematical progress made by the small number of children who participate in a teaching experiment. However, we strive to build models that are general as well as specific. On the one hand, the model should be general enough to account for other children’s mathematical progress. On the other hand, it should be specific enough to account for a particular child’s progress in a particular instructional setting. We attempt to attain these seemingly contradictory objectives by ensuring that there is a dialectical interaction between the theoretical and empirical aspects of our work. Although one aspect may be more prominent for a time, it should never completely dominate the other. The interaction can be seen in the following brief account of a sequence of teaching experiments.

One objective of our research program was to build a viable model to account for children’s construction of numerical concepts and operations. An initial model of children’s counting (Steffe, Richards, & von Glasersfeld, 1978) was formulated on the basis of the experience of teaching six-year-old children in two yearlong teaching experiments. Although we constantly attempted to organize and make sense of these experiences, our emphasis in the initial phase was empirical. The next phase of the modeling process involved reformulating the initial model, aided by a theoretical model of the construction of units and number (von Glasersfeld, 1981). The reformulation of the initial model into a developmental model of counting types was also aided by analyses of videotapes of children solving arithmetical tasks. Our constant return to children’s behavior via the videotapes both stimulated and modified our thinking. But it was the theoretical work that predominated during this phase.

Even though the developmental model of counting types had an experiential basis, it did not indicate how we might help children make progress. A new teaching experiment was called for to fill out and, if necessary, refine the skeleton of some possible courses of development specified by the developmental model. When we formulated the model, we were primarily concerned with the form and structure of the counting scheme at various points in its development. The teaching experiment allowed us to conduct an analysis of the steps the children took when making progress in the construction of more sophisticated unit items. A cursory *experiential* analysis of children's progress is exemplified by the above discussion of Jason's progress in counting. Analyses of teaching episodes made possible the reconstitution of what had before been couched in terms of theoretical constructs only. The human activities that might constitute "doing mathematics," as referred to by Plunkett (1982, p. 46), were used to specify the construction of the counting scheme. The theoretical aspect of our methodology is apparent in the final phase of the modeling process in that the theoretical model served as a guiding framework. However, the primary emphasis was, once again, empirical.

In summary, we attempt to account for the observed regularities in children's progress by developing abstract, theoretical constructs. This constitutes our quest for generality. We strive for specificity by filling out these constructs with experiential content. At any point in the modeling process, novel, unexpected observations can lead to a reformulation of the theoretical constructs. Conversely, a theoretical reformation can lead to the novel interpretation of previous observations. The interdependence of theory and observation is consistent with Lakatos' (1970) analysis of a scientific research program. We firmly believe that this feature of our method is essential and contributes enormously to the understanding of how children might construct their mathematical realities. However, we should not forget that a model is no more than a plausible explanation of children's constructive activities. One can never claim a correspondence between the model and children's inaccessible mathematical realities. Although a model can be viable, it can never be verified.

The Educational Significance of Models

Hawkins (1973a) offers the following characterization of the activity of teaching:

> The teacher begins to assemble...information over a variety of children although for thirty children the task is enormous, and even the best teachers will confess to omissions. Then there is a trial-and-error of communication, further observation, a gradual and still tentative sort of portraiture involving the child's style, strengths, weakness, skills, fears, and the like.... What [the teacher] finds himself doing is beginning to build what I would call a map of each child's mind and of the trajectory of his life. It is fragmentary, fallible, but it is subject always to corrections. (p. 13)

Hawkins (1973b) emphasizes the commonality of research and teaching:

> The really interesting problems of education are hard to study. They are long-term and too complex for the laboratory, and too diverse and nonlinear for the comparative method. They

> require longitudinal study of individuals.... The investigator who can do that and will do it is, after all, rather like what I have called a teacher. (p. 135)

The researcher who conducts a teaching experiment attempts to perform the same activities as Hawkins' teacher. The single difference between the researcher and the teacher is that the researcher interacts with fewer children and has greater opportunity and more time to make sense of their behavior.

Essentially, our models are the results of attempts to explicate our understanding of children's constructions. These models, which were developed by intensively analyzing children's behavior, capture our knowledge of recurrent features of the activity of doing mathematics. They also embody our suggestions on how to aid children as they attempt to construct mathematical knowledge. The counting-types model, for example, constitutes the organization that we use when interpreting children's behavior and when planning interactions with children on the basis of those interpretations.

We believe that nothing could be more useful to teachers than the type of knowledge represented by a model. However, just as children construct mathematical knowledge, so teachers construct their own understanding of children's mathematical realities. We can no more give teachers our counting-type model than we can give children our knowledge that subtraction is the inverse of addition. The question of how to help teachers as they strive to understand children's mathematical realities is of critical importance.

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Part II
Social Constructivism

Chapter 4
Introduction

Paul Cobb with Erna Yackel

I completed my dissertation studies with Les Steffe and Ernst von Glasersfeld at the University of Georgia in 1983 and then accepted a faculty position at Purdue University in Indiana. The first study that I conducted at Purdue University was built on my dissertation work and focused on the psychological contexts within which young children interpret and attempt to solve arithmetical tasks in school (see Chapter 2). In this study, I interviewed approximately 40 first-grade students from two classrooms at the beginning, middle, and end of the school year. In the initial interviews, most of the children attempted to solve all types of arithmetical tasks presented by reasoning about quantities. However, in the interviews at the end of the school year, most of the same children attempted to solve all interview tasks that were similar to those in their school textbook by either using very elementary counting methods or by focusing on patterns in numerals regardless of whether they made sense in terms of relations between quantities. In this respect, the children's solutions were reminiscent of those that Erlwanger (1973) had documented in his influential case of study of a fifth-grade student's conception of mathematics.

I called the two distinct personal contexts within which the first-graders' approached arithmetical tasks the context of pragmatic numerical problem solving and the context of academic arithmetic, respectively (Cobb, 1986). One of the most striking observations made during the interviews conducted at the end of the school year was that it was possible to influence the context within which children interpreted tasks by making seemingly superficial changes to tasks formats. For example, the children almost invariably acted in the context of academic arithmetic and used relatively primitive counting methods when addition and subtraction tasks were presented in a vertical format similar to their school textbooks. In contrast, they acted in the context of pragmatic numerical problem solving and used more sophisticated methods when addition and subtraction tasks involving exactly the same number combinations were presented as horizontal number sentences. Furthermore, the children did not see any contradiction if they arrived at different answers to

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tasks involving the same number combinations. I took this to indicate that tasks that differed only in the format of presentation were not the same task for the children when they interpreted them within different personal contexts. The conclusions I drew from this study were that the two contexts were separate domains of activity for the children, and that typical US first-grade instruction supported children's construction of academic arithmetic contexts that were divorced from their everyday lives and that had little to do with numbers as quantities.[1]

This interview study served to emphasize that typical US instruction was a poor setting in which to investigate students' development of increasingly sophisticated ways of reasoning about numerical quantities. It was against this background that Grayson Wheatley, the senior mathematics educator at Purdue University, and I submitted a successful proposal to the US National Science Foundation in which we planned to extend the one-on-one constructivist teaching experiment methodology to the classroom. Our goal in proposing a study of this type was to support students' development of the forms of mathematical reasoning that we wanted to study. As part of our preparations for what would now be called a classroom design experiment, we recruited Graceann Merkel, an experienced second-grade teacher who worked in a rural/suburban school, to the project team.

Wheatley took a leave of absence from Purdue University during the year in which the design experiment was conducted to work in Malaysia (1986–1987). I therefore invited Erna Yackel and Terry Wood to work with me in Merkel's classroom. Yackel had completed her dissertation under Wheatley's supervision and was teaching statistics courses in the Statistics Department at Purdue University. Wood's background was in educational psychology and she was teaching mathematics pedagogy courses for future elementary teachers.

Our original plan was to conduct the design experiment during the first three months of the school year and to focus exclusively on arithmetic. However, as the experiment progressed, Merkel came to value the positive nature of the learning environment established in her classroom and also became intrigued by the various ways in which her students reasoned about quantity. During a project team meeting, she stated that she could not return to using the textbook adopted by her school district and asked the researchers directly what she was supposed to do for the remainder of the school year. We viewed Merkel's concerns as legitimate and eventually decided to continue the experiment for the entire school year, in the course of which we developed a complete set of instructional activities for second-grade mathematics.

Our initial findings about the quality of the learning environment established in Merkel's classroom were encouraging and, on this basis, the National Science Foundation supported a request for additional funding to enable Yackel and Wood to contribute to retrospective analyses of the video-recordings of classroom sessions. While the experiment was in progress, Merkel shared her positive views of the

[1]These conclusions are consistent with the findings of a series of studies that Schoenfeld (1983) conducted to investigate the beliefs that high-school students developed as a consequence of typical US instruction.

developments in her classroom with other teachers and with school administrators. A number of them observed classroom sessions and were typically impressed by the nature of students' engagement in classroom activities. As a consequence of these positive reports, we subsequently supported the learning of approximately 20 second-grade teachers who used the instructional activities we had developed as the basis for their mathematics instruction. In addition, we conducted a third-grade design experiment in collaboration with another teacher in the same school. At about this same time, Yackel accepted a faculty position at another Purdue University campus and conducted a series of design experiments in an urban district. Wood accepted a faculty position at Purdue University and continued to work with groups of teachers in order to study their classroom practices after I moved to Vanderbilt University in 1992. Sadly, she died in january 2010 after a long battle with pancreatic cancer.

Our original research goal when we began the design experiment in Merkel's classroom was to study the development of individual children's arithmetical reasoning. Our theoretical perspective on mathematical learning when preparing for the experiment acknowledged social interactions as a source of learning opportunities for individual children. We had been influenced by neo-Piagetian studies that focused on the interpersonal conflicts that occur between children as they work together to solve tasks (Doise & Mugny, 1979; Doise, Mugny, & Perret-Clermont, 1975; Perret-Clermont, 1980). These studies indicated that interpersonal social conflicts in task interpretations could give rise to intrapersonal cognitive conflicts or perturbations for individual students. From a constructivist perspective, conflicts or perturbations of this type are considered to be critical precipitators of substantial learning in which students reorganize their reasoning. Thus, at the outset of the design experiment, we viewed social interactions between students, and between the teacher and students, as catalysts for individual students' cognitive development. The significance that we attributed to social interactions as a source of learning opportunities was evident in the way that we organized classroom activities during the design experiment: the children solved tasks in pairs and then Merkel led a whole class discussion of their interpretations and solutions.

Our research focus changed significantly during the first days of the design experiment as a consequence of events that we observed in Merkel's classroom. As we noted in the chapter reprinted in this part of the book, Merkel's expectation that her students should explain how they had interpreted and attempted to solve tasks ran counter to the students', prior experiences of class discussions in school. Entirely on her own initiative, Merkel initiated a process that we subsequently came to call the renegotiation of classroom social norms. In the reprinted chapter, we describe an incident in which a student became embarrassed during a discussion when he realized that the answer he had given was incorrect. As we report, Merkel used this incident to clarify her expectations for the students. In doing so, she characterized mistakes both as normal and as opportunities for learning. Exchanges of this type took on immediately significance against the background of my prior work on the personal contexts within which students approach arithmetical tasks. In particular, we realized that Merkel was supporting her students to interpret tasks within the context of pragmatic numerical problem solving.

The reprinted chapter was published just 2 years after we completed the year-long design experiment in Merkel's classroom. The chapter documents that we developed an initial approach for analyzing the negotiation of classroom social norms relatively quickly. We were able to make relatively rapid progress because we could build directly on the work of a number of scholars. Erickson's (1986) discussion of what, at the time, we referred to as the social dimension of the classroom convinced us of the importance of understanding the interactional grammar of classroom life if we were to develop adequate analyses of the children's mathematical learning. We were also influenced by a distinction that Bateson (1973) made between learning within an established context and learning to act in a new context. This distinction is evident in the reprinted chapter in the distinction that we drew between the teacher and students talking about mathematics, and talking about how to talk about mathematics. We also took seriously Maturana's (1980) argument that individual cognitive processes and collective social processes are non-intersecting domains of analysis. In a formulation that we were later to learn from Gotz Krummheuer, an interactional analysis documents the conditions for the possibility of learning, whereas a cognitive analysis documents learning process as they are located within those social situations.

In addition to drawing on work outside mathematics education, we were also profoundly influenced by two papers written by mathematics education researchers. The first was Bauersfeld's (1980) discussion of patterns in classroom interactions. Bauersfeld's arguments led us to conclude that our view of social interactions as catalysts for otherwise autonomous cognitive developments was unduly restricted. As he demonstrated, the nature of classroom social interactions influences not merely the process of mathematical learning but also its products, the forms of mathematical reasoning that students develop. Voigt (1985) extended Bauersfeld's line of argument in a paper published 5 years later in which he described and illustrated an approach for analyzing classroom interaction patterns in terms of the largely implicit obligations that the teacher and students attempt to fulfill in particular situations, and the expectations that they have for each others activity. It was readily apparent that this approach had (and continues to have) considerable explanatory power and was well suited to our immediate purpose of accounting for the negotiation of social norms in Merkel's classroom. As we indicated in the reprinted chapter, our analysis of social norms drew directly on Voigt's work.

Our focus in the reprinted chapter was on the children's emotional acts in the design experiment classroom. As background, Yackel and I had attended a meeting on affect and mathematical problem solving organized by Douglas McLeod in San Diego in June 1987, near the end of the design experiment. We were surprised when the other mathematics educators who participated in the meeting reported that students' affective responses to challenging mathematical problems were almost universally negative. The research question as they framed it was to explain why this was the case. Our observations in Merkel's classroom directly contradicted the assumption that such reactions are an inevitable aspect of mathematical problem solving. We therefore volunteered to write a chapter for a book that McLeod was editing on this topic. The resulting chapter was shaped by a number of contributions to a book edited by Rom Harré (1986), *The Social Construction of Emotion*. A key insight that we took from this book was that, within the microculture established in a particular classroom, certain emotional acts are warranted when attempting to solve challenging mathematical tasks. This insight in turn implies that the warranted emotional acts can differ significantly from one classroom to another depending on the nature of the social norms that have been established. We therefore concluded that we had observed generally positive responses to mathematical problem solving in Merkel's classroom because she had initiated and guided the negotiation of social norms that contrasted sharply with those established in most US mathematics classroom. The reprinted chapter was out of step with the approaches to affect that mathematics educators took at the time and made little impact. It did, however, represents a significant advance in our thinking in that we were beginning to move beyond our initial view of context solely as a personal orientation to mathematical tasks. In this chapter, we made a first attempt to treat the local world of the classroom as a social context that was established jointly by the teacher and the students.

Yackel, Wood, and I began a 3-year collaboration with Heinrich Bauersfeld, Gotz Krummheuer, and Jorg Voigt in 1990, shortly after the reprinted chapter was published. I met (and shared a room with) Bauersfeld at a conference on cybernetics in 1986. I subsequently visited Bauersfeld and his colleagues at the University of Bielefeld in Germany, and the idea of a sustained collaboration was developed during this meeting. The explicit goal of this collaboration, which was supported by the Spencer Foundation, was to formulate an approach for integrating social and cognitive perspectives on mathematical learning. To this end, we met for a week approximately every 9 months to try and hammer out agreed-upon theoretical constructs. Typically, we began these meetings by viewing a short video-recorded episode from Merkel's classroom. The differences in the theoretical perspectives of the two groups of researchers were such that the resulting discussions of core suppositions and assumptions typically continued for several hours. We eventually concluded that it would not be possible to develop a single, overarching set of constructs and instead attempted to achieve the more modest goal of developing a way of coordinating social and cognitive perspective on mathematical learning.[2] The collaboration was particularly valuable to us as US researchers in that it gave us access to ideas of symbolic interactionism (e.g., Blumer, 1969; Schutz, 1962) and ethnomethodology (e.g., Mehan & Wood, 1975). These ideas proved to be critical in our subsequent work in which we elaborated our nascent social perspective on mathematics classrooms.

The first major step in this elaboration involved the development of the notion of sociomathematical norms (Yackel & Cobb, 1996). This notion originated in an analysis Yackel carried out of a design experiment she had conducted with Willie King,

[2]This collaboration resulted in an edited book (Cobb & Bauersfeld, 1995) in which the six participating researchers each reported an analysis that they had conducted of the video-recordings of Merkel's classroom.

a second-grade teacher who worked in a predominantly African-American inner city school.[3] Yackel's analysis focused on shifts in what counted as an acceptable mathematical explanation and as a legitimate challenge. As the analysis continued, we identified two further norms, what counted as a different and as a sophisticated mathematical solution. It was at this point that we began to realize that these norms were instances of a particular type or class that was distinct from general classroom social norms because they were all specific to mathematical activity. We called norms of this type sociomathematical norms to indicate both that they are jointly constituted by the teacher and students and that they are specifically mathematical.

The final step in our elaboration of a social perspective on mathematics classrooms is discussed in the fourth part of this book and involved the development of the notion of a classroom mathematical practice. We called the resulting perspective the emergent perspective to emphasize that the classroom social context or microculture emerges from (and is continually regenerated by) the teacher's and students' coordinated actions (Cobb & Yackel, 1996).

At the time that we wrote the chapter reprinted in this part of the book, we differentiated between what we termed the social aspects of the classroom, which included classroom social norms, and the cognitive aspects that included students' mathematical reasoning. We subsequently revised this view while developing the notions of sociomathematical norms and classroom mathematical practices, and in doing so questioned our assumption that some aspects of the classroom are inherently social and other aspects are inherently cognitive. We instead came to the view that any aspect of the classroom can be analyzed from either a social or a cognitive perspective. This revised view is central to the resulting emergent perspective as it involved the explicit coordination of a social perspective on classroom events with a cognitive perspective on the teacher's and students' individual interpretations as they participate in those events.

In looking back, we now regard the classroom design experiment that we conducted in Merkel's classroom as limited in two important respects. First, as we discuss in the next chapter of this book, our design of instructional tasks was somewhat ad hoc and, as a consequence, the tasks we developed do not constitute coherent instructional sequences. Second, we did not organize the whole class discussions conducted in Merkel's classroom systematically to achieve a mathematical agenda. To be sure, we worked to ensure that students explained their reasoning, attempted to understand others' reasoning, asked clarifying questions, and so forth. However, we did not build systematically on students' solutions to make sure that key mathematical ideas emerged as the focus of discussion. In later classroom design experiments, we planned more carefully for class discussions by first documenting the range of solutions that students were developing as they worked individually or in groups, and then deciding which students should present their solutions so that mathematically significant issues would emerge as topics of

[3] In this experiment, Yackel and King adapted the instructional tasks that had been developed in Merkel's classroom to the inner city setting in which King worked.

conversation. The discussions in these latter design experiments were often less animated than those in Merkel's classroom, but were superior in terms of the learning opportunities that arose for significant mathematical learning (Cobb, 1998).

The observations that we made in Merkel's classroom during the first days of the design experiment initiated a productive line of inquiry that continued for more than a decade. Certainly, we read widely and worked long hours when conducting subsequent classroom design experiment and when analyzing the resulting classroom video-recordings. In addition, the collaboration with Bauersfeld and his colleagues enabled us to become familiar with a range of theoretical ideas that were largely unknown to US mathematics education researchers. However, more than a little luck was also involved, not least our good fortune to collaborate with two teachers whose classroom practices gave us rich data with which to work. The first was Graceann Merkel, who had a remarkable feel for classroom social relationships and who repeatedly made decisions on her own initiative that supported the development of productive classroom social norms.[4] The second was Willie King, who was skilled in recognizing the potential mathematical significance of his students' contributions to whole class discussions and who made in-the-moment decisions that supported the development of what we later called (productive) sociomathematical norms.[5] The fact that we developed an approach for analyzing classroom social norms while analyzing data from Merkel's classroom and then developed the notion of sociomathematical norms while analyzing data from King's classroom is surely not coincidence. The advances that we made are attributable in large measure to Merkel's and King's complementary strengths as mathematics teachers.

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[4] Merkel continues to teach in the same elementary school but has moved from second grade to first grade.

[5] Sadly, Mr. Willie King died shortly after the design experiment that he conducted with Yackel. At Yackel's suggestion, Heinrich Bauersfeld and I dedicated the book that we edited together (Cobb & Bauersfeld, 1995) to his memory.

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Chapter 5
Young Children's Emotional Acts While Engaged in Mathematical Problem Solving

Paul Cobb, Erna Yackel, and Terry Wood

Several authors in this book have eloquently argued that affective issues in mathematics teaching and learning have long been under-represented themes in research. Our interest in the emotional acts of teachers and children is due in part to Doug McLeod's gentle prodding. In addition, we have recently conducted a teaching experiment in a second-grade classroom for an entire school year. We and others observed that many "nice things" happen in this classroom. The children were generally excited about doing mathematics, were very persistent, did not become frustrated, frequently experienced joy when they completed solutions to personally challenging problems, and did not evidence either embarrassment or jealousy. These observations contrast with the findings of Goodlad's (1983) study of over 1,000 classrooms. He concluded that "affect—either positive or negative—was virtually absent. What we observed could only be described as neutral, or perhaps 'flat'"(p. 467). In fact, the emotional tone of the classroom we observed seemed to contribute substantially to the favorable opinions of the mathematics instruction formed by classroom observers such as parents, other teachers, and administrators. At a minimum, the nurturing of positive emotional experiences for children would seem to have immediate propaganda value.

From the beginning, the classroom teaching experiment gave explicit attention to the children's noncognitive development. The general instructional goals included the promotion of intellectual and moral autonomy (Kamii, 1985) and task-involvement (Nicholls, 1983) as a form of motivation. It was not until we observed and tried to understand what was happening in the classroom that the children's emotional acts in specific situations began to take on greater significance. Initially, we viewed these emotional acts as desirable outcomes and as indicators that things were working out in the classroom as we had hoped. As the year progressed, we

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have increasingly come to view these acts as essential features of the dynamic, self-organizing social system that characterized life in the classroom.

This chapter is divided into four general sections, the first of which presents the current framework within which we are analyzing the children's emotional acts. We then outline the classroom teaching experiment, focusing primarily on methodology, the rationale for the instructional activities, the classroom organization, and the nature of typical classroom social interactions. Next, we present examples of analyses of the emotional acts of children as they occurred within the context of social life in the classroom. Finally, we consider possible implications of this analysis.

Emotional Acts, Beliefs, and Social Context

Like other contributors to this book, we subscribe to the cognitive and constructivist approach to emotions. More specifically, we believe that emotional acts are based on cognitive appraisals of particular situations (Mandler, this volume, Chapter 1). From this perspective, emotions are not uncontrollable impulses that just happen to passive sufferers. Instead, "our capacity to experience certain emotions is contingent upon learning to make certain kinds of appraisals and evaluations. . . . It is learning to interpret and appraise matters in terms of norms, standards, principles, and ends or goals judged desirable or undesirable" (Pritchard, 1976, p. 219). The observation that emotions are generated by cognitive appraisals of particular situations makes it possible to talk meaningfully about the construction of emotions.

As Mandler (this volume, Chapter 1) noted, emotional experiences or feelings typically involve the perception of visceral arousal in concatenation with cognitive appraisals. These two aspects of emotions correspond to the distinction between emotion viewed as a state and emotion viewed as an act (Armon-Jones, 1986a). Emotion as state is concerned primarily with the phenomenological aspect of emotional experiences, with emotions as inner feelings. Emotion as act acknowledges the performatory aspect of emotions, which conveys appraisals relating to some standard or value. Our focus in this chapter will be almost exclusively on emotions as acts. As Harré (1986) observed, "emotion words cannot be the names for the [distinct physiological] agitation since it has been clearly demonstrated that qualitatively one and the same agitation can be involved in many different emotions" (p. 8). In other words, introspection does not reveal the existence of a multitude of distinct feelings that correspond to the subtle linguistic differentiation of our vocabulary for discussing emotions (Beford, 1986). The situation is much the same when we focus on the behavior of someone who is having an emotional experience. The behavior associated with anger, for example, differs across people and occasions; conversely, the same observed behavior is interpreted differently depending on the circumstances. The distinction between shame and embarrassment, for example, depends on whether an individual interprets that he or she is at fault in a situation (Beford, 1986). More generally, we agree with Coulter's (1986) claim that we cannot "*identify* the emotion we are dealing with unless we take into account how a person is appraising an object or situation" (p. 121).

Social Norms

As the example of shame and embarrassment illustrates, emotions have an underlying rationale. Within a culture in general and a local social world, such as an elementary mathematics classroom, in particular, certain emotions are not only warranted in specific situations but, at times, ought to occur (Armon-Jones, 1986a). The teacher in our project capitalized on this aspect of emotions by attempting to teach the children how they ought to feel in particular situations during mathematics instruction. The cognitive basis of emotions is also indicated by the observation that expressions of emotion are open to criticism by reference to the way in which the situation has been interpreted (Armon-Jones, 1986b). We might, for example, attempt to defuse a confrontation by explaining to the angry party that the transgressor did not intentionally infringe on his or her rights. As this example of anger suggests and other analyses show, "the study of emotions . . . will require careful attention to the details of the local systems of rights and obligations, of criteria of value and so on. In short . . . emotions cannot be seriously studied without attention to the local moral order. . . . What is at issue in differentiating emotions are the rights, duties, and obligations of . . . people, *in that culture*" (Harré, 1986, p. 6). In other words, "emotions achieve their qualitative character by being contextualized in the social reality that produces them" (Bruner, 1986, p. 117). It was therefore essential that we pay careful attention to the social norms that the teacher and children mutually constructed when we analyzed emotional acts as they occurred in the project classroom. These acts must be placed in the social context within which they were performed and within which they take on meaning. We will in fact argue that it was because the teacher and children established social norms that contrast sharply with those of typical classrooms that we observed generally desirable emotional acts.

Our concern with the unfolding social world within which we observed emotional acts implies that we will not attempt to abstract particular emotions and treat them as objects that can be studied as independent, detachable objects. We will not, for example, analyze particular emotions, such as joy, but will instead focus on joyful acts as they occur in the concrete world of contexts and activities. Further, because emotional acts have a rationale with respect to the local social order, individuals

> can offer an account of their conduct through an examination of "reasons." The causality of internal and external forces becomes irrelevant. Instead of asking, "What caused me to feel ashamed?" the actor asks, "What were my *reasons* for being ashamed?" The scientific observer may be guided by the same perspective (Sarbin, 1986, p. 92).

In other words, emotions are often used to account for someone's actions. Emotion words, therefore, "set the action to be explained, not merely in the context of the rest of the individual's behavior, but in a social context. . . . Emotion words explain by giving one sort of reason for action, i.e., by giving a justification, or partial justification, for it" (Beford, 1986, p. 30). Consequently, we make an inference about the child's appraisal of a situation each time we attribute a particular emotional quality to his or her actions.

The kinds of appraisals that lead to emotional acts follow the occurrence of some perceptual or cognitive discrepancy in which expectations are violated (Mandler, this volume, Chapter 1). McLeod (1985) has noted that this viewpoint is relevant for researchers interested in students' mathematical problem solving because discrepancies or blockages are precisely what characterize true problem solving. Mandler also observed that emotional acts following a discrepancy are less intense if they are considered to be a routine part of life. Similarly, Hundeide (1985) suggested that "what is interesting and problematic is always related to a *standard* of what is taken for granted as typical and normal . . . and *it is the deviations from that standard that create reactions.* . . . As this standard changes, new experiences become habituated and taken for granted" (p. 311). This insight is particularly relevant to our work because all areas of second-grade mathematics, including arithmetical computation, were taught through small-group problem solving that was followed by discussions involving the whole class. Therefore, it is possible that the blockages that occurred during problem solving were not construed as discrepancies by the children because they took the occurrence of blockages for granted—that is, they expected to encounter difficulties. Such expectations would, of course, be expressions of the children's beliefs about the nature of mathematical activity (Cobb, 1986c; Confrey, 1984). The children's beliefs would therefore seem to be a crucial aspect of what Hundeide called the "standard of normative expectancies" (1985, p. 311).

Sources of Beliefs

If emotional acts are influenced by the nature of the beliefs used to interpret situations, then questions concerning the origin of beliefs immediately become relevant. In our view, students' construction of their beliefs about the nature of mathematical activity involves drawing on paradigm cases to thematize their experiences of doing mathematics. These experiences include interactions with the teacher and peers in the classroom. As Balacheff (1986) noted:

> We have to realize that most of the time the pupil does not act as a *theoretician* but as a *practical man.* His job is to give a solution to the problem the teacher has given, a solution that will be acceptable with respect to the classroom situation. In such a context the most important thing is to be effective (p. 12).

A student's realization that he or she has failed to fulfill an obligation can itself give rise to a problematic situation for the student. To the extent that the student wants to fulfill the obligation, situations of this sort can precipitate strong emotions. It should be noted that this case is distinct from that in which the student has experiences that confound what he or she takes for granted. For example, a student might repeatedly fail to meet the teacher's expectations. Failure (as the teacher defines it) is the norm for this student, and he or she expects to fail in the future. Nonetheless, each future failure might be quite traumatic for the student, and give rise to feelings of guilt or embarrassment depending on the circumstances. Here, the discrepancy is with respect to the student's understanding of what he or she is expected to accomplish rather than with respect to his or her understanding of what is typical. In fact, anticipations of typical experiences can lead to emotional acts.

Consideration of students' obligations emphasizes the claim that emotional acts occur within the context of the local social world. This point is crucial to our analysis because the teacher we worked with actively attempted to place the children under certain obligations that differed markedly from those of typical classrooms. In other words, the teacher and children mutually established a nonstandard working consensus (Hargreaves, 1975) or didactical contract (Brousseau, 1984). We are attempting to understand this contract by first identifying the regularities or patterns that occurred in classroom social interactions. For the most part, these patterns are outside the conscious awareness of both the teacher and the students and are repeatedly constructed in the course of interactions (Voigt, 1985). Thus, although the teacher and students did not have a blueprint of the interaction patterns, each subconsciously knew how to act appropriately in specific situations as they arose. By teasing out these patterns, one can infer the largely implicit social norms negotiated by the teacher and students, the norms that structured the local social reality within which they taught and learned mathematics and from which emotional acts derived meaning.

The interaction patterns and associated social norms can themselves be analyzed in terms of both the largely implicit, taken-for-granted obligations that the teacher and students accepted in particular situations and the expectations that they had for each other (Voigt, 1985). Such an analysis simultaneously addresses the teacher's and student's beliefs about their own and each others' roles as they were played out while interacting in the classroom. These beliefs, together with beliefs about the nature of mathematical activity, would seem to constitute a substantial part of the standard of normative expectancies. It is perceived discrepancies with these expectations that give rise to emotional acts.

Thus far, we have argued that emotional acts are generated by cognitive appraisals of situations, and that these appraisals are influenced by the local social order. The appraisals involve a comparison of the interpreted situation with expectations. As Averill (1986) stated, "In cognitive terms, emotions may be conceived of as belief systems or schemas that guide the appraisal of situations, the organization of responses, and the self-monitoring (interpretation) of behavior" (p. 100). With regard to mathematical problem solving, beliefs about the nature of mathematical activity and about one's own and others' roles in the classroom would seem to be particularly relevant. These beliefs are constructed in an attempt to make sense of classroom life during mathematics instruction. Our emphasis on the cognitive basis of emotions as acts in no way denies that people feel emotions or that they may, on occasion, feel gripped by a particular emotion that is beyond their control. In our view, these sometimes intense emotional experiences are generated by subjective cognitive interpretations of particular situations.

As a further point, emotional acts are "functional in that they are constituted and prescribed in such a way as to sustain and endorse cultural systems of beliefs and value" (Armon-Jones, 1986b, p. 57). In other words, emotional acts play a role in the development and regeneration of the obligations and expectations that regulate activity in such situations as a classroom during mathematics instruction. We have noted that emotional acts have a rationale, and that this rationale derives in part from the local order (or at least the acting individual's understanding of it as represented by his or her beliefs). Emotional acts that are warranted in one social context might well be completely inappropriate in another. This would seem to be the case with the second-grade classroom we studied when compared with a more typical classroom. Occasions when the children in our class make socially appropriate emotional acts (e.g., rushing excitedly to the teacher to tell her about their solution to a personally challenging problem) served to sustain and endorse the beliefs about mathematical activity and themselves that the teacher had attempted to nurture. In effect, these emotional acts serve the social function of helping to keep the communal story about doing mathematics alive. The emotional sentiment of their actions indicated not only their sincerity but also the significance and importance they attributed to the story. Further, the social appropriateness of their emotions made them autonomous adherents to the story in a way that mere rational comprehension of it would not (Armon-Jones, 1986b, p. 81). This simultaneously served to regulate socially undesirable behavior. It is one thing for a child to understand that some act would transgress particular norms, and quite another to know that he or she will feel guilty after doing it.

Thus far, we have claimed that appropriate emotional acts sustain social norms. Conversely, socially inappropriate emotional acts indicate either that the student has misinterpreted others' intentions or that the student's beliefs are incompatible with social norms that are acceptable to the teacher and other students. Because these acts are open to criticism by reference to norms, their occurrence constitutes opportunities for the teacher and other students to initiate a dialogue about beliefs and values. Further, socially undesirable acts can be criticized by explaining that their construal by others will constitute a cognitive basis for negative emotions. In other words, the student can be told that his or her actions in this social context will probably make other people feel bad. Finally, the teacher does not have to wait for students to evidence a particular emotion, whether positive or negative, before ascribing that emotion (Armon-Jones, 1986b). For example, delight can be prescribed in those situations in which the teacher believes that students *ought* to feel pleased with their accomplishments (e.g., situations in which they persisted and solved a personally challenging problem through their own efforts). In attempting to understand the prescription and feel delighted, students have the opportunity to reorganize their beliefs about the nature of mathematical activity. In the last analysis, however, it is the students who have to make construals that constitute a cognitive basis for delight.

In summary, emotional acts not only support but can actually play a role in the teacher's and students' mutual construction of social norms. This certainly appears to be the case with the classroom that we observed. We discuss the essential features of the teaching experiment in the following section before analyzing examples of emotional acts as they occurred in the classroom.

Overview of the Teaching Experiment

The teaching experiment was conducted in one second-grade public school classroom for the entire school year as part of a 3-year research and development project. The experiment had a strong pragmatic emphasis in that we were responsible for the mathematics instruction of the 20 children in the classroom; thus, we had to accommodate a variety of institutional constraints while developing and implementing instructional activities in a manner compatible with constructivist learning theory (Cobb & von Glasersfeld, 1984; Piaget, 1970, 1980; von Glasersfeld, 1983, 1984). Not surprisingly, the constraints profoundly influenced the ways in which we attempted to develop a form of practice compatible with both constructivism as a general theory of knowledge and specific models of early number learning (Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe, Cobb, & von Glasersfeld, 1988). We were fortunate in that the classroom teacher was a member of the project staff. Her practical wisdom and insights proved to be invaluable.

Methodology

The teaching experiment conducted in the classroom is a natural extension of the constructivist teaching experiment methodology, in which the researcher interacts with a single child and attempts to guide the constructive activities of the child (Cobb & Steffe, 1983; Steffe, 1983). In our view, the two methodologies are appropriate for different phases of a research program (Cobb, 1986a). Both methodologies allow the researcher to focus on the critical moments when children make cognitive restructurings as they develop increasingly powerful ways of knowing mathematics. In the case of the classroom teaching experiment, these restructurings occur as the children interact with the teacher and their peers rather than with the researcher. The methodology also allows the researcher to address a variety of related issues; the most important is to embed the children's learning of mathematics in social context. To this end, the mutually constructed social norms are analyzed in terms of the teacher's and children's obligations and expectations in the classroom. The children's emotional acts both take meaning from and contribute to the construction and continual regeneration of these norms.

The classroom teaching experiment also bears certain resemblances to a type of Soviet experiment that Menchinskaya (1969) called a macroscheme: "Changes are studied in a pupil's school activity and development as he [or she] makes the transition from one age level to another, from one level of instruction to another" (p. 5). However, there is a crucial difference between our approach and that of the Soviet researchers. Typically, Soviet investigators construct the instructional materials before the experiment begins (e.g., Davydov, 1975). We, in contrast, developed samples of a wide range of possible instructional activities in the year preceding the experiment, but the specific activities used in the classroom were developed, modified, and in some cases, abandoned while the experiment was in progress. To aid this process, two video cameras were used to record every mathematics lesson for the school year. While working in groups, eight children faced the cameras. Consequently, it was possible to record the problem-solving activity of approximately half of each of the four pairs of children. Initial analyses of both the whole class dialogues and the small group interactions focused on the quality of the children's mathematical activity and learning as they completed and discussed their solutions to specific instructional activities. These analyses, together with the classroom teacher's observations, guided the development of instructional activities and, on occasion, changes in classroom organization for subsequent lessons. Thus, the processes of developing materials, conducting a formative assessment, and developing an initial explanation of classroom life were one and the same.

Rationale for Instructional Activities

From the constructivist perspective, substantive mathematical learning is an active problem-solving activity (Cobb, 1986b; Confrey, 1987; Thompson, 1985; von Glasersfeld, 1983). This is the case even when students receive direct recitation instruction. In this context, substantive learning refers to cognitive restructuring as opposed to accretion or tuning (Rumelhart & Norman, 1981). Consequently, our primary focus as we developed, implemented, and refined instructional activities was on that aspect of cognitive development that is both the most significant and the most difficult to explain and influence.

At the risk of oversimplification, an immediate implication of constructivism is that mathematics, including the so-called "basics," such as arithmetical computation, should be taught through problem solving. Admittedly, the application of an efficient computational algorithm, once constructed, can be a routine task for a child; however, the process of constructing such algorithms is characterized by active problem solving (Cobb & Merkel, 1989; Kamii, 1985; Labinowicz, 1985) in which conceptual and procedural developments should ideally go hand in hand (Cobb, Wood, & Yackel, 1991b; Hiebert & Lefevre, 1986). The concern with mathematical problem solving does not mean that the instructional activities necessarily emphasize what are traditionally considered to be problems—stereotypical textbook word problems. The general notion that readymade problems can be given to students is highly questionable; instead, teaching through problem solving acknowledges that problems arise for students as they attempt to achieve *their* goals in the classroom. The approach respects that students are the best judges of what they find problematic and encourages them to construct solutions that are acceptable to them given their current ways of knowing. The situations that children find problematic take a variety of forms, including resolving obstacles or contradictions that arise when they use their current concepts and procedures, accounting for surprising outcomes (particularly when two alternative procedures lead to the same result), verbalizing their mathematical thinking, explaining or justifying solutions, and resolving conflicting points of view. As these examples make clear, genuine mathematical problems can arise from classroom social interactions as well as from attempts to complete the instructional activities. (A more detailed discussion of problematic situations can be found in Cobb, Wood, & Yackel, 1991a.)

Classroom Organization

The instructional activities were of two general types: teacher-directed wholeclass activities and small-group activities. To the extent that any lesson can be considered typical (Erickson, 1985), the teacher would first spend at most 5 minutes introducing the small-group activities to the children to clarify the intent of the activities. She would, for example, ask the children what they thought a particular symbol meant or ask them how they interpreted the first activity. In doing so, she did not attempt to steer the children towards an official solution method, but instead tried to ensure that the children's understanding of what they were to do was compatible with the intent of the activity as she understood it. Any suggested interpretation or solution, however immature, was acceptable provided it indicated the child had made appropriate suppositions.

Next, a child gave an activity sheet to each group of two or, occasionally, three children. As the children worked in groups for perhaps 25 minutes, the teacher moved from one group to the next, observing and frequently intervening in their problem-solving efforts. Children moved around the classroom on their own initiative. Some went to a table to get one of the available manipulatives that they had decided was needed. Others got additional activity sheets or perhaps a piece of scrap paper. Some groups completed four or five activity sheets while others completed only one, with the teacher's assistance. Finally, the teacher told the children when there was only 1 minute of work time remaining. Most of the children began to put away the manipulatives and prepared for the discussion of their solutions.

The teacher started the discussion by asking the children to explain how they solved the first activity. Sometimes she asked follow-up questions to clarify the explanation or to help the child reconstruct and verbalize his or her solution. Occasionally, a child would become aware of a problem with his or her solution while explaining it to the class. Because of the accepting classroom atmosphere, the child did not become embarrassed or defensive but might simply say "I disagree with my answer" and sit down. It was immediately apparent that the teacher accepted all answers and solutions in a completely nonevaluative way. If, as frequently happened, children proposed two or more conflicting answers, she would frame this as a problem for the children and ask them how they thought the conflict could be resolved. Children volunteered to justify particular answers and, typically, the class arrived at a consensus. On the rare occasions when they failed to do so, the teacher wrote the activity statement on a chalk board so that the children could think about it during the following few days. Although the discussion might have continued for 15 or 20 minutes, the time was sufficient to consider only a small proportion of the activities completed by some groups (the children have much to say about *their* mathematics). Eventually, the teacher terminated the discussion owing to time constraints. She collected the children's activity sheets and glanced through them before distributing them to parents; however, she did not grade their work or in any way indicate whether or not their answers were correct.

In the remaining 10 minutes of the 1-hour lesson, the teacher introduced a whole-class activity and posed one or more questions to the children. She was again nonevaluative when the children offered their solutions and, as before, attempted to orchestrate a discussion among the children.

Classroom Social Interactions

The teacher's overall intention as she led the class in discussions was to encourage the children to verbalize their solution attempts. Such dialogues give rise to learning opportunities for children as they attempt to reconstruct their solutions (Levina, 1981), distance themselves from their own activity in an attempt to understand alternative points of view (Sigel, 1981), and resolve conflicts between incompatible solution methods (Perret-Clermont, 1980). However, the teacher's expectation that the children should verbalize how they actually interpreted and attempted to solve the instructional activities ran counter to their prior experiences of class discussions in school (Wood, Cobb, & Yackel, 1988). The teacher, therefore, had to exert her authority in order to help the children reconceptualize their beliefs about both their own roles as students and her role as the teacher during mathematics instruction. She and the children initially negotiated obligations and expectations at the beginning of the school year, which made possible the subsequent smooth functioning of the classroom. Once established, this mutually constructed network of obligations and expectations constrained classroom social interactions in the course of which the children constructed mathematical meanings (Blumer, 1969). The patterns of discourse served not to transmit knowledge (Mehan, 1979; Voigt, 1985) but to provide opportunities for children to articulate and reflect on their own and others' mathematical activities.

The teacher's and students' mutual construction of social as well as mathematical realities was reflected in the dual structure of classroom dialogues. At one level, they talked about mathematics; at another level, they talked about talking about mathematics. As in a traditional classroom, the teacher was very much an authority figure who attempted to realize an agenda. The difference resided in the way she expressed her authority in action (Bishop, 1985). When she and the children talked about talking mathematics, the teacher typically initiated and attempted to control the conversation. When they talked about mathematics, however, she limited her role to that of orchestrating the children's contributions. These two conversations were conducted at distinct logical levels (Bateson, 1973), one in effect serving as a framework for the other; therefore, it makes sense to say that the teacher exerted her authority to enable the children to say what they really thought.

The following dialogue, which is taken from an episode that occurred at the beginning of the school year, illustrates the development of the basic pattern of interaction during whole-class discussions. The dialogue centers on word problems that were shown on an overhead projector.

Teacher:	There are 6 more tulips behind the rock [4 are in front of the rock]. How many in all? What do we have to do to figure it out? Kara.
Kara:	10.
Teacher:	How did you figure this out? . . . Kara, how did you get your answer?
Kara:	I got 6 and then added 4 more.
Teacher:	She got 6 and added 4 more. Did anybody else get that answer or maybe did it a different way? Yes, Andrew.
Andrew:	11.
Teacher:	You had 11. How did you get 11?

Because the teacher wanted to make the children aware that she respected their solutions to the problems and, at least implicitly, help them come to believe that mathematical solutions should be justifiable, her response at this point deviated distinctly from typical patterns of classroom interaction (Mehan, 1979). Instead of evaluating Andrew's response, she asked him for an explanation.

Andrew:	Well. Um. Wait a minute. There would be 10 flowers.
Teacher:	How did you discover that?
Andrew:	If it was 4 flowers, 6 flowers in front and 6 flowers in back. That would equal up to 12. If you took 2 away to make 4 in front and 6 in back it would make 10.
Teacher:	Did anybody else do it a different way? Lisa.
Lisa:	5, 6, 7, 8, 9, 10 (counting on her fingers).

The teacher's nonauthoritarian, nonevaluative role as she orchestrated the children's contributions to discussions about mathematics was in contrast to her directive interventions when she initiated and guided conversations in which she and the children talked about talking about mathematics. The following incident occurred later in the same episode.

Teacher:	Take a look at this problem. "The clown is first in line. Which animal is fourth?" Peter.
Peter:	The tiger.
Teacher:	How did you decide the tiger? . . . Would you show us how you got the fourth?
Peter:	(Goes to the screen at the front of the room.) I saw the clown and then . . . (He counts the animals.) Oh, the dog [is fourth]. (He hesitates.) Well, I couldn't see from my seat. (He looks down at the floor.)
Teacher:	Okay. What did you come up with?
Peter:	I didn't see it. (He goes back to his seat quickly.)

The teacher realized that in making Peter obliged to explain his solution, she had put him in the position of having to admit that his answer was wrong in front of the entire class. Peter construed this as a situation that warranted embarrassment. Crucially, the teacher was immediately directive in her comments as she talked about talking about mathematics.

Teacher: That's okay, Peter. It's all right. Boys and girls, even if your answer is not correct, I am most interested in having you think. That's the important part. We are not always going to get answers right, but we want to try.

She expressed her expectations for the children by telling them how she as an authority interpreted the situation. She emphasized that Peter's attempts to solve the problem were appropriate in every way, and simultaneously expressed to the other children her belief that it was more important in this class to think about mathematics than to get right answers.

As the preceding example illustrates, it was the teacher who typically initiated the mutual construction of obligations and expectations in the classroom. In doing so, she simultaneously had to accept certain obligations for her own actions. If she expected the children to honestly express their current understandings of mathematics, then she was obliged to accept their explanations rather than to evaluate them with respect to an officially sanctioned method of solution; thus, the teacher had obligations to the children, just as they did to her. The interlocking obligations and expectations established by the teacher and her class constituted a trusting relationship. The teacher trusted the children to resolve their mathematical problems, and they trusted her to respect their efforts.

These obligations and expectations influenced the children's activity as they worked in small groups, because the children anticipated that they would have to explain and, if necessary, justify their solutions. In addition, the teacher attempted to place the children under the obligation of solving problems in a cooperative manner and of respecting each others' efforts when they worked in groups. As with the whole-class setting, the teacher was explicit about what she expected of the children as they worked together in small groups (Wood & Yackel, 1988). The children were obligated to explain their solution methods to each other and, at a minimum, to agree on a common answer. When possible, the teacher also encouraged them to agree on a solution method.

Our discussion of obligations and expectations directly addresses the children's evolving beliefs about their own and the teacher's role. In addition, we have implicitly dealt with certain aspects of the children's beliefs about the activity of doing mathematics, another crucial aspect of the standard of normative expectancies. For example, the whole-class obligations nurtured the belief that mathematical activity should be explainable, justifiable, and rationally grounded. Further, the children had the opportunity to view mathematics as an activity under their control rather than as disembodied, objectified, subject matter content. The children also came to realize that mathematical problems can have multiple solutions. With regard to small-group work, the children's acceptance of the obligation to think through their problems for themselves, as indicated by their persistence, evidenced the belief that it sometimes takes hours (literally) rather than minutes to solve mathematical problems.

Examples from the Classroom

The teacher's insistence that the children reach a consensus as they worked in groups meant that the children had two distinct types of problems to solve. The first concerned the mathematical problems that arose as they attempted to complete the instructional activities; the second involved the negotiation of a viable cooperative relationship that would make it possible for them to solve *their* mathematical problems. Emotional acts occurred as the children attempted to resolve each of these two types of problems. In this chapter, our primary focus is on the emotional acts related to doing mathematics rather than those related to the problem of cooperation. With regard to the pragmatics of the classroom, a basic level of cooperation was necessary if the children were to construct mutually acceptable solutions to the mathematics activities (Cobb, Wood, & Yackel, 1991a). Consequently, children were assigned to different partners if they were unable even with teacher intervention to solve the problem of cooperation over an extended period of time. This became increasingly rare as the year progressed.

