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Causal and Stochastic Elements in Business Cycles

An Essential Extension of Macroeconomics Leading to Improved Predictions of Data

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Preface

1. Facts:

A leisure term ***with an unbounded value function***, when added to utility in the Lucas (1988) 'mechanics of economic development', expands enormously the range of data covered by the theory. To explain this we have to ask two questions. First: why leisure would be so much desired? Perhaps because leisure is one's own time and such a leisure term means an unbounded value of ***individual freedom***. But why leisure is economically productive, as implied by the results obtained in this study? Perhaps because cognitive innovations often occur during the time which in economics is registered as leisure? Then an unbounded leisure term would also make room for an unbounded ***creation of knowledge***, as distinguished from the mere transmission of knowledge in education and training. In any case the leisure term seems to act as if it where the'hole' through which strong nonmaterial values affect economics.

The ensuing 'extended mechanics' is derived in Chapters 4-6 and proves to involve an extension of growth theory as well as a theory of the causal part of business cycles. Their empirical verification is given by showing

(i) that the existence of the two Basic Growth Paths derived from this theory, defining its Growth Type 1 and Growth Type 2, respectively, is verified already by the statistics collected by Solow (1957) but ignored so far (see Chapter 5 of the present study); one of them, viz. that of Growth Type 1, is the familiar balanced-growth path that exists in other growth theories as well; the other one, that of Growth Type 2, is new and shows a logistically increasing level in real interest rate (or in output/capital ratio);

(ii) that the nonstochastic Basic Business Cycles derived from this theory predict the correlations, autocorrelations, and standard deviations of principal real economic variables over detrended 'ordinary' business cycles better than do the existing other real cycle models (of Kydland and Prescott, Hansen and Rogerson, or Danthine and Donaldson; see Chapter 7). The ordinary business cycles correspond to cycles appearing in Growth Type 1 around the balanced-growth path. It will be also shown

(iii) that the anomalous correlations over detrended business cycles, observed in the U.S. economy in the period 1914-50, are well predicted by this theory, which accordingly gives their first (roughly) ***quantitative*** explanation (see Chapter 9). They correspond to cycles in Growth Type 2 around its cycle center that moves along the path mentioned above.

Since the correlations and variances of such economic variables over a detrended cycle are the essential available data concerning business cycles, the results (ii) and (iii) imply a challenge to the modern methodology in macroeconomics based on stochastic optimization. According to the method of stochastic optimization, the technological or other shocks are introduced and coupled with economic variables ***before the optimization***. The method implies that the economic agents are able to react rationally (as expected by theory) to the shocks. This hypothesis is obviously involved in an optimization of an utility including the shocks. But there is the other possibility that the agents in general react to trends, observable over some longer interval of time, rather than to the shocks. The results (ii) and (iii) suggest that *in general* they indeed do so: this possibility is realized by the Basic Business Cycles, which according to these results predict the essential data better than do the models based on stochastic optimization.

To test this conclusion technological shocks, of the size usually applied, were superposed (see Chapter 12) upon the Basic Business Cycles (BBC) of the ordinary type, i.e. upon those of Growth Type 1. This means that the shocks are introduced ***after the optimization*** of utility, as perturbations on the strictly causal Basic Cycles given by the foregoing optimization. The results tell (see Chapter 13) that the effect of shocks on correlations and variances over a detrended cycle is small but not negligible. Of the empirical variance of output the BBC account for 90 % and the shocks for 10 %. What is more they tell (ibid.) that the BBC with *and* without shocks predict far better the correlations and variances over a detrended cycle than do the models based on stochastic optimization. This final result is illustrated by Figures 9 and 10 (p. 96-97) and in Table 12 (p.95). They show

(iv) that the BBC approaches with and without shocks both follow the patterns of empirical values of correlations and standard deviation proportions, respectively, while those models, marked as K-P, H-R and D-D, follow each other but stray together off from the empirical values. This suggests a common methodical error in those models: stochastic optimization.

2. Conclusions:

The usual stochastic approach to development *in the short run*, where material values prevail, is not necessarily affected by the above criticism provided that some dominant economic agents exist, which are able to react quickly, as assumed, to shocks. Material values and value functions are of course finite: you can increase your welfare by consumption but only up to a finite limit. However in view of the above results even the elegant Lucas (1987) macroeconomic formalism, more general than the "Keynesian" ones but involving (finite) stochastic optimization, is a short-term approach.

In the business cycles already nonmaterial values are involved, according to the above results (ii)-(iv). Nonmaterial values, such as individual freedom or the pursuit of objective knowledge, have as a matter of principle no upper limit. Their growth may continue, with many historical setbacks and however slowly, but without any finite limit. The inclusion of nonmaterial values in this study, by means of the unbounded value of leisure, explains the results (i)-(iv) and suggests a superposition of stochastic shocks, after the optimization of the extended utility, upon the strictly causal Basic Business Cycles. The Basic Cycles in this theory are caused by a mutual interference of material and nonmaterial values, which is *not reducible to microeconomics.*

The trend of economic development in the long run, i.e. over many business cycles, is in this theory represented by the Basic Growth Paths mentioned above. The development of real economic variables along these paths is determined by three functions of time indicating the development of human capital, the strength of the wish of greater individual freedom, and the natural talents in population, respectively (see Chapter 10).

Less trivial is a result concerning the mutual relations of certain constant parameters, necessary for the existence of solutions in this theory. The same parametric relations are necessary also for the existence of solutions in the Solow and Lucas growth models (but ignored so far). These relations imply that, counter to those authors (Solow, 1956; Lucas,1988, p.12), *savings rate has growth effects*, and also a stability effect (see Chapter 11).

A further unorthodoxy is the use of differential instead of difference equations so that economic variables get their usual meaning when integrated over a finite interval of time. This choice was made for convenience (and better tools of analysis). The concept of *invariance group* (see Chapter 6) is an important new tool of interpretation: The differential equations of Basic Business Cycles are invariant in a group of linear transformations including time. This gives a new degree of freedom in operations with finite times.

3. Is this economics?

Economics is still more a profession than a science. Practising economists have to advise politicians, just like astrologists used to advise kings, about the nearest future. Only part of their predictions can be based on their science. The rest is based on thumb rules and routines established in practice. The practical routines, kind of professional fashions of the day, often petrify into strong convictions about what is "economics" and what is not. Medicine is another example of a practical science, where fashionable methods play a certain role. It follows that the definition of what is "economics", or "medicine", not to speak of "sociology", will still be subject to many changes in the future, much more so than is the case in exact natural sciences. Can one hope that economics, in a not too distant future, could cope also with nonmaterial values?

Such an extension of scope would not harm economics as a science. Let me give an example, in fact the reason why this study was started. The collapse of the East-European economic and political system in 1989 and the disintegration of the Soviet Union in 1991 took by surprise a great majority of economists as well as general public. In textbooks of economics the 'great experiment' of socialism was mostly given the treatment of a benign observer. In the eve of the fall of East-European socialism *the* leading textbook of mainstream economics stated: "... several points about Soviet communism should be clear from the outset. First, contrary to what its early critics believed, the Soviet economy has grown rapidly, has expanded its influence, and has won many allies... From the point of view of economics, perhaps the most significant lesson is that a command economy can function." (Samuelson and Nordhaus, 1985, p.771).

Many reasons for this glaring example of mistaken economic prediction by an outstanding economist authority are easy to give, afterwards. But one fundamental reason undoubtedly is the traditional emphasis in economics on labour and physical capital as the main factors of production – a point that has been lately criticized (I am referring here to P. Romer, 1986, 1987, and to Lucas, 1988). In the long run human capital, i.e. knowledge and skills, is the decisive factor. But this observation leads us inevitably to the domain of nonmaterial values in the end.

We have to ask wherefrom come *new* knowledge and skills. My favourite answer is that they come mainly from the free wanderings and experiments of thought during the leisure time of people. But if this is true, the leisure that allows creative faculties to bloom is the most important factor of all kinds of production in the long run. If the economic relevance of the unbounded human pursuits of larger individual freedom (i.e. leisure) and deeper objective knowledge would have been in some form incorporated in economic theory, there would have been nothing to surprise one in the break-down of the Soviet Union. As to the general relevance of nonmaterial values in economic development, this of course is the *Leitmotiv* of the present work.

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Contents

I

THE MATHEMATICAL TOOLS

In the present work the classical theory of mathematical dynamics underlying dynamic economics will be applied paying special attention to the *parameter conditions of existence of solutions* in the applications of dynamics to economic theory. This has not been always done in economic applications, which suggests that a detailled introduction to the mathematical tools as applied here may be useful. It will be given in the present Part I.

1. The Hamilton-Jacobi Theory

1. The Hamiltonian function. Assuming all the necessary properties of existence and continuity we write the *Hamiltonian equations* as usually,

$$\dot{q}_\nu = H_{p_\nu}, \quad \dot{p}_\nu = -H_{q_\nu}, \qquad (\nu = 1, 2, ..., r) \tag{1.1}$$

for the position co-ordinates q_ν and the momentum components p_ν. The Hamiltonian function $H(q_\nu, p_\nu)$ *does not depend explicitly on time.* It follows that it is a constant of motion:

$$\dot{H} = \sum_{\nu=1}^{r} (H_{q_\nu}\dot{q}_\nu + H_{p_\nu}\dot{p}_\nu) = 0. \tag{1.2}$$

A formal solution of (1) can be written, obviously, as

$$\begin{aligned} q_\nu &= e^{tD} q_\nu(0), \quad p_\nu = e^{tD} p_\nu(0), \qquad (\nu = 1, 2, ..., r), \\ D &= \sum_{\nu=1}^{r} (H_{p_\nu}\partial_{q_\nu} - H_{q_\nu}\partial_{p_\nu}). \end{aligned}$$

The differential operator D gives the time derivative of any function $z(q_\nu, p_\nu)$: $\dot{z} = Dz$. The total state of the Hamiltonian system is defined by the totality of the components q_ν and p_ν.

A system of n mass points, moving in the 3-dimensional physical (Euclidean) space according to the laws of Newtonian mechanics, is the prototype of a Hamiltonian system. Each of the mass points, consisting of matter concentrated at a point, has three position co-ordinates and three momentum components, thus $r = 3n$. The momentum components of each mass point are defined as the product of its mass and its respective component of velocity, and can be accordingly written as

$$p_\nu = m_\nu \dot{q}_\nu, \qquad (\nu = 1, 2, ..., r) \tag{1.3}$$

where the same mass m_ν, of course, appears three times. Then the equations of motion are given by (1), with the Hamiltonian function

$$H = T + V, \quad T = \sum_{\nu=1}^{r} \frac{1}{2m_\nu} p_\nu^2, \quad V = V(q_\nu). \tag{1.4}$$

Here T, V and H are the kinetic, potential and total energy, respectively, of the system.

2. The Lagrangian function. Because of (3) the laws of motion of a system of mass points can also be expressed in terms of the q_ν and $\dot{q}_\nu$ instead of the q_ν and p_ν. Instead of the Hamiltonian equations (1) we then have the *Euler equations of motion*,

$$\frac{d}{dt} L_{\dot{q}_\nu} = L_{q_\nu}, \qquad (\nu = 1, 2, ..., r). \tag{1.5}$$

The Lagrangian function L gives the difference between the kinetic and potential energies, i.e.

$$L(q_\nu, \dot{q}_\nu) = \frac{1}{2} \sum_{\nu=1}^{r} m_\nu \dot{q}_\nu^2 - V(q_\nu). \tag{1.6}$$

The formulae (1) and (5) of course are mutually equivalent expressions of the equations of motion of the mass point system. The transformations $H \to L$ and $L \to H$, or from the variables (q_ν, p_ν) to the variables $(q_\nu, \dot{q}_\nu)$ and vice versa, can be given the following symmetric form:

$$L(q_\nu, \dot{q}_\nu) + H(q_\nu, p_\nu) = \sum_{\nu=1}^{r} \dot{q}_\nu p_\nu, \tag{1.7}$$

$$L_{\dot{q}_\nu} = p_\nu, \quad H_{p_\nu} = \dot{q}_\nu. \tag{1.8}$$

The latter equations obviously follow from (1) and (5), while (7) has the double kinetic energy on both sides. From (7) we immediately obtain the further relation

$$L_{q_\nu} + H_{q_\nu} = 0.$$

3. The action principle. We now consider an $r + 1$-dimensional space of the variables u_ν and the time t. Let the u_ν be functions of time, continuous and continuously differentiable. Let A and B be two points of this space, defined by $A = (u_\nu(0), 0)$ and $B = (u_\nu(t_1), t_1)$, $t_1 > 0$. We let B move freely on the surface $T(u_\nu(t_1), t_1) = 0$.

For any function $L(u_\nu, \dot{u}_\nu, t)$, continuous and at least twice differentiable with respect to its variables, and satisfying the condition

$$|L_{\dot{u}_\nu \dot{u}_\mu}| \neq 0,$$

where the expression $|\cdot|$ means determinant, the *geodetic distance* from A to B is defined as the smallest value of the integral

$$J(u_\nu(t_1), t_1) = \int_0^{t_1} L(u_\nu, \dot{u}_\nu, t)\, dt. \tag{1.9}$$

A necessary condition of an extremal value of J is obtained by variation of the path $u_\nu(t)$ from A to B and requiring that $\delta J = \int_0^{t_1} \delta L dt = 0$. But

$$\delta L = \sum_{\nu=1}^{r} \left(L_{u_\nu} \delta u_\nu + L_{\dot{u}_\nu} \delta \dot{u}_\nu \right),$$

where, by partial differentiation, we have

$$L_{\dot{u}_\nu} \delta \dot{u}_\nu = \frac{d}{dt} \left(L_{\dot{u}_\nu} \delta u_\nu \right) - \frac{d}{dt} L_{\dot{u}_\nu} \delta u_\nu .$$

This gives, since $\delta u_\nu(0) = 0$:

$$\delta J = \sum_{\nu=1}^{r} \int_0^{t_1} \left(L_{u_\nu} - \frac{d}{dt} L_{\dot{u}_\nu} \right) \delta u_\nu \, dt + \sum_{\nu=1}^{r} L_{\dot{u}_\nu}(t_1) \delta u_\nu(t_1) = 0.$$

Since this must hold good for any points on the path and since $\delta u_\nu(t_1)$ is different from zero at least for some value of ν, the first term gives the Euler equations now written for the function L *with an explicit time-dependence*,

$$\frac{d}{dt} L_{\dot{u}_\nu}(u_\nu, \dot{u}_\nu, t) = L_{u_\nu}(u_\nu, \dot{u}_\nu, t), \qquad (\nu = 1, 2, ..., r)$$

while the second term defines what is called the *natural boundary conditions*:

$$L_{\dot{u}_\nu}(t_1) = 0. \qquad (\nu = 1, 2, ..., r) \tag{1.10}$$

If we drop the explicit time-dependence of the *action function* L, and if we put $u_\nu = q_\nu$, we can identify L with the Lagrange function of a system of mass points. Thus interpreted the proof given in this section has shown that the motion of particles in such a system takes place along the "least action" path.

4. The Legendre function. In imitation of the equations (7) and (8) holding good for a mass point system, we can define for any given action function $L(u_\nu, \dot{u}_\nu, t)$ now the *canonical momentum* v_ν, associated with the function u_ν, by writing

$$v_\nu \stackrel{def}{=} L_{\dot{u}_\nu}, \qquad (\nu = 1, 2, ..., r) \tag{1.11}$$

and the *Legendre function* G by the formula

$$G(u_\nu, v_\nu, t) + L(u_\nu, \dot{u}_\nu, t) \stackrel{def}{=} \sum_{\nu=1}^{r} \dot{u}_\nu v_\nu. \tag{1.12}$$

In the imitation, the rest of the transformation (7)-(8) gives

$$\dot{u}_\nu = G_{v_\nu}. \qquad (\nu = 1, 2, ..., r) \tag{1.13}$$

By derivation, (12) immediately gives

$$G_{u_\nu} + L_{u_\nu} = 0. \tag{1.14}$$

It follows that the Euler equations of the action principle are equivalent to the equations

$$\dot{u}_\nu = G_{v_\nu}, \quad \dot{v}_\nu = -G_{u_\nu}, \qquad (\nu = 1, 2, ..., r) \tag{1.15}$$

called the *canonical equations of motion.*

It is to be emphasized that in these equations both functions G and L in the general case depend explicitly on time.

Thus the canonical dynamical system is in the general case *not a Hamiltonian system* and it has necessarily *no constant of motion.*[1]

However, just like in a Hamiltonian system to every Lagrangian function there corresponds a Hamiltonian function and vice versa, in a canonical system *to every variational integrand or the action function $L(u_\nu, \dot{u}_\nu, t)$ there corresponds a Legendre function $G(u_\nu, v_\nu, t)$ and vice versa.*

[1] In dynamic economics the dynamical systems usually are canonical systems, not Hamiltonian ones. Therefore no constant of motion is necessary.This has been sometimes forgotten, for instance when P.Mirowski (1990,p.302) regrets that the economists "have never made up their minds about what precisely it is that should be conserved in their theoretical system".

5. Transversality conditions. For a small time-displacement δt_1 in (9) we get, in view of (12):

$$\delta_{t_1} J = L(t_1)\delta t_1 = \left[\sum_{\nu=1}^{r} \dot{u}_\nu(t_1) v_\nu(t_1) - G(t_1)\right] \delta t_1 .$$

Similarly, for a small displacement of the path $u_\nu(t)$ we first get by applying (12):

$$\delta_u J = \int_0^{t_1} \sum_{\nu=1}^{r} [v_\nu \delta \dot{u}_\nu + \dot{u}_\nu \delta v_\nu - G_{u_\nu} \delta u_\nu - G_{v_\nu} \delta v_\nu]\, dt .$$

Here the canonical equations (15) can be applied to give

$$\delta_u J = \sum_{\nu=1}^{r} \int_0^{t_1} \frac{d}{dt}(v_\nu \delta u_\nu)\, dt = \sum_{\nu=1}^{r} v_\nu(t_1)\delta u_\nu(t_1). \tag{1.16}$$

(Note that $\delta u_\nu = 0$ for $t = 0$.)

Written for a variable time t instead of the fixed time t_1 the variational formulae give:

$$\frac{dJ}{dt} = \sum_{\nu=1}^{r} J_{u_\nu} \dot{u}_\nu - G, \quad \text{thus } \partial_t J = -G.$$

Hence we get the *Hamilton-Jacobi equation*

$$\partial_t J + G(J_{u_\nu}, u_\nu, t) = 0. \tag{1.17}$$

It defines the geodetic distance from A to B as a function of the moving end point $B = (u_\nu(t), t)$.

On the other hand, the surface of equivalence of the geodetic distance J from the fixed initial point A to the point B is given by

$$J(u_\nu(t_1), t_1) = \text{Constant},$$

or, written for the variation δJ:

$$\delta J(t_1) = \sum_{\nu=1}^{r} (J_{u_\nu})_{t_1} \delta u_\nu(t_1) + (\partial_t J)_{t_1} \delta t_1 = 0. \tag{1.18}$$

In order for the extremal path $u_\nu(t)$ to be transverse with respect to a given boundary surface $T(u_\nu(t_1), t_1) = 0$ through the point B, the equation of the surface T, in a differential form

$$\delta T(t_1) = \sum_{\nu=1}^{r} (T_{u_\nu})_{t_1} \delta u_\nu(t_1) + (\partial_t T)_{t_1} = 0,$$

must be equivalent to (18) in a neighbourhood of B, i.e. there must be a proportionality of the coefficients:

$$(\partial_t J)_{t_1} : (J_{u_\nu})_{t_1} = (\partial_t T)_{t_1} : (T_{u_\nu})_{t_1} . \qquad (\nu = 1, 2, ..., r)$$

These relations are called the *conditions of transversality* of the extremal path $u_\nu(t)$ with respect to the given surface $T = 0$. In view of (16) and (17) they can be written in the form

$$(1.19) \qquad \left(\frac{-G}{v_\nu}\right)_{t_1} = \left(\frac{\partial_t T}{T_{u_\nu}}\right)_{t_1} . \qquad (\nu = 1, 2, ..., r)$$

They too are, of course, only necessary but not sufficient conditions of the attainment of the minimal distance from A to B.

A further necessary but not sufficient condition of the minimal distance is the *Legendre condition*

$$(1.20) \qquad \sum_{\nu,\mu=1}^{r} \dot{u}_\nu |L_{\dot{u}_\nu \dot{u}_\mu}| \dot{u}_\mu \geq 0 \quad \text{for} \quad 0 \leq t < t_1 .$$

For the derivation of this condition a consultation of the well-known advanced textbook of Courant-Hilbert, Methods of Mathematical Physics I-II (in its English translation first printed by Interscience Publishers, New York, in 1953) is suggested.

6. The "principle of the largest action". If instead of a minimum J a maximum of J is wanted, we simply have to replace the variational integrand L by $-L$ in all the above formulae. The *Legendre transformation* (11)-(13) then reads:

$$(1.21) \qquad v_\nu = -L_{\dot{u}_\nu}, \quad \dot{u}_\nu = G_{v_\nu}, \qquad (\nu = 1, 2, ..., r)$$

$$(1.22) \qquad G(u_\nu, v_\nu, t) = L(u_\nu, \dot{u}_\nu, t) + \sum_{\nu=1}^{r} \dot{u}_\nu v_\nu .$$

Thus we have now

$$(1.23) \qquad G_{u_\nu} = L_{u_\nu}, \qquad (\nu = 1, 2, ..., r)$$

and the canonical equations of motion retain their form (15).

The transversality conditions (19) and the natural boundary conditions (10) are unchanged, while the Legendre condition (20) obviously is changed to

$$\sum_{\nu,\mu=1}^{r} \dot{u}_\nu |L_{\dot{u}_\nu \dot{u}_\mu}| \dot{u}_\mu \leq 0 \quad \text{for} \quad 0 \leq t < t_1. \tag{1.24}$$

It is in the form of the "principle of the largest action" that the Hamilton-Jacobi theory is applied to growth theory.

2. Maximization of Accumulating Utility

1. Canonical equations for discounted utilities. Suppose we have to maximize the function

$$J(u_\nu(t_1), t_1) = \int_0^{t_1} e^{-\rho t} U(u_\nu, \dot{u}_\nu, t) dt, \quad \rho > 0,$$

where U is a *current-time utility* depending on a number of *factors of production* u_ν, their time derivatives and the time t, ρ being the *discount rate*.

The problem is one of the action principle applied to discounted utility

$$L(u_\nu, \dot{u}_\nu, t) = e^{-\rho t} U(u_\nu, \dot{u}_\nu, t). \tag{2.1}$$

For L the Legendre transformation (1.21)-(1.22) is valid: the canonical momentum v_ν is now defined by

$$v_\nu = -L_{\dot{u}_\nu} = -e^{-\rho t} U_{\dot{u}_\nu}, \qquad (\nu = 1, 2, ..., r) \tag{2.2}$$

the Legendre function G by

$$G(u_\nu, v_\nu, t) = e^{-\rho t} U(u_\nu, \dot{u}_\nu, t) + \sum_{\nu=1}^{r} \dot{u}_\nu v_\nu, \tag{2.3}$$

and the canonical equations are given by

$$\dot{u}_\nu = G_{v_\nu}, \quad \dot{v}_\nu = -G_{u_\nu} = -L_{u_\nu}. \tag{2.4}$$

Let us introduce, with Kurz (1968), the variables

$$\pi_\nu \stackrel{def}{=} -U_{\dot{u}_\nu} = e^{\rho t} v_\nu, \qquad (\nu = 1, 2, ..., r) \tag{2.5}$$

$$G^* \stackrel{def}{=} e^{\rho t} G(u_\nu, v_\nu, t). \tag{2.6}$$

In terms of these variables the canonical equations (1.15) become

$$\dot{u}_\nu = e^{-\rho t} G^*_{v_\nu} = G^*_{\pi_\nu}, \qquad (\nu = 1, 2, ..., r) \tag{2.7}$$

$$\dot{\pi}_\nu = \rho \pi_\nu - G^*_{u_\nu}. \qquad (\nu = 1, 2, ..., r) \tag{2.8}$$

These equations are in economics often called the "modified Hamiltonian equations", which is misleading in two ways: first, it misleads one to believing that the function G^* is a Hamiltonian function, which it is not; secondly, it wrongly suggests that the system considered is a Hamiltonian system, which it is not. The system is a non-Hamiltonian canonical system, and the function G^* of course belongs to the category of Legendre functions. Let us call it the *current-time Legendre function*.

The transversality conditions retain their form (1.19), only the definitions of the variable v_ν and the function G have changed to the forms (2) and (3), respectively. When expressed in terms of the Kurz momentum π_ν the conditions are:

$$\left(\frac{-G}{e^{-\rho t}\pi_\nu}\right)_{t_1} = \left(\frac{\partial_t T}{T u_\nu}\right)_{t_1}. \qquad (\nu = 1, 2, ..., r) \tag{2.9}$$

The Legendre condition is that of the "principle of the largest action", i.e. (1.24). The natural boundary conditions (1.10) too are unchanged but can be now written in the form

$$\left(e^{-\rho t} U_{\dot{u}_\nu}\right)_{t_1} = 0. \qquad (\nu = 1, 2, ..., r) \tag{2.10}$$

Mostly in applications to growth theory $t_1 = \infty$.

2. The Solow growth model revisited. We have a single factor of production, the physical capital K, together with a given labour force $N = N(0)e^{nt}, n > 0$, to produce the output Y and the consumption per capita c by means of the following production function and growth equation:

$$Y = AK^\beta N^{1-\beta}, \quad \dot{K} = sY = Y - cN, \quad \dot{A}/A = g > 0, \tag{2.11}$$

$$c = \frac{1}{N}\left(AK^\beta N^{1-\beta} - \dot{K}\right) = c(K, \dot{K}, t), \quad 0 < \beta < 1. \tag{2.12}$$

We maximize

$$J(K, t_1) = \int_0^{t_1} e^{-\rho t} N V(c(K, \dot{K}, t))dt = N(0)\int_0^{t_1} e^{(n-\rho)t} V(c)dt, \quad t_1 = \infty, \tag{2.13}$$

where ***current-time utility per capita*** V will be determined so that the Legendre condition and the natural boundary condition are satisfied.

The canonical Kurz momentum p associated with the variable K, according to (2),(5) and (11), is

$$p = -N\,(V_c)\,c_{\dot{K}} = V_c, \tag{2.14}$$

By imposing on $V(c)$ the further assumption that it is a bijection, we have

$$c = V_c^{-1}(p), \tag{2.15}$$

which expresses the consumption per capita in terms of the Kurz momentum p.

The current-time Legendre function G^* is then, according to (3), (6),(13) and (14), given by

$$G^*(K,p,t) = NV(c(p)) + p\dot{K}(K,p,t), \tag{2.16}$$

where, in view of (11),

$$\dot{K}(K,p,t) = Y(K,t) - c(p)N(t).$$

It follows that the canonical equations of motion, (7) and (8), now have the form

$$\begin{aligned} &(2.17) \quad \dot{K} &= \; & Y(K,t) - c(p)N(t), \qquad \text{(growth equation of capital)} \\ &(2.18) \quad \dot{p} &= \; & \rho p - \beta p Y/K. \qquad \text{(Euler equation)} \end{aligned}$$

The natural boundary condition and the Legendre condition now become

$$\lim_{t\to\infty} e^{-\rho t} V_c = 0, \tag{2.19}$$

$$V_{cc} < 0 \quad \text{for finite times } \; t \geq 0, \tag{2.20}$$

respectively.

The "balanced-growth" substitutions

$$Y \to Y^* = Y^*(0)e^{\lambda t}, \quad K \to K^* = K^*(0)e^{\lambda t},$$

where

$$\lambda = s^* b^* = n + \frac{g}{1-\beta}, \tag{2.21}$$

s^* and b^* being positive constants, viz. the balanced-growth net savings rate and the balanced-growth net output/capital ratio, respectively, obviously satisfy the growth equation in (11), equivalent to (17). This gives

$$c \to c^* = b^*(1 - s^*)[K^*(0)/N(0)]e^{(\lambda - n)t} \to \infty \quad \text{with } t \to \infty$$

since, because of (21),

$$\lambda > n. \tag{2.22}$$

It follows that the choice

$$V(c) = \frac{1}{1-\sigma}\left(c^{1-\sigma} - 1\right) \tag{2.23}$$

for the function V satisfies (19) and (20) for any positive σ. (For $\sigma = 1$ it gives $\log c$.) Here the positive constant σ can be interpreted as the *coefficient of risk aversion* (Lucas,1988). With this choice for the function $V(c)$ we have, in view of (14),

$$p = c^{-\sigma}. \tag{2.24}$$

To satisfy also the Euler equation (18) on the balanced-growth path we must have, along with (24), the further parameter condition

$$\rho + \sigma(\lambda - n) = \beta b^*. \tag{2.25}$$

When solved for σ this gives

$$\sigma = \frac{\beta b^* - \rho}{s^* b^* - n}. \tag{2.26}$$

Thus the risk aversion coefficient is not an independent parameter but is determined by β, b^*, s^*, ρ and n. In (26) both the nominator and denominator are positive, because of (21) and (22).

3. The parameter conditions of transversality in the Solow model. Let us study under which conditions the balanced-growth solution satisfies the transversality condition. It is natural to choose the surfaces of equivalence of output, $Y(K,t) =$ Constant, as the boundary surfaces $T = 0$. This gives, in view of (9) and (11), the following transversality condition:

$$\left(\frac{-G}{e^{-\rho t}p}\right)_{t_1} = \left[\frac{g + n(1-\beta)}{\beta}\right] K(t_1), \quad \text{for } t_1 = \infty. \tag{2.27}$$

In this form the condition is not valid in the Solow model, since for $t_1 = \infty$ the left-hand side gives negative infinity, while the right-hand side gives the positive one. By rewriting it as

$$\left[\frac{g + n(1-\beta)}{\beta}\right] e^{-\rho t} p(t_1) K(t_1) = -G(t_1), \quad \text{for } t_1 = \infty, \tag{2.28}$$

which for a finite t_1 is equivalent to (27) this problem is avoided.

First we shall study under what conditions, to be imposed on parameters, the transversality condition (28) of the Solow model can be given in its usual "textbook form"

$$e^{-\rho t} p(t) K(t) \to 0 \quad \text{with } t \to \infty. \tag{2.29}$$

By applying (6),(16) and (23) we get

$$G = N(0) e^{(n - \rho)t} \left(\frac{c^{1-\sigma} - 1}{1 - \sigma} \right) + e^{-\rho t} p \dot{K}. \tag{2.30}$$

There are two alternative cases:

Case (i). On the condition that

$$\sigma > 1 \tag{2.31}$$

we have

$$\frac{c^{1-\sigma} - 1}{1-\sigma} \to \text{Constant} \quad \text{with } t \to \infty.$$

On this condition the first term in (30) approaches zero with time, provided that the further condition

$$\rho > n \tag{2.32}$$

is also satisfied. From (31) and (32) it follows, in view of (26), that also the following parameter relation holds good:

$$s^* < \beta. \tag{2.33}$$

But this is just the condition under which the second term in (30) vanishes in the limit $t \to \infty$. Thus (31) and (32) express the necessary and sufficient conditions under with the transversality condition of the Solow growth model assumes the "textbook form" (29).

Case (ii). If (31) is not valid, the maximal utility per capital function will be obtained for $\sigma = 0$, in which case

$$\frac{c^{1-\sigma} - 1}{1-\sigma} \to c - 1.$$

This approaches infinity on the balanced-growth path like the exponential function $\exp(\lambda - n)t$. It follows that the first term in (30) will approach zero with increasing time on the condition that we have now, instead of (31),

$$\rho > \lambda. \tag{2.34}$$

This is a very strong condition, stronger than (32) and, in view of empirical evidence, also stronger than (31). In order that the "textbook form" would express the transversality condition in the Case *(ii)*, both parameter relations (33) and (34) must hold good.