Construction of Classroom Norms

The emotional acts displayed in a specific situation depend on the interpretation given to the situation, which in turn depends on the social norms (or at least understanding of the norms, as represented by beliefs). The teacher played a crucial role in initiating the mutual construction of the classroom norms. On numerous occasions, she brought specific situations to the attention of the whole class and asked the children to elaborate on their feelings in that situation. In effect, the teacher told the children how they ought to construe the situation. For example, the following episode occurred at the beginning of a class discussion that followed small-group work. One pair of children volunteered that they had spent the entire 20 minutes allocated to group work on a single problem.

Kara and Julie: Because at first we didn't understand it.
Teacher: How did you feel when you finally got your solution?
Kara and Julie: Good!

Kara and Julie's excitement at having solved the activity was indicated by the manner in which they stood up and almost jumped up and down on the spot during this interchange with the teacher. Julie went on to explain that she had wanted to go on to another activity but that Kara had insisted that they continue working until they had solved the problem. By calling the attention of the entire class to this incident, the teacher demonstrated that the two girls had construed the situation appropriately. In doing so, she implicitly ruled out as inappropriate construals that would lead to feelings of embarrassment, inadequacy, or stupidity at having completed only one activity, even though a number of groups had completed several. At the same time, she illustrated that the classroom social norms are such that "feeling good"—that is, a feeling of satisfaction and pride in one's own accomplishments—is a socially acceptable emotional response to situations in which the children fulfill their obligation of persisting in problem solving and complete a personally challenging task. By way of contrast, the teacher never drew attention to a group that had completed a relatively large number of activities.

The teacher repeatedly emphasized that figuring out problems for yourself ought to make you "feel good." The following episode occurred less than a week after the one just reported. Children had been working in pairs on several problem-solving tasks and found one activity particularly challenging. The following dialogue is part of the total-class discussion that followed small-group work.

Teacher: One of the problems you're having is figuring out what you're expected to do. (The teacher then said that they would talk about the problem to clarify what is expected.)
Andy: Wow! I figured it out.
Teacher: What if someone asks you for the answer?
Andy: I won't tell them.
Teacher: Good for you. Let them figure it out for themselves and get the enjoyment out of figuring it out for themselves. It makes us feel so good when we do something.

In both of these episodes, the teacher used the children's emotional acts to endorse and to sustain the construals from which the emotional acts were derived. The children's positive emotional acts indicated that their understandings of the classroom norms were appropriate and, through the teacher's interventions, served to sustain and perpetuate the norms.

The teacher also capitalized on situations that arose naturally in the classroom in an attempt to indicate to the children that some acts were socially undesirable in this classroom because they might make people feel bad. The following episode is extracted from a dialogue that occurred at the beginning of a whole-class discussion about an activity that was designed to encourage the construction of increasingly sophisticated concepts of 10 and increasingly sophisticated thinking strategies. The episode began with the teacher talking about incidents that she had observed and that she considered inappropriate with respect to the social norms of the classroom.

Teacher: Now another thing I noticed was happening, and it is something I *don't* like and I *don't* want to hear because it makes me feel bad, and if it makes me feel bad it probably makes someone else in here feel bad. It's these two words. (She writes "that's easy" on the chalkboard and draws a circle around the phrase.) These words are no, no's starting today. What are these two words, Mark?
Mark: That's easy.
Teacher: That's right. When we are working in math, I've had kids come up to me and say, "Oh, that's easy!" Well, maybe I look at it and say, "Gee

whiz, I don't think that's very easy." How do you think that's going to make me feel?

Brenda: Bad.

The teacher listened as several other children offered their suggestions and then explicitly told them her interpretations.

Teacher: . . .It hurts my feelings when someone says, "Oh that's easy!" (She points to the words on the board.) When I am struggling and trying so hard, it makes me feel kind of dumb or stupid. Because I am thinking, gosh, if it's so easy why am I having so much trouble with it?

She closed the conversation by explicitly telling the children that saying "that's easy" violates a social norm.

Teacher: So that's going to be something we are not going to say. You can think it if you like, but I don't want you to say it out loud because that can hurt other people's feelings. And what's one of our rules in here? It's to be considerate of others and their feelings.

Interpreting Situations

Another way in which the teacher initiated the mutual construction of the social norms that determined the appropriateness of emotional acts was to articulate alternative interpretations of situations. In the following example, the children were working on an activity about time in which they were to answer questions individually about their daily schedules. Andy had completed his own answers and began to tell John what to write for his answers, but John refused to accept Andy's suggestions. At this point, the teacher arrived to observe the pair's activity.

Teacher: How are you gentlemen doing here? Okay, whose side is whose here? . . . Okay, it says, "What time do you each go to bed?" You [John] go to bed at 8 o'clock. You [Andy] go to bed at 8:30. And what time do you get up? You both get up at 7. How many hours do each of you sleep?

John: I'm still figuring that out.

Andy: I sleep 11 hours.

Teacher: (To Andy.) Will his [John's] time be the same time as your time?

Andrew: Un-uh. I told him 111/2 hours for him. But he doesn't believe me.

John: I don't know.

Teacher: (To John.) You just don't know. (To Andy.) It's not that he doesn't believe you. Maybe he's just not really sure.

John: I'm not really sure. (The teacher leaves and John writes in his answer.)

Andy tried to meet his obligation of helping his partner work out the solution to the problem. In Andy's view, John rejected his attempt to be helpful, and Andy showed severe irritation. But the teacher suggested an alternative explanation. In this classroom, children were also obligated to think things through for themselves and try to make sense of them. John was still trying to figure out how to solve the problem and did not want to simply accept Andy's answer. By proposing this alternative explanation, the teacher encouraged Andy to construe the situation in a different way. In effect, she told Andy that John's rejection of his answer did not warrant irritation and that he ought to consider the possible reasons for John's rejection.

The teacher also attempted to defuse confrontations that arose in the total-class setting by suggesting interpretations that would make a child's emotional acts unwarranted. The following brief episode is taken from a whole-class discussion in which one child complained angrily that another had provided the answer to a problem.

Teacher: Well, I think what was happening—I think he [Ronnie] was proud he had the answer because we had worked that out.

In this situation the "offending" child, Ronnie, was the weakest student in the class. The teacher's comment reflects an interpretation that he may have been so proud that his group had solved the problem that he wanted to share the accomplishment with others. Interpreted in this way, Ronnie's intent was not to violate the classroom norms that obliged children to figure out problems for themselves; consequently, a negative sanction, such as expression of anger, against Ronnie was unwarranted.

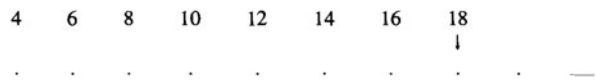
Thus far, we have focused on reinterpretations initiated by the teacher. The children also responded to the emotional acts of their partners by suggesting alternative construals and, in doing so, sustained the classroom social norms. The following episode is taken from a class period in which the children solved story problems in small groups. After 20 minutes of intense effort, Andy and Rodney were still working on their first problem.

Rodney: I've been sitting here all day not figuring out this problem. Look at John and them. They're on their third problem.
Andy: They didn't get this one.

Andy's remark expressed the belief that persistence and figuring out personally challenging problems were expected in this class, and that fulfilling this obligation was more important than completing as many problems as possible. His statement implicitly indicated to Rodney that he should not interpret the situation as warranting negative emotions such as frustration or anxiety.

Another example from small group work early in the year shows how a task-motivated child, Kara, responded to an ego-motivated child, Lois, by suggesting an alternative interpretation of a situation. Kara and Lois had just completed the following problem.

The problem is to determine the number indicated by the arrow and to indicate at the right a relationship between successive numbers (e.g., plus 2, add 2, or skip 1 number). The children's paper looked like this:



Lois: Erase those [numbers above the intermediate dots] because then they can't tell that you did that.

Kara: We should leave them in because they help you.

Lois' comment indicates the public self-awareness that is typical of ego-involvement (Nicholls, 1983) because she would be embarrassed if others ("they") saw that she and Kara had used what she considered to be a relatively immature method. Such embarrassment would be appropriate in a situation in which children are publicly compared with others on the basis of their solution methods. Kara's comment implies that these concerns were misplaced in this classroom and that whatever method they used to solve a problem was acceptable. With regard to Kara's interpretation, embarrassment was not warranted in the event that others saw their paper.

Beliefs About the Teacher's Role

Unlike traditional mathematics classes in which children typically experience frustration when they encounter situations in which they do not know what to do (McLebd, this volume, Chapter 2), children in the project classroom quickly learned that not knowing what to do was routine. Also, the process of genuine problem solving became an overriding feature of mathematical activity. The children's understanding of the teacher's role as one of framing problematic situations and facilitating solution processes developed simultaneously with their beliefs about the nature of mathematical activity. The teacher frequently responded to students' questions about what they were "supposed to do" with "I don't know" or "You are going to have to figure that out for yourself." Students accepted the teacher's response from the outset but interpreted it differently as the year progressed. To illustrate the students' evolving beliefs about the teacher's role, we first consider an episode that occurred early in the year. One pair of children was working on an arithmetical task in which they had to decide what number to put in an empty box to equilibrate a pan balance.

Ann: I'm going to ask Mrs. M if we're supposed to add or subtract. (Ann goes off to talk to the teacher. She returns and reports to her partner.)

Ann: She doesn't know either.

Later in the year, Ann came to understand that the teacher's failure to tell her what to do was not an indication of ignorance, but instead implied that she expected the children to figure things out for themselves. This was indicated by Ann's recommendation to another child; "Don't ask her [Mrs. M.]. She won't tell you." Once the children had reconceptualized their understanding of the teacher's role, they had no reason to become angry or frustrated because she would not tell them what they were to do. At the same time, the children realized that they were not expected to use any particular method that the teacher had in mind.

Beliefs About Doing Mathematics

At the beginning of the school year, the teacher guided the mutual construction of social norms that made it possible for the children to freely express their mathematical ideas for the remainder of the year. The following episode occurred during the third day of the school year. The discussion centered on the word problem "How many runners altogether? There are six runners on each team. There are two teams in the race."

Teacher: Jack. What answer-solution did you come up with?
Jack: 14
Teacher: 14. How did you get that answer?
Jack: Because 6 plus 6 is 12. 2 runners on 2 teams . . . (Jack stops talking, puts his hands to the side of his face and looks down at the floor. Then he looks at the teacher and then at his partner, Ann. He turns and faces the front of the room with his back to the teacher, and mumbles incoherently.)
Teacher: Would you say that again. I didn't quite get the whole thing. You had . . . Say it again, please.
Jack: (Softly, still facing the front of the room.) It's 6 runners on each team.
Teacher: Right.
Jack: (Turns to look at the teacher.) I made a mistake. It's wrong. It should be 12. (He turns around and faces the front of the room.)

The teacher realized that Jack had interpreted the situation as warranting acute embarrassment. His concern with social comparison confounded the teacher's desire that the children should feel free to publicly express their thinking. For her purposes, it was vital that children feel no shame or embarrassment when they present solutions in front of others. Consequently, she immediately responded:

Teacher: (Softly.) Oh, okay. Is it okay to make a mistake?
Andrew: Yes.
Teacher: Is it okay to make a mistake, Jack?
Jack: (Still facing the front of the class.) Yes.
Teacher: You bet it is. As long as you're in my class it is okay to make a mistake. Because I make them all the time, and we learn from our mistakes, a lot. Jack already figured out, "Oops. I didn't have the right answer the first time" (Jack turns and looks at the teacher and smiles), but he kept working at it and he got it.

Later in the year, as the following episode illustrates, admissions of mistakes were no longer construed as warranting embarrassment or shame but were instead viewed simply as events that occur in the normal course of classroom life.

Charles: 67.
Teacher: 67. (She starts to write the answer.)
Joel: Disagree.
Teacher: All right, Joel. What do you think?
Joel: 72.
Teacher: You think it's 72 (several students disagree).
Joel: Well . . . (he stands up and walks to the front of the class).
Teacher: Let's listen to Joel's explanation.
Joel: (Stands looking at the board.) I used 25 and 10 makes 35. And another 10 makes 45 and another 10 makes 55. (He stops and looks at his paper in his hand.) Another makes 65 (pause) [and 2 more make] 67. (He turns and looks at the teacher.) I disagree with my answer. (He smiles.)
Teacher: (Laughing) I like that. I disagree with my answer. That's great. Raise your hand if you ever disagree with your own answer. It happens to all of us.

Because children came to believe that doing mathematics is essentially a problem-solving activity, typical negative emotions such as anxiety, embarrassment, and shame, which accompany the obligation of producing publicly evaluated solutions to a large number of tasks in a quick, error-free manner by using prescribed methods, did not occur in this classroom. As in any classroom, however, occasions when a child transgressed the social norms were construed by others as situations warranting negative emotional acts. For example, because the children felt obliged to figure things out and to be able to justify their answers, they did exhibit frustration, disappointment, and sometimes anger with their peers when they were denied the opportunity to do just that. In these cases, the negative emotional acts were directed at other children and not at mathematics or the teacher. The following episode illustrates this point. Connie and Rodney worked on the problem of determining which number goes in place of the arrow in the following number-dot sequence:

Rodney had figured out that 18 should go above the first dot after 13, but Connie did not understand how he arrived at this result. Ann, a member of a neighboring pair, leaned over to tell them the answer that she and her partner had came up with.

Ann: The answer is 21.
Connie: (To Rodney) The answer is 21.
Rodney: (Angrily) No it isn't. This is—look! I'm going to figure it my *own* way.
Connie: I already told you.
Rodney: I don't want to copy.

(At this point, the teacher comes by to observe the group working.)

Rodney: (To the teacher) I don't want them to tell me the answer!

Teacher: They want to—I know you're trying to work—

Rodney: (Interrupting her) I want to try to work it out myself, but they're over here telling me the answer and everything. I think it should be 18 because . . . (Rodney explains his thinking to the teacher.)

The anger Rodney displayed in this episode was appropriate according to the norms established in this classroom. Children were expected to construct justifiable solutions and not simply fill in answers. Rodney's display of anger sustained those norms and served to remind both Ann and Connie that their actions had deprived him of the opportunity to fulfill his obligations. The teacher's affirmation of the rationale for his anger gave his interpretation of the situation further credence.

Later in the year, the following incident occurred during a whole-class discussion. The episode began as the teacher called on Dan and his partner Brenda for a solution.

Teacher: Dan.

Dan: They [another pair] were bothering us.

Brenda: They were telling us the answers.

Teacher: Oh. . . . You know when people give you the answers, boys and girls, does that really help you understand what you're doing? You don't know how you got it, you might as well just not waste your pencil.

The teacher recognized the appropriateness of Dan and Brenda's complaint and immediately initiated a conversation in which she reminded the children that understanding what they were doing mathematically was paramount.

Teacher: If you don't know what you are doing, it isn't going to help you get the answer. It's like saying, "Yup the answer is 7. Yes, I got it right. How did I get 7? I *don't* know." That doesn't help you one single bit. I know you are all friends. I know you want to help each other, but you help each other more by helping each other figure out an answer, rather than saying "7. Just write down 7. That's the answer. Trust me." You have to try and understand it.

By explicitly telling the children that it was wrong to give others the answer, the teacher used the incident to indicate to the rest of the class that Dan's and Brenda's indignation was warranted. She further sustained the social norm by posing a problematic situation for Dan and Brenda to solve.

Teacher: How did you handle the situation?

Brenda: We just said we didn't want the answer. . . . We were on the same question, and they were telling us the answer. We didn't pay any attention, because we wanted to figure it out for ourselves.

Teacher: Good! Good for you. I'm proud of you. It's easy to take someone else's answer, isn't it, than to think about it yourself?

Students: Yeah.

Teacher: Sure it is. I could just fill these all in (points to the problems) and say, "These are the answers, kids." But would you learn anything?

Brenda: (Interrupts) We have to think for ourselves. We can't have other people think for us. . . .They might be wrong.

Teacher: That's right.

Positive Emotional Acts

The project classroom was characterized by both a general absence of negative emotional acts and the frequent occurrence of positive emotional acts when solving mathematical (as opposed to social) problems. Visitors to the project classroom invariably remarked about the excitement for mathematics displayed by the children as they solved the activities. Children frequently jumped up and down, hugged each other, and rushed off to tell the teacher when they solved a particularly challenging problem. Significantly, the positive emotional acts occurred when the children completed personally challenging tasks or constructed mathematical relationships. Because doing mathematics is thought by many, including many mathematics educators, to be associated with negative emotion (McLeod, 1985), it is especially important to clarify that, in the project classroom, positive emotional acts were not reactions to extraneous factors, such as receiving extrinsic rewards or ego satisfaction, but stemmed directly from mathematical activity. In this sense, the emotional acts of the children parallel those of the mathematician when solving a problem or developing an elegant proof (Silver & Metzger, this volume, Chapter 5). To illustrate, we present examples from both whole-class discussions and small-group work.

The whole-class episode began with a problem from an activity called *number-line*.

The children were to figure out what number goes above the dot indicated by the arrow and to construct a relationship between successive numbers.

Teacher: We have a 3, 8, 13, and then nothing. Ann, how did you and Alex do this?

Ann: We got 25.

Teacher: Okay. This is 25. (On an overhead transparency, she writes 25 over the dot where the arrow is pointing.)

Alex: The pattern was add 5.

Teacher: Plus 5. Good. I'm going to keep going very quickly. If you disagree, shoot up your hand and say, "I disagree." (Several students raise their hands and say disagree.) Oh, my gosh! Good thing I stopped when I did. Okay, Jeff what did you want to say?

Jeff: It should be 22.
Teacher: 22. (She writes 22 under the 25. Dan, Brenda, Lisa, and Johanna wave their hands frantically in the air saying "No, Uh! Uh!" John turns and looks at Jeff and shakes his head no.)
Teacher: Gee. Here we go (laughs). Dan, What do you say?
Dan: 18.
Teacher: You say 18. Okay. (She writes 18 under 22 and pauses. Hands wave in the air.) Kirsten.
John: (Speaks out.) He wants to change his answer.
Peter: (Shouts out.) I disagree.
Teacher: I know you do. I hear you loud and clear.
Lisa: (Standing up waving her hand frantically.) It's 23! It's 23!

The children continued giving their answers. Finally, the teacher asked:

Teacher: How are we going to figure this out very quickly? Lisa?
Lisa: Count on our hands, 13, 14, . . . 23.
Teacher: What's another way of checking your work?
Andrew: Well, the pattern is plus 5. (He stands up and rushes excitedly to the front of the class and gestures at the screen.) This dot is 13. (He counts the next two dots.) 5 plus 5 is 10. Just add 10 to 13.
Teacher: 13 plus 10 is . . .
Students: (In unison) 23.
Teacher: 23. You bet.

As the relative merits of solutions were never discussed in the whole-class setting, it is unlikely that Andrew was excited because his solution was the most sophisticated. Rather, he had reconceptualized his understanding of the task and construed his insight as warranting excitement. Furthermore, with respect to the classroom social norms, he gave his explanation to share this insight, not to show how clever he was. The other children seemed genuinely pleased by his breakthrough.

The following dialogue once again illustrates the excitement that the children typically experienced when they constructed mathematical relationships. The dialogue is between two children as they solved a sequence of multiplication tasks corresponding to the sentences $10 \times 4 = \_\_$, $9 \times 4 = \_\_$, $8 \times 4 = \_\_$, and $8 \times 5 = \_\_$. (The children's use of the term *sets* in talking about multiplication derives from the teacher's use of the term when she first introduced "×" as the mathematical symbol for multiplication.) They are working on 10 x 4 = __ after having found $5 \times 4 = 20$.

John: It's five more sets [of 4]. Look. Five more sets than 20.
Andy: Oh! 20 plus 20 is 40. So its gotta be 40. No.
John: Yeah!
Andy: No. 4, 8, 12, 16, 20, 24, 28, . . . (keeping track on his fingers).
John: 40.
Andy: 40.
John: Yeah, I know . . . 'cause ten 4 s make 40.

Andy: Like five 4s make 20.
John: Four sets of 10 makes 40. Just turn it around.
Andy: Five sets of 4s make 20 and so five more than that.
John: Yeah, just turn it around. Just turn it around.
Andy: 5 times 4 is 20, so 20 more than that makes 40.
John: Just switch them around.

John's visible excitement did not stem from the fact that he had solved the problem because he had previously arrived at 40 as an answer by relating $10 \times 4 = \_\_$ to $5 \times 4 = 20$. Instead, it derived from his construction of the principle of commutativity of multiplication, which allowed him to develop a second, more satisfying solution to the problem. John's reaction was analogous to that of the mathematician who succeeds in proving a theorem by a particularly elegant method. At this point in the episode, Andy did not display any emotion. John's repeated comment, "Just turn it around," is an indication that he is trying to convey both his excitement and his insight to Andy, but Andy is oblivious to John's intent. As the episode continued, John again exhibited excitement when he constructed a relationship between successive tasks.

$9 \times 4 = \_\_$
John: Just take away 4 from that [10×40].
Andy: 36.
John: (pause) Yeah!
$8 \times 4 = \_\_$
John: Look! Look! Just take away 4 from that [9×4] to get that [8×4]. See! Just take away 4 from there [9×4]. (The dialogue continues as Andy solves the problem using a different method.)
8 x 5 = __
Andy: Five more than that [8×4] is 37.
John: Eight sets of 4. Eight sets of 5.
Andy: No. 9, 39, I think.
(Both children pause to reflect for a few moments.)
John: (Very excitedly) It's 40.
Andy: It is?
John: Yeah, it's 40! Yeah, look!

The episode concluded as John demonstrated his method and Andy verified it. Throughout, John repeatedly displayed excitement at having constructed mathematical relationships. It was his construction of mathematical knowledge, in and of itself, that gave rise to this excitement.

Reflections

In presenting the sample episodes from the classroom we have attempted to illustrate that children's beliefs, their emotional acts, and the network of obligations and expectations that constitute the social context within which they do mathematics are all intimately related. Consider, for example, beliefs and emotional acts. The process of attributing a particular emotional quality to someone's actions necessarily involves inferences about his or her construal of the situation. This construal in turn reflects underlying beliefs. Consequently, emotional acts can be viewed as expressions of beliefs and, from the observer's perspective, are valuable sources of insight into the possible nature of those beliefs (Cobb, 1986c). In other words, if someone acts in an emotional way, we know that they really mean it, and a "willingness to act and . . . the assumption of some risk and responsibility for action in relation to a belief represent essential indices of actual believing" (Smith, 1978, p. 24). In short, emotional acts depend on beliefs.

The converse, that beliefs depend on emotional acts, is at least partially true. We have seen how the teacher frequently capitalized on the children's emotional acts to renegotiate the classroom social norms. In doing so, she explicitly discussed obligations that she expected the children to fulfill. To the extent that the children accepted these obligations—and there is every indication that they did, for the most part—they reorganized their beliefs about their own role, the teacher's role, and the activity of doing mathematics. Thus, the teacher intuitively agreed with Smith's dictum and spontaneously focused on the children's emotional acts as prime indicators of their beliefs. Within the unfolding stream of classroom life, the teacher acted on the basis of her interpretations of the children's emotional acts and gave the children opportunities to reorganize their beliefs. We use the phrase "gave them opportunities" for the simple reason that beliefs can no more be transmitted from one person to another than can conceptual knowledge of mathematics. The teacher helped the children construct their own beliefs by, in effect, setting puzzles for them to solve. For example, she told the children in no uncertain terms that the phrase "that's easy" was inappropriate. In the last analysis, however, the children had to figure out for themselves why it was inappropriate. They had to understand that this could make other people feel bad, and that making people feel bad is morally wrong. Their ability to reorganize their beliefs in a way that was compatible with the teacher's expectations depended on their current conceptions of morality—that is, on how they interpreted the teacher's statements. We note in passing that the puzzles set by the teacher also gave opportunities for moral growth and, thus, the development of moral autonomy. This was one of the initial noncognitive goals of the project.

It has been argued that beliefs span the cognitive and affective domains (Schoenfeld, 1985). Our analysis expresses the alternative view that beliefs are basically cognitive but that they function in the construals that generate emotional acts. We suggest that beliefs span the individual and social domains. We argued previously, for example, that one has to consider unfolding classroom social life in order to appreciate the partial dependence of beliefs on emotional acts. A child's beliefs about his or her own role and the teacher's role are, in fact, the child's understanding of the classroom social norms. It should be clear that here we include both the implicitly and the explicitly held beliefs that give rise to obligations and expectations in the course of social interactions in the classroom. In fact, to infer that a child holds particular beliefs is a way of summarizing the child's inferred obligations and expectations in a variety of concrete situations.

The claim that beliefs span the individual and the social domain is an instance of the more general contention that neither the individual nor the social domain is primary (Bauersfeld, 1988; Cobb, 1986c; Cobb, Wood, & Yackel, 1991a; Voigt, 1985). One cannot adequately analyze one without considering the other, because the activities of individuals (including their emotional acts) serve to construct the social norms that constrain those very same activities. Conversely, the norms constrain the activities that construct the norms. Thus, to acknowledge that social context is an integral part of an individual's cognition and affect does not imply that social norms are taken as solid, independently existing bedrock upon which to anchor analyses of learning and teaching.

In the first section of this chapter, we proposed that children's beliefs about their own role, the teacher's role, and the nature of mathematical activity comprise a vital core of children's standards of normative expectancies. Emotional acts occur when experience is incompatible with these expectancies. At several points in the analysis of sample episodes, we hinted that the three types of beliefs develop together. In fact, it was for this reason that we avoided language that might suggest that they are three independent components. Instead, we view them as mutually dependent aspects of a self-organizing system. It is readily apparent that a child's beliefs about his or her own role and the teacher's role are intimately connected. In our project classroom, for example, the children's beliefs about the teacher's role changed with the realization that they were obliged to resolve their problems for themselves and that they were not obliged to use any particular solution method. Their beliefs about the nature of mathematical activity also changed once they accepted and attempted to fulfill this obligation. Similarly, the belief that mathematical solutions should be justifiable evolved together with beliefs about their own role and the teacher's role during both small-group work and whole-class discussions. Consideration of these relationships suggests a way to get at the system in belief systems. As we have seen, emotional acts played a crucial role in the children's construction of these systems.

Finally, the reader may recall that nurturing the development of the children's intellectual autonomy was a major noncognitive goal of the project. The astute reader may have noted that many of the sample episodes presented were strong indicators of intellectual autonomy. In the course of the teaching experiment, we came to realize that the manner in which the children reorganized their beliefs was synonymous with the growth of both intellectual and moral autonomy, developments that involved emotional acts. We now prefer to talk of beliefs rather than autonomy because the latter is a more global, less easily differentiable construct.

Implications

The sample episodes serve as prototypical cases of the ways in which the teacher capitalized on the children's emotional acts to initiate, guide, and sustain the mutual construction of what Silver (1985) called a *problem-solving atmosphere.* The episodes exemplify our view of what constitutes effective mathematics teaching. More far-reaching implications become apparent when we relate our analysis of the project classroom to current theories of achievement motivation, particularly that of Nicholls (1983, 1984, 1987).

Nicholls distinguishes between two conceptions of ability. The more differentiated conception is that embodied in standard ability-testing procedures, in which ability is defined with reference to the performance of others. When this conception of ability is operative, effort is considered to improve mathematical learning and problem solving only up to the limit of one's present capacity; "that is, ability is conceived as capacity—an underlying trait that is not observed directly but is inferred from both effort and performance, in a context of social comparison" (Nicholls, 1984, p. 41). When two students are equally successful in solving mathematical problems, for example, but one had to expend much greater effort than the other, we would conclude with respect to the differentiated conception that the more conscientious student was less able. Students who assess their performance in terms of this conception and believe that their ability is low come to believe that they lack capacity. They believe that persisting and doing their best will frequently not be good enough because of inadequacies that are beyond their control.

When the undifferentiated conception of ability is operative, "high ability is implied by learning or by success at tasks they [students] are uncertain of being able to complete. They do not judge ability with reference to performance norms or social comparison" (Nicholls, 1984, p. 41). A sense of being able comes from persisting in solving a personally challenging problem. The fact that others might have solved the problem with less effort is irrelevant. As Nicholls (1984, p. 42) stated: "Ability does not, in this case, imply an inferred trait . . . When more effort is needed for success, this implies more learning, which is more ability . . . The subjective experience of gaining insight or mastery through effort is the experience of competence of ability." Individuals do not typically assess their performance uniformly in terms of one or the other conception of ability; for example, a student might typically employ the differentiated conception during mathematics lessons and the undifferentiated conception in art lessons. In other words, use of the conceptions is contextual.

The promotion of task-involvement as a form of motivation (and thus employment of the less differentiated conception of ability) was the third major noncognitive goal of the project. We have provided initial documentation elsewhere (Cobb, Wood, & Yackel, 1991b) that the project was extremely successful in this regard. Furthermore, the emotional acts presented in the sample episodes are consistent with the inference that the children became increasingly involved in their mathematical problems; for example, they became excited and reported that they felt good when they solved personally challenging problems, irrespective of whether other groups had already completed the same instructional activities. Negative emotional acts occurred when they were deprived of the opportunity to think things through for themselves, as when another child told them the answer, but not when they struggled to solve a problem. This success is directly attributable to the manner in which the teacher initiated and guided the mutual construction of the classroom social norms. Thus, the children's reorganization of their beliefs occurred concomitantly with the development of task-involvement as a form of motivation. (See Silver, 1982, and Cobb, 1985, for more detailed discussions of the relationship between beliefs and motivations.)

The most general implication of our work is that the teacher should renegotiate the social context within which children attempt to solve mathematical problems and thus influence their beliefs about their own and the teacher's roles and the nature of mathematical activity. The objective is for both the teacher and the students to create a social context in which construals that warrant detrimental negative emotions such as frustration are simply not made while solving mathematical problems. This recommendation can be contrasted with those that attempt to help students cope with emotions that are warranted in ego-involving situations.

Consider, for example, recommendations derived from Weiner's (1979) casual attribution theory. Weiner conducted an analysis of the possible causes of (what he takes for granted is) success and failure and classified them according to whether the causes are internal or external to the individual, stable or unstable, and controllable or uncontrollable. Within this scheme, ability is considered to be an uncontrollable, stable, internal cause. This is the differentiated conception of ability—ability as capacity. With regard to this conception, "if students attribute their failures in problem solving to their lack of ability, they are likely to be unwilling to persist in problem-solving tasks very long" (McLeod, 1985, p. 275). As long as the differentiated conception of ability is accepted unquestioningly as the way things are and must be, interventions are limited to persuading students to make alternative attributions. We, in contrast, suggest that the problem of deleterious attributions disappears if social norms are renegotiated to encourage task-involvement and the undifferentiated conception of ability. Students would then "have no reason to consider the role of factors such as ability, difficulty, or luck because effort can be perceived directly. . . . Thus, if effort attributions were the prime mediators of achievement affect, causal attribution theory would be irrelevant to achievement motivation" (Nicholls, 1984, p. 62).

To clarify this point, consider frustration as an emotional act. Frustration is generally warranted in situations in which one is unable to achieve one's purposes. In terms of Weiner's scheme, this emotion is appropriate when students attribute failure to lack of ability, in the differentiated sense, because it is beyond their control. In line with the claim that the children in the project classroom became increasingly task-involved during mathematics instruction, we were unable to identify a single instance during the second semester in which a child became frustrated and gave up because he or she could not complete an instructional activity. We observed children who, when compared with their peers, failed repeatedly day after day, yet these children continued to persist during small-group work, contributed to whole-class discussions, and achieved personal satisfaction by doing so. In terms of the undifferentiated conception of ability, these children were not failing. Their purpose was not to demonstrate superior capacity, avoid looking stupid, or put one over on the teacher. Engaging in mathematical activity was an end in itself, and as long as they did their best they considered themselves to be succeeding. Indeed, they were succeeding in a classroom in which the social norms obliged them to think their problems through for themselves and take responsibility for their own learning. With respect to these norms, then, frustration was not warranted.

In conclusion, we contend that well-meaning attempts to either persuade students to make alternative attributions of the cause of failure or to teach students about detrimental affective variables that might influence them miss an essential point. The students' undesirable attributions and affects are appropriate only with respect to the social norms established in settings that are characterized by competition, social comparison, and public self-awareness and, thus, induce ego-involvement (Nicholls, 1983). The same can be said of recommendations to develop blockage-free instruction or to design computer-based learning environments that alleviate blockages and thus reduce frustration. (We believe that such environments can have educational value. We are merely questioning one proposed rationale for their use.) Here again, it is assumed that blockages are inevitably construed as warranting negative emotions that interfere with learning. If one believes, as we do, that substantive mathematical learning is a problem-solving process, them attempts to exorcise or ameliorate blockages or problematic situations reduce students' opportunities to learn. Surely, the solution to the problem of students' negative emotions during mathematical problem solving is not to quash mathematical problem solving. We suggest instead that it is more productive to initiate and guide the construction of alternative social norms with respect to which deleterious emotions and attributes are not warranted. The sample episodes illustrate how one teacher capitalized on children's emotional acts to do just this and, in the process, achieved greater psychic rewards (Lortie, 1975) as she observed the children learn in her classroom.

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Part III
Symbolizing and Instructional Design – Developing Instructional Sequences to Support Students' Mathematical Learning

Chapter 6
Introduction

Paul Cobb with Koeno Gravemeijer, and Erna Yackel

The emergent perspective described in Chapter 4 serves to orient the analysis of the teacher's and students' actions and interactions in particular mathematics classrooms. As noted in Chapter 4, this framework was developed while analyzing the data generated in the course of two year-long classroom design experiments (an initial experiment conducted with Erna Yackel and Terry Wood in Graceann Merkel's second-grade classroom in a rural suburban school, and a follow-up experiment that Erna Yackel conducted in Willie King's second-grade classroom in an urban school). One of the key characteristics of design research is that instructional design and research are interdependent.[1] Current descriptions of the methodology emphasize that the design of classroom learning environments serves as the context for research and, conversely, ongoing and retrospective analyses are conducted in order to inform the improvement of the design. It is fair to say that our primary focus while conducting the design experiment in Merkel's classroom was on the research aspect of design research – on attempting to understand what was going on in the classrooms in which we worked. To be sure, Yackel, Wood, and I developed a complete set of instructional activities for second-grade mathematics. However, our motivation for doing so was pragmatic: instructional activities were a means to the end of supporting the forms of mathematical learning that we wanted to study given our view about what was worth knowing and doing mathematically.

It is important to note that broad theoretical perspectives such as radical constructivism, social constructivism, and sociocultural theory orient the process of accounting for students' mathematical learning. They offer at best relatively global and ill-focused design heuristics for guiding the development of sequences of

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[1] The basic tenets of design research that serve to differentiate it from other methodologies have been discussed in some detail by Cobb, Confrey, diSessa, Lehrer, and Schauble (2003) and Design-Based Research Collaborative (2003).

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instructional activities.[2] The same can be said of interpretive frameworks that are narrower in scope such as the emergent perspective on mathematics classrooms. As a consequence, our efforts to develop instructional tasks in the design experiment conducted in Merkel's classroom had the characteristics of a craft activity rather than of a principled effort to gain insight into the means of supporting students' learning in particular mathematical domains. We attempted to develop instructional activities that would give rise to cognitive perturbations for individual students, thereby providing occasions for significant mathematical learning. In doing so, we took account of the diversity in the students' reasoning and tried to ensure that every child would be able to engage in each task, and that every task had the potential to advance the sophistication of each child's mathematical reasoning. In addition, we focused on social interactions in classroom. However, as described in Chapter 4, we conjectured that interactions in which conflicts in students' interpretations became apparent would give rise to individual cognitive perturbations, and that students might reorganize their reasoning as they attempted to resolve these perturbations. In addition, we attended to the types of social norms[3] that were established while the design experiment was in progress with the intent of ensuring that the types of social interactions that occurred would give rise to learning opportunities.

In retrospect, it is apparent that the sets of instructional activities we developed in the course of the experiment did not, by any stretch of the imagination, constitute coherent instructional sequences. It was against this background that I read Adrian Treffers's (1987) book, *Three dimensions: A model of goal and theory description in mathematics instruction – The Wiskobas Project.* In this book, Treffers discussed two decades of instructional design work and classroom experimentation that had been conducted at the Freudenthal Institute in the Netherlands. In doing so, he illustrated a number of positive heuristics for instructional design that underpinned the design theory of Realistic Mathematics Education (RME). It was apparent from Treffers's account that RME was a detailed, empirically grounded design theory that was compatible with our constructivist perspective on mathematical learning. For example, RME's basic tenets of mathematics as a human activity and of mathematical learning as the progressive reorganization of activity were both consistent with our general viewpoint. In addition, we valued the manner in which RME placed students' mathematical reasoning at the center of the design process while simultaneously proposing specific means for systematically supporting the development of their reasoning. Furthermore, Treffers's account of RME clarified that the

[2] As the currently fashionable notion of constructivist teaching indicates, efforts have frequently been made to derive pedagogical recommendations from broad theoretical perspectives. In our view, these attempts involve a fundamental category error in which theoretical assumptions are erroneously translated into pedagogical prescriptions.

[3] As we indicated in the previous part of this book, social norms emerged as significant while conducting design experiment in Merkel's second-grade classroom. Yackel and I developed the notion of sociomathematical norms while conducting a retrospective analysis of video-recordings of King's classroom.

purpose for conducting design experiments was not limited to developing explanatory constructs, but could also include developing, testing, and revising sequences of instructional activities.

These insights led us to realize that we had, to this point, attempted to study students' mathematical learning in situations in which the supports for learning had been less than optimal. The design decisions that Treffers illustrated involved a relatively fine-grained level of detail that had been absent in our work. The types of design decisions that we especially noted included both the careful selection of the problem situations that were used during the first part of an instructional sequence and the explicit attention given to the design of non-standard notation schemes as a means of supporting students' reorganization of their mathematical activity. In addition to taking account of these and other aspects of RME theory, we also came to appreciate some of the limitations of the relatively global design heuristics on which we had relied when compared with the finely honed heuristics that Treffers described and illustrated.

Our interest in RME design theory led Terry Wood and myself to visit the Freudenthal Institute in 1989. In the course of this visit, I met Koeno Gravemeijer for the first time and was impressed by his perspective on issues of instructional design. I therefore invited him to spend the spring 1991 semester at Purdue University and to participate in a year-long design experiment that Wood and I were conducting in a third-grade classroom.[4] Gravemeijer had been introduced to us as a researcher and instructional designer who was still working on his doctorate. It was not until he had been at Purdue for several months that I learned that he was the Research Coordinator at the Freudenthal Institute. When I made this discovery, I was especially relieved that I had been able to bend the rules for appointments at Purdue University so that he had a visiting faculty position.

Gravemeijer's stay at Purdue led to a 10-year collaboration that proved to be productive for all involved, in large measure because the strengths of the Dutch and US researchers[5] were complementary. Gravemeijer wanted to enhance the research component of the RME work at the Freudenthal Institute, whereas the US researchers wanted to improve their approach to instructional design. The initial basis for communication that we developed was grounded in the perceived fit between RME and constructivism in terms of their perspectives on mathematical learning, and in a common background of working intensively in classrooms. The face validity that RME had for the US researchers reflected the fact that it had emerged from and yet remained grounded in the activities of designing and

[4]This experiment was a follow up to the experiment that we had conducted in Merkel's second-grade classroom and was conducted in collaboration with Kathy Fahlsing, a third-grade teacher in the same school. Yackel had moved to another Purdue University campus by the time we began this experiment and was developing relationships with teachers (including Willie King) at a nearby urban district. She visited one day each week throughout the third-grade experiment.

[5]The US researchers were initially Yackel, Wood, and myself, and then later Yackel, Kay McClain, and myself after I moved from Purdue University to Vanderbilt University in 1992.

experimenting in classrooms.[6] There were nonetheless differences in viewpoints that became an explicit focus of discussion and negotiation over a period of years.[7] In the course of our collaboration, we conducted three classroom design experiments that focused on arithmetic and linear measurement in the early elementary grades, and two experiments that focused on statistical data analysis in the middle grades. Practically, we developed seven instructional sequences in the course of these experiments.[8] Theoretically, we developed a joint perspective on RME and on instructional design that is described in the chapter reprinted in this part.

In introducing the paper, it is important to note that the US researchers did not accept RME instructional design theory readymade and simply attempt to apply it. Instead, we adapted and elaborated some of the key ideas of RME in the course of our collaboration. We were aware at the time that in doing so, we were developing what we came to think of as our version of RME. The chapter reprinted in this part describes this version of RME and illustrates key constructs by referring to an instructional sequence that was developed in a first-grade design experiment that I conducted with Kay McClain and Koeno Gravemeijer after I had moved to Vanderbilt University. As an orientation to the chapter, it is worth noting two of the most important adaptations we made to RME theory. The first adaptation concerns the way in which we formulated instructional goals. As we have indicated, the intent of the illustrative instructional sequence was to support students' in coming to reason flexibly with numbers up to 20. Reasoning of this type is indicated observationally by students' flexible use of thinking or derived fact strategies to solve a

[6] The Dutch researchers called their methodological approach "developmental research" to emphasize that it involved both instructional development and research on mathematical learning (Freudenthal, 1988; Gravemeijer, 1994b; Streefland, 1991). This methodology anticipated many of the key tenets of design research.

[7] McGatha's (2000) analysis of the year-long process of formulating a hypothetical learning trajectory for a design experiment conducted after we had worked together for several years documents that differences in viewpoint continued to emerge even at this point in our collaboration. In retrospect, our design meetings can be viewed as a trading zone for sharing, appropriating, and adapting ideas across research traditions (cf. Gorman, 2002).

[8] These sequences were (1) the Patterns and Partitioning sequence that focused on adding and subtracting numbers up to 10 (McClain & Cobb, 2001b; Whitenack, 1995) and was informed by Neuman's (1987) analysis of young children's numerical finger patterns, (2) the Structuring Numbers sequence that focused on adding and subtracting numbers up to 20 (Gravemeijer, Cobb, Bowers, & Whitenack, 2000) and was based on the Arithmetic Rack developed by Treffers (1990), (3) the Candy Shop sequence that focused on the addition and subtraction of numbers up to 100 (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997), (4) the Candy Factory sequence that focused on the addition and subtraction of numbers up to 1,000 (Bowers, 1996; Bowers, Cobb, & McClain, 1999), (5) a Measurement sequence that focused on elementary linear measurement as a precursor to mentally adding and subtracting number up to 100 (Stephan, 1998; Stephan, Bowers, & Cobb, 2003), (6) a Statistics Sequence that focused on the analysis of univariate data (Cobb, 1999; McClain & Cobb, 2001a), and (7) a Statistics Sequence that focused on the analysis of bivariate data (Cobb, McClain, & Gravemeijer, 2003).

wide range of tasks. For example, students might solve a task interpreted as 14 – __=6 by reasoning that 14–4=10, and 10–4=6, so the answer is 8 (i.e., a going-through-ten strategy). Alternatively, they might reason that 7+7=14, so 14–7=7, and 14–6=8 (i.e., a doubles strategy). However, in formulating the instructional goals for the instructional sequence, we also went beyond this focus on observable solution methods by emphasizing that our intent was that students would come to act in a quantitative environment structured by relationships between numbers up to 20 (cf. Greeno, 1991). In other words, our intent was that the students would come to act in what Gravemeijer (1999) termed a new mathematical reality in which relationships between numbers are ready-to-hand for them. This framing of instructional goals in terms of the nature of the students' (socially situated) mathematical experience was a significant adaptation of RME because researchers at the Freudenthal Institute tended to characterize students' mathematical development in terms of students' development of increasingly sophisticated solution methods (see for instance Treffers, 1991).[9]

This adaptation in how we conceptualized instructional goals has implications for the nature of classroom discourse and thus for students' learning opportunities. In the chapter reprinted in the next part of this book, we distinguish between calculational and conceptual classroom discourse. The contrast between these two types of discourse concerns the norms or standards for what counts as an acceptable mathematical argument. It should therefore not be confused with Skemp's (1976) well-known distinction between instrumental and relational understandings. In calculational classroom discourse, contributions are acceptable if students describe how they produced a result and they are not obliged to explain why they used a particular method. The formulation of instructional goals primarily in terms of students' development of increasingly sophisticated solution methods orients instructional designers and teachers to view students' calculational explanations of their solutions as acceptable. In contrast to this focus on solution methods, conceptual explanations involve an account of both solution methods and the reasons for using those methods (cf. Thompson, Philipp, Thompson, & Boyd, 1994). As we clarify in the next part of this book, the issues that can (and should) become explicit topics of conversation in conceptual discourse include the task interpretations that underlie different solution methods and that constitute their rationales. Conceptual explanations therefore provide greater support than calculational explanations for students as they attempt to understand each others' (and the teacher's) mathematical reasoning. Furthermore, conceptual explanations facilitate the teachers' task of ensuring that central mathematical ideas become an explicit topic of classroom discussions

[9]Our concern for the mathematical realities in which students were coming to act was influenced in large measure by Sfard's (1994) notion of a virtual mathematical reality, and Thompson's (1993, 1994, 1996) theory of quantitative imagery.

(e.g., numerical part – whole relations in the case of adding and subtracting numbers up to 20).

The second adaptation that we made to RME was to reframe the individualistic notion of a learning trajectory in collectivist terms. In the approach to instructional design developed at the Freudenthal Institute, the designer first conducts an anticipatory thought experiment in which he or she envisions both a route or trajectory for students' mathematical learning,[10] and the specific means by which their learning can be supported. The focus of designers at the Freudenthal Institute when they described initial design conjectures developed in this way appeared to be on the mathematical learning of individual students. To be sure, they aimed to develop tasks that had multiple points of entry and allowed for a range of solution methods. However, the acknowledged diversity in students' reasoning tended to fade in the background when they outlined long-term trajectories that constituted the rationale for instructional sequences. This way of characterizing trajectories gave the impression that students would engage in tasks in similar ways and develop the same insights as they did so. Although the general notion of a learning trajectory appeared to be valuable, we found the implicit implication that students' learning would be relatively uniform to be problematic. As the chapters reprinted in this and the next parts illustrate, we attempted to resolve this difficulty by conceptualizing a learning trajectory as a sequence of collective or communal mathematical practices, each of which emerges as a reorganization of prior practices. The actual learning trajectory realized in the classroom as the teacher and students enact an instructional sequence in a classroom can then be analyzed as a sequence of successive mathematical practices.

The analysis of mathematical learning in collectivist terms represented a major shift in my position when compared with the radical constructivist position illustrated in the first part. The notion of a classroom mathematical practice extended our effort described in the previous part to break down the distinction between the social and mathematical aspects of the classroom. As the reprinted chapter makes evident, we considered it both sensible and useful (for the purposes of instructional design) to speak of the mathematical learning of the classroom community as well as of individual students. In taking this view, we conceptualized the classroom mathematical practices in which students participated as constituting the immediate social contexts of their learning. We realized that this position challenged an assumption that was widely held in the mathematics education research community: explanations of mathematical learning are primarily the province of cognitive and developmental psychology.

In retrospect, it is fair to say that our conceptualization of hypothetical and actual learning trajectories in terms of evolving mathematical practices has had remarkably little impact in mathematics education research. This contrasts with the way in which the constructs of classroom social norms and sociomathematical norms were

[10]The term learning trajectory was coined by Simon (1995) and fits well with the approach to instructional design developed at the Freudenthal Institute.

taken up and continue to be used relatively widely. Several mathematics educators have critiqued our conceptualization of learning trajectories for implying that all students in a classroom should follow a single learning path. As the section "In Praise of the Individual" in Chapter 7 indicates, this was an unfortunate interpretation given that our primary reason for recasting the notion of a learning trajectory in collectivist terms was to take account of the diversity in students' mathematical reasoning. In my view, these critiques (and much of the mathematics education literature on learning trajectories) reflect a fundamental miscommunication. Although we attempted to be as explicit as possible in emphasizing that students participate in the practices established in their classroom in a range of different ways, it seems that many readers of the reprinted paper and other of our papers nonetheless interpreted the notion of a learning trajectory in precisely the individualistic terms that we sought to challenge. In other words, our talk of a collective or communal learning trajectory that constitutes the evolving social setting of the students' learning appears to have been interpreted as statements about a single learning trajectory that all students should follow in lock step.[11] The apparent prevalence of this interpretation indicates that despite the widespread use of sociocultural and situated perspectives in mathematics education, individualistic perspectives frequently hold sway when the focus is on issues of mathematical content.