We have so far studied under which conditions the transversality relation of the Solow model can be given its "textbook form" (29). But the same conditions already certify the validity of this relation on the balanced-growth path. This is because the parameter condition (33) is also sufficient to make the expression $e^{-\rho t}p^*K^*$ to approach to zero asymptotically. This evokes the queston: why to write at all the "textbook version" of the transversality condition, since its validity is already guaranteed when writing it down?

A potential objection to the above question could be that the conditions of convergence might be different in the cases, where the initial state $K(0)$ is not on the balanced-growth path. But this is not true. On the contrary, it has been proved by D.Cass that for any initial capital $K(0) > 0$ the general solution of the Solow model will converge to the balanced-growth path asymptotically.

The importance of the parameter relations (22),(25),(26),(31) and (32) has been emphasized above, since the same relations will appear also in the Lucas growth model and in its generalization later on. A further relation of considerable interest is obtained when solving the balanced-growth Euler equation (25) for the parameter b^*:

$$b^* = \frac{n - \rho/\sigma}{s^* - \beta/\sigma}. \tag{2.35}$$

We shall have reason to return also to this formula later.

Comparison with empirics. Lucas (1988) used estimates based on the Denison (1961) numbers calculated from the U.S. economy in the period 1909-57. These numbers are in accordance with the theoretical parameter conditions obtained in Case *(i)*. The relevant empirical estimates are:

$$\beta b^* \approx .0675 > \lambda \approx .027 > n \approx .013, \quad s^* \approx .1 < \beta \approx .25. \tag{2.36}$$

For the discount rate ρ a value between .02 and .04 is regarded reasonable. This would give for the risk aversion coefficient the limits 3.96 and 1.39, respectively, in accord with Case *(i)* above.

There are five independent parameters in the Solow model. They can be chosen in several ways, for instance: n, β, ρ, s^* and b^*.

4. The Arrow-Kurz generalization. From (16),(17) and (24) we get:

$$G_c^* = N(V_c - p) = 0 \quad \text{for } p = V_c,$$
$$G_{cc}^* = NV_{cc} < 0.$$

Thus the Legendre function G^* attains a maximum with respect to c at the point, where V_c is equal to the Kurz momentum p.

If we forget now the origin of this Legendre function, and *define* a surrogate function H^* by writing

$$H^*(K,p,t;c) \stackrel{def}{=} NV(c) + p\dot{K}(K,t,c), \quad \dot{K} = Y(K,t) - cN, \tag{2.37}$$

we can perform the maximization of the accumulated utility

$$J = \int_0^{t_1} e^{-\rho t} NV(c)dt$$

in the following simple way.

First choose $c = \hat{c}$, where $\hat{c}$ solves $H_c^* = 0$ and $H_{cc}^* < 0$, to get a function H^o of the variables K, p and t:

$$H^o(K,p,t) = NV(\hat{c}) + p\dot{K}(K,t,\hat{c}). \tag{2.38}$$

Then, obviously,

$$H_p^o = \dot{K}. \tag{2.39}$$

Consider then the Euler equation

$$\frac{d}{dt}L_{\dot{K}} = L_K,$$

which is a necessary condition of the maximization of the accumulated utility J, provided that

$$L(K,\dot{K},t) = e^{-\rho t}NV(c). \tag{2.40}$$

Hence we get successively:

$$\begin{aligned}
\frac{d}{dt}L_{\dot{K}} &= \frac{d}{dt}\frac{\partial}{\partial \dot{K}}e^{-\rho t}NV(c) \\
&= \rho e^{-\rho t}V_c - e^{-\rho t}\frac{d}{dt}V_c, \\
L_K &= e^{-\rho t}NV_c c_K = e^{-\rho t}V_c Y_K.
\end{aligned}$$

For $c = \hat{c}$ this gives

$$\dot{p} = \rho p - H_K^o. \tag{2.41}$$

Thus we can maximize the accumulated utility also by 1) maximizing H^o with respect to the *control parameter c* and 2) writing down the "modified Hamiltonian equations" for the surrogate function H^o, in this connection often called the "current-time Hamiltonian".

Arrow and Kurz (1970) showed that this shorthand method can be as well used in a many-factor and many-parameter case, to maximize

$$J = \int_0^{t_1} e^{-\rho t} U(u_\nu, \dot{u}_\nu, t; a_\mu) dt, \tag{2.42}$$

subject to the growth equations

$$\dot{u}_\nu = f^{(\nu)}(u, t; a_\mu), \qquad (\nu = 1, 2, ..., r; \mu = 1, 2, ..., m), \tag{2.43}$$

where $u = (u_1, u_2, ..., u_r)$. Thus one must first write down the equations $\partial H^o / \partial a_\mu = 0$, and then add to the list of equations the "modified Hamiltonian equations" for each pair of the canonical variables (u_ν, π_ν). The three kinds of necessary conditions, viz. the natural boundary conditions, the Legendre condition and the transversality conditions must of course been finally added to the list. This generalized method will be applied in the next Chapter.

II

THE LUCAS GROWTH THEORY AND ITS GENERALIZATION TO BUSINESS CYCLES

3. The Lucas Growth Theory

The Lucas growth theory (Lucas (1988) has a particular structure that reflects the idea of rational expectations. First there is a situation, in which the households and firms react to what is generally *expected* to be common knowledge, i.e. a sort of average level of human capital in society. In the theory this level is exogeneously given, just as the exogeneous factor of technological progress in the Solow model. The fundamental equations of the theory are constructed in this situation, which will be here called the "reaction of the market to common knowledge". Then market clearing creates a second situation, in which the exogeneously given and the endogeneously produced average levels of human capital coincide. The solution of the fundamental equations has to take place in the second phase, to be called here the "market clearing".

1. The reaction of the market to common knowledge. We maximize

$$(3.1)\quad J = \int_0^\infty L dt = \int_0^\infty e^{-\rho t} N\left(\frac{c^{1-\sigma}-1}{1-\sigma}\right) dt, \quad N(t) = N(0)e^{nt},$$

subject to

$$(3.2)\quad \dot{K} = sY = Y(K,h,t;u) - cN,$$

$$(3.3)\quad Y = AK^{\beta}(huN)^{1-\beta}h_a^{\kappa},$$

$$(3.4)\quad \dot{h} = k(1-u)h,$$

where n, A, κ, k and β are positive constants there being $\beta < 1$ as usual. What is new when compared to the Solow model is the ***human capital*** as a new factor of production. It appears in two versions, viz. as the human capital h to be endogeneously determined by the theory, and as the exogeneously given 'common knowledge' h_a. The factor h represents the actual average level of knowledge and skills per worker in society, while h_a stands for the generally expected average level.

A new control parameter also appears, viz.u, which is the share of the total working time $N(t)$ used in production in the period of production (i.e.

the year) t. The remaining part $(1-u)N$ of the total working time is spent in education and thus devoted to the accumulation of human capital. Learning by doing is not included in this Lucas theory, but it will be included in the generalization to be given in Chapter 4. The factor of production huN represents the *labour input* to production, where the impact of knowledge and skills is taken into account.

The surrogate H^* for the Legendre function G^*, which appears in the Arrow-Kurz generalization, is now given by

$$H^*(K,h,p,q,t;c,u) = N\left(\frac{c^{1-\sigma}-1}{1-\sigma}\right) + p\dot{K}(K,h,t;c,u) + q\dot{h}(h,t;u),$$

where p and q are the Kurz momentums associated with K and h, respectively. The equations of motion accordingly include the following ones:

$$\begin{aligned} &(3.5) \qquad & H^*_c &= 0, \quad \text{i.e. } p = c^{-\sigma}, \\ &(3.6) \qquad & H^*_u &= 0, \quad \text{i.e. } u = (1-\beta)pY/kqh. \end{aligned}$$

The "modified Hamiltonian equations" can then be written for the "current-time Hamiltonian" H^o. They add to the growth equations (2) and (4) the Euler equations

$$\begin{aligned} &(3.7) \qquad & \dot{p}/p &= \rho - (1/p)H^o_K = \rho - \beta Y/K \\ &(3.8) \qquad & \dot{q}/q &= \rho - (1/q)H^o_h = \rho - k. \end{aligned}$$

The latter is obtained immediately from (6) and from the fact that H^o is equal to H^*, where c and u solve the equations (5) and (6), respectively.

We can see at once that the Legendre condition is satisfied, since in view of (1),(2),(3) and (4) we have:

$$\left\| \begin{matrix} L_{\dot{K}\dot{K}} & L_{\dot{K}\dot{h}} \\ L_{\dot{h}\dot{K}} & L_{\dot{h}\dot{h}} \end{matrix} \right\| = \left\| \begin{matrix} -(\sigma/N)e^{-\rho t}c^{-(\sigma+1)} & 0 \\ 0 & 0 \end{matrix} \right\|.$$

Thus the quadratic form (1.24) of Chapter 1 now reduces to

$$-(\sigma/N)e^{-\rho t}c^{-(\sigma+1)}(\dot{K})^2 < 0 \quad \forall t.$$

The natural boundary conditions are

$$L_{\dot{K}} = -e^{-\rho t}c^{-\sigma} \to 0, \quad L_{\dot{h}} \to 0 \quad \text{with } t \to \infty.$$

The first one is satisfied, if $c \to \infty$, which is the case again, at least on the balanced-growth path, of which we are interested. The latter condition is fulfilled, because $L_{\dot{h}} = 0$.

2. The market clearing. The markets now adjust the exogeneously given and the endogeneously produced levels of human capital to coincide:

$$h(t) = h_a(t) \;\; \forall t. \tag{3.9}$$

Thus instead of (3) we now have

$$Y = AK^{\beta}(uN)^{1-\beta}h^{1-\beta+\kappa}. \tag{3.10}$$

The transversality conditions in their original form, of course, are the following:

$$\left(\frac{-G}{e^{-\rho t}p}\right)_{t_1} = \left(\frac{\partial_t Y}{Y_K}\right)_{t_1}, \quad \left(\frac{-G}{e^{-\rho t}q}\right)_{t_1} = \left(\frac{\partial_t Y}{Y_h}\right)_{t_1},$$

with $t_1 = \infty$. Here

$$\partial_t Y = n(1-\beta)Y, \;\; Y_K = \beta Y/K, \;\; Y_h = (1-\beta+\kappa)Y/h.$$

But again, as in the Solow model, this original form of transversality conditions would give $-\infty = +\infty$, and it has to be replaced by the form

$$\lim_{t\to\infty} e^{-\rho t}pK = -\frac{\beta}{n(1-\beta)}\lim_{t\to\infty} G, \tag{3.11}$$

$$\lim_{t\to\infty} e^{-\rho t}qh = -\frac{1}{n}\left(\frac{1-\beta+\kappa}{1-\beta}\right)\lim_{t\to\infty} G. \tag{3.12}$$

To see on which conditions they are valid we have to study the solution of the defining equations of motion (2)-(8) of the Lucas growth theory.

Again the balanced-growth path will be the most interesting part of the solution of the equations (2),(4)-(8) and (10), in fact the only part of solution which can be actually worked out. It is defined by

$$Y^* = Y^*(0)e^{\lambda t}, \;\; K^* = K^*(0)e^{\lambda t}, \;\; h^* = h^*(0)e^{\nu t}.$$

This together with (2) gives again

$$c^* = c^*(0)e^{(\lambda - n)t},$$

but for ν and λ we now get, from (4) and (10), respectively, the conditions

$$\begin{aligned}\nu &= k(1-u^*), \quad \text{thus } u^* = \text{Constant},\\ \lambda &= \beta\lambda + (1-\beta)n + (1-\beta+\kappa)\nu.\end{aligned}$$

Hence three new parameter conditions emerge: first

$$u^* = \frac{k-\nu}{k}, \quad 1-u^* = \frac{\nu}{k}, \tag{3.13}$$

then, because we must have $1 < u^* < 1$,

$$k > \nu > 0, \tag{3.14}$$

and finally:

$$\lambda - n = \left(\frac{1-\beta+\kappa}{1-\beta}\right)\nu > 0. \tag{3.15}$$

With (10)and (13)-(15) the above balanced-growth substitutions solve the growth equations (2) and (4). The balanced-growth equation (2) of physical capital again gives the further relation

$$\lambda = s^* b^*. \tag{3.16}$$

Turning to the Euler equations (7) and (8), the former one together with (5) gives on the balanced-growth path the important parameter condition

$$\rho + \sigma(\lambda - n) = \beta b^*, \tag{3.17}$$

met in the Solow model already. From this together with (16) we have again:

$$\sigma = \frac{\beta b^* - \rho}{s^* b^* - n}, \quad b^* = \frac{n - \rho/\sigma}{s^* - \beta/\sigma}. \tag{3.18}$$

As to the equation (8) it of course is already a general solution and holds good on the balanced-growth path as well.

Of the two conditions (5) and (6) to be imposed on the respective control parameters c and u, the equation (6) gives, as a consequence of a constant balanced-growth share u^* and (17):

$$k = \beta b^* - (\lambda - \nu). \tag{3.19}$$

It remains to consider the transversality conditions (11) and (12). Now the Legendre function G comprises three terms:

$$G = e^{-\rho t} N V(c) + e^{-\rho t} p\dot{K} + e^{-\rho t} q\dot{h}. \tag{3.20}$$

We again have, just like in the Solow model, two possible ways of satisfying the transversality conditions:

Case (i). Here we have the parameter conditions

$$\sigma > 1 \quad \text{and} \quad \rho > n, \tag{3.21}$$

from which, in view of (16) and (18),

$$s^* < \beta \tag{3.22}$$

follows. The conditions (21) make the first term in G to approach asymptotically zero and the condition (22) makes the second term to do so. In order for the third term in G to approach zero asymptotically we need the further parameter condition (14), which of course must be valid for other reasons too. On the conditions (14),(21) and (22) we can thus write the transversality relations in their "textbook form"

$$e^{-\rho t} pK \to 0 \ \text{ and } \ e^{-\rho t} qh \to 0 \ \text{ with } t \to \infty. \tag{3.23}$$

Actually, just like in the Solow model, there is no reason to do so, since these relations are already satisfied on the same conditions under which they could be written down.

Case (ii). If we reject the condition that $\sigma > 1$, we have to impose on ρ the very strong condition that

$$\rho > \lambda. \tag{3.24}$$

This together with (14) and (22) will in this case allow us to write down the transversality relations in the "textbook form" (23). And again the same conditions already suffice to satisfy them.

Comparison with empirics. The Denison estimates (US 1909-57) can again be used to check the order of magnitudes. As an addition to the Solow model we can now make use also of the average annual rate of growth of education, which in the U.S. in that period gives $\nu \approx .009$. This together with the earlier mentioned numbers $\beta \approx .25, \lambda \approx .027$ and $n \approx .013$ gives, in view of (15), $\kappa \approx .417$. Lucas (1988) chose $k = .05$, which by (13) gives $u^* \approx .82$ and $1 - u* \approx .18$, which are not too bad numbers for approximating the percentage of American grown-ups in production and in education, respectively. But the coefficient k is not an independent parameter but depends, as shown by (19), on the values of $\beta b^*, \lambda$ and ν. This gives $k = .0635$, which indeed is not far from the value .05 suggested by the Lucas intuition. The value .0635 of k gives for the share of the people in education the value .14.

Lucas (1988) also considered the values of ρ between .02 and .04 as resonable. They give, by (18), for the risk aversion coefficient σ the corresponding limits of variation, between 3.39 and 1.96, respectively. Here the numbers $s^* \approx .1$ and $b^* \approx .27$, calculated from the Denison estimates, were used in addition to the already mentioned numbers. Again the empirical estimates suggest that Case *(i)* has to be preferred to Case *(ii)* as the source of the parameter conditions to be applied.

There are six independent parameters in the Lucas theory, for instance $n, \rho, \beta, \kappa, s^*$ and b^*.

4. Generalization to Include Business Cycles

1. The first axiom of generalization. If $N(t)$ is the grand total of the living times of all grown-up people in society during the period of production, i.e. the year, t, we write

$$N(t) = v(t)N(t) + [1 - v(t)])N(t) = N_o e^{nt}, \quad n > 0, N_o > 0 \tag{4.1}$$

$$v(t)N(t) = u(t)v(t)N(t) + [1 - u(t)]v(t)N(t) \tag{4.2}$$

Here $v(t)N(t)$ is the grand total of working times of all those people during the year t, $[1 - v(t)]N(t)$ being the grand total of their leisure times in that year. The coefficient $v(t)$ thus is the share of working time of the grand total of their living times in that year. The coefficient $u(t)$ tells the share of the time spent in technical or manual kind of work in the total working time vN. The rest of the total working time, $[1 - u(t)]v(t)N(t)$ has been devoted to the accumulation of human capital, i.e. of knowledge and skills.

Here a difference when compared with the concepts of Lucas is evident: now both the time spent in school or in listening to some teachings, that is, in education *and the time at work during which learning-by-doing has taken place are included* in the time devoted to the accumulation of human capital. Lucas exluded the latter part of it, since it cannot be measured. We shall include it because of its importance in the learning process. Thus the share u is here accepted as a *hidden variable*, which cannot itself be observed but which affects observable variables.

The total human time $N(t)$ is an important economic variable, because it necessarily appears in the economic process both as an input and as an output. Here, however, its production by the economic process is only implicitly involved, while its effects on economic development will be explicitly analysed. This implies that, in addition to the total working time vN, we shall study also the economic effects of the leisure time $(1-v)N$.

The effects of leisure time upon economic development are here assumed to be so remarkable that leisure is represented by a term in the current-time utility function. Thus our *Axiom No.1* is the following value function:

$$J = \int_0^\infty L dt = \int_0^\infty e^{-\rho t} \left[N \left(\frac{c^{1-\sigma} - 1}{1-\sigma} \right) + \xi(1-v)N \right] dt,$$

where ξ is a weight coefficient of leisure, to be endogeneously determined.

This weight thus measures the strength of wish for leisure in population, that is, the wish for the control of one's own time, or the wish for individual freedom if you like. Thus the second term introduces a nonmaterial value to the economic utility function, in addition to the material value as expressed by the consumption term.

The idea behind the inclusion of the second term is that people do not use leisure only for being lazy. Or, if they are lazy, their thoughts may fruitfully wander and search new domains they would never find in their routine work. In fact the thesis can be promoted that people are at their most innovative just during the time, which in economic statistics must be registered as leisure. Innovations, on the other hand, are essential for scientific, technological and economic progress alike.

2. The fundamental equations. In accordance with the two-phase process of the Lucas theory we shall first maximize the accumulated utility given above subject to

$$\begin{aligned}
&(4.3) \qquad & \dot{K} &= sY = Y(K,h,t;c,u,v) - cN, \\
&(4.4) \qquad & Y &= AK^{\beta}(huvN)^{1-\beta}h_a^{\kappa}, \\
&(4.5) \qquad & \dot{h} &= k(1-u)vNh, \\
&(4.6) \qquad & k &= k_o e^{mt}, \quad k_o > 0, \quad m > 0,
\end{aligned}$$

the factor h_a being exogeneously given. Here the growth equation (5) of human capital differs essentially from the corresponding Lucas equation. This *Axiom No.2* will be explained in the next paragraph.

The maximization can again be done by means of the Kurz-Arrow method, starting with the "current-time Hamiltonian"

$$H^*(K,h,p,q,t;c,u,v) = N \left(\frac{c^{1-\sigma} - 1}{1-\sigma} \right) + \xi(1-v)N + p\dot{K} + q\dot{h}.$$

It gives first:

$$(4.7) \qquad H_c^* = 0, \quad \text{i.e. } p = c^{-\sigma},$$

$$(4.8) \qquad H_u^* = 0, \quad \text{i.e. } u = \frac{(1-\beta)pY}{kqhvN},$$

$$(4.9) \qquad H_v^* = 0, \quad \text{i.e. } v = \frac{(1-\beta)pY}{N[\xi - kqh(1-u)]}.$$

The growth equations of both factors of production having been given already, by (3) and (5), respectively, we still have to write the Euler equations, in which the function H^o is equal to the function H^* in which the control parameters satisfy their equations (7)-(9). Thus we get:

$$(4.10) \qquad \dot{p}/p = \rho - (1/p)H_K^o = \rho - \beta Y/K,$$

$$(4.11) \qquad \dot{q}/q = \rho - (1/q)H_h^o = \rho - (1-\beta)pY/qh - k(1-u)vN.$$

Finally, before the solution of the equations (3)-(11), we have to indicate the market clearing by making

$$h(t) = h_a(t) \ \forall t,$$

thus replacing (4) by the *ex post* production function

$$(4.12) \qquad Y = AK^{\beta}(uvN)^{1-\beta}h^{1-\beta+\kappa}.$$

The formulae (1)-(3) and (5)-(12) together with the natural boundary conditions, the Legendre condition and the transversality relations define the generalized dynamics to be studied in the rest of this book.

3. The second axiom of generalization. Let us stop for a while to consider the second essential deviation from the Lucas theory, viz. the new growth equation (5) of human capital.

The main reason for introducing the formula (5) is the wish to emphasize the character of the accumulation of knowledge as a social activity. Lucas himself was far from denying this, as he wrote: "human capital accumulation is a social activity, involving groups of people in a way that has no counterpart in the accumulation of physical capital" (Lucas, 1988, p.19). But this property of human capital accumulation is better represented by (5), *not reducible to microeconomics*, than by the Lucas formula. The latter, when expressed in our variables, would be

$$(4.13) \qquad \dot{h} = k'(1-u)vh.$$

It is impossible to find any direct evidence or observation to support the form (5) rather than (13), because of the impossibility to observe directly, at least with the presently available scientific means, the processes, social or individual, in which knowledge is accumulated. However, it is possible to reject an objection that the formula (5) may easily evoke. The idea, involved in (5), that the growth of human capital also depends on the magnitude of the "population" N or, more exactly, on the grand total of time spent by all individuals in society to the accumulation of knowledge and skills, is by no means unrealistic. Consider for instance the contribution to scientific civilization of advanced nations, such as the Americans, British, Germans and French. If we study the national distribution of great innovations among these nations, we can see the advantages of a large civilized population quantitatively confirmed. In the period 1840-1971 studied by van Duijn (1981), the U.S. led by 49 innovations (56%), before Germany's 20 or 23%, Britain's 10 (11%) and France's 9 (10%). But important innovations contribute to the *growth* of common knowledge, i.e. the average human capital h.

The numbers evidently are much higher than the shares of the corresponding populations in the world, which suggests some necessary distribution of education or wealth, or both, before we can speak of a systematic accumulation of average human capital in a given society. It is also understood that the innovations on the highest level of some Einstein or Newton or Galilei or Hamilton are done by individuals. What is suggested here is that a large and dense civilized community creates an intellectually activating environment, where those rare individuals capable of real innovations find stimulation and incentive. Just like cities often are more intellectually activating than the countryside.

There is also a formal argument favouring the 'mass effect' formula (5), and thus the idea that in human capital formation indeed "the effect of a whole may be greater than the sum of the effects of the parts", if you like. One can show that, while the differential equations of the Basic Business Cycles (paragraph 4 below) can be derived with both (5) or (13), the existence of solutions requires the form (5). Indeed, if (13) is used, the existence of acceptable solutions demands that $k' = kN$, which returns (13) to (5).

4. The derivation of a general solution algorithm. Let us use the short notations

$$\psi \stackrel{def}{=} \rho + m - \dot{\xi}/\xi, \tag{4.14}$$

$$\Phi \stackrel{def}{=} (\dot{\xi}/\xi) + (\dot{\psi}/\psi) \tag{4.15}$$

We can then write the following equations:

$$qh = \xi/k, \tag{4.16}$$

$$uvN = \psi/k, \tag{4.17}$$

$$pY = \xi\psi/(1-\beta)k. \tag{4.18}$$

The first of them is obtained by dividing the left and right side of (8) by the corresponding side of (9). The next is then obtained from (8) and (11), when using the notation (14). The last equation is given by (8),(16) and (17).

From (18), by using (1),(3),(6),(7) and (15), we get successively:

$$Y = \left[\frac{(1-\beta)k}{\xi\psi}\right]^{\frac{1}{\sigma-1}} \left[\frac{N}{1-s}\right]^{\frac{\sigma}{\sigma-1}}, \tag{4.19}$$

$$\frac{\dot{Y}}{Y} = \left(\frac{\sigma}{\sigma-1}\right)\left(n + \frac{m-\Phi}{\sigma} + \frac{\dot{s}}{1-s}\right). \tag{4.20}$$

By inserting Y from (18) into (10) we have first:

$$K = \frac{Y}{\frac{1}{\beta}\left(\frac{\dot{Y}}{Y} + \rho + m - \Phi\right)}. \tag{4.21}$$

By derivation with respect to time and equating the result with (3) this gives:

$$\frac{\dot{Y}}{Y} - s\frac{Y}{K} = \frac{\frac{d}{dt}\left(\frac{\dot{Y}}{Y} + \rho + m - \Phi\right)}{\frac{\dot{Y}}{Y} + \rho + m - \Phi}.$$

By inserting here $\dot{Y}/Y$ from (20) and Y/K from (21) we get for the *net savings rate* s the following second-order differential equation:

$$\frac{d}{dt}\log\left(a\frac{\dot{s}}{1-s} + b\right) = (\beta - s)\left(a\frac{\dot{s}}{1-s} + b\right) - \alpha, \tag{4.22}$$

where

$$a = \frac{\sigma}{\beta(\sigma-1)}, \tag{4.23}$$

$$b = a(\alpha + n - \rho/\sigma), \tag{4.24}$$

$$\alpha = \rho + m - \Phi. \tag{4.25}$$

The variable whose logarithm is taken in (22) will be denoted by w. It represents the ***net output/capital ratio*** and satisfies, obviously, the following identities:

$$w \equiv a\frac{\dot{s}}{1-s} + b \equiv \frac{1}{\beta}\left(\frac{\dot{Y}}{Y} + \rho + m - \Phi\right) \equiv \frac{Y}{K}. \tag{4.26}$$

In terms of the variables s and w we can rewrite (22) in the form of two mutually coupled first-order differential equations:

$$\dot{s} = \frac{1}{a}(1-s)(w-b), \quad \dot{w}/w = (\beta - s)w - \alpha. \tag{4.27}$$

The time functions N and k having been given by (1) and (6), respectively, a general solution algorithm of the fundamental equations can now be given in terms of the time functions ξ and ψ and the constants of this dynamics:

(1) Solve the savings rate s from (22), choose the parameters a, b and α to satisfy the conditions (23)-(25).

(2) Choose the (because of (16)-(18)) positive-valued functions ψ and ξ in accordance with (24) and (25).

(3) Find the (net) output Y from (19).

(4) Compute the physical capital K from (21).

(5) Get the human capital h from (12), after inserting there uvN from (17).

(6) Find the total working time vN and its share v in the total life time N from (5), after inserting uvN from (17) into (5),

(7) Compute the other allocation function u then from (17).

Of the remaining two fundamental variables p and q the price function p, related to the expected future values of physical capital, can be found immediately after the third step of the above algorithm, by using (18). The price q telling about the expected future values of human capital can be computed after the fifth step from (16).

The dependent economic variables, such as the total consumption $C = (1-s)Y$, investment sY, employment i.e. the labour input $E = huvN = h\Psi/k$ to the production function, and the productivity of labour i.e. the real wage level $W = Y/E$ etc. can be of course then easily computed on the basis of the known fundamental variables.

5. The natural boundary conditions. In the generalized dynamics they obtain the form of the following conditions:

$$L_{\dot{K}} = -e^{\rho t}p \to 0, \tag{4.28}$$

$$L_{\dot{h}} = -e^{\rho t} q \to 0, \tag{4.29}$$

with $t \to \infty$. The former one is the same as in the Lucas theory. The latter needs some attention since now, otherwise than in the Lucas case, we have $L_{\dot{h}} \neq 0$.

Now also the control parameter v, and not only c, is included in the utility function. But this parameter is a function of $\dot{h}$, since in view of (5) the equation (9) can be written as

$$v = \frac{(1-\beta)pY}{\xi N - q\dot{h}/v}.$$

Solved for v we get

$$v = \frac{(1-\beta)pY + q\dot{h}}{\xi N}, \quad \text{thus } v_{\dot{h}} = q/\xi N, \tag{4.30}$$

which gives (29).

6. The Legendre condition. We have again, as in the Lucas theory,

$$L_{\dot{K}\dot{K}} = -\frac{\sigma}{N} e^{-\rho t} c^{-\sigma - 1}. \tag{4.31}$$

Obviously $L_{\dot{K}\dot{h}} = 0$ and $L_{\dot{h}\dot{K}} = 0$, since c and thus p does not depend on $\dot{h}$, and q does not depend on $\dot{K}$. But, in view of (5) and (16) we have:

$$q = (v - uv)\xi N/\dot{h}. \tag{4.32}$$

Here uv according to (17) is equal to ψ/kN and does not depend on $\dot{h}$. Thus we get, by applying (30):

$$q_{\dot{h}} = \frac{\xi N}{\dot{h}} v_{\dot{h}} - \frac{\xi N}{\left(\dot{h}\right)^2}(1-u)v = (1/\dot{h})[q - \xi N(1-u)v/\dot{h}].$$

This expression however, in view of (5) and (16), is equal to zero. Thus $L_{\dot{h}\dot{h}} = 0$ and the quadratic form in the Legendre condition is determined by the same 4x4 matrix as in the Lucas theory. Hence this condition is satisfied in the generalized dynamics as well.

7. Transversality conditions. It follows from (6),(12) and (17) that we have now, as a deviation from the Lucas theory,

$$\partial_t Y = (1-\beta)\left(\frac{\dot{\psi}}{\psi} - m\right),$$

while the derivatives Y_K and Y_h are the same as in the Lucas case. It will be shown in the next Part that there will be either $\dot{\psi}/\psi = 0$ or then $\dot{\psi}/\psi \to 0$ with $t \to \infty$. Thus the transversality conditions of the generalized dynamics will assume the form

$$\lim_{t\to\infty}\left(\frac{-G}{e^{-\rho t}p}\right) = -\left[\frac{m(1-\beta)}{\beta}\right]\lim_{t\to\infty} K, \tag{4.33}$$

$$\lim_{t\to\infty}\left(\frac{-G}{e^{-\rho t}q}\right) = -\left[\frac{m(1-\beta)}{1-\beta+\kappa}\right]\lim_{t\to\infty} h. \tag{4.34}$$

It can be seen from these expressions that the generalized dynamics indeed differs very much from the Lucas or Solow theories, which in this respect were close to each other. The ultimate reason for this great difference is the appearance of nonmaterial values in the generalized dynamics. Such values are implied by the introduction of the leisure term into the current-time utility. This affects the character of economic theory in an essential way, as we shall see in Part 3 and especially in Part 4 of this study. It also improves the way in which the transversality conditions are satisfied: in the generalized dynamics they will be fulfilled in their above original form.

Note (The mutual interference of material and nonmaterial values as the cause of the business cycles). We have seen that the introduction to the current-time utility of the leisure term $\xi(1-v)N$, standing for individual freedom and thus for nonmaterial values in general (as assumed), produced the second-order differential equation (4.22) or, equivalently,the two mutually coupled first-order differential equations (4.27). Such equations did not appear in the Lucas growth theory and they disappear, if we eliminate the leisure term in utility by putting $\xi = 0$. Thus the mentioned differential equations are produced by the mutual interaction of the material and the nonmaterial parts of utility, the former being of course represented by the consumption term in the value function J. It will be seen in the next chapter that these differential equations define what we shall call the Basic Business Cycles. Thus these cycles are produced by the *continuous mutual interference* between material and non-material values, just like the electromagnetic waves are produced by the continuous mutual interference of electricity and magnetism. Thus, in the present theory, the economic boom and recession have the *same* reason, not different ones.

III

THE GENERAL THEORY OF ECONOMIC GROWTH AND BUSINESS CYCLES

5. The Basic Growth Paths

1. The balanced-growth path: Growth Type 1. Following the algorithm it is easily verified that a special solution of the equations of motion (4.3),(4.5)-(4.12), and of the relations (4.14)-(4.15) and (4.23)-(4.25) associated with them,is given by

(5.1) $\quad 1 > s = s^* = \text{Constant} > 0, \quad \text{thus } w = b,$

(5.2) $\quad a > 0 \ \text{ i.e. } \sigma > 1, \quad b = b^* = \text{Constant} > 0, \quad \psi = \text{Constant},$

(5.3) $\quad \alpha = \alpha^* = (\beta - s^*)b^* = \text{Constant} > 0 \quad i.e. \ \ s^* < \beta,$

(5.4) $\quad \psi = \alpha^*, \quad \xi = \xi_o e^{(\rho + m - \alpha^*)t}, \quad \xi_o > 0,$

(5.5) $\quad Y = Y^* = Y_o^* e^{\lambda t}, \quad K = K^* = K_o^* e^{\lambda t}, \quad h = h^* = h_o^* e^{\nu t},$

(5.6) $\quad \lambda = s^* b^*, \quad \nu = \left(\dfrac{1-\beta}{1-\beta+\kappa}\right)(\lambda + m),$

(5.7) $\quad v = v^* = v_o^* e^{-(m+n)t}, \quad v_o^* = (\alpha^* + \nu)/k_o N_o < 1,$

(5.8) $\quad 1 > u = u^* = \alpha^*/(\alpha^* + \nu) = \text{Constant} > 0,$

(5.9) $\quad p = p^* = p_o^* e^{(\rho - \beta b^*)t},$

(5.10) $\quad q = q^* = q_o^* e^{(\rho - \alpha^* - \nu)t}.$

After the choices (1) and (2) the choices (3)-(10) are uniquely determined.

Again the Euler equation for the "price" p gives the important parametric relation

(5.11)
$$\rho + \sigma(\lambda - n) = \beta b^*,$$

met already in both the Solow and Lucas growth theories. Solved for σ or for b^* this gives, in view of (6),

(5.12)
$$\sigma = \frac{\beta b^* - \rho}{s^* b^* - n}, \quad b^* = \frac{n - \rho/\sigma}{s^* - \beta/\sigma},$$

respectively.

The natural boundary conditions (4.28)-(4.29) are satisfied because of (9) and (10). It follows from (5) and from (4.1) that

$$c = c^* = (1 - s^*)(Y_o^*/N_o)e^{(\lambda - n)t} \to \infty \quad \text{with } t \to \infty,$$

provided that

$$\lambda > n. \tag{5.13}$$

Coming to the transversality conditions (4.33)-(4.34) we see at once that the right-hand sides go on the balanced-growth path to negative infinity with $t \rightarrow \infty$. Consider then the left-hand sides

$$-G/e^{-\rho t}p^* = -G^*/p^* \quad \text{and} \tag{5.14}$$

$$-G/e^{-\rho t}q^* = -G^*/q^*, \tag{5.15}$$

where

$$G^* = NV + \xi(1-v)N + p^*\dot{K}^* + q^*\dot{h}^*$$

We shall study the behaviour of each term when $t \rightarrow \infty$ in detail. There is now, because of (2), $\sigma > 1$, in which case the function $V(c)$ approaches asymptotically a positive constant. The absolute values of the two first terms in (14) go to infinity with $t \rightarrow \infty$ as follows:

$$NV/p^* \sim e^{[n + \sigma(\lambda - n)]t}, \quad \xi(1-v)N/p^* \sim e^{(m + n + \lambda)t},$$

the first term because of (9) and (13), the second one because of (3), (4), (7), (9) and (11). The absolute values of the third and fourth terms go to infinity in a similar way as shown by the formulae

$$\dot{K}^* \sim e^{\lambda t}, \quad q^*\dot{h}^*/p^* \sim e^{\lambda t},$$

the former one because of (5) and the latter one because of (5),(9) and (9)-(11). Thus also the left-hand side of the transversality condition (4.33) goes to negative infinity with $t \rightarrow \infty$, which that this condition is satisfied in its original form discussed in Part 1.

Consider then the left-hand side (15) of the transversality condition on human capital. The absolute value NV/q^* of the first term behaves asymptotically as shown by

$$NV/q^* \sim e^{(\alpha^* + \nu + n - \rho)t},$$

because of (10). For $\rho < \alpha^* + \nu + n$ this goes to infinity, otherwise it approaches asymptotically a constant, possibly zero. Its behaviour is not important for the validity of transversalitu conditions. This is because the absolute values of the other terms in (15) always rise to infinity, as shown by

$$\xi(1-v)N/q^* \sim e^{(m+n+\nu)t}, \quad p^*\dot{K}/q^* \sim e^{\nu t}, \quad \dot{h}^* \sim e^{\nu t},$$

the first of them because of (4),(7) and (10), the second one because of (5),(9) and (10), and the third one because of (5).

It follows that also the second transversality condition (4.34) is fulfilled on this path of the generalized dynamics. Both transversality conditions are now satisfied in their original forms studied in Part I.

Let it be remarked that also the formulae

$$e^{-\rho t}p^*K^* \to 0, \quad e^{-\rho t}q^*h^* \to 0$$

are valid in the generalized dynamics too, the first one because of (3)-(5) and (9), the second one because of the above formula (4) and the formulae (4.6) and (4.16). This time, however, these formulae do not guarantee the validity of the transversality conditions.

2. The path of logistically rising productivity of capital: Growth Type 2. Again we follow the algorithm by choosing first:

$$1 > s = s^* = \text{Constant} > 0, \quad \text{thus } w = b, \tag{5.16}$$

$$a > 0 \text{ i.e. } \sigma > 1, \quad b \neq \text{Constant}, \quad \psi \neq \text{Constant}. \tag{5.17}$$

From (4.27) we see that the parameter b has now to obey the equation

$$\dot{b}/b = (\beta - s^*)b - \alpha.$$

In view of (4.24) this is equivalent to

$$\dot{b}/b = (n - \sigma/\rho) - (s^* - \beta/\sigma)b.$$

Here we can express $n - \rho/\sigma$ in terms of the solution b^* the Growth Type 1, obeying the equation (12) above, to get:

$$\dot{b}/b = (s^* - \beta/\sigma)(b^* - b). \tag{5.18}$$

This is solved by

$$b = \frac{b^*}{1 + Be^{-\gamma t}}, \quad \text{where } \gamma = n - \rho/\sigma, \quad B = \frac{b^* - b(0)}{b(0)}. \tag{5.19}$$

This is a logistic function with two branches. For $\gamma < 0$ it would give a downward curve approaching asymptotically zero, which predicts an economic catastrophe but is not an interesting solution from the point of view of growth theory. The other option $\gamma > 0$ is more interesting and gives the branch of the logistic function rising asymptotically toward the Growth Type 1 solution b^*. Choosing this solution we thus have in Growth Type 2, in addition to the b-equations (18) and (19), also the conditions

$$(5.20) \qquad \dot{b} > 0, \quad \gamma > 0 \quad \text{i.e. } s^* - \beta/\sigma > 0.$$

The last condition of course is in view of (12) equivalent to the second one, since b^* is positive.

Next consider the parameter α. By applying (4.23) and (4.24), and for γ again (12), we obtain

$$(5.21) \qquad \alpha = (\beta - \beta/\sigma)b - \gamma = \alpha^* - (\beta - \beta/\sigma)(b^* - b),$$

which expresses the relation between the Growth Type 2 parameter α and the corresponding Growth Type 1 parameter α^*.

For the auxiliary ψ we now get, with the help of (4.14),(4.15) and (4.25), the differential equation

$$(5.22) \qquad \dot{\psi}/\psi = \psi - \alpha.$$

This is solved, in view of (18) and (21), by

$$(5.23) \qquad \psi = (\beta - s^*)b > 0 \; \forall t.$$

This gives, in view of (4.14):

$$(5.24) \qquad \xi = \xi_o e^{(\rho + m)t - \int_0^t \psi \, dt}, \quad \xi_o > 0.$$

Thus we have settled the two first steps in the general solution algorithm. For the next steps we first get from (4.20), by applying $\dot{s} = 0$, (4.25) and the above formulae (21) and (6) in this order,

$$(5.25) \qquad \dot{Y}^\dagger/Y^\dagger = \lambda - (\beta/\sigma)(b^* - b),$$

where λ is the growth rate of output in Growth Type 1. This gives

$$(5.26) \qquad Y^\dagger = Y_o^\dagger e^{\lambda t - (\beta/\sigma)\left(b^* t - \int_o^t b dt\right)}, \quad \text{and } K^\dagger = Y^\dagger/b,$$

which settles the third and fourth steps of the algorithm.

To do the fifth step we first calculate, with the help of the ex post production function (4.12) and the above formulae (18),(21),(23) and (25),

$$\text{(5.27)} \qquad \frac{\dot{h}^\dagger}{h^\dagger} = \nu + \left(\frac{\beta}{1-\beta+\kappa}\right)(s^* - \beta/\sigma)(b^* - b),$$

where ν is the growth rate of human capital in Growth Type 1. This gives:

$$\text{(5.28)} \qquad h^\dagger = h_o^\dagger \exp\left\{\nu t + \left(\frac{\beta}{1-\beta+\kappa}\right)\left[\gamma t - (\gamma/b^*)\int_0^t b\,dt\right]\right\}.$$

To get the sixth step done we can write, by inserting (4.17) into (4.5) and solving for the share v, the generally valid equation

$$\text{(5.29)} \qquad v = \frac{1}{kN}\left(\frac{\dot{h}}{h} + \psi\right).$$

It can of course be developed further by inserting $\dot{h}/h$ from (27) and ψ from (23), but this is left to the reader, as they say. The other share parameter u is then in view of (4.17) given by the generally valid equation

$$\text{(5.30)} \qquad u = \frac{\psi}{\psi + \dot{h}/h}.$$

It remains to study the Euler equations (4.10) and (4.11). They give:

$$\text{(5.31)} \qquad \dot{p}^\dagger/p^\dagger = \rho - \beta b, \quad \dot{q}^\dagger/q^\dagger = \rho - \psi - \dot{h}^\dagger/h^\dagger.$$

Here b is to be taken from (19), ψ from (23) and $\dot{h}^\dagger/h^\dagger$ from (27), after which the function $q^\dagger(t)$ can be calculated in a way similar to the calculations of $Y^\dagger$ in (26) and $h^\dagger$ in (28).

After the choices (16), (17) and (20) have been made, the above Growth Type 2 solution for the basic growth path is uniquely determined.

The natural boundary conditions (4.28) and (4.29) are satisfied without further conditions, since it follows from (31) that $e^{-\rho t}p^\dagger$ and $e^{-\rho t}q^\dagger$ both approach asymptotically zero. The Legendre condition was proved to hold good generally in the generalized dynamics (see Part 2).

Since it has been already proved that the transversality conditions (4.33) and (4.34) are satisfied on the balanced-growth path, i.e. in Growth Type 1 of the generalized dynamics, the case of Growth Type 2 is trivial. This is because the basic growth path of Growth Type 2, studied in this paragraph, asymptotically approaches that of Growth Type 1, as is evident from (19),(21), (23),(24),(25),(27),(29),(30) and (31). Thus we have already proved the validity of the transversality conditions also in the case of the basic growth path of Growth Type 2.

The remark at the end of paragraph 1 can be repeated here: the functions $e^{-\rho t}pK$ and $e^{-\rho t}qh$ vanish asymptotically also on the basic growth path of Growth Type 2, but this does not guarantee the validity of the transversality conditions, which are now valid in their original form derived in Part 1.

3. Verification by the Solow (1957) material. At the very beginning of the modern era of growth theory, in Solow's seminal works in the 1950s already, an empirical material was published which leaves no doubt about the existence of the two Growth Types just described. This material in Solow's 1957 paper was then, of course, published to serve another purpose, viz. that of supporting Solow's representation of technological progress by the exponential function $A(t)$ in the production function (2.11).

Finding statistical materials useful for the verification of a theory was then, and still is, a demanding task if only because economic statistics normally has been collected mainly for variable practical uses and keeps rarely the same standards and units over any longer period. "The capital time series is the one that will really drive a purist mad. For present purposes, 'capital' includes land, mineral deposits, etc. Naturally I have used Goldsmith's estimates (with government, agricultural, and consumer durables eliminated). Ideally what one would like to measure is the annual flow of capital services. Instead one must be content with a less utopian estimate of the stock of capital goods in existence... Lacking any reliable year-to-year measure of the utilization of capital I have simply reduced the Goldsmith figures by the fraction of the labor force unemployed in each year, thus assuming that labor and capital always suffer unemployment to the same percentage." (Solow, 1957, p.314)

The first and second columns of Table 1 were composed and published as the fifth and sixth column, respectively, in the corresponding Table of Solow (ibid., p.315). By dividing the numbers in the column of "Private nonfarm GNP per manhour" by the corresponding numbers in the column "Employed capital per manhour" (the titles used by Solow), one gets the third column of our Table 1. It gives estimates of the annual output/capital ratios Y/K in the U.S. economy from 1909 to 1949, and their graphical illustration is given in Fig.1.

The business cycles in employment are seen as the oscillations of the variable Y/K in the picture. What is interesting for the present purpose,

TABLE 1. – THE SOLOW DATA FOR CALCULATION OF OUTPUT PER CAPITAL*

Year	Priv.nonfarm GNP per manhour (1)	Employed capital per manhour (2)	Output per capital (3)	Year	Priv.nonfarm GNP per manhour (1)	Employed capital per manhour (2)	Output per capital (3)
1909	.623	2.06	.302	1930	.880	3.06	.288
1910	.616	2.10	.293	1931	.904	3.33	.271
1911	.647	2.17	.298	1932	.897	3.28	.273
1912	.652	2.21	.295	1933	.869	3.10	.280
1913	.680	2.23	.305	1934	.921	3.00	.307
1914	.682	2.20	.310	1935	.943	2.87	.329
1915	.669	2.26	.296	1936	.982	2.72	.361
1916	.700	2.34	.299	1937	.971	2.71	.358
1917	.679	2.21	.307	1938	1.000	2.78	.360
1918	.729	2.22	.328	1939	1.034	2.66	.389
1919	.767	2.47	.311	1940	1.082	2.63	.411
1920	.721	2.58	.279	1941	1.122	2.58	.435
1921	.770	2.55	.302	1942	1.136	2.64	.430
1922	.788	2.49	.316	1943	1.180	2.62	.450
1923	.809	2.61	.310	1944	1.265	2.63	.481
1924	.836	2.74	.305	1945	1.296	2.66	.487
1925	.872	2.81	.310	1946	1.215	2.50	.486
1926	.869	2.87	.303	1947	1.194	2.50	.478
1927	.871	2.93	.297	1948	1.221	2.55	.479
1928	.874	3.02	.289	1949	1.275	2.70	.472
1929	.895	3.06	.292				

* Numbers in columns (1) and (2) are given in 1939 dollars.

however, is the behaviour of the level of Y/K, which is to be compared with the theoretical level as represented by the variable b in the generalized dynamics. The numbers and the picture suggest that there was indeed a clear-cut balanced-growth path, i.e. a constant $b = b^*$ from 1909 until the Great Depression of 1929-33. But after that there was an equally clear-cut rise of the logistic type in the level of the output/capital ratio that lasted some twenty years, approaching again a constant but higher value toward the 1950s.

Thus the qualitative behaviour of the U.S. economy in the period 1909-30 in terms of the present theory corresponds to Growth Type 1, while the period 1930-50 was that of Growth Type 2, with its logistic rise of the level of the output/capital ratio. The lines drawn in Fig.1 approximately represent the constancy of b prevailing before the Great Depression and the approximately logistic rise during 1930-50, respectively.

The appearance in the above statistics of the two different kinds of Growth Types predicted by the present unified theory of growth and business cycles is an important fact in favour of this theory: The Solow growth model and the Lucas 1988 growth mechanics are both of them of the Growth Type 1, with a balanced-growth path as the basic growth path.

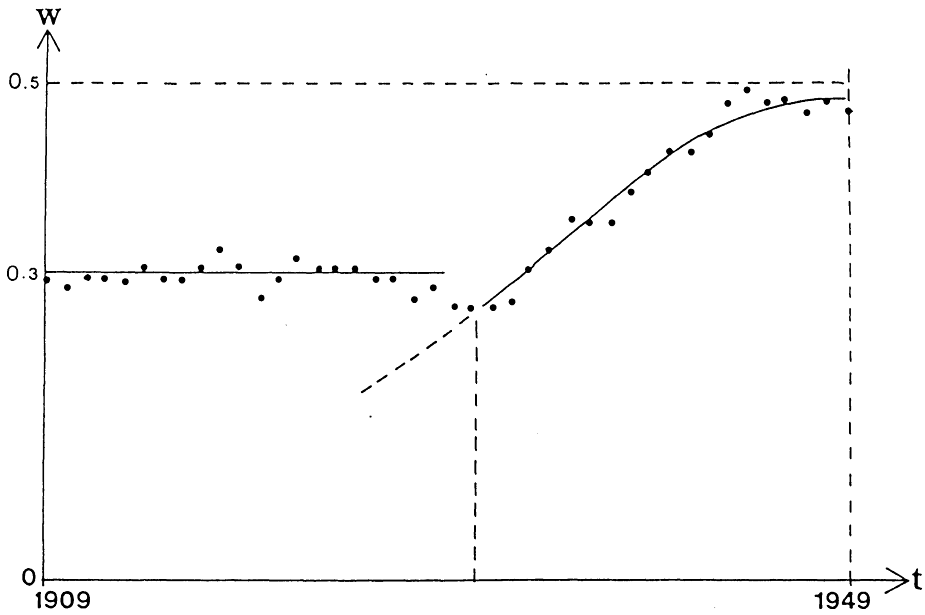

FIGURE 1. – GROWTH TYPES 1 AND 2 VERIFIED BY SOLOW DATA

If the data support the existence of the both Growth Types 1 and 2 so clearly, as it seems to the present author, the question may arise: Why this evidence has been ignored ever since 1957 when they were first published? The answer, to draw from the experience in physics, is simply that the data have been ignored since no theory has been there to explain them. Even scientists are prone to look elsewhere, if they encounter phenomena that are not explicable by the wisdom of the day.

6. The Basic Business Cycles

1. The state-plane and the cycle center. The equations (4.27), where w obeys (4.26), are equivalent with the equations (4.22) and thus start the general solution algorithm of generalized dynanmics given in Chapter 4. Obviously we can accept only the values of s and w such that

$$s(t) < 1, \quad w(t) > 0 \ \ \forall t. \tag{6.1}$$

Under these conditions (4.27) gives:

$$(6.2)\quad \dot{s}\begin{cases} >0 & \text{for } s<1 \text{ and } w>b,\\ =0 & \text{for } s=1 \text{ or } w=b,\\ <0 & \text{for } s<1 \text{ and } w<b.\end{cases}\qquad \dot{w}=\begin{cases} >0 & \text{for } w>\frac{\alpha}{\beta-s},\\ =0 & \text{for } w=\frac{\alpha}{\beta-s},\\ <0 & \text{for } w<\frac{\alpha}{\beta-s}.\end{cases}$$

This tells that the state (s,t) revolves clockwise around the *cycle center* $P=(s^*,b)$ on the plane (s,w). For the Growth Type 1, in which case P is a fixed point, the situation is schematically illustrated in Fig.2. The boundaries Γ_1 and Γ_2 of the state space,

$$\Gamma_1=\{(s,w);\ s=1, w>0\},\quad \Gamma_2=\{(s,w);\ s<1, w=0\},$$

are also indicated.

2. The decreasing relative size of the cycles. It follows from the negative Bendixson criterion that the oscillations of the state (s,t) obeying (4.27) do not approach any limit cycle asymptotically. As the equations (4.27) cannot be solved, they cannot tell more about the asymptotic behaviour of the the state (s,w). But since the Growth Type 2 approaches asymptotically the Growth Type 1 the linearized equations (4.27),

$$(6.3)\quad \dot{s}=(\beta-\beta/\sigma)(w-b^*),\ \ \dot{w}=-(b^*)^2(s-s^*)+\alpha^*(w-b^*),$$

valid in a neighbourhood of the fixed cycle center P, can help us one step further in this question. The solution of (3) can be written as

$$\begin{aligned}(6.4)\quad s-s^* &= e^{\alpha^*t/2}\left(\cos\omega t-\frac{\alpha^*}{2\omega}\sin\omega t\right)[s(0)-s^*]\\ &\quad + e^{\alpha^*t/2}\left(\frac{1-s^*}{\alpha^*\omega}\right)\sin\omega t\,[w(0)-b^*],\\ (6.5)\quad w-b^* &= -e^{\alpha^*t/2}\left[\frac{a}{(1-s^*)\omega}\right]\left[\frac{(\alpha^*)^2}{4}+\omega^2\right]\sin\omega t\,[s(0)-s^*]\\ &\quad + e^{\alpha^*t/2}\left(\cos\omega t+\frac{\alpha^*}{2\omega}\sin\omega t\right)[w(0)-b^*].\end{aligned}$$

We can see that the cycles expand and that the size of the parameter α^* determines how much. The smaller is α^* the less they expand, and in the limit $\alpha^*=0$ we get closed cycles and *stability in the Liapunov sense.* This value of α is however denied because of the conditions $s^*<\beta$ and $b^*>0$

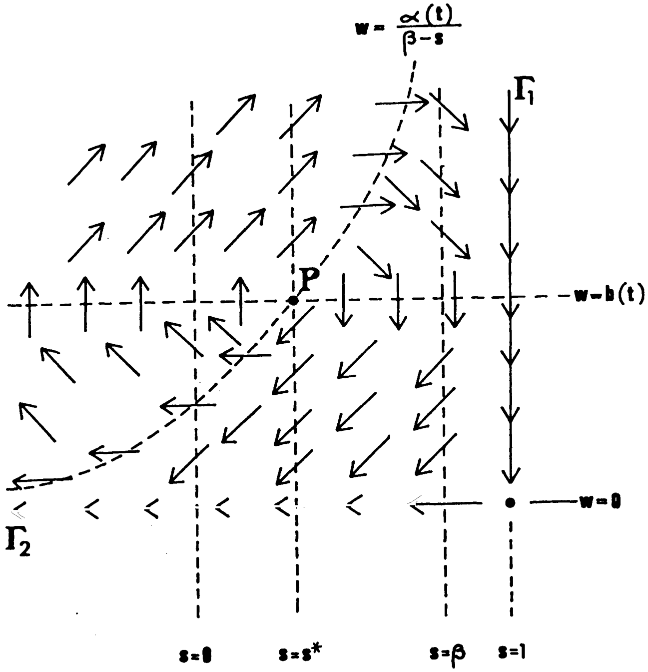

FIGURE 2. – SCHEMATIC ILLUSTRATION OF THE VELOCITY FIELD OF THE STATE (s, w).

valid in generalized dynamics. For a small α we can speak of an approximate Liapunov stability.

What is important from the point of view of economics, however, is not the behaviour of the cycles as separated from the rest of the economic dynamics. The relevant thing is the expansion of the cycles in proportion to the expansion of economy. The development of this proportion is indicated by the coefficient

(6.6) $$A(t) \sim e^{\alpha^* t/2}/e^{\lambda t},$$

where the nominator indicates the growth of the cycles and the denominator that of the output of economy. This gives

$$A(t) \sim e^{(\beta - 3s^*)(b^*/2)t} \to 0 \quad \text{for} \quad s^* > \beta/3.,$$

when $t \to \infty$. The condition is satisfied in all advanced economies.

Thus the relative significance and size of the cycles in proportion to the growth of economy decreases indicating an increasing relative stability of the economic system. It follows that for instance the needs of monetary theory are satisfied.

3. The existence of well-behaving general solutions. So far only the existence of the two special solutions that defined the basic growth paths in the respective Growth Types 1 and 2 has been proved, by means of a detailled construction. Now we have to see to it that the general solution algorithm given in Chapter 4 leads to well-behaving general solutions, i.e solutions of the fundamental equations that retain their properties as meaningful economic variables also over the cycles. (The reader may have expected this at an earlier stage, however, we could not give it before studying the expansion of the cycles in Section 2 above.)

The first of these properties concerned the limits of the savings rate s and the output-capital ratio w imposed on them by the formula (1). Next we have to check that the second step of the algorithm gives an output variable that retains its positivity through all the cycles, i.e. that (cf. (4.19))

$$(6.7) \qquad Y(t) = \left[\frac{(1-\beta)k(t)}{\xi(t)\psi(t)}\right]^{\frac{1}{1-\sigma}} \left[\frac{N(t)}{1-s(t)}\right]^{\frac{\sigma}{\sigma-1}} > 0 \quad \forall t.$$

Here the positivity of k is ascertained by (4.6), that of ξ by (5.4) and (5.24), that of ψ by (5.23), that of N by (4.1) and that of $1-s$ by the above formula (1). The positivity of the physical capital K,

$$(6.8) \qquad K(t) = Y(t)/w(t) > 0 \quad \forall t,$$

is guaranteed over all the cycles by the formulae (1) and (7).

The positivity of the human capital h, as defined by (4.12) and (4.17) in the fourth step of the algorithm, i.e.

$$(6.9) \qquad h(t) = \left(\frac{k(t)}{\psi(t)}\right)^{\frac{1-\beta}{1-\beta+\kappa}} \left(\frac{Y(t)}{AK(t)^{\beta}}\right)^{\frac{1}{1-\beta+\kappa}} > 0 \quad \forall t,$$

is confirmed through the cycles by the positivity of Y, K and N, if the share variables u and v are well-defined i.e. with values between 0 and 1.

We shall first prove that the values of u as given by the general algorithm, obey

$$(6.10) \qquad 0 < u(t) = \frac{\psi(t)}{\psi(t) + \dot{h}(t)/h(t)} < 1$$

over the cycles. Since ψ is a positive function without cycles, the values of u are between 0 and 1, provided that $\dot{h}/h > 0$ always holds good, i.e. that human capital grows all the time even over the cycles. From (9), in view of (4.6) and (5.23), we get first:

$$(1-\beta+\kappa)\frac{\dot{h}}{h} = (1-\beta)(m-\dot{b}/b) + (\dot{Y}/Y - \beta\dot{K}/K).$$

In view of (4.20) and (4.25)-(4.27) we have here

$$\dot{Y}/Y + \beta\dot{K}/K = \beta(1-s)w - \alpha,$$

and, since $\dot{b}/b$ can be written as $(\beta - s^*)b - \alpha$, we get:

$$(6.11) \qquad \begin{aligned} (1-\beta+\kappa)\dot{h}/h \;&=\; (1-\beta)[m-(\beta-s^*)b+\alpha)] \\ &\quad -\alpha+\beta(1-s)w > 0 \quad \forall t \end{aligned}$$

on the condition that

$$(6.12) \qquad (1-\beta)m > \mathrm{Max}[(1-\beta)(\beta-s^*)b + \beta\alpha] = \alpha^*.$$

The positivity of u guarantees in view of (4.17) also the positivity of the other share parameter v. It remains to show that the values of v too, as given by the general algorithm, are also smaller than one, i.e. that

$$(6.13) \qquad v(t) = \frac{1}{k(t)N(t)}\left[\frac{\dot{h}(t)}{h(t)} + \psi(t)\right] < 1 \quad \forall t.$$

From (11) we get, in view of (13) and (5.23):

$$v = \frac{1}{kN}\left\{\beta(1-s)w + (1-\beta)m - \beta[(\alpha+(\beta-s^*)b]\right\}.$$

The other terms within the braces are finite, while $(1-s)w$ has no upper limit. What we have to prove, accordingly, is that the function $(1-s)w/kN$ remains finite, after which a suitable choice of k_o and N_o always keeps the values of v below one.

The cycle equations (4.27) together with (4.23) and (5.21) give:

$$(6.14) \qquad \frac{d}{dt}(1-s)w = (1-s)w\left[\left(n-\frac{\rho}{\sigma}\right) - \left(s-\frac{\beta}{\sigma}\right)w\right].$$

It follows that the relative maximums $(1-\hat{s})\hat{w}$ of the function $(1-s)w$ during the cycles are on the curve

$$w = \frac{n - \rho/\sigma}{s - \beta/\sigma}. \tag{6.15}$$

The fixed cycle center (s^*, b^*) of course is on this curve. For $s^* > \beta/\sigma$, which in view of (5.20) is valid in generalized dynamics, the curve runs entirely between the straight lines $s = \beta/\sigma$ and $s = 1$. It follows that the values $1 - \hat{s}$ are smaller or equal to $1 - \beta/\sigma$ and thus finite. The value $\hat{w}$ approaches infinity when $\hat{s}$ approaches the value β/σ. We accordingly have to study the behaviour of the function $(1 - \hat{s})\hat{w}$ when $\hat{s}$ approaches that value.

Let the points of intersection of the cycle trajectories on the (s, w)-plane with the straight line $s = \beta/\sigma$ be denoted by $(\overline{s}, \overline{w})$. Since these points, because of the diverging nature of the cycle, rise along the curve (15), which approaches asymptotically the straight line $s = \beta/\sigma$ along which the relative maximum points $(\hat{s}, \hat{w})$ rise, we have (Fig.3)

$$\lim_{t\to\infty} \hat{w} = \lim_{t\to\infty} \overline{w}. \tag{6.16}$$

But at the points $(\overline{s}, \overline{w})$, which are on the straight line $s = \beta/\sigma$, (14) gives

$$\frac{d}{dt}\log[(1 - \overline{s})\overline{w} = n - \frac{\rho}{\sigma} < n + m = \frac{d}{dt}\log(kN). \tag{6.17}$$

It follows from (16) and (17) that the function kN grows faster than the relative maximums $(1-\hat{s})\hat{w}$ of the function $(1-s)w$, and we have the result:

$$\frac{(1 - \hat{s})\hat{w}}{kN} \to 0 \quad \text{with} \quad t \to \infty.$$

From this and from the finiteness of $(1 - s)w$ for finite times it follows that also this cycle term in the function v keeps finite for all points of time. By a suitable chose of k_o and N_o we can then always guarantee the validity of (13).

This completes the proof that the cycles do not destroy the properties of the dynamical variables of generalized dynamics as well-behaving economic variables. This is because the positivity, over the cycles, of the "imputed prices" p and q is immediately clear from the equations (4.18) and (4.16), respectively, in which all the variables have only positive values.

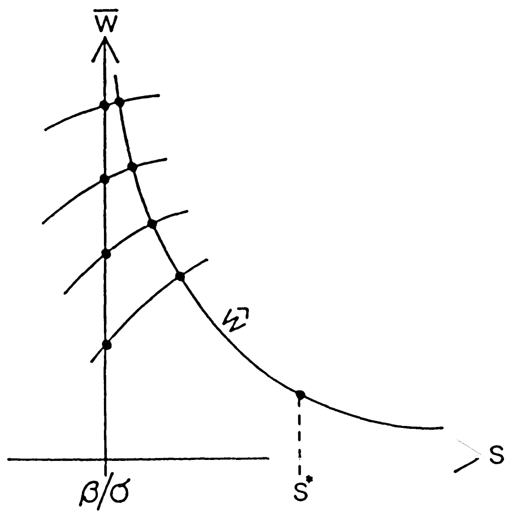

FIGURE 3. – THE RELATIVE MAXIMUMS OF THE FUNCTION $(1-s)w$.

4. The invariance group and the time scale. As soon as continuous growth models are tried to be applied to problems involving finite time intervals, problems appear because the periods of production are themselves indicated by points of zero extension in such models. The solution of this problem, inherent in all economic theories with continuous time, seems to imply the existence of what in physics is called an invariance group involving time. This however presupposes that the fundamental equations of the theory are invariant with respect to the transformations of such a group. The condition is fulfilled in the case of the Basic Business Cycles defined by the equations (3.27), i.e.

$$\dot{s} = (\beta - \beta/\sigma)(1-s)(w-b), \quad \dot{w}/w = (\beta - s)w - \alpha. \tag{6.18}$$

An invariance group is a commonplace in theoretical physics but has to my knowledge not appeared in economic theory so far. Since its significance is essential for the understanding of the theory of Basic Business Cycles (BBC), it will be here studied in detail. It is easy to see that the equations (18) are invariant with respect to the transformations

$$s \to s' = s, \quad w \to w' = gw, \quad t \to t' = t/g, \quad g > 0 \tag{6.19}$$

of the (s, w, t)-space. These transformations can be interpreted as changes of the time scale.