The joint perspective on instructional design presented in the reprinted chapter indicates that the collaboration with Gravemeijer was particularly productive for the US researchers. The content-specific design heuristics outlined in the first part of the chapter contrast sharply with the global, content-independent heuristics that were evident in the second-grade design experiment conducted in Merkel's classroom. A doctoral student who made notes of our project meetings recorded that one thing we did quite explicitly while conducting the experiment reported in the reprinted chapter was "to select initial situations such that students' interpretation of and activity in them can constitute a basis for subsequent mathematizing while, at the same time, the situation in imagery can serve as a paradigm case or prototype for students." In Ball's terms, we attempted to "keep one eye on the mathematical horizon and the other on students' current understandings, concerns, and interests" (1993, p. 377). It is worth noting in passing that the perspective on modeling described in the chapter was a primary means of supporting students' mathematization of their initially informal mathematical activity (cf. Gravemeijer, 1999). The prominent role of modeling and, more generally, of tools (including informal and conventional notations) in RME challenged the US researchers' initial conception of mathematical reasoning as a process grounded in sensory-motor activity and bounded by the skin. Our initial constructivist position had conspicuously little to say about the use of tools as an integral

[11] It is important to distinguish these critiques based on a fundamental miscommunication from an alternative approach to instructional design that contends that students should each be encouraged to follow up on their personal findings and interests. The grounding metaphor for approaches of this latter type is that of a learning landscape rather than a learning trajectory (Bakker, 2004). In my view, the two approaches reflect a genuine difference in design philosophy.

part of mathematical activity. As a consequence, I found it essential to develop a theoretical position on use of tools that was useful for the purposes of instructional design. The chapter reprinted in this part gives an account of the resulting position.

As a final observation, it is worth noting that RME is an inter-related set of practices that has been developed by a community of designers and researchers over an extended period of time. As the US researchers came to realize, it is one thing to attempt to understand a sophisticated form of practice such as RME by reading about it and quite another to become competent in developing instructional designs for supporting students' mathematical learning. The competence that the US researchers developed as consequence of reading books by some of the leading contributors to RME (e.g., Gravemeijer, 1994a; Streefland, 1991; Treffers, 1987) was that of commentating on the design theory (cf. Fish, 1989). However, we had to co-participate in the process of formulating, testing, and revising designs before we could begin to contribute to the development of adequate designs. In this regard, the collaboration with Gravemeijer was, in many respects, a process of apprenticeship in the course of which the US researchers were legitimate peripheral participants in the RME research community (cf. Lave & Wenger, 1991). The following chapter reports the results of this collaboration in the course of which we adapted RME to fit with our emerging collectivist perspective on mathematical learning. At the same time, our engagement in the process of developing, testing, and revising instructional sequences in classrooms spurred the development of this collectivist perspective.

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Chapter 7
Learning from Distributed Theories of Intelligence

Paul Cobb

The analysis reported in this article is grounded in the practice of classroom-based developmental or transformational research and focuses on the distributed views of intelligence developed by Pea (1993) and by Hutchins (1995). The general areas of agreement with this theoretical perspective include both the nondualist orientation and the critical role attributed to tool use. Against this background, I focus on two aspects of the distributed view that I and my colleagues have found necessary to modify for our purposes. The first concerns the legitimacy of taking the individual as the unit of analysis, and here I argue that the distributed view implicitly accepts key tenets of mainstream American psychology's characterization of the individual even as it explicitly rejects it. The second modification concerns distributed intelligence's characterization of tool use. Drawing on a distinction made by Dewey, I argue that it is more useful for the purposes of instructional design to focus on activity that involves using the tool as an instrument, rather than focusing on the tool itself.

As the title of this article implies, I see much value in recent analyses of activity that stress the distributed nature of intelligence. My overall purpose is both to clarify the contributions of these analyses and to discuss the adaptations and modifications that I and my colleagues have found necessary to make as we have attempted to come to grips with the basic assumptions of this theoretical orientation. I focus specifically on accounts of the distributed nature of intelligence developed by Pea (1987, 1992, 1993) and by Hutchins (1995). Further, I approach this theoretical orientation from the perspective of a mathematics educator who conducts classroom-based developmental or transformational research in collaboration with teachers. I therefore make no pretense at offering a neutral or distanced appraisal but instead address issues that appear pertinent from the viewpoint of someone who is interested in instructional

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design and reform at the classroom level. As a consequence, the focus of the article is not on the theories as things-in-themselves, as constellations of constructs that have careers of their own in traditional academic discourse. Instead, engagement in design and reform constitutes the setting within which I discuss distributed accounts of intelligence.

As background, I should clarify that I originally approached instructional design from the essentially individualistic perspective of psychological constructivism. During the past 10 years, I have modified my position considerably as I have struggled to develop ways of making sense of what might be going on in the classrooms in which I and my colleagues[1] have worked. These shifts in basic theoretical commitments have been profoundly influenced by the writings of a number of sociocultural and sociolinguistic theorists. However, this process has not involved either a wholesale conversion or a conscious attempt to apply the central tenets of sociocultural theory to mathematics education. Instead, it has involved some resistance to what might be called the official sociocultural narrative, or at least neo-Vygotskian versions of it. Wertsch's (1994b) discussion of the narrative construction of identity strikes a resonant chord as I reflect and attempt to describe this process of change. In Wertsch's account, the process of constructing the self is not merely one of appropriating the official narratives of the community by figuring out how one fits into them. It can also involve resistance in which a series of counter-claims are made to parts of the official story. Wertsch observed that, even in opposition, resistance is profoundly influenced by the official narrative in that it occurs on what he termed the semiotic territory of the other. The self constructed in the course of resistance can therefore be seen to emerge during a dialogical encounter with the official narrative. In the case of me and my colleagues, the issues we have delineated in the course of our research practice are located, to a considerable extent, on the semiotic territory of sociocultural theory and include semiotic mediation, communication, and interaction.

This metaphor of locating oneself within and, at times, resisting an official narrative also helps clarify our attempts to come to terms with distributed accounts of intelligence. On one hand, the claim that the tools with which students act profoundly influence the mathematical ways of knowing that they develop has come to orient our approach to instructional design. On the other hand, we have resisted certain parts of the distributed intelligence narrative as developed by Pea and Hutchins. These concern the individual as a unit of analysis and the way in which tool use is characterized. In addressing these issues, we have operated on the semiotic territory of distributed accounts of intelligence and have learned from these accounts in the process.

In the following sections of this article, I first ground the discussion by clarifying why tool use has emerged as the central focus of our classroom-based practice.

[1] I repeatedly use the first-person plural to acknowledge the collaborative nature of our research. From 1986 until 1991, these colleagues were Ema Yackel and Terry Wood, and from 1991 until the present they are Ema Yackel, Koeno Gravemeijer, Janet Bowers, Kay McClain, and Joy Whitenack.

Against this background, I then briefly outline areas of agreement with what might be called the official narrative of distributed intelligence. Finally, I describe two points of resistance and contrast the adaptations we have made with the basic tenets of the distributed perspective.

Classroom-Based Instructional Design

The methodology we use when collaborating with teachers is that of the classroom teaching experiment (Cobb, 2000; Confrey & Lachance, 2000; Yackel, 1995). The duration of these experiments extends from a few months to 1 year, in the course of which the collaborating teacher is a full member of the research and development team. As part of the process of preparing for an experiment, we identify possible global goals for the students' mathematical development and outline provisional sequences of instructional activities.[2] Gravemeijer (1994) described the initial phase of this design process in some detail. As he put it, the designer carries out an anticipatory thought experiment that involves envisioning how the instructional activities might be realized in interaction in the classroom and how students' interpretations and solutions might evolve as the students participate in them. In approaching design in this manner, the designer formulates conjectures about both the course of the classroom community's mathematical development and the means of supporting and organizing it. The domain-specific instructional theory that we draw on when conducting these orienting thought experiments is that of Realistic Mathematics Education (RME; Gravemeijer, 1994; Streefland, 1991; Treffers, 1987). The issue of tool use comes to the fore when we consider the general heuristics of this approach to design.

A first heuristic of RME is that the starting points of an instructional sequence should be experientially real to students. One indication that the initial activities are appropriate is that only a minimal negotiation of task conventions is required before students can engage in what might colloquially be called meaningful mathematical activity. This heuristic is consistent with recommendations derived from investigations that have compared and contrasted mathematical activity in school with that in out-of-school settings (e.g., Nunes, Schliemann, & Carraher, 1993; Saxe, 1991). As a point of clarification, it should be stressed that the term *experientially real* means only that the starting points should be experienced as real by students given their prior participation in both in-school and out-of-school practices, not that they should necessarily involve so-called real-world situations. Further, we take it as self-evident that even when everyday scenarios are used, they are necessarily different from the situations as students might experience them out of school (Lave, 1993; Walkerdine, 1988).

[2] These provisional sequences provide an initial orientation for the teaching experiment. Adaptations and modifications are made on a daily basis throughout the experiment so that the actual sequences realized in the classroom typically differ significantly from those envisioned at the outset.

A second heuristic of RME is that the initial instructional activities should also be justifiable in terms of the global endpoints of the conjectured learning sequence. This implies that as students participate in and contribute to the evolving classroom mathematical practices, their initially informal activity should constitute a basis on which they can reorganize and construct increasingly sophisticated mathematical understandings.

The third heuristic of RME focuses on the means of supporting the evolution of classroom mathematical practices and the constructive activities of students as they participate in them. This heuristic proposes that instructional sequences should involve settings in which students are expected to develop and elaborate models of their informal mathematical activity. This modeling activity might involve acting with physical devices, or it might involve making drawings, diagrams, or tables, or developing nonstandard notations and using conventional mathematical notations. The conjecture underlying this third heuristic is that, with the teacher's guidance, students' models of their informal activity will evolve into models for increasingly sophisticated mathematical reasoning. With regard to communal practices, this heuristic involves the conjecture that a shift will occur such that means of symbolizing initially developed as protocols of action (Dörfler, 1989) will subsequently take on a life of their own and become integral to mathematical reasoning in a range of settings.

This transition from a model of informal mathematical activity to a model for mathematical reasoning can be illustrated by referring to an instructional sequence developed during a first-grade teaching experiment we conducted with 6- and 7-year-old students. The intent of the sequence was, in terms of Greeno's (1991) environmental metaphor, that the students would come to act in a quantitative environment structured by relations between numbers with sums up to 20. Observationally, this would be indicated by their flexible use of thinking or derived fact strategies to solve a wide range of additive tasks. For example, they might solve a task interpreted as $14 - = 6$ by reasoning that $14 - 4 = 10$, and $10 - 4 = 6$, so the answer is 8. Alternatively, they might reason that $7 + 7 = 14$, so $14 - 7 = 7$, and $14 - 8 = 6$. Our global goal was that the numerical relations implicit in these and other observable strategies would be ready-to-hand for the students. In other words, they would not have to consciously figure out appropriate strategies to use. Instead, our intent was that the students would come to have the experience of directly perceiving relations as they interpreted tasks. Needless to say, coming to act in such an environment is a major intellectual achievement that requires proactive developmental support.

The sequence of instructional activities developed in the course of the teaching experiment involved the use of a physical device called the arithmetic rack designed by Treffers (1990). It consists of two parallel rods on each of which are five red and five white beads. To use the rack, the students move beads from right to left either by counting individual beads or by moving several beads at once. For example, a student might show seven (beads) by simultaneously moving three beads on the

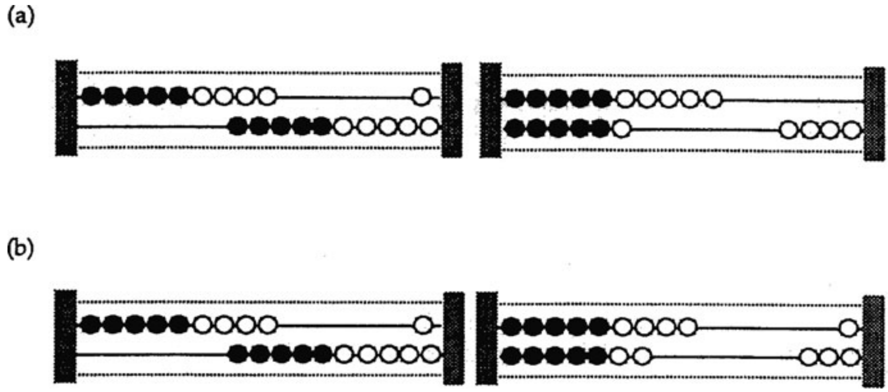

Fig. 7.1 Two ways of acting with the arithmetic rack to solve 9+7=–: (**a**) Going through-10 and (**b**) doubles

top rod and three on the bottom rod, and then moving an additional bead on the top rod.[3]

The rationale for introducing this device can be clarified by considering possible solutions to a specific task such as one interpreted as $9 + 7 = 16$. Suppose that a student has moved 9 beads on the top rod and intends to add 7 more. To complete the solution, the student might move 1 bead on the top rod and then 6 on the bottom rod (see Fig. 7.1a). From the observer's perspective, a going-through-10 strategy is implicit in the student's activity (i.e., $9 + 1 = 10$, $10 + 6 = 16$).

Alternatively, a student might first show 9 beads on the top rod as before, but then move 7 beads on the bottom rod and read the resulting configuration as 7 and 7 is 14, and 2 more is 16 (see Fig. 7.1b). In this case, a doubles strategy is implicit in the student's activity. The instructional challenge is then to organize and support the emergence of these aspects of students' ways of acting with the rack as explicit topics of conversation in the classroom.

A longitudinal analysis of the teaching experiment completed by Whitenack (1995) revealed that acting and reasoning with the rack first emerged as a model of problem-solving activity that centered on a scenario involving a double-decker bus. In particular, the students moved beads on the top and bottom rods to show the number of passengers on the top and bottom decks of the bus.

At a later point in the sequence, the teacher used both conventional and non-standard notations to symbolize the ways in which the first graders reasoned with

[3]Coming to reason with groups of beads rather than counting beads one by one is itself a developmental achievement for young children. We supported the development of these ways of reasoning in the teaching experiment by designing an instructional sequence called Patterning and Partitioning. This sequence was enacted in the classroom immediately before the arithmetic rack was introduced.

Fig. 7.2 Ways of notating the two alternative solutions to 9 + 7 = 16

$9 + 1 = 10$

$10 + 6 = 16$

$9 + 7 = 16$

1 6

7 2

$9 + 7$

$7 + 7 = 14$

$14 + 2 = 16$

the rack. For example, the two solutions to 9 + 7 described previously might be symbolized as depicted in Fig. 7.2.

At this juncture, references to the scenario of the double-decker bus had ceased. At least in the public discourse of the classroom, acting with the rack now served as a model for reasoning with written notations. As the instructional sequence progressed, reasoning in this way without either acting with the rack or writing notations became increasingly routine. In individual interviews conducted after the instructional sequence was completed, 10 of the 18 students used thinking strategies that reflected the ways they had acted with the rack to solve all of a wide variety of number sentences and story problems posed to them. Three more students used thinking strategies to solve at least half of the tasks presented to them in the interviews. Viewed as a group, both the path of these students' mathematical learning and the mathematical understandings they developed differ markedly from those typically reported in the psychological research literature (cf. Fuson, 1992).[4]

This example of the arithmetic rack and the more general discussion of RME illustrate that designing tools is central to our attempts to support students' mathematical development. Parallel examples could have been given from any one of several recent teaching experiments, including those in which we designed computer microworlds as components of coherent instructional sequences (cf. Bowers, Cobb, & McClain, 1999). As a consequence, Vygotsky's (1987) claim that the tools with which people act profoundly influence the understandings they develop is more than a theoretical conjecture for us. It is a description of the pedagogical reality in which we act when conducting teaching experiments. As van Oers (1996) put it, the struggle for mathematical meaning can be seen in large part as a struggle for means of symbolizing. Given these considerations, it is readily apparent that theories that emphasize the distributed nature of intelligence are relevant to our work in classrooms. I focus on this issue in the remainder of the article by first outlining broad areas of agreement between our perspective and theories of distributed intelligence, and then discussing points of resistance.

[4]Psychological analyses almost universally report a developmental progression of increasingly sophisticated counting methods (e.g., counting all and counting on) followed by thinking strategies. The purview of these analyses is, of course, restricted to students who participate in currently institutionalized classroom practices.

Distributed Intelligence

It is important to note at the outset that although Pea (1993) and Hutchins (1995) both drew on sociocultural theory, their distributed accounts of intelligence evolved from American mainstream psychology. A central assumption of both accounts is that intelligence is distributed "across minds, persons, and the symbolic and physical environments, both natural and artificial" (Pea, 1993, p. 47). Hutchins's (1995) analysis of a team navigating a warship into a harbor illustrates this general perspective. As he observed, when the unit of cognitive analysis is moved out beyond the skin,

> communication among the actors is seen as a process internal to the cognitive system. Computational media, such as diagrams and charts, are seen as representations internal to the system, and the computations carried out on them are more processes internal to the system, (p. 128)

Dörfler (1993) developed the general implications of this theoretical approach for mathematics education by noting that thinking

> is no longer considered to be located exclusively within the human subject. The whole system made up of the subject and the available cognitive tools and aids realizes the thinking process. Mathematical thinking for instance not only uses those cognitive tools as a separate means but they form a constitutive and systematic part of the thinking process. The cognitive models and symbol systems, the sign systems, are not merely means for expressing a qualitatively distinct and purely mental thinking process. The latter realizes itself and consists in the usage and development of the various cognitive technologies.

Similarly, Hutchins argued that the relevant cognitive system when a mathematician went about solving problems

> was a person actually doing the manipulation of the symbols with his or her hands and eyes. The mathematician or logician was visually and manually interacting with the material world. A person is interacting with the symbols and that interaction does something computational, (p. 361)

The discussion of the first-grade teaching experiment exemplifies the contention that the use of cognitive technologies is central to intellectual development. As Pea (1993) and Hutchins (1995) both noted, tools are not mere amplifiers of human capabilities. Instead, their use is integral to the creation and reorganization of those capabilities. In the case of the teaching experiment, the students came to act in a quantitative environment of organized number relations as they participated in classroom mathematical practices that involved reasoning with the arithmetic rack and with written notations. More generally, this distributed view of intelligence is at least partly consistent with RME in that the conjectured model-of/model-for transition is premised on the assumption that modeling and symbolizing are directly implicated in the reorganization of mathematical activity.

As has been frequently noted, the distributed view of intelligence attempts to transcend the traditional philosophical dualism between the cognizing individual and the world about which he or she cognizes. The description given of the first-grade classroom can also be cast in nondualist terms. In particular, the students can be seen to have participated in and contributed to the development of classroom mathematical practices that involved reasoning with tools. By virtue of this participation, they can also be viewed as acting in a taken-as-shared world of signification that constituted what Lemke (1997) called the semiotic ecology of the classroom community. As a consequence, the relation between the students' activity and the world in which they acted can be characterized as one of mutual constitution, a position consistent with Pea's and Hutchins's positions. Hutchins (1995), for example, made the point directly when he stated that "humans create their cognitive powers by creating the environments in which they exercise those powers" (p. 169). Similarly, Pea (1985) argued that "our productive activities change the world, thereby changing the ways in which the world can change us. By shaping nature and how our interactions with it are mediated, we change ourselves" (p. 169). As Whitson (1997) observed, theoretical approaches of this type that begin with activity in a world of signification simply bypass a number of philosophical issues including the classical problem of reference.

Points of Resistance

The areas of consensus identified previously serve to situate the contrasts between Pea's and Hutchins's distributed perspectives and the interpretive stance that has emerged in the course of our classroom-based work. The first of the two points of resistance that I discuss deals with the individual as a unit of analysis, whereas the second concerns the way in which tool use is characterized.

In Praise of the Individual

Pea has been outspoken in delegitimizing analyses that take the individual as a unit of analysis. In his view, the functional system consisting of the individual, tools, and social contexts is the appropriate unit of analysis. Pea's admonition contrasts sharply with the explicit attention that I and my colleagues give to individual students' interpretations and meanings. Hutchins (1995) was more forgiving and identified a range of cognitive systems that include (a) the processes internal to a single individual, (b) an individual in coordination with a set of tools, and (c) a group of individuals in interaction with one another and with a set of tools (p. 373). In his theoretical approach, analyses of more encompassing systems composed of individuals manipulating tools serve to specify the cognitive tasks actually facing the individuals. I suggest here that, despite this difference in their positions. Pea and Hutchins subscribed to similar characterizations of the individual that are at odds with the view that had emerged in our work. Thus, the issue at hand is not that of siding with either Pea or Hutchins, but instead it concerns the way in which the individual is treated in both their accounts. To set the stage, I first illustrate that this is a pragmatic issue for me and my colleagues by explaining why we find it essential to analyze individual students' mathematical interpretations when we experiment in classrooms.

The lessons conducted in the course of a teaching experiment typically involve periods in which the students work either in pairs or individually but with the proviso that they can move around the classroom to discuss their ongoing activity with peers of their choosing. This small-group or individual work is typically followed by a teacher-orchestrated whole-class discussion that takes the students' interpretations and solutions as its starting point. During the pair and individual work, the teacher usually circulates around the classroom to gain a sense of the diverse ways in which the students are attempting to solve the tasks.[5] For our part, I and a graduate research assistant each observe and interact with two students to document the process of their mathematical development throughout the teaching experiment. In doing so, we consciously attempt to infer their individual mathematical interpretations on an ongoing basis.

Towards the end of pair or individual work, the teacher, the graduate assistant, and I "huddle" in the classroom to discuss our observations and to plan for the subsequent whole-class discussion. In these conversations, we routinely focus on individual students' qualitatively different interpretations and solutions to develop conjectures about mathematically significant issues that might emerge as topics of discussion. In this opportunistic approach, our intent is to capitalize on prior individual or small-group activity by identifying specific students whose explanations might give rise to substantive mathematical discussions that will advance our pedagogical agenda. At times, the discussions might focus on one student's mathematical activity, whereas on other occasions, the discussions might involve a comparison of two or more solutions. It is important to emphasize that our intent in proactively organizing discussions in this manner is not to confront solutions so that students who initially agree with the one classified as less sophisticated in some purely psychological scheme will come to appreciate the superiority of the other solution. Instead, our primary concern is with the quality of the discussion as a social event in which the students will participate. It is for this reason that we attempt to anticipate mathematically significant issues[6] that might emerge as topics of conversation with the teacher's guidance. In doing so, we conjecture that participating in discussions

[5] Although atypical in the United States, this approach of monitoring students' activity to plan for the subsequent whole-class discussion is routine in Japan (Stigler, Fernandez, & Yoshida, 1996).

[6] Judgments of mathematical significance are made with respect to current conjectures about the classroom community's learning trajectory and the means of supporting it. In other words, although goals and conjectures continue to evolve throughout the experiment, one has in mind an envisioned learning trajectory at any particular point in the experiment. The currently anticipated learning trajectory both provides a sense of direction and constitutes the broader setting in which judgments of mathematical significance are made. In particular, an issue is judged to be mathematically significant if it contributes to the realization of the currently envisioned learning trajectory. Our observations of the subsequent whole-class discussion can, however, lead us to revise this judgment and to modify the conjectured learning trajectory. As a consequence, the actual learning trajectory realized in the classroom is enacted jointly by the students, the teacher, and the researchers. To paraphrase Varela, Thompson, and Rosch (1991), it is much like a path that exists only as it is laid down by walking, even though we have a sense of where we are going and how we might get there at each moment.

in which such issues emerge constitutes a supportive situation for the students' mathematical development.

To be as clear as possible, it is important to stress that the analyses we conduct on a daily basis throughout a teaching experiment involve a psychological constructivist perspective that focuses squarely on individual students' mathematical reasonings. In a recently completed experiment that dealt with measuring, for example, we developed key terms and explanatory constructs by drawing heavily on individualistic constructivist accounts of children's early number development (Steffe, Cobb, & von Glasersfeld, 1988). To be sure, when we step back, we realize that we are analyzing individual students' activity as they participate in and contribute to the development of the mathematical practices of the classroom community. However, these practices fade into the background when we actually observe and interact with individual students in the classroom and attempt to understand their personal meanings. In effect, both we and the students are, at that moment, "inside" the communal practices.

The approach I have described for planning whole-class discussions emerged during a series of teaching experiments conducted over a 10-year period. Our focus on individual students' diverse meanings is a central aspect of our classroom practice in that it has enabled us to be more effective as we collaborate with teachers to investigate ways of supporting their students' mathematical development. However, analyses of this type are prime examples of those disallowed by Pea. In addition, the way in which we view individual students as they participate in classroom practices clashes with Hutchins's characterization of individuals as processors of symbolic structures, some of which are external and others of which are internal representations of external symbols. Given that our classroom-based practice itself constitutes the ultimate justification for our theoretical approach, we find it necessary to resist theoretical arguments that delegitimize this way of working in classrooms.

Characterizing the Individual

I have noted that the distributed theories of intelligence proposed by Pea and Hutchins evolved from mainstream American psychology. More precisely, these theories have been developed in part by resisting central tenets of mainstream psychology.[7] Foremost among these is the traditional separation between internal representations in the head and external representations in the world. However, as we have seen, resistance begins on the semiotic territory of the other. In this regard, Pea's and Hutchins's accounts of intelligence carry the vestiges of their development from mainstream psychology even as they react against it. This is particularly apparent in the debate between Pea (1993) and Salomon (1993) on the legitimacy of taking the individual as a unit of analysis. Salomon contended that, in distributed accounts of intelligence,

[7] This is particularly the case with Hutchins (1995), whose book is directed primarily toward cognitive scientists.

> the individual has been dismissed from theoretical consideration, possibly as an antithesis to the excessive emphasis on the individual by traditional psychology and educational approaches. But as a result the theory is truncated and conceptually unsatisfactory, (p. 111)

Salomon went on to argue that some competencies are not distributed but are instead solo achievements and that the individual is the appropriate unit of analysis in such cases. Pea, for his part, countered that many tools and social networks are invisible and that intelligence is distributed even in the case of apparently solo intelligence and purely mental thinking processes.

Despite these differences in perspective. Pea and Salomon appear to agree on at least one point. The individual of whom they both speak is the disembodied creator of internal representations who inhabits the discourse of mainstream psychology. It is this theoretical individual who features in Pea's claim that intelligence is distributed across the individual, tools, and social context. In developing his viewpoint. Pea, in effect, attempted to equip this mainstream character with cultural tools and place it in social context. However, in doing so, he implicitly accepted the traditional characterization of the individual and preserves it as a component of tool-person systems even as he rejected it. The choice that Pea and Salomon offer us is that of either accepting one specific characterization of the individual—that of mainstream psychology—or rejecting the very notion of the individual as a legitimate unit of analysis.

Hutchins's detailed analysis of navigation is helpful in clarifying the role of this mainstream character within a sociocultural system. Hutchins observed that it is often possible to follow the trail of directly observable representations quite a long way when analyzing sociocultural systems. However, Hutchins (1995) went on to note that "from time to time the stream of representational state disappears inside the individual actors and is lost to direct observation" (p. 129). It is in these situations that Hutchins admitted analyses of internal symbol processing. In doing so, he took care to emphasize the differences between his theoretical approach and traditional cognitive science. In the case of navigation, for example, he stressed that

> not all the representations that are processed to produce the computational properties of this [sociocultural] system are outside the heads of the quartermasters. Many of them are in the culturally constituted material environment that the quartermasters share with and produce for each other, (p. 360)

As a consequence

> although some of the representations are internal, they are still all cultural in the sense that they are the residua of a process enacted by a community of practice rather than idiosyncratic inventions of their individual users, (p. 130)

The theoretical model that emerges on my reading is again that of the mainstream character of cognitive science equipped with tools and located in social context. The differences between Pea's and Hutchins's positions on the legitimacy of taking the individual as a unit of analysis appears to be primarily one of terminology. Pea might point to a history of participation in cultural practices to argue that Hutchins's focus on internal symbolic processing is not entirely individualistic.

Hutchins's (1996) recent exchange with Latour (1996) served to illustrate how the mainstream character is implicit in his account of cognition. Latour described the numerous merits he saw in Hutchins's work but Latour was "disturbed by the idea, frequent in the book, that on one side there is the world and on the other cognitive skills. Distribution, in my [Latour's] view does not go all the way" (p. 60). In responding, Hutchins (1996) countered that Latour "would dissolve the individual and the psychology of the individual as well" (p. 64). He continued, "One cannot empty the person by delegating cognitive activity to 'something or someone else.' The work must be done somewhere, and some of the work will be done in regions that lie inside the bounds of persons" (p. 65). Thus, the debate as both Hutchins and Latour framed it is about how much of the cognitive activity of a sociocultural system is internal to persons and how much is external. For them, as for most others who focus on the legitimacy of various units of analysis, it is the boundary of the skin that is decisive. In casting the issue this way, Hutchins and Latour, like Pea and Salomon, conducted their exchange on the semiotic territory of mainstream psychology. If we accept Hutchins's arguments, we take an approach that seems to involve partitioning rather than distributing intelligence. If we follow Latour's line of reasoning, we push cognition out beyond the skin, thereby distributing intelligence by "emptying the person." Against the background of my own and my colleagues' work in classrooms, not one of these alternatives is particularly attractive. As was the case with Pea and Salomon, the issue at hand is not that of choosing sides in such exchanges. Instead, we need to scrutinize a central assumption that is taken for granted in these debates: Mainstream psychology offers the only possible conception of the individual.

As a starting point, I first note that, in the discussion of the first-grade teaching experiment, the students were not the putative creature of mainstream psychology. More generally, the psychological orientation that I and my colleagues take when analyzing individual students' activity is not part of the mainstream story, but is instead part of an alternative European tradition that draws on aspects of Piaget's genetic epistemology (von Glasersfeld, 1991). In this tradition, there is no talk of cognitive skills or processing symbols, or creating internal representations. Instead, intelligence is seen to be embodied, or to be located in activity (Johnson, 1987; Piaget, 1970; Winograd & Flores, 1986). Further, rather than representing a world, people are portrayed as individually and collectively enacting a taken-as-shared world of signification (Varela, Thompson, & Rosch, 1991). The goal of analyses conducted from this perspective is therefore not to specify internal cognitive behaviors located in the head that intervene between perceptual input from the world and observed output responses. Instead, it is to infer the quality of individuals' experience in the world and to account for developments in their ways of experiencing in terms of the reorganization of activity and of the world acted in.

Once the shift is made from characterizing the individual in terms of cognitive behavior to activity-in-the-world, it no longer makes sense to talk of intelligence being stretched over individuals, tools, and social contexts. To anticipate the discussion of the tool metaphor, the physical devices and notations that people use are not considered to stand apart from or outside the individual but are instead viewed as constituent parts of their activity. It was for this reason that I spoke of the first graders reasoning with the arithmetic rack and with notations. What, from Pea's and Hutchins's distributed perspectives, is viewed as a student-rack system is, from the perspective I have outlined, characterized as an individual student engaging in mathematical activity of which the arithmetic rack is a constituent part. Thus, although the focus of this psychological viewpoint is explicitly on individual activity, its emphasis on tools is generally consistent with the notion of mediated action (John-Steiner, 1995; Meira, 1995; Ueno, 1995; Wertsch, 1994a).

With regard to the remaining component of the functional system posited by distributed intelligence, the social context, I have already suggested that a student's individual activity is an act of participating in the collective mathematical practices of the classroom community. Consequently, although our focus when actually conducting a psychological analysis is on individual students' activity per se, we can subsequently step back and view it as an analysis of students' qualitatively different ways of participating in communal classroom practices. In effecting this reconceptualization, we explicitly coordinate psychological analyses of individual students' acts of participation with an analysis of the evolving mathematical practices in which they participate[8] (cf. Bowers, Cobb, & McClain, 1999; Cobb, 1996; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). This latter analysis of communal practices, it should be noted, serves simultaneously to delineate the learning of the classroom community and the evolving social situation of the individual students' mathematical development. In such an approach, the basic relation between the communal practices and the activity of the students who participate in them is one of reflexivity.[9] This is an extremely strong relation in that it does not merely mean that individual activity and communal practices are interdependent. Instead, it implies that one literally does not exist without the other (cf. Mehan & Wood, 1975). On one hand, participation in communal practices is seen to support, enable, and constrain the ways in which students reorganize their individual acts of participating. On the other hand, in reorganizing their activity, students are seen to contribute to the development of the practices that enable and constrain their reasoning. Cast in these terms, both the process of individual students' mathematical development and its products, increasingly sophisticated ways of mathematical knowing are seen to be social through and through.[10] As a

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[8]Some years ago, we described this relation as dialectical. However, German colleagues noted that, in their country, the use of the term *dialectical* is often viewed as indicating a commitment to the philosophy of dialectical materialism. To avoid such confusions, we prefer to speak of reflexive rather than dialectical relations.

[9]It is important to stress that the issue at hand is not that of coordinating two sets of separate processes—one psychological and the other communal. Instead, we coordinate different ways of interpreting and describing classroom activity.

[10]As stated at the outset, the interests that motivate this discussion are those of instructional design and reform at the classroom level. It should however be acknowledged that mathematical knowing is social through and through in a second sense. In particular, generally accepted beliefs about what counts as normal in development and as more and less sophisticated are themselves social constructions that are reflexively verified in practice (Cobb & Yackel, 1996; Walkerdine, 1988).

consequence, although psychological analyses are an essential part of our practice in classrooms, they do not by themselves result in adequate accounts even of individual students' mathematical development. By the same token, an analysis that focuses only on communal practices is also inadequate for our purposes. Given our agenda, we find it necessary to focus on both the practices in which students participate and the quality of their individual acts of participating.

In the approach that I have sketched, the analytic focus is not on activity at the expense of the individual. Instead, the individual has been reconceptualized such that his or her activity is necessarily located in social context that is not assumed to exist apart from that activity. As a consequence of this reflexive relation, it does not make sense to talk of intelligence as being stretched over the individual and the social context because this formulation implies that they are separate entities that need to be brought together. Concerns about whether it is legitimate to take the individual as a unit of analysis arise only if one accepts mainstream psychology's characterization of the individual. The issue dissolves if social context is viewed as an inseparable aspect of individual activity, and vice versa. Given the alternative view of the individual, there is no need to equip individuals with tools or to place them in social contexts for the simple reason that individuals are not seen to exist apart from tools and contexts.

The approach I have sketched of coordinating analyses of individual activity and communal practices is closely related to several other proposals. These include Hatano's (1993) call to synthesize constructivist and Vygotskian perspectives, Saxe's (1991) discussion of the intertwining of cultural forms and cognitive functions, and Rogoff's (1995) distinction between three planes of analysis that correspond to personal, interpersonal, and community processes. In the case of me and my colleagues, the need to coordinate perspectives is rooted in and has emerged from our classroom-based practice. Recall, for example, the manner in which we plan for whole-class discussions during teaching experiments. A psychological focus comes to the fore when we draw on analyses of individual students' activity, and the communal practices in which the students are participating fade into the background. This focus can be contrasted with that which we take when we plan whole-class discussions by focusing on the social event in which students will participate. At this moment, our focus is on the discussion as a collective activity, and the students' individual interpretations and meanings have now faded into the background. In addition, once the classroom discussion begins, we find ourselves monitoring both the nature of the discussion as a social event and individual students' qualitatively distinct contributions to it. Thus, in describing how I and my colleagues coordinate analytical perspectives, I have in effect attempted to explicate a relation that is implicit in our practice in the classroom. As a consequence, this theoretical approach can be viewed as a report from the field that outlines a way of working that we currently find useful when collaborating with teachers. It should therefore be differentiated from discussions in which various types of processes characteristic of different perspectives (e.g., personal processes, communal processes) are sometimes spoken of as though they have an existence independent of our interests and purposes as researchers. Such attempts to specify the legitimate universe of inquiry for sociocultural investigation involves what Shotter (1995) called the lure of cosmology. The tenet of self-reflexive consistency mandates that this temptation should be strongly resisted.

Tool Use

In discussing characterizations of the individual, I hinted at a second point where I and my colleagues find it necessary to resist the theories of distributed intelligence formulated by Pea and Hutchins. This concerns the way in which tool use is dealt with in their accounts. To sharpen the contrast between perspectives, I focus on an individual acting with a tool. As a consequence, the communal practices in which individuals participate fade into the background. I would, however, note in passing that my view is consistent with that articulated by Ueno (1995) when he argued that the usefulness of a tool is not a property of the tool itself Instead, the formulation of a tool as useful "presupposes accomplishment of a social construction of reality in a specific practice" (Ueno, 1995, p. 233).

We have seen that, in distributed accounts, intelligence is said to be stretched over individual tool-context systems. In this scheme, tools are typically treated in instrumental terms that separate ends from means. This contention can be clarified by referring to Dewey's (1977) discussion of two different ways of thinking about tool use. The particular example that Dewey considered was that of the role of scaffolding in the construction of a building. In one characterization, the scaffolding is viewed as an external piece of equipment, and in the other it is viewed as integral to the activity of building.

> Only in the former case can the scaffolding be considered a mere tool. In the latter case, the external scaffolding is not the instrumentality; the actual tool is the action of erecting the building, and this action involves the scaffolding as a constituent part of itself. (Dewey, 1977, p. 362, as cited in Prawat, 1995, p. 20)

The view of the individual implicit in both Pea's and Hutchins's distributed accounts of intelligence leads to the first of these characterizations in which people are equipped with tools. The separation of means from ends is apparent, for example, in Hutchins's (1995) contention that the evolution of material means "permits a task that would otherwise be difficult to be recoded and represented in a form in which it is easy to see the answer" (p. 367). In general, analyses cast in distributed terms provide compelling demonstrations that the introduction of a tool results in changes in forms of activity. For example, it has frequently been noted that when students are equipped with computers, they "off load" computational processes and engage in planning and problem-solving activities to a greater extent. Illustrations of this type make an important contribution by clarifying that tools are not mere amplifiers of activity. However, accounts based on the first of the two characterizations identified by Dewey typically limit their focus to that of documenting the reorganizations that occur when people are equipped with tools by contrasting before and after snapshots. Although analyses of this type might be appropriate for many purposes, they do not address an issue central to my interests as a mathematics educator. This issue concerns the process by which activity evolves. In the case of the first-grade teaching experiment, for example, it was not sufficient to demonstrate that the first graders' mathematical activity was qualitatively different from that of students who were not equipped with arithmetic racks. When we planned the sequence of instructional activities, we found it essential to envision the process by which ways of reasoning with the arithmetic rack might evolve. Further, when we planned whole-class discussions, we focused on the various qualitatively distinct ways in which individual students acted with the rack.[11] In doing so, we adopted the second of Dewey's two characterizations of tool use by viewing the rack as a constituent part of the students' activity that was itself the instrumentality.

Bateson's (1973) example of a blind person using a stick provides perhaps the most well-known illustration of this second characterization of tool use.

> Suppose I am a blind man, and I use a stick. I go tap, tap, tap. Where do I start? Is my mental system bounded at the handle of the stick? Is it bounded by my skin? Does it start halfway up the stick? Does it start at the top of the stick? (p. 459)

For Bateson, the person acting and the artifact-acted-with are inseparable. Significantly, in making this point, Bateson approached activity from the inside rather than from the position of someone observing a blind person. He asked us to pretend that we are blind and to imagine the nature of our experience when using the stick. This actor's viewpoint stands in sharp contrast to the observer's orientation inherent in Pea's and Hutchins's distributed accounts wherein an artifact and a person using it are treated as separate components of a functional system. For the actor, however, the two are inseparable. In the case of Bateson's illustration, the tool is the act of tapping with the stick, not the stick per se.

Our preference for an actor's rather than an observer's viewpoint is not restricted to the issue of tool use, but instead runs throughout the discussion of the first-grade teaching experiment. For example, in stating the overall intent of the arithmetic rack instructional sequence in terms of Greeno's environmental metaphor, I described a world in which students might come to act by paying particular attention to the quality of their experience[12] as they acted in it. Similarly, in differentiating our psychological orientation from mainstream cognitive science approaches, I emphasized that our purpose is to infer the quality of individual students' experiences in the world and to account for developments in their ways of experiencing in

[11] In doing so, we took a psychological perspective and focused on individual students' qualitatively distinct ways of participating in classroom mathematical practices. In contrast, we adopted a social perspective when we planned the instructional sequence and envisioned the evolution of taken-as-shared, communal ways of reasoning with the arithmetic rack.

[12] Strictly speaking, this focus on experience is redundant: The world acted in is the world experienced. I have used the term *experience* to differentiate the world-acted-in from what an observer might take to be the environment that can be analyzed independently of activity in propositional terms.

terms of the reorganization of activity and of the world acted in.[13] In the most general terms, this adoption of the actor's viewpoint is central to our activity as mathematics educators who coparticipate in learning-teaching processes with teachers and their students. To coparticipate is to engage in communicative interactions that involve a reciprocity of perspectives characteristic of the actor's viewpoint (cf. Rommetveit, 1992; Schutz, 1962). Our commitment to the second of the two characterizations of tool use identified by Dewey is symptomatic of this broader interest.

Conclusion

In this article, I have delineated general areas of agreement with distributed theories of intelligence and have identified two points of resistance. In doing so, I have described my learning in this dialogical encounter, thereby acknowledging my debt to the developers of distributed theories. The challenges that these theories pose for those of us who see value in constructivist analyses of individual students' activity is particularly apparent in the case of tool use. As Walkerdine (1988) and Kaput (1991) both noted, semiotic processes in general and symbolizing in particular have often played little if any role in constructivist analyses of mathematical development. There are, to be sure, several notable exceptions (e.g., Bednarz, Dufour-Janvier, Poirier, & Bacon, 1993; Kaput, 1987; Mason, 1987; Nemirovsky, 1994; Pirie & Kieren, 1994; Sfard & Linchevski, 1994; Thompson, 1992). Nonetheless, there has been a tendency to view mathematical reasoning as occurring apart from mediational means and to treat symbols as mere vehicles used to express its results. Therefore, much remains to be learned from distributed analyses of activity. The challenge as I have framed it is to view mediational means as constituent parts of individual students' qualitatively distinct ways of acting when we take a psychological perspective. The analyses that I and my colleagues have conducted of recent teaching experiments represent one attempt to move in this direction.

In this article, I have focused on accounts of distributed intelligence that evolved from mainstream psychology and have not considered those located more directly within the sociocultural tradition. In closing, I note that some of the distributed views of intelligence developed within the sociocultural tradition exemplify the observer's viewpoint on tool use and others the actor's viewpoint. For example, Cole and Engeström (1993) adopted the observer's viewpoint when they argued that the mediation of activity through artifacts "implies a distribution of cognition among individual, mediator, and environment" (p. 13). In contrast, Meira (1995), like Ueno (1995), adopted the actor's viewpoint when he questions the treatment of individuals and tools as distinct components of

[13] Critiques of IP psychology that elaborate the contrast between the actor's and the observer's viewpoints can be found in Cobb (1987, 1990).

an encompassing system. He instead stressed the "dialectical [or reflexive] relation between notations-in-use and mathematical sense making" (p. 270) and proposed

> an activity-oriented view that takes cultural conventions, such as notational systems, to shape in fundamental ways the very activities from which they emerge, at the same time that their meanings are continuously transformed as learners produce and reproduce them in activity, (p. 270)

It should be clear that accounts of this latter type are of greater relevance to my interests as a mathematics educator. I also suggest that many of the recent contributions to sociocultural theory that involve the observer's perspective have emerged at least implicitly in opposition to mainstream cognitive science. Although these contributions might well be useful for many purposes, I nonetheless contend that the development of sociocultural theory would benefit by acknowledging a broader range of psychological theories, some of which are concerned with meaning in the world rather than with cognitive behaviors that intervene between perceptual input and response output. Once we move beyond the semiotic territory of mainstream cognitive science, several oppositions, including that between the individual and more encompassing systems as units of analysis, dissipate, thus enabling us to transcend a range of dichotomies.

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Part IV
Classroom Mathematical Practices

Chapter 8
Introduction

Paul Cobb with Michelle Stephan, and Janet Bowers

The notion of a classroom mathematical practice was introduced in the chapter reprinted in the previous section of this book. It is fair to say in retrospect that my initial use of this construct was largely intuitive: I had not defined a classroom mathematical practice with any precision or clarified how other researchers might identify the mathematical practices interactively constituted by the teacher and students in other classrooms. The primary goal of the chapter reprinted in this part was to overcome some of these limitations. The process of attempting to explicate and refine this notion was lengthy and occurred as Janet Bowers[1] and Michelle Stephan[2] conducted retrospective analyses of two classroom design experiments for their dissertation studies (Bowers, 1996; Stephan, 1998). A report of Bowers' analysis was subsequently published in *Cognition and Instruction* (Bowers, Cobb, & McClain, 1999) and a revised version of Stephan's dissertation was published as a *Journal for Research in Mathematics Education* monograph (Stephan, Bowers, & Cobb, 2003). The chapter reprinted here framed part of Stephan's dissertation analysis as a case in which to explicate the process of analyzing the collective mathematical learning of a classroom community.[3]

I can best clarify the phenomenon that we sought to account for when analyzing the mathematical learning of a classroom community by asking the reader to consider the following thought experiment. The analysis reported in this reprinted chapter is of a classroom design experiment conducted in a first-grade classroom

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[1] Janet Bowers is currently a member of the mathematics education faculty at San Diego State University.

[2] Michelle Stephan is currently a middle-school mathematics teacher in Seminole County, Florida, where she recently received the award of teacher of the year.

[3] Stephan and Rasmussen (2002) subsequently identified an additional evidentiary criterion for claiming that a particular type of mathematical activity has been constituted as normative in a classroom.

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that focused on linear measurement.[4] Suppose that we had interviewed not only the students in the design experiment classroom at the end of the school year but also the students from another first-grade classroom in the same school. I have no doubt that if the video-recordings of these interviews were shuffled, the reader could nonetheless identify which classroom each student came from with few errors. An analysis of the mathematical practices established in the two classrooms documents differences in the classroom learning environments. These content-specific differences in the classroom social settings in turn account for the contrasts in the forms of mathematical reasoning that the two groups of students developed.

To continue the thought experiment, suppose that we now focus only on video-recorded interviews conducted with the students in the teaching experiment classroom. The contrast set is then not one group of students compared with another, but is instead the mathematical reasoning of students in the same classroom. This comparison brings to the fore the (socially situated) diversity in the reasoning of students who have participated in the same classroom mathematical practices. As we illustrated in the reprinted chapter, analyses of individual students' mathematical activity as they participate in collective practices complement analyses of collective mathematical learning and provide an account of the diversity in their mathematical reasoning. In the reprinted chapter, we also emphasize that we view an individual student's mathematical reasoning as an act of participating in collective practices. As a consequence, students are seen to actively construct their mathematical understandings as they participate in (and contribute to the development of) collective classroom practices. It is in this sense that we regard students' participation in collective practices as constituting the conditions for the possibility of their mathematical learning.

In the reprinted chapter, we state that a mathematical practice comprises three inter-related types of mathematical norms:

- normative purpose for engaging in mathematical activity;
- normative standards of mathematical argumentation;
- normative ways of reasoning with tools and symbols.

In arriving at this characterization, we intentionally kept the number of constituent elements of a mathematical practice to a minimum. The key criterion used when formulating this definition was that there was evidence that variations in each constituent element significantly influenced the quality of participating students' learning. The inclusion of normative standards of mathematical argumentation as a constituent element extended prior work on sociomathematical norms that I had conducted with Erna Yackel (Yackel & Cobb, 1996). The inclusion of normative ways of reasoning with tools and symbols was a direct consequence of our efforts to draw on the theory of Realistic Mathematics Education (RME) when designing instructional sequences (see the chapter reprinted in the previous part). The inclusion of a normative purpose as an integral aspect of a mathematical practice

[4]This experiment was conducted in collaboration with Beth Estes, a first-grade teacher. The members of the research team in addition to Estes and myself were Koeno Gravemeijer, Michelle Stephan, Kay McClain, and Beth Petty.

was inspired by the philosopher Philip Kitcher's (1984) analysis of mathematical knowledge and reflects prior work that I had conducted on the orientations to mathematics that students typically develop in school (see the introduction to the second chapter reprinted in this book).