For instance, in a transformation with $g > 1$ the real number t, representing in the theory of the BBC a certain interval of time, obviously becomes smaller. This is equivalent to the introduction of a larger *time unit* [TU] in this theory. It follows that the output Y', obtained during the new and larger time unit [TU] from a given capital K tied up to production, and thus the new output/capital ratio Y'/K are both of them larger than those obtained from this capital during the old and shorter time unit [TU], in the way expressed by the formulae

$$(6.20) \qquad Y' = gY > Y, \quad w' = Y'/K = gw > w = Y/K,$$

respectively.

The parameter b is an output/capital variable and accordingly transforms like w. But so does also the paramneter α, since it in view of (5.12) (the second equation in this formula) and (5.21) can be expressed as

$$(6.21) \qquad \alpha = (\beta - \beta/\sigma)b - (s^* - \beta/\sigma)b^*.$$

Note that the parameter σ, in view of (5.12) (the first equation of this formula), is invariant since in this transformation of time scale we of course have $\rho \longrightarrow g\rho$ and $n \longrightarrow gn$. The parameters β and s are share variables and accordingly invariant.

It follows that the variables in the differential equations (18) transform as follows:

$$(6.22) \qquad \dot{s} \longrightarrow g\dot{s}, \quad \dot{w} \longrightarrow g^2\dot{w}, \quad b \longrightarrow gb, \quad \alpha \longrightarrow g\alpha.$$

This leaves the form of the equations (18) invariant.

From the existence of an invariance group involving a transformation of time scale it follows that there is a new degree of freedom to cope with problems of time in the theory in question. We can choose freely the real time interval equivalent to the theoretical time unit [TU]. If the real time in years is indicated by τ, the scale constant g_o determined by the equation

$$(6.23) \qquad T(t)/g_o = T(\tau) \text{ [years]}$$

must then be used in calculations to be compared with empirical data involving a finit time interval $T(\tau)$.

For instance if, in an application to business cycles, the observed length of period of a cycle is four years, and the length of period obtained from the theory is $T(t) = 2\pi/\omega$, where ω is the angular velocity of the cycle as seen from the cycle center, we have to compute g_o from the equation $g_o = \pi/2\omega$. This scale constant must then be used consequently in the application of the theoretical business cycles to that case.

5. The cycle functions. The theory of the Basic Business Cycles will be verified in Part 4 by comparing the correlations and variances of some of the most important real economic variables over detrended cycles, calculated from the theory, with the corresponding empirical correlations and variances. For this purpose the *cycle functions* Q_X of those variables X, defined by

$$\text{(6.24)} \qquad \frac{\dot{X}}{X} = \left(\frac{\dot{X}}{X}\right)_P + Q_X(s, w),$$

are needed. Here $(\dot{X}/X)_P$ is the growth rate of the variable X on the basic growth path through the cycle center P. It may thus represent either the growth rate on the balanced-growth path or on the path of a logistically rising output/capital ratio, depending on the Growth Type.

For the fundamental growth variables Y (output), K (physical capital), h (human capital), p (imputed future price of physical capital) and q (imputed future price of human capital) the cycle functions are obtained by direct calculation from the defining equations of these variables in the general solution algorithm (Chapter 4) and from the differential equations of the Basic Business Cycles. We then get successively:

$$\text{(6.25)} \qquad Q_Y = \beta(w - b), \quad Q_K = sw - s^*b,$$

$$\text{(6.26)} \qquad Q_h = \left(\frac{\beta}{1 - \beta + \kappa}\right)[(1 - s)w - (1 - s^*)b],$$

$$\text{(6.27)} \qquad Q_p = -Q_X, \quad Q_q = -Q_h.$$

Derived variables that play important roles in empirical comparisons are the total consumption $C = (1 - s)Y$, the total (net) investment $I = sY$, the employment or the employed labour force $E = huvN = h\psi/k$ and the productivity of labour $W = Y/E$. Their cycle functions are readily obtained from the above cycle functions and from the differential equations of the Basic Business Cycles:

$$\text{(6.28)} \qquad Q_C = \frac{\beta}{\sigma}(w - b),$$

$$\text{(6.29)} \qquad Q_I = \left[\beta + \left(\beta - \frac{\beta}{\sigma}\right)\left(\frac{1 - s}{s}\right)\right](w - b),$$

$$(6.30) \qquad Q_E = \left(\frac{\beta}{1-\beta+\kappa}\right)[(1-s)w-(1-s^*)b] = Q_h,$$

$$(6.31) \qquad Q_W = Q_Y - Q_E.$$

The cycle functions of employment and human capital are identical, of course, since the working time $uvN = \psi/k$, devoted to work of purely manual or physical character, in view of (4.17) has no Basic Business Cycles.

IV

THE BASIC BUSINESS CYCLES AS THE CAUSAL PART OF BUSINESS CYCLES

7. The Predictive Power of the Basic Cycles Compared With That of the Stochastic Models: Ordinary Business Cycles

1. The linear approximation of the Basic Business Cycles. The linear approximation (6.4)-(6.5) of the Basic Business Cycles, valid in a neighbourhood of the balanced-growth point $P = (s^*, b^*)$, can be written in the form

$$(7.1) \quad s - s^* \stackrel{Lin}{=} C\, e^{ut} + \bar{C}\, e^{\bar{u}t}, \quad C = \left(\frac{1}{2} + \frac{i\alpha}{4\omega}\right)[s(0) - s^*],$$

$$(7.2) \quad w - b^* \stackrel{Lin}{=} D\, e^{ut} + \bar{D}\, e^{\bar{u}t}, \quad D = \left[\frac{ia(\alpha^2 + 4\omega^2)}{8(1 - s^*)\omega}\right][s(0) - s^*].$$

Here $u = \alpha/2 + i\omega$ and $\bar{u} = \alpha/2 - i\omega$,

$$(7.3) \quad \alpha = (\beta - s^*)b^*, \quad \omega = b^*\sqrt{\left(\frac{1 - s^*}{a}\right) - \left(\frac{\alpha}{2b^*}\right)^2}, \quad a = \frac{\sigma}{\beta(\sigma - 1)},$$

and the state of the system at time 0 has been taken, for convenience, to be such that $w(0) = b^*$.

Direct calculation gives the 'basis integrals':

$$I_s = \int (s - s^*) = \left(\frac{\alpha}{\omega^2 + \alpha^2/4}\right) \cdot \left(e^{\alpha\pi/\omega} - 1\right)[s(0) - s^*],$$

$$I_w = \int (w - b^*) = \left(\frac{a}{1 - s^*}\right) \cdot \left(e^{\alpha\pi/\omega} - 1\right)[s(0) - s^*],$$

$$I_{ss} = \int (s - s^*)^2 = F_{ss}(\alpha, \omega) \cdot \left(e^{2\alpha\pi/\omega} - 1\right)[s(0) - s^*]^2,$$

$$(7.4) \qquad F_{ss} = \left[\frac{\alpha^2 + 4\omega^2}{8\alpha\omega^2} + \alpha\left(\frac{\frac{3}{2} - \alpha^2/8\omega^2}{\alpha^2 + 4\omega^2}\right)\right],$$

$$I_{sw} = \int (s - s^*)(w - b^*) = F_{sw}(a, s^*) \cdot \left(e^{2\alpha\pi/\omega} - 1\right)[s(0) - s^*]^2,$$

$$(7.5) \qquad F_{sw} = \left[\frac{a}{2(1 - s^*)}\right],$$

$$I_{ww} = \int (w-b^*)^2 = F_{ww}(a,\alpha,\omega,s^*)\cdot\left(e^{2\alpha\pi/\omega}-1\right)[s(0)-s^*]^2,$$

$$F_{ww} = \frac{a^2(\alpha^2+4\omega^2)^2}{32(1-s^*)^2\omega^2}\left(\frac{1}{\alpha}-\frac{\alpha}{\alpha^2+4\omega^2}\right). \tag{7.6}$$

All the integrals are over the cycle, i.e. from $t=0$ to $t=T=2\pi/\omega$.

The linear approximation will be used also for the cycle functions (6.25) and (6.28)-(6.31), by expressing each of them in terms of the powers of $s-s^*$ and $w-b^*$ and accepting only the linear terms. This gives a linear function

$$Q_X \overset{Lin}{=} L_X(s-s^*, w-b^*) = A_X(s-s^*) + B_X(w-b^*)$$

for each variable X, with the coefficients

$$A_Y = 0, \quad B_Y = \beta, \quad A_C = 0, \quad B_C = \beta/\sigma, \tag{7.7}$$

$$A_I = 0,\ B_I = \beta\left[1+\left(\frac{\sigma-1}{\sigma}\right)\left(\frac{1-s^*}{s^*}\right)\right],\ A_E = -\left(\frac{\beta}{1-\beta+\kappa}\right)b^*; \tag{7.8}$$

$$B_E = \left(\frac{\beta}{1-\beta+\kappa}\right)(1-s^*),\ A_W = A_Y - A_E,\ B_W = B_Y - B_E. \tag{7.9}$$

The mean values, variances, covariances and correlations of our variables over the detrended cycle are defined by the formulae

$$E(s-s^*) = \frac{1}{T}I_S = m_S, \quad E(w-b^*) = \frac{1}{T}I_W = m_W,$$

$$E(Q_X) \overset{Lin}{=} L_X(m_S, m_W) = m_X,$$

$$E[Q_X - m_X)]^2 \overset{Lin}{=} \frac{1}{T}\int_0^T L_X^2\, dt - m_X^2 = \sigma_X^2,$$

$$E[Q_X - m_X)]\,[Q_Y - m_Y)] \overset{Lin}{=} \frac{1}{T}\int_0^T L_X L_Y\, dt - m_X m_Y = \operatorname{cov}(Q_X, Q_Y),$$

$$r_{XY} = \operatorname{cov}(Q_X, Q_Y)/\sigma_X\sigma_Y.$$

The relevant functions in comparisons with data and with other models are the σ_X/σ_Y and the correlations r_{XY}. The following formulae are suitable for calculations:

$$\left(\frac{\sigma_X}{\sigma_Y}\right)^2 \overset{Lin}{=} \frac{(\omega/2\pi)\left[A_X^2 I_{SS} + B_X^2 I_{WW} + 2A_X B_X I_{SW}\right] - m_X^2}{(\omega/2\pi)B_Y^2 I_{WW} - m_Y^2}, \tag{7.10}$$

$$r_{XY} \overset{Lin}{=} \frac{(\omega/2\pi)\left[A_X B_Y I_{SW} + B_X B_Y I_{WW}\right] - m_X m_Y}{\sigma_X\sigma_Y}. \tag{7.11}$$

The shortcut formulae obtained for $m_X = m_Y = 0$, i.e.

$$(7.12)\quad \left(\frac{\sigma_X}{\sigma_Y}\right)^2 \approx \left(\frac{A_X}{B_Y}\right)^2 \frac{F_{ss}}{F_{ww}} + \left(\frac{2A_X B_X}{B_Y^2}\right) \frac{F_{sw}}{F_{ww}} + \left(\frac{B_X}{B_Y}\right)^2,$$

$$(7.13)\quad r_{XY} \approx \left[\left(\frac{A_X}{B_Y}\right) \frac{F_{sw}}{F_{ww}} + \frac{B_X}{B_Y}\right] (\sigma_X \sigma_Y)^{-1},$$

with the coefficients A and B from (7)-(9) and the F-functions from (4)-(6), do very well. A busy reader can use (12)-(13) to check roughly the numerical calculations. Let it be mentioned that the unknown initial state $s(0) - s^*$ cancels out in the formulae (10)-(13).

2. Comparisons with empirical correlations and variances. The prediction formulae, both the (10)-(11) and (12)-(13), include ten parameters. Only six of them are independent of each other. They will be chosen to be the parameters $\beta, \kappa, \rho, n, b^*$ and s^*, mainly because they can be given realistic numerical values in advance, obtained either from empirical experience or earlier theoretical usage. Thus the estimates

$$\beta = .25, \quad \kappa = .417, \quad \rho = 2\%, \quad n = 1.5\%, \quad b^* = .3$$

are chosen, since these values of β, n and b^* are either identical or very close to the Denison (1961) estimates (as reported by Lucas,1988) based on the averages in the U.S. economy in the period 1909-57, while the number representing κ is exactly the one used by Lucas (ibid.), who also accepted the above value of ρ to be a realistic one. As to the numerical value of the balanced-growth net savings rate s^*, several possibilities are left open. The Denison value .1 is too small for the present comparisons, since the latter concern a period after the second world war, when the level of the savings rate has risen (cf. Mankiw et al.,1990, but note that their numbers are those of the gross savings rate). Thus three numerical values of s^* will be here experimented with, viz. .13, .15 and .17.

The remaining four parameters are determined by the above six, α and ω as indicated in (3), where the parameter a was given as a function of β and σ. It remains to give the dependence of the risk aversion coefficient σ on the independent parameters:

$$(7.14)\quad \sigma = \frac{\beta b^* - \rho}{s^* b^* - n}.$$

This dependence is a consequence of the balanced-growth Euler equation $\rho + \sigma(\lambda - n) = \beta b^*$ and the balanced-growth equation of the growth of physical capital, $\lambda = s^* b^*$. Thus it is valid in the Lucas (1988) model as well and even in his modification of the Solow model (Lucas,ibid.).

Table 2 gives the numerical predictions of the Basic Business Cycles (BBC) as far as the correlations (with output) and the variances of consumption (C), investment (I), employment (E) and the real wage level (W) are concerned. These predictions have been calculated from (10)-(11). To facilitate the reader's checking of the calculations, the numerical values of

TABLE 2. – THE NUMERICAL CALCULATION OF CORRELATIONS AND STANDARD DEVIATIONS[1]

$\underline{s^* = .13}$	$\alpha = .036$	$\omega = .1035$	$\sigma = 2.2917$
$a = 7.0968$	$F_{SS} = 15.5204$	$F_{Sw} = 4.0786$	$F_{ww} = 10.1061$
$m_Y = .0666$	$I_{SS} = 122.5726$	$I_{Sw} = 32.2108$	$I_{ww} = 80.5236$
$m_C = ..0291$	$m_I = .3178$	$m_E = .0428$	$m_W = .0238$
$A_C = 0$	$A_I = 0$	$A_E = -.0643$	$A_W = .0643$
$B_C = .1091$	$B_I = 1.1930$	$B_E = .1864$	$B_W = .0636$
$r_{CY} = 1.00$	$r_{IY} = 1.00$	$r_{EY} = .9060$	$r_{WY} = .7654$
$\sigma_C/\sigma_Y = .4364$	$\sigma_I/\sigma_Y = 4.7720$	$\sigma_E/\sigma_Y = .7094$	$\sigma_W/\sigma_Y = .4668$
$\underline{s^* = .15}$	$\alpha = .03$	$\omega = .0920$	$\sigma = 1.8333$
$a = 8.8$	$F_{SS} = 18.3922$	$F_{Sw} = 5.1765$	$F_{ww} = 15.5290$
$m_Y = .0677$	$I_{SS} = 124.2441$	$I_{Sw} = 34.9685$	$I_{ww} = 104.9027$
$m_C = .0369$	$m_I = .2419$	$m_E = .0435$	$m_W = .0242$
$A_C = 0$	$A_I = 0$	$A_E = -.0643$	$A_W = .0643$
$B_C = .1364$	$B_I = .6439$	$B_E = .1821$	$B_W = .0679$
$r_{CY} = 1.00$	$r_{IY} = 1.00$	$r_{EY} = .9205$	$B_W = .0679$
$\sigma_C/\sigma_Y = .5455$	$\sigma_I/\sigma_Y = 3.5758$	$\sigma_E/\sigma_Y = .6982$	$\sigma_W/\sigma_Y = .4496$
$\underline{s^* = .17}$	$\alpha = .024$	$\omega = .0794$	$\sigma = 1.5278$
$a = 11.5789$	$F_{SS} = 22.6934$	$F_{Sw} = 6.9753$	$F_{ww} = 26.1567$
$m_Y = .0698$	$I_{SS} = 128.8443$	$I_{Sw} = 39.6028$	$I_{ww} = 148.5073$
$m_C = .0457$	$m_I = .1876$	$m_E = .0449$	$m_W = .0249$
$A_C = 0$	$A_I = 0$	$A_E = -.0643$	$A_W = .0643$
$B_C = .1636$	$B_I = .4217$	$B_E = .1778$	$B_W = .0722$
$r_{CY} = 1.00$	$r_{IY} = 1.00$	$r_{EY} = .9395$	$r_{WY} = .8124$
$\sigma_C/\sigma_Y = .6545$	$\sigma_I/\sigma_Y = 2.6867$	$\sigma_E/\sigma_Y = .6841$	$\sigma_W/\sigma_Y = .4208$

[1]The initial state factor $[s(0) - s^*]$ has been omitted in the mean values m, as well as its square in the I-integrals.

the dependent parameters, of the mean values and the A- and B-coefficients of the mentioned variables, as well as of the F- and I-functions, such as they have been obtained and used in my calculations, are mentioned for the purpose of comparison.

Table 3 compares these predictions with the U.S. data and with the predictions of three models based on stochastic shocks, viz. that of Danthine and Donaldson (1990-1993), that of Hansen and Rogerson (1985), and that of Kydland and Prescott (their later, 1986 version). The success of each model is expressed by the sum of error squares in each case. This sum is for the versions of the Basic Business Cycles included in this table, viz. BBC $s^* = .13$ and BBC $s^* = .15$, smaller than in the models based on stochastic shocks. The greater though not perfect success of the BBC is obvious both in the prediction of standard deviations and in that of correlations.

TABLE 3. – COMPARISONS OF THE PREDICTIONS OF FIVE MODELS WITH U.S.DATA[2]

Standard deviations:

X :	*The U.S. economy* :	*BBC 1* $s^* = .13$:	*BBC 2* $s^* = .15$:	*Danthine–Donaldson* :	*Hansen–Rogerson* :	*Kydland–Prescott* :
Y	1.00	1.00	1.00	1.00	1.00	1.00
C	.73	.44	.55	.19	.29	.25
I	4.89	4.77	3.58	3.45	3.24	3.07
E	.94	.71	.70	.72	.77	.68
W	.67	.47	.45	.35	.28	.40
$\sum \Delta^2$:		.1961	1.8645	2.5185	3.1071	3.6933

Correlations with output:

X :	*The U.S. economy* :	*BBC 1* $s^* = .13$:	*BBC 2* $s^* = .15$:	*Kydland–Prescott* :	*Hansen–Rogerson* :	*Danthine–Donaldson* :
C	.85	1.00	1.00	.85	.87	.69
I	.92	1.00	1.00	.88	.99	.99
E	.76	.91	.92	.95	.98	.98
W	.42	.77	.80	.86	.87	.91
$\sum \Delta^2$:		.1739	.1989	.2313	.2562	.3190

[2]The numbers other than those in the BBC columns are taken or computed from those given by Danthine and Donaldson (1993) who, for Hansen-Rogerson and Kydland-Prescott models, quoted Prescott (1986). The last row indicates the sum of error squares for each model.

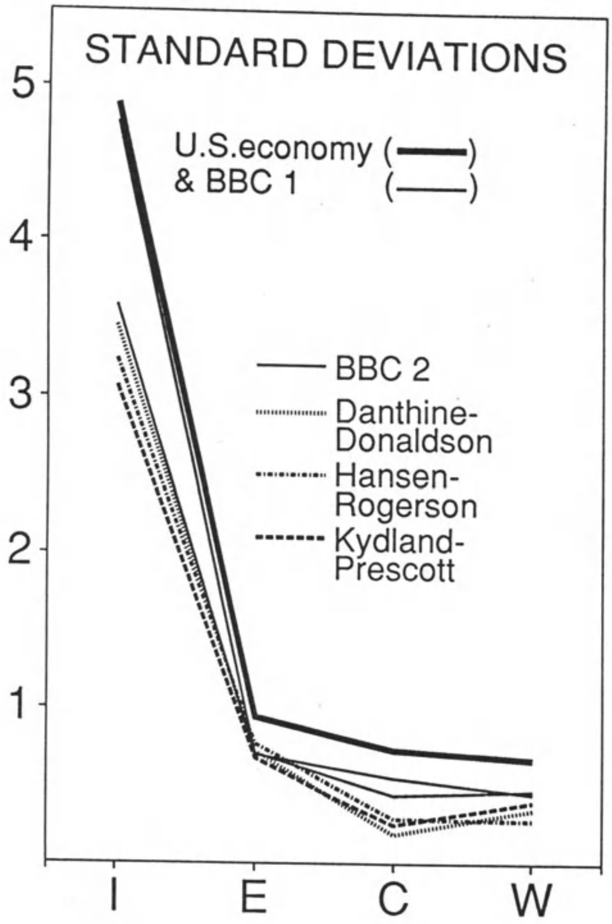

FIGURE 4.

Figures 4 and 5, which illustrate the numbers shown in Table 3, confirm the impression given by the numbers and indeed addsomething to it: what makes the difference is that the BBC predictions follow the *pattern* of empirical correlations and standard deviations better than do the predictions derived from the stochastic models.

Table 4 on page 52 shows a similar comparison of two BBC versions, viz. one with $s^* = .15$ and one with $s^* = .17$, with the original Kydland-Prescott model and with the data they used (Kydland and Prescott,1982). Again the

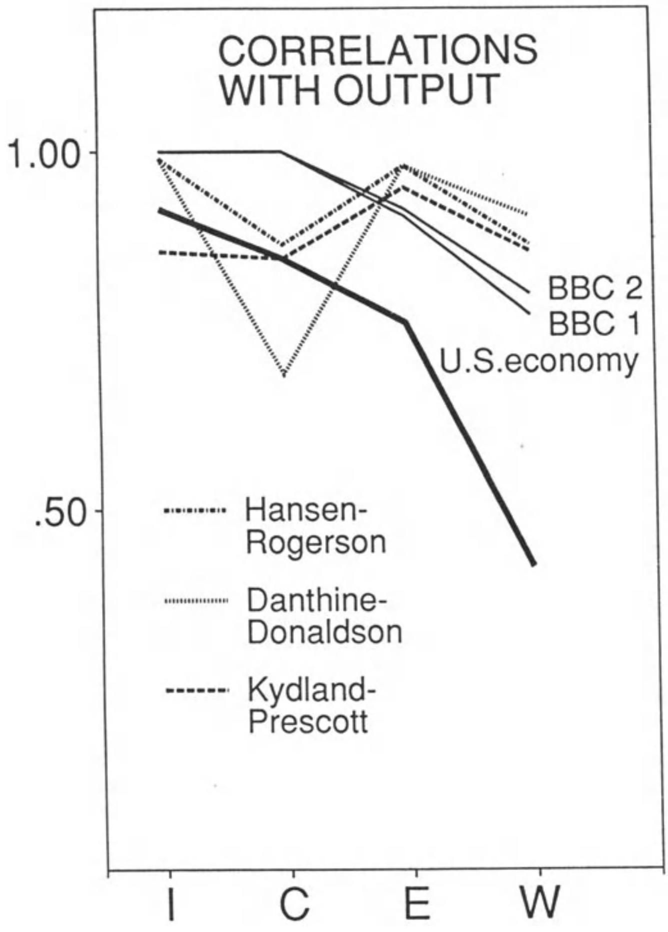

FIGURE 5.

nonstochastic BBC models are better in their predictions, even though they are not perfect either.

Taken together Tables 3 and 4 show that the BBC version with $s^* = .15$ is the best when both comparisons are considered. The version with $s^* = .13$ takes best into account the peculiarities in the U.S. economy as displayed by the material in Table 3, while the version with $s^* = .17$ brings home the victory in the Table 4 game.

Scarce though the material is in these comparisons, I have deliberately

TABLE 4. – COMPARISONS WITH THE KYDLAND-PRESCOTT (1982) MODEL AND WITH THE U.S. ECONOMY IN THE PERIOD 1950-79[3]

Standard deviations:

X :	*The U.S. economy* :	*BBC 3* $s^* = .17$:	*BBC 2* $s^* = .15$:	*Kydland–Prescott* :
Y	1.00	1.00	1.00	1.00
C	.33	.65	.55	.50
I	2.83	2.69	3.58	3.58
E	1.11	.68	.70	.58
W	.55	.42	.45	.50
$\sum\Delta^2$:		.3238	.7890	.8748

Correlations with output:

X :	*The U.S. economy* :	*BBC 2* $s^* = .15$:	*BBC 3* $s^* = .17$:	*Kydland–Prescott* :
C	.94	1.00	1.00	.66
I	.71	1.00	1.00	.80
E	.85	.92	.94	.93
W	.10	80	.84	.90
$\sum\Delta^2$:		.5826	.6434	.7329

made it even scarcer by leaving out capital (Lucas,1987, also discarded it in his Kydland-Prescott Table). The problem with quantitative comparisons involving the dynamics of capital still comes from the fact that the capital stock, which can be measured, is not the same thing as the capital input to production, i.e. the capital services, which is relevant in dynamics. One can easily understand that capital stock does not vary much during a cycle, and that it has almost a zero correlation with output over a cycle. But 'capital' in growth theory is a different thing, whose measurement is difficult.

3. Comparisons with empirical autocorrelations. The autocorrelation of output Y with its value Y' delayed by a time D is given by

$$r_{YY'} = \mathrm{cov}\,(Y,Y')/\sigma_Y^2, \quad \mathrm{cov}\,(Y,Y') = I_{YY'} - m_Y m_Y'$$
$$I_{YY'} = \frac{1}{T}\int_0^T Q_Y(t) Q_Y(t-D)\,dt,$$

[3]The data and the predictions of the Kydland-Prescott model as given in Kydland and Prescott (1982). The last row indicates the sum of error squares in each model.

$$m'_Y = \frac{1}{T}\int_0^T Q_Y(t-D)\,dt, \;\; \sigma_Y^2 = \operatorname{cov}(Y, Y') \;\text{ for } D = 0.$$

Here $m'_Y \neq m_Y$ because of the divergence of the cycles defined by (1) and (2), which we shall use. Direct calculation gives, by applying (6.25) and (2):

$$\begin{aligned} I_{Y,Y'} &= \frac{1}{32}\left[\frac{\beta a\left(\alpha^2+4\omega^2\right)}{(1-s^*)\omega}\right]^2 \left(e^{2\pi\alpha/\omega}-1\right) e^{-\alpha D/2}\,. \\ &\left[\frac{\cos\omega D}{\alpha} - \frac{\alpha\cos\omega D - 2\omega\sin\omega D}{\alpha^2+4\omega^2}\right][s(0)-s^*]^2, \end{aligned} \tag{7.15}$$

$$m'_Y/m_Y = \frac{e^{(2\pi/\omega\, -\, D)\alpha/2}\left(\cos\omega D + \frac{\alpha}{2\omega}\sin\omega D\right) - 1}{e^{\pi\alpha/\omega} - 1}. \tag{7.16}$$

Again the unknown initials state $s(0) - s^*$ cancels out in the final formulae.

In a comparison of the predictions of the BBC with those of the Kydland-Prescott (1982) model, which is so far the only case where a comparison of a shock model with autocorrelation data has been performed, the theoretical equivalent of a quarter of a year is needed. By taking the average length of

TABLE 5. – COMPARISONS OF PREDICTED AUTOCORRELATIONS WITH THOSE OF THE KYDLAND-PRESCOTT MODEL AND WITH DATA[4]

j :	*The U.S. economy* :	*BBC 2* $s^* = .15$:	*BBC 4* $s^* = .18$:	*BBC 0* $s^* = .11$:	*Kydland–Prescott* :
1	.84	.90	.92	.93	.71
2	.57	.71	.72	.73	.45
3	.27	.44	.45	.46	.28
4	–.01	.15	.14	.16	.19
5	–.20	–.13	–.19	–.11	.02
6	–.30	–.35	–.39	–.33	–.13
$\Sigma\Delta^2$:		.0851	.0951	.1077	.1487

[4]The data and the Kydland-Prescott predictions are those reported in Kydland and Prescott, 1982, and again in Lucas, 1987. The last row indicates the sum of error squares for each model.

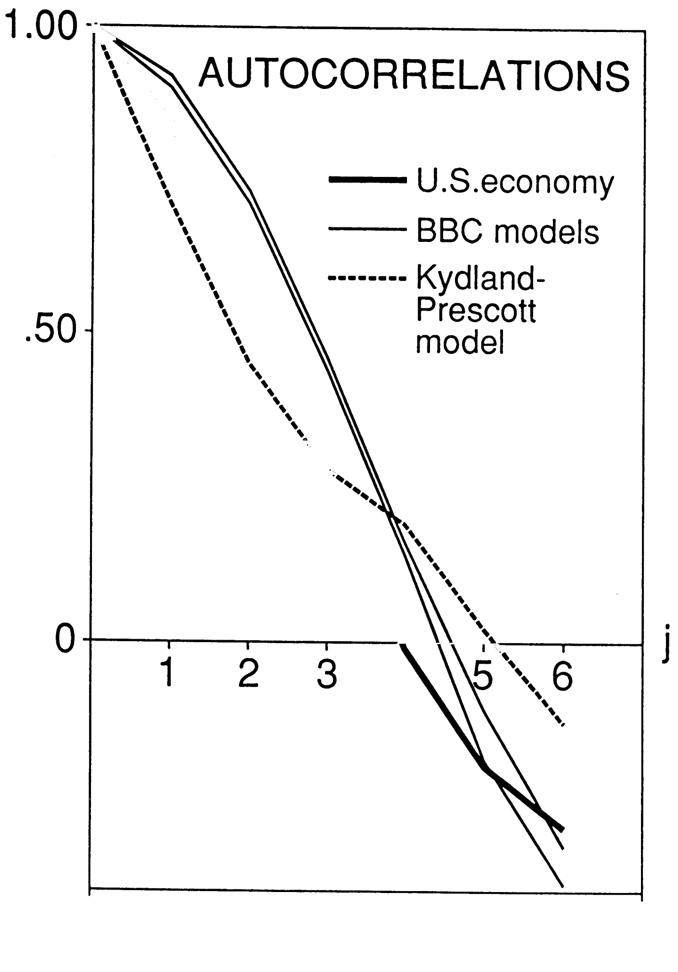

FIGURE 6.

period of the business cycles in the U.S. economy in the period 1950-79, studied by Kydland and Prescott, to be 4 years, we have $g_o = T/4$, where $T = 2\pi/\omega$ is the length of period in terms of the theoretical unit of time [TU] (for the time scales, see Chapter 6, Section 4). Thus $D_1 = T/16 = \pi/8\omega$ is the theoretical equivalent of a quarter of a year.

The comparison with the Kydland-Prescott (1982) model and with the data from the U.S. economy in the period 1950-79, which they used, are given in Table 5. Three versions of the Basic Business Cycles were again

TABLE 6. – THE NUMERICAL CALCULATION OF AUTOCORRELATIONS[5]

$s^* = .11$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
$D = jD_1$	3.4563	6.9126	10.3690	13.8253	17.2816	20.7379
$e^{-\alpha D/2}$	.9300	.8649	.8043	.7480	.6959	.6469
$\cos\omega D$	.9239	.7071	.3826	0	−.3826	−.7071
$\sin\omega D$	.3826	.7071	.9239	1	.9239	.7071
$m_Y m_Y'$	.0040	.0027	.0009	−.0011	−.0030	−.0045
$\mathrm{cov}\,(YY')$	.0652	.0515	.0324	.0115	−.0080	−.0234
$r_{YY'}$	.9272	.7326	.4612	.1632	−.1142	−.3331
$s^* = .15$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
$D = jD_1$	4.2674	8.5348	12.8023	17.0697	21.3371	25.6045
$e^{-\alpha D/2}$	.9380	.8798	.8253	.7741	.7261	.6811
$\cos\omega D$	.9239	.7071	.3826	0	−.3826	−.7071
$\sin\omega D$	.3826	.7071	.9239	1	.9239	.7071
$m_Y m_Y'$	.0040	.0026	.0006	−.0017	−.0038	−.0054
$\mathrm{cov}\,(YY')$	.0848	.0669	.0417	.0138	−.0124	−.0333
$r_{YY'}$	.8957	.7065	.4403	.1456	−.1311	−.3514
$s^* = .18$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
$D = jD_1$	5.4162	10.8323	16.2485	21.6647	27.0808	32.4970
$e^{-\alpha D/2}$	.9447	.8925	.8432	.7965	.7525	.7110
$\cos\omega D$	.9239	.7071	.3826	0	−.3826	−.7071
$\sin\omega D$	.3826	.7071	.9239	1	.9239	.7071
$m_Y m_Y'$	.0054	.0035	.0007	−.0024	−.0053	−.0076
$\mathrm{cov}\,(YY')$	.1185	.0933	.0576	.0178	−.0198	−.0500
$r_{YY'}$	.9197	.7241	.4471	.1386	−.1537	−.3879

experimented with, this time determined by the values .11, .15 and .18 of the balanced-growth net savings rate s^*. As shown by the sums of error squares reported in Table 5 the value $s^* = .15$ gives again the best prediction of data. But also the other versions of the BBC predict the data better than does the model based on stochastic shocks. And again it is the *pattern* that singles out the BBC predictions from the predictions of the shock model: the former ones reproduce the pattern of data better (Fig.6).

To facilitate the reader's checking of the results, some phases of the numerical calculation are again indicated (Table 6).

[5]The initial state factor $[s(0) - s^*]^2$ has been deleted in $m_Y m_Y'$ and in $\mathrm{cov}\,(YY')$. It is canceled out in $r_{YY'}$. Note that $\omega D = (j/16)360^o$.

8. Conclusions and Challenges

1. Is the Lucas-Bellman formalism too narrow? The linear approximation certainly does not full justice to the theory of the Basic Business Cycles, neither to the differential equations (4.27) nor to the detrended cycle functions (6.25)-(6.31). Still its predictions are better than those of the stochastic shock models in all the comparisons performed so far. This non-stochastic simple theory also gives correctly *all the patterns* of data in Figures 4-6, which cannot be said of any of the considered stochastic models.

The conclusion is near that the dynamics of the Basic Business Cycles reflects some fundamental characteristics of the real dynamics of business cycles, fundamental enough to be taken into account in macroeconomic theory.

On the other hand, we know that stochastic shocks do exist and affect human life in most of its aspects, and certainly they influence economic development and the business cycles. But the success of the BBC, if repeatedly observed, suggests that

the effects of technological and other shocks upon business cycles should be considered as disturbances to be superposed upon the Basic Business Cycles due to the mutual interference of material and nonmaterial values.

If this reformulation of the problem of business cycles is accepted, it necessitates a reinterpretation of the elegant Lucas 1987 vision of macroeconomic theory (Lucas, 1987). This vision, based on finite values of welfare functions as determined by the Bellman equation, was formulated for material values. They are of course the main subject of economic theory. But economic development in the long run seems to be affected by nonmaterial values too. Two of them are involved in the theory of the Basic Business Cycles.

One of them is expressed by the 'mass effect principle' underlying the growth equation (4.5) of human capital in the present unified theory of economic growth and business cycles. The appearance in this growth equation of the total time $[(1-u(t)]v(t)N(t)]$ devoted to the accumulation of knowledge and skills in society during the period of production t, instead of its share in total working time (the Lucas assumption (3.4)), means that a mass-scale social interaction between a great number of intelligent people is necessary to produce the accumulation of true and exact knowledge.The *truth* of course is a fundamental nonmaterial value.The mass effect principle underlying (4.5) of course makes our theory nonreducible to microeconomics (cf.p.22).

The other one is the inexhaustable ***pursuit of individual freedom***, as expressed by the leisure term in the current-time utility in the present unified theory. The inclusion of the freedom aspect makes it impossible to apply the finite Bellman formalism, since there is no finite measure of the value of freedom, which is infinite. This differs greatly from the situation met if only material values, such as consumption, are taken into account.

Indeed, the need for consumption per capita as expressed by the first term divided by N in the current-time utility,

$$\frac{c(t)^{1-\sigma}-1}{1-\sigma},$$

is bounded for $\sigma > 1$, which is strictly true in the unified theory. (In fact it is in the normal case true also in the Lucas theory and the Solow model because of parametric transversality conditions: see pages 11 and 19.) This is realistic: no one can consume food or other commodities more than up to some limit, after which further consumption does no more increase one's welfare. In an infinite horizon the (discounted) value of consumption goes to zero. This indeed takes place in all the three mentioned growth theories under the condition that $\rho > n$, which on the other hand is true because of parametric transversality conditions.

On the other hand the wish for freedom expressed in the unified theory by the endogeneously determined weight ξ of the second term of current-time utility,

$$\xi(t)\,[1 - v(t)]\,N(t),$$

is because of (5.4) and (5.24) unbounded in a way that even the discounted utility never becomes zero. I think this is realistic. One's material needs are limited, but the wish for freedom is not: whatever practical limits economical and other factors may impose on the actual freedom as realized at any particular time, the value of freedom does not have any upper limit.

It follows that a macroeconomic theory, where the Basic Business Cycles have a role to play, has two levels: one, reducible to microeconomics by means of the Bellman-Lucas formalism, on which the economic game is played according to the rules of that formalism, and another one, where the long-term human pursuit of ever greater individual freedom and deeper objective knowledge set the tone. The theory of the Basic Business Cycles can be understood only within the framework of such a wider conception of mathematical economics as sketched in the present study.

2. Economic stability. Another general consequence of the present unified theory of economic growth and business cycles concerns the prospects of economic stability. In a world where the Basic Business Cycles are true, economic development does not automatically lead to a stable economic system in the end. The negative Bendixson criterion denies it. Thus there is no free lunch, but people have to earn it by hard saving, risky investment – and by having patience to wait. This is because an approximately stable economic system, stable in the sense of Liapunov, can be reached only by such paths in the parameter space along which ρ/n (always larger than one), σ (always larger than one) and s^*/β (always smaller than one) approach very close to one in the end.

However,

the relative significance of the business cycles with respect to output approaches asymptotically zero,

as shown in Chapter 6, Section 2. The relative asymptotic stability defined by

$$\sigma_Y/Y^* \propto e^{(\alpha/2 - \lambda)t} \longrightarrow 0 \text{ for } s^* > \beta/3.$$

holds good in advanced economies, since the parameter condition is fulfilled in all of them. This indeed is sufficient, as far as I can see, for the existing monetary theory to make sense.

3. Are real business cycle theories outdated? Quite recently serious doubts about real business cycle theories were expressed by Hairault and Portier (1993), who suggest that "the typical modern business cycle cannot be reduced to the real business cycle archetype." They showed that "a monopolistic competition model with price adjustment costs, affected by technological *and* monetary shocks, better mimics the economic fluctuations in two countries with very different cyclical properties, namely France and the United States." It is interesting to test the goodness of predictions of the entirely non-stochastic theory of Basic Business Cycles by comparing them with the predictions of the Hairault-Portier model.

TABLE 7. – THE BASIC BUSINESS CYCLES AND THE HAIRAULT-PORTIER MODEL COMPARED.

Prediction of:	BBC:	H-P:
U.S. economy	.2057	.2013
French economy	1.511	3.4929

If we compare (cf. Table 7) the predictions of these two theories concerning the correlations with output over detrended cycles *and* the standard deviations in proportion to the standard deviation of output, we find that the Hairault-Portier and the BBC do on the average equally well in the case of the U.S. economy (1959-90, the period of Hairault and Portier), while the BBC do better than the Hairault-Portier model in the case of the French economy (the same period). The sum of error squares, as a grand total of all those items, is .2013 for the Hairault-Portier and .2057 for the BBC in the prediction of the U.S. economy. In the case of the French economy there is, however, a clear difference in favour of the BBC predictions, their sum of error squares being 1.1511, to be compared with the number 3.4929 of the Hairault-Portier model. .

In these comparisons the numbers of Table 3 for the BBC 1 ($s^* = .13$) were used. Thus the parameter values were kept fixed. The data were those reported by Hairault and Portier (ibid.). Their (benchmark) model, used in the above comparison, had been separately calibrated for the U.S. and French economies, while no such thing was done with the BBC model. This shows in the better achievement of the Hairault-Portier model, if only correlations are taken into account. However, the numbers just quoted tell that the theory of Basic Business Cycles, formally deduced from two axioms imposed on the Lucas theory, still explains the whole pool of data somewhat better than does what seems to be a very sophisticated shock model. Ergo: conclusions as above.

9. The Dynamics of Anomalous Basic Business Cycles and Its Quantitative Verification

1. Fundamental theory vs model construction in economics. In this chapter the anomalous cycles observed in the period 1914-50 in the U.S. and U.K. economies will be studied. Characteristic of this period in these countries was

- that the normally highly procyclical consumption and investment lost their procyclicality;

- that the fall in the procyclicality of investment was still larger than that in the procyclicality of consumption, the correlation with output over detrended business cycles being reduced in the case of investment to .16 (U,S.) or to -.41 (U.K.), and in the case of consumption to .51 (U.S.) or to -.33 (U.K.);

- that as a contrast to these anomalous phenomena employment retained its usual high procyclicality, its correlation with output over detrended cycles being .78 in the U.S. economy and .92 in the U.K. economy; and

- that in the U.K. economy the real wage level, usually moderately procyclical or neutral, turned into a highly anticyclical variable, with the correlation -.61 with output over detrended cycles.

All the mentioned numbers come from Correia, Neves and Rebelo (1992). In this paper a causal explanation of these phenomena is given in terms of the Growth Type 2 of Basic Business Cycles. A more specific quantitative model is constructed for their study in the U.S. economy, the numbers derived from this model being close to the empirical values.

There is an aspect of the theory of the Basic Business Cycles that makes it more suitable for discussions of anomalous phenomena than are the usual methods applied in current economics. In the construction of an economic model usually a *great number of specific hypotheses* pertaining to different fields of economics are brought together. The result is a sophisticated and more or less balanced combination of various aspects of current economic thinking: often a sort of compendium of the economic wisdom of the day. But this implies, methodologically, that only what is average and ordinary – according to the common reason – may be covered by the models thus constructed.

What is exceptional and anomalous is easily ignored in this kind of model approach, which involves numerous hypotheses, each of which must sound reasonable enough to satisfy common sense. Still the anomalous too may be methodical – it only follows a logic different from the obvious one. Such anomalous cases indeed play a remarkable role for instance in the selection of correct physical theories from the wrong ones. One can even say that they offer the essential arguments for or against any general theory. The history of science knows many examples of this. The small deviations from perfect circular symmetry in the orbits of planets settled the dispute in favour of the Keplerian theory and against the Ptolemaian one. The small anomalies in the perihelic motion of Mercurius testified for the general theory of relativity, etc.

The present unified theory of economic growth and business cycles was constructed by following the methodology of generalization of earlier theories, much applied in theoretical physics. What is considered as a fundamental theory, is generalized by means of *the least possible number of hypotheses.* The general idea behind this method is to derive as much as possible from as little as possible. This is somewhat quite contrary to what is involved in the typical construction of an economic model: there many hypotheses are used to derive often not so many consequences.

In saying so the current practice of dynamical model construction in economics was meant. This leaves out the wast literature on mathematical economic theory concerning competitive equilibrium, which of course is general and fundamental. But this theory is static in character. Compared with it any attempt to generalize economic dynamics must be tentative.

The critical issue in the methodology of generalization in economic dynamics is the choice of the fundamental theory to be generalized. In physics the choice tends to be too evident to evoke discussion. In economics the situation is different. The economic litterature knows many attempts at general theories, but until quite recently they were mostly verbal and ambiguous. Hence, the first criterion of a fundamental theory is its exact mathematical nature. Or, as expressed by Robert Lucas: "I prefer to use the term 'theory' in a very narrow sense, to refer to an explicit dynamic system, something that can be put on a computer and run." (Lucas, 1988, p.5).

As a matter of fact the mentioned first criterion leaves very little to choose about in economic theory. As a mathematical science using explicit dynamical systems as its theoretical foundation economics is very young. If this criterion is combined with the required generality of the theoretical ideas expressed, only a few candidates for a fundamental theory are left. In the present study the Lucas 1988 'mechanics of economic development' was chosen as the starting point of generalization, since the pursuit of a general dynamical theory in economics is characteristic of Lucas himself – and his 'mechanics' can be considered to be already a generalization itself, viz. a generalization of the seminal Solow growth model.

The generalization of the Lucas 1988 mechanics is quite essential for the present theoretical discussion of the anomalous business cycles. This generalization produced the Growth Type 2, in terms of which such a discussion becomes possible. The Solow model and also the Lucas 1988 mechanics are both of them examples of the Growth Type 1, where the basic growth path is the balanced-growth path. Only the Growth Type 2 gives the method and logic obeyed by the observed anomalous cycles. The difference from Growth Type 1 is that now the cycle center P is not a fixed point on the (s, w)–plane but moves slowly upwards in the positive w–direction. The cycles are changed accordingly, and this produces the observed anomalous correlations over detrended cycles.

We shall first, in Section 2 below, construct a method of calculation useful in the applications of Growth Type 2. After that a specific model for the present purpose will be chosen within the general framework of the theory, and the fall in procyclicality of consumption and investment will be considered by means of this model (Section 3). The retaining high procyclicality of employment and the plunge in procyclicality of the real wage level are then considered in terms of the same model (Section 4). Finally, a brief appraisal of the results is given (Section 5).

2. The method of calculation. We can again start with the solutions of the linearized equations of the Basic Business Cycles, valid in a neihgbourhood of the fixed cycle center P and expressible in the form (7.1)-(7.2). This is because we can reduce the detrended cycle functions of Growth Type 2 to those in Growth Type 1 plus a change variable.

2.1. The detrended cycle functions in Growth Type 2. By writing $b = b^* - (b^* - b)$ and introducing the symbol $\Delta = b^* - b$ we can reduce each cycle function V_X of Growth Type 2, detrended over a loglinear trend, to a sum

$$V_X = Q_X^* + \Delta_X. \tag{9.1}$$

Here the first term is the cycle function of Growth Type 1. The second term is a *change variable* that describes how the moving cycle center P_2 of Growth Type 2 affects the cycle. The change variable is obviously given by

$$\Delta_X = \left[\left(\frac{\dot{X}}{X}\right)_{P_2} - \left(\frac{\dot{X}}{X}\right)_{P_1} \right] + Q_X(s - s^*, \Delta). \tag{9.2}$$

For V_Y and V_C we get at once:

$$\begin{aligned}
&(9.3) \qquad & V_Y &= Q_Y^* + \Delta_Y, \text{ where} \\
&& & Q_Y^* = \beta(w - b^*) \text{ and } \Delta_Y = (\beta - \beta/\sigma)\Delta, \\
&(9.4) \qquad & V_C &= Q_C^* = (1/\sigma)Q_Y^*, \ \Delta_C = 0.
\end{aligned}$$

To get V_I we first expand $(1-s)/s$ in a series by the Cauchy rule:

$$\begin{aligned}
\frac{1-s}{s} &= \frac{1-s^*-(s-s^*)}{s^*\left[1+\frac{s-s^*}{s^*}\right]} = \left[\frac{1-s^*}{s^*} - \frac{s-s^*}{s^*}\right]\left[1 - \frac{s-s^*}{s^*} + \ldots\right] \\
(9.5) \qquad &= \frac{1-s^*}{s^*} - \frac{s-s^*}{(s^*)^2} + \text{higher terms.}
\end{aligned}$$

The infinite series in $s-s^*$ converges strongly for very small values of $s(0)-s^*$. As we shall be interested only in such initial states very near to the point P_1, the above two first terms will be sufficient for the present purposes. The resulting detrended cycle function V_I of investment in Growth Type 2 is now easily constructed according to the rules (1) and (2). It can be expressed, after some calculation, in the following useful form:

$$\begin{aligned}
&(9.6) \quad & V_I &= (F-G)Q_Y^* - F^2(\beta - \beta/\sigma)(s-s^*)(w-b^*) + F\Delta_Y \\
&(9.7) \quad & &- F^2(s-s^*)\Delta_Y, \ F = \left[\frac{1}{(s^*)^2}\right], \ G = \frac{1}{\sigma}\left(\frac{1-s^*}{s^*}\right).
\end{aligned}$$

Finding the detrended cycle function V_E of employment in Growth Type 2 offers no problems. First we write $w = (w - b^*) + b^*$ and $1 - s = 1 - s^* - (s - s^*)$ in Q_E. The rules (1) and (2) then give, after some calculation:

$$(9.8) \qquad V_E = Q_E^* + \left(\frac{1}{1-\beta+\kappa}\right)\left[1 + \frac{\kappa(s^* - \beta/\sigma)}{\beta - \beta/\sigma}\right]\Delta_Y.$$

The corresponding function of the productivity is $V_W = V_Y - V_E$.

2.2. Parameters. I suppose it is a good methodological advise to keep the parameter values fixed as much as possible when applying to empirical data many-parameter models, such as necessarily appear in growth theory. Otherwise the accusation is near that one has used the available many parameters to justify "theoretical predictions" of whatever data. To avoid this criticism in advance a policy of fixed parameter values, following closely the estimates calculated by or from Denison (1961), has been followed in the applications of the unified theory: strictly in Aulin, 1992, while allowing an experimentation with three different values of the net savings rate s^* in Aulin, 1993a, and in Chapter 7 of the present study. In this chapter like in Aulin, 1993b, all the parameter values are fixed from the very beginning:

$$\underline{n} = .015, \;\; \underline{\rho} = .02, \;\; \underline{\beta} = .25 \;\; \Longrightarrow \sup s^* = .1875, \;\; \inf \sigma = 1.3333,$$
$$\underline{b^*} = .5, \;\; \underline{s^*} = .13 \;\; \Longrightarrow \sigma = 2.1, \;\; \alpha^* = .06, \;\; \omega = .1660786,$$
$$T = 2\pi/\omega = 37.832599, \;\; \gamma = .0054761, \;\; \underline{\kappa} = .417, \;\; \underline{B} = .75.$$

The independent parameters have been underlined. The other independent parameter values have appeared in earlier applications too, except for the value chosen above for the balanced-growth output/capital ratio b^*. The value .5 of this parameter, as well as the value .75 of B, will be explained in the next Section. The length of period of a cycle, T, is indicated in theoretical time units and corresponds to 4 years in real time (for time scales see Chapter 6, Section 4). This length of period in real time has been chosen in earlier applications too. The numerical value of κ has been taken from Lucas (1988).

2.3. Integrations over the period of a cycle. To get correlations we have to integrate, over the period of a cycle, products of powers of $(w - b^*), (s - s^*)$ and Δ, all of which are time functions. Products of the two first factors have already been taken care of in Chapter 7: we have the solution (7.1)-(7.2) of the cycle equations, which makes integrations over time easy. As to the last factor we expand it for those integrations in an infinite series with exponential terms, i.e.

$$\Delta = b^* - b = b^* \sum_{r=1}^{\infty} (-1)^{r+1} B^r e^{-r\gamma t}. \tag{9.9}$$

The functional expressions and numerical values of the 'basis integrals' over a cycle, needed in the application to the anomalous cycles, can now be easily computed by using the formulae (7.1) and (7.2). In the following formulae $a = (\beta - \beta/\sigma)^{-1}$ and $x = s(0) - s^*$. First we need the following integrals over the first cycle including the first and second degree in the Growth Type 1 basic variables $(w - b^*)$ and $(s - s^*)$:

$$\frac{1}{T}\int_0^T (w - b^*)\, dt = \left(\frac{e^{\alpha^* T/2} - 1}{T}\right)\left(\frac{a}{1 - s^*}\right) x = (.489791)\, x,$$

$$\frac{1}{T}\int_0^T (s-s^*)\,dt = \left(\frac{e^{\alpha^* T/2}-1}{T}\right)\left(\frac{4\alpha^*}{(\alpha^*)^2+4\omega^2}\right) x = (.11755)\,x,$$

$$\frac{1}{T}\int_0^T (w-b^*)^2\,dt = \left(\frac{e^{\alpha^* T}-1}{T}\right) 2|D|^2\left(\frac{1}{\alpha^*}-\frac{\alpha^*}{(\alpha^*)^2+4\omega^2}\right) x^2 = (4.194957);$$

$$\frac{1}{T}\int_0^T (s-s^*)^2\,dt = \left(\frac{e^{\alpha^* T}-1}{T}\right)\left[\frac{(\alpha^*)^2+4\omega^2}{8\alpha^*\omega^2}+\alpha^*\left(\frac{1.5-(\alpha^*)^2/8\omega^2}{(\alpha^*)^2+4\omega^2}\right)\right] x^2$$

$$= (2.1533352)\,x^2,$$

$$\frac{1}{T}\int_0^T (w-b^*)(s-s^*)\,dt = \left(\frac{e^{\alpha^* T}-1}{T}\right)\left[\frac{a}{2(1-s^*)}\right] x^2 = (1.0067913)\,x^2.$$

Then we have to calculate the integrals over the first cycle of the following functions, in which also the first or second degree of the change variable Δ appear:

$$\frac{1}{T}\int_0^T (w-b^*)\Delta\,dt = \left(\frac{b^*}{T}\right) 2\omega|D|\sum_{r=1}^{\infty}(-1)^{r+1}B^r\left[\frac{e^{(\alpha^*/2-r\gamma)T}-1}{(\alpha^*/2-r\gamma)^2+\omega^2}\right]_o$$

$$= (.0875566)\,x, \quad \text{here } \Sigma = 26.5,$$

$$\frac{1}{T}\int_0^T (s-s^*)\Delta\,dt = \left(\frac{b^*}{T}\right)\sum_{r=1}^{\infty}(-1)^{r+1}B^r(\alpha^*-r\gamma)[\,\cdot\,]_o$$

$$= (.0198241)\,x, \quad \text{here } \Sigma = 1.50,$$

$$\frac{1}{T}\int_0^T (s-s^*)^2\Delta^2\,dt = S_1+S_2+S_3 = (.0827938)\,x^2, \quad \text{where}$$

$$S_1 = \frac{2(b^*)^2|C|^2}{T}\sum_{r=2}^{\infty}(-1)^r B^r(r-1)\left[\frac{e^{(\alpha^*-r\gamma)T}-1}{\alpha^*-r\gamma}\right]_1 = (.0767655)\,x^2,$$

here $\Sigma = 22.5$,

$$S_2 = \frac{(b^*)^2\alpha^*}{T}\sum_{r=2}^{\infty}(-1)^r B^r(r-1)\left[\frac{e^{(\alpha^*-r\gamma)T}-1}{(\alpha^*-r\gamma)^2+4\omega^2}\right]_2 = (.004222512)\,x^2,$$

here $\Sigma = 10.65$,

$$S_3 = \frac{(b^*)^2(.5-(\alpha^*)^2/8\omega^2)}{T}\sum_{r=2}^{\infty}(-1)^r B^r(r-1)(\alpha^*-r\gamma)[\,\cdot\,]_2 = (.0018058)\,x^2,$$

here $\Sigma = .565$,

$$\begin{aligned}\frac{1}{T}\int_0^T (w-b^*)(s-s^*)\Delta\,dt &= \frac{b^*|D|}{T}\left\{\frac{\alpha^*}{2\omega}\left(\sum_{r=1}^{\infty}\right)_a (-1)^{r+1}B^r[\,\cdot\,]_1\right.\\ &\quad+\left.\left(\sum_{r=1}^{\infty}\right)_b (-1)^{r+1}B^r\left[2\omega-\frac{\alpha^*}{2\omega}(\alpha^*-r\gamma)\right][\,\cdot\,]_2\right\}\\ &= (.2043341)\,x^2, \text{ here } \Sigma_a = 62.9,\ \ \Sigma_b = 9.18,\end{aligned}$$

$$\begin{aligned}\frac{1}{T}\int_0^T (s-s^*)\Delta^2\,dt &= \frac{(b^*)^2}{T}\sum_{r=2}^{\infty}(-1)^r B^r(r-1)(\alpha^*-r\gamma)[\,\cdot\,]_o\\ &= (.0032643)\,x, \text{ here } \Sigma = .494,\end{aligned}$$

$$\begin{aligned}\frac{1}{T}\int_0^T \Delta\,dt &= \frac{b^*}{\gamma T}\log\left(\frac{1+B}{1+Be^{-\gamma T}}\right) = (.2017471),\\ \frac{1}{T}\int_0^T \Delta^2\,dt &= \frac{(b^*)^2}{\gamma T}\left[\log\left(\frac{1+B}{1+Be^{-\gamma T}}\right)+\frac{1}{1+B}-\frac{1}{1+Be^{-\gamma T}}\right]\\ &= (.0407535).\end{aligned}$$

In the computation of integrals, where the integrand includes an infinite series, 20 first terms of the series were used, except for the two first such integrals in the above list, in which cases 12 first terms were considered to be sufficient.

3. The fall in procyclicality of consumption and investment. The general theory tells nothing specific concerning the period 1914-50, in which the anomalous phenomena appeared. To be specific we have to choose a model for this particular period, to be constructed on the basis of the general theory of the Basic Business Cycles.

3.1. The choice of model for the anomalous period. The modern level of output/capital ratio in advanced countries has been reported to be generally about 1/3 (Mankiw, Romer and Weil, 1990). In the period 1909-30 it was approximately .3 in the U.S. economy, as is evident from Fig.1 (on page 35), based on the numbers Solow (1957) gives. Therefore in previous applications of the theory of Basic Business Cycles, where Growth Type 1 was applied, the value $b^* = .3$ was chosen. The so obtained predictions, concerning output-correlations and variances over detrended business cycles (see Chapter 7, or Aulin, 1993a), as well as autocorrelations of output (ibid.), were rather well in agreement with available empirical data. The empirical results concerning periods before the first and after the second world wars (Correia et al.,1992; Danthine and Girardin.1989) all show the high procyclicality of C, I and E, and thus the ordinary cycles of Growth Type 1.

Now the period 1914-50 is in question. We can see from Fig.1 that now the value $b^* = .5$ must be adopted: this is the level to which the approximately logistic rise in b leads in the U.S. economy during the period between the Great Depression and 1950. Secondly, this logistic rise suggests Growth Type 2. Thirdly, we can choose,in the exactly logistic growth function b of Growth Type 2, the value $B = (b^* - b(0))/b(0) = .75$. This implies, with $b^* = .5$, the value $b(0) \approx .28$, which in turn locates the start of Growth Type 2 (see Fig.1) in the middle of the Great Depression in the U.S.

It follows from the form of the logistic function b, roughly represented by the rising line in Fig.1, that the characteristic phenomena of Growth Type 2 are most outstanding during the first business cycle after the start of this Growth Type: the movement of the cycle center P_2 is fastest there. Thus a model with a time span from $t = 0$ to $t = T$, i.e. over the period of the first cycle after the change, is most revealing. This will accordingly be the model in terms of which we shall quantitatively analyse the anomalous phenomena encountered in the period 1914-50 in the U.S. economy.

3.2. The fall in procyclicality of consumption. Using for the integrals the numerical values given above we first calculate, denoting the unknown initial state $s(0) - s^*$, as before, by x, the expectation values

$$\begin{aligned} m^*(Y) &= \beta\frac{1}{T}\int_0^T (w - b^*)\,dt = (.1224477)\,x, \\ m(\Delta_Y) &= (\beta - \beta/\sigma)\frac{1}{T}\int_0^T \Delta dt = (.0264192)\,x, \end{aligned}$$

then the variances and the mutual covariance of Q_Y^* and Δ_Y:

$$(\sigma_Y^*)^2 = \beta^2\frac{1}{T}\int_0^T (w - b^*)^2\,dt - [m^*(Y)]^2 = (.2471914)\,x^2,$$

$$\sigma_{\Delta_Y}^2 = (\beta - \beta/\sigma)^2\frac{1}{T}\int_0^T \Delta^2\,dt - [m(\Delta_Y)]^2 = .00000089,$$

$$\begin{aligned} \mathrm{cov}(Q_Y^*, \Delta_Y) &= \beta(\beta - \beta/\sigma)\frac{1}{T}\int_0^T (w - b^*)\Delta\,dt - m^*(Y)m(\Delta_Y) \\ &= (-.00036857)\,x. \end{aligned}$$

Hence we get the expressions

$$\begin{aligned}\sigma_Y^2 &= (\sigma_Y^*)^2 + \sigma_{\Delta_Y}^2 + 2\mathrm{cov}\,(Q_Y^*, \Delta_Y) \\ &= \left(.2471914 + \frac{.0000008900}{x^2} - 2\frac{.00036857}{x}\right) x^2,\end{aligned}$$

$$(\sigma_Y^*)^2 + \mathrm{cov}(Q_Y^*, \Delta_Y) = \left(.2471914 - \frac{.00036857}{x}\right) x^2.$$

What we want is the correlation

$$r_{CY} = \frac{\mathrm{cov}(C, Y)}{\sigma_C \sigma_Y} = \frac{(\sigma_Y^*)^2 + \mathrm{cov}(Q_Y^*, \Delta_Y)}{\sigma_Y^* \sigma_Y}.$$

But this correlation obviously has different values for different initial states x. Table 8 shows these values for a number of magnitudes of x. The function $r_{CY}(x)$ is illustrated in Fig.7. A curious new phenomenon appears. While in normal situations, to which in this theory there corresponds Growth Type 1, the correlations over detrended business cycles are (at least to the second degree) entirely independent of the initial state x (see Chapter 7), they now heavily depend on it.

TABLE 8. – THEORETICAL CORRELATIONS WITH OUTPUT OVER DETRENDED CYCLES IN GROWTH TYPE 2[1]

x	r_{CY}	r_{IY}	r_{EY}	r_{WY}
.000	−.78	1.00	1.00	1.00
.0005	––	––	.99	.40
.001	−.38	.78	.96	−.01
.0011	−.32	––	.95	––
.0015	.01	.58	.89	.08
.002	.40	.37	.77	.32
.0023	––	.28	––	––
.0025	.65	.24	.76	.45
.003	.79	.20	.76	.55
.004	––	.21	.78	.64
.005	.95	.26	.81	.68
.0075	––	.36	.85	.72
.01	.99	.41	.86	.73

[1]These correlations are illustrated by points in Figures 7 or 8.