It is of course possible to conduct analyses of classroom mathematical activity that focus just on one of the three aspects of a mathematical practice. However, as we illustrate in the reprinted chapter, accounts of how a single aspect evolves over time are incomplete and lack explanatory power unless they relate changes in the focal aspect to changes in the other two aspects. This claim that the three aspects of a mathematical practice are interdependent is consistent with Stigler and Hiebert's (1999) observation that classrooms are systems of mutually influencing elements rather than collections of independent variables that investigators can manipulate at will.

In addition to being more explicit about the nature of a classroom mathematical practice, the second goal that we sought to address in the reprinted chapter was to understand how one practice emerges as a reorganization of prior practices. As the analysis of data generated in the course of the measurement design experiment illustrates, there is often a (sometimes protracted) period of emergence during which novel forms of mathematical reasoning are introduced by one or more students or by the teacher, and are negotiated and legitimized in public classroom discourse. We had initially assumed that the emergence of a new mathematical practice would be tied to the introduction of a new type of instructional activity, but the analysis of the measurement sequence makes it clear that this is not necessarily the case. After completing this analysis, we in fact came to regard it as entirely natural for a new practice to emerge as a reorganization of prior practices as students worked on a single type of instructional activity.

The third goal of this chapter was to illustrate the relationship between the evolution of collective practices and the development of the participating students' mathematical reasoning. We therefore included analyses of two children's learning as the measurement instructional sequence was enacted in the classroom. In doing so, we interwove these analyses with our account of the evolution of collective practices to highlight the reflexive relation between individual and collective mathematical learning. Reflexivity is an extremely strong relationship and implies not only that the two phenomena are interdependent, but also that one does not exist without the other. On the one hand, the ways in which individual students participate in collective practices change as they learn, thereby contributing to the evolution of those practices. On the other hand, the collective practices in which the students participate both support and constrain their individual learning.[5]

[5]This characterization of the relation between individual and collective learning makes contact with Sfard's (2008) recent analysis of mathematical learning from a strong discourse perspective. Sfard stresses the process of both individualizing collective practices and collectivizing novel individual contributions. In addition, it is consistent with the sociologist Anthony Giddens' (1984) characterization of the relation between social structure and human agency. Giddens argues that social structure both enables and constrains the ways in which agency can be exercised, and that structure is created, reproduced, and potentially transformed as people exercise agency.

We refined the notion of a classroom mathematical practice over a period of several years. The order in which this and the previous section appear in this book is somewhat arbitrary as Gravemeijer, Yackel, and I were working out the version of Realistic Mathematics Education (RME) described in the previous part during this same time period. In retrospect, it is apparent that our version of RME and the interpretive constructs that we developed for making sense of what was going on in mathematics classrooms were interdependent. On the one hand, the construct of a classroom mathematical practice grew out of our efforts to test and revise the design conjectures inherent in instructional sequences. On the other hand, the interpretations of classroom events that we made while using the emerging interpretative framework fed back to inform the ongoing instructional development effort. It is worth highlighting that in this process, we proposed and modified theoretical constructs in response to problems and issues encountered in the classroom. The strength of this approach to theory development is that the resulting constructs do not stand apart from the reality of working in classrooms but were instead grounded in our efforts to support students' learning in particular mathematical domains.[6]

As a point of clarification, I should stress that the notion of a classroom mathematical practice and, more generally, the interpretive framework did not emerge from classrooms per se, but from our activity of experimenting in classrooms. As I have indicated, we initially used the notion of a mathematical practice intuitively. In other words, the precursor to this construct was an interpretive routine that we enacted both when formulating hypothetical learning trajectories as we planned for design experiments, and when documenting the actual learning trajectory jointly enacted by the teacher and students in a classroom. The process of reifying this construct by defining it explicitly proved to be challenging, in part because we had to explicate how, in action, we made sense of specific classroom episodes. Our goal in doing so was to describe in general terms how we came to grips with and made judgments in concrete cases. In my view, the most important contribution that theory can make to educational practice is to inform the process of making pedagogical and design decisions and judgments in particular cases. As the reprinted chapter makes clear, our commitment to ground constructs in the reality of educational practice results in strongly situated accounts of mathematical learning.

Toward the end of the reprinted chapter, we argued that analyses of the type that we present together with the associated sequence of instructional activities could serve as resources for supporting the learning of teacher communities. This conjecture oriented a subsequent research project in which we collaborated with two groups of middle-school mathematics teachers who worked in two different

[6]This approach has much in common with grounded theory as described by Glaser and Strauss (1967). The primary difference is that grounded theory emphasizes the importance of grounding constructs empirically when analyzing data. The design research approach that I and my colleagues take also emphasizes the importance of developing constructs in response to problems encountered while intervening to support and understand students' mathematical learning.

urban districts to support their learning over a 5-year period.[7] In the course of these collaborations, we investigated whether and in what ways instructional sequences developed in two prior design experiments that had focused on statistical data analysis could serve as supports for the teacher's learning.[8] Two issues that emerged in the course of this project bear directly on the analysis of classroom mathematical learning and the design of instructional sequences.

With regard to the first of these issues, we found that we were able to support the middle-school teachers in reconstructing the learning trajectory that underpinned the statistics instructional sequences. However, the process of doing so was not as straightforward as we had anticipated and it proved essential to address a range of additional issues including the school and district settings in which the teachers worked, the deprivatization of the teachers' classroom practices, and the cultivation of students' mathematical interests (Cobb, Zhao, & Dean, 2009; Dean, 2005; Visnovska, 2009; Visnovska, Zhao, & Gresalfi, 2007; Zhao & Cobb, 2007). We did not formally introduce the term mathematical practice while working with the teachers. Nonetheless, the practices that we had identified when conducting a retrospective analysis of the two statistics design experiments proved useful in delineating major phases of the statistics instructional sequences. In addition, it was also important for the teachers to tease out qualitative differences in students' ways of participating in these collective practices during each phase of the two sequences. In retrospect, this is eminently reasonable given that students' diverse interpretations and solutions constituted the primary resource on which the teachers capitalized in their classrooms as they attempted to move the entire group forward and thus achieve their mathematical agendas. In particular, the visions of high-quality mathematics instruction that the two groups of teachers developed with our support involved building on students' solutions to initiate and guide whole-class discussions that focused on significant statistical ideas. Based on this experience, if I were to revise the reprinted chapter, I would augment the detailed analysis of the two students' mathematical learning by giving a broad overview of the full range of ways in which the students participated in each of the classroom mathematical practices.

The second issue that emerged while we were collaborating with the two groups of teachers concerns instructional designers' intent when developing instructional tasks. In our experience, designers frequently assume that they are developing tasks and associated resources such as computer tools in order to support students' learning directly. The implicit model is that of students learning as they

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[7]This project was conducted by Paul Cobb, Kay McClain, Chrystal Dean, Jana Visnovska, Qing Zhao, Teruni Lamberg, Melissa Gresalfi, and Lori Tyler. In their dissertation studies, Dean (2005) documented the gradual transition of one of these groups into a professional learning community during the first 2 years of our collaboration with the teachers, and Visnovska (2009) documented the collective learning of this community during the final 3 years of our collaboration.

[8]Reports of the classroom design experiments in which these two instructional sequences were developed can be found in Cobb (1999), Cobb, McClain, and Gravemeijer (2003), and McClain and Cobb (2001).

complete tasks either on their own or in groups. In this model, the teacher's role is to ensure that instructional tasks are enacted in the classroom as the designer intends. We had questioned this model while conducting classroom design experiments by emphasizing the importance of whole-class discussions that focused on central mathematical issues. However, it was during our collaboration with the two groups of teachers that our framing of instructional tasks as tools that teachers can use to achieve an instructional agenda became increasingly explicit. In this framing, teachers' use of instructional tasks and associated resources can be seen to involve adaptations that de Certeau (1984) characterized as a second process of creation or production (cf. McLaughlin, 1987; Wertsch, 1998). In my view, this perspective is potentially productive for both instructional design and teacher professional development because it orients us to focus our efforts on supporting the development of both teachers' capabilities and social resources such as professional networks and communities on which they can draw as they adapt instructional sequences to their local setting.

The increasing importance that we came to attribute to the teachers' central mediating role is at odds with the way in which the teacher is backgrounded in the reprinted chapter. It is apparent from the transcribed excerpts included in the chapter that the teacher's actions as a more knowledgeable were crucial. However, the teacher's initiatives and her responses to students are treated as ancillary to the focus on the students' learning. At one level, this is justifiable in that choices necessarily have to be made about what to focus on when studying social settings as complex as classrooms. Nonetheless, a companion analysis that focused explicitly on the teacher's role in interaction with the students would have been valuable.[9]

In the final paragraphs of the reprinted chapter, we discuss possible limitations of the analytical approach that we have illustrated. One of the limitations we identified is that the focus is restricted to the social context of the classroom and ignores the institutional setting in which the classroom was located. The broader school and districts setting in which teachers develop and refine their instructional practices became a major focus of my subsequent work and is the subject of the final chapter reprinted in this book. As a further limitation, we note that the analytical approach does not address issues of equity in students' access to significant mathematical ideas explicitly. When we conducted the measurement design experiment, we attempted to ensure that we supported the learning of all the participating students. However, we treated this as a pragmatic concern that was ancillary to our primary research focus on the students' individual and collective mathematical learning. In contrast, we framed issues of equity in learning opportunities as an explicit research focus in the subsequent design experiments that focused on statistical data analysis. A paper that grew out of this work is reprinted in the next part of this book.

[9] Kay McClain did subsequently analyze aspects of her role as teacher in the later classroom design experiments that focused on statistics (e.g., McClain, 2002).

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Chapter 9
Participating in Classroom Mathematical Practices

Paul Cobb, Michelle Stephan, Kay McClain, and Koeno Gravemeijer

In this article, we describe a methodology for analyzing the collective learning of the classroom community in terms of the evolution of classroom mathematical practices. To develop the rationale for this approach, we first ground the discussion in our work as mathematics educators who conduct classroom-based design research. We then present a sample analysis taken from a 1st-grade classroom teaching experiment that focused on linear measurement to illustrate how we coordinate a social perspective on communal practices with a psychological perspective on individual students' diverse ways of reasoning as they participate in those practices. In the concluding sections of the article, we frame the sample analysis as a paradigm case in which to clarify aspects of the methodology and consider its usefulness for design research.

In his introduction to this special issue, Barab (Barab, & Kirshner, 2001) clarifies why new methodologies are needed that capture learning in rich environments and that feed back to inform design and instruction. We contribute to this effort by describing an approach that involves analyzing the collective mathematical learning of the classroom community in terms of the evolution of classroom mathematical practices. In doing so, we follow Skemp (1982) in interpreting methodology broadly to include background assumptions as well as specific analytical methods. As Saxe (1994) observed, discussions of methodology typically take one of two forms. One is technique based and focuses on data gathering techniques, procedures for analyzing data, and so forth. The second form is framework based and focuses on the framing of questions about a general class of phenomena by relating the methodological approach to central epistemological assumptions. In concert with Saxe, we attempt to blend the two types of discussions by describing both basic theoretical tenets and analytical techniques.

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In the first of the eight sections of the article, we provide a grounding for the analytic approach we use by describing the type of classroom-based design research that we conduct. As we clarify, this research involves teaching experiments of up to a year duration in the course of which we develop sequences of instructional activities. In the second section of the article, we present the interpretive framework that we use to organize our analyses of classroom events. We illustrate that this framework coordinates a social perspective on communal activities with a psychological perspective on the reasoning of the participating students. In the third section of the article, we turn our attention to the specific method we use when analyzing the relatively large corpus of classroom video recordings and other data sources generated during a classroom teaching experiment. We clarify that, as the approach we take is an adaptation of Glaser and Strauss's (1967) constant comparison method, it falls within the interpretivist tradition and is concerned with meaning and context. We then go on to describe how we systematically work through the data by continually testing and revising conjectures. The sample analysis that we present in the fourth section of the article to illustrate our analytic approach is taken from a first-grade teaching experiment that focused on linear measurement. Although we primarily describe the classroom social and the sociomathematical norms, we concentrate on explicating the process by which we analyze the evolution of both the classroom mathematical practices (social perspective) and the reasoning of individual students (psychological perspective).

Following our discussion and application of the methodology, in the sixth section of the article we consider the traditional methodological issues of trustworthiness and replicability. As we note, it is the grounding of final claims and assertions in the data that makes the analytic approach trustworthy; final claims and assertions can be justified by backtracking through the various phases of the analysis, if necessary to the original data sources. With regard to replicability, we argue that an approach of this type, which brings context and meaning to the fore, can facilitate disciplined, systematic inquiry into instructional innovation while simultaneously doing justice to the complexity of the classroom. In the seventh section of the article, we consider the usefulness of the analytic approach. In doing so, we return to one of the criteria that we discussed in the first part of the article, namely that an analytic approach that is appropriate for our purposes should feed back to inform the ongoing instructional design effort. Finally, in the last section of the article, we discuss the limitations of our methodological approach.

Design Research

As an initial orientation, it is important to note that the analytical approach we use to develop accounts of students' mathematical learning as it occurs in the social context of the classroom has emerged over a 12-year period as we have conducted a series of teaching experiments in elementary and middle-school classrooms.[1] In

[1] A discussion of the purposes for conducting a classroom teaching experiment and of the theoretical and pragmatic issues involved can be found in Cobb (2000b), Confrey and Lachance (2000), Simon (2000), and Yackel (1995).

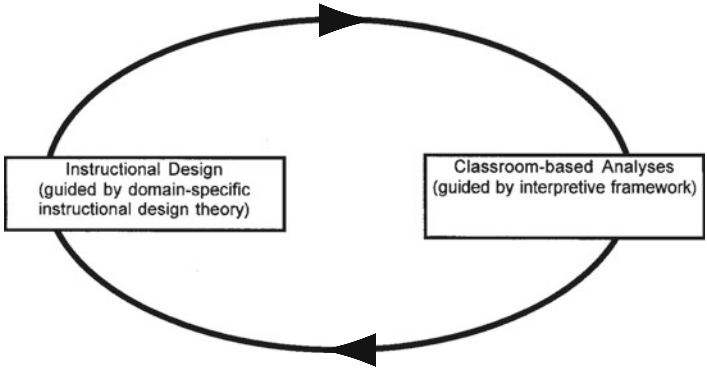

Fig. 9.1 The instructional design research cycle

the course of these experiments, which are up to a year in duration, we develop sequences of instructional activities and analyze both students' mathematical learning and the means used to support that learning. Research of this type falls under the general heading of *design research* in that it involves both instructional design and classroom-based research.[2] The basic design–research cycle is shown in Fig. 9.1.

The first aspect of the cycle involves developing instructional sequences as guided by a domain-specific instructional theory. In our case, we draw on the theory of Realistic Mathematics Education developed at the Freudenthal Institute in the Netherlands (Gravemeijer, 1994a; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Streefland, 1991; Treffers, 1987). Gravemeijer (1994a, 1994b) has written extensively on the process of instructional design within the design–research cycle and clarifies that the designer initially conducts an anticipatory thought experiment. In doing so, the designer envisions how mathematical activity and discourse may evolve as proposed types of instructional activities are enacted in the classroom, thereby developing conjectures about both possible trajectories for mathematical learning, and the means that may be used to support and organize that learning. It is important to stress that these conjectures are tentative and provisional, and are tested and modified on a daily basis once the teaching experiment begins. The process of adapting and revising the conjectures is informed by an on-going analysis of classroom events and it is here that the second aspect of the design–research cycle, classroom-based analyses, comes to the fore.

Our purpose in outlining the design–research cycle is to clarify the purposes that motivate our attempts to make sense of what is happening in the classrooms in which we work. It is with respect to these concerns and interests that we have become aware of several criteria that an analytical approach should satisfy if it is to enable us to contribute to reform in mathematics education as an ongoing, iterative process of improvement. These criteria for an appropriate analytical approach include that

[2]Following Gravemeijer (1994a), we have previously called this type of research developmental research. Our purpose in relabeling it design research is to reduce the possibility that it may be confused with either child development research or with noninterventionist research into the development of particular mathematical concepts.

1. It should enable us to document the collective mathematical development of the classroom community over the extended periods of time covered by instructional sequences.
2. It should enable us to document the developing mathematical reasoning of individual students as they participate in the practices of the classroom community.
3. It should result in analyses that feed back to inform the improvement of our instructional designs.

These criteria can be thought of as constituting design specifications for an appropriate analytic approach. In the following paragraphs, we discuss the rationale for each in turn.

The first of these criteria stems from the observation that the conjectures the designer develops when conducting an anticipatory thought experiment cannot be about the trajectory of each and every student's learning for the straightforward reason that there are significant qualitative differences in their mathematical thinking at any point in time. In our experience, descriptions of planned instructional approaches written so as to imply that all students will reorganize their thinking in particular ways at particular points in an instructional sequence involve, at best, questionable idealizations. An issue that has arisen for us is therefore that of clarifying what the envisioned learning trajectories that are central to our (and others') work as instructional designers may be about. The resolution we propose involves viewing a hypothetical learning trajectory as consisting of conjectures about the collective mathematical development of the classroom community. This proposal in turn indicates the need for a theoretical notion or construct that enables us to talk explicitly about collective mathematical learning, and it is for this reason that we have developed the notion of a classroom mathematical practice. Cast in these terms, an envisioned learning trajectory then consists of an envisioned sequence of mathematical practices together with the means of supporting and organizing the emergence of each practice from prior practices.

The second of the three criteria focuses on the qualitative differences in individual students' mathematical reasoning. The rationale for this criterion is again deeply rooted in our work in classrooms. In particular, the classroom sessions we conduct during a teaching experiment are frequently organized so that students initially work either individually or in small groups before convening for a whole-class discussion of their solutions. A pedagogical strategy that we have found productive involves the teacher and one or more of the project staff circulating around the classroom during individual or small-group work in order to gain a sense of the diverse ways in which students are interpreting and solving instructional activities. Toward the end of the individual or small-group work, the teacher and project staff members then confer briefly to prepare for the whole-class discussion. In doing so, they routinely focus on the qualitative differences in students' reasoning in order to develop conjectures about mathematically significant issues that may, with the teacher's proactive guidance, emerge as topics of conversation. Their intent is to capitalize on the diversity in the students' reasoning by identifying interpretations and solutions that, when compared and contrasted, may lead to substantive mathematical discussions.

Given this pragmatic focus on individual students' reasoning, we require an analytic approach that takes account of the diverse ways in which students participate in communal classroom practices. As becomes apparent when we present the sample analysis, this diversity is, in the hands of a skillful teacher, a primary motor of the collective mathematical learning of the classroom community.

The rationale for the third criterion, that analyses should feed back to inform the improvement of instructional designs, is readily apparent given our interest in design research (see Fig. 9.1). This criterion in turn implies that analyses should enable us to document individual and collective mathematical learning over, say, the 3-month period of an instructional sequence. Although this requirement may appear innocuous at first glance, we found it challenging to develop an analytical approach that enables us to step back and view in broad relief what has transpired in a classroom over a time period of this length. In making this comment, we note that analyses that locate students' mathematical activity in a social context often deal with a small number of lessons, or perhaps focus on just a few minutes within one lesson. One of us has in fact contributed to entire articles that deal with classroom episodes that last 10 min (e.g., Cobb, Wood, & Yackel, 1992). Although detailed microanalyses of this type can make an important contribution to design research, they do not enable us to achieve the broad perspective on individual and collective mathematical development that we require.

It should be clear from this discussion that we view a theoretical idea such as that of classroom mathematical practice as a conceptual tool whose development reflects particular interests and concerns. We mention this because academic discourse about education often reflects the assumption that instructional approaches should be derived from theory in a top–down manner. The design–research cycle involves an alternative view of the relation between theory and instructional practice in which neither is taken as primary. Instead, the basic relation is one of reflexivity in which the development of theoretical ideas is driven by and remains rooted in instructional practice that is itself guided by current theoretical ideas (cf. Cobb & Bowers, 1999). From this point of view, the relevant criterion when assessing the value of a theoretical construct is whether it enables us to be more effective in supporting students' mathematical learning.

Interpretative Framework

The interpretative framework that we use to organize our analyses of individual and collective mathematical learning is shown in Fig. 9.2. As the column headings "Social Perspective" and "Psychological Perspective" indicate, the analytical approach involves coordination two distinct theoretical viewpoints on mathematical activity. The entries in the column under social perspective indicate three aspects of the classroom microculture[3] that we have found useful to differentiate, and the

[3] For a justification of the notion of a classroom microculture, see Cole (1995).

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical interpretations and reasoning

Fig. 9.2 An interpretive framework for analyzing communal and individual mathematical activity and learning

entries in the column under psychological perspective indicate three related aspects of individual students' activity in the classroom.

Social and Psychological Perspectives

The social perspective indicated in Fig. 9.2 is concerned with ways of acting, reasoning, and arguing that are normative in a classroom community. From this perspective, an individual student's reasoning is framed as an act of participation in these normative activities. In contrast, the psychological perspective focuses squarely on the nature of individual students' reasoning or, in other words, on his or her particular ways of participating in communal activities. Thus, whereas the social perspective brings to the fore normative taken-as-shared[4] ways of talking and reasoning, the psychological perspective brings to the fore the diversity in students' ways of participating in these taken-as-shared activities. Together, these two perspectives therefore address the first two criteria that an analytical approach should satisfy if it is to be appropriate for our purposes. An issue central to the methodology being discussed is that of coordinating these two distinct perspectives on mathematical activity, and, as described later, we take the relation between them to be reflexive.[5]

In terms of intellectual lineage, the social perspective draws inspiration from sociocultural theory (e.g., Cole, 1996; Lave, 1988; Rogoff, 1997) and from

[4]We speak of normative activities being taken as shared rather than shared to leave room for the diversity in individual students' ways of participating in these activities. The assertion that a particular activity is taken as shared makes no deterministic claims about the reasoning of the participating students, least of all that their reasoning is identical.

[5]For a number of years, we spoke of the relation between the two perspectives as being dialectical. However, German colleagues pointed out that, in their country, this term was associated with neo-Marxist theories of dialectical materialism. It was to avoid this miscommunication that we began to describe the relation as reflexive.

ethnomethodology and symbolic interactionism (Blumer, 1969) as they have been adapted to problems and issues in mathematics education (cf. Bauersfeld, Krummheuer, & Voigt, 1988). The lineage of the psychological perspective can be traced to both constructivism (Piaget, 1970; Steffe & Kieren, 1994; Thompson, 1991) and to distributed accounts of intelligence (e.g., Hutchins, 1995; Pea, 1993). Given this relatively comprehensive list of intellectual sources, we should clarify that our goal is not to achieve some grand theoretical synthesis.[6] Instead, our focus is pragmatic and centers on supporting and organizing students' mathematical learning. As a consequence, in drawing on the theoretical sources we have listed, we have adapted and modified ideas to suit our purposes.[7] Although space limitations prevent a complete discussion of these modifications, we briefly touch on two assumptions that have been adapted and that are central to this discussion.

We have already indicated that a key theoretical construct we use when we take a social perspective is that of a classroom mathematical practice—a construct adapted from sociocultural theorists' notion of a cultural practice (cf. Axel, 1992; Minick, 1989). In sociocultural theory, this notion typically refers to normative ways of acting that have emerged during extended periods of human history. We found this idea attractive in that it makes it possible to characterize mathematics as a complex human activity and in that it brings meaning to the fore by eschewing a focus on socially accepted ways of behaving in favor of an emphasis on the development of taken-as-shared meanings (van Oers, 1996, 2000). Despite these advantages, sociocultural theorists' notion of a cultural practice is not a completely adequate conceptual tool given our interest in changes in the normative activities of classroom communities. For example, in sociocultural theory, the historically developed practices of the discipline are seen to exist prior to and independently of teachers and their students. In contrast to preexisting, historically developed disciplinary practices, we view the normative practices of a local classroom community as being constituted by the teacher and his or her students in the course of their ongoing interactions (cf. Boaler, 2000). Thus, when we take the local classroom community rather than the discipline as our point of reference, a practice is seen to be an emergent phenomenon rather than an already-established way of reasoning and communicating into which students are to be inducted.

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[6] Given our discussion of social and psychological perspectives, we need to say explicitly that we are not attempting to reconcile Vygotskian and Piagetian theory as some have assumed (e.g., Lerman, 1996).

[7] In reflecting on his activity as an instructional designer, Gravemeijer (1994b) argued that design resembles the thinking process that Lawler (1985) characterized by the French word *bricolage*, a metaphor taken from Claude Levi–Strauss. A bricoleur is a handy man who invents pragmatic solutions in practical situations. The bricoleur has become adapt at using whatever is available. The bricoleur's tools and materials are very heterogeneous: Some remain from earlier jobs; others have been collected with a certain project in mind. Extending this metaphor, we would add the interpretative framework we use can also be viewed as a bricolage. In developing it, we have acted as bricoleurs who have drawn on and adapted ideas from a range of theoretical sources for pragmatic ends. Casting our work in these down-to-earth terms serves to differentiate it from more grandiose efforts that aim to fashion theoretical cosmologies (cf. Shotter, 1995).

We made similar modifications when fashioning a psychological perspective that is appropriate for our purposes, in this case by drawing on constructivism and distributed theories of intelligence. For example, our debt to distributed accounts of intelligence acknowledged (Pea, 1992), we could not accept this theoretical orientation ready made given its rejection of analytical approaches that focus explicitly on the nature of individual students' reasoning. In the psychological perspective that we take, the tools and symbols that students use are not considered to stand apart from or outside the individual, but are instead viewed as constituent parts of their activity (cf. Dewey, 1981). There is no talk of processing information or creating internal representations. Instead, intelligence is seen to be embodied, or to be located in activity (Bateson, 1973; Johnson, 1987; Winograd & Flores, 1986). Rather than representing a world, people are portrayed as individually and collectively enacting a taken-as-shared world of signification (Varela, Thompson, & Rosch, 1991).

The goal of analyses conducted from this psychological perspective is therefore not to specify cognitive mechanisms located inside students' heads. Instead, it is to infer the quality of individual students' reasoning in, with, and about the world, and to account for developments in their reasoning in terms of the reorganization of activity and the world acted in.[8] Consequently, what is viewed as a student–tool system from the perspective of distributed intelligence is, from our psychological perspective, an individual student engaging in mathematical activity that involves reasoning with tools and symbols. Thus, although the focus of this psychological viewpoint is explicitly on the quality of individual students' reasoning, its emphasis on tools is generally consistent with the notion of mediated action as discussed by sociocultural theorists (cf. Kozulin, 1990; van der Veer & Valsiner, 1991; Wertsch, 1994). Furthermore, as we have seen, the remaining component of the functional system posited by distributed theories of intelligence, social context, becomes an explicit focus of attention when this psychological perspective is coordinated with the social perspective of the framework.[9] An approach of this type is fundamentally nondualist in that learning involves the reorganization of the world acted in as well as of ways of acting in the world (cf. Roth & McGinn, 1998).

In summary, there is an extremely strong relation between what we have described as the social and psychological perspectives that does not merely mean that the two perspectives are interdependent. Instead, it implies that neither perspective exists without the other in that each perspective constitutes the background against which mathematical activity is interpreted from the other

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[8]The characterization of the individual is generally consistent with Roth's use of the individual's life world or the individual in the world as his unit of analysis (Roth & McGinn, 1998). In his terms, our concern when we adopt the psychological perspective of the interpretive framework is to delineate the ontology of the world in which an individual student acts.

[9]This approach of coordinating psychological and social analyses is closely related to several other proposals. These include Hatano's (1993) call to synthesize constructivist and Vygotskian perspectives, Saxe's (1991) discussion of the intertwining of cultural forms and cognitive functions, and Rogoff's (1995) distinction between three planes of analysis that correspond to personal, interpersonal, and community processes.

perspective. For example, normative activities of the classroom community (social perspective) emerge and are continually regenerated by the teacher and students as they interpret and respond to each other's actions (psychological perspective). Conversely, the teacher's and students' interpretations and actions in the classroom (psychological perspective) do not exist except as acts of participation in communal classroom practices. When we take a social perspective, we therefore locate a student's reasoning within an evolving classroom microculture, and when we take a psychological perspective, we treat that microculture as an emergent phenomenon that is continually regenerated by the teacher and students in the course of their ongoing interactions. As a consequence the coordination is not between individual students and the classroom community viewed as separate, sharply defined entities. Instead, the coordination is between two alternative ways of looking at and making sense of what is going on in classrooms. The resulting analytical approach brings the diversity in students' reasoning to the fore while situating that diversity in the social context of their participation in normative classroom activities.

Aspects of the Classroom Microculture and Individual Students' Reasoning

Having described the social and psychological perspectives that underpin our analytical approach, we now focus on the details of the interpretative framework shown in Fig. 9.2. We should again clarify that this framework summarizes the way in which we organize our analyses of classroom events. Thus, the framework does not itself give rise to direct implications for educational change and improvement. Instead, it is a conceptual tool that we use to understand what is going on in the classrooms in which we work. The insights that we develop in the process do, however, have practical consequences in that they lead to conjectures about how we can improve our instructional designs.

In Fig. 9.2, the three entries under the column headed "Social Perspective" indicate three distinct aspects of the classroom microculture. The first of these aspects is *classroom social norms*. In general, an analysis that focuses on social norms serves to delineate the classroom participation structure (Erickson, 1986; Lampert, 1990). Social norms are characteristics of the classroom community and document regularities in classroom activity that are jointly established by the teacher and students. Examples of social norms include explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement or disagreement, and questioning alternatives when a conflict in interpretations had become apparent (Cobb, Yackel, & Wood, 1989). When developing conjectures about social norms, we look for instances where a student appears to violate a proposed norm and check to see whether his or her activity is constituted in the classroom as legitimate or illegitimate. In the former case we, of course, have to revise our conjecture whereas, in the latter case, the observation that the students' activity was constituted in the classroom as a breach of a norm provides evidence in support of the conjecture. This approach of focusing on regularities in joint activity can be contrasted with an alternative approach that casts criteria in terms of the proportion of students who act accord with a proposed norm. Criteria of this type are neither helpful nor relevant in our view because they are framed from a psychological perspective that is concerned with individual students' activity rather than from a social perspective that is concerned with how students' activity is constituted in the classroom. Similarly, because social norms document regularities in joint activity, we question accounts framed in individualistic terms in which the teacher is said to establish or specify social norms for students. To be sure, the teacher is an institutionalized authority in the classroom (Bishop, 1985). He or she expresses that authority in action by initiating, guiding, and organizing the renegotiation of classroom social norms. However, the students also play their part in contributing to the evolution of social norms. One of our primary conjectures is in fact that, in making these contributions (social perspective), students reorganize their individual beliefs about their own role, others' roles, and the general nature of mathematical activity (psychological perspective). As a consequence, we take these beliefs to be what, for want of a better term, we refer to as the psychological correlates of classroom social norms (see Fig. 9.2). We therefore conjecture that in guiding the establishment of particular classroom social norms, teachers are simultaneously supporting their students' reorganization of these beliefs. Furthermore, in line with our discussion of the reflexive relation between the social and psychological perspectives, we give primacy to neither the social norms nor individual students' beliefs. This implies that it is neither a case of a change in social norms causing a change in students' beliefs, nor a case of students first reorganizing their beliefs and then contributing to the evolution of social norms. Instead, social norms and the beliefs of the participating students coevolve in that neither is seen to exist independently of the other.

Our interest in classroom social norms emerged within the context of design research as we attempted to further our agenda of supporting students' mathematical learning, and it was within this context that we subsequently came to view our preoccupation with classroom social norms as inadequate. In particular, we came to realize that these norms are not specific to mathematics, but apply to any subject matter area. For example, one may hope that students would explain and justify their reasoning in science or history classes as well as in mathematics. We attempted to address this limitation by shifting our focus to normative aspects of students' activity that are specific to mathematics (Lampert, 1990; Simon & Blume, 1996; Voigt, 1995; Yackel & Cobb, 1996). Examples of these so-called *sociomathematical norms* include what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation.

Pragmatically, the analysis of sociomathematical norms has proven useful in helping us understand the process by which the teachers with whom we work foster the development of intellectual autonomy in their classrooms. This issue is particularly significant to us given that the development of student autonomy was an explicitly stated goal of our work in classrooms from the outset. However, we originally characterized intellectual autonomy in individualistic terms and spoke of students' awareness of and willingness to draw on their own intellectual capabilities when making mathematical decisions and judgments (Kamii, 1985; Piaget, 1973). As part of the process of supporting the growth of autonomy, the teachers with whom we have collaborated initiated and guided the development of a community of validators in their classrooms such that claims were established by means of mathematical argumentation rather than by appealing directly to the authority of the teacher or textbook. For this to occur, it was not sufficient for the students to merely learn that they should make a wide range of mathematical contributions. It was also essential that they became able to judge both when it was appropriate to make a mathematical contribution and what constituted an acceptable contribution. This required, among other things, that the students could judge what counted as a different mathematical solution, an insightful mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation. However, these are precisely the types of judgments that are negotiated when establishing sociomathematical norms. We therefore conjectured that students develop specifically mathematical beliefs and values that enable them to act as increasingly autonomous members of the classroom mathematical community as they participate in the negotiation of sociomathematical norms (Yackel & Cobb, 1996). Furthermore, we took these specifically mathematical beliefs and values to be the psychological correlates of the sociomathematical norms (see Fig. 9.2). In doing so, we conjecture that, in guiding the establishment of particular sociomathematical norms, teachers are simultaneously supporting their students' reorganization of the beliefs and values that constitute what may be called their mathematical dispositions. Once again, this conjecture is open to empirical investigation.

It is apparent from this discussion of sociomathematical norms that we revised our conception of intellectual autonomy as we worked in classrooms. At the outset, we defined autonomy in purely individualistic terms as a characteristic of students. However, as the notion of sociomathematical norms emerged, we came to view autonomy as a characteristic of an individual's way of participating in a community. In particular, the development of autonomy can be equated with a gradual movement from relatively peripheral participation in classroom activities to more substantial participation in which students increasingly rely on their own judgments rather than on those of the teacher (cf. Forman, 1996; Lave & Wenger, 1991). The example of autonomy is paradigmatic in this regard in that it illustrates the general shift we have made in our theoretical orientation during the 12 years that we have worked in classrooms away from an initially individualistic position toward one that involves coordinating social and psychological perspectives.

Returning to the interpretive framework (see Fig. 9.2), we have already clarified that our motivation for teasing out a third aspect of the classroom microculture, *classroom mathematical practices*, stems directly from our concerns as instructional designers. In particular, we argued that the conjectures inherent in a learning trajectory formulated while planning a teaching experiment cannot be about the anticipated mathematical learning of each and every student in a class. We then suggested that it is feasible to view a conjectured learning trajectory as consisting of an envisioned sequence of classroom mathematical practices together with conjectures about the means of supporting their evolution from prior practices. Extending this line of reasoning, we note that, in analyzing the evolution of classroom mathematical practices, we can document the actual learning trajectory of the classroom community as it is realized in interaction. Such analyses therefore draw together the two general aspects of design research, instructional design and classroom-based analyses, and can therefore feed back to inform ongoing design efforts (see Fig. 9.1). In addition, analyses of this type bear directly on the issue of accounting for students' mathematical learning as it occurs in the social context of the classroom. Viewed against the background of classroom social and sociomathematical norms, the mathematical practices established by a classroom community can be seen to constitute the immediate, local situations of the students' development. Consequently, in delineating sequences of such practices, the analysis documents the evolving social situations in which students participate and learn. We take individual students' mathematical interpretations and actions to be the psychological correlates of these practices and view the two as reflexively related (see Fig. 9.2). What is seen from one perspective as an act of individual learning in which a student reorganizes his or her mathematical reasoning is seen from the other perspective as an act of participation in the evolution of communal mathematical practices. In coordinating social and psychological perspectives, the approach we propose therefore seeks to analyze the development of students' mathematical reasoning in relation to the local social situations in which they participate and to whose emergence they contribute.

We conclude this discussion of the interpretative framework by giving an illustration, which brings the social perspective to the fore in order to further clarify the distinction between the three aspects of the classroom microculture. For the purposes of the illustration, consider the social norm of explaining and justifying interpretations. As we have noted, this and other social norms deal with facets of the classroom participation structure that are not specific to mathematical activity. In contrast, the related sociomathematical norms for argumentation deal with criteria that the teacher and students establish in interaction for what counts as an acceptable mathematical explanation and justification. For example, a criterion that became established during a teaching experiment that focused on place value numeration was that explanations had to be clear in the sense that the teacher and other students could interpret them in terms of actions on numerical quantities rather than, say, the mere manipulation of digits (Bowers, Cobb, & McClain, 1999). Due to the fact that sociomathematical norms are concerned with the evolving criteria for mathematical activity and discourse, they are not specific to any particular mathematical idea. Thus, the criterion that mathematical explanations should be clear could apply to elementary arithmetical word problems or to discussions about relatively sophisticated mathematical ideas that involve proportional reasoning. Classroom mathematical practices, in contrast, focus on the taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas. Consequently, if sociomathematical norms are specific to mathematical activity, then mathematical practices are specific to particular mathematical ideas. In the case of the teaching experiment that focused on place value numeration, the analysis of mathematical practices were concerned with the specific arguments and ways of reasoning about quantities that were treated as being clear and beyond further justification. In addition, the analysis described how each mathematical practice delineated in this way emerged as a reorganization of prior practices, thereby providing an account of the process by which these taken-as-shared ways of reasoning about place value numeration became established in this particular classroom.

This illustration serves to emphasize the analyses of classroom mathematical practices that take mathematics seriously by documenting the emergence of what is traditionally called mathematical content. We return to this point in the concluding sections of the article, once we have presented the sample analysis.

Methodological Considerations

Our primary focus when we present the sample analysis is on the evolution of classroom mathematical practices, as this is the least developed aspect of the interpretive framework. Our unit of analysis, therefore, is that of a classroom mathematical practice and students' diverse ways of participating in and contributing to its constitution. It should be noted that in making reference to both communal practices and individual students' reasoning, this unit captures the reflexive relation between the social and psychological perspectives on mathematical activity. The data corpus for an analysis of this type is relatively large and typically consists of video recordings of all classroom sessions conducted during a teaching experiment, copies of all the students' written work, and video recordings of student interviews conducted before and after the teaching experiment. Given this wealth of data, an obvious concern is that of developing a method for analyzing the data corpus in a systematic manner. This issue has been the focus of considerable debate in the interpretative social sciences in the course of which a range of methods has been proposed. The approach that we take follows Glaser and Strauss's (1967) constant comparison method as it has been adapted to the needs of design research (Cobb & Whitenack, 1996). Glaser and Strauss's method treats data as text and aims to develop coherent, trustworthy accounts of their possible meanings. They argued that, to produce such accounts, investigators must immerse themselves in the social situation they wish to understand by acting as participant observers. In doing so, they document incidents of the participants' activity which, when compared against one another, give rise to general themes or patterns. A hallmark of their method is that, as new data are generated, they are compared with currently conjectured themes or categories. This process of constantly comparing incidents leads to the ongoing refinement of the broad theoretical categories developed from the data. As Glaser and Strauss noted, negative cases that appear to contradict a current category are of particular interest and are used to further refine the emerging categories.

As we clarify shortly, the method we follow to document the emergence of classroom mathematical practices and the development of the reasoning of the participating students is consistent with this process. Like Glaser and Strauss (1967), we look for regularities and patterns in the ways that the teacher and students act and interact as they complete instructional activities and discuss solutions. The primary difference between our approach and that of Glaser and Strauss concerns the way in which we capitalize on the results of prior analyses when we begin a new analysis. In Glaser and Strauss's method, theoretical categories or constructs are developed anew from the data in each investigation. In contrast, we had developed and refined the general notion or category of a classroom mathematical practice before we began the sample analysis that we present in this article. To be sure, this notion was itself open to refinement and elaboration during the analysis as we interrogated data to delineate the specific practices that emerged during one teaching experiment. Thus, whereas Glaser and Strauss seem to imply that the development of theoretical categories or constructs is limited to the investigation at hand, for us this process spans a series of investigations.

The emphasis that our method, like that of Glaser and Strauss (1967), gives to intention and meaning serves to distinguish it from alternative approaches that take a strong social point of view by equating rational action as a matter of acting in accord with the norms or standards of a community. The difficulty for us with such an approach is that mathematical learning is treated exclusively as a process of coming to use conventional tools and symbols in socially accepted ways. This reduction of meaning to social use leads to an epistemological behaviorist position, which ignores what Sfard (2000) called the experiential aspect of meaning including imagery and emotion. The analytic approach we take is interpretivist precisely in the sense that we go beyond the observed social use of tools and symbols by inferring both the taken-as-shared intentions and meanings established by the classroom community (social perspective) and the interpretations that individual students make as they participate in communal practices (psychological perspective). This attention to both individual and taken-as-shared meaning is important given our agenda as mathematics educators who coparticipate in the learning–teaching process with teachers and their students. As Rommetveit (1992) and Schutz (1962) both observed, to coparticipate is to engage in communicative interactions that involve a reciprocity of perspectives characterized by a concern for socially situated meaning.

With regard to the actual process of analyzing data, the first phase of the method involves working through the data chronologically episode by episode. We are able to clarify what constitutes an episode in some detail once we have presented the sample analysis. For the present, we note that the determining characteristic of an episode is that a single mathematical theme is the focus of activity and discourse. When working through the data in this way, we develop conjectures both about the ways of reasoning and communicating that might be normative at a particular point in time and about the nature of selected individual students' mathematical reasoning. These conjectures together with the evidence that supports them are explicitly documented as part of a permanent log of the analysis. This log also records the process of testing and revising conjectures as subsequent episodes are analyzed. When we first attempted to coordinate the two perspectives some years ago, we found it useful to keep two parallel records, one labeled *social* and the other *psychological*. This strategy was helpful in that it enabled us to avoid confounding the two distinct perspectives for the most part. It was as we included cross references that indicated how inferences made from one perspective informed those made from the other perspective that we gradually improved our ability to coordinate the two view points. On reflection, constructing two parallel records in this manner proved to be an important means by which we supported and organized our own learning.[10]

The result of this first phase of the analysis is a chain of conjectures, refutations, and revisions that is grounded in the details of the specific episodes. In the second phase of the analysis, the logs of the first phase themselves become data that are meta-analyzed to develop succinct yet empirically grounded chronologies of the mathematical learning of the classroom community and of selected individual students. It is during this phase of the analysis that the conjectures developed during the first phase about the possible emergence of classroom mathematical practices are scrutinized from a relatively global viewpoint that looks across the entire teaching experiment. The results of the analysis are then cast in terms of the unit that was alluded to earlier, namely mathematical practices and students' diverse ways of participating in them.

As part of the process of testing and refining conjectures about mathematical practices, we differentiate between three types of mathematical norms: (a) a taken-as-shared purpose, (b) taken-as-shared ways of reasoning with tools and symbols, and (c) taken-as-shared forms of mathematical argumentation. The first of these aspects of a practice, the taken-as-shared purpose, is concerned with what the teacher and students are doing together mathematically. The second aspect is concerned both with the ways of using tools and symbols that are treated as legitimate and with the taken-as-shared meanings that these actions with tools and symbols come to have in the classroom. It is important to stress that these taken-as-shared meanings do not correspond to an overlap in the teacher's and students' individual interpretations (Voigt, 1996). Any attempt to delineate an overlap of this type involves the psychological perspective in that the focus is on the relation between individual interpretations. In contrast, inferences about taken-as-shared interpretations are made from the social perspective and concern the ascription of meanings that are constituted as legitimate and beyond question in the classroom. Such meanings are collective rather than individual accomplishments in that the status of legitimacy is established in the course of ongoing classroom interactions.

We have found Toulmin's (1969) scheme of justification to be a useful tool when we analyze the third aspect of mathematical practices, taken-as-shared forms of argumentation (see Fig. 9.3).

In the case of the sample analysis that we present, which deals with linear measurement, the data is the physical extension of an object and the conclusion is the

[10] In psychological terms, this learning can be described as the internalization and interiorization of the activity of creating two coordinated logs. This activity was socially situated in that we were participating in the practices of our research community as we conducted the analyses.

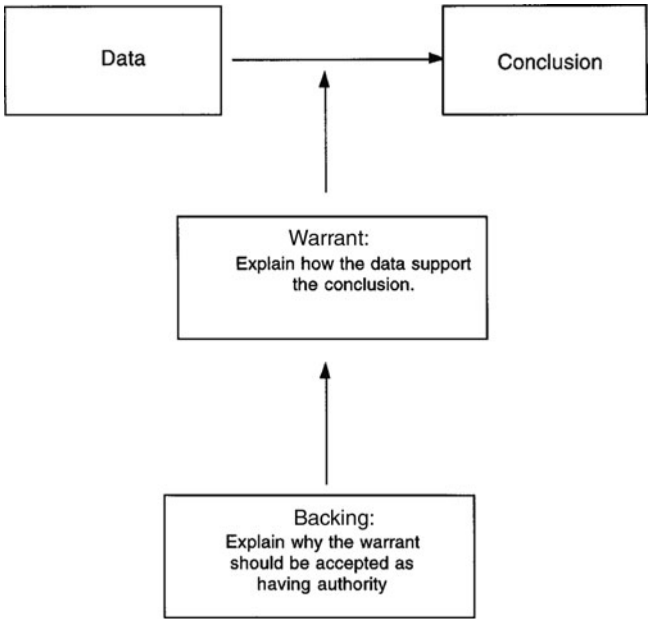

Fig. 9.3 Toulmin's (1969) scheme of argumentation

numerical measure of that physical extension. An appropriate warrant that serves to explain why the data support the conclusion may involve demonstrating how a measurement tool was used to produce the numerical result. This warrant can of course be questioned, thus obligating the student to give a backing that indicates why the warrant or measurement procedure should be accepted as having authority. A backing that may be treated as legitimate could involve explaining how the use of the measurement tool structures the physical extension of the object into quantities of length.

The illustration we have given is relatively general and constitutes a sociomathematical norm in that it applies to any measuring activity in which a serviceable tool is used. In contrast to this general scheme of argumentation for measuring, the specific types of arguments that we discriminated when conducting the sample analysis take account of the taken-as-shared ways in which particular tools were used to measure. In both the sample analysis and in other analyses that we have conducted (e.g., Bowers et al., 1999; Cobb, 1999, 2000a; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer et al., 2000), we have often found it fruitful to initially focus on changes in forms of arguments when attempting to delineate mathematical practices. Observations made by Yackel (1997) help clarify why this analytical strategy can be productive. We can best elaborate her insight by again referring to the measurement teaching experiment. As we document when we present the sample analysis, students were typically obliged to give backings for their measuring activity when they first used a new measurement tool. This need for a backing provides indication that the way in which the use of the tool structured the physical extent of objects into quantities was still being negotiated. However, it frequently happened that in later episodes it was no longer necessary for students to give backings to justify their use of the new tool. Shifts of this type provide one indication that a taken-as-shared meaning for the use of the tool may have become established (i.e., the way in which the use of the tool structures the physical extent of an object into qualities of length has become taken-as-shared). Further evidence would be provided by instances in which members of the classroom community object when they perceive that the conjectured normative meaning has been violated (cf. Much & Shweder, 1978). Consequently, we test our conjectures as we analyze subsequent episodes by looking for occasions when a student's activity appears to be at odds with a proposed normative meaning and examine whether the student's contribution is treated as illegitimate by the other members of the classroom community. In the event that the contribution is treated as legitimate, we of course have to revise our conjecture.

The methodological points that we have made thus far provide an initial orientation to our analytical approach. We should acknowledge that the trustworthiness and replicability of interpretivist approaches of this type have sometimes been questioned. We address these issues directly once we have teased out additional aspects of the approach by using the sample analysis as a paradigm case. For the present, we note that an analysis developed in this manner contributes to the development of what may be termed a local instructional theory by documenting both the actual learning trajectory of the classroom community and the means by which that learning was supported and organized (cf. Gravemeijer, 1998).

Measurement Practices

The sample analysis was taken from a 14-week teaching experiment that was conducted in a first-grade classroom with 16 students. Two instructional sequences were enacted during the experiment. The first sequence spanned 31 classroom sessions over a 7-week period and focused on linear measuring. The second sequence, which lasted the remaining 7 weeks of the experiment, built directly on the first sequence and focused on mental computation with numbers up to 100. Stephan (1998) analyzed the measurement sequence as it was enacted in the classroom and argued that it can be divided into five phases, each of which involves the emergence of a distinct mathematical practice. For the purposes of this article, we briefly summarize the process by which the first two practices emerged and then focus in some detail on the emergence of the third practice and on two students' reasoning as they participated in and contributed to it.

Background to the Teaching Experiment

The classroom teacher with whom we collaborated was a full member of the research team and assumed primary responsibility for instruction in all classroom sessions. However her relationship with other members of the research team was such that they were free to intervene by posing questions to students both during individual and small-group work and during whole-class discussions. The primary data sources generated during the teaching experiment were video recordings of all classroom sessions made with two cameras, copies of all the students' written work and other products, three sets of field notes made by observing researchers, video recordings of individual interviews conducted with all students at the beginning and end of the experiment, video recordings of interviews conducted with five target students midway through the measurement sequence, and audio recordings of the weekly meetings of the research team.

On the basis of the initial interviews, which focused on the students' numerical reasoning, we selected five students to track throughout the experiment. The selection criteria we used were that there was considerable diversity in the ways that they were reasoning in the social setting of the interview and that they were willing to try and explain their reasoning. During individual and small-group work, the two cameras followed two researchers as they documented how the five students were attempting to complete instructional activities. These researchers intervened to ask clarifying questions when they inferred that a viewer of the classroom video recordings would have difficulty in understanding how the students were approaching particular instructional activities.[11] At the end of individual or small-group work, the two researchers conferred with the teacher to plan the subsequent whole-class discussion. In addition, they recorded their ongoing conjectures about the target students' reasoning in reflective journals after each classroom session. When we describe the emergence of the third mathematical practice, we focus on the reasoning of two of these students, Nancy and Megan.

Our initial objective when we planned the measurement sequence was that the activity of measuring by iterating a tool along the physical extent of an object would come to signify the accumulation of distance (cf. Piaget, Inhelder, & Szeminska, 1960; Thompson & Thompson, 1996). In other words, if students were measuring by pacing heel to toe, we wanted it to become taken as shared that the number words said while pacing each signified the measure of the distance paced thus far rather than the single pace taken while saying a particular number word. In the case of the number word *five*, for example, we hoped that it would come to signify the measure of the distance from the beginning of the object's physical extension to the end of the fifth pace rather than merely the fifth pace. Beyond this, we also hoped that practices involving the use of tools that signified multiple units would become established (e.g., a paper strip that was five paces long). Participation in such a practice would involve interpreting the numerical results of measuring in a variety of different ways as the need arose (e.g., 25 paces, 5 strips, 2 distances of 10 paces and a distance of 5 paces, etc.). Described more generally, our instructional

[11] These interventions clearly changed the social interactions in which the target students engaged. We justify the interventions on the pragmatic grounds of generating information that we needed to make pedagogical decisions.

intent was therefore that a taken-as-shared spatial environment would become established in which distances are quantities of length whose numerical measures can be specified by actually measuring (Greeno, 1991). In such an environment, it would be self-evident that although distances are invariant, their numerical measures vary according to the size of the measurement unit used (i.e., if a larger measurement unit is used, then the resulting numerical measure will be smaller).

In sociocultural terms, this statement of our instructional intent specifies the historically developed mathematical practices into which we planned to induct the students. It is readily apparent from this description of our intent that we did not view learning to measure as solely a matter of coming to use tools in socially accepted ways. Instead, we were also vitally concerned with the taken-as-shared meanings that the use of the tools may come to have or, in other words, with how they may become structured as measurement practices evolve in the classroom. This emphasis on measuring as the structuring of space is clearly value laden and reflects what we consider is important in the doing of mathematics in school.[12]

The Classroom Microculture

The instructional activities used during the teaching experiment were embedded in ongoing narratives that extended over several classroom sessions. The teacher typically began each classroom session by engaging the students in a story in which the characters in the narrative either encountered a problem or needed to complete a task that involved measurement. The students worked either individually or in pairs to solve the problem or complete the task before returning to the whole-class setting to discuss their solutions. Stephan (1998) presented evidence that indicates that, even though social norms were renegotiated throughout the experiment, the general classroom participation structure was relatively stable. The social norms that she identified for whole-class discussions can be summarized as follows:

1. Students were obliged to explain and justify their reasoning.
2. Students were obliged to listen to and attempt to understand others' explanations.
3. Students were obliged to indicate nonunderstanding and, if possible, to ask the explainer clarifying questions.
4. Students were obliged to indicate when they considered solutions invalid, and to explain the reasons for their judgment.

Stephan also notes that the two overriding values that characterized whole-class discussions and that the teacher frequently discussed explicitly with her students were those of attempting to understand and of being actively involved at all times.

[12]The emphasis that we give to quantitative reasoning can be contrasted with the almost exclusive emphasis on numerical reasoning both in traditional classrooms and in many classrooms where instruction is compatible with recent reform recommendations (cf. Smith, 1997; Thompson, 1993, 1994; Thompson & Thompson, 1996).

A sociomathematical norm that is particularly important for the purposes of this article is that of what counted as an acceptable explanation. One of our objectives was that classroom discourse should be conceptual rather than calculational in nature (Cobb, 1998b; Lampert & Cobb, 1998b; Thompson, Philipp, Thompson, & Boyd, 1994). In calculational discourse, the focus of classroom conversations is on the calculational method or process for producing results. In terms of Toulmin's (1969) scheme of justification, calculational explanations of measuring activity involve giving a warrant by demonstrating how a measurement tool was used to produce a numerical result. However, calculational explanations would not involve a backing for the measurement procedure that describes how the enactment of the procedure structures the physical extent of objects into quantities of length.

To avoid misunderstanding, we should stress that students' measuring activity can be meaningful (i.e., it can involve structuring space into quantities of length) even if they give calculational explanations. The key point is that the ways in which students structure space as they use a measurement tool do not become topics of conversation when the classroom discourse is calculational. In contrast, the hallmark of conceptual discourse about measuring is that students are obliged to give a backing by explaining how they structured space as they measured. Elsewhere, we have argued that classroom discussions in which the discourse is conceptual rather than calculational can be particularly productive settings for mathematical learning. Stephan's analysis documents that the teacher with whom we collaborated was generally successful in ensuring that, during whole discussions, students were obliged to give a backing by explaining how they structured the space measured.[13] The establishment of this sociomathematical norm had implications for our analysis in that it facilitated our task of differentiating between cases in which their measuring activity involved the structuring of space. This distinction would have been much harder to make had the classroom discourse been calculational rather than conceptual in nature.

The Emergence of the First Two Mathematical Practices

Our purpose in briefly outlining the emergence of the first two practices is to both give some indication of the scope of an analysis of classroom mathematical practices and to provide a backdrop against which to give a more detailed account of the third practice and of two students' reasoning as they participated in it. We therefore reserve our methodological comments for the discussion of the third practice.

The first of five classroom mathematical practices emerged during Sessions 1 through 3. The instructional activities were all posed within a narrative about a king who measured items in his kingdom by pacing heel to toe. During Session 1 of the experiment, we observed two distinct ways of counting paces. The contrast between

[13] This is not to say that all explanations involved a backing. It was not necessary for a student to give a backing when other students indicated that they understood a particular method of measuring structured space.

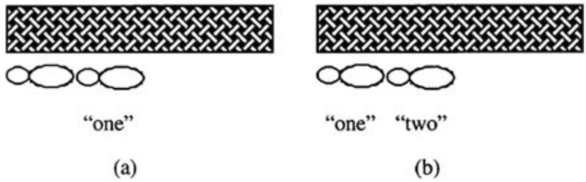

Fig. 9.4 Two methods of counting as students paced the length of a rug

these methods became the focus of the concluding whole-class discussion when the students paced along the edge of a rug at the front of the classroom. In one method, some students placed one foot such that the heel was aligned with the beginning of the rug but did not count "one" until they placed the other foot heel to toe with the first (see Fig. 9.4a).

In the other method, students counted the placement of one foot aligned with the beginning of the rug as "one" and then continued "two" as they placed their other foot (see Fig. 9.4b). During the whole-class discussion, the teacher made a record of a student's paces along the edge of the rug by placing pieces of masking tape on the floor. This intervention proved crucial in that it supported a conceptual conversation in which arguments focused not merely on the two methods of counting paces but on how these methods structured the physical extension of the rug. For example, several students reasoned with the record to argue that those who did not count the first pace had "missed" a piece of the rug. In doing so, they gave backings to explain why the method of counting the first pace should be accepted as having authority. These arguments proved to be decisive in that all the students counted their first pace in the remaining two sessions of this first phase of the teaching experiment. Furthermore, forms of argumentation in which backings focused on amounts of space (or pieces of the rug) became taken as shared. Issues that emerged as topics of conversation in the final two sessions during which the first practice emerged included the differences in numerical results when students with different size feet measured the physical extension of the same object and the adjustments that should be made when the final pace extended beyond the end of the object being measured. In the course of these discussions, the numerical results obtained when measuring were spoken about in public discourse as signifying the last pace (or the amount of space it defined) rather than the space covered by the entire sequence of paces. It therefore appeared to be taken as shared that measuring by pacing structured the physical extension of an object into a chain or sequence of individual paces, each of which defined an amount of space. Furthermore, the taken-as-shared purpose of measuring appeared to be to find out how many paces were needed to traverse the physical extension of the object. These assertions about the first mathematical practice that emerged, *measuring by pacing*, were significant given our instructional agenda in that they indicated that measuring did not involve the accumulation of distance.

The second mathematical practice emerged as a reorganization of the practice of measuring by pacing during the next seven classroom sessions. The teacher continued the narrative about the king who used his foot to measure by explaining that the king was receiving too many requests to measure things in his kingdom and could not be everywhere at once. The problem the students addressed was therefore that of devising ways in which the king's foot could be used as the unit of measure even though the king would not do the measuring himself. The solution that emerged during several classroom sessions involved tracing five feet heel to toe on a strip of paper that the students called a *footstrip*. The students initially constructed footstrips by tracing around their feet and were given standard footstrips once the issue of obtaining different numerical results when using different sized footstrips had been resolved.

A taken-as-shared method for using the footstrip emerged with little need for discussion in Session 1, in which the students used it to measure objects in the classroom. This method involved iterating the footstrip along the physical extension of objects while counting "5, 10," and so forth. The ease with which this normative method emerged led us to conjecture that it was taken as shared that each placement of the footstrip was equivalent to taking five paces. This conjecture proved viable when we analyzed subsequent sessions. For example, explanations in which students spoke of taking one "big step" of five and of measuring now being faster (i.e., it was faster to count five paces at a time than to count single paces by ones) were treated as legitimate.

It was not until 2 days after the students first used the footstrip as a measuring tool that differences in their personal meanings became an explicit topic of conversation. This occurred during a whole-class discussion in Session 8 when two students demonstrated how they had measured a cabinet at the side of the classroom that abutted a wall (see Fig. 9.5).

The students iterated the footstrip three times while counting "5, 10, 15" but found that there was not enough room to place the footstrip for a fourth time. They attempted to resolve the dilemma by placing the footstrip so that one end touched the wall and then counting the individual paces that filled the gap beyond the third placement of the footstrip. However, they had difficulty in completing their solution

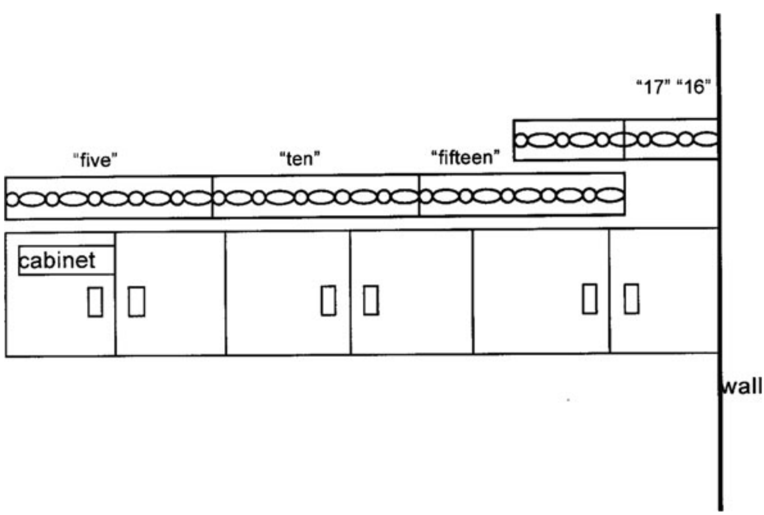

Fig. 9.5 The method of measuring the cabinet using a footstrip

and a researcher intervened by suggesting that they move the footstrip so that it extended up the wall. A number of students argued that this method would not work, contending that the space being measured would then extend up the wall to the end of the footstrip. It appeared that, for these students, the placement of the footstrip defined the space being measured. Their arguments appeared to reflect their prior participation in the first mathematical practice in that it had been taken as shared that each physical act of taking a pace defined an amount of space and that, consequently, the final pace could not extend beyond the end of the object being measured.

As the exchange continued, one student argued against the rejection of the researcher's proposal by explaining that he imagined cutting the footstrip at the point where the cabinet met the wall. Several students then reversed their position and, the following day, none of the students appeared to encounter any difficulty when part of the footstrip extended beyond the end of an object being measured. Furthermore, this approach of mentally cutting the footstrip at any point as the need arose seemed to be beyond justification. It now appeared to be taken as shared that the structured space created while measuring was no longer tied to the physical acts of placing the footstrip (or taking a pace) but was instead treated as a property of the object being measured. This was the most crucial advance made by the classroom community as the second mathematical practice of *measuring by iterating a footstrip* emerged.

A final point that we need to clarify concerns the structure of the space that was now attributed to objects. Two competing interpretations became the explicit focus of discussion on several occasions, but the conflict between them was not resolved during the time that the students used the footstrip. Several students' arguments indicated that, for them, measuring with the footstrip structured the physical extension of an object into a sequence or chain of individual paces much as had measuring by pacing. In contrast, for other students the physical extension of the object being measured constituted what they called a "whole space" that was partitioned into paces. Thus, in this latter interpretation, "the whole 12" meant the distance from the beginning of the 1st pace to the end of the 12th pace whereas, in the former interpretation, it meant the whole 12th pace. Furthermore, in the latter interpretation, a measure of $12\frac{1}{2}$ meant 12 paces and an additional half of a pace whereas, in the former, it meant half of the 12th pace. The contrast between these two interpretations constituted a mathematically significant issue given our instructional intent that measuring would come to signify the accumulation of distance. However, as a resolution was yet to be achieved, the most that we can claim is that the two ways of structuring space, both of which were treated as properties of the object being measured, emerged as an explicit focus of public discussion when students participated in the practice of measuring by iterating the footstrip.

The Emergence of the Third Mathematical Practice

The two students on whose reasoning we will focus as we discuss the third mathematical practice had participated in the second practice in different ways. Nancy interpreted measuring with the footstrip in terms of the accumulation of distance whereas, for Megan, the result of measuring was a sequence of individual paces. The third practice emerged over a period of 11 classroom sessions and commenced on the 11th day of the reaching experiment. In these sessions, the teacher developed a narrative about a village of Smurfs (dwarf-like creatures from the television cartoon *The Smurfs*). In this narrative, the Smurfs encountered various situations in which they needed to measure objects. According to the narrative, the Smurfs measured by taking empty food cans and placing them end to end to determine the length or height of an object. The students solved the problems the Smurfs encountered by using Unifix cubes as substitutes for food cans. With little discussion, all the students including Megan and Nancy clipped cubes together to make a rod that spanned the physical extension of the object being measured and then counted the cubes by ones.

We should clarify that the students measured by making rods of Unifix cubes only during Sessions 11 and 12. In Session 13, the teacher explained that the Smurfs found it problematic to carry a large number of food cans with them when they wanted to measure something and needed to develop a new method. The solution that emerged with the teacher's guidance during a whole-class discussion was to use a bar of 10 cubes that the students called a Smurf bar as a measurement tool. The taken-as-shared method that became established with little need for discussion involved iterating a Smurf bar along the physical extension of an object while counting by tens. This development would appear to be made possible by the students' prior participation in the second mathematical practice in which they iterated a footstrip while counting by fives.

It was against this background that an issue that relates directly to our instructional focus on measuring as the accumulation of distance became an explicit topic of conversation in subsequent sessions. This is illustrated by a series of exchanges that occurred between Megan, Nancy, and a researcher as they worked together to measure objects in the classroom on the 14th day of the experiment and the first on which they used a Smurf bar as a measuring tool. Nancy first measured the length of a table while Megan watched her. A researcher intervened after Nancy had iterated the Smurf bar twice while counting "10, 20" to clarify what "20" signified for her. Nancy responded by gesturing to the space covered by the first two iterations while explaining, "Twenty is how many little cubes we've done so far."[14] She then continued iterating the Smurf bar until she reached the end of the table and obtained a numerical result of exactly 80. This exchange is consistent with all other observations of Nancy's measuring activity during this period of the experiment

[14]Nancy's gesturing as she explained, "Twenty is how many little cubes we've done so far" illustrates Roth's (2001, this issue) arguments concerning the integration of word and gesture in establishing the entities that are spoken about. Several other examples of this phenomenon are apparent in our account of the emergence of the first three mathematical practices. In this connection, it is worth noting that we attended to the teacher's gesturing as part of our instructional design. In particular, we conjectured that it would be important for the teacher to indicate by gesture the entire distance that had been measured when iterating a Smurf bar rather than the successive placements of the bar.

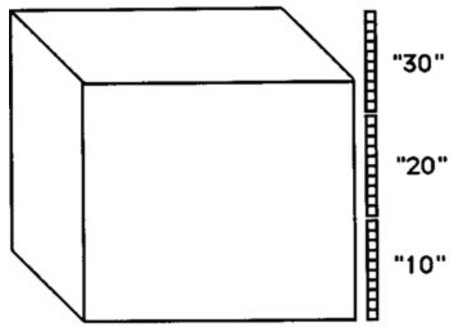

Fig. 9.6 Megan's method for measuring the animal tank

and indicates both that measuring involved the accumulation of distance for her and that iterating the Smurf bar structured the physical extension of an object into a single spatial quantity that was itself partitioned into spaces each of which was 10 cubes long.

Megan next used a Smurf bar for the first time and began to measure the height of an animal tank at the side of the classroom. She iterated the bar twice while counting "10, 20" and then iterated it for a third time while saying "30" even though the bar extended beyond the top of the tank (see Fig. 9.6).

She then held the Smurf bar in this position and counted the individual cubes in the bar by ones, starting at 31 until she reached the top of the tank. We conjectured on the basis of this observation that interacting with the Smurf bars for a third time while saying "30" indicated to her that the cubes that composed the bar were in the 30 s decade. This conjecture implied that, in contrast to Nancy, measuring with the Smurf bar was not a curtailment of measuring with individual cubes and counting them by ones for Megan. Instead, the relation between the two methods appeared to be based primarily on a number word relation for her. This in turn led us to infer that when Megan iterated the Smurf bar, it structured the height of the tank into a chain of spaces, each of which was composed of ten cubes long, rather than into a single spatial quantity.

Nancy, for her part, challenged Megan's reasoning by iterating the Smurf bar twice while counting "10, 20" before moving it a third time and counting individual cubes "21, 22, 23." Megan responded by measuring as she had before and again obtained 33 as a numerical result, indicating that she did not understand Nancy's calculational explanation. This led Nancy to elaborate the backing for her argument by counting the individual cubes in each iteration of the Smurf bar, "1, 2, . . .10; 11, 12,. . .20; 21, 22, 23." She then continued, "If the top [of the third iteration] ended here (points to the top of the tank), then it would be 30. But it [the third iteration] ended way up there (points above the tank) so it would not be 30." In making this final argument, Nancy may have been attempting to explain how iterating the Smurf bar structured the height of the tank for her (i.e., "it would not be 30" may, for her, have signified that 30 individual cubes would not fit into the space between the bottom and top of the tank). However, because she spoke in calculational terms, the differences in the meaning that measuring with the Smurf bar had for the two girls did not become a topic of conversation. Instead, Megan continued to measure as she had before for the remainder of the session even though she accepted Nancy's result of 23 in this instance. In the subsequent whole-class discussion, the teacher asked Nancy and Megan to demonstrate how they would measure the length of the white board at the front of the classroom. The episode serves to further corroborate the assertion that measuring with the Smurf bar involved the accumulation of distance for Nancy. In addition, subsequent events indicate that this exchange was particularly significant for Megan. (*T* indicates the teacher, and *R* indicates a researcher.)

Both: 10 [the teacher marks the end of the first iteration with the numeral 10, as shown in Fig. 9.7], 20 [the teacher marks the end of the second iteration with the numeral 20].

R: Where's the 20? What does 20 mean?
Megan: 20 means 20 food cans.
R: That means 20 food cans. How much space would that be? Can somebody show me how much space 20 cans would take up there? Mitch?
Mitch: About that long [indicates the space between 10 and 20].
Nancy: No [indicates the space from the edge of the white board to 20]. This is, because he [Mitch] did 10, not 20. [Mitch indicates he has changed his mind.]

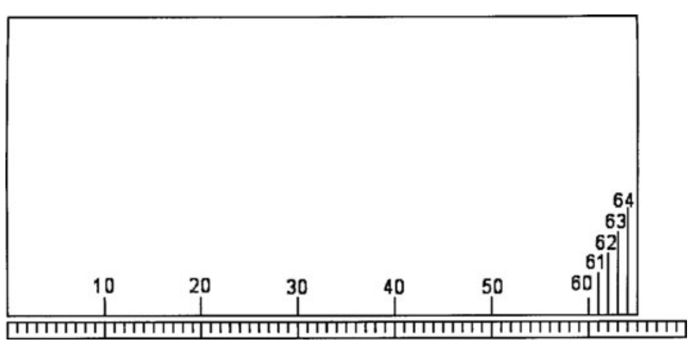

Fig. 9.7 The teacher's record of Nancy and Megan's measuring activity

R: Oh, so it's the whole 20.

Megan: [Continues to measure with the Smurf bar as the teacher marks each iteration.] 21,22, . . . 30; 40; 50; 60; 61, 62, 63, 64.

In this exchange, the questions the researcher posed together with the record the teacher made of the girls' measuring activity served to initiate a conceptual discussion that focused on how Megan and Nancy's method of measuring structured the physical extension of the white board. This contrasts with the calculational conversations that Nancy and Megan had during small-group work. Their small-group interactions were typical in this respect in that the students rarely gave backings for their methods of measuring when they first used the Smurf bar unless an adult gave support. The pattern that emerged across the experiment was in fact one in which the teacher and researcher had to intervene at the beginning of the emergence of each new practice to initiate conceptual discussions.[15] The sociomathematical norm of what counted as an acceptable explanation was relatively stable during whole-class discussions only because of these interventions. However, as each mathematical practice emerged, students assumed increasing responsibility for answering their peers' questions that required a conceptual response.

Turning now to consider the substance of the episode, it is apparent from the exchange between Nancy and Mitch that measuring as the accumulation of distance was not taken as shared. Our point in making this claim is not that we, as observers, can identify differences in Mitch and Nancy's individual interpretations. Instead, we have adopted a social perspective from which we note that the issue of how measuring with the Smurf bar structured space became an explicit topic of conversation. In this regard, we remind the reader that our concern when identifying taken-as-shared ways of reasoning is not to delineate an overlap in individual students' interpretations. Instead, we are concerned with normative ways of reasoning, with ways of reasoning that are beyond justification. As a consequence, we focus on the status that individual students' contributions come to have in public classroom discourse. In the case at hand, Nancy's explanation was constituted as legitimate, whereas Mitch's proposal was treated as illegitimate. In responding to Mitch, Nancy therefore contributed to the emergence of the third mathematical practice, that of *measuring by iterating the Smurf bar.* Similarly, Mitch contributed by making his proposal and by indicating that he had changed his mind.

A final observation about the episode concerns the manner in which Megan completed the task of measuring the white board. In contrast to her activity during the prior small-group work with Nancy, she continued to count individual cubes "61, 62, 63, 64" when she iterated the Smurf bar for a final time and it extended beyond the edge of the white board. This is particularly noteworthy given that she and Nancy were the first students to demonstrate during the whole-class discussion how they had measured. When we first analyzed this episode, we conjectured that she may

[15] Bowers and Nickerson (1998) reported a similar pattern in their analysis of a university mathematics course that focused on quantitative reasoning.

have reorganized her reasoning with the Smurf bar as she listened to the exchange between the researcher, Mitch, and Nancy. However, this conjecture was extremely speculative and it was not until we continued to work through the data that we were able to substantiate it and to clarify the nature of the reorganization she made. Switching to the social perspective, we can also note that, in measuring as she did, Megan, like Nancy, contributed to the emergence of the third mathematical practice. The fact that we can make this observation before we can substantiate our inferences about her reasoning serves to emphasize that we are not concerned with an overlap in individual interpretations when we delineate taken-as-shared meanings.

Events that occurred the following day during the Session 16 proved particularly helpful as we tested and refined our conjectures about Megan's reasoning. The instructional activities involved helping the Smurfs cut lumber a certain length so that they could use the boards to build a house. Students were given long pieces of adding machine tape as the lumber and asked to produce boards of given lengths. During small-group work, Megan and Nancy recorded each iteration of the Smurf bar on the adding machine tape by writing numerals much as the teacher had during the whole-class discussion the previous day. Furthermore, measuring seemed to involve the accumulation of distance for both students. During the subsequent whole-class discussion, the issue of how measuring with a Smurf bar structures space again emerged as a topic of conversation. Midway through this conversation, a student alternated between two methods of measuring and was unable to resolve the conflict in the numerical results she produced. One method was consistent with measurement as the accumulation of distance, whereas the other involved counting within a decade as Megan had the previous day when she measured the animal tank. At this juncture, Megan volunteered that she could help but said that she would need the teacher's assistance. She then measured with a Smurf bar while the teacher marked each iteration at her request. A researcher interrupted her after she had iterated the bar twice and asked her how many cans would fit into the spaces the teacher had marked. Megan placed one hand at the beginning of the adding machine tape and the other at the end of the second iteration and replied that 20 cans would fit into the whole space and that 10 cans would fit into each of the small spaces. She then completed her solution by iterating the Smurf bar for a third time and counting three cubes to find where the adding machine tape should be cut to produce a board 23 cans long. In doing so, she again contributed to the emergence of the third mathematical practice.

Megan's activity during this session is consistent with the conjecture that she had reorganized her reasoning and that measuring with a Smurf barnow involved the accumulation of distance for her. However, these observations also led us to modify our conjecture by speculating that measuring with a Smurf bar only had this meaning for her when she made or had access to symbolic records of her activity. As it transpired, this inference proved to be viable when we continued to analyze data during the remainder of this period of the experiment. For example, in an individual interview conducted after Session 16, Megan counted within a decade as she had when measuring the animal tank and became visibly confused. However, she resolved the difficulty by writing numerals after each iteration of the Smurf bar on her own initiative. Significantly, she appeared to be aware of the role that making records played in her reasoning both in this instance and on other occasions.

With regard to the learning of the classroom community, the last occasion on which an explanation consistent with measuring as the accumulation of distance was publicly questioned and thus had to be justified was during Session 15. In addition, explanations that were at odds with this way of structuring space were consistently challenged throughout the remainder of the experiment. These two observations together provide the basis for our assertion that a new mathematical practice had emerged.

In reflecting on the sequence of events we have described, an issue of immediate interest given our agenda is that of teasing out the conditions that may have supported Megan's reorganization of her reasoning with the Smurf bar. Three points are worth mentioning in this regard. The first concerns Megan's (and the other students') prior participation in the first two mathematical practices, measuring by pacing and measuring with the footstrip. We contend that it is only against the background of this prior participation that we can understand how iterating the Smurf bar emerged as a taken-as-shared method almost immediately when the tool was introduced. Megan's learning was therefore situated historically with respect to an emerging sequence of mathematical practices.

Second, we conjecture that the researcher's intervention during the whole-class discussion to support a conceptual conversation was crucial. As a consequence of this intervention, the issue of Megan and Nancy's method of measuring structured space became a focus of discussion. We note that, in contrast, Nancy's calculational explanations during the prior small-group work did not lead Megan to modify her reasoning. It would of course be premature to make an assertion about the importance of conceptual discourse on the basis of this single case of learning. However, our conjecture about the supportive role of this type of classroom discourse did hold up when we reviewed our chronological analysis of the entire data set. In Megan's case, it is therefore reasonable to suggest that her learning was situated with respect to the quality of the immediate conversation.

Third, the records that the teacher made of Megan and Nancy's measuring activity appeared to play a critical role. From the social perspective, these records supported the conceptual conversation about their measuring activity.[16] From the psychological perspective, the records made it possible for Megan to reflect on and interpret her ongoing activity in terms of the accumulation of distance when she listened to the exchange between Nancy, Mitch, and the researcher. Her learning was therefore situated with respect to the inscriptions that were available for her to reason with.

These claims about the various ways in which Megan's learning was situated indicate why we find it useful to view her act of reorganizing her reasoning (psychological perspective) as simultaneously an act of participation in the emergence of

[16] See Thompson et al. (1994) for a more extensive discussion of the role of inscriptions in supporting conceptual conversations.

a new mathematical practice (social perspective). We can further clarify this point by noting that the teacher and researcher both assumed that Megan had already developed aspects of the type of reasoning that they wanted to engender.[17] For example, the possibility that Megan may have been merely moving the Smurf bar end to end while reciting the number word sequence "10, 20," and so forth was not considered by the researcher. Instead, his intervention was premised on the assumption that she and Nancy were structuring space. As a consequence of this intervention, Megan became a participant in a conversation in which the taken-as-shared purpose for measuring may have differed from her initial intentions. What Megan appeared to learn in the course of this exchange was how to create records of her measuring activity so that she could reason in a way compatible with explanations that were treated as legitimate in public classroom discourse. Her learning was therefore supported by her participation in the emergence of the very practice to which she contributed by learning. It is in this sense that we elevate neither individual students' learning nor the emergence of communal classroom practices above the other but instead see them as reflexively related.

In closing this discussion of the emergence of the third mathematical practice, it is important to emphasize that the analytic approach we have illustrated is tailored to our purposes as mathematics educators. For example, each of the three ways in which Megan's learning was situated has implications for the assessment and revision of our instructional design. First, our observation of the students' prior participation in the first two mathematical practices indicates that the modifications we made to the first part of the instructional sequence during the teaching experiment are generally viable. To complete this assessment, we would of course have to consider the subsequent mathematical practices that emerged during the remainder of the experiment as reorganizations of the third practice. As it transpired, this assessment is favorable in that a fourth practice that involved using paper strips 100 cans long as a measurement tool emerged with the teacher's guidance. In the fifth practice, this tool was used not as a measurement device but as a means of reasoning about measures (e.g., the students found the difference between the heights of two items that they were told measured 49 cans and 73 cans). This in turn provided a basis for the emergence of the subsequent practices that involved mental computation with numbers up to 100 (cf. McClain, Cobb, & Gravemeijer, 2000).

Second, the assertion we made about the importance of conceptual discourse has implications for both the teacher's role in teaching mathematics for understanding and for teacher education (cf. Bowers & Nickerson, 1998; Thompson & Thompson, 1996). For example, it indicates the importance of teachers themselves engaging in conceptual mathematical conversations and reflecting on such conversations as situations for mathematical learning. Third, our claims about the supportive role played by symbolic records of measuring activity have more specific implications for the

[17] The important role of interactions in which an adult attributes greater competence to a child than the child is displaying features prominently in the Vygotskian literature (e.g., Newman, Griffin, & Cole, 1989; Stone, 1993).

teacher's role when teaching children to measure with understanding. In particular, it indicates the importance of teachers making such records and intervening to ask students about the way in which space is being structured as they measure.

We return to this discussion of the potential contributions of the analytic approach we have illustrated when we consider its usefulness in more general terms. First, however, we tease out additional aspects and discuss its trustworthiness and replicability.

Methodological Reflections

A methodological issue about which we can be more explicit in light of the sample analysis is that of what constitutes an episode during the first phase of an analysis when we work through the data chronologically. The determining characteristic of an episode is that a single mathematical theme is the focus of mathematical activity and discourse. As an example, we viewed the exchange between Megan and Nancy as they measured the height of an animal tank in Session 14 as a single episode. In this case, the theme was that of how to count individual cubes when the final iteration of the Smurf bar extends beyond the object being measured. Had this theme continued when they moved on to measure another object, we would have treated their subsequent activity as part of this episode. As a second example, recall the whole-class exchange that occurred during the emergence of the second mathematical practice when a pair of students used a footstrip to measure the length of a cabinet that abutted a wall. In this instance, the theme that we identified when delineating a single episode was that of how to resolve the difficulty that arose when there was not enough room to place the footstrip for a final time and thus complete the measurement of the cabinet. As it so happened, the focus of discussion when a previous pair of students had demonstrated how they measured the cabinet was on the number of paces corresponding to a certain number of iterations of the footstrip. Thus, in contrast to the exchange that occurred as Megan and Nancy measured the animal tank, we viewed the discussion that occurred while measuring the physical extension of a single object (the cabinet) as consisting of two distinct episodes.

As the two examples we have given illustrate, an episode does not necessarily correspond to either a single measuring solution or to a single instructional activity. Instead, in keeping with our interpretivist approach, episodes are determined by the mathematical themes that we identify in the students' and teacher's activity and discourse. As a related issue, it is worth clarifying the characteristics of episodes that prove to be particularly critical when we move to the second phase of the analysis and meta-analyze the results of the initial episode-by-episode analysis. In general, the critical episodes are those that prove pivotal in either refuting a conjecture or substantiating an assertion. We should add that these episodes might initially appear to be of little significance when viewed in isolation. Their critical role only becomes apparent when they are located within the chain of conjectures, refutations and revisions that result from the first phase of the analysis. The first example we have given is a critical episode when viewed in light of Megan's measuring activity during the remainder of the teaching experiment in that it documents that measuring with the Smurf bar was yet to involve the accumulation of distance for her. This episode is therefore central to our claim that she reorganized her reasoning as she participated in the whole-class discussion that followed this episode. The second example in which a pair of students measured with a footstrip proved critical when we developed an empirically grounded chronology of the classroom community during the second phase of the analysis. We juxtaposed this episode with both prior and subsequent episodes to claim that it was during this exchange that it became taken as shared that the structured space created while measuring was a property of the object being measured. These two examples are representative in that all of the episodes we described when presenting the sample analysis proved critical and were in fact included for this reason.

A second important methodological issue that we have yet to address concerns the criteria we use when we delineate mathematical practices. The sample analysis is helpful in this regard in that it clarifies that the distinctions we draw between different practices are oriented by our instructional agenda. Recall that we wanted measuring to come to involve the accumulation of distance and that we cast our instructional intent in terms of the emergence of a taken-as-shared spatial environment in which distances are quantities whose numerical measures can be specified by actually measuring. Given this focus, the distinctions we drew when we delineated the three mathematical practices in the sample analysis go beyond changes in observed measuring strategies by focusing on the emergence of taken-as-shared ways of structuring space. The hallmark of the first practice was that it was taken as shared that measuring by pacing structured the physical extension of an object into a chain of single paces. As the second practice of measuring with the footstrip emerged, it became taken as shared that the structured space created by measuring was no longer tied to the physical measuring activity but was instead treated as a property of the object being measured. In the case of the third practice, it became taken as shared that measuring with a Smurf bar involved the accumulation of distance.

We can highlight the contrast between this analytic approach and one that focuses on observed solution methods by considering the emergence of the third mathematical practice. Recall that, at the beginning of the third phase of the teaching experiment, the students measured by making rods of Unifix cubes that spanned the physical extension of the object being measured and then counted them by ones. Clearly, this is a distinct measuring procedure when compared with both measuring with a footstrip and with a Smurf bar. However, the emergence of this taken-as-shared method did not appear to involve an advance in the taken-as-shared way of structuring space when compared with the second practice. As a consequence, we did not treat it as a distinct practice but instead viewed it as a step in the emergence of the third practice, measuring with a Smurf bar. We would note that, in the last analysis, we would justify this approach of focusing on the evolution of taken-as-shared mathematical meanings by referring to our values as mathematics educators. Given our arguments about what is important in the knowing and doing of mathematics, an analytic approach that behaviorizes communal mathematical practices by equating them with taken-as-shared procedures is ill suited to our purpose.

A third methodological issue that the sample analysis enables us to further clarify is that of what is involved in taking a social perspective. Clearly, when we view classroom video recordings, we do not see the classroom community as the discrete, concrete entity in the same way that we see the teacher and students as distinct physical beings. As a consequence, we cannot observe mathematical practices directly any more than we can directly observe the meanings that individual students' measuring activity had for them. Instead, we develop and test conjectures about both communal practices (social perspective) and individual students' reasoning (psychological prospective) as we analyze what the teacher and individual students say and do in the classroom. The distinction between the two interpretative perspectives resides in what may be termed the grain size with reference to which we characterize what they are doing. In the case of the psychological perspective, we view the teacher and students as a group of individuals who engage in acts of reasoning as they interpret and respond to each other's actions. In contrast, when we take the social perspective, we view the teacher and students as members of a local community who jointly establish communal norms and practices. As an example, consider again Nancy's explanation to Mitch in the whole-class discussion during Session 14 of the teaching experiment. Nancy argued that 20 cans would take up the space from the beginning of the white board to the end of the second iteration of the Smurf bar. In taking a psychological perspective, we were concerned with the quality of Nancy's reasoning and saw her activity in the exchange with Mitch as substantiating our conjecture that measuring with a Smurf bar involved the accumulation of distance for her. In contrast, when we interpreted this same episode from the social perspective, we were concerned with the status that her explanation came to have and argued that it was in fact constituted as a legitimate explanation. The important point to note is that its constitution as a legitimate explanation was not an individual act but was instead a collective accomplishment. It was from this perspective that we viewed Nancy as contributing to the emergence of the third mathematical practice. We established our basic unit of analysis, that of classroom mathematical practices and students' diverse ways of participating in them, when we coordinated interpretations of what the teacher and students were doing from the two perspectives. This diversity in the students' reasoning was illustrated in the sample analysis by the contrast between Megan and Nancy's measuring activity. As we saw, measuring with a Smurf bar involved the accumulation of distance for Nancy from the outset. However, measuring with a Smurf bar had this meaning for Megan only when she generated or had access to records of her measuring activity.

A fourth methodological issue that we can clarify by referring to the sample analysis concerns our treatment of the role of tools in individual and collective mathematical learning. The sample analysis addresses this issue directly in that the students reasoned with tools as they participated in all three of the mathematical practices that we have described. At first glance, it may in fact appear that the use of a new tool leads to the emergence of a new mathematical practice. This, however, is not always the case. The fourth and fifth mathematical practices that we described briefly both involved the use of a paper strip that was 100 cans long. In one practice, the strip was used as a measuring tool whereas, in the other, it was used as a tool for reasoning about the measures of objects that the students had not actually measured. As a further example, suppose that it had become taken as shared that measuring with the footstrip involved the accumulation of distance. In such a scenario, use of the Smurf bar may not have involved a significant change in how measuring structured the physical extension of objects into quantities of space. We would therefore have treated use of the Smurf bar as a part of the second practice, rather than as a distinct practice. A final example draws on the observations we have made about the use of a rod of Unifix cubes as a measurement tool. As we noted, the use of this tool did not appear to involve a restructuring of space, and therefore did not, in our view, constitute a distinct practice.

Our purpose in presenting these examples is not to dismiss the important role that students' use of tools plays in their mathematical learning, but instead to rule out a simplistic view of the relation between tool use and the emergence of mathematical practices. The fact that the three practices we have described involved the use of a different tool is attributable in part to the viability of our instructional design. These tools were specifically designed so that their use would involve significant mathematical learning. We can best clarify our position on tool use by focusing on a notion that has common currency in distributed accounts of intelligence, that of affordances. We have no doubt that it would in fact be possible to develop an analysis of our data in which great emphasis is placed on the affordances of the various measuring tools that the students used. To give but one example, we saw that it became taken as shared when the students used the footstrip that the structured space created by measuring was a property of the object being measured. A whole-class discussion that focused on what to do when the last iteration of a footstrip extended beyond the end of a cabinet appeared to be particularly crucial in this regard. It is noteworthy that a similar issue, that of what to do when the measuring tool extends beyond the end of the object being measured, had emerged earlier when the students measured by pacing. However, in that case, some students resolved the difficulty to their satisfaction by turning their foot sideways so that the last pace did not extend beyond the end of the object. Such a solution was not possible when the students used the footstrip because of its physical structure. Instead, the idea of mentally cutting the footstrip became an explicit topic of conversation. It may therefore be argued that the footstrip afforded ways of reasoning in which structured space is treated as a property of the object being measured.

A central feature of this explanation is that it treats the affordances of the footstrip and of feet as intrinsic characteristics of the tools. Such an account embodies what Meira (1998), following Roschelle (1990), called the epistemic fidelity view in that it emphasizes the perceived relation between the material features of tools and particular mathematical ideas. In focusing on the physical characteristics of tools, this approach ignores both students' prior participation in particular mathematical practices and the taken-as-shared purposes for using the tool. For example, we saw that it was as a consequence of the students' participation in the first mathematical practice (measuring by pacing) that iterating the footstrip involved structuring the physical extension of the objects into a chain of paces from the outset. Furthermore, the taken-as-shared purpose for using the footstrip was to specify how many paces it would take to transverse the physical extension of the object being measured. Against this background, a number of potential ways of resolving the difficulty of what to do when the last iteration extended beyond the end of the object simply failed to materialize. For example, none of the students suggested aligning the footstrip with the end of the cabinet that was being measured and then counting all five paces, presumably because they understood that some of these paces would overlap with those of the second last iteration. It was also against this background that some of the students immediately saw value in the proposal of mentally cutting the footstrip at the point where the cabinet met the wall. They presumably understood that the paces on the footstrip up to the cut would fill the gap beyond the last iteration and complete the process of traversing the physical extension of the cabinet. Their contributions to the emergence of the second mathematical practice, like those of the student who made the proposal, reflect both the taken-as-shared purposes that have been established for using the footstrip and their prior participation in the first practice.

We should emphasize that our rejection of the epistemic fidelity view does not imply a complete abandonment of the notion of affordances. It is self-evident, for example, that the central mathematical idea of measurement as the accumulation of distance would not have emerged but for the students' use of particular tools. Although the approach of equating affordances with particular material features of tools is problematic, they can reasonably be characterized as both individual and collective accomplishments. There is every reason to doubt that the footstrip would have afforded the emergence of structured space as a property of the object being measured had the students used it from the very beginning of the teaching experiment. Instead, in more general terms, the affordances that tools come to have are constituted as they are used for particular purposes against the background of participation in previously established practices. This view is highly compatible with Meira's (1998) contention that "instructional devices should be thought of in connection to some task, system of activities, and cultural context in which they make sense" (p. 140). For our purposes as instructional designers, this implies that we should not focus on the material features of tools per se, but on how students reason with them as they participate in an evolving sequence of mathematical practices.

As a final observation about our treatment of tool use, the reader may have noticed that we did not follow the standard sociocultural approach when speaking of students appropriating ways of reasoning with tools. Megan's learning in the course of the exchange between Nancy, Mitch, and a researcher during Session 14 would seem open to this kind of explanation. One may argue, for example, that Megan appropriated a way of reasoning with the Smurf bar from a publicly accessible conversation. Our reason for eschewing this relatively straightforward explanation is not that we believe that the general idea of appropriation is misguided. Instead, our rationale relates to our purposes of design research. At a gross level, one can plausibly speak of Megan appropriating a way of reasoning from an ongoing conversation. In such an account, the process of appropriation is, in effect, a black box.

We find it essential for our purposes to look inside the box by analyzing how teachers and students mutually adjust their interpretations and actions in the course of ongoing interactions. We took this psychological perspective when we analyzed the exchange involving Megan by focusing on what it may have meant to her. This in turn enabled us to identify aspects of the ongoing events that may have supported her learning, and thus to consider the implications of the episode for the improvement of our instructional design. We contend that we were able to develop these conjectures about possible improvements only because we "unpacked" the appropriation process. As we have noted elsewhere, explanations cast in terms of appropriation are adequate for many purposes in that they provide a broad overview of the process of induction into a culture (Cobb, Jaworski, & Presmeg, 1996). However, given our interest in classroom-based design research, it is important to complement a strong social perspective with an equally strong psychological perspective.

Trustworthiness, Replicability, And Commensurability

Turning now to consider the legitimacy of the methodological approach we have illustrated, we first note that the standard convention when reporting interpretivist analyses is to clarify general assertions by presenting a limited number of critical episodes (Atkinson, Delamont, & Hammersley 1988; Taylor & Bogdan, 1984). However, a difficulty arises in that the interpretations of these episodes frequently do not seem justified if they are considered in isolation from the rest of the data. For example, we ruled out the possibility that, during Session 14, Megan was merely imitating a measuring procedure she knew was valued only when we tested and refined conjectures while working through the remainder of the data corpus. Our assertion that she significantly reorganized her reasoning during Session 14 is substantiated by regularities we observed in her mathematical activity during the remainder of the teaching experiment. In this analytical approach, the interpretation of specific episodes and the delineation of general regularities are therefore interdependent in that each informs the other. As a consequence, the interpretation of a particular episode is located within a network of mutually reinforcing inferences that span the entire data set. In such an approach, a central methodological concern is that of the trustworthiness of the analysis.

The notion of trustworthiness acknowledges that a range of plausible analyses may be made of a given data set for a variety of different purposes. The issue at hand is that of the reasonableness and justifiability of the inferences and assertions. The most important criterion in this regard is the extent to which the analysis of a longitudinal data set of this type generated during a teaching experiment is both systematic and thorough. The hallmark of an analytical approach that satisfies this criterion is that inferences are treated as provisional conjectures that are continually open to refutation. We attempted to give a sense of this process of testing and refining conjectures during the sample analysis when we discussed the emergence of the third mathematical practice. To satisfy the criterion, we find it essential to document all phases of the analysis including that of positing and testing initial conjectures.

Final interpretations and assertions can then be justified by backtracking through the various phases of the analysis, if necessary to the video recordings and other data sources. This procedure provides a way of differentiating systematic analysis in which sample episodes are used to illustrate robust claims from questionable analyses in which a few possibly atypical episodes are used to support unsubstantiated assertions. Additional criteria that enhance the trustworthiness of an analysis include both the extent to which it has been critiqued by other researchers, some but not all of whom are familiar with the teaching experiment from which the data were generated, and the extent to which it derives from a prolonged engagement with teachers and students. This latter criterion is typically satisfied in the case of design research and can be viewed as a strength of the methodological approach we take.

The issue of replicability is relevant to the type of design research we conduct in that the approach rests on the assumption that the mathematical practices and associated patterns of learning documented during a teaching experiment can emerge when the instructional sequence is enacted in other classrooms. However, as we know only too well, the history of educational research in general, and in mathematics education in particular, is replete with more than its share of disparate and irreconcilable findings. In our view, a primary source of difficulty is that the independent variables of traditional experimental research are often relatively superficial and have little to do with either context or meaning. The conceptualization of the classroom as a matrix of variables is at odds with the approach we have taken in which the classroom microculture is viewed as a semiotic ecology that involves meaning making in which one thing is taken as a sign for another.[18] We have attempted to illustrate that, from this point of view, students are seen to perceive, act, and learn as they participate in and contribute to the development of a system that is larger than themselves, the classroom community. As Lemke (1997) put it, learning can be characterized as "an aspect of self organization, not just of the human organism as a biological system, but of ecosocial systems in which the organism functions as a human being" (p. 49). It is this sense of participation in an evolving community of practice that typically falls beyond the purview of traditional experimental research.