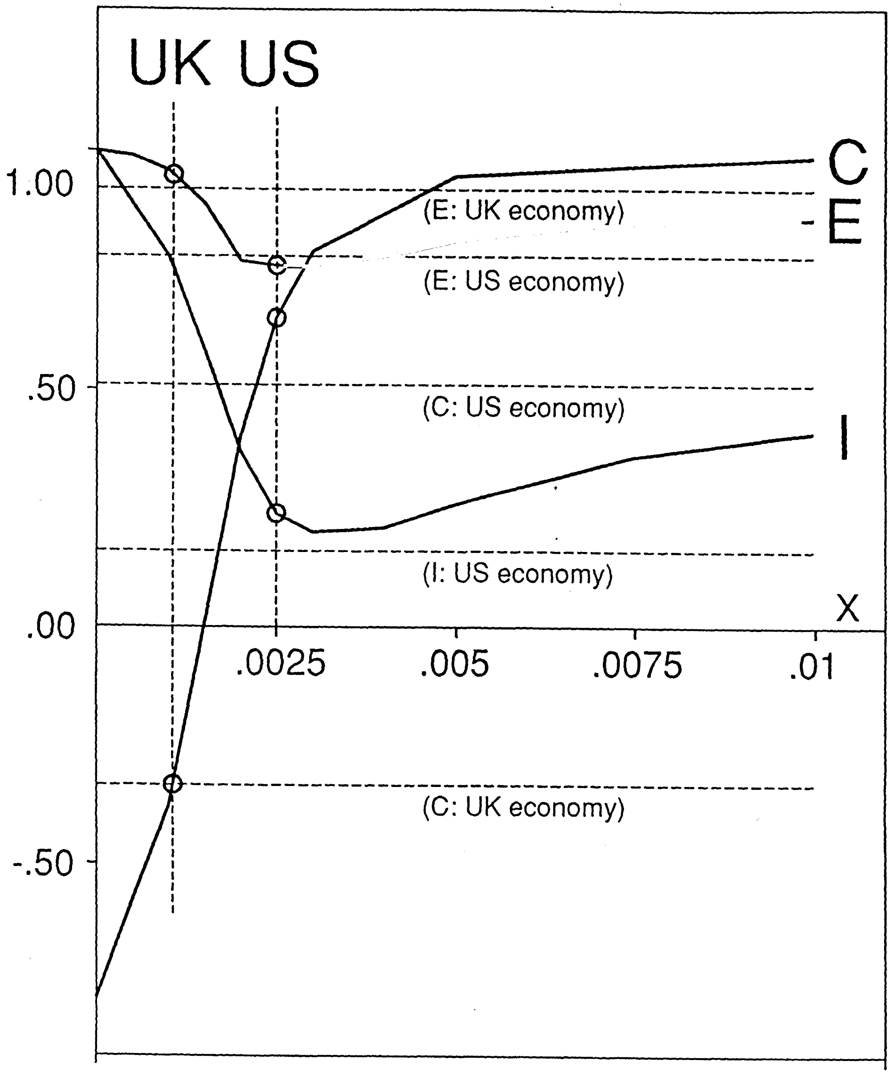

FIGURE 7. – THE PREDICTED ANOMALOUS CORRELATIONS OF CONSUMPTION, INVESTMENT AND EMPLOYMENT AS FUNCTIONS OF THE INITIAL STATE, COMPARED WITH DATA FROM THE U.S. AND U.K. ECONOMIES

We can see that the fall in the procyclicality of consumption is the larger the smaller is the difference x, i.e. the closer to the fixed cycle center P_1 of Growth Type 1 start the cycles of Growth Type 2. This suggests the following interpretation of the anomalous correlations: 1) they are due to a change from the ordinary cycles of Growth Type 1, with its fixed cycle center P_1, to the cycles of Growth Type 2 with a moving center, and 2) the change takes place when the ordinary cycles have collapsed, as a consequence of which the state (s, w) has come to the immediate vicinity of the fixed cycle center P_1. The decline in procyclicality of consumption is (cf. Fig.7) the larger, the larger is the collapse of ordinary cycles, i.e. the closer to the point P_1 is the initial state of the resulting Growth Type 2 cycles.

What, then, may have caused the assumed collapse of the ordinary business cycles? In the case of the U.S. economy, the collapse must have taken place as a consequence of the Great Depression, since according to Fig.1 (on page 35) the transformation of Growth Type in the U.S. economy has taken place at that time. On the other hand, the numbers, given by Solomou (1990) concerning the British economy, suggest that a change from Growth Type 1 to Growth Type 2 started in Britain earlier, as a consequence of world war one already.

3.3. The still larger observed fall in procyclicality of investment. The form of the detrended cycle function V_I of investment, (6), gives immediately the variance and covariance we need:

$$\begin{aligned}\sigma_I^2 &= (F-G)^2(\sigma_Y^*)^2 + F^2\sigma_{\Delta_Y}^2 + F^4\sigma_{s\Delta}^2 \\ &+ 2F(F-G)\mathrm{cov}(Q_Y^*, \Delta_Y) - 2F^2(F-G)\mathrm{cov}(Q_Y^*, s\Delta_Y) \\ &- 2F^3\mathrm{cov}(s\Delta_Y, \Delta_Y) - 2F^3(\beta - \beta/\sigma)\mathrm{cov}(sw, \Delta_Y),\end{aligned}$$

$$\begin{aligned}\mathrm{cov}(I, Y) &= (F-G)(\sigma_Y^*)^2 + F\sigma_{\Delta_Y}^2 + (2F-G)\mathrm{cov}(Q_Y^*, \Delta_Y) \\ &- F^2\mathrm{cov}(Q_Y^*, s\Delta_Y) - F^2\mathrm{cov}(s\Delta_Y, \Delta_Y), \\ &- F^2(\beta - \beta/\sigma)\mathrm{cov}(sw, \Delta_Y).\end{aligned}$$

In both expressions the terms of third and higher orders in x have been omitted, and the short notations $(s-s^*)\Delta_Y = s\Delta_Y$ and $(s-s^*)(w-b^*) = sw$ have been used.

The expectation value, variance and covariances, not encountered before but needed in these formulae, are easily computed from the numerical values of basis integrals given in Section 2 above:

$$\begin{aligned}
m(s\Delta_Y) &= (\beta - \beta/\sigma)\frac{1}{T}\int_0^T (s - s^*)\Delta\, dt \\
&= (.00259600)\, x, \\
\sigma^2_{s\Delta_Y} &= (\beta - \beta/\sigma)^2 \frac{1}{T}\int_0^T (s - s^*)^2 \Delta^2\, dt - [m(s\Delta_Y)]^2 \\
&= (.001412961)\, x^2,
\end{aligned}$$

$$\begin{aligned}
\mathrm{cov}(Q_Y^*, s\Delta_Y) &= \beta(\beta - \beta/\sigma)\frac{1}{T}\int_0^T (w - b^*)(s - s^*)\Delta\, dt - m^*(Y)m(s\Delta_Y) \\
&= (.0063716)\, x^2, \\
\mathrm{cov}(s\Delta_Y, \Delta_Y) &= (\beta - \beta/\sigma)^2 \frac{1}{T}\int_0^T (s - s^*)\Delta^2\, dt - m(s\Delta_Y)m(\Delta_Y) \\
&= (-.000012607)\, x, \\
\mathrm{cov}(sw, \Delta_Y) &= (\beta - \beta/\sigma)\frac{1}{T}\int_0^T (s - s^*)(w - b^*)\Delta\, dt - m(sw)m(\Delta_Y) \\
&= (.000159400)\, x^2.
\end{aligned}$$

By combining the above results we get the following numerical expressions:

$$\sigma_I^2 = \left(6.548721 + \frac{.000047336}{x^2} - \frac{.01407100}{x}\right) x^2,$$

$$\mathrm{cov}(I, Y) = \left(.7354666 + \frac{.0000061538}{x^2} - \frac{.0037506}{x}\right) x^2.$$

This gives for the correlation of investment with output over a detrended cycle, $r_{I,Y} = \mathrm{cov}(I, Y)/\sigma_Y \sigma_I$, the numerical values listed in Table 8 for a number of values of the initial state x. These results are illustrated in Fig.7. Again we can see the fall in procyclicality depending heavily on x. By comparing the curves with the dashed lines indicating the empirical values of $r_{C,Y}$ and $r_{I,Y}$ in the U.S. economy, as calculated by Correia, Neves and Rebelo (ibid.) from the period 1914-50, we can make a couple of interesting observations: 1) the fall in procyclicality of investment is indeed larger than that of consumption as soon as $x > .0022$, and 2) the closest fit with the empirical values $r_{C,Y} = .51$ and $r_{I,Y} = .16$ is obtained for the distance .0025 of the initial state from the cycle center of Growth Type 1.

The conclusion to be drawn from the result accordingly is that the theory suggests a collapse in the U.S. of ordinary business cycles, in which the state

(s, w) in the Great Depression reached as small a distance as .25 percentage units (on the scale of net savings rate) from the fixed center P_1.

4. The retained high procyclicality of employment. As a contrast to the fall in procyclicality of the usually highly procyclical consumption and investment, employment retained its high procyclicality even in the anomalous period 1914-50 in the U.S. (and also in the U.K). This result too is predicted correctly by the present model.

4.1. The nearly perfect fit of theory and the U.S. data. We first compute, using the numerical values of the basis integrals given in Section 2 above, the here needed magnitudes of the expectation value, the variance and the covariance with output of employment in the ordinary detrended cycles of Growth Type 1:

$$\begin{aligned}
m_E^* &= \left(\frac{\beta}{1-\beta+\kappa}\right)\left\{(1-s^*)\frac{1}{T}\int_0^T (w-b^*)\,dt\right.\\
&\quad \left. -b^*\frac{1}{T}\int_0^T (s-s^*)\,dt\right\} = (.0786939)\,x,\\
(\sigma_E^*)^2 &= \left(\frac{\beta}{1-\beta+\kappa}\right)^2\left\{(1-s^*)^2\frac{1}{T}\int_0^T (w-b^*)^2\,dt\, +\right.\\
&- \;2(1-s^*)b^*\frac{1}{T}\int_0^T (s-s^*)(w-b^*)\,dt\\
&+ \;\left.(b^*)^2\frac{1}{T}\int_0^T (s-s^*)^2\,dt\right\} - (m_E^*)^2 = (.1240301)\,x^2,\\
\operatorname{cov}(E,Y)^* &= \left(\frac{\beta^2}{1-\beta+\kappa}\right)\left\{(1-s^*)\frac{1}{T}\int_0^T (w-b^*)^2\,dt\right.\\
&- \;\left. b^*\frac{1}{T}\int_0^T (s-s^*)(w-b^*)\,dt\right\} - m_E^* m_Y^* = (.1588632)\,x^2.
\end{aligned}$$

Here again the terms of higher than second order in x have been omitted.

We need also the magnitude

$$\begin{aligned}
\operatorname{cov}(Q_E^*, \Delta_Y) &= \left(\frac{\beta}{1-\beta+\kappa}\right)\left\{\left(\frac{1-s^*}{\beta}\right)\operatorname{cov}(Q_Y^*, \Delta_Y)\right.\\
&- \;\left. b^*\left[(\beta-\beta/\sigma)\frac{1}{T}\int_0^T (s-s^*)\Delta\,dt - m(s)m(\Delta_Y)\right]\right\}\\
&= \;(-.000220195)\,x.
\end{aligned}$$

This completes the preliminary work necessary for the calculation of the following functions of Growth Type 2:

$$\begin{aligned}
\mathrm{cov}(E,Y) &= \mathrm{cov}(E,Y)^* + \mathrm{cov}(Q_E^*, \Delta_Y) + \left(\frac{1}{1-\beta+\kappa}\right) \cdot \\
&\cdot \left[1 + \frac{\kappa(s^* - \beta/\sigma)}{\beta - \beta/\sigma}\right]\left[\sigma^2_{\Delta_Y} + \mathrm{cov}(Q_Y^*, \Delta_Y)\right] \\
&- \left[\frac{\beta}{(1-\beta+\kappa)^2}\right]\left[1 + \frac{\kappa(s^* - \beta/\sigma)}{\beta - \beta/\sigma)}\right] \mathrm{cov}(sw, \Delta_Y) \\
&= \left(.1588329 + \frac{.000000789237}{x^2} - \frac{.0005470367}{x}\right) x^2,
\end{aligned}$$

and

$$\begin{aligned}
\sigma_E^2 &= (\sigma_E^*)^2 + \left(\frac{1}{1-\beta+\kappa}\right)^2 \left[1 + \frac{\kappa(s^* - \beta/\sigma)}{\beta - \beta/\sigma}\right]^2 \sigma^2_{\Delta_Y} \\
&+ \left(\frac{2}{1-\beta+\kappa}\right)\left[1 + \frac{\kappa(s^* - \beta/\sigma)}{\beta - \beta/\sigma}\right] \mathrm{cov}(Q_E^*, \Delta_Y) \\
&- \left[\frac{2\beta}{(1-\beta+\kappa)^2}\right]\left[1 + \frac{\kappa(s^* - \beta/\sigma)}{\beta - \beta/\sigma}\right] \mathrm{cov}(sw, \Delta_Y) \\
&= \left(.1239696 + \frac{.00000069988}{x^2} - \frac{.0003905305}{x}\right) x^2.
\end{aligned}$$

The resulting correlations over a detrended cycle in Growth Type 2, i.e. $r_{EY} = \mathrm{cov}(E,Y)/\sigma_E\sigma_Y$, are given in Table 8 and illustrated in Fig.7 and Fig.8 for certain values of the initial state x. What strikes one when looking at these correlations, after having studied the corresponding correlations of consumption and investment, is their continuously high values. Never do they fall below the value .76 obtained for x =.002, .0025 and 003. This is in harmony with the empirical value .78 reported by Correia, Neves and Rebelo (1992) for the period 1914-50 in the U.S. economy.

We can complete the present analysis by constructing also the function $r_{WY}(x)$ of the correlations of real wage level (or labour productivity) with output. By using the formulae

$$\begin{aligned}
\sigma_W^2 &= \sigma_Y^2 + \sigma_E^2 - 2\mathrm{cov}(E,Y) \text{ and} \\
\mathrm{cov}(W,Y) &= \sigma_Y^2 - \mathrm{cov}(E,Y)
\end{aligned}$$

we get for the mentioned function the values shown in Table 8 and illustrated in Fig.8. The value obtained at the point x = .0025 is .45. Unfortunately this result cannot be compared with an empirical correlation, since the real wage level was not included in the U.S. material analysed by Correia, Neves and Rebelo (ibid.).

TABLE 9. – COMPARISON OF MODEL CORRELATIONS WITH THE EMPIRICAL ONES[2]

Variable	Model (0025)	The U.S. economy	Model (0011)	The U.K. economy
Consumption	.65	.51	-.32	-.33
Investment	.24	.16	!!	-.41
Employment	.76	.78	.95	.92
Productivity	.45	–	!!	-.61

The fit of theory and the available empirics of anomalous correlations shown in Table 9 is good in the case of the U.S. economy. Quite obviously the strange phenomena encountered in the period 1914-50 in the U.S. economy can be explained in terms of Growth Type 2 of the Basic Business Cycles.

4.2. The empirical evidence about the birth of anomalous cycles. The result also agrees with the earlier result obtained from the analysis of consumption and investment, according to which the change from the ordinary business cycles (of Growth Type 1) to the anomalous business cycles (of Growth Type 2) in the aftermath of the Great Depression took place at the distance $x = .0025$ from the fixed cycle center P_1.

On the other hand Fig.7 suggests that, as far as consumption and employment are concerned, the fit is good also in the case of the U.K. economy of that period, but for a smaller value of x. The distance $x = .0011$ gives for the value of the correlation of consumption with output over detrended business cycles the number $-.32$ and for the value of the corresponding correlation of employment the number .95. Both are close to the empirical correlations, which are $-.33$ and .92, respectively, as reported by Correia et. al.. Such a fit can hardly be a product of chance, but suggests that the anomalies in these correlations also in the U.K. are due to the same cause as in the case of the U.S. If this is to be believed, the collapse of ordinary business cycles in Britain, as a consequence of the first world war, must have been still more drastic as it was in the U.S. in connection of the Great Depression.

[2]The empirical correlations are those reported by Correia, Neves and Rebelo (1992). The points where the present model is unfit for the U.K. economy are denoted by !!

The evidence, given by the appearance of the (nearly) correct values of the correlations with output of consumption, investment and employment (in the U.S. economy) at the same initial distance $x = .0025$ from the fixed cycle center, is of course indirect. The same holds for the evidence given by the appearance of the (nearly) correct values of the correlations with output of consumption and employment at the same distance $x = .0011$ from the fixed cycle center. Direct evidence about the collapse of the ordinary business cycles at those distances from the their center point in the U.S. and U.K. economies is of course hard to get: the present econometric analysis is still insufficient to make such sophisticated distinctions.

Why should the business cycles collapse in those cases? The answer in view of the present theory is simple. During a long duress, such as the Great Depression in the U.S. and the first world war in the case of Britain, the importance of leisure is naturally diminished while people are focused on the major material value of survival. In such a situation accordingly the current-time utility function approaches unidimensionality, as a consequence of which the business cycles collapse.

5. An appraisal of the results.

5.1. Why the fit of theory with the U.K. data is only 50 %*?* We have seen (Table 9) that in the case of the U.K. economy there is

1) a perfect fit of theory and data as far as the anomalous correlations with output of consumption and employment are concerned – the errors are only .01 and .03,

2) a perfect failure of the present theory to cope with the anomalous output correlation of investment, while

3) only a qualitative explanation is given by this theory of the anomalous output correlation of the productivity of labour.

The empirical correlations of investment and productivity with output over detrended cycles, as reported by Correia et al. in the case of the U.K. economy in the anomalous period 1914-50, were as low as $-.41$ and $-.61$, respectively. These numbers are very far indeed below the curves in Fig.7 and Fig.8 illustrating the functions $r_{IY}(x)$ and $r_{WY}(x)$ given by the present model.

However the predicted qualitative behaviour of the dependence of the correlation from the distance x is correct in the case of real wages. We can see from Fig.8 how the curve of this correlation indeed plunges deep down at the correct value $x = .001$. One may ask whether the quantitative prediction

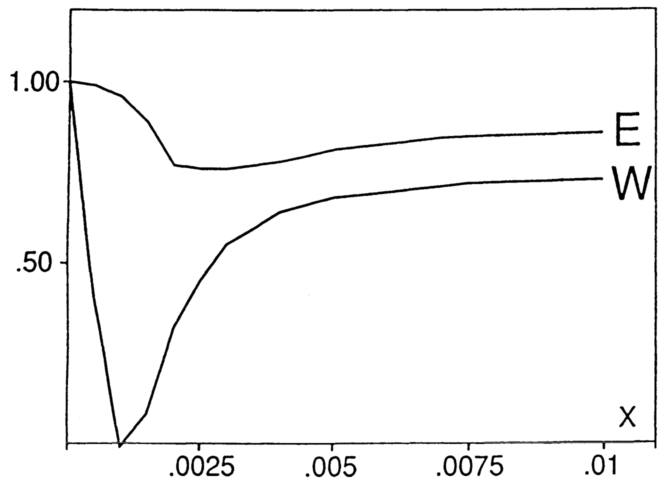

FIGURE 8. – THE PREDICTED OUTPUT CORRELATIONS OF EMPLOYMENT AND PRODUCTIVITY OF LABOUR OVER ANOMALOUS BUSINESS CYCLES AS FUNCTIONS OF THE INITIAL STATE.

would be better, say, for some other value of the parameter s^*. However, there is no gain to be obtained in this way: the values $s^* = .11$ and $s^* = .15$ do not improve the quantitative predictions given by the value $s^* = .13$ (see Appendix). Nevertheless the same deep plunge down in the curve of the output correlation of productivity (of labour) is equally clear in these other cases, with about the same depth as in the present model.

The conclusion is that for the anomalous value $-.41$ of the output correlation of investment over detrended cycles in the U.K. economy in the period 1914-50 a totally different explanation must be looked for. The present theory, which assumes perfect markets, cannot explain the behaviour of investment in the U.K. economy in that period, not even qualitatively. An obvious other explanation would be the command economy during the world wars: it must have dominated the smaller British economy much more than it could dominate the U.S. economy.

As to the anomalous value $-.61$ of the output correlation of the productivity of labour, only qualitative explanation is offered by the present theory. It follows that something has to be added to get the whole story. Maybe the wartime economy can again be invoked, as the military efforts had to be supported by a steady productivity of labour at home. Again Britain as a smaller country felt the impact more than the United States.

5.2 Conclusions. When judging the results obtained in this chapter, it is good to notice that, even in the case of the U.S. economy, neither the model used in Sections 3 and 4 nor the reported empirical correlations do concern exactly the period (1930-50 in the U.S.), which should be the object of study of the anomalous correlations. The model is based on the first theoretical cycle of that period, since the anomalities can be expected to be most outstanding in the first cycle of Growth Type 2. The empirical correlations reported by Correia, Neves and Rebelo (ibid.) on the other hand have been calculated from the period 1914-50. But nearly one half of this period, from 1914 to about 1930, was a time with quite ordinary business cycles according to the testimony of Fig.1 on page 35, based on the Solow (1957) data over the years 1909-49. According to this testimony the anomalities started about 1930 and ended about 1950.

Therefore an exact quantitative agreement between the model and the empirical correlations should not be expected even in the U.S. case. Against this background an approximate agreement with the empirical correlations in the U.S. economy, such as shown by Table 9, must be considered very good. The data from the U.S. economy are in a better agreement with the present model than are the data from the U.K. economy. In the latter economy investment and wages pose in the period 1914-50 problems that are not solved by the present model, but they are just two cases of seven. There can be hardly any doubt, that the theory of Basic Business gives an essential part of the causal explanation of anomalous correlations. This explanation is quantitative and covering in the case of the U.S. economy. In the case of the U.K. economy its coverage is only 50 %, if a quantitative fit with empirics is required, while a qualitative fit can be said to cover 75 % of the (admittedly scarce) data.

What is the cause we are speaking about? Can it be explained in any less formal way than by the mathematical formulae constructed in Sections 3 and 4? I think that one can illustrate the cause of the anomalous correlations, embedded in the formulae, by a visual image. The ultimate cause, in terms of the theory of Basic Business Cycles, of course is the moving cycle center P_2 of Growth Type 2. It rises along a straight line, viz. the line $s = s^*$ on the plane (s, w), where the cycles are originated. The cycles "leave behind" in this movement of their center, and thus produce a phenomenon of retardation similar to that we meet in autocorrelations: the correlations fall. But this concerns only such economic variables whose cycle function has a strong component in the direction of the w-axis, i.e. in the direction of the movement of the cycle center.

V

THE EFFECTS OF NONMATERIAL VALUES AND OTHER IGNORED FACTORS UPON ECONOMIC GROWTH

10. Primary Causal Factors of Economic Growth

1. The reduction to human capital. The output $(Y)_P$ of an economy on a basic growth path P, whether in Growth Type 1 or 2, can be expressed in terms of the average human capital $(h)_P$ of population on that growth path:

$$(10.1) \qquad (Y)_P = A^{1/(1-\beta)} b^{-\beta/(1-\beta)} (\psi/k)\,(h)_P^{(1-\beta+\kappa)/(1-\beta)}.$$

Here the formulae (4.12),(4.17) and (4.26) were used, together with the equation $w = b$ holding good on the basic growth paths.

Similar reductions to human capital are easily given for each of the aggregate *growth variables* C (total consumption), I (total net investment), K (physical capital), E (employment, i.e. labour input) and W (the productivity of labour= real wage level divided by $1-\beta$), as well as for the *costate price variables* p and q, since we have:

$$\begin{aligned}
&(C)_P = s^*(Y)_P, \quad (I)_P = (1-s^*)\,(Y)_P, \quad (K)_P = b^{-1}(Y)_P, \\
&(E)_P = (\psi/k)\,(h)_P, \quad (W)_P = A^{1/(1-\beta)} b^{-\beta/(1-\beta)}\,(h)_P^{\kappa/(1-\beta)}, \\
&(p)_P = [\xi\psi/(1-\beta)k]\,(Y)_P^{-1}, \quad (q)_P = (\xi/k)\,(h)_P^{-1}.
\end{aligned}$$

Here the expressions of consumption and investment of course follow from the constancy of the net savings rate, $s = s^*$ on the basic growth paths. The formula (4.26) and the equation $w = b$ were used for K, the formula (4.17) for $E \equiv huvN$, and the definition formula $W \equiv Y/E$ for W.

A little different kind of reduction to human capital is obtained for the time allocation variables v and u directly from their defining equations on the basic growth paths, (5.29) and (5.30), respectively:

$$(v)_P = (1/kN)[\psi + (\dot{h})_P/(h)_P], \quad (u)_P = \psi/[\psi + (\dot{h})_P/(h)_P].$$

This of course gives for the total working time vN and the leisure time $(1-v)N$ the corresponding reductions.

Thus all the essential real economic variables have been given a reduction to one of them, viz. human capital, on the basic growth paths. Of course we could have singled out some other of those variables, and express the rest of them in terms of that one. The reason why human capital was chosen as the "reference variable" is that it comes naturally to consider human capital as the prerequisite of all economic development in any human society.

2. The freedom factor. What is most interesting in the above reductions is the coefficient of human capital for each growth variable. In these coefficients, obviously, further reductions can be made. First of all, ψ is not an independent variable but was in (4.14) introduced as a short notation of the variable $\rho + m - \dot{\xi}/\xi$. Thus it is a function of ξ and $\dot{\xi}$:

$$\psi = \psi(\xi, \dot{\xi}).$$

It follows that $\dot{\psi}$, thus the other auxiliary function Φ defined by the formula (4.15) as well as the function α defined by the formula (4.25), are functions of $\xi, \dot{\xi}$ and $\ddot{\xi}$:

$$\Phi = \Phi(\xi, \dot{\xi}, \ddot{\xi}), \quad \alpha = \alpha(\xi, \dot{\xi}, \ddot{\xi}).$$

But the parameter b, defined by the formula (4.24), then is also defined as a function of these three time functions:

$$b = b(\xi, \dot{\xi}, \ddot{\xi}). \tag{10.2}$$

We can accordingly rewrite (1) as

$$(Y)_P = f^{(Y)}(k, \xi, \dot{\xi}, \ddot{\xi})\,(h)_P^{(1-\beta+\kappa)/(1-\beta)}, \quad \text{with} \tag{10.3}$$

$$f^{(Y)} = A^{1/(1-\beta)}\, b(\xi, \dot{\xi}, \ddot{\xi})^{-\beta/(1-\beta)}\, [\psi(\xi, \dot{\xi})/k].$$

Similar coefficients $f^{(X)}$ are easily obtained, by using the formulae given in Section 1, for each growth variable or price variable X:

$$(X)_P = f^{(X)}(k, \xi, \dot{\xi}, \ddot{\xi})\,(h)_P^{a^{(X)}(\beta,\kappa)}.$$

Here the exponents $a^{(X)}$ can be functions of only the constants β and κ, as indicated. In the functions $f^{(X)}$ also the constants A, ρ, σ, m, n and s^* may appear.

But because of $\psi = \psi(\xi, \dot{\xi})$ we can include also the time allocation variables u and v in the reduction of real economic variables to the time functions

k, ξ and $(h)_P$ by constructing, for each Z of our economic variables, the formula

$$(Z)_P = F^{(Z)}(k, \xi, \dot{\xi}, \ddot{\xi}, (h)_P, (\dot{h})_P). \tag{10.4}$$

Thus the development of each real economic variable on the basic growth paths is determined, as soon as the time functions k, ξ and $(h)_P$ (and the constants) are given. Here the function ξ, i.e. the weight of leisure term in the current-time utility, is important as it together with human capital appears in the reduction formulae of all our variables. It can be called the *freedom factor* of economic development, since it indicates the strength of the pursuit of one's own time and thus that of individual freedom.

3. The three ultimate determinants of the level of national economy. The reduction just completed tells that, according to the present theory, the level of economic achievement of a national economy is in the last analysis determined

1) by the level of average knowledge and skills in society, as represented by the human capital $(h)_P$ on the basic growth path,

2) by the level of natural talents in population, as represented by the average efficiency k of learning new things, and

3) by the level of the pursuit of individual freedom in society, as represented by the strength ξ of the pursuit of leisure time.

4. The form of the function $b(\xi, \dot{\xi}, \ddot{\xi})$. To be more specific let us find out the function $b(\xi, \dot{\xi}, \ddot{\xi})$, even though this does not add anything to the general result just stated in Section 3 above.

Beginning with the equation (4.24) we can write:

$$b(\xi, \dot{\xi}, \ddot{\xi}) = a[\alpha(\xi, \dot{\xi}, \ddot{\xi}) + n - \rho/\sigma].$$

By applying (4.25),(4.15) and (4.14) in this order we then get:

$$b(\xi, \dot{\xi}, \ddot{\xi}) = a\left[\rho + m - \frac{\dot{\xi}}{\xi} - \frac{\ddot{\xi}/\xi - (\dot{\xi}/\xi)^2}{\rho + m - \dot{\xi}/\xi} + n - \rho/\sigma\right].$$

Here we can substitute for the constants a and $n - \rho/\sigma$ their values obtained from (4.23) and from the parametric Euler equations (5.12), respectively, to get the final form:

$$b(\xi, \dot{\xi}, \ddot{\xi}) = (\beta - \beta/\sigma)^{-1}\left\{\frac{(\rho + m - \dot{\xi}/\xi)^2 - \ddot{\xi}/\xi + (\dot{\xi}/\xi)^2}{\rho + m - \dot{\xi}/\xi} + (s^* - \beta/\sigma)b^*\right\}. \tag{10.5}$$

Check. It is easy to verify, after some calculation, that for the Growth Type 1 substitution

$$\dot{\xi}/\xi = \rho + m - \alpha^*, \text{ with } \alpha^* = (\beta - s^*)b^*,$$

the formula (10.5) gives the correct result $b = b^*$.

With a little longer but still trivial calculation one can also verify that for the Growth Type 2 substitution

$$\dot{\xi}/\xi = \rho + m - \psi, \text{ with } \psi = (\beta - s^*)b,$$

the same formula leads to the equation

$$\dot{b}/b = (s^* - \beta/\sigma)(b^* - b)$$

and thus to the correct logistic time function b of the form (5.19).

11. The Growth Effects of Savings Rate

1. Which is the causal order of parameters? Of the eight parametric constants ρ, σ, β, κ, m, n, A and s^* involved in the functions $F^{(Z)}$ of the formula (10.4), five are connected with the balanced-growth net output/capital ratio b^* by the balanced-growth Euler equation

$$\rho + \sigma(s^*b^* - n) = \beta b^*. \tag{11.1}$$

In this form of the Euler equation the balanced-growth equation of the growth of physical capital, $\lambda = s^*b^*$, has been already used. Both of these balanced-growth equations and accordingly the relation (1) are valid also in the Solow model and in the Lucas 1988 growth theory, as was indicated in Chapters 2 and 3, respectively.

The relation (1) tells that each one of the parameters involved depends on the other ones. As far as pure mathematics is concerned, any one of those constant parameters can be chosen as the dependent one. We have used this freedom of choice in Chapters 7 and 8 especially, by choosing the dependent parameter in a way that best helps the calculations.

But as soon as a causal dependence is meant, the choice is not free. In this case we have to think about the real process and try to guess which parameter in that process is determined after all the other ones. It cannot be any of the parametric constants included in the utility function, neither the current-time utility nor the discounted utility, since utility expresses the aims to be pursued by the economic process, and the aims of course precede the attempts at their achievement. This excludes the discount rate ρ, the risk aversion coefficient σ and the growth rate n of population in all the mentioned growth theories including the present one. The capital's share β must be also excluded, since it is one of the dominating constant parameters in the production function, which must be there before we can think of any economic process. Only two parameters of (1) remain, viz. the net savings rate s^* determining the basic growth paths and the balanced-growth net output/capital ratio b^*. Of these two the former must be thought of as preceding causally the latter one, since investment (=saving) decisions must precede the production process which has to take place before the productivity of capital can get any definite value, for instance b^*.

It follows that the relation

$$b^* = \frac{n - \rho/\sigma}{s^* - \beta/\sigma}, \tag{11.2}$$

obtained by solving (1) for b^*, can be interpreted as a causal relation, the causes being on the right-hand side and the effect on the left-hand side. In the present theory both the nominator and the denominator have to be positive (cf.(5.20)):

$$n - \rho/\sigma > 0, \quad s^* - \beta/\sigma > 0. \tag{11.3}$$

2. The existence of the growth effects of savings rate. From (2) and (3) it follows that, against common belief, the *savings rate has growth effects*, viz. negative ones:

$$\frac{\partial b^*}{\partial s^*} = -\frac{b^*}{(s^* - \beta/\sigma)} < 0, \quad \frac{\partial \lambda}{\partial s^*} = \left(\frac{\beta}{\sigma}\right) \frac{\partial b^*}{\partial s^*} < 0. \tag{11.4}$$

Here the balanced-growth equation for the growth of capital, $\lambda = s^* b^*$ has been again used.

For the growth rate e^* of the employment $E = h\psi/k$ and the growth rate $g^* = \lambda - e^*$ of the productivity of labour $W = Y/E$ we get similarly, in view of (5.6) and (4):

$$\frac{\partial e^*}{\partial s^*} = \left(\frac{1-\beta}{1-\beta+\kappa}\right) \frac{\partial \lambda}{\partial s^*} < 0, \quad \frac{\partial g^*}{\partial s^*} = \left(\frac{\kappa}{1-\beta+\kappa}\right) \frac{\partial \lambda}{\partial s^*} < 0. \tag{11.5}$$

Thus the following theorems are valid in the present theory:

Theorem 1. In a period during which the economy follows the ordinary Growth Type 1 an increase in the level of the net savings rate causes a decrease in the level of the productivity of capital and in the levels of growth rates of output, employment and the productivity of labour.