We contend that the central issue is not so much that past findings have been disparate, but that they have been irreconcilable: It has not been possible to account for differences in findings when different groups of students have received supposedly the same instructional treatment. In contrast to traditional experimental research, the challenge as we see it is not that of replicating instructional treatments by ensuring that instructional sequences are enacted in exactly the same way in different classrooms. The conception of teachers as professionals who continually adjust their plans on the basis of ongoing assessments of their students' reasoning would in

[18]This semiotic ecology can be made explicit by delineating the chain of signification (Lacan, 1977; Walkerdine, 1988) that is constituted as successive classroom mathematical practice emerge. Stephan (1998) described the chain of signification that was constituted during the measurement teaching experiment. Examples of other analyses of this type can be found in Cobb et al. (1997) and Gravemeijer et al. 2000).

fact suggest that complete replicability is neither desirable nor, perhaps, possible (Ball, 1993; Carpenter & Franke, 1998; Gravemeijer, 1994b). The challenge for us is instead to develop ways of analyzing treatments so that their realizations in different classrooms can be made commensurable. We contend that the approach we have illustrated in the sample analysis offers this possibility. The key point to note is that an analysis of the mathematical practices that are established when an instructional sequence is enacted in a particular classroom documents the sequence as it is realized in interaction. Furthermore, when it is viewed against the background of classroom social and sociomathematical norms, the instructional sequence as enacted by a particular classroom community can also be seen to constitute the evolving social situation in which the students' mathematical learning occurred. Consequently, an analysis of two different enactments of the same instructional sequence enables us to relate the differing patterns of what are traditionally called learning outcomes to the differing situations of learning as they were actually constituted in the two classrooms. In such an analysis, the focus on the practices in which the students actually participated as they reorganized their mathematical reasoning brings context and meaning to the fore. It is this that makes it possible to compare and contrast critical aspects of different enactments of a treatment, thereby making them commensurable. We therefore claim that an analytical approach of this type can lead to greater precision and control by facilitating disciplined, systematic inquiry into instructional innovation and change that embraces the messiness and complexity of the classroom.

Usefulness

Throughout this article, we have indicated that a primary criterion by which an analytic approach should be judged is its usefulness. In this regard, there are three additional points that should be considered. The first is to note that, in documenting the actual learning trajectory of the classroom community, the approach we have taken simultaneously documents the emergence of what is traditionally called mathematical content as it occurs in particular classrooms. In the case of the sample analysis, for example, we described how a taken-as-shared mathematical environment in which space was structured in relatively sophisticated ways gradually emerged during the first part of the teaching experiment. The fact that this approach takes mathematics seriously is obviously important given our purposes. This in turn enables us to address two questions that Roth (2001, this issue), in his contribution to this special issue, contends should be asked: How does being in the world change in the course of activity? What are the long-term effects of students' engagement in individual instructional activities? The sample analysis is paradigmatic in that, in Roth's terms, it documents both the emergence of a taken-as-shared ontology of space and students' learning as they participated in and contributed to its emergence. Furthermore, in line with Roth's questions, the account we gave of this process of emergence is grounded in the students' use of particular tools to complete specific instructional activities.

A second observation about the usefulness of the analytical approach centers on the manner in which it situates students' mathematical activity and learning. We have noted that what we need in order to improve our instructional designs are accounts of students' learning that are tied to analyses of what happened in the classrooms where that learning occurred. Analyses of classroom mathematical practices, when coordinated with psychological analyses of individual students' reasoning, provide situated accounts of students' learning in which the process of their learning is directly related to the means by which it was supported. As a consequence, a difficulty that often arises when more individualistic approaches are followed, that of figuring out what the results of an analysis may imply for instruction, simply fails to materialize. Instead, we are in a position to immediately develop testable conjectures about how we might be able to improve the means of supporting students' learning. For our purposes as instructional designers, the situated nature of this analytical approach is a strength when compared with alternative approaches that aim to produce context free descriptions of cognitive development that apparently unfold independently of history, situation, and purpose.[19] The all-to-familiar gulf between theoretical analyses and instructional practice is side stepped because theoretical insights about the means of supporting students' learning in a particular domain are rooted in the practice of attempting to support that learning. At a minimum, situated approaches of the type that we have illustrated support a process of educational innovation in which change is a process of continual iterative improvement.

Our third observation about the usefulness of the analytical approach touches on an issue about which we have said little thus far, that of collaborating with teachers. In this regard, we conjecture that analyses of the type that we have illustrated together with the associated instructional sequences can serve as important means of supporting the development of professional teaching communities (cf. Ball & Cohen, 1996; Hiebert & Wearne, 1992).[20] As we have seen, the analyses justify the instructional sequences that are developed in the course of teaching experiments in terms of the trajectory of the classroom community's mathematical learning, and the

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[19] Wertsch (1991) made a similar point when he observed that much contemporary research in psychology does not in fact have the practical implications claimed for it: "In my view, a major reason is the tendency of psychological research, especially in the United States, to examine human mental functioning as if it exists in a cultural, institutional, and historical vacuum" (p. 2).

[20] It could be argued that the forms of instruction developed in the course of a teaching experiment are unfeasible for any teacher working alone. We would acknowledge, for example, that the entire research team in effect constitutes a collective teacher with some members of the team actually teaching, whereas others observe and analyze classroom events. The demands of this collective activity are, however, balanced by the possibility that the collaborating teachers will be able to capitalize on our learning as represented by instructional sequences and learning trajectories. This conjecture about the proposed role of instructional sequences as a means of supporting the development of professional teaching communities is discussed in some detail by Cobb and McClain (2001). As part of our work with teachers, we are currently developing a series of companion CD-ROMs to support their learning that are based on the video recordings and other data sources. Readers who are interested in the practical implications of our work are referred to a series of articles and book chapters that we have developed for practitioner audiences (e.g., McClain & Cobb, 1999; McClain, Cobb, & Gravemeijer, 1999; McClain, Cobb, Gravemeijer, & Estes, 2000).

means of supporting that learning. If the sequences were justified solely with traditional experimental data, teachers would know only that the sequences had proved effective elsewhere but would not have an understanding of the underlying rationale that would enable them to adapt the sequences to their own instructional settings. In contrast, the type of justification that we favor offers the possibility that teachers will be able to adapt, test, and modify the sequences in their classrooms. In doing so, they can then contribute to both the improvement of the sequences and the development of local instructional theories, rather than merely being the passive consumers of instructional innovations developed by others. In this view, implementation may better be seen as a process of idea-driven adaptation in which pedagogical approaches that have proven effective elsewhere are tested and refined.

Limitations

In focusing on the limitations of our analytical approach, we first note that it was developed while analyzing data from classrooms in which instruction was generally compatible with current reform recommendations. Although we see no reason, a priori, why this approach could not be applied to more traditional instructional settings, a difficulty does arise that concerns the process of documenting individual students' learning. We noted when introducing the sample analysis that the predominance of conceptual discourse in which students were obliged to explain how measuring with a particular tool structured space greatly facilitated our analysis of their reasoning. In contrast, the participation structure in traditional instructional settings is typically such that students are rarely expected to articulate their mathematical interpretations. It would therefore be necessary to supplement classroom data by repeatedly interviewing target students in order to assess their developing reasoning. However, this strategy leads to a further difficulty in that interviews can be learning situations for students despite the interviewer's intentions (Mishler, 1986). This implies that the interviews should be included in the chronological data set that is analyzed to document the process of the students' learning. Furthermore, to tie their learning to the means by which it was supported, it would be important to coordinate a psychological analysis with a social analysis that focuses on the obligations and expectations jointly established by the student and researcher during each interview (cf. Schoenfeld, 1982; Voigt, 1995).

A second limitation of the type of analysis we have illustrated is that its treatment of social context is restricted to norms and practices that are established in the course of face-to-face classroom interactions. The approach, therefore, does not take account of the location of classrooms within schools and local communities, and ultimately within the activity system that constitutes schooling in the United States. The work of scholars who look beyond the classroom demonstrates that schooling involves a number of taken-for-granted policies and practices that foster inequity due to race, gender, class, and economic status (Apple, 1995; Zevenbergen, 1996). Furthermore, as Lave (1996) observed, school as a social institution involves an inherent contradiction between the functions of universal socialization on the one hand and those of the unequal distribution of particular ways of knowing a cultural capital on the other hand. We have acknowledged these and other global, structural characteristics of schooling elsewhere by arguing that our analytical approach should itself be complemented by a strong sociocultural perspective that places the classroom in broader sociopolitical context (Cobb & Yackel, 1996). This is, in our view, essential if we are to pay more than lip service to the pressing concerns of equity and diversity. Minimally, it will be important to take account of teachers' and students' participation in practices located outside the classroom at the school and community levels. When we do, a variety of other methodologies including ethnography can be seen to be appropriate.

A further limitation of our analytic approach stems from the observation that it cannot sensibly be proceduralized because it does not limit its focus to observable activity but is also concerned with mathematical meaning. As the sample analysis illustrates, the approach assumes a relatively deep grasp of the mathematics being taught that transcends the topics, methods, and procedures institutionalized in traditional school mathematics. In addition, the adoption of a social perspective on teachers' and students' classroom activity is a nontrivial accomplishment that is at odds with the graduate education of most mathematics educators and psychologists. We are therefore under no illusion that readers will immediately be able to use the methodology to analyze data they have generated even though we have been relatively direct in describing the approach. Although this may seem unsatisfactory to some, we take comfort from the observation that the situation is similar in the physical, chemical, and biological sciences. The stipulation is inherently conservative in that a methodology can be stripped of conceptual content and reduced to mere method only when researchers share basic theoretical commitments and assumptions. Consequently, although we value explicitness, we contend that pronouncements which equate methodology with method should be challenged in much the same way that we would question the proceduralization of mathematics in school.

Conclusion

Throughout this article, we have stressed that our overall goal is to be increasingly effective in developing instructional designs that support student's mathematical learning. We reiterate this point to emphasize that our commitment to a situated viewpoint on mathematical activity is not ideological. When we described our theoretical position in the first part of this article, for example, we balked at the claim that approaches that involve a focus on individual students' reasoning should be delegitimized. Our reasons for doing so are pragmatic and relate directly to the purposes of design research. As we clarified, we find it essential to focus on the particular ways in which individual students are reasoning when we make instructional decisions in the classroom. In addition, we illustrated how the analysis of individual students' reasoning can lead to conjectures about how we can improve our instructional designs. The importance of focusing on individual students' reasoning acknowledged, we also clarified why we reject purely individualistic approaches and instead find it useful to view students' reasoning as acts of participation in communal practices that they and the teacher establish in the course of their ongoing interactions. The reasons we gave were again pragmatic and relate to the process of planning a teaching experiment. In particular, we discussed why we need a language in which to develop conjectures about the envisioned learning of the classroom community. We also illustrated how this approach has the benefit of enabling us to develop analyses of individual students' mathematical learning that are tied directly to the means of supporting that learning.

Given these considerations, the methodology we have presented may be best viewed as a report from the field rather than a contribution to the ongoing debate between adherents of situated cognition and those who subscribe to more standard psychological paradigms. For us, the methodology is nothing more than a potentially revisable solution to the concrete problems and issues that we have encountered while experimenting in classrooms. We can therefore readily accept that alternative methodologies may be more appropriate for other purposes. We leave it to the reader to judge whether aspects of the analytical approach we have described are relevant to the problems of interest to them. In doing so, we extend the view of implementation as an idea-driven adaptation to our fellow researchers as well as to the teachers with whom we collaborate.

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Part V
Diversity and Equity

Chapter 10
Introduction

Paul Cobb with Lynn Liao Hodge, and Melissa Gresalfi

Supporters of US current reform recommendations argue that the classroom instructional practices they advocate are more equitable than traditional instructional practices in giving all students access to significant mathematical ideas. The approach to instructional design outlined in the previous part of this book is broadly compatible with the influential set of reform recommendations proposed by the National Council of Teachers of Mathematics (2000). The arguments of professional organizations such as NCTM not withstanding, I nonetheless took the view that reform advocates' claims about equity should be scrutinized carefully. My doubts stemmed from prior work with groups elementary teachers in two different school districts in the late 1980s and early 1990s. The first of these districts was rural/suburban, whereas the second served an almost exclusively inner-city student population. Erna Yackel, Terry Wood, and I collaborated with teachers in first district for several years. Our overall goal was to help these teachers reorganize their classroom instructional practices in ways consistent with reform recommendations. To this end, we formulated an initial approach to teacher professional development while working with teachers at this site that proved to be reasonably effective (Cobb, Wood, & Yackel, 1990).

Our initial goal when working with the teachers in this district was to help them make aspects of their current instructional practices problematic, so that they might then have reason and motivation to want to reform their instructional practices. To this end, the teachers analyzed video-recordings of both individual interviews and classroom episodes to explore the consequences of traditional instruction. In doing so, they differentiated between students' correct adherence to prescribed mathematical procedures and their development of forms of reasoning that have quantitative significance. Once the teachers began to question the adequacy of their current ways of teaching, they became willing to consider alternative instructional activities designed to support the development of quantitative reasoning. We noted at the time that

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> we did not have to convince them that children should learn with understanding. Rather, they had assumed that this kind of learning was occurring in their classrooms. *A shared desire to facilitate meaningful learning and a general concern for children's intellectual and social welfare constituted the foundation upon which we and the teachers began to mutually construct a consensual domain*. (Cobb et al., 1990, p. 140, emphasis added)

Shortly after the above passage was written, Yackel began working with a group of teachers in the inner-city district. It soon became apparent that our initial approach to teacher development was not viable in this setting. For the most part, an exploration of the consequences of traditional instruction did not lead these teachers to question their current instructional approaches. Yackel's subsequent efforts to support these teachers were more successful and several of them did in fact develop forms of practice that were compatible with current reform recommendations in mathematics education.

In accounting for these experiences, Yackel and I concluded that what counts as students' intellectual and social welfare reflects particular culturally specific beliefs and values (Cobb & Yackel, 1996). We argued that the teachers in the rural/suburban district shared our view that instruction that emphasizes formal procedures at the expense of quantitative meaning is not in students' long-term interest. However, it was apparent from observations of classroom lessons and teacher induction sessions at the inner-city site that the teachers who worked in this district were also committed to providing instruction that was beneficial to their students. Crucially, the teachers in this district appeared to take a different view of students' intellectual and social welfare. In particular, teachers and administrators in the inner-city district seemed to value a highly regulated environment. We also noticed that the reasons for school and classroom regulations were not discussed with students. Although the issue of whether a regulation had been violated in a particular instance was discussed, the appropriateness of the regulations and reasons for complying did not become topics of conversation.

Yackel and I argued on the basis of these observations that "what it means to be a student in school" is constituted by school and school district staff in the course of their ongoing interactions (cf. Banks, 1995; Walkerdine, 1988). In other words, assumptions about the legitimate ways that student's can exercise agency informs teachers' and administrators' interpretations of individual students' actions. These assumptions are not fixed and universal, but is instead continually regenerated by members of educational institutions as they participate in the practices of schooling. In the inner-city district, for example, to be a student in school was to follow specific rules and instructions. Further, to understand was to be able to verbalize relevant rules. Consequently, adults showed their concern for students' welfare by helping them learn to follow and verbalize the relevant rules. It might well have been the case that in guiding the development of a regulated school environment, the inner-city teachers were attempting to provide students with a safe and secure setting for learning. The crucial point for my purposes is that there was no conflict for the teachers at this site between the consequences of traditional mathematics instruction and the institutionalized view about what it meant to be a student in school. The teachers therefore had no reason to revise their current instructional practices.

The analysis that Yackel and I conducted revealed that core assumptions and values implicit in current reform recommendations were compatible with those of the teachers at the rural/suburban site but conflicted with those of teachers at the inner-city site (Cobb & Yackel, 1996). This conclusion implied that reform efforts in which mathematics educators assume that their culturally situated commitments are universal might well result in even greater disparities in the types of mathematics education that children experience than is currently the case. It was against this background that Lynn Liao Hodge[1] and I submitted a very modest proposal to the US National Science Foundation (NSF) in 1998 to develop a synthesis of then current research literature on issues of equity and diversity in mathematics education. The proposal received favorable reviews and, much to our chagrin, we were asked to expand the scope of the project by organizing two working meetings that brought together leading researchers in mathematics education and in the field of educational equity.[2] These meetings provided the grounding for a special double issue of *Mathematical Thinking and Learning* and an edited book (Nasir & Cobb, 2007). The chapter reprinted in this part is a longer version of a chapter included in the edited book. It represents a temporary resting place in my and Hodge's ongoing efforts to develop a position on issues of equity and diversity that can inform the formulation, testing, and revision of designs for instructional improvement at the classroom, school, and district levels.

As Hodge and I made clear near the beginning of the reprinted chapter, we were under no illusion that we would be able to make a seminal contribution to research on issues of equity in mathematics education. Our more modest goal was to explore how a focus on equity in students' learning opportunities can become an integral aspect of what might be termed mainstream mathematics education research. As we dug into the literature, we realized relatively quickly that the term equity was being used in a range of differing and often not entirely compatible ways. We were also struck by the general lack of acknowledgement of these differences.[3] The three

[1] At that time, Hodge was a doctoral student in the mathematics education program at Vanderbilt University. She subsequently completed her dissertation study that focused on issues of equity and diversity in 2001 and is currently a member of the mathematics education faculty at the University of Tennessee.

[2] The principal investigators for this project were Paul Cobb, Lynn Liao Hodge, and Carol Lee. Lee was a faculty member at Northwestern University and later became president of the American Educational Research Association. Her work focuses on equity in student learning opportunities in the field of language and literacy.

[3] For a notable exception, see Gutiérrez (2002).

aspects of the definition that we proposed in the reprinted chapter emerged as we worked to understand various conceptions of equity against the backdrop of our concerns and interests as design researchers.

The first aspect of the definition we proposed concerns students' development of forms of mathematical reasoning that have authority and pull beyond the classroom. This aspect of the definition was influenced by Delpit's (1988) argument that students from underserved groups should be inducted into what she termed the languages of power within society. In addition, it reflects design researchers' concern that what students learn in the classroom should enable them to become relatively substantial participants in significant practices beyond school (cf. Brown, Collins, & Duguid, 1989; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). The second aspect of the definition was influenced by Secada's (1995) critique of progressive reform proposals for failing to take account of the current organization and functions of schooling. This aspect acknowledges that instruction that gives students access to the language of power is inequitable if it does not also enable them to enroll and succeed in future mathematics courses, particularly those that serve as gatekeepers to future educational and economic opportunities. The final aspect of the definition focuses on what has traditionally been called student motivation and highlights the importance of cultivating students' mathematical interests as an explicit instructional goal. This aspect of the definition draws on Dewey's (1913/1975) analysis of mathematical interests as an integral facet of mathematical literacy, and on Nicholls' (1989) critique of standard US instructional practices as a primary source of inequities in student motivation.

Hodge and I developed the proposed definition of equity at the same time that we and others prepared for and conducted two classroom design experiments that focused on statistical data analysis.[4] The first experiment focused on the analysis of univariate data and was conducted with an intact class of 13-year-old students in an urban school, and the second experiment focused on the analysis of bivariate data and was conducted with some of the same students a year later. It is apparent in retrospect that these experiments illustrate the design challenges involved in attempting to address all three aspects of equity as we defined it.

The first aspect of our definition came to the fore when we were preparing for the two experiments and asked ourselves why statistics should be included in the school mathematics curriculum. As we indicate in the reprinted chapter, the justification that we found compelling builds on the observation that debates about public policy issues typically involve presenting and critiquing arguments that involve the analysis of data (G. Cobb, 1997). In many respects, this discourse has become the language of power in the public policy arena. Inability to participate necessarily entails disenfranchisement and, potentially, alienation from the political process. Cast in these terms, statistical literacy that involves reasoning with data in relatively

[4] These design experiments were conducted by Paul Cobb, Kay McClain, Koeno Gravemeijer, Erna Yackel, Clifford Konold, Jose Luis Cortina, Lynn Liao Hodge, Maggie McGatha, Beth Petty, Carla Richards, and Michelle Stephan.

sophisticated ways bears directly on both equity and participatory democracy. The image that guided our preparation for the statistics design experiments was therefore that of students as increasingly substantial participants in public policy discourse. This image led us to frame students' formulation and critique of data-based arguments as a central instructional goal. This framing influenced our design efforts by orienting us to develop tasks in which the students analyzed data sets that they viewed as legitimate for reasons that they considered significant from the outset of the first design experiment.[5]

With regard to the second aspect of our definition of equity, it was essential that the instructional sequences we designed supported students' development of key statistical ideas that are typically addressed in school instruction and assessed by high-stakes tests. In the middle grades, these notions included mean, mode, and median as well as a range of statistical graphs culminating with histograms, box-and-whiskers plots, and scatter plots. The challenge inherent in addressing the first two aspects of our definition was therefore to transcend what Dewey (1980) termed the dichotomy between process and content by supporting the emergence of key statistical ideas while simultaneously ensuring that students engaged in genuine data analysis. This non-trivial challenge proved tractable once we identified distribution as an overarching statistical idea that could orient our instructional design effort. In the approach that we took, mean, mode, median, skewness, spread-outness, and relative frequency emerged as ways of describing how specific data sets were distributed. Further, various statistical graphs emerged as different ways of structuring distributions in order to understand a phenomenon and thus make a consequential decision or judgment (cf. Cobb, 1999; Cobb et al., 2003).

The third aspect of our definition, cultivating students' interest in analyzing data, emerged as an explicit research focus while the design experiments were in progress. As the teacher in whose classroom we conducted the first of the two design experiments[6] put it, the students were initially not at all interested in analyzing data, then became somewhat interested, and by the latter part of the experiment had become extremely interested. When Hodge and I conducted a retrospective analysis of the experiments in collaboration with Jana Visnovska and Qing Zhao, we found it essential to differentiate between what we termed pragmatic interests and statistical interests (Cobb, Hodge, Visnovska, & Zhao, 2007). We classified an interest as pragmatic if the students' developed and critiqued analyses in order to justify pragmatic decisions or judgments. In contrast, we classified an interest as statistical if students' attempts to understand an analysis presented during a class discussion centered on clarifying a statistical idea rather than making a pragmatic decision. This retrospective analysis documented the process by which the teacher cultivated students' pragmatic interest at the beginning of the first experiment, and then subsequently supported the students' development statistical interests as the first experiment

[5]A detailed report of our design decisions can be found in Cobb (1999) and Cobb, McClain, and Gravemeijer (2003).

[6]McClain served as the teacher in both experiments and was assisted in this role by Cobb.

progressed. The analysis implicated all major aspects of the classroom learning environment in this process: the structure of lessons (including the social norms established in each phase of a lesson), the key characteristics of instructional tasks, the data analyses tools the students used, and the nature of classroom discourse (including norms of statistical argumentation). The analysis therefore indicates that surface-level modifications to an instructional design will not be sufficient to support students' development of mathematical interests. Efforts to addresses inequities in motivation should attend to all aspects of the classroom learning environment if they are to be effective.

The process of formulating this definition of equity and, indeed, of writing the chapter reprinted in this part was a long and tortuous process in the course of which our synthesis of the literature went through nine major revisions. Many of the studies we reviewed focused on differences between the out-of-school practices in which students engage and the practices established in the classroom as potential sources of inequities in students' learning opportunities. In organizing this literature, we found it useful to distinguish between two general lines of scholarship, one that focuses on the practices of students' local, home communities and a second that focuses on the practices of broader groups within society. The importance of students' local, home communities was highlighted by studies that identified significant differences between mathematical reasoning in school and in various out-of-school settings such as grocery shopping (Lave, 1988), packing crates in a dairy (Scribner, 1984), selling candies on the street (Nunes, Schliemann, & Carraher, 1993; Saxe, 1991), laying carpet (Masingila, 1994), farming sugar cane (de Abreu, 1995), and playing basketball (Nasir & Cobb, 2002). We viewed the work of de Abreu (1995; 2000) as particularly significant within this line of scholarship because she emphasized that the issue at hand is not merely one of cultural difference but also concerns the manner in which the forms of mathematical reasoning associated with particular out-of-school communities are frequently treated as illegitimate in school. Against this background, we saw considerable value in Moll's (1997) and Civil's (2002) efforts to frame the practices of the students' home communities as funds of knowledge on which teachers can capitalize in the classroom.

The second line of scholarship complements the work of de Abreu, Moll, and Civil by illustrating the value of analyzing the practices of broader groups in wider society as potential instructional resources. We viewed Lee's (1995) work in supporting the learning of high school English students as paradigmatic in this regard. Lee illustrated that there are important continuities between signifying, a form of social discourse prominent in the African-American community, and the process of interpreting literary texts in that both involve figurative language, irony, and double meanings. Lee demonstrated in a series of intervention studies that instruction that capitalizes on these continuities is indeed feasible. We also saw considerable value in a related program of research conducted by Warren and Rosebery in science education (Warren, Ballenger, Ogonowski, Rosebery, & Hudicourt-Barnes, 2001; Warren & Rosebery, 1995). As part of the process of developing their instructional designs, Warren, Rosebery, and the teachers with whom they collaborated looked for potential continuities between the out-of-school discourse practices of Haitian immigrant students and key aspects of scientific argumentation. Thus, like Moll, Civil, and Lee, they recast ways of knowing that are typically devalued in school as potential instructional resources. Furthermore, they reported encouraging empirical findings concerning students' development of a relatively deep understanding of the specific scientific phenomena and the role of hypotheses and the use of evidence.

In reviewing these two lines of work, we came to view diversity and equity as emerging from the relations between students' participation in the practices of the mathematics classroom, the local home community, and broader groups in wider society. The key point to emphasize is that in this relational perspective, equity concerns the continuities and discontinuities between the out-of-school ways of reasoning and talking into which students have been enculturated and the norms and practices established in the mathematics classroom (Cobb & Hodge, 2002).

For many mathematics educators, this view of equity necessarily implies that classroom activities should be aligned with students' out-of-school practices. As we continued to explore the literature, we began to question whether this is the most productive approach for two reasons. First, an approach of this type is problematic if students in a particular classroom are members of a number of distinct out-of-school groups and communities whose practices differ significantly from each other. We were struck by the fact that Moll, Civil, Lee, and Warren and Rosebery all portrayed the students with whom they worked as coming from a single group that was culturally homogeneous. Second, we knew from our own work as instructional designers that it is extremely difficult to develop effective designs of this type even in cases where students are members of a single identifiable group. Doing justice both to students' informal ways of reasoning that are grounded in out-of-school practices and to central disciplinary ideas is no easy task. We therefore appreciated Civil's (2007) and Enyedy and Mukhopadhyay's (2007) frank accounts of the difficulties they encountered while pursuing approaches of this type in the classroom. The problems they described only increased our admiration for Lee's and Warren and Rosebery's pioneering efforts. We did, however, question whether such approaches are feasible in all areas of mathematics and with truly diverse groups of students.

The alternative approach that we have come to value also views students as cultural beings and attends explicitly to the possibility that the norms and practices established in the classroom might disadvantage certain groups of students. In this approach, the teacher's responsibilities include identifying instances in which particular groups of students might be disadvantaged and adjusting instruction accordingly. As an illustration, the teacher would not restrict the selection of task scenarios to out-of-school situations with which the students are already familiar. However, the teacher would be aware that task scenarios necessarily involve cultural-specific suppositions and assumptions, and that some of these assumptions might be foreign to some or all the students (cf. Ladson-Billings, 1995). The teacher would therefore support students' access to tasks by leading a discussion of task scenarios in the classroom with the goal of ensuring that they become real in imagery for all students (Boaler, 2002). In this alternative approach, the range of potential tasks is much broader. Task scenarios would only be rejected if they cannot become real for students as they participate in a discussion in which grounding assumptions become explicit topics of conversation. The challenge for the teacher in such an approach is to pursue a significant mathematical agenda while ensuring that all students can participate substantially in classroom activities. Yackel followed such an approach with considerable success while working with elementary teachers and their students at the urban site. The statistics design experiments also indicate that an approach of this type can be reasonably effective. In adopting this approach, we follow Boaler and Staples (2008) in noting that there are different routes to equity in learning opportunities, not all of which involve aligning instruction with out-of-school practices. In our view, an approach of this type is potentially feasible with diverse groups of students in a range of mathematical domains.

In the last few paragraphs, we have focused on the development of instructional designs that are equitable in terms of student learning opportunities. From the perspective of design research, a set of coherent design heuristics that can guide the development of classroom learning environments is one of two essential conceptual tools. The second essential tool is an interpretive framework for making sense of what is going on in the classroom in a manner that can feedback to inform the ongoing instructional design effort (Cobb et al., 2003). In the latter part of the chapter reprinted, we outlined an interpretive scheme of this type that enables researchers to document the personal identities that students are developing as they participate in classroom activities. Hodge and I subsequently collaborated with Melissa Gresalfi[7] to revise and elaborate this interpretive scheme. The revised scheme is described and illustrated in some detail in an article published in *Journal for Research in Mathematics Education* (Cobb, Gresalfi, & Hodge, 2009).

The phenomenon the scheme is designed to explain is that of why, in particular classrooms, some students come to identify with classroom mathematical activity whereas others merely cooperate with the teacher, and still others actively resist engaging in classroom activities. This scheme builds on and extends the prior work on classroom social and sociomathematical norms. In the reprinted chapter, we introduce the notion of the normative identity established in the mathematics classroom. As we clarify, the normative identity is a communal or collective construct and refers to key aspects of the classroom learning environment with which students would have to identify in order to develop a sense of affiliation with classroom mathematical activity. We argue that these key aspects include norms, which we define

[7] Gresalfi accepted a 2-year post doc position at Vanderbilt University in 2004 to work with Cobb on issues of equity and student identity in mathematics classrooms. She is currently a faculty member in the learning sciences program at Indiana University.

as recurrent patterns in joint activity that is regulated by the expectations that the teacher and students have for each other's actions in particular situations (Searing, 1991). We then go on to illustrate how the classroom learning environment can be analyzed in terms of the expectations that the teacher and students have for others' actions, and the obligations that they attempt to fulfill (or resist) by acting in accord with expectations.

This perspective on the mathematics classroom aligns well with Holland, Skinner, Lachicotte, and Cain's (1998) characterization of identification as a process whereby communal activities "in which one has been acting according to the directions of others becomes a world that one uses to understand and organize aspects of one's self and at least some of one's own feelings and thoughts" (p. 121). An analysis of the classroom learning environment conducted using the revised scheme presented in the *JRME* article focuses on the general and specifically mathematical obligations that a student would have to fulfill in order to be an effective student in that classroom. In addition, this analytical approach documents whether these obligations-to-others become obligations-to-oneself[8] for students who come to identify with classroom mathematical activity, or whether they remain obligations-to-others for students who merely cooperate with the teacher, or become obligations-for-others for students who resist engaging in classroom activities. The strength of this interpretive approach is that the resulting analyses of the identities that students are developing in the classroom are situated with respect to key aspects of the classroom learning environment. The analyses can therefore feed back to inform instructional design and teaching because the resulting accounts of the identities that students are developing are situated with respect to the classroom learning environment, and because they focus on specifically mathematical aspects of the classroom learning environment.

As the various parts of this book indicate, I have contributed to investigations of a relatively wide range of phenomena that relate to mathematics teaching and learning over the years. Along the way, I have analyzed the development of individual students' mathematical reasoning in particular mathematical domains, the evolution of learning environments established in particular classrooms, the learning of professional teaching communities, and the school and district settings in which mathematics teachers develop and revise their instructional practices. As a point of reference, the effort to develop useful ways of understanding issues of diversity and equity as they play out in the mathematics classroom constitutes the most complex and challenging set of problems that I have attempted to address. I will be more than satisfied if my work in this area contributes to a issues of equity becoming a routine aspect of mainstream research in mathematics education.

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[8] This formulation of obligations-to-others becoming obligations-to-oneself directly parallels Sfard (2008) argument that learning involves turning discourse-for-others into discourse-for-oneself. The account we have given of the process of identifying is consistent with Sfard's participationist viewpoint, the basic tenet of which is that "patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual" (p. 43).

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Chapter 11
Culture, Identity, and Equity in the Mathematics Classroom

Paul Cobb and Lynn Liao Hodge

The motivation for this chapter stems in part from the relatively marginalized status of issues of diversity and equity within mathematics education research. As Lubienski (2002) documents, research on equity is underrepresented in the mathematics education literature. Furthermore, as Secada (1995) observes, the relatively limited number of studies with an equity focus have, for the most part, been constituted as peripheral to the field. Against this background, we have struggled with the challenge of making a concern for issues of diversity and equity integral to our ongoing research for the last several years. We do not pretend that we have either the experience or expertise to make a seminal contribution to research on diversity and equity in mathematics education. Instead, the issue that we have sought to address is how a focus on diversity and equity can become part and parcel of mainstream research that involves the development of instructional designs and the analysis of the learning and teaching of significant mathematical ideas.

In this chapter, we first offer a definition of equity that reflects our focus on classroom processes of mathematics learning and teaching. We then differentiate between two views of culture that can be discerned in the mathematics education literature. In one view, culture is treated as a characteristic of readily identified and thus circumscribable communities, whereas in the other view it is treated as a set of locally instantiated practices that are dynamic and improvisational. We clarify the relation between these two characterizations of culture and argue that both are relevant to the goal of ensuring that all students have access to significant mathematical

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ideas. We then focus on the second, more recent view of culture as local and improvisational in the remainder of the chapter and consider its potential relevance. We argue that the manner in which this perspective brings the identities and interests that students develop in mathematics classrooms to the fore make it directly relevant to researchers who focus on instructional design, learning, and teaching at the classroom level. We then go on to develop this perspective in the latter part of the chapter by proposing an interpretive scheme for analyzing the identities that students develop in mathematics classrooms that can inform instructional design and teaching.

A Provisional Definition of Equity

As R. Gutierrez (2002, this volume) observes, ongoing debates over how equity might be usefully construed in mathematics education constitute important contexts within which to articulate both immediate and longer-range goals in the field. Our purpose in attempting to clarify what we mean by equity is not to close down these debates but to offer a potentially revisable definition that reflects our interest in instructional design, learning, and teaching at the classroom and school level. The concept of equity encompasses a complex range of concerns that emerge when people who are members of various local communities and broader groups within society act and interact in the mathematics classroom. Foremost among these is the issue of students' access to opportunities to develop forms of mathematical reasoning that, as Bruner (1986) puts it, have clout. Bruner went on to clarify that, in his view, forms of reasoning have clout to the extent that they enable students to participate in significant out-of-school practices in relatively substantial ways. As an illustration, it is apparent that public policy discourse increasingly involves the formulation and critique of data-based arguments. Students' development of the relatively sophisticated forms of statistical reasoning that are implicated in such arguments therefore have clout in that they enable them to participate in a type of discourse that is central to what Delpit (1988) termed the *culture of power* (cf. Cobb, 1997; Cobb, 1999).

In offering this perspective on what it means for particular forms of reasoning to have clout, Bruner viewed the societal function of schools to be that of inducting students into what he referred to as *culture as lived*. In developing a working definition of equity, it is essential that we also consider a second societal function of schooling, that of comparing and differentiating between students in ways that have direct consequences for their future educational and economic opportunities. The pervasiveness of this function of schooling indicates the need to broaden what it means for forms of reasoning to have clout by taking account of criteria that are internal to the school (cf. Secada, 1995). Foremost amongst these is that of students' access to future mathematics courses (Tate & Rousseau, 2002). As an illustration, Moses and Cobb (2001) clarify that the goal of the Algebra Project is to make it possible for *all* students to have access to and to succeed in high school algebra courses that function as gatekeepers to college-preparatory tracks. Moses and Cobb also alert us to a third aspect of equity that concerns the cultivation of students' interests in and feelings of equity about the future study of mathematics. As we discuss later in this chapter, this aspect of equity brings to the fore the identities that students develop as they engage in classroom mathematical activities. For the present, it suffices to note that a perspective on equity is inadequate if it is limited to students' participation in out-of-school practices and to their access to particular school mathematics courses. The definition that we propose also includes what are traditionally referred to as students' motivations to continue to study mathematics and their persistence while doing so. Thus, equity as we construe it encompasses students' development of a sense of efficacy (empowerment) in mathematics together with the desire and capability to learn more about mathematics when the opportunity arises.

Two Views of Culture

Two lines of scholarship that are grounded in differing views of culture can be discerned in research on issues of equity in mathematics education. The first line of research reflects the view of culture as a *way of life* that is characteristic of a bounded community. In this view, culture comprises a network of relatively stable practices that capture daily life within a group or community that are passed on from one generation to the next. This view of culture is prominent in the mathematics education literature and is consistent with the typical use of the term in everyday discourse. The second line of research reflects a more recent view of culture that has emerged within the past 20 years to capture the changing aspects of contemporary life. In this second view, culture is viewed as a network of locally instantiated practices that are dynamic and improvisational (Bauman, 1999; Calhoun, 1996; Gutierrez, Baquedano-Lopez, & Tejeda, 1999; Eisenhart, 2001). This perspective emphasizes people's participation in multiple communities or groups and considers the boundaries between these groups or communities to be blurred and permeable.

The changes that precipitated the emergence of the second view of culture include technological advances in communication and travel that have made the world a much smaller place. These advances have made possible a dramatic increase in immigration in many parts of the world, thereby altering demographic patterns that once seemed relatively stable. A second set of changes concern the role of women and the composition of the work force. Children frequently grow up in a variety of social settings (e.g., in day care, with babysitters, in school, and among peer groups) that function together with the family and home communities to raise them (Gutierrez & Rogoff, 2003). As Eisenhart (2001) observes, it is difficult to capture these and other aspects of contemporary life when culture is viewed as a way of life within a bounded community.

> [I]t is no longer straight-forward for anthropologists to plan to study "cultural groups," i.e., designated groups of people with coherent, shared value systems, households or communities with clearly defined boundaries, or shared funds of knowledge transmitted primarily from adults to their children. Conventional assumptions of culture as coherent

> and co-terminous with social background, language use, region, or ethnicity have become impossible to sustain. (p. 16)

In this more recent view, culture is grounded in shifting social networks and relationships as people who are members of a variety of communities present themselves to and are recognized by others (Clifford, 1986; Eisenhart, 2001). It is in the course of often-contested interactions that people identify themselves and are identified by others. As Calhoun (1996) notes, an explicit concern for issues of identity in both everyday life and in the social sciences is a defining aspect of the modern age.

> It is not simply – or even clearly the case – that it matters more to us than to our forebears to be who we are. Rather, it is much harder for us to establish who we are and maintain this own identity satisfactorily in our lives and in the recognition of others. (p. 32)

Calhoun goes on to clarify that the difficulties that we frequently face in establishing who we are stem both from the disintegration of all-encompassing identity schemes and from changes in discourse about identity. As an illustration, Calhoun observes that identity schemes such as kinship within a bounded community offered clear notions of who particular people were in relation to others and how they should participate in social relations with each other. However, the modern age has brought about a questioning of social categories and social networks that were once taken for granted, thereby problematizing the process of determining who we are in relation to others. Calhoun demonstrates that this process is further complicated by socially sustained discourses that center on identity. It is not merely that how we are recognized often does not fit with who we consider ourselves to be. Discourse about who it is important to be and who it is possible to become is continually changing and may be in conflict with who people view themselves to be and who they want to become. A number of scholars have in fact coined the metaphor of people existing in the *borderlands* of various communities to capture their struggle to construct or maintain who they are (Gutierrez et al., 1999; Rosaldo, 1989).

The first view of culture as a way of life within a bounded community is far more prominent in the mathematics education research on equity. Research oriented by this view typically identifies discontinuities between the out-of-school practices in which students participate and those established in the mathematics classroom as the primary source of inequities. In more sophisticated investigations of this type, the significance of the discontinuities is clarified by locating them within the context of broader sociostructural processes that encompass race, ethnicity, and social class and that account for the major fault lines within society. Researchers who take this approach emphasize that the relative value attributed to a particular practice in school typically reflects the differential position that the group with which the practice is associated occupies in society (cf. de Abreu, 1995, 2000). The work of Moll (1997) and Civil (1998, this volume) is paradigmatic of one body of research that attempts to reorganize traditional patterns in schooling by taking the practices of students' home communities as its point of reference for classroom design. Civil describes how she and her colleagues collaborate with the mathematics teachers of predominantly Latino students to reduce conjectured sources of inequities by analyzing the practices of the students' home communities in terms of funds of knowledge. The goal in doing so is to develop innovative instructional activities and practices that build on students' out-of-school mathematical experiences. The work of Warren and Rosebery (Warren & Rosebery, 1995; Warren et al., 2001) illustrates a second body of work that takes the Discourse of broader groups within society as its point of reference. The Discourse on which Warren and Rosebery focus is that of Haitian immigrants in the United States. As part of the process of developing their instructional designs, Warren and Rosebery collaborate with teachers to identify potential continuities between this out-of-school Discourse and the Discourse of scientific argumentation.

Willis's (1977) seminal analysis of how British working-class students typically end up in working-class jobs is paradigmatic of investigations oriented by the second view of culture as local and dynamic. His ethnographic analysis of a group of working-class boys demonstrates how manifestations of the boys' working-class backgrounds were devalued in school. Thus, like Moll, Civil, and Warren and Rosebery, Willis places discontinuities in practices in the context of sociostructural processes. However, in contrast to researchers who are oriented by the view of culture as a way of life, Willis did not assume that the delegitimization of out-of-school practices in school necessarily leads to lack of academic success. It was not self-evident to him, for example, why the boys and their families did not demand better treatment so that they could move into the middle class. He also sought to understand why the boys did not follow some of their working-class peers in attempting to accommodate the expectations of the school. Rather than assuming that the boys were passive bearers of a working-class culture that had been passed down to them by their parents, he examined the meaning that the discontinuities he identified had for the boys. His analysis focused on the boys' identities and revealed that they could not reconcile accommodation to the school's expectations with who they were and who they wanted to be. He documents that the boys actively constructed a positive sense of their lives in school by drawing on a number of sources that included popular culture and their parents' shop-floor culture. Willis therefore concludes that the boys' resistance to the school was not predetermined by their socialization into a monolithic working-class culture. Instead, the boys actively contributed to the reproduction of their relatively low status in society by constructing a local counterculture and fashioning oppositional identities that involved a sense of self-worth and status. As he makes clear, this local culture was both dynamic and improvisational.

Local Cultures and Broader Discourses

An issue that arises when culture is viewed as local and dynamic is that of how to account for the types of relatively broad and enduring macro patterns in people's individual and collective activity that are of interest to sociologists. Willis's analysis is again relevant as he did not set out to develop a narrative about a particular group of boys in Britain in the 1970's. Rather, he framed the boys' activity as a paradigm case to understand students' resistance to schooling more generally. Two aspects of his analysis contribute to its potential generalizability. First, he took account of the boys' position within class-stratified British society and documented that the school devalued manifestations of their working-class backgrounds. Second, he stressed that the cultural resources on which the boys could draw as they constructed their local counterculture and their oppositional identities were constrained by their positioning within broader sociostructural processes, the most evident of which is social class. On this basis, he argued that working-class students in other British high schools might be treated similarly and that some would attempt to make positive sense of their lives in school by drawing on similar cultural resources to create local countercultures that, while not identical, shared family resemblances. For Willis, the resulting pattern of resistance in different schools is an emergent phenomenon situated within but not directly caused by class stratification in British society. As Erickson (1992) notes, an explanation of this type would appear to be relevant to societies such as the United States in which the major sociostructural distinctions fall along lines of race and ethnicity as well as class (Erickson, 1992).

In this account of the production of relatively stable macro patterns, it is tempting to interpret the resources such as popular culture and their parents' shop floor culture on which the boys drew as ways of life that are characteristic of bounded communities. However, we have argued elsewhere (Cobb & Hodge, 2002) that it is more useful to treat these broader and more enduring practices as aspects of a Discourse. It is important to stress that a Discourse involves much more than linguistic practices. Gee (1997) offers the following definition:

> Discourses are sociohistorical coordinations of people, objects (props), ways of talking, acting, interacting, thinking, valuing, and (sometimes) writing and reading that allow for the display and recognition of socially significant identities, like being a (certain type of) African American, boardroom executive, feminist, lawyer, street-gang member, theoretical physicist, 18th-century midwife, 19th-century modernist, Soviet or Russian, schoolchild, teacher, and so on through innumerable possibilities. If you destroy a Discourse (and they do die), you also destroy its cultural models, situated meanings, and its concomitant identities. (pp. 255–256)

The crucial differences between culture and Discourse as theoretical constructs concerns their origins and what they take as central. The notion of culture has its origins in anthropology and sociology and emphasizes activities that transform the world and that involve the use of physical and symbolic artifacts. In contrast, the notion of a Discourse has its origins in linguistics and semiotics and emphasizes communication together with everything that makes it possible. Our proposal to follow Gutierrez et al. (1999) in viewing culture as a set of locally instantiated practices that are dynamic and improvisational in nature emphasizes people's mutual engagement in joint activities that involve the directly negotiated use of artifacts. In viewing broader practices that extend beyond the scope of mutual engagement as constituting a Discourse, we bring processes of communication beyond direct interaction to the fore.

As an illustration close to the experience of most mathematics educators, the various Standards documents produced by the National Council of Teachers of

Mathematics (NCTM, 1989, 1991, 2000) can be viewed as proposing an educational Discourse. Thus, a group of teachers who are members of a local professional teaching community might also view themselves as members of the broader community of mathematics education reformers. In such a case, the Standards documents serve as a primary resource on which the teachers draw as they jointly construct a local culture of mathematics teaching. As this illustration indicates, Discourses such as that of reform teaching tie local communities of practice into broader configurations (Wenger, 1998). It is only as people actively draw on a Discourse as a resource when improvising a local culture that the Discourse can touch their experience and be given new life (Holland et al., 1998; Wenger, 1998). In the illustration, the Discourse of reform in mathematics education touches teachers' experience only to the extent that they actually attempt to change their instructional practices. The illustration also serves to clarify the relation between local, improvizational cultures and broader Discourses. On the one hand, Discourses constitute resources for the construction of local cultures. On the other hand, people contribute to both the vitality of a Discourse and to its ongoing evolution as they use it as a resource.

Structural and Situational Rationales for Learning in School

Erickson (1992) clarifies that achievement and motivation in school are explicitly political processes "in which issues of institutional and personal legitimacy, identity, and economic interest are central" (p. 33).

> Students in school, like other human beings, learn constantly. When we say they are "not learning" what we mean is that they are not learning what the school authorities, teachers, and administrators intend for them to learn as a result of intentional instruction . . . Learning what is deliberately taught can be seen as a form of political assent. Not learning can be seen as a form of political resistance. (Erickson, 1992, p. 36)

For his part, D'Amato (1992) distinguishes between two ways in which learning in school can have value for students. D'Amato refers to the first of these ways as extrinsic value or *structural significance*, in that achievement in school has instrumental value as a means of attaining other ends such as entry to college and high-status careers, or acceptance and approval in the household and other social networks. D'Amato contrasts this source of value with what he terms intrinsic value or *situational significance*, in which students view their engagement in classroom activities as a means of maintaining valued relationships with peers and of gaining access to experiences of mastery and accomplishment. The crucial point to note for our purposes is that students' participation in Discourses that give them access to a structural rationale varies as a consequence of family history, race or ethnic history, class structure, and caste structure within society (D'Amato, 1992; Erickson, 1992; Mehan et al., 1994).

> Where school success has been associated with social mobility, as in the case of the middle and upper classes, the need to succeed in school [and in mathematics in particular] is emphasized in home-life networks, and children take for granted the value to their futures and to

> present social relationships of positive teacher evaluations and other markers of school success. . . . School, however, tends to have little credible structural significance for castelike minority children (Ogbu, 1978) and for the majority of children of lower socioeconomic strata. (D'Amato, 1992, p. 191)

In our terms, Discourses that inscribe the achievement ideology wherein society is seen to reward hard work and individual effort with future educational and economic opportunities constitute a resource on which some students but not others can draw as they attempt to make positive sense of their lives in school. From our perspective as mathematics educators interested in instructional design, the resulting inequities in motivation (Nicholls, 1989) emphasize the importance of ensuring that all students have access to a situational rationale for learning mathematics. It is here, we contend, that issues of equity can potentially intersect with mathematics educators' traditional focus on instructional design, teaching, and learning. In our view, supporting students' development of a sense of affiliation with mathematics as it is realized in their classrooms should be an explicit goal of both instructional design and teaching. Elsewhere, we have reported an initial attempt to address issues of instructional design by documenting an approach for cultivating students' mathematical interests (Cobb & Hodge, 2003a). In the remainder of this chapter, we focus on the challenge of analyzing classroom actions and interactions in a manner that can feed back to inform the improvement of such designs. The interpretive scheme that we outline focuses on the identities that students develop in mathematics classrooms.