Theorem 2. In a period during which the economy follows the ordinary Growth Type 1 an increase in the level of the net consumption rate causes an increase in the level of productivity of capital and in the levels of growth rates of output, employment and the productivity of labour.

Thus by increasing the net consumption rate one can raise the growth rate of those economic variables. This is the good news. The bad news is that *simultaneously the stability of the economic system falls.*

Stability of the system decreases when the parameter α^* increases, as was shown in Chapter 6. But this takes place when the net savings rate decreases, that is, when the net consumption rate increases, since we get:

$$\frac{\partial \alpha^*}{\partial s^*} = (\beta - s^*)\frac{\partial b^*}{\partial s^*} - b^* < 0. \tag{11.6}$$

It follows that we have to choose between 1) raising the growth rates of output, employment and the productivity of labour while decreasing the stability of the economic system and 2) reducing the mentioned growth rates while increasing the economic stability. In economic policy this means that a good balance, suitable for each economic situation, has to be found between the good and bad consequences in each case by means of trade-off between growth and stability. Let it be remarked that stability cannot be neglected in this equation, since increasing instability means loosing the governability of the economic process.

3. An empirical test. The study of the Solow material in Chapter 5 left us to a situation in the U.S. economy toward the end of the 1940s, in which the observed level of output/capital ratio had risen to the anomalous height of 1/2 (see Fig.1 on p.35). According to the OECD statistics published in 1985 (Patel and Soete,1985), the average annual growth rate of output/capital ratio (= productivity of capital) in the U.S. economy was $-1.9\,\%$ in the period 1955-82. This gives

$$(1 - .019)^{27}(.50) = .298 \approx .30. \qquad \text{(period 1955-82)}$$

Thus the normal level of the productivity of capital (cf. Fig.1) was back in 1982.

According to the same source (Patel and Soete, ibid.) the periods 1955-73 and 1973-82 show drastically different behaviour of the productivity of capital, its average annuel growth rates being

$$\begin{aligned} \text{av}\left(\frac{\dot{b}}{b}\right) &= -.01\,\% \quad \text{(period 1955-73)}, \\ \text{(11.7)} \qquad \text{av}\left(\frac{\dot{b}}{b}\right) &= \underline{-5.7\,\%} \quad \text{(period 1973-82)}, \end{aligned}$$

respectively. This gives:

$$\begin{aligned} b_{1973} &= (1-.0001)^{18}(.50) = .499, \\ b_{1982} &= (1-.057)^{9}(.499) = .295 \approx .30. \end{aligned}$$

Thus the level of output/capital ratio did not fall from 1955 to 1973 but started then to decline, falling between 1973 and 1982 to its normal value .30.

Following the suggestions made in Section 2 above, we can search for an explanation of the fall observed in the period 1973-82 in terms of an increasing level of savings rate in the U.S. economy. For this purpose we need some statistics about the behaviour of the growth rate of output in the U.S. economy during this period, given by the OECD 1990 statistics (OECD,1990) reported in Table 10.

TABLE 10. – GROWTH RATES OF OUTPUT IN U.S. ECONOMY 1973-82.

Year	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982
λ %	5.2	−.5	−1.3	4.9	4.7	5.3	2.5	−.2	1.9	−2.5

Hence we calculate the average annual rate of decline in the growth rate of output in the period 1973-82:

$$\text{(11.8)} \qquad \text{av}\left(\frac{\dot{\lambda}}{\lambda}\right) = \frac{1}{9}\sum_{n=1973}^{1981}\left(\frac{\lambda_{n+1}-\lambda_n}{\lambda_n}\right) = \underline{-1.8\,\%}.$$

The numbers −5.7 % and −1.8 % in the equations (7) and (8), respectively, are the empirical observations to be compared with theory.

The period 1973-82 will be taken to be one of Growth Type 1. Hence there must be, according to the present theory, $b = b^*$, where b^* is the balanced-growth output/capital ratio. For the average annual rise in the level of the net savings rate, now represented by the balanced-growth net savings rate s^*, we get from (7) and (8):

$$\text{(11.9)}\quad \text{av}\left(\frac{\dot{s}^*}{s^*}\right) = \text{av}\left(\frac{\dot{\lambda}}{\lambda}\right) - \text{av}\left(\frac{\dot{b}^*}{b^*}\right) = -1.8\,\% + 5.7\,\% = \underline{+3.9\,\%}.$$

This value, derived from the observed values $-1.8\,\%$ and $-5.7\,\%$, can also be considered as empirically given. The question is: can the two empirical values (7) and (8), and accordingly the value (9) deduced from them, be derived from the present theory, i.e. from the theoretical equations (4)?

The equations (4) first give:

$$\text{(11.10)}\qquad \frac{\dot{b}^*}{b^*} = \frac{1}{b^*}\left(\frac{\partial b^*}{\partial s^*}\right)\dot{s}^* = -\left(\frac{s^*}{s^*-\beta/\sigma}\right)\frac{\dot{s}^*}{s^*},$$

$$\text{(11.11)}\qquad \frac{\dot{\lambda}}{\lambda} = \frac{1}{\lambda}\left(\frac{\partial \lambda}{\partial s^*}\right)\dot{s}^* = -\left(\frac{\beta/\sigma}{s^*-\beta/\sigma}\right)\frac{\dot{s}^*}{s^*}.$$

Since 1.8 is roughly half of 3.9 and 5.7 is roughly equal to 1.5 times 3.9, the theoretical equations (10) and (11) give correctly the orders of magnitude of the observed values, if the conditions

$$\text{(11.12)}\qquad A = \frac{\beta/\sigma}{s^*-\beta/\sigma} \approx 1/2 \quad\text{and}\quad B = \frac{s^*}{s^*-\beta/\sigma} \approx 3/2,$$

can be satisfied by some values $\hat{s}$ and $\hat{\sigma}$, which when inserted for s^* and σ in (12) give the annual averages av (A) and av (B). To satisfy (12) these values must obey the equation

$$\text{(11.13)}\qquad \hat{s}\hat{\sigma} = 3\beta.$$

Which can be those values? Taking into account that the value s^* from 1973 to 1982 increases according to the law

$$s^*_{1982} = (1.039)^9\,(s^*_{1973}) = (1.411)s^*_{1973},$$

the value $s^*_{1973} = .13$ for instance gives $s^*_{1982} = .18$. Taking the value $\hat{s} \doteq .15$ near the average between .13 and .18, and the value $\hat{\sigma} = 5$, the condition (13) is satisfied with the usual capital's share $\beta = .25$. These values of parameters are realistic enough.

It then follows from (10),(11) and (12) that the theoretical predictions for the annual averages of $\dot{\lambda}/\lambda$ and $\dot{b}/b$ are the following:

$$\text{(11.14)}\qquad \left(\frac{\dot{\lambda}}{\lambda}\right)_{theor} = -1.95\,\% \quad\text{and}\quad \left(\frac{\dot{b}^*}{b^*}\right)_{theor} = -5.85\,\%.$$

Compared with the observed values $-1.8\,\%$ and $-5.7\,\%$, respectively, the predictions are not too bad. Thus the present theory of the growth effects of savings rate can give a causal explanation of the observed decreases in the level of output and in the level of the productivity of capital, at least as far as the orders of magnitude are concerned.

Let it be remarked that the inverse proportionality of savings rate and the risk aversion coefficient in the condition (13) is natural indeed: the higher the rate of investment (=savings rate), the lower the aversion to risk taking tends to be.

VI

AN ALTERNATIVE VISION OF THE STOCHASTIC ELEMENT IN BUSINESS CYCLES

12. Stochastic Shocks as Perturbations Superposed Upon the Basic Business Cycles

1. Are the business cycles purely stochastic processes? Stochastic shocks do not essentially influence the long-term economic development. This is an accomplished fact in current growth theories (of Solow and Lucas, for instance). But ever since an important paper of the Russian mathematician Eugen Slutsky from the year 1927 was translated and published in English in a completed form (Slutsky, 1937), the economists have been fascinated by the idea that the business cycles may be purely stochastic processes. This would surely account for the ragged outlook of most economic time series. What Slutsky showed, however, was something more, viz. that random series are capable of forming cyclic phenomena. In fact this follows already from the symmetry of the Gauss curve around the mean of the series, and the effect can be made more visible by summations of certain sequences in a random series.

The idea is today applied in economics typically by multiplying the production function by a random variable, which undergoes, say, a Markov process. The values of this variable form a random series, each member of which is called a "technological shock". Such a stochastic production function is then treated just like a deterministic production function in the optimization process described in Chapter 2. It is this kind of treatment that underlies each of the stochastic shock models of business cycles mentioned in Chapter 7 above. The so produced oscillations of output and other economic variables around a trend, a usually loglinear one, are then considered as models of the business cycles, with the results quoted in Chapter 7.

What seems questionable to the present author in the method just described is that economic agents are supposed to react to the shocks: this idea is of course involved in the maximization of utilities which already include the stochastic shock variables. So far as the business cycles are thought to be produced mainly by the shocks, no other method of constructing the cycles seems to be available. However, isn't it too much to expect that all the economic agents are capable of reacting rationally to the shocks? Wouldn't it be more natural to assume that the agents react to the trends, observable over a certain interval of time, rather than to the irregular shocks?

In the theory of the Basic Business Cycles, which is nonstochastic, we can easily realize the latter situation. We already have the "cycles". What we need is to complete the theory by superposing upon the Basic Cycles the technological (or other) shocks, since the latter too surely appear in the real world. This is a problem different from that met with in the usual stochastic models of the business cycles. In the theory of the BBC the shocks have a minor function: they appear only as perturbations on the nonstochastic Basic Cycles. How much do such shocks affect the correlations and standard deviations of economic variables over detrended cycles: this is the problem that will be illuminated here by means of an example.

In this purpose technological shocks of the usually applied size will be superposed upon the Basic Business Cycles in order to see how great is their influence on correlations and standard deviations. In the example given in Chapter 13 such shocks give only a minor contribution to the results obtained in Chapter 7 by means of the nonstochastic BBC. For instance, the technological shocks account only 5 % of the standard deviation of output over detrended business cycles, while the theory of the Basic Business Cycles accounts for the rest 95 %. But first we have to construct the shocks.

2. The production of random series with a definite mean and standard deviation. We can construct a random series in many ways. Here the following method is used. Take the seven first decimals of two irrational numbers, say $\pi - 3$ and $\sqrt{3} - 1$. Multiply each of the so obtained two numbers from the interval [0,1] by a prime number, say 317. Take only the decimal part of each product. Multiply them again by 317 and take the decimal parts. Go on until you have in both series 305 numbers belonging to the interval [0,1]. We can name them: let they represent the two random variables u_1 and u_2, with a constant distribution in the interval [0,1].

Hence we can proceed by constructing the new random variables

$$\begin{aligned} x_1 &= \sqrt{2\log(1/u_1)}\,\sin 2\pi\, u_2, \\ x_2 &= \sqrt{2\log(1/u_1)}\,\cos 2\pi\, u_2. \end{aligned}$$

The number series constructed in this way give two variables with the normal distriburtion $\mathbf{N}(0,1)$. Random variables having a normal distributiuon with any desired means μ_i and standard deviations σ_i are then given by

$$z_1 = \mu_1 + \sigma_1 x_1, \quad z_2 = \mu_2 + \sigma_2 x_2.$$

This is a quick method of getting two random series with given means and standard deviations. Here we shall need only one of them, say z_1. The random series u_i and z_i, each of them including 305 numbers, are given in Appendix 2. Also the value of the ordinary cycle function $Q_Y = \beta(w - b^*)$ at the points of time $t_n = (.2)n$ for $n = 0, 1, 2, ..., 304$ are given there. These times correspond to an approximate division of the full cycle of the function $w - b^*$ in equal intervals having the length of .02 in terms of the theoretical time unit [TU] (for this time units see Chapter 7, Section 4).

3. How the technological shocks affect each economic variable? It is usual in current shock models of business cycles to assume a technological shock variable with a lognormal distribution having the mean one. The multiplication of a given output Y by such a stochastic factor is equivalent to adding to the growth rate of Y a random variable of the type z, with the mean zero. Let us call it z_Y and identify it with the variable z_1 constructed above. This gives

$$z_Y = \sigma_1 x_1, \tag{12.1}$$

with a standard deviation σ_1 to be chosen later.

To consider the effects of technological shocks on the consumption C and the investment I we have to study the expressions indicating the relations of their growth rates with that of output:

$$\frac{\dot{C}}{C} = \frac{\frac{d}{dt}(1-s)}{1-s} + \frac{\dot{Y}}{Y}, \quad \frac{\dot{I}}{I} = \frac{\dot{s}}{s} + \frac{\dot{Y}}{Y}.$$

The shock variables of these economic variables must obey the same equations:

$$z_C = z_{1-s} + z_Y, \quad z_I = z_s + z_Y.$$

With the approximation

$$\frac{\dot{s}}{s} \equiv -\frac{1-s}{s}\left(\frac{\frac{d}{dt}(1-s)}{1-s}\right) \approx -\frac{1-s^*}{s^*}\left(\frac{\frac{d}{dt}(1-s)}{1-s}\right)$$

and assuming no consumption shocks, $z_C = 0$, this gives:

$$z_s = -\left(\frac{1-s^*}{s^*}\right) z_{1-s}.$$

Accordingly we have:

$$z_C = 0, \quad z_I = \frac{1-s^*}{s^*} z_Y + z_Y = \frac{\sigma_1}{s^*} x_1. \tag{12.2}$$

To find the influence of shocks upon the employment E and the productivity of labour W we have to discuss the production function (4.12) when written for the growth rates. In view of (4.17) this reads:

$$\frac{\dot{Y}}{Y} = \beta\frac{\dot{K}}{K} + (1-\beta+\kappa)\frac{\dot{h}}{h} - (1-\beta)m.$$

It follows that we have to put:

$$z_Y = \beta z_K + (1-\beta+\kappa)z_h - (1-\beta)m.$$

Here we have to decide to which extent we shall now assume the technological shocks to be caused by shocks in physical capital (e.g. new oil fields found, or old ones closed, or oil price changed) and to which extent we suppose they are due to shocks in human capital (e.g. new methods of production discovered). For the sake simplicity we can make the assumption that

$$\text{(12.3)} \qquad z_K = 0, \quad \text{thus } z_h = \frac{\sigma_1 x_1}{1-\beta+\kappa} + \frac{(1-\beta)m}{1-\beta+\kappa}.$$

This gives, since $E = h\alpha^*/k$ (in the Growth Type 1) and $W = Y/E$:

$$\text{(12.4)} \qquad z_E = \frac{\sigma_1 x_1}{1-\beta+\kappa} - \frac{\kappa m}{1-\beta+\kappa}, \quad z_W = z_Y - z_E.$$

The stochastic shock variables (1)-(4) have to be added to the corresponding cycle functions Q_X. This gives the stochastic economic variables we need:

$$\text{(12.5)} \quad Y_{st} = Q_Y + \sigma_1 x_1, \; C_{st} = Q_C = \frac{1}{\sigma}Q_Y, \; I_{st} = Q_I + \frac{\sigma_1 x_1}{s^*},$$

$$\text{(12.6)} \quad E_{st} = Q_E + \frac{\sigma_1 x_1}{1-\beta+\kappa} - \frac{\kappa m}{1-\beta+\kappa}, W_{st} = Y_{st} - E_{st}.$$

We can call these functions X_{st} the *stochastic cycle functions.* Note that in the expression for C_{st} the symbol σ of course means the risk aversion coefficient, not a standard deviation (to make the distinction clear the latter are always equipped with a subscript).

13. Final Result: Both the Stochastic and Nonstochastic BBC Versions Predict Better Than Any of the Models Based On Stochastic Optimization

1. Correlations and variances of stochastic cycle functions over a cycle: the formulae. By inserting the ordinary cycle functions given by the equations (6.25), (6.28)-(6.31) in the stochastic cycle functions (12.5)-(12.6) we can easily deduct the following formulae for the variances of these stochastic variables over a detrended cycle:

$$(13.1) \quad \sigma^2_{Y_{st}} = \sigma^2_Y + 2\sigma_1 \text{cov}(x_1, Q_Y) + \sigma_1^2,$$

$$(13.2) \quad \sigma^2_{C_{st}} = \sigma^2_C = \left(\frac{\sigma_Y}{\sigma}\right)^2,$$

$$(13.3) \quad \sigma^2_{I_{st}} = \sigma^2_I + (2B\sigma_1/s^*)\,\text{cov}(x_1, Q_Y) + (\sigma_1/s^*)^2,$$
$$\text{with } B = \sigma_I/\sigma_Y;$$

$$(13.4) \quad \sigma^2_{E_{st}} = \sigma^2_E + \left[\frac{2\sigma_1(1-s^*)}{(1-\beta+\kappa)^2}\right] \text{cov}(x_1, Q_Y)$$
$$- \left(\frac{2\sigma_1\beta b^*}{1-\beta+\kappa}\right) \text{cov}(x_1, s-s^*) + \left(\frac{\sigma_1}{1-\beta+\kappa}\right)^2,$$

$$(13.5) \quad \sigma^2_{W_{st}} = \sigma^2_{Y_{st}} + \sigma^2_{E_{st}} - 2\,\text{cov}(E_{st}, Y_{st}).$$

For their covariances with output we get likewise:

$$(13.6) \quad \text{cov}(I_{st}, Y_{st}) = \text{cov}(I, Y) + (B + 1/s^*)\sigma_1\,\text{cov}(x_1, Q_Y) + \sigma_1^2/s^*,$$

$$(13.7) \quad \text{cov}(E_{st}, Y_{st}) = \text{cov}(E, Y) + \left(\frac{\sigma_1(2-s^*)}{1-\beta+\kappa}\right) \text{cov}(x_1, Q_Y)$$
$$- \left(\frac{\sigma_1\beta b^*}{1-\beta+\kappa}\right) \text{cov}(x_1, s-s^*) + \frac{\sigma_1^2}{1-\beta+\kappa},$$

$$(13.8) \quad \text{cov}(W_{st}, Y_{st}) = \sigma^2_{Y_{st}} - \text{cov}(E_{st}, Y_{st}).$$

For the consumption C, to which no shock variable is attached, we get directly the correlation:

$$(13.9) \qquad r_{C_{st},Y_{st}} = \frac{\sigma_Y}{\sigma_{Y_{st}}} + \sigma_1 \frac{\text{cov}(x_1, Q_Y)}{\sigma_Y \sigma_{Y_{st}}}.$$

2. Preliminary steps of calculation. *(i).* The first term in each of the equations (1)-(4) and (6)-(7) is the variance or covariance, respectively, of the corresponding nonstochastic variable in Growth Type 1. We can take their numerical values directly from the calculations made in Chapter 7.

*(ii).*In the same way we get the numerical values of the constants

$$\sigma = 2.2916666,\ \ B = 4.7702278,\ \ s^* = .13,\ \ 1-\beta+\kappa = 1.167,\ \ \beta = .25,\ \ b^* = .3.$$

(iii). The computation of $\text{cov}(x_1, Q_Y)$. Here we must first of course construct the random series x_1 as well as the corresponding series of the values of the cycle function $Q_Y = \beta(w - b^*)$. The lattercan be constructed on the basis of the equation (6.5). When written for the initial value $w(0) = b^*$ of the output/capital variable, this equation reads:

$$(13.10) \qquad Q_Y = -e^{\alpha^* t/2}\left[\frac{\beta a(\alpha^2 + 4\omega^2)}{4(1-s^*)\omega}\right](\sin \omega t)x_o.$$

Here the unknown initial state $s(0) - s^*$ has been denoted by x_o, and the constants, again obtained directly from Chapter 7, have the values:

$$\alpha^* = .036,\ \ \beta = .25,\ \ \omega = .1034851,\ \ a = 7.0967745.$$

To match the values of Q_Y with the series x_1 the length $T = 2\pi/\omega$ of a period of its cycle (in theoretical time units) must be divided in 304 equal intervals. Since $T = 60.715844$ this gives the following series of time points:

$$(13.11) \qquad t_n = (.2)n \quad \text{with} \quad n = 0, 1, 2, ..., 304.$$

The values of Q_Y at these points of time are easily computed by means of the formula obtained by inserting (11) and the above values of constants in the equation (10), which gives:

$$(13.12) \qquad Q_Y(n) = (-.217421)\, e^{(.0036)n} \sin(1.1858519n)^\circ x_o.$$

Here the angle has been already transformed to degrees. Both series $Q_Y(n)$ and $x_1(n)$ are given in Appendix 2. The series of $x_1(n)$ is long enough to bring the mean very close to its expectation value zero, and we can compute the covariance by using the formula

$$(13.13) \qquad \text{cov}(x_1, Q_Y) = \frac{1}{305}\sum_{n=0}^{304} x_1(n)Q_Y(n) = -(.0242242)\, x_o.$$

(iv). The computation of $\mathrm{cov}(x_1, s-s^*)$. Here we can shorten the calculation by using the formula

$$(13.14)\qquad s-s^* = \left(\frac{\alpha^*}{2\beta\omega A}\right) Q_Y + Rx_o,$$
$$A = .8696843, \quad R = e^{\alpha^* t/2}\cos\omega t,$$

obtained from (6.4) and (6.5). Appendix 2 gives also the series $R(n)$ for $n = 0, 1, 2, ..., 304$ needed in this computation. We compute first:

$$(13.15)\qquad \mathrm{cov}(x_1, Rx_o) = \frac{1}{305}\sum_{n=0}^{304} x_1(n)R(n)x_o = (-.0107423)\, x_o.$$

Combining the results (13)-(15) we then get:

$$(13.16)\qquad \mathrm{cov}(x_1, s-s^*) = (-.0301217)\, x_o.$$

3. Calibration. To tell what there is to be calibrated we need to know how far can take us the numerical knowledge we already have. After the above preliminary calculations we can give the numerical expressions, where only two unknown constants appear, viz. the standard deviation σ_1 of the random variable x_1 and the initial state x_o of the cycle function $Q_Y(s, w)$:

$$(13.17)\qquad \sigma^2_{Y_{st}} = (.0784541)\, x_o^2 - (.0484484)\, \sigma_1 x_o + \sigma_1^2,$$

$$(13.18)\qquad \frac{\sigma_C}{\sigma_Y} = (6.944552)\, x_o, \quad r_{C_{st},Y_{st}} = (15.914585)\, x_o$$
$$-(4.9139279)\, \sigma_1,$$

$$(13.19)\qquad \sigma^2_{I_{st}} = (1.7865789)\, x_o^2 - (1.7784393)\, \sigma_1 x_o + (59.171597)\, \sigma_1^2,$$

$$(13.20)\qquad \sigma^2_{E_{st}} = (.0398651)\, x_o^2 - (.027632)\, \sigma_1 x_o + (.7342742)\, \sigma_1^2,$$

$$(13.21)\qquad \sigma^2_{W_{st}} = (.017479)\, x_o^2 - (.0023186)\, \sigma_1 x_o + (.8773762)\, \sigma_1^2.$$

The remaining expressions of covariances are:

$$(13.22)\qquad \mathrm{cov}(I_{st}, Y_{st}) = (.3743856)\, x_o^2 - (.3019385)\, \sigma_1 x_o$$
$$+(7.6923076)\, \sigma_1^2,$$

$$(13.23)\qquad \mathrm{cov}(E_{st}, Y_{st}) = (.0504201)\, x_o^2 - (.0368809)\, \sigma_1 x_o$$
$$+(.856898)\, \sigma_1^2,$$

$$(13.24)\qquad \mathrm{cov}(W_{st}, Y_{st}) = (.028034)\, x_o^2 - (.0115675)\, \sigma_1 x_o$$
$$+(.143102)\, \sigma_1^2.$$

To calibrate as close to empirical values as possible, we first of course choose the standard deviation of the stochastic output variable Y_{st} to be equal to the observed standard deviation, which in the U.S. economy used as empirical comparison in the stochastic optimization models is .0176. The same calibration was of course used in those stochastic optimization models (of Kydland and Prescott, Hansen and Rogerson, and Danthine and Donaldson).

According to Hansen (1985, p.320), " A data analysis suggests that [the standard deviation of the technological shock] could reasonably be expected to lie in the interval [.007,.010]." To keep as close as possible to the choices made in the stochastic optimization models, with whose results the outcome of the BBC approach will be compared here, the exact Hansen-Rogerson value $\sigma_1 = .00712$ is chosen here for the constant σ_1. Thus the *technological shocks here applied are exactly of the size of the Hansen shocks.*

Thus we have the following calibration:

$$\sigma_{Y_{st}} = .0176, \quad \sigma_1 = .00712. \tag{13.25}$$

These values give, in view of (17):

$$x_o = .0597035. \tag{13.26}$$

4. The small but not negligible effect of shocks. After this calibration we can first find out which proportions of the standard deviation of output are explained by the nonstochastic and stochastic BBC approaches, respectively. We get:

$$\sigma_Y = (.2800967)\,x_o = .0167227 = 95\% \text{ of.}0176, \tag{13.27}$$

leaving only 5% of the empirical value .0176 to be explained by the technological shocks.

By inserting the values (25) and (26) in the expressions (18)-(24) we get the other numerical predictions given by the stochastic BBC-approach. They are compared in Table 11 with the numerical predictions of the nonstochastic BBC approach given in Chapter 7. We can see that the shocks have contributed little to the predictions of the BBC approach, with the exception perhaps of the standard deviation of investment. But even the change in this case is in terms of precentages only 10% (see the next page).

TABLE 11. STANDARD DEVIATIONS AND CORRRELATIONS OVER A CYCLE PREDICTED BY THE NONSTOCHASTIC AND STOCHASTIC BBC.

	Standard deviation proportions :			*Correlations with output :*		
Variable	*BBC*	BBC_{st}	Δ%	*BBC*	BBC_{st}	Δ%
Output	1.00	1.00	–	1.00	1.00	–
Investment	4.77	5.27	10%	1.00	.98	2%
Consumption	.44	.41	7%	1.00	.92	8%
Employment	.71	.73	3%	.91	.91	0%
Productivity	.47	.45	4%	.77	.74	4%

5. The final result in numbers: Table 12. The sums of error squares show that both BBC versions, with or without shocks, do far better than the Danthine-Donaldson, Hansen-Rogerson or Kydland-Prescott models in standard deviations and clearly better also in correlations.

TABLE 12. THE PREDICTION SUCCESS OF THE BBC VERSIONS AS COMPARED WITH THAT OF STOCHASTIC OPTIMIZATION MODELS.

Standard deviation proportions:

Variable	*BBC*	BBC_{st}	$D-D$	$H-R$	$K-P$	*empirical*
Investment	4.77	5.27	3.45	3.24	3.07	<u>4.89</u>
Consumption	.44	.41	.19	.29	.25	<u>.73</u>
Employment	.71	.73	.72	.77	.68	<u>.94</u>
Productivity	.47	.45	.35	.28	.40	<u>.67</u>
$\sum \Delta^2$	.1961	.3393	2.5185	3.1071	3.6933	

Correlations with output:

Variable	BBC_{st}	*BBC*	$K-P$	$H-R$	$D-D$	*empirical*
Investment	.98	1.00	.86	.99	.99	<u>.92</u>
Consumption	.92	1.00	.85	.87	.69	<u>.85</u>
Employment	.91	.91	.95	.98	.98	<u>.76</u>
Productivity	.74	.77	.86	.87	.91	<u>.42</u>
$\sum \Delta^2$	.1334	.1739	.2333	.2562	.3190	

6. The final result illustrated: Figures 9 and 10. The lines of the predictions of the BBC versions follow – at a certain distance – the empirical line, those of the other models together stray off that line (p.96-97).

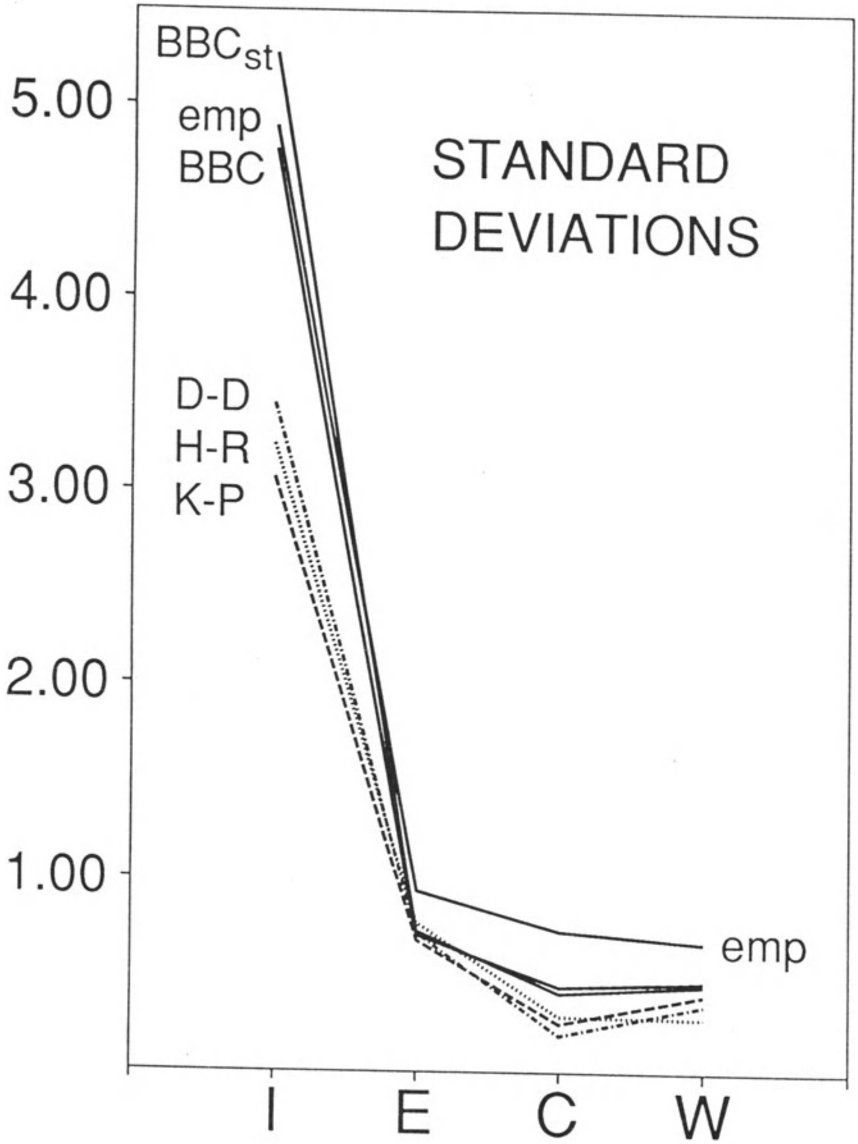

FIGURE 9: STANDARD DEVIATION PROPORTIONS.
TWO DISTINCT PATTERNS APPEAR: THE PREDICTIONS OF THE BBC, BOTH STOCHASTIC (BBC_{st}) AND NONSTOCHASTIC (BBC), FOLLOW THE PATTERN OF EMPIRICAL VALUES, WHILE THOSE OF THE STOCHASTIC OPTIMIZATION MODELS TOGETHER STRAY OFF FROM THOSE VALUES.