Identity and Learning

The notion of identity has become increasingly prominent in the mathematics education research literature in recent years (de Abreu, 1995; Boaler & Greeno, 2000; Cobb & Hodge, 2002; Gutstein, 2002a, 2002b; Sfard, 2002). Part of the appeal of this construct is that it enables researchers to broaden the scope of their analyses beyond an exclusive focus on the nature of students' mathematical reasoning by also considering the extent to which they have developed a sense of affiliation with and have come to see value in mathematics as it is realized in their classrooms. The notion of identity as it is used in mathematics education therefore encompasses a range of issues that are typically subsumed under the heading of affective factors. These include students' persistence, interest in, and motivation to engage in classroom mathematical activity. As Nasir (2002) clarifies, the development of students' classroom identities is intimately related to the development of their mathematical reasoning.

> [On the one hand,] as members of communities of practice experience changing (more engaged) identities, they come to learn new skills and bodies of knowledge, facilitating new ways of participating which, in turn, helps to create new identities relative to their community. . . [On the other hand,] increasing identification with an activity or with a community of practice motivates new learning. In this sense, identities can act as a motivator for new learning, prompting practice participants to seek out and gain the new skills they need to participate in their practice more effectively. (p. 239–240)

This interrelation underscores the importance of cultivating students' identification with mathematical activity as a goal for both instructional design and teaching.

The interpretive scheme that we propose for analyzing the identities that students develop in mathematics classrooms involves three primary constructs: normative identity, core identity, and personal identity. We introduce these constructs by drawing on seminal studies conducted by Boaler and Greeno (2000) and Martin (2000) and then discuss each construct in turn. As part of an investigation of students' valuations of classroom mathematical activity, Boaler and Greeno (2000) interviewed students from four high schools who were enrolled in advance placement calculus classes in which they were expected to complete tasks by applying methods and strategies presented by the teacher. Many of these students indicated both that they found their experiences of engaging in mathematical activity in these classes distasteful and that they had come to dislike mathematics and would choose not to study it further. Boaler and Greeno's analysis of these student interviews revealed that the students' viewed themselves as having to give up agency and creativity if they were to become mathematical persons. Boaler and Greeno account for this finding by arguing that for these students, the identity that they would have to develop in order to become mathematical persons was in conflict with who the students viewed themselves to be and who they wanted to become. In terms of the interpretive scheme that we propose, the identity that the students would have to develop in order to become mathematical persons corresponds to the *normative identity* as a doer of mathematics established in their classrooms, whereas who the students viewed themselves to be and who they wanted to become corresponds to their *core identities*. This distinction is crucial to Boaler and Greeno's analysis in that they isolate the irreconcilable differences that the students experienced between their core identities and the normative identities established in their classrooms as the source of their alienation from mathematics.

The notion of identity also plays a central role in Martin's (2000) investigation of mathematically successful and failing African American students in an urban middle school. In the course of his analysis, Martin identified two distinct groups of students. In one group that he calls the dominant group, learning mathematics had a negative connotation and the students in this group promoted norms of underachievement that involved resisting mathematics instruction, frequently by being disruptive. In contrast, the students in the second group, most of whom were succeeding in mathematics, had high levels of confidence in their mathematical ability, viewed their teachers positively, and regarded achievement in mathematics as necessary to fulfill their long-term goals that involved careers in high status occupations. In terms of the interpretive scheme that we propose, these students' envisioned life trajectories are aspects of their core identities, of who the students viewed themselves to be and who they wanted to become. Martin's analysis demonstrates that it is both possible and useful to distinguish between students' core identities and the *personal identities* that they develop as they participate in (or resist) the activities of particular groups and communities, including those of the mathematics classroom. It is also apparent from Martin's analysis that personal identity is a relational construct and concerns the extent to which students have reconciled their core identities with the normative identity as doers of mathematics established in their classroom. The succeeding students that he studied subscribed to the achievement ideology and had reconciled this aspect of their core identities with the normative identities as doers of mathematics established in their classrooms by viewing mathematics achievement as a means of social and economic advancement. In contrast, the oppositional personal identities that the failing students were developing indicate that they experienced irreconcilable conflicts between their core identities and the normative classroom identities established in their classrooms. Like the boys that Willis (1977) studied, these students were active contributors to the processes that delimited their access to significant mathematical ideas.

Normative Identity

We have indicated that normative identity concerns the identity that students would have to develop in order to become mathematical persons in a particular classroom. In order to develop this sense of affiliation, a student would have to identify with the obligations that he or she would have to fulfill in order to be an effective and successful mathematics student in that classroom. Operationally, the process of analyzing the normative identity established in a classroom therefore involves documenting the obligations that the teacher and students interactively constitute and continually regenerate in the course of their ongoing classroom interactions. The obligations that proved relevant in a previously completed investigation (Cobb & Hodge, 2003b) include general norms for classroom participation as well as several sociomathematical norms that are specific to mathematical activity: (1) norms for what counts as an acceptable mathematical argumentation, (2) normative ways of reasoning with tools and written symbols, (3) norms for what counts as mathematical understanding, and relatedly, (4) the normative purpose for engaging in mathematical activity. It is important to note that these specifically mathematical norms collectively serve to specify what counts as mathematical competence in a particular classroom. The level of specificity that we propose when documenting the normative identity established in a classroom moves beyond characterizations of classrooms as traditional or reform in nature. It can therefore be viewed as a response to Boaler's (2002) call for investigations of classroom practices that promote equity to "pay attention to a level of detail in the enactment of [mathematics] teaching that has been lacking from many analyses" (p. 243).

Ideally, the data generated to document the normative identity established in a particular classroom should include classroom video-recordings. Elsewhere, we have described in some detail the types of evidence that we use to determine whether a particular norm has been established in a classroom (Cobb et al., 2001). For our current purposes, it suffices to clarify that the teacher and students *jointly constitute* both general and specifically mathematical norms. The process of delineating classroom norms therefore involves identifying patterns or regularities in the teacher's and students' ongoing interactions. Consequently, the conjectures that are substantiated or refuted in the course of an analysis apply not to individual students' actions but to patterns in collective activity and to students' obligations as they contribute to the regeneration of these patterns. We would therefore question accounts in which the teacher is portrayed as inviting students to adopt a normative identity as a doer of mathematics that exists independently of their classroom participation. Instead, in the perspective that we propose, students are seen to develop their personal identities in particular classrooms as they contribute to (or resist) the initial constitution and ongoing regeneration of the normative identity as a doer of mathematics.

Core Identity

Normative identity is concerned with the immediate social context of the classroom, whereas core identity is concerned with students' more enduring sense of who they are and who they want to become. We developed the notion of core identity by drawing directly on the work of Gee (2001, 2003). Gee observes that students each have a unique trajectory of participation in the activities of various groups and communities both in and out of school. As a consequence of this personal history of engagement, they have had a unique sequence of specific experiences of presenting themselves and being recognized in particular ways, some of which have recurred. "This trajectory and the person's narrativization ... of it are what constitute his or her (never fully formed and always potentially changing) 'core identity'" (Gee, 2001, p. 111). Two aspects of this definition make it particularly relevant to our purposes as mathematics educators. First, in emphasizing students' active role in developing their life stories, Gee acknowledges personal agency as well as the social structures inherent in the activities in which they participate. It is therefore conceivable that students with similar life histories might develop markedly different core identities at any particular point in time. Second, Gee emphasizes that students' development of new personal identities in particular settings can involve changes in their core identities. This is important and alerts us to the possibility that students' development of particular personal identities in specific classroom settings might, over time, influence their more enduring sense of who they are and who they want to become.

A primary consideration when documenting students' core identities in relation to schooling is to determine whether they have access to a structural rationale for learning in school and subscribe to the achievement ideology. Investigating this issue might involve using questionnaires or interviews that focus on a range of issues including (1) students' long-term aspirations, (2) their commitment to learning in school and in their mathematics classes, and (3) their assessments and explanations of other students' commitment to and perceptions of the benefits of succeeding in school and in their mathematics classes. As this proposal for data generation indicates, we question the common assumption that students' core identities can be equated with their membership of particular racial and ethnic groups (Gutierrez & Rogoff, 2003). Our intent in doing so is not to deny that a sense of affiliation with the common ancestry and cultural patterns of an ethnic group can be an important source of identity (Nasir & Saxe, 2003). Instead, it is to highlight students' personal agency in constructing multifaceted core identities while also acknowledging that their core identities are informed by who others say they are based on racial and ethnic group membership (Gee, 2003).

In taking this approach, we follow Martin (2000) in questioning Ogbu's (1992, 1999) influential thesis that children of historically oppressed groups become skeptical about their prospects for social advancement as they are socialized into a collective cultural identity. It is important to note that in developing his thesis, Ogbu adopted the view of culture as a way of life characteristic of a bounded community. This view is apparent in his contention that historically oppressed groups have developed a monolithic cultural identity in opposition to institutions such as schools that are equated with assimilation into dominant social groups. His thesis is sociostructurally deterministic in that it implies that children of marginalized groups will resist instruction regardless of the teacher's actions in order to maintain a sense of affiliation with their cultural group. However, as Martin (2000) notes, Ogbu's appeal to the family as the locus of socialization into cultural values does not adequately account for the manner in which successful African American students come to identify with academic achievement. In contrast to Ogbu's notion of a collective cultural identity, the notion of core identity that we have presented reflects the view that cultures are local and dynamic and are constructed by using broader Discourses as resources. This perspective capitalizes on Ogbu's crucial insight about the importance of sociostructural processes but also acknowledges personal agency and treats the classroom as the immediate social context in which sociostructural processes play out in face-to-face interaction. It is therefore a perspective that offers some hope to the instructional designer and the teacher by questioning the claim that interactions in mathematics classrooms necessarily have to unfold in a sociostructurally determined manner.

Personal Identity

While core identity is concerned with students' relatively enduring sense of who they are and who they want to become, personal identity is concerned with who students are becoming in particular mathematics classrooms. The goal in analyzing students' personal identities is to document the extent to which they have reconciled their core identities with participation in the ongoing regeneration of the normative identity as a doer of mathematics established in their classroom. Personal identity as we define it is therefore an ongoing process of being a particular kind of person in the local social world of the classroom. The data generated might include questionnaires, surveys, and interviews that focus on students' understandings of both their general and specifically mathematical obligations in the classroom, and on their valuations of those obligations. The intent in generating these data is to document (1) students' understandings of what counts as effectiveness and mathematical competence in their classrooms, and (2) whether and to what extent they identify with those forms of effectiveness and competence. The analysis of these data can therefore inform the interpretation of additional data that document students' assessments of their own and other students' mathematical competence in the classroom.

We can glean several distinctions in the types of personal identities that students might develop in their mathematics classes by synthesizing the available literature in mathematics education and related fields (Cobb & Hodge, 2003b). For example, students might reconcile their core identities with participation in the ongoing regeneration of the normative identity as doer of mathematics established in the classroom, thereby identifying with classroom mathematical activity (Cobb & Hodge, 2003b; Gutstein, 2002a, 2002b; Nasir & Hand, 2003). A second possibility is that students might reconcile their core identity with a broader goal for which succeeding in their mathematics classes is the means, such as going on to college and having a high-status career (Martin, 2000; Mehan et al., 1994; Nasir & Hand, 2003). In this case, striving to succeed in mathematics classes may not involve an experienced conflict, but neither does it involve identification with mathematical activity. A third possibility is that students might be unable to reconcile their core identity with the normative classroom identity but might nonetheless be willing to cooperate with the teacher in order to maintain relationships at home or with the teacher. In such cases, students experience an inner conflict or tension even as they strive to succeed in their mathematics classes, in the process becoming disenchanted with or alienated from mathematical activity (Boaler & Greeno, 2000; Cobb & Hodge, 2003b). A final possibility is that students might actively resist contributing to the establishment of the normative identity as a doer of mathematics, in the process developing oppositional classroom identities (Gutierrez, Rymes, & Larsen, 1995; Martin, 2000).

The first three of these four possibilities correspond to key distinctions made by self-determination theorists (Deci & Ryan, 2000; Grolnick, Deci, & Ryan, 1997; Ryan & Deci, 2000). Self-determination theory seeks to account for the inner adaptations that occur in the course of socialization such that children eventually accept and endorse the values and behaviors advocated by parents, experiencing them as their own. The three corresponding distinctions are: (1) *regulation through integration*, in which the value of the activity has been fully integrated with the person's core identity; (2) *regulation through identification*, in which the person sees the activity as instrumentally important for his or her own goals; and (3) *introjected regulation*, in which the source of regulation is internal but has not been integrated with the self and thus gives rise to tensions and inner conflicts. To these distinctions, we add *regulation through opposition* to take account of the fourth possibility, in which students develop oppositional classroom identities. We should acknowledge that there are significant theoretical difference between self-determination theory and the perspective on identity that we have presented. Self-determination theory focuses on core identity and accounts for its development in terms of the internalization of preestablished norms. We, in contrast, differentiate between core identity and the personal identities that people construct as they participate in the activities of particular groups and communities. In our view, people reconstruct their core identities as they attempt to reconcile who they are and who they want to be with participation in particular groups and communities. Despite these differences, the parallels between the forms of regulation identified by self-determination theorists and the types of personal classroom identities that we have discerned add credibility to the latter.

To conclude this discussion of the interpretive scheme, we note that most prior investigations of the personal identities that students are developing in mathematics classrooms have restricted their focus to general norms of participation and to the degree of openness of instructional tasks. Although these analyses open up new, potentially productive lines of inquiry, the constructs employed are not specific to mathematics and could be employed to analyze the learning environments established in science or in social studies classes. The resulting characterizations of classroom environments are therefore relatively global and provide only limited guidance for instructional design and the improvement of mathematics instruction. In contrast, the analytic scheme that we have outlined focuses on the extent to which students have reconciled their core identities with several specifically mathematical norms. The scheme is therefore designed to produce analyses of students' engagement (or the lack thereof) in the mathematics classrooms that take account of both their core identities and of critical features of the learning environments established in those classrooms. The relatively detailed, targeted nature of the analyses contributes to their potential to inform instructional designers' and teachers' efforts to support students' development of a sense of affiliation with classroom mathematical activity.

Conclusion

We have said little about instructional design in this chapter, as our primary focus has been on understanding who students are becoming in mathematics classrooms. We can clarify the general implication of the perspective we have developed for design by noting with Dewey that the process of identifying with an activity is synonymous with the development of what he termed a true interest in that activity. Dewey (1913/1975) explicated this relation between students' interests and their personal identities in particular settings by observing that "true interests are signs that some material, object, mode or skill (or whatever) is appreciated on the basis of what it actually does in carrying to fulfillment some mode of action with which a person has identified him [- or her]self" (p. 43). He also emphasized that motivation "expresses the extent to which the end foreseen is bound up with an activity with which the self is identified" (p. 60). In conceptualizing interests in this way, Dewey took an explicitly developmental perspective and repeatedly emphasized that the evolution of students' interests is a deeply cultural process. This viewpoint implies that the cultivation of students' interest in engaging in mathematical activity should be an explicit goal of instructional design. Elsewhere, we have followed diSessa (2001) in making an initial contribution to the development of a theory of this type (Cobb & Hodge, 2003a). A theory of this type would orient the development of designs, whereas the perspective that we have presented on identity guides analyses that inform the improvement of such designs.

In the first part of this chapter, we noted that research on equity has generally been marginalized within mainstream mathematics education research. The interpretive scheme we have outlined is the product of our efforts to make a concern for issues of equity integral to our ongoing research. Theoretically, it is premised on the view of broad Discourses as resources on which people draw to construct local, dynamic cultures. It therefore reflects a shift away from the more established view of culture as a way of life that is characteristic of a bounded community. Pragmatically, the interpretive scheme is premised on the assumption that supporting students' development of a sense of affiliation with classroom mathematical activity should be an explicit goal of instructional design and teaching. Although this proposal complicates the process of developing instructional designs, the potential pay off is substantial. In complementing the traditional focus on students' mathematical reasoning with a concern for who they are becoming in mathematics classrooms, we necessarily make an interest in issues of equity an integral aspect of mainstream research in mathematics education. In our view, this opportunity is too important to pass up.

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Part VI
The Institutional Setting of Mathematics Teaching and Learning

Chapter 12
Introduction

Paul Cobb with Chrystal Dean, Teruni Lamberg, Jana Visnovska, and Qing Zhao

The chapter reprinted in this part of the book outlines an analytical approach for documenting the school and district settings in which teachers develop and revise their instructional practices. The institutional setting of mathematics teaching as we conceptualize it encompasses district[1] and school policies for mathematics instruction. It therefore includes both the adoption of curriculum materials and guidelines for using those materials (e.g., pacing guides that specify a timeline for completing instructional units) (Ferrini-Mundy & Floden, 2007; Remillard, 2005; Stein & Kim, 2006). The institutional setting also includes the people to whom teachers are accountable and what they are held accountable for (e.g., expectations for the structure of lessons, the nature of students' engagement, as well as assessments of students' learning) (Elmore, 2004). In addition, the institutional setting includes supports that give teachers access to new tools and forms of knowledge together with incentives to take advantage of these supports (e.g., opportunities to participate in formal professional development activities and in informal professional networks, assistance from a school-based mathematics coach, or a principal who is an effective instructional leader) (Bryk & Schneider, 2002; Coburn, 2001; Cohen & Hill,

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[1] As background for non-US readers, we should clarify that each US state is divided into a number of independent school districts. In rural areas, districts might serve less than 1,000 students whereas a number of urban districts serve more than 100,000 students. Larger districts typically have a central office whose staff are responsible for selecting curricula and for providing teacher professional development in various subject matter areas including mathematics. In the US, the district is an important administrative unit whose policies can have a significant influence on teachers' instructional practices.

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2000; Horn, 2005; Nelson & Sassi, 2005). The findings of a substantial and growing number of studies document that teachers' instructional practices are partially constituted by the instructional materials and resources that they use in their classrooms, the institutional constraints that they attempt to satisfy, and the formal and informal sources of assistance on which they draw (Cobb, McClain, Lamberg, & Dean, 2003; Coburn, 2005; Spillane, 2005; Stein & Spillane, 2005).

The importance of taking account of the school and district setting when working with mathematics teachers to support their learning was first made clear to me during a teacher development project in which I participated in the late 1980s. In this project, Erna Yackel, Terry Wood, and I worked with a group of approximately 20 second-grade teachers to support their reorganization of their classroom practices.[2] One year after we began working with the second-grade teachers, several members of the newly elected school board responsible for the overall governance of the district questioned the changes that were occurring in how mathematics was being taught and learned. A 2-year struggle then ensued in the district that centered on the issue of who controlled the mathematics curriculum (see Dillon, 1993, for an account of these events). As a consequence of their collaboration with us, the teachers had become able to justify their new instructional practices in terms of the quality of their students' mathematical learning and therefore believed that they were more qualified than the school board members to make decisions about mathematics instruction (Simon, 1993). The teachers, supported by their principals, eventually prevailed and were given considerable autonomy in making curricular and instructional decisions in mathematics.

The role of the research team in this sequence of events was largely reactive and involved responding to a series of unanticipated crises that threatened the continuation of the project. The lesson that I drew from this experience for any future collaborations with teachers was that it would be crucial to attend to what transpires outside the teachers' classrooms as well as what occurrs within them by documenting the institutional settings in which they work (Cobb & McClain, 2001). I reasoned that in doing so, my colleagues and I would be able to anticipate potential tensions and conflicts, and could then adjust our plans accordingly before they escalated into full-scale crises. This would enable us to be proactive rather than merely reactive. In addition, I anticipated that analyses of the school and district settings in which the collaborating teachers worked might allow us to better understand the teachers' activity in both professional development sessions and their classrooms, thus enabling us to be more effective in supporting their learning.

Kay McClain and I developed an initial approach for documenting the institutional setting of teaching in the late 1990s as we prepared for what Simon (2000) termed a teacher development experiment[3] conducted with two groups of

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[2] These teachers worked in the rural/suburban district referred to in the introductions to several previous parts of this book.

[3] A teacher development experiment is a design experiment that aims to support and understand the learning of a group of teachers.

middle-school mathematics teachers who worked in two different urban districts.[4] We then tested and refined this approach as we worked with the teachers to support their learning. We initially planned to collaborate with one or more researchers in educational policy or educational leadership on this aspect of the experiment. However, it soon became apparent that researchers in these fields typically conduct observational studies in which they investigate others' efforts to support instructional improvement. As a consequence, we were not able to find a policy or leadership researcher who was willing to work with us to develop analyses that would actually inform a teacher development effort while it was still in progress. As one prominent leadership researcher put it, providing feedback about the institutional setting in order to inform work with teachers would involve "messing with the intervention." With some trepidation, I took the lead in attempting to develop a way of analyzing the institutional setting of mathematics teaching that was tailored to our purposes as mathematics educators. In doing so, I read extensively in both the policy and leadership literature, and in the sociocultural literature with the goal of identifying potentially relevant constructs. As the reprinted chapter makes clear, I found Wenger's (1998) book, *Communities of Practice*, to be especially helpful. An article by Ueno (2000) published in *Mind, Culture, and Activity* also proved to be an extremely valuable source of ideas. Both pieces have been criticized for being unnecessarily abstract (Wenger, personal communication, January, 2001). However, I found them to be relatively concrete as I looked at the schools in which the middle-school teachers worked through these texts, and found that both gave rise to insights that were immediately relevant to the task of support for the teachers' learning.

As we clarify in the reprinted chapter, the overall approach constitutes a general way of documenting the institutional settings in which specific groups of teachers work. The approach involves first identifying the groups of people within a school and district that are pursuing an agenda for how mathematics should be taught, and then documenting the interconnections between these groups. We illustrate the usefulness of this approach in the reprinted chapter by presenting a sample analysis of one of the districts in which the collaborating middle-school teachers worked. As our analysis indicates, this district was organized to support teachers' ongoing improvement of their instructional practices. One important aspect of this organization was the presence of teacher networks in which interactions focused directly on instructional practice. Crucially, the teachers' interactions typically went beyond the sharing of materials and involved both discerning the mathematical potential of instructional tasks and identifying the relative sophistication of student reasoning strategies. As a consequence, participation in these networks constituted a supportive context for teachers' learning.

The second significant aspect of the district organization concerned the development of a common vision of high-quality mathematics instruction across the district. Importantly, this vision was generally consistent with current reform

[4]This experiment was conducted by Paul Cobb, Kay McClain, Chrystal Dean, Teruni Lamberg, Melissa Gresalfi, Lori Tyler, Jana Visnovska, and Qing Zhao.

recommendations (e.g., National Council of Teachers of Mathematics, 2000) and with current research on learning in mathematics education and related fields. Research in educational leadership indicates that teacher networks are more likely to emerge and be sustained if the vision of high-quality mathematics instruction that they promote is consistent with the instructional vision of school leaders. This research also indicates that collegiality between teachers and school leaders is rarely effective unless it is tied to a common vision of high-quality instruction that gives their work purpose and direction (Elmore, Peterson, & McCarthey, 1996; Newman & Associates, 1996; Rosenholtz, 1985, 1989; Rowan, 1990). In the reprinted chapter, we describe the interconnections between various groups in the school district that made possible the ongoing regeneration of a common vision for mathematics instruction across teachers, school leaders, and district leaders even as they and indeed the district as an organization continued to learn. As our account indicates, the development of a common instructional vision is a significant accomplishment even for a relatively small school district given that teachers, school leaders, and district leaders have different responsibilities, engage in different forms of practice, and have different professional affiliations and identities. In the reprinted chapter, we also highlight the critical role of district leaders in framing the challenges posed by a state-mandated high-stakes accountability program[5] primarily in terms of improving instructional quality (and thus student learning) rather than gaming the student testing system[6] (cf. Confrey, Bell, & Carrejo, 2001). In my view, the case on which we focused in the reprinted chapter has important implications for the process of supporting the improvement of mathematics teaching and learning at scale.

Elmore's (2006) observation that most US schools and districts are clueless about how to respond productively to high-stakes accountability makes it clear that this district was atypical. In a companion paper (Cobb et al., 2003), we reported an analysis of the second urban district in which we collaborated with a group of middle-school teachers. This district was, like the first, located in a US state that had implemented a high-stakes accountability program. However, it proved to be much more typical of urban districts in the US. In contrast to leaders of the first

[5] As background for non-US readers, the US Congress passed a national policy called No Child Left Behind (NCLB) in 2001 with the overwhelming support of both Republicans and Democrats. The intent of NCLB is to enable all students to meet high performance standards in language arts and mathematics. The legislation provides financial incentives for States to design and enact the three central components of NCLB policy: content standards, tests aligned with the standards, and mechanisms for holding schools accountable for increasing test scores. The resulting state accountability policies constitute key aspects of the settings within which district and school leaders formulate local policies for mathematics instruction. The resulting local policies as they are actually enacted in schools in turn constitute key aspects of what we have termed the institutional setting of mathematics teaching.

[6] Heilig and Darling-Hammond (2008) document some of the strategies that districts use to increase test scores by exploiting loopholes in State accountability systems. As Cohen, Moffitt, and Goldin (2007) and Elmore (2004) observe, this gaming of the accountability system is to be expected when school and district leaders are held accountable of boosting test scores but do not know how to improve the quality of instruction.

district, leaders of this district viewed mathematics teaching as a relatively routine activity that does not require specialized knowledge or expertise. They framed accountability challenges in terms of raising test scores rather than improving the quality of instruction by supporting teachers' learning. School leaders, for their part, responded to the district leaders' focus on test scores by attempting to monitor and regulate teachers' instructional practices. In doing so, they focused on whether students were on task and whether instruction targeted objectives that would be assessed by the state test. In the companion paper, we also documented that the mathematics curriculum specialists in this district were attempting to support mathematics teachers' improvement of their instructional practices in line with the current reform recommendations. As a consequence, school leaders and mathematics specialists used conflicting criteria to assess the quality of instruction when they visited classrooms. Not surprisingly, the institutional setting as experienced by the teachers with whom we collaborated was characterized by tension and struggle.

The contrast between the institutional settings in which the two groups of middle-school teachers worked had a profound influence on our efforts to support their learning. For example, the teachers in the first district routinely observed each other's instruction and were already collaborating to improve their instructional practice when we began working with them. It therefore took only 3 months for the teacher group to become a full-fledged professional teaching community whose joint enterprise was to improve their instructional practices. In contrast, the frequent classroom observations that school leaders in the second district conducted to monitor and evaluate instruction had resulted in mathematics teaching becoming highly privatized (Cobb et al., 2003). It was not until we had worked with the teachers in this district for 19 months that teaching became fully deprivatized and the group became a professional teaching community that worked collectively on problems of practice (Dean, 2005). Our analysis of the institutional setting of teaching in this district proved to be an important means of supporting this development.

Interviews that we conducted with the teachers in this district to understand their perceptions of the settings in which they worked revealed that they viewed the school leaders solely as managerial or administrative leaders who gave instructional issues a low priority (Cobb et al., 2003). However, the interviews conducted with school leaders consistently indicated that they considered instructional leadership in mathematics to be an important part of their work. The reasons for this discrepancy in perspectives are beyond the scope of this introduction. We did, however, conclude at the time that it was important for the teachers and the mathematics leaders to appreciate that the school leaders' ineffectiveness as instructional leaders in mathematics was primarily a matter of competence rather than will. We conjectured that if they understood the school leaders' current view of high-quality mathematics instruction, they might be better positioned to influence them on issues central to mathematics teaching and learning. We therefore shared our analyses of the school leaders' practices with the teachers. In doing so, we described how the school leaders' view that effective mathematics teaching is a relatively routine activity influenced their approach to instructional leadership. This analysis constituted a primary point of reference during a series of activities that we subsequently conducted with the teachers. Dean's (2005) analysis of the learning of this teacher group during the first two years of our work in the district indicates that the teachers' developing understanding of how the institutional setting influenced their instructional practices made possible the deprivatization of instructional practice. This was in turn a crucial step in the group becoming a professional teaching community whose members collaborated to address problems of practice.

As I have indicated, I initially anticipated that analyses of the institutional setting would inform our plans for supporting the teachers' learning, and could contribute to our accounts of the teachers' activity in both professional development sessions and their classrooms. The realization that analyses of this type could also be a useful support for the learning of a teacher group therefore constituted significant learning on our part.

In the course of our collaboration with the two groups of middle-school teachers, we identified a further way in which documenting the institutional setting of teaching enabled us to be more effective in supporting the teachers' learning. Early in our work at both sites, we conducted individual interviews and group conversations with the teachers that focused on their perceptions of the school and districts settings. It soon became apparent that these interviews and conversations constituted excellent contexts in which to begin developing collaborative relationships with the teachers. Although our immediate purpose was to generate data, we necessarily attempted to understand some of the problems with which the teachers had to cope on a daily basis. This appeared to be a relatively novel experience for the teachers in the second district in particular, and seemed to indicate to them that we took their viewpoints seriously. Further, as we came to understand their concerns, we were better able to explicitly negotiate a joint agenda with them that reflected their priorities and concerns. Based on this experience, I now routinely initiate conversations of this type when I first begin working with a group of teachers.

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Chapter 13
The Collective Mediation of a High-Stakes Accountability Program: Communities and Networks of Practice

Paul Cobb and Kay McClain

This article describes an analytic approach for situating teachers' instructional practices within the institutional settings of the schools and school districts in which they work. The approach treats instructional leadership and teaching as distributed activities and involves first delineating the communities of practice within a school or district whose enterprises are concerned with teaching and learning and then analyzing three types of interconnections between them: boundary encounters, brokers, and boundary objects. We illustrate the analytic approach by focusing on one urban school district in which we have conducted an ongoing collaboration with a group of middle school teachers. In doing so, we clarify the critical role that school and district-level leaders can play in mediating state and federal high-stakes accountability policies. We conclude by discussing the implications of the analysis for the process of upscaling and the diffusion of instructional innovations.

Our purpose in this article is twofold.[1] The first is to describe an analytic approach for situating teachers' instructional practices in the institutional settings of the schools and districts within which they work. The approach treats instructional leadership and teaching as distributed activities and involves delineating the communities of practice within a school or district whose enterprises are concerned with teaching and learning. Our second purpose is more pragmatic and involves demonstrating the critical role that school and district-level leaders can play in mediating state and federal high-stakes accountability policies. To address these two purposes, we illustrate the analytic approach by focusing on one urban school district in which we have collaborated with a group of middle school mathematics teachers for the past 4 years. The district is of interest because school and district leaders have

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responded to a state-mandated accountability program not by attempting to regulate teachers' instructional practices but by giving teachers access to material resources (e.g., instructional materials, joint planning time, release time, outside consultants) and by supporting their development of social and personal resources. These social resources encompass collaborative relationships in which teachers jointly address issues that emerge from their instructional practices. The concomitant personal resources include the teachers' conceptualizations of particular content domains, their understanding of the development of students' reasoning in these domains, and the possibilities they see in their students' solutions and explanations. This case is significant because it is sometimes assumed that high-stakes accountability policies necessarily delimit opportunities for teachers to develop instructional practices that focus on supporting the development of conceptual understandings of significant ideas.

In illustrating the analytic approach, we focus both on the leadership practices of school and district leaders and on teachers' instructional practices. As will become apparent when we document the interconnections between the various communities of practice to which they belong, teachers and leaders constitute significant aspects of the environment for each other (see McDermott, 1976). The members of each community therefore afford and constrain the practices developed by members of other communities. It is in this sense that we speak of the practices of each community being partially constituted by the institutional setting in which its members act and interact. In focusing on the communities of practice in which the functions of teaching and instructional leadership are actually accomplished, we are primarily concerned with what Engeström (1998) described as a middle level between the formal structures of schools on the one hand and the content and methods of instruction on the other.

> The middle level consists of relatively inconspicuous, recurrent, and taken-for-granted aspects of school life. These include grading and testing practices, patterning and punctuation of time, uses (not contents) of textbooks, bounding and uses of the physical space, grouping of students, patterns of discipline and control, connections to the world outside the school, and interactions among teachers as well as between teachers and parents [and administrators]. (p. 76)

Engeström characterized these middle-level features as sense- and identity-building processes and argued that they largely determine the sense of schoolwork and thus the experience of what it means to be an instructional leader, teacher, or student within the institutional setting of a particular school and district. This orientation steers us away from a structural perspective on the school as an institution and toward a focus on leaders' and teachers' activities as they participate in what he terms the taken-for-granted aspects of school life. In adopting this latter orientation, our focus is on schools and school districts viewed as lived organizations rather than as formal structural systems that have been abstracted from the activities of the persons who constitute them.

In the first part of the article, we provide an overview of the district and our collaboration with the middle school mathematics teachers. We then discuss the methodology that we used for analyzing a school or district as a configuration of communities of practice. Against this background, we present the results of an analysis of both the relevant communities of practice in the collaborating district and the interconnections between them. In describing the critical role of school and district leaders in mediating a state-mandated accountability program, we document in some detail that their leadership practices were not solo accomplishments but were instead partially constituted by the institutional setting in which leaders developed and refined such practices. In the final section of the article, we broaden our purview beyond the district by considering the prospects for diffusing and upscaling the innovative leadership and instructional practices that we document. In doing so, we introduce the notion of networks of practice that have considerable spatial reach and that link groups of teachers and leaders in numerous schools and districts whose enterprises and practices are broadly compatible. A defining feature of such a network is that innovations developed in one community of practice can diffuse rapidly and be assimilated readily by members of other communities within a network. As we clarify, analyses of the topology of networks and communities of practice can provide guidance for efforts at improvement that aim to transform rather than merely augment currently institutionalized instructional and leadership practices.

Background to the District and to Our Collaboration

The district in question, which we call Washington Park, is located in a large city in the southwest United States and serves a 42% minority student population with 46% of the students on free or reduced-cost lunches and 36% of the students receiving special services. The district's seven schools serve Grades K–8. Students in Grades 6–8 attend three of these schools. Two of the schools span Grades 5–8, and the third spans Grades 3–8.[2] The remaining four schools each serve a configuration of Grades K–4. There is considerable variation in the composition of the student population across the three middle schools. In addition, there is high turnover in the student enrollment, even within an academic year. As an example, the student turnover rate at one of the middle schools was 29% during the 2002–2003 academic year, and the English-language learner population doubled during a 2-week period.

A state-mandated testing program was in place when we began our collaborations. In this program, students are tested in mathematics, reading, and language arts at each grade level on a nationally norm-referenced test. The results of these annual assessments are disseminated widely in the local media, and school and district leaders are held accountable for student performance. An effort to improve mathematics and language arts instruction was underway before our collaboration. With respect to mathematics, the district had, for example, adopted a National Science Foundation (NSF) funded middle school curriculum compatible with current reform recommendations—for example, the National Council of Teachers of

[2] Although the term *middle school* is not entirely accurate in all three settings, we use it nonetheless in the remainder of the article to designate the three schools that housed Grades 6–8.

Mathematics (1999) document *Professional Standards for Teaching Mathematics*—and had received an NSF implementation grant. In addition, the district routinely hired university mathematics educators, many with national reputations, as consultants to conduct professional development sessions with teachers. Similar efforts had been conducted in language arts.

Scores on the norm-referenced state-mandated test have remained consistently high because the district has undertaken these efforts to reform mathematics and language arts instruction. Further, there has been improvement in students' mathematics scores in Grades 6, 7, and 8 over the 4 years of our collaboration with teachers in the district, even in the context of high student turnover and rapidly changing demographics. Teachers and administrators in the district almost uniformly attributed their success in mathematics as assessed by the norm-referenced state-mandated test to the implementation of the NSF curriculum. Comments such as "If we just teach the curriculum, the test will take care of itself" reflect teachers' and administrators' confidence in these instructional materials. However, teachers and administrators also acknowledged that it is "difficult for teachers to really do [the curriculum] well." One teacher described the process of learning to use the materials effectively as follows:

> It is not like computation, you know, you have the formulas and computation here and go. And so this takes a lot of work and a lot of thinking. . . . And so, it's an extremely difficult program to teach. So for me, the first two years were literally just keep your head above water and learn something about it. Make sure you got through day by day. Then at the third year I felt I started feeling a little more comfortable knowing what is coming, what were some problems the kids ran into, and that I could fill in a little bit better so that they didn't have those problems.

Because of the perceived challenges of implementation, school and district leaders invested considerable material resources (e.g., both time and money) in professional development activities so that teachers might learn to use the instructional materials as the developers intended.

Our initial classroom observations revealed that most of the collaborating mathematics teachers' instructional practices exhibited fidelity to the materials when we began working with them. In particular, their instruction was guided by the investigations around which the curricular materials are organized, and their sequencing of activities mirrored that advocated by the developers. As an example, the text resources typically engage students in an exploratory activity that is intended to provide them with an opportunity to generalize a process or strategy, such as finding a missing quantity through the use of proportions. In these instances, the teachers carefully guided the students' explorations and then worked to support the emergence of the mathematical generalization in a subsequent discussion. The regularities that we identified in the teachers' instructional practices therefore centered on the manner in which they guided the students through the investigations and activities. Variations in their instructional practices became apparent when we analyzed the ensuing whole-class discussions.

The purpose of whole-class discussions for some teachers was to teach a particular solution method, whereas, for others, the intent was to enable their students to share what they had learned during the investigation. In the first of these two approaches, the teachers seemed to view the prior investigation as a pretext for their introduction of a predetermined solution method, and, as a consequence, whole-class discussions were somewhat disconnected from the prior investigations. In the second approach, the teachers expected that their students would develop relevant mathematical understandings as they engaged in the investigation and, as a consequence, did not intervene to support their learning during whole-class discussions. As an example, in one instance students analyzed graphs to investigate rate of change in a graph as a precursor to understanding slope. Teachers who focused on a predetermined solution method introduced the formula for slope without attempting to relate it to students' activity during the prior investigation. Teachers who focused on students' discoveries assumed that their students had deepened their understanding of slope as they engaged in the prior investigation.

It was apparent from our initial observations that most of the teachers assessed their students' reasoning in terms of their completion of instructional activities and their contributions to whole-class discussions. However, the adjustments that they made when they judged that a significant proportion of their students did not understand typically involved either explaining the process for a second time or asking their students to engage in a second, similar investigation. In doing so, most of the teachers did not seem to view their students' interpretations and solutions as resources on which they could build. Instead, they took an implicit deficit view of their students by using the instructional goal of an investigation or activity as a norm against which to assess their performance. One teacher's explanation of her approach of repeating an instructional activity when she judged that students did not understand as she had hoped is representative in this regard:

> And see, that's kind of like this program is because I do what [the developers] call a launch, like an introduction, and then the kids start working. So I kind of like clue them in on what's coming, this is what you're going to learn, this is what I hope you figure out. I just kind of give them, I know they don't completely understand, so then when they start working with it, then as I walk around, if I notice that there's too many people, sometimes I regroup, and I kind of, teach the whole class again, and go, everybody's asking me how to do number two, so everybody stop, if you understand number two, you can keep working quietly, but everybody else is, and do that a lot, and then, if I notice that nobody's got it, I'll do it again the next day.

At the time that we began working with them, the teachers' efforts to use the curriculum materials as intended were supported by a variety of informal professional networks both within and across schools. Participation in these networks involved conversations that focused on instructional issues and frequent visits to each other's classrooms. At the beginning of our collaboration with them, the teachers all indicated that they viewed their colleagues as resources on whom they could draw when questions or issues arose in the course of their instruction. Mary Jean's response to the question of how she resolved problems that arose in her classroom is typical in this respect:

> Mary Jean: . . .that is something that I've never been afraid to say, wait a minute, something is rotten here, I'm not understanding it. I'll find about it, and then I would ask. . . . and that's when I go hunt Pamela down real quick, and find another example, another idea.

Comments by Joan, Beth, and Julian indicate a similar reliance on colleagues.

> Joan: I think I'll probably always go to [Pamela] because I know she's taught the program, she's the more experienced person. So at the beginning of the year, I was asking her about everything, about assemblies, how do I get the kids to lunch, so she was there for that. Now I just go to her for the math stuff. . . . The questions the kids ask me, then I'm out of time, I'm going, you guys I really don't know, I'll get back to you tomorrow, then I find Pamela, go find Pamela.

> Beth: No, I'll go talk, usually, I will go to [another teacher] first cause she has been here long as I have. We've taught in the program the same amount of years. I'll go and say, you know, how did it go with you, what's going on. And usually her and I will figure something out, and if it's still a big issue, then we bring it to the math meetings that we have once a month.

> Julian: I talk to other sixth-grade math teachers. Other fifth-grade math teachers. I talk to Joseph about it. He's a wealth of information. I go to him. Mari is over at [another school] and I know Mari so I go to Mari. You know, how did you do it? Does this sound appropriate?

It is apparent from these and other teachers' comments that they drew on an array of social resources as a routine part of their practice. Our intent in collaborating with a group of teachers in the district has been to initiate and support the development of a professional teaching community by capitalizing on these resources. To this end, during each of the 4 years of the collaboration, we have conducted a summer work session and half-day work sessions each month during the school year. At the outset, we worked with 14 mathematics teachers representing all three middle schools, and 15 teachers are currently involved in the collaboration. Our pragmatic goal in working with the teachers has been to support their eventual development of instructional practices in which they place students' reasoning at the center of their instructional decision making. In the envisioned forms of instructional practice to which the collaboration aims, students' interpretations and solutions are viewed as resources on which the teachers can capitalize to achieve their instructional agenda. Instructional materials would then serve not as blueprints for instruction but as resources that teachers adapt to the context of their classroom as informed by conjectures about both students' reasoning and the means of supporting its development.[3] Furthermore, implementation would become a process of conjecture-driven adaptation rather than one of fidelity of reproduction. However, the complex and demanding nature of instructional practices of this type indicate the importance of social resources, such as those on which we planned to build when supporting the development of a professional teaching community

[3] We construe these means of support broadly so that they include the nature of classroom discourse and the classroom activity structure as well as instructional materials and associated tools.

(Gamoran et al., 2003). When situated within such a community, the process of instructional improvement then becomes a collaborative problem-solving activity in which teachers generate knowledge about students' mathematical reasoning and the process of supporting its development (Franke, Carpenter, Levi, & Fennema, 2001).

This overview of the district, the teachers' initial instructional practices, and our proposed collaboration with them serves to describe the setting in which we pursued our primary research goal of investigating conjectures about the means of supporting and sustaining the development of professional teaching communities.[4] To achieve this goal, we conducted a design experiment (cf. Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) in which we tested and revised conjectures about both a learning trajectory for a professional teaching community and the specific means that might be used to support that learning. A detailed discussion of the conjectures that informed our initial plans for working with the collaborating teachers can be found in Cobb and McClain (2001). The analyses that we conducted to inform the ongoing revision of these initial conjectures track the evolution of the activities of the professional teaching community and changes in the participating teachers' instructional practices.

The data we generated to document the learning of the professional teaching community throughout our collaboration with the teachers include semistructured interviews conducted with the teachers each year, video recordings of all work sessions, and copies of all material artifacts produced by the teachers.[5] To document the teachers' instructional practices, we generated modified teaching sets (Simon &Tzur, 1999) each year for each teacher. A modified teaching set consists of classroom observations followed by an audio-recorded semistructured interview with the teacher that focused on instructional planning and on reflections of lessons. Analyses of these data indicate that we had some success in supporting the learning of the professional teaching community and the participating teachers. Within the professional teaching community, we were able to document growth not only in the teachers' mathematical understandings (McClain, 2003) but also in their understanding of students' reasoning as a resource on which they could capitalize to achieve their instructional agendas. In particular, in the course of discussions within the professional teaching community, the teachers analyzed student work samples to delineate distinct types of mathematical reasoning and focused on how

[4]We are in fact investigating these conjectures by collaborating with groups of teachers in two contrasting urban districts. A description of the second district can be found in Cobb, McClain, Lamberg, and Dean (2003).

[5]The process of documenting the learning of a professional teaching community involved identifying the successive norms that became established for general participation, mathematical reasoning, pedagogical reasoning, and strategic norms (i.e., the ways of understanding the institutional setting for mathematics teaching that have become normative within the professional teaching community). A discussion of the criteria that need to be satisfied when identifying communal norms can be found in Cobb, Stephan, McClain, and Gravemeijer (2001).

they can build on their students' mathematical thinking during whole-class discussions. Thus, rather than introduce a predetermined method or expect students to develop relatively sophisticated mathematical understandings without support, some of the teachers assumed a proactive role in which they use their students' reasoning as a resource.

The Institutional Setting in Which the Teachers Revised Their Instructional Practices

The analyses that we have described to this point are internal to the professional teaching community. However, a number of investigations have documented that teachers' instructional practices are profoundly influenced by the institutional constraints that they attempt to satisfy, the formal and informal sources of assistance on which they draw, and the materials and resources that they use in their classroom practice (Ball, 1996; Brown, Stein, & Forman, 1996; Feiman-Nemser & Remillard, 1996; Nelson, 1999; Senger, 1999; Stein & Brown, 1997). The findings of these studies indicate the need to take account of the institutional setting in which the collaborating teachers developed and refined their instructional practices. It is only when we do so that we can adequately explain both our success in supporting the teachers' development of increasingly sophisticated instructional practices and the district's success as assessed by student performance on state-mandated achievement tests. We have therefore complemented our focus on the activities of the professional teaching community and the teachers' instructional practices with analyses of the institutional setting in which the collaborating teachers work.

The approach that we have taken when conducting these analyses involves identifying the communities of practice within a school or district whose missions or enterprises are concerned with the teaching and learning of mathematics. We build from Wenger (1998) by using his three interrelated dimensions that serve to characterize a community of practice: a joint enterprise, mutual relationships, and a well-honed repertoire of ways of reasoning with tools and artifacts. In the following, we clarify each dimension of the professional teaching community in Washington Park by example.

A Joint Enterprise

In the case of the professional teaching community, the joint enterprise was to ensure that students come to understand central mathematical ideas while performing more than adequately on high-stakes assessments of mathematics achievement. This entailed the teachers' developing a relatively deep understanding of the mathematical intent of instructional activities so that they could achieve their instructional agendas by capitalizing on students' reasoning.

Mutual Relationships

In the case of the professional teaching community, these relationships encompass general norms of participation as well as norms that are specific to mathematics teaching, such as norms of mathematical reasoning and the standards to which the members of the community hold each other accountable when they justify pedagogical decisions and judgments. As an illustration, when sharing the results of instructional activities, the teachers routinely challenged arguments that take the goal of instruction as a normative point of reference and characterized students' reasoning in terms of deficits. In doing so, the teachers held each other accountable to the norm of attempting to tease out differences in students' current capabilities.

A Well-Honed Repertoire of Ways of Reasoning with Tools and Artifacts

In the case of a professional teaching community, this repertoire includes (a) normative ways of reasoning with instructional materials and other resources when planning and organizing for mathematics instruction and (b) normative ways of using instructional activities and other resources to make students' mathematical reasoning visible. The normative ways of reasoning with instructional materials that have emerged within the professional teaching community encompass both the mathematical domain that is the focus of instruction and the diverse ways that students might approach and solve instructional activities. These norms became apparent in discussions as the teachers worked together to plan instructional activities that they would use in their classrooms. In these conversations, the teachers who characterized instructional goals solely in terms of processes that students were to learn for producing answers were typically challenged for failing to explicate the underlying mathematical ideas. Furthermore, in the course of these conversations, the teachers typically envisioned the nonstandard approaches that students might take.

Methodologically, we used what Spillane (2000) referred to as a *snowballing strategy* and Talbert and McLaughlin (1999) termed a *bottom-up strategy* to delineate the communities of practice within the Washington Park district whose missions or enterprises were concerned with the teaching and learning of mathematics. The first step in this process involved conducting audio-recorded semistructured interviews with the collaborating teachers to identify people within the district who influenced their classroom instructional practices in some significant way. The issues addressed in these interviews included the professional development activities in which the teachers have participated, their understanding of the district's policies for mathematics instruction, the people to whom they are accountable, their informal professional networks, and the official sources of assistance on which they can draw. To corroborate these interview data, we also administered a survey that addressed these same issues to all the mathematics teachers in the Washington Park district who taught Grades 6–8. The second step in this bottom-up, or snowballing, process involved interviewing the people identified in the teacher interviews and surveys to understand their agendas as they related to mathematics instruction and the means by which they attempted to achieve those agendas. We then continued this process as we identified additional people in this second round of interviews who actively attempted to influence how mathematics is taught in the district.

The communities of practice that we identified in addition to the professional teaching community as we analyzed these data were the districtwide mathematics leadership community and the school leadership communities in the three schools in which the teachers work.

The Mathematics Leadership Community

The core members of the mathematics leadership community were the mathematics teacher leaders in each of the three middle schools who received 50% release time from teaching to lead the district's instructional improvement effort in mathematics. A number of teachers were also members of this community but had peripheral roles. The mathematics teacher leaders were, for their part, full members of the professional teaching community and participated in all sessions.

In addition to the semistructured interviews conducted with the core members, the data generated to document the activities of the mathematics leadership community include a series of follow-up interviews, scheduled monthly meetings, frequent informal discussions, and an ongoing e-mail exchange as well as observations of professional development sessions that the mathematics teacher leaders conducted in the district. These data consistently indicate that the mathematics teacher leaders viewed themselves as members of a broader community of mathematics education reformers and had a relatively deep understanding of and a commitment to the general intent of proposals for mathematics teaching and learning. For example, they attempted to organize mathematics instruction around central mathematical ideas and viewed mathematical communication not merely as a possible instructional strategy but as an important instructional goal in its own right.

The analysis of the data that we generated consistently indicates that the joint enterprise of this community was to improve the mathematics understanding of all students by assisting teachers in developing a relatively deep understanding of both the mathematical ideas addressed in the NSF textbook series and the ways in which students' reasoning might evolve as they complete instructional activities. As part of the process of supporting teachers' learning, the mathematics leaders had developed a district curriculum guide that correlated the NSF text resources with the state standards and provided a pacing guide to ensure coverage of the text resources. The tools with which the members of the mathematics leadership community reasoned as they organized for mathematics teaching and learning therefore included the state standards, the district curriculum guide, the NSF instructional materials, the pacing guide, and samples of students' work that served to document their mathematical reasoning. In contrast to this array of tools, the mathematics teacher leaders all indicated that they made students' learning visible by relying almost exclusively on scores on the state-mandated test.

As part of the process of organizing for mathematics instruction, the mathematics teacher leaders conducted biweekly meetings with the teachers at a particular grade level in each school. Although the mathematics teacher leaders gave priority to the implementation of the curriculum and adherence to the state standards in these grade-level meetings, their larger goal was to support the teachers' development of instructional practices that would support students' development of mathematical understanding as reflected on state test scores. To achieve this goal, the mathematics teacher leaders focused on the teachers' understanding of the mathematical intent of instructional activities when they addressed implementation issues. To this end, they and the teachers worked together to complete instructional activities and examined student work on these and similar activities.

One classroom teacher described this emphasis in the grade-level meetings as follows:

> I would call it a grade-level learning group. It's a grade level math meeting where you go in and you usually pick the topic at the prior meeting [based] on where you'll be. That's where you go in and really look at what are you studying, how close your students are getting. We take a bit of time doing that, then we may tear apart the [mathematics] book. We may sit and look at a fraction book, we have two fraction books. We may say, this is really redundant, these are the same lessons, let's do one and take the other out for expediency's sake. Or we may say, you know, this is really crucial so we need more lessons. I know people are reading this and that, but look at this lesson over here and how relates to it. And this one and this one, and this one. We may bring in articles that we found were valuable. Or we may say, you know what, I have no idea what the point is, I have no idea what the form or the function is. We can sit and discuss how important that is, and how it works.

To make mathematics learning visible, the mathematics teacher leaders (MTLs) spent time analyzing standardized test scores, typically in collaboration with the leadership community in each school. Their primary purposes in doing so were to monitor achievement levels in each school at each grade and to identify potential weaknesses in the curriculum strands both within and across grades. The MTLs then typically analyzed the fit between potential areas of weakness and the curriculum to investigate whether there had been "adequate coverage" during the year. It was apparent from these analyses that the MTLs viewed the state test scores as an assessment of not only the students but also the curriculum and the teachers' ability to implement it effectively. For this reason, their first step when investigating a drop in scores was to determine which parts of the curriculum had been implemented the previous year. In conducting these analyses, the MTLs assumed that fidelity to the curriculum correlated strongly with high test scores.

School Leadership Communities

The school leadership communities in each of the three schools in which the collaborating teachers work consist of the principal and the assistant principal. In addition, the mathematics teacher leader and one or more teachers in each school were peripheral members. We relied on semistructured interviews conducted with the school leaders to document the activities of these communities, and we triangulated these interviews with the collaborating teachers' descriptions of the settings of their work. These data document that the joint enterprise of each of the school leadership communities is to support mathematics teachers' efforts to improve the quality of mathematics teaching and learning in the district while remaining vigilant about student test scores on state-mandated tests. The interviews indicate that the school leaders, like the mathematics teacher leaders, viewed fidelity to the curriculum as evidence of effective instructional practice. The school leaders at each school characterized an effective classroom as one in which

- There is more than one way to solve a problem.
- There are ample opportunities for students to explain their thinking.
- There is enthusiastic interaction among the students around mathematical ideas.
- Teachers are using formative assessment to plan instruction.

They pursued their agenda for mathematics teaching and learning by providing resources (e.g., texts, materials, release time, attendance at professional meetings), arranging schedules to facilitate collaboration, and modifying observation forms so that they supported reflection rather than assessment.

The primary tools that members of the school leadership communities used as they organized for mathematics teaching and learning were the state standards and the NSF curriculum. These tools were frequently the focus of discussion in regularly scheduled meetings between the leaders in each school and the mathematics teacher leaders and during the classroom observations that the school leaders conducted. The focus in these settings was on fidelity to the curriculum, on the assumption that this entailed alignment with state standards. One principal confirmed this sentiment when he described his job by noting, "If you teach the curriculum, then the test scores will go up. My job is to make sure they teach the curriculum."

It is important to note that the school leaders had a relatively deep understanding of the general intent of current reform proposals in mathematics education. This reflected their engagement in activities conducted as part of an implementation grant funded by the NSF. Their participation included numerous professional development seminars as well as 120 contact hours of mathematics content courses during the past 4 years. In this process, the school leaders had experienced instruction consistent with the vision articulated in National Council of Teachers of Mathematics reform documents. Furthermore, they had come to see these competencies as being crucial to their role as instructional leaders in their schools. For example, the school leaders devoted a portion of their districtwide biweekly meetings to mathematics. In these settings, they completed an instructional activity from the NSF curriculum to develop their own mathematical reasoning and to appreciate the mathematical intent of the curriculum. These experiences supported their belief that fidelity to the curriculum was the primary means of improving student learning as indicated by test scores.

To make mathematics teaching and learning visible, the members of the school leadership communities analyzed state test scores in the context of districtwide administrative meetings and at the school level with members of the mathematics leadership community. As they sought explanations for the scores, the school leaders acknowledged and relied on the expertise of the MTLs. As an example, school leaders typically brought drops in test scores to the attention of the MTLs but left the analysis of possible causes to them. The development of plans to address these deficiencies was also the purview of the MTL. The school leaders, for their part, made resources available to the MTLs so that they could pursue courses of action with respect to the scores.

It is apparent from these descriptions of the mathematics leadership and school leadership communities that the visions they attempted to realize for mathematics teaching and learning were broadly compatible. For the mathematics teacher leaders and the school leaders, mathematics teaching is a complex and demanding activity that requires a deep understanding of students' mathematical reasoning and the mathematical ideas that are the focus of instruction. Furthermore, the mathematics teacher leaders and the school leaders conceptualized instructional goals in terms of mathematical ideas and pursued agendas that were not limited to instructional methods or strategies but also encompassed the nature of students' engagement in classroom activities and the forms of mathematical reasoning that they were developing. Although the school leaders in particular were attentive to student scores on state-mandated tests, it is significant that they and the mathematics teacher leaders participated primarily in the discourse of reform in mathematics education rather than the discourse of high-stakes testing and accountability (see Confrey, Bell, & Carrejo, 2001).

It should also be apparent from the account that we have given of the various communities in the Washington Park district that teachers developed and refined their instructional practices in an institutional setting in which they were consistently supported to implement the NSF curriculum with fidelity. We saw, for example, that teachers' participation in grade-level meetings conducted by mathematics teacher leaders supported their development of relatively deep understandings of central mathematical ideas. The consistently supportive nature of the institutional setting was particularly evident when aspects of a teacher's instructional practice were perceived to be problematic. In such cases, the teacher and a school leader (e.g., the assistant principal or the principal) jointly constructed an improvement plan. The teachers often talked openly about these plans in the professional teaching community and solicited advice about how they might best approach the identified problems they were experiencing in their classrooms. This deprivatization of instructional practices was made possible by and contributed to an institutional setting in which teachers viewed the mathematics teacher leaders, school leaders, and each other as resources for their learning.

Interconnections Between Communities of Practice

To this point, we have documented that the practices of the professional teaching community, mathematics leadership community, and school leadership communities were in broad alignment. However, we have not explained how this alignment was sustained or how the practices of the mathematics and school leaders related to and influenced teachers' instructional practices. To address these issues, we have to take the analysis one step further by delineating the interconnections between the various communities that we have identified. In doing so, we distinguish between three types of interconnections: boundary encounters, brokers, and boundary objects.

Boundary Encounters

The first type of interconnection arises when teachers' or leaders' routine participation in the practices of their community involves boundary encounters in which they engage in activities with members of another community. As an illustration, boundary encounters occurred in the Washington Park district when mathematics leaders and school leaders conducted classroom observations. Additional boundary encounters included the grade-level meetings that the mathematics teacher leaders conducted with teachers and the regularly scheduled meetings between the school leaders and the mathematics teacher leader in each school. The mathematics teacher leaders' institutionalized role as authorities with expertise in the teaching and learning of mathematics was readily apparent in these two series of meetings.

Brokers

The second type of interconnection that we documented when analyzing the institutional settings in which the collaborating teachers developed and revised their instructional practices concerns the activities of brokers who were at least peripheral members of two or more communities of practice. Brokers can bridge between the activities of different communities by facilitating the translation, coordination, and alignment of perspectives and meanings (Wenger, 1998). Their role can therefore be important in developing alignment between the enterprises of different communities of practice. In the Washington Park district, the mathematics teacher leaders were the most visible brokers. As we have noted, they were not only members of the mathematics leadership community but were also core members of the professional teaching community and peripheral members of the school leadership community. In this pivotal role as brokers between their own and the other communities, the mathematics teacher leaders had at least partial access to the practices of the professional teaching community and the school leadership community. This in turn enabled them to provide the school leaders and teachers with access to the practices of each other's communities. One important consequence of their activity as brokers was that they could therefore provide teachers with a voice in the school leadership communities.

Boundary Objects

The third type of interconnection between the communities of practice involves the use of a common boundary object by members of two or more communities as a routine part of their activities. In the Washington Park district, boundary objects include the curriculum materials, the state standards, and reports of students' test scores. As Wenger (1998) noted, boundary objects are based on what he termed *reification* rather than participation.[6] Wenger defined reification as "the process of giving form to our experience by producing objects that congeal this experience into 'thingness'" (p. 58). He argued that in creating reifications, "we project our meanings into the world and then we perceive them as existing in the world, as having a reality of their own" (p. 58). However, as he went on to emphasize, reifications cannot capture the richness of lived experience precisely because they are frozen into a concrete form such as a text. As a consequence, although a reifying object is a relatively transparent carrier of meaning for members of the community in which it was created, there is the very real possibility that these objects will be used differently and come to have different meanings when they are incorporated into the practices of other communities. Even when this occurs, common boundary objects that are used differently in different communities can nonetheless enable the members of these communities to coordinate their activities. Consequently, as Star and Griesemer (1989) demonstrated, successful coordination does not require that members of different communities achieve consensus. Boundary objects do not therefore carry meanings across boundaries but instead constitute focal points around which interconnections between communities emerge.

In our experience, the role of boundary objects is typically far less visible to leaders and teachers than are interconnections that involve boundary encounters and brokers. Their inclusion in an analysis of the institutional setting in which a group of teachers have developed their instructional practices is therefore crucial if we are to document the inconspicuous, recurrent, and taken-for-granted aspects of school life. In the case of the Washington Park district, the curriculum materials and the state standards were constituted as boundary objects between all three communities as their members organized for mathematics instruction. However, only reports of students' scores on the state-mandated test were constituted as boundary objects as

[6]Reification as Wenger (1998) defined it should not be confused with Sfard's use (1991, 1994) of this same term. For Sfard, reification is the process by which mathematical objects are constructed from operational mathematical processes. Wenger's use of the term is less technical and refers to the process by which members of a community create objects that, for them, carry particular practice-based meanings. As he made clear, the process of reification complements participation in the sense that mutual engagement typically involves the use of artifacts that are the product of prior reifications.

the members of the various communities made mathematics teaching and learning visible.

Given Star and Griesemer's observation (1989) that boundary objects do not carry meanings across boundaries, the compatibility in the ways that members of the various communities used the state standards, curriculum, and test scores needs to be explained. Two observations are relevant in this regard. First, boundary encounters, particularly those in which the mathematics teacher leaders acted as brokers, constituted contexts in which the members of the different communities could explicitly negotiate how they used the state standards, curriculum materials, and students' test scores. For example, the mathematics teacher leaders had access to the ways in which the school leaders used instructional materials and could give the school leaders access to the ways in which they and the teachers used these materials. As a consequence, differences in uses of this and other boundary objects could become an explicit topic of conversation in the meetings between the school leaders and mathematics teacher leaders. Similar comments apply to the mathematics teacher leaders' participation in the professional teaching community and to grade-level meetings that they conducted with the teachers in each school. In each of these cases, the boundary objects supported brokering and the bridging of perspectives, thereby contributing to the alignment of the enterprises of the various communities. More generally, Wenger (1998) noted that mutual engagement and reification offer two complementary ways of attempting to shape the future and that one is rarely effective without the other.

Our second observation concerns the practices in which the various boundary objects originated within the district. For example, the state standards and the reports of test scores were primarily grounded in the practices of school leaders as they monitored students' performance on the state-mandated test. The constitution of the state standards and test scores as boundary objects enabled teachers and mathematics teacher leaders to contribute to the enterprise of the school leadership communities. In contrast, the adopted curriculum was primarily grounded in the practices of teachers and the mathematics teacher leaders. Its constitution as a boundary object enabled the school leaders to contribute to the enterprises of the mathematics leadership and professional teaching communities. The alignment that we have documented between the practices of the various communities was continually regenerated as the members of particular communities contributed to the enterprises of other communities in this manner.

In concluding this characterization of the Washington Park district, we note that the analysis of interconnections based on boundary objects is pragmatically useful in that it can inform our efforts to support the learning of the professional teaching community. As an illustration, we consider it significant that the only tool for making mathematics teaching and learning visible that was constituted as a boundary object was grounded primarily in the practices of the school leadership community. As we have seen, the negotiation of the ways in which the members of various communities used students' test scores enabled teachers and the mathematics teacher leaders to contribute to the enterprise of the school leaders. In contrast, the samples of students' work that teachers generated to document their mathematical reasoning were a focus of discussion only in grade-level meetings with the teacher leaders. As a consequence, school leaders had few opportunities to negotiate their understandings of students' reasoning with the members of the other communities. This in turn delimited their contributions to the enterprises of the professional teaching and mathematics leadership communities. In our future work in the district, we will therefore endeavor to support the constitution of student work as a boundary object between all three communities.

As this example illustrates, analyses of the institutional setting of teaching that include a focus on interconnections between communities can provide a perspective from which to consider whether collaborations with teachers should entail concerted attempts to bring about changes in the settings in which they have developed their current instructional practices. In addition, analyses of this type can inform the development of testable conjectures about the means of bringing about such changes. The approach that we have illustrated was in fact developed as a general way of documenting and analyzing the specific institutional settings in which particular groups of teachers work that can feed back to inform efforts to support their learning. The potential value of such an approach is that it can support teacher development efforts by enabling researchers and teacher educators to monitor the institutional settings of the sites in which they are working on an ongoing basis.

Discussion

In stepping back from the Washington Park district, we first foreground key characteristics of the analytical approach and then draw together the central aspects of the sample analysis. We conclude by discussing the implications of the analysis for the process of working in multiple districts and the diffusion of instructional innovations.

It should be clear that in identifying relevant leadership communities, we do not assume that school and district leadership resides exclusively with the individuals who occupy designated leadership positions. Instead, we follow Spillane, Halverson, and Diamond (1999, 2001) by discerning how various leadership functions are actually accomplished with the expectation that we will find that many are in fact distributed across several people who use a range of tools to accomplish those functions. Similarly, the analytic approach that we have illustrated characterizes teaching as a distributed activity. At first glance, this assumption might seem highly questionable for districts where, in contrast to Washington Park, teachers work in relative isolation and have limited opportunities for collaboration with each other. However, this contention becomes plausible when we note that the approach that we have taken focuses not on actions of individual teachers working alone in their classrooms but on the functions of teaching as they are accomplished in schools and school districts. As we have illustrated, these functions are not restricted to interacting with students in the classroom to support their mathematical learning but also include

Organizing for mathematics teaching and learning by, for example, delineating instructional goals and by selecting and adapting instructional activities and other resources.

Making mathematics learning and teaching visible by, for example, posing tasks designed to generate a record of students' mathematical reasoning.

When we analyze how these latter two functions are actually accomplished in specific cases, it almost invariably proves to be the case that a number of persons in various designated positions within the school and district are involved in accomplishing them and that they use a variety of tools as they do so. As an illustration, the mathematics teacher leaders used the state standards and the curriculum materials as they organized for mathematics teaching and learning by conducting grade-level meetings and by meeting regularly with the members of the leadership community in each school.

As a point of clarification, we should stress that this distributed perspective on teaching does not imply that people within a school or district necessarily coordinate their activities seamlessly or smoothly. Although this was the case in the Washington Park district, we have reported an analysis of a second district elsewhere (Cobb et al., 2003) in which mathematics teaching was a site of tension and struggle as people within that district pursued conflicting agendas. More generally, the immediate institutional setting within which teachers develop and refine their instructional practices is constituted as members of different communities of practice pursue sometimes-conflicting instructional visions and gauge the extent to which their visions have been realized in classrooms.

Consistent with the distributed perspective that we have proposed on mathematics teaching, the analytical approach characterizes individual teachers' instructional practices as situated and as partially constituted by the institutional setting in which they work. For example, the instructional practices of the collaborating teachers were situated in that they involved

- Reasoning with the NSF instructional materials and with work samples that served as records of students' reasoning.
- Having coordinated schedules and joint planning time.
- Having access to computer labs and other physical resources.
- Having access to peers when difficulties arose in their classrooms.
- Drawing on their understanding of their students' mathematical reasoning and the intent of the NSF materials to justify their pedagogical decisions to members of the professional teaching community and the mathematics leadership community.
- Receiving assistance rather than assessment from mathematics teacher leaders and school leaders during classroom observations.

Taken together, these aspects of the institutional setting in which the collaborating teachers developed their instructional practices provided them with access to resources for improving their instructional practices while simultaneously insulating them from high-stakes accountability pressures.

Similarly, the analytical approach also produced situated accounts of the activities of the mathematics teacher leaders and the school leaders. For example, the mathematics teacher leaders' activity was situated in that they

- Were constituted as content experts as they interacted with members of both the professional teaching community and the school leadership communities.
- Reasoned with the NSF instructional materials and state standards when organizing for mathematics teaching and learning.
- Had access to and could influence the practices of school leaders.
- Had the autonomy and material resources to investigate and address perceived problems with test scores.
- Had access to and could influence teachers' instructional practices during classroom visits and grade-level meetings.

The responsibilities of mathematics teacher leaders as they were constituted in the Washington Park district involved supporting teachers to improve their instructional practices while collaborating with school leaders to ensure that test scores continued to be acceptable.

The school leaders' activity was situated in that they

- Were held accountable by district leaders and the community for test scores.
- Reasoned with the NSF instructional materials and state standards when organizing for mathematics teaching and learning and with test scores when making mathematics teaching and learning visible.
- Negotiated their interpretations of the instructional materials and test scores with the mathematics teacher leader in their school.
- Deepened their understanding of the mathematical intent of the NSF instructional materials during biweekly meetings of school leaders in the district.
- Observed teachers' instructional practices during classroom visits and addressed perceived difficulties by formulating an improvement plan in collaboration with the teacher.

The responsibilities of school leaders as they were constituted in the Washington Park district involved collaborating with mathematics teacher leaders in their efforts to improve the quality of mathematics teaching and learning while remaining vigilant about test scores.

In the first part of this article, we note that the Washington Park district is of particular interest because school and district leaders have responded to a state-mandated accountability program not by attempting to regulate teachers' instructional practices but by giving teachers access to material resources and by supporting their development of social and personal resources. The analysis that we present demonstrates that the roles of individual school leaders in mediating the state accountability program were not solo accomplishments but were instead partially constituted by the institutional setting in which they worked. For example, we point to the opportunities for the school leaders to deepen their understanding of the learning and teaching of mathematics through courses that they had been required to take. We also saw that in meeting regularly with the mathematics teacher leader in their school, they had the opportunity to negotiate their interpretations of the NSF instructional materials with a person who was constituted in the district as a content expert. These and other aspects of the institutional setting in which the school leaders worked afforded and constrained their development of leadership practices that involved supporting teachers' learning by giving them access to resources and by engaging in the discourse of educational reform rather than of high-stakes testing when they interacted with them. In a very real sense, what it meant to be a school leader in the Washington Park district was partially constituted by the institutional setting in which the school leaders developed and refined their leadership practices.

The analysis that we present of the Washington Park district substantiates research in the fields of educational policy and educational leadership that has sought to identify characteristics of schools in which innovative instructional practices are likely to be sustained. As Newman and Associates (1996) documented, facets of the organizational capacity for change and improvement of such schools include knowledge and skills, shared visions, collaboration among staff, classroom autonomy, and collective responsibility for students' learning. The analysis of various communities that comprise the Washington Park district and the interconnections between them serve to specify the underlying processes that give rise to these characteristics in this particular case. For example, we illustrate how the institutional niches in which the members of each community developed and refined their practices involved considerable autonomy while simultaneously giving them access to new skills and forms of knowledge. Similarly, we clarify how an alignment between the enterprises and practices of the different communities involved shared visions, collaboration, and collective responsibility for student learning was generated and sustained. More generally, analyses of the type that we have illustrated complement those conducted in the fields of educational policy and educational leadership by documenting the processes by which teachers and leaders in particular schools and districts collectively generate characteristics of schools associated with a high capacity for change and improvement in mathematics teaching and learning.

Given the success of teachers and leaders in the Washington Park district in supporting students' mathematical learning, a question that naturally arises is that of how innovative aspects of their practices might be disseminated to other districts. In addressing this issue, we draw on Brown and Duguid's work (2000) to introduce the notion of a *network of practice*. Brown and Duguid clarified that networks of practice have considerable spatial reach that transcend the constraints of direct interaction. A defining feature of such a network is that innovations developed in one community of practice can diffuse rapidly and be assimilated readily by members of other communities. As an illustration, the mathematics teacher leaders and most members of the professional teaching community in the Washington Park district considered themselves to be members of a nationwide community of mathematics education reformers. This broader community is not a tight-knit community of practice in which people negotiate meanings directly as they interact while engaging in joint activities. Instead, it is a network of practice that links groups of teachers and leaders in numerous schools and districts whose enterprises and practices are broadly compatible.

Brown and Duguid (2000) emphasized that people whose local community of practice is part of such a network are separated from other people whose local communities are oriented by different enterprises and are thus part of different networks in terms of dispositions, attitudes, and knowledgeability. Brown and Duguid also clarified that networks of practice often correspond to occupational groups. In the case of the Washington Park district, for example, the mathematics teacher leaders, school leaders, and teachers differed from each other in terms of their concerns and interests even though the enterprises of their respective communities were aligned.

The relevance of the notion of networks of practice to the issue of upscaling becomes apparent once we note that innovations are not taken up uniformly but instead diffuse according to what Brown and Duguid (2000) referred to as the topology of networks and communities of practice. As they put it, innovations "leak" along networks of practice while sticking between communities of practice in different networks. Brown and Duguid's account of diffusion indicates that it might be possible to disseminate instructional and leadership innovations in the Washington Park district successfully to other districts that already have a high capacity for change and improvement. In contrast, their analysis indicates that these innovations are unlikely to diffuse to school districts with a limited capacity for change even when concerted efforts that involve newsletters, websites, listservs, and so forth are made to support this process. This is particularly the case for other urban districts in high-stakes testing environments where administrators have responded to accountability pressures by attempting to monitor and regulate teachers' instructional practices.

The analysis that we present documents that the Washington Park district differs significantly from most other urban districts in terms of what it means to be a teacher and an instructional leader. The analysis also illustrates that teachers' and leaders' attitudes and dispositions are not solo achievements but are partially constituted by the institutional setting in which they work. Taken together, these observations imply that upscaling an innovation from a high-capacity district such as Washington Park to other urban districts cannot be accomplished merely by attempting to develop more effective ways to reify the innovation. This is the case even if new information technologies are used (Brown & Duguid, 2000). To be successful, the dissemination process would have to involve the restructuring of the target districts as lived organizations such that the communities of practice that constitute them might become part of the same networks of practice as the corresponding communities of the high-capacity district. This required restructuring process is both profound and daunting in that it penetrates the inconspicuous, recurrent, and taken-for-granted aspects of teaching and leadership. However, in the absence of such a restructuring, it is highly probable that even if objects that reify the innovation are seen as being relevant, they will be used in very different ways and come to have different and quite possibly conflicting meanings when they are incorporated into the practices of communities in the target districts. As the term *travel* is frequently used to describe the process of upscaling, it is worth noting that this metaphor is grounded in cases where innovations “leak” from one community of practice to another within a network of practice. It is only in such instances that reifying objects are perceived to function as relatively transparent carriers of meaning by members of different communities.

The notion of networks of practice goes some way toward explaining the relative ineffectiveness of most large-scale educational reform efforts. It also serves to clarify the daunting challenges involved in upscaling instructional and leadership practices that place students’ mathematical reasoning at the center of decision making. The perspective that this construct offers on the process of dissemination is consistent with Rogers’s seminal analysis (1995) of the features of an innovation and the mechanisms of communication that influence the success or failure of diffusion. In summarizing Rogers’s findings, Zaritsky, Kelly, Flowers, Rogers, and O’Neill (2003) noted that “among the factors relevant to successful innovation is perceived relative advantage: the degree to which an innovation is perceived as better than the idea, product, or technique it hopes to supersede” (p. 33). Zaritsky et al. explained that measures of relative advantage include complexity, or the degree to which the innovation is perceived as being relatively difficult to understand and use, and compatibility, or the extent to which the innovation is perceived as being compatible with existing values and needs. In our terms, *complexity* and *compatibility* are measures of the extent to which the practices of different communities are aligned. It is therefore understandable that, on our reading, the cases of successful innovation that Rogers discussed to illustrate his perspective are cases of “travel” within established networks of practice.

As Zaritsky et al. observed, the application of Rogers’s analysis (1995) of diffusion to educational settings would restrict dissemination efforts to innovations that fit with teachers’ and administrators’ current practices. In contrast, current reform efforts in mathematics education aim to penetrate what Gamoran et al. (2003) termed the *instructional core* of basic suppositions and assumptions about learning, teaching, and mathematics. Gamoran et al.’s observation strongly suggests that Rogers’s analysis cannot, by itself, provide adequate guidance for reform efforts that seek to transform rather than merely augment the instructional core by supporting teachers’ development of increasingly sophisticated instructional practices. Instead, it will also be important to attend to the local topology of networks and communities of practice. The analysis that we present demonstrates an approach for doing so that is specifically tailored to the needs and interests of researchers and teacher educators. In this regard, the analytic approach can best be viewed as a tool that is designed to support transformative educational change as iterative processes of continual improvement.

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Chapter 14
Epilogue: On the Importance of Looking Back

Anna Sfard

It may be a useful exercise to look at the story of Paul Cobb's life-long quest for an ever better understanding of educational processes as a modern representative of the same genre as the ancient tales of Argonauts attempting to recover the Golden Fleece or of Sir Galahad pursuing the phantom of the Holy Grail. As I will be arguing in this epilogue, much of what can be learned from those old myths is certainly relevant to what can be found on the preceding pages.

Much, but not all. The success of some of the mythological heroes was predicated on their ability to refrain from looking back. For instance, Orpheus lost Eurydice forever when, on his way out of Hades, he turned back to his beloved and threw her a wistful glance. This book is an attempt to do what was not to be done by the mythological protagonist. I wish to claim that in our present case such looking back may have important payoffs. By saying so, I forestall the question that might have been asked by a prospective reader of this book: Why should a group of busy people decide to take the trouble of reprinting formerly published articles? Be Paul Cobb's work in general, and these six chapters in particular, as important and influential as they might, what value can be added in such enterprise? I hope that no doubt about the worthiness of the endeavor is left in the minds of those who are reaching these summarizing passages after a thorough walk through the preceding pages. As with any long journey rich in consequential revelations and memorable experiences, constructing a retrospective travelogue of one's odyssey through the world of ideas is a creative undertaking, likely to bring new insights and understandings simply by putting each episode of the decades-long wandering and wonderings in the context of all the others.

Such enlightening experience was my own unanticipated reward for engaging in this project. Having been in regular contact with Paul over the last 20 years, and thus being able to follow his exploits from their early stages, the last thing I expected was that preparing this book would turn, for me, into an opportunity for substantial learning. I have definitely underestimated the power of recombining

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and recontextualizing. While putting the familiar pieces together, I arrived at two types of insights. First, I was suddenly able to see the big picture. Over the years, Paul has been releasing his work piece by piece, and my vision of his project, so far, was like the proverbial blind person's idea of an elephant. Once illuminated by the logic of the whole, the fragments grew in meanings and associations, as does a single note when heard as a part of a concerto. The other type of learning occurred at the meta-level, the level of reflection on the art of researching. Indeed, Paul's life project is an exemplary case of its genre, to be studied by those who wish to deepen their understanding of what research is all about, where its power to innovate comes from, and what makes it possible for the researcher to change lives of people who have never heard as much as his name. In the rest of this closing piece let me share my vision of the contribution the present retrospective of Paul's work can make to the project of answering these important questions.

What Is New – Seeing the Big Picture

Considering what was said along these pages by Paul himself, one may wonder whether the six different images of educational processes to be found in this volume really combine into one "big picture". Paul, the conceptual nomad, has been changing places and vantage points with considerable frequency. Some of these changes would amount to a wholesale overhaul, with transformations occurring in all aspects of research, including in what he was looking at and in the way he navigated his gaze. In the span of two decades, Paul has revolutionized the epistemological foundations of his research by moving from radical constructivism to pragmatic realism; he gradually extended the object of his study from the individual learner to classroom community to institution; and he reformed his research method when abandoning teaching experiment for the sake of design experiment. The latter type of experiment, by the way, has been recently scaled-up from 1-year classroom study to district-wide years-long project.

Can the different pieces produced in this way be really parts of one consistent whole? Being grounded in seemingly conflicting epistemologies, shouldn't these six images be rather treated as mutually exclusive alternatives? I wish to claim that even if Paul finds faults in his past creations, he does not dissociate himself from them altogether. On the contrary, judging from his own stories, his guiding principle can be described as "Try to reconcile differences before you trash the different." Reconciling differences, by the way, does not mean the kind of cutting and bending that would be necessary if one was to squeeze all the pieces into a single theoretical "shoe." Paul deeply believes that one can live with differences and even thrive on them. As he said more than a decade ago in the midst of our email conversation, "[my] attempt is NOT to develop one single encompassing perspective. Instead, it is to acknowledge perspectival relativity and to explicate on possible way of coordinating them that might be helpful for our purposes" (emphasis as in the original).

If one opens oneself to Paul's call for coexistence of incommensurables, one may also agree that the six pictures of educational processes republished in this volume constitute, as it were, a series of camera takes with a successively increased frame. As Paul has been moving along, changing his lenses and broadening his field of vision, his eyes stayed fixed on the same final destination. Because of the successive zooming out and the simultaneous increase in the amount of fine details, it may sometimes be difficult to see that all the images target the same slice of reality, albeit with different resolution and at different levels of complexity.

The overall change that occurs because of this gradual scaling-up results, above all, in the reversal in the degree of visibility and in the roles of the individual and the collective. Whereas in the first of these pictures the individual is the primary focus and the construction of the collective version, to be done by a simple reiteration of this basic unit, is left to the reader, the first thing that meets the eye in the second, third, and fourth study is the classroom community and its collective actions. The individual does not disappear altogether from these latter pictures; it only loses its hitherto unquestionable primacy. Rather than being foregrounded and immediately visible, as it was in the first part of this travelogue, it requires some deliberate effort to be dug out from the latter collective pictures. This digging out is actually done in the fifth part, where the authors speak about students' identities, those special entities which, although inextricably tied to the individual, are nevertheless the product of collective doing, be the collective a classroom community, a community defined by school, or the one delineated by school district.

Using Paul's own terms (Cobb, 2007), one can say that his and his colleagues' work took us as far as one can wish from the *collective individual*, the statistically constructed protagonist of positivistically minded psychological studies, also known as the "typical representative" of his kind. After a brief affair with the *epistemic individual*, the diligent builder of her own conceptual structures, Paul decided to devote his studies to *individual-in-cultural-practice*, the learner whose activity cannot be understood unless considered as a part of an activity of a collective. In this latter version, individual shifts are the result of and a cause for the collective motion. Today, Paul and his colleagues view the development of a child's mathematical activity and the emergence of classroom mathematical practices as two sides of the single process of learning mathematics in class. What happens in the class as a whole is, in turn, in a similar reflexive relation with what happens in the school at large. Needless to say, the school is not an isolated island either; it is subject to an analogous, albeit larger-scale dynamics of district-wide processes. All in all, Paul's work draws a compelling multilayer picture of mathematics learning and teaching. In this picture, the "learning subject" may be a single person or a collective of any size and complexity, but whatever the nature of this basic unit, the change called learning is always the same: It is a process of intricate multilevel interactions that transform all parties involved.

Let me make one final remark on the added value of re-reading Paul's classic papers. It regards our ability to appreciate the novelty of his ideas. Since these ideas were first published, they became a canon and, as such, acquired the appearances of self-evident truths.[1] Today, terms coined by Paul and his associates a decade or two ago, such as *sociomathematical norms* and *classroom mathematical practices*, are so much a part of our discourse that taking them away would likely make at least some of us fall silent. This book helps us to put Paul's work in its historical context and re-appreciate its pioneering character. While doing so, one is also in a position to get a fresher, deeper insight into some of his ideas.

How Does Innovation Happen

What could be learned from Paul and his colleagues about classroom learning can now be retold as a story of their own work. After all, mathematics learners and educational researchers aim at comparable goals and are engaged in a similar type of activity: mathematics students and the students of mathematical learning alike pursue new understandings and, while doing so, they develop new collective practices. Both types of learning, the individual and the societal – and research can be seen as this latter type of learning – change the narratives the learners are able to tell about the world. These stories may be innovative in two ways: First, their authors may be saying new things; second, the very way the stories are told may be novel.[2] One of the factors that make Paul's project unique is its being always geared toward both kinds of innovation, toward object- and meta-level learning, alike.[3] Whereas the majority of us just talk and rarely stop to reflect on the way we do it, Paul never moves his gaze from the discourse itself and, as a rule, looks not just for new things to say, but also for ever better ways to say them. Let us take a closer glance at how the resulting multilevel innovation happens.

When a child tries to become a participant in the discourse of a community, the process of innovating begins with her attempt at *individualization* of that discourse, that is, with the effort to turn the discourse of the community into her own. More often than not, this process entails *modification*, when the learner develops her signature way of participating. Some of these individual modifications will often end up being individualized by other participants, thus contributing to the *communalization* of the altered form of discourse. I am tempted to call these three types of occurrence, individualization, modification, and communalization, "phases" of innovation, but this name would imply a serial order, which is not the way things are. The three steps are in fact inextricably intertwined. Moreover, my work has taught me that the

[1]To be sure, some of the declared adherents of this vision may, in fact, be closer to the position that Paul, after Lave (1997), likes to call "cognition plus"; this, however, only reinforces my claim that today, it may not be easy to fully appreciate how innovative Paul and colleagues' approach was only two decades ago.

[2]In the context of learning, where the "new" is new for the students even if it is not new to other people.

[3]Object- and meta-level learning correspond to phenomena that inspired Piaget's ideas of, respectively, *assimilation* and *accommodation*. In the context of research, where the name of the game is to create stories never told before, these two types of development bring to mind Kuhn's ideas of *normal science* and *scientific revolution*, respectively.

individualization–modification–communalization cycle may repeat itself on many levels, inflicting any kind of change, from a cosmetic amendment to a complete transformation. A major innovation is likely to occur through evolution rather than revolution – through a series of individualization–modification–communalization cycles, with each consecutive round refining the outcome of the previous one (Sfard, 2002). While reading Paul's historical account, one realizes that the vision of innovation as a combination of the complementary processes of individualization of the collective and communalization of the individual is as true for Paul's own quest for ideas as it is for the class developing its mathematical discourse. Each of the six reports from Paul's intellectual journey presents an episode of substantial meta-level learning. With some assistance from Paul himself, each one of them reveals its tripartite structure. Let's have a look.

In the six studies, the trigger for the cycle of meta-learning is always the same: A sense of dissatisfaction with at least some aspects of what had been done before. It is remarkable how critical Paul and his colleagues have always been and how closely they have been monitoring their own moves. They were gauging what had been done against standards either clearly stated in advance or ones that kept emerging as the work went on. Thus, in the introduction to the first episode Paul recalls that while trying to account for his interviewees' discursive actions, he and his colleagues could feel that the picture they were able to draw was lacking some potentially consequential elements. At hindsight, he ascribes this problem to his rather impoverished initial idea of context ("[we were] bracketing out ... social and cultural dimensions"). In a similar way, in the second study the team realized post factum that the instructional tasks they used and the whole class discussion led by the teacher were "somewhat ad hoc." In the introduction to the fifth episode, Paul tells us about how he and Erna Yackel learned through their work with teachers about pitfalls of the objectified discourse on "welfare of the student," of this omnipresent form of talk that provides no tools for dealing with the cultural situatedness of what counts as "good for the student." These examples should suffice to show the workings of Paul's tightly observed principle of constant self-monitoring and persistent doubting.

Faced with a shortcoming of their project so far, Paul and his colleagues would typically shift their glance to the meta-level – they would scrutinize their discourse for limitations. Once they identified those weaknesses that could be seen as responsible for the team's failure to deal effectively with the object-level problem, the deliberate search for a new discourse would begin. The sense of having exhausted the potential for new understandings attainable with their present discursive tools, Paul and his travel companions would turn to others for ideas about how their own tools might be usefully modified. And this is where the process of individualization would typically commence.

Reading Paul's first person account of his forays into other researchers' discourses is a particularly educative experience. One could not wish for a better instantiation of what a genuine attempt at individualization is. Throughout my ongoing decades-long conversation with Paul, I had an opportunity to watch this process many times and in great detail. His approach to other people's discourses from which he hoped to learn brings to mind a traveler arriving in a new place and considering the possibility to settle: Paul would be acting in the new discursive setting as if it was to become his own. Never mind that his critical mind was only too likely to prevent him from using ready-made solutions. At least for the time being, he would give the authors a fair chance to convince him. True to his motto "people's practices are reasonable from their perspective" (Introduction to Part I), but also aware that seeing the logic of the other may require abandoning one's own, he would suspend disbelief and apply the same Kuhnian advice which has probably been guiding him in the task of interpreting classroom interactions:

> [L]ook first for the apparent absurdities in the text and ask yourself how a sensible person could have written them. When you find an answer ... when those passages make sense, then you may find that more central passages, once you previously thought you understood, have changed their meaning. (Kuhn, 1977, p. xii; quoted on Cobb, 2007, p. 32)

The exercise in potential permanent residency never prevented Paul from being critical. Work of others, once understood, as much as possible, on its own terms, would invariably become subject to the same close scrutiny as his own. Thus, as he testifies himself, as keen as he was to learn from sociocultural discourse of neo-Vygotskians, from the Dutch school of Realistic Mathematics Education, from the social philosophy of Alfred Schütz as brought to his attention through the work of his German associates, and from distributed theories of learning, he could not help finding limitations in each one of them. His subsequent return to his own discourse with the intention to incorporate what he was able to learn from others was never an act of wholesale acceptance. Rather, as is not uncommon for a traveler impressed by the culture and language of a new place, Paul would preserve much of his old way of talking, while also making his narratives unmistakably tinted with the local dialect. True, some of the new words, expressions and routines would appear as if simply transferred from the original. Nevertheless, while amalgamated with Paul's earlier discourse, they would undergo an adaptation, sometimes quite radical. And thus, the words *norms*, *hypothetical learning trajectory*, or *practice*, although borrowed from others, became the signature elements of Paul's work precisely because of the special way he applied them in his discourse. The change in use was always explicit and well argued. This, indeed, is what individualization is all about: taking from others, but making it unmistakably one's own.

Two characteristics of the way in which Paul and his colleagues worked toward the desired modification of their discourse are particularly noteworthy. First, theirs was a truly collective work, a work of people who more often than not would hold differing views, but whose agreed goal "was not to win the argument as an end in itself, but to develop and refine theoretical constructs" (Introduction to Part I). What makes Paul and his teams' work special is their approach to the task of attaining consensus. Although the process of modifying was not considered to be completed unless all parties involved arrived at an agreement, this did not mean that the team was determined to end up with a single well-defined discourse. Rather, conflicts were seen as resolved when a platform had been created for communicating across participants' differing discourses. Thus, the US–German team satisfied itself with "developing a way of coordinating social and cognitive perspective on mathematics learning" (Introduction to Part II) and the US–Dutch group's work was described in retrospect as an ongoing and never fully completed process of "sharing, appropriating, and adapting ideas across research traditions" (Introduction to Part III).

The second distinctive feature of Paul's techniques is that his modifications have been grown in the field not any less than in the university halls. The evolving discourse was incessantly tested by being applied in one design experiment after another. This process can be described in terms I used a few years ago while investigating learning in one of Paul's classes: as a process of intermittent intimations and implementations, that is, of forging proposals for refinement and then testing them empirically. As I was able to show, children's intimations about what was possible were coming mainly from intuitions spurred by metaphors. In Paul's case, the intimating was always a systematic process, where the ideas for refinement were analytically substantiated theoretical principles. The implementation, in turn, was not just an application of the refined discursive tools in the hope of producing a better story about processes of teaching and learning, but also an attempt at refining the processes themselves. Undeterred by the risk of being criticized for "messing with intervention" (Introduction to Part VI), Paul was always determined to reciprocate to the participants of his studies by leaving their place better than it was upon his arrival. His implementation was thus an act of altering the investigated phenomena before becoming an act of telling a new story about them. This is what design experiment is all about: changing ways of talking to transform classroom practices and allowing the latter transformation, in return, to alter the ways of talking.

This brings one last remark about the mechanism of modification, as can be gleaned from the preceding pages. Paul often speaks about his being quite lucky. He claims to have come across various opportunities for learning fortuitously, by chance encounter, incidental reading or a choice randomly made. And yet, while it is certainly true that contingency rather than necessity is responsible for what we find on our way, what we eventually choose from the available options is usually a matter of our own reasoned decisions. As a witness of Paul's long travel, I can attest to his being a much more thoughtful decision-maker than the majority of people I know. The odds are that much of what he noticed on his way and deemed as an opportunity to grab, many other travelers would have passed without paying any attention.

When it comes to the last element in the cycle of innovation, to the communalization of the modified discourse, Paul and his fellow researchers' practices prove, once again, to be a source of valuable insights. Perhaps the most important lesson to learn is that the work of spreading the message, far from being a final act in the process of innovating, plays an all important role in this process all along. From the earliest stages of their travels, Paul and his colleagues would be sending interim reports back home, to the community. Their stories, usually compiled at some "resting place" (Cobb, Introduction to Part V), have been told in real time, as things were happening. These frequent bulletins constituted an important trigger for modifications, since they provided the researchers with opportunities to hear themselves, to assess what has been said, and to decide on the need for a subsequent refinement. After all, trying to be understood by others may be the best way to understand oneself.

Having presented the mechanism of innovation that worked extremely well for Paul's teams, it is only natural to follow with the question: What is it that made this cyclic process of innovation so productive? In the search for an answer one should, no doubt, turn to the research practice developed by Paul and his associates over the years. This practice can be described as a set of norms he and his colleagues have been observing while traveling in the world of ideas. The metaphor of traveling is particularly apt in our present case, since Paul, as I know him, is almost as keen on trotting the globe as on exploring intellectual landscapes. Even if none of the norms that define both his traveling practices is quite unique, the special power of Paul's research practice lies in these norms particular combination and in the determination with which they have always been observed. Counting them all would not be easy. Following is just the beginning of the long list of the dos and don'ts of Paul's traveling practice which I managed to glean from all I know about his work.

- Don't embark on a new expedition without consulting those who were there before you. Whatever new discovery you subsequently make, it has to be clearly related to what was known before you began.
- Don't travel alone. Traveling together is not only safer, but also richer in opportunities. A good team is much more than the sum of its parts.
- While visiting foreign places remember that the otherness of its inhabitants is your problem, not theirs. If you cannot understand what they are saying, the odds are that it is not because they make no sense but because you insist on speaking your language, which is obviously not the one they use.
- Reciprocate to those whom you meet on your way. Don't be just a visitor. Try to make the location you explore a better place to live. Your commitment to the locals is not any lesser, perhaps even greater, than is your obligation toward your own community, which is waiting to hear your story.
- While traveling, keep your eye on your tools as much as on what you are supposed to use them for. Improving the tools for exploring is part and parcel of your mission as an explorer.
- Never stay in one place for too long. If you do, you expose yourself to the risk of going in circles. You are more likely to arrive at new insights by moving to another vantage point.
- While reporting on the results of your expedition, be multivocal. Avoid the pitfalls of monological discourse. Never talk as if you were the spokesperson of the world itself. In the insights you offer to others, seek to be of help rather than to be right.
- Watch your stories. Remember that the narratives inspired by your travels, far from being merely shaped by phenomena they describe, have the power of shaping these phenomena in return.
- Always bear in mind that it is your responsibility to be understood. The value of your travelogue depends not only on its being a collection of original insights, but also on its potential for becoming a solid basis on which others will be able to

build through their own explorations. For this to happen, you have to make your discourse as clear and immune to misunderstandings as humanly possible.

After giving this book the close read it deserves, I am feeling more clearly than ever that traveling with Paul, or even just witnessing his pursuits from afar, is a humbling experience. Having compiled the above list of norms that define his practice I can also see why. There is much I am taking home, to my own studies, from this new exposure to the familiar body of work.

Coda

In the beginning of these summarizing remarks, I claimed that Paul's story belongs to the same genre as the ancient tales of great quests. After having a close look at Paul's intellectual adventures so far, I can close the circle and justify the analogy. Paul's is a story of life on the road with a clear destination but no end in sight. Just like Jason and Sir Galahad he is surrounded by a team of people who share his goals and perseverance. Like Argonauts, he does not mind changing places every so often and never spends in one location more than absolutely necessary to make a major advance toward his ultimate goal. In trying to achieve his destination, he is as persistent as any of the mythological pursuers, and the fact that the goal he set to himself is not any less elusive than the Holy Grail does not undermine his determination. On the contrary, the inherent unattainability may be this goal's principal merit: Thanks to its tantalizing nature, it will never stop spurring novelty and insights. There is thus no reason to suspect that this book is any more than an interim summary of Paul's ongoing travel.

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