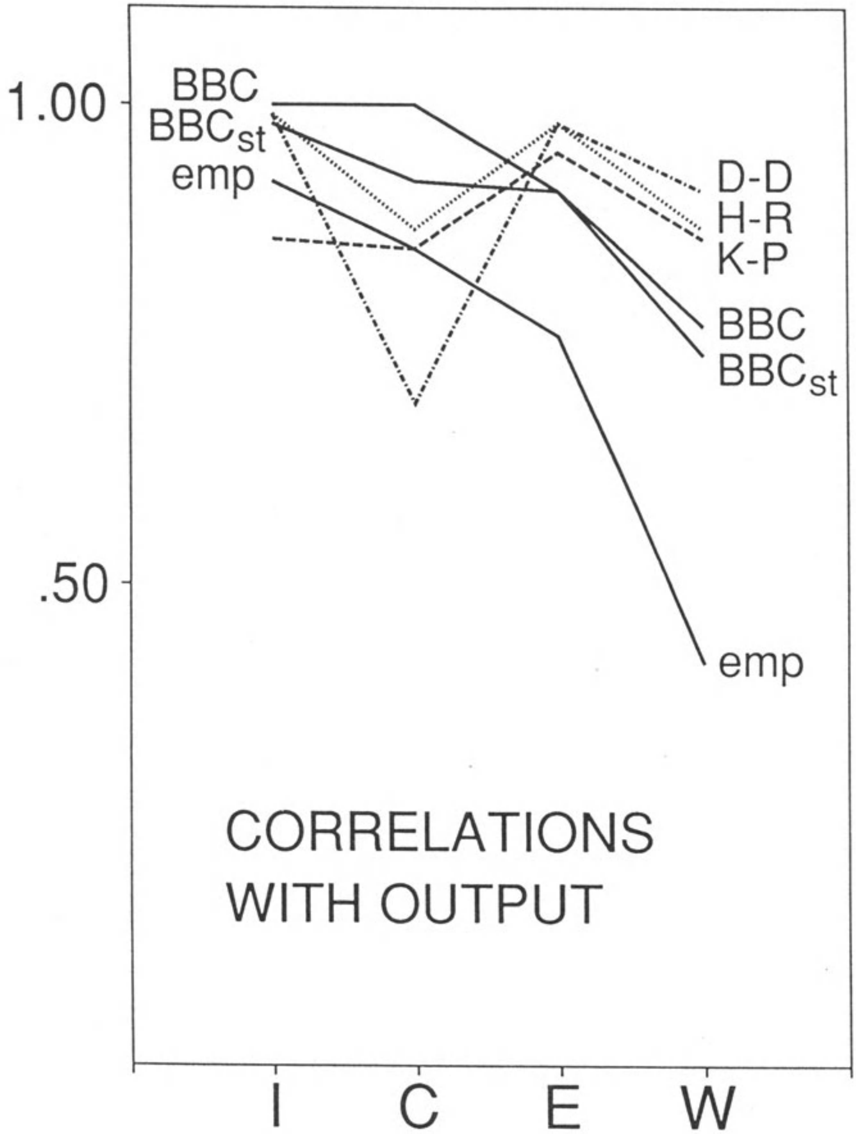

FIGURE 10: CORRELATIONS WITH OUTPUT.
AGAIN TWO PATTERNS APPEAR: THE PREDICTIONS OF BASIC BUSINESS CYCLES, BOTH STOCHASTIC (BBC_{st}) AND NONSTOCHASTIC (BBC), FOLLOW – AT A CERTAIN DISTANCE – THE PATTERN OF EMPIRICAL VALUES, WHILE THOSE OF THE STOCHASTIC OPTIMIZATION MODELS TOGETHER STRAY OFF FROM THOSE VALUES.

7. Final comments. There are three of them:

1. There can be no doubt about the testimony given by Figures 9 and 10 and, in a numerical form, by Table 12. They confirm the success of the theory of Basic Business Cycles in the prediction of what is generally considered as the essential available data on business cycles, viz. standard deviations and correlations with output of economic variables over a detrended cycle. In particular, these Figures and that Table add to the evidence given in Chapters 7 and 9 the important point that the BBC theories , *both the stochastic and nonstochastic versions*, predict clearly better than the current models constructed by the method of stochastic optimization.

2. These results thus give the strongest evidence, possible with the so far available data, for the decisive influence of nonmaterial values also upon the business cycles, and not only upon economic growth. The decisive effect of the pursuit of cognitive innovations, also without any material rewards (e.g. in science), upon economic growth is obvious and widely recognized. But the most important nonmaterial values, like individual freedom and the pursuit of knowledge, can be represented only by unbounded utility functions, of which the general theory constructed in Chapters 4,5 and 6 gives an example. It follows that in macroeconomic theory we must have two levels. First of them operates with the (always finite) material values and bounded utility functions. This level of macroeconomics is reducible to microeconomics, and represents the level on which the economic game is played in the short run. This, in other words, is the macroeconomics as understood in economics today. The long-term development, however, is dominated by nonmaterial values expressing the long-term human pursuits of ever larger individual freedom and ever greater objective konwledge of the world we are living in. This accordingly is the suggested new level of macroeconomics on which nonmaterial values are the decisive ones.

3. The gap that still remains between the empirical estimates and the values predicted by the BBC theory may be just a result of the use of linear approximation of the nonlinear equations (4.27) in calculations, and also of the corresponding linear approximations of the cycle functions. Only a numerical computer solutions of the nonlinear equations and nonlinear cycle functions could settle this question. But no theory ever invented can explain everything. Even supposing that the true nonlinear solutions of the BBC equations and cycle functions will bring the predictions still closer to the empirical data, there will always be plenty of room for other and better future theories to come.

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Appendix 1: The Dependence of Predicted Anomalous Correlations on Savings Rate

TABLE 13.
THE PREDICTED ANOMALOUS OUTPUT CORRELATIONS OF CONSUMPTION AND INVESTMENT.[1]

x	r_{CY}			r_{IY}		
	.11	.13	.15	.11	.13	.15
.0000	−.784	−.786	−.785	1.000	1.000	1.000
.0005	−.701		−.456			
.0006	−.680		−.340		+.886	+.911
.0010	−.575	−.386		+.946	+.786	+.654
.0011		−.316				
$.0012_5$	−.490		+.527			+.486
.0013	−.471		+.571			+.460
.0014			+.641			+.417
.0015	−.389	+.008	+.706		+.584	+.384
.0020	−.141	+.398	+.868	+.754	+.372	+.338
.0023					+.284	
.0025		+.652	+.929		+.245	+.375
.0030	+.382	+.789	+.956	+.476	+.200	+.425
.0033	+.501					
.0035	+.572			+.369		
.0040	+.699		+.979	+.298	+.216	+.507
.0050	+.838	+.948	+.987	+.230	+.265	+.560
.0060	+.903					
.0075	+.947		+.995	+.219	+.362	+.630
.0100	+.974	+.990	1.000	+.247	+.417	+.662

[1] Only the $C-$ and $I-$points marked in Figures 11 and 12, respectively, are indicated in Table 13. The numbers .11,.13 and .15 are the chosen values for the savings rate parameter s^*.

TABLE 14.
THE PREDICTED ANOMALOUS OUTPUT CORRELATIONS OF EMPLOYMENT AND PRODUCTIVITY.[2]

x	r_{EY}			r_{WY}		
	.11	.13	.15	.11	.13	.15
.0000	1.000	1.000	1.000	1.000	1.000	1.000
.0005	+.997	+.994	+.978	+.675	+.404	+.067
.0006			+.963			+.025
.0008				+.278		+.085
.0010	+.986	+.963	+.863	+.108	−.009	+.229
.0011		+.952				
.0012		+.939				
$.0012_5$			+.810	−.003		
.0013			+.804			
.0014			+.795			
.0015		+.891	+.791	−.043	+.086	+.523
$.0017_5$	+.938			−.037		
.0020	+.912	+.769	+.806	−.000	+.323	+.645
.0025	+.850	+.763	+.831		+.451	
.0030		+.757			+.547	
.0035	+.750		+.864	+.370		+.743
.0040	+.732	+.783	+.874	+.454	+.639	
.0045	+.728					
.0050	+.733	+.810	+.887	+.555	+.680	+.744
.0075	+.779	+.848	+.894	+.649	+.720	+.775
.0100	+.810	+.865	+.910	+.681	+.735	+.789

[2] Only the E- and W–points marked in Figures 13 and 14, respectively, are indicated in Table 14. The numbers .11,.13 and .15 are the chosen values for the savings rate parameter s^*.

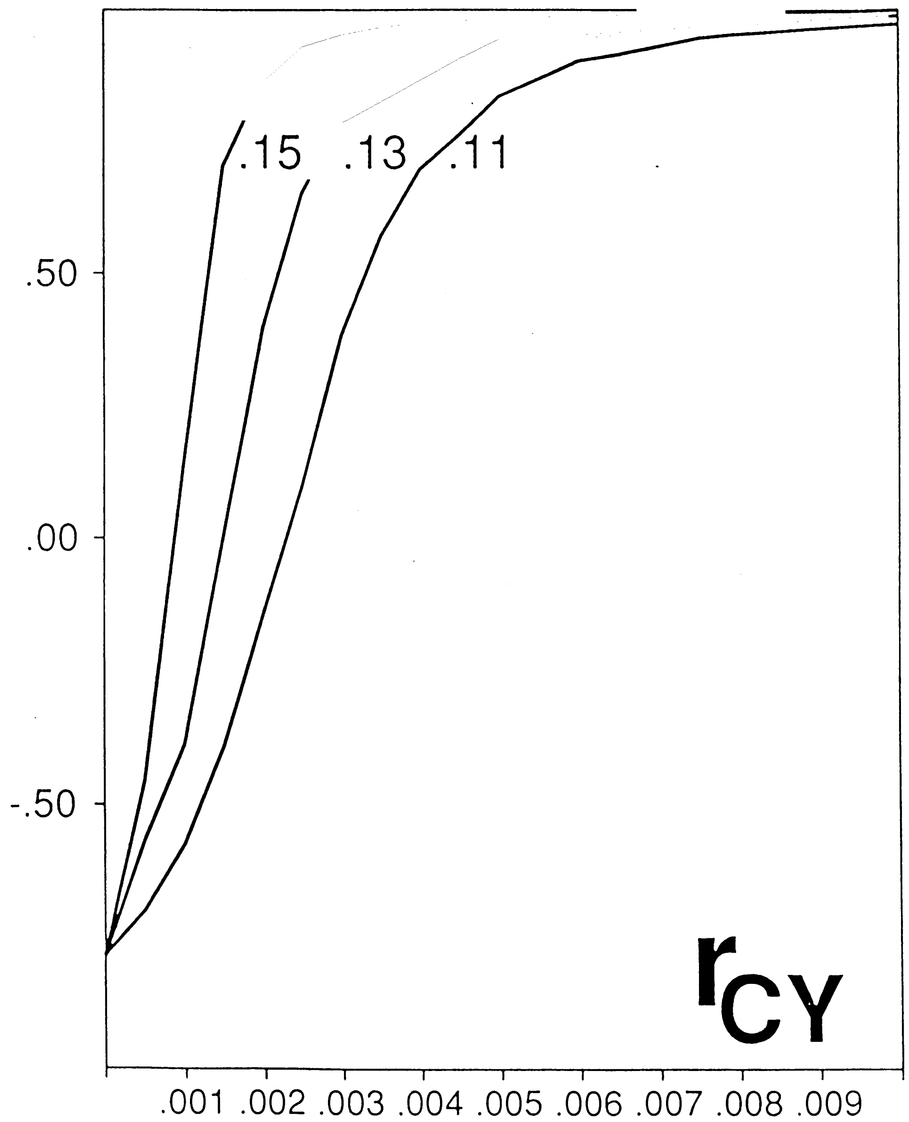

FIGURE 11.
AN ILLUSTRATION OF THE DEPENDENCE OF THE ANOMALOUS OUTPUT CORRELATION OF CONSUMPTION ON THE SAVINGS RATE PARAMETER.

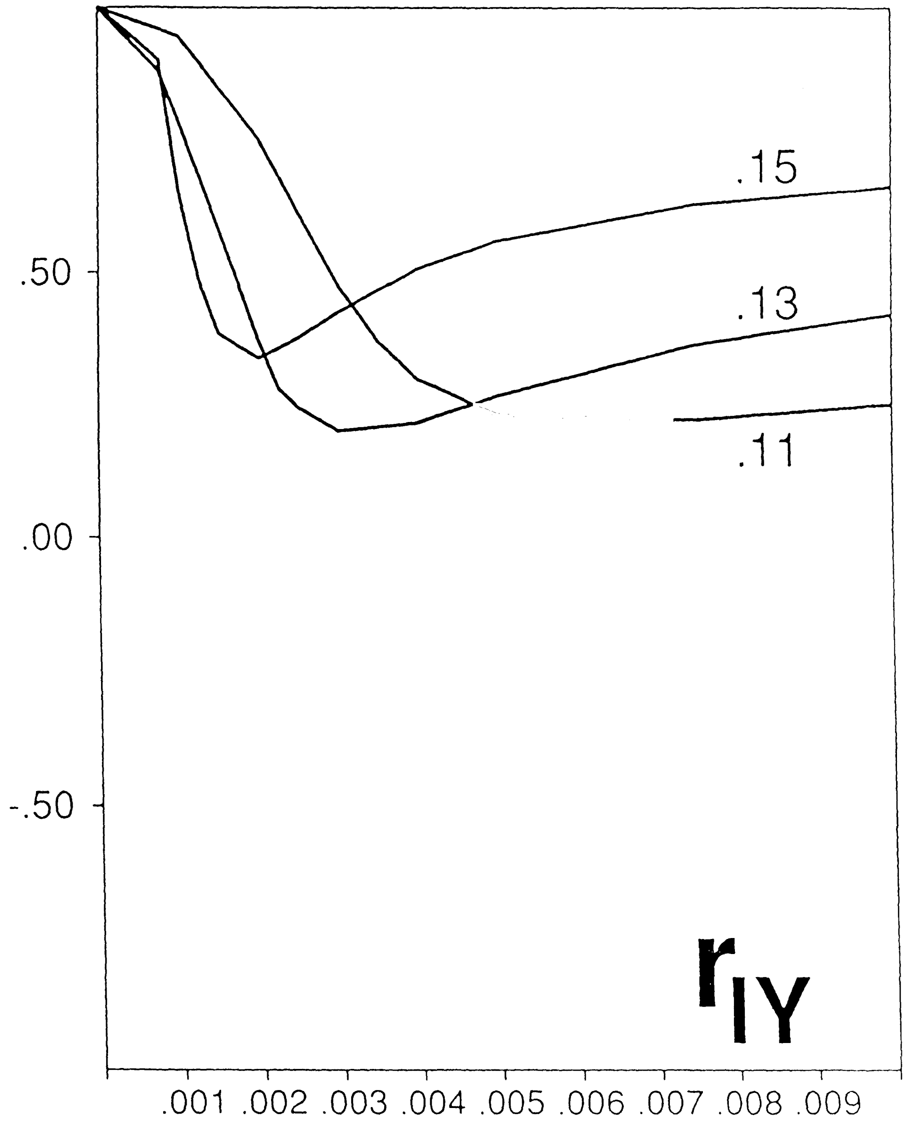

FIGURE 12.
AN ILLUSTRATION OF THE DEPENDENCE OF THE ANOMALOUS OUTPUT CORRELATION OF INVESTMENT ON THE SAVINGS RATE PARAMETER.

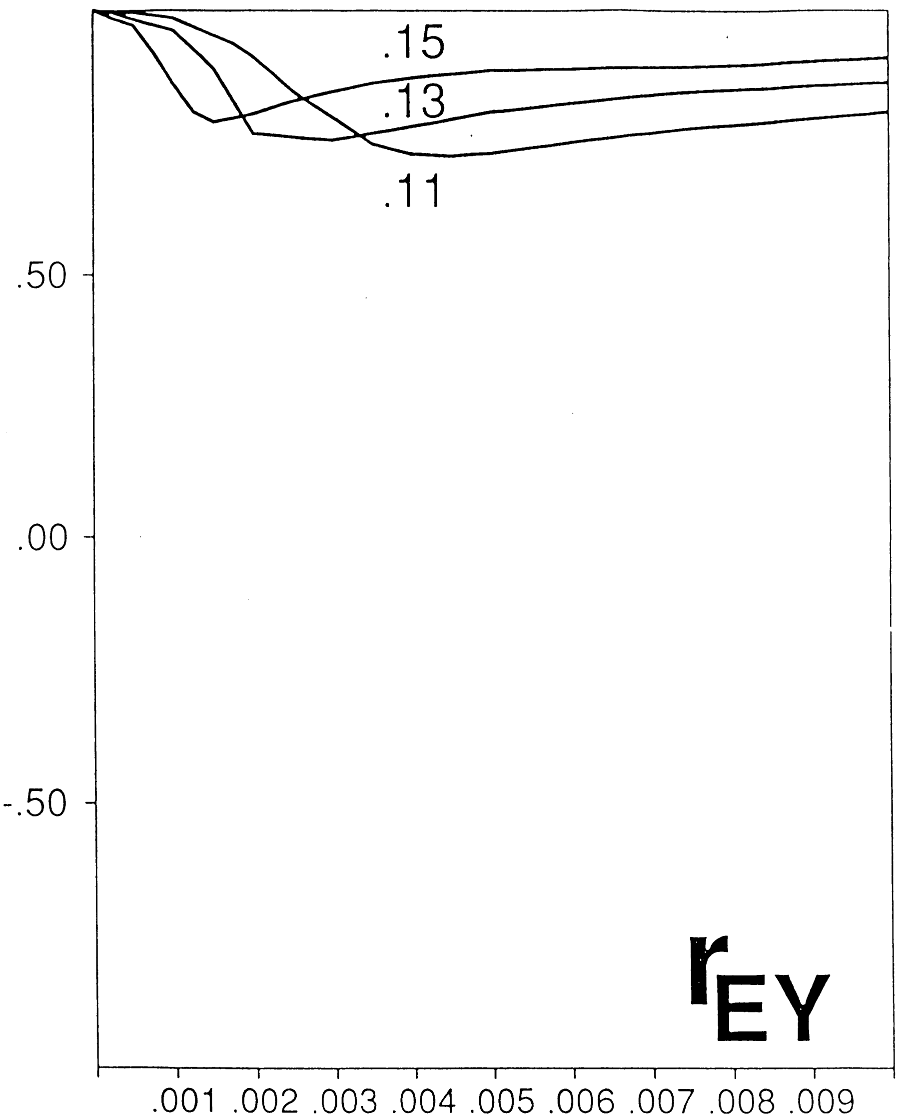

FIGURE 13.
AN ILLUSTRATION OF THE DEPENDENCE OF THE ANOMALOUS OUTPUT CORRELATION OF EMPLOYMENT ON THE SAVINGS RATE PARAMETER.

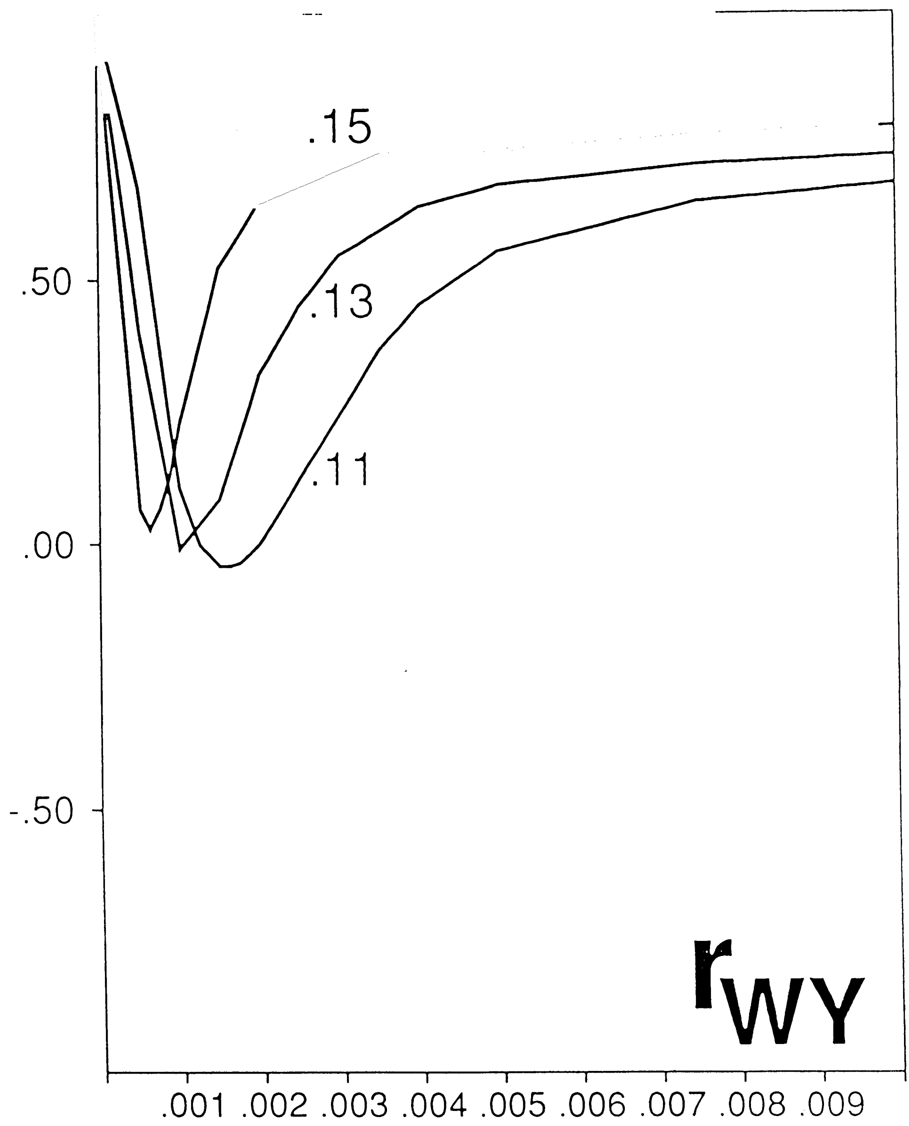

FIGURE 14.
AN ILLUSTRATION OF THE DEPENDENCE OF THE ANOMALOUS OUTPUT CORRELATION OF THE PRODUCTIVITY OF LABOUR ON THE SAVINGS RATE PARAMETER.

Appendix 2: Numerical Tables

TABLE 15.
THE RANDOM SERIES $u_i(n)$ AND $x_1(n)$, AND THE CORRESPONDING VALUES OF THE FUNCTIONS Q_Y/x_o AND R.[1]

n	$u_1(n)$	$u_2(n)$	$x_1(n)$	$Q_Y(n)/x_o$	$R(n)$
0	.1415926	.7320508	−1.9647102	.0000000	1.0000000
1	.8848542	.0601036	.1823869	−.0045158	1.0033915
2	.4987814	.0528412	.3844457	−.0090623	1.0063630
3	.1137038	.7506604	−2.0852436	−.0136377	1.0089105
4	.0441046	.9593468	−.6312724	−.0182400	1.0110295
5	.9811582	.1129356	.1270773	−.0228676	1.0127159
6	.0271494	.8005852	−2.5511509	−.0275184	1.0139661
7	.6063598	.7855084	−.9754896	−.0321907	1.0147761
8	.2160566	.0061628	.0677678	−.0368825	1.0151426
9	.4899422	.9536076	−.3432899	−.0415918	1.0150621
10	.3116774	.2936092	1.1266953	−.0463168	1.0145315
11	.8017358	.0741164	.2985173	−.0510554	1.0135476
12	.1502486	.4948988	.0623949	−.0558056	1.0121079
13	.6288062	.8829196	−.6464034	−.0605654	1.0102093
14	.3315654	.8855132	−.9790354	−.0653328	1.0078497
15	.1062318	.7076844	−2.0432002	−.0701057	1.0050266
16	.6754806	.3359548	.7597361	−.0748820	1.0017381
17	.1273502	.4976716	.0296999	−.0796597	.9979820
18	.3700134	.7618972	−1.4061796	−.0844366	.9937568
19	.2942478	.5214124	−.2098076	−.0892107	.9890610
20	.2765526	.2877308	1.5584978	−.0939797	.9838931
21	.6671742	.2106636	.8723315	−.0987416	.9782521
22	.4942214	.7803612	−1.1657046	−.1034941	.9721372
23	.6681838	.3745004	.6369640	−.1082352	.9655475
24	.8142646	.7166268	−.6270039	−.1129625	.9584825
25	.1218782	.1706956	1.8022195	−.1176740	.9509419

[1] For the construction of the random series see Chapter 12, Section 2; for the functional expressions of Q_Y and R see the equations (13.10) and (13.14), respectively.

n	$u_1(n)$	$u_2(n)$	$x_1(n)$	$Q_Y(n)/x_o$	$R(n)$
26	.6353894	.1105052	.6093982	−.1223675	.9429256
27	.4184398	.0301484	.2485553	−.1270407	.9344338
28	.6454166	.5570428	−.3282650	−.1316913	.9254668
29	.5970622	.5825676	−.5035684	−.1363172	.9160251
30	.2687174	.6739292	−1.4394879	−.1409162	.9061096
31	.1834158	.6355564	−1.3857575	−.1454860	.8957209
32	.1428086	.4713788	.3528892	−.1500244	.8848606
33	.2703262	.4270796	.7154284	−.1545291	.8735298
34	.6934054	.3842332	.5689930	−.1589979	.8617303
35	.8095118	.8019244	−.6158205	−.1634286	.8494639
36	.6152406	.2100348	.9547266	−.1678190	.8367325
37	.0312702	.5810316	−1.2831537	−.1721668	.8235386
38	.9126534	.1870172	.3945056	−.1764696	.8098847
39	.3111278	.2844524	1.4924406	−.1807255	.7957732
40	.6275126	.1714108	.8500693	−.1849320	.7812074
41	.9214942	.3372236	.3451510	−.1890871	.7661905
42	.1136614	.8998812	−1.2270501	−.1931884	.7507256
43	.0306638	.2623404	2.6320198	−.1972337	.7348164
44	.7204246	.1619068	.6889139	−.2012211	.7184669
45	.3745982	.3244556	1.2507874	−.2051480	.7017345
46	.7476294	.8524252	−.6101264	−.2090125	.6844630
47	.9985198	.2187884	.0533864	−.2128123	.6668175
48	.5307766	.3559228	.8853319	−.2165453	.6487490
49	.2561822	.8275276	−1.4584093	−.2202095	.6302628
50	.2097574	.3262492	1.5683945	−.2238024	.6113638
51	.4930958	.4209964	.5663474	−.2273223	.5920575
52	.3113686	.4558588	.4182644	−.2307670	.5723494
53	.7038462	.5072396	−.0381095	−.2341342	.5522454
54	.1192454	.7949532	−1.9805970	−.2374222	.5317515
55	.8007918	.0001644	.0006885	−.2406288	.5108740
56	.8510006	.0521148	.1827011	−.2437519	.4896195
57	.7671902	.5203916	−.0930239	−.2467897	.4679945
58	.1992934	.9641372	−.4013022	−.2497401	.4460059
59	.1760078	.6314924	−1.4452538	−.2526012	.4236610
60	.7944726	.1830908	.6192787	−.2553711	.4009669

n	$u_1(n)$	$u_2(n)$	$x_1(n)$	$Q_Y(n)/x_o$	$R(n)$
61	.8478142	.0397836	.1421450	−.2580479	.3779312
62	.7571014	.6114012	−.4805603	−.2606297	.3545617
63	.0011438	.8141804	−3.3853679	−.2631149	.3309236
64	.3625846	.0951868	.8020293	−.2655014	.3068532
65	.9393182	.1742156	.3144772	−.2677875	.2825306
66	.7638694	.2263452	.7258813	−.2699717	.2579071
67	.1465998	.7514284	−1.9595377	−.2720519	.2329914
68	.4721366	.2028028	1.1716657	−.2740268	.2077926
69	.6673022	.2884876	.8732286	−.2758945	.1823197
70	.5347974	.4505692	.3419233	−.2776535	.1565818
71	.5307758	.8304364	−.9848241	−.2793023	.1305886
72	.2559286	.2483388	1.6508835	−.2808393	.1043499
73	.1293662	.7233996	−1.9942463	−.2822630	.0778753
74	.0090854	.3176732	2.7932558	−.2835719	.0511748
75	.8800718	.7024044	−.4830387	−.2847648	.0242588
76	.9827606	.6621948	−.1588238	−.2858402	−.0028622
77	.5351102	.9157516	−.5647015	−.2867967	−.0301782
78	.6299334	.2932572	.9261041	−.2876331	−.0576782
79	.6888878	.9625324	−.2013715	−.2883481	−.0853514
80	.3774326	.1227708	.9731757	−.2889407	−.1131866
81	.6461342	.9183436	−.4587510	−.2894094	−.1411729
82	.8245414	.1149212	.4105582	−.2897535	−.1692990
83	.3796238	.4300204	.5924445	−.2899716	−.1975532
84	.3407446	.3164668	1.3412791	−.2900630	−.2259242
85	.0160382	.3199756	2.6015556	−.2900265	−.2544001
86	.0841094	.4322652	.9186701	−.2898614	−.2829691
87	.6626798	.0280684	.1591564	−.2895665	−.3116192
88	.0694966	.8976828	−1.3844398	−.2891413	−.3403383
89	.0304222	.5654476	−1.0564577	−.2885851	−.3691142
90	.6438374	.2468892	.9382331	−.2878970	−.3979346
91	.0964558	.2683764	2.1545009	−.2870764	−.4267868
92	.5764886	.6488188	−.8445187	−.2861227	−.4556587
93	.7468862	.6755596	−.6819377	−.2850354	−.4845374
94	.7629254	.1523932	.6015928	−.2838140	−.5134100
95	.8473518	.3086444	.5369349	−.2824580	−.5422640

n	$u_1(n)$	$u_2(n)$	$x_1(n)$	$Q_Y(n)/x_o$	$R(n)$
96	.6105206	.8402748	−.8378531	−.2809672	−.5710863
97	.5350302	.3671116	.8290528	−.2793412	−.5998639
98	.6045734	.3743772	.7121589	−.2775796	−.6285838
99	.6497678	.6775724	−.8340856	−.2756825	−.6572328
100	.9763926	.7904508	−.2115664	−.2736495	−.6857976
101	.5164542	.5729036	−.5083620	−.2714806	−.7142653
102	.7159814	.6104412	−.5227975	−.2691758	−.7426222
103	.9661038	.5098604	−.0162599	−.2667352	−.7708552
104	.2549046	.6257468	−1.1746033	−.2641587	−.7989508
105	.8047582	.3617356	.5032416	−.2614466	−.8268954
106	.1083494	.6701852	−1.8486685	−.2585992	−.8546758
107	.3467598	.4487084	.4609675	−.2556167	−.8822784
108	.9228566	.2405628	.3999986	−.2524993	−.9096898
109	.5455422	.2584076	1.0993503	−.2492476	−.9368962
110	.9368774	.9152092	−.1834148	−.2458620	−.9638843
111	.9901358	.1213164	.0972340	−.2423432	−.9906405
112	.8730486	.4572988	.1381351	−.2386915	−1.0171513
113	.7564062	.9637196	−.1688646	−.2349077	−1.0434032
114	.7807654	.4991132	.0039199	−.2309925	−1.0693825
115	.5026318	.2188844	1.1505976	−.2269467	−1.0950761
116	.3342806	.3863548	.9695100	−.2227711	−1.1204703
117	.9669502	.4744716	.0414073	−.2184668	−1.1455516
118	.5232134	.4074972	.6249222	−.2140346	−1.1703070
119	.8586478	.1766124	.4944212	−.2094755	−1.1947227
120	.1913526	.9861308	−.1582765	−.2047908	−1.2187860
121	.6587742	.6034636	−.5529849	−.1999814	−1.2424830
122	.8314214	.2979612	.5802661	−.1950488	−1.2658008
123	.5605838	.4537004	.3085919	−.1899941	−1.2810977
124	.7050646	.8230268	−.7495486	−.1848187	−1.3112481
125	.5054782	.8994956	−.6895942	−.1795242	−1.3333509
126	.2365894	.1401052	1.3089680	−.1741117	−1.3550230
127	.9988398	.4133484	.0249569	−.1685831	−1.3762514
128	.6322166	.0314428	.1879610	−.1629399	−1.3970237
129	.4126622	.9673676	−.2708938	−.1571837	−1.4173272
130	.8139174	.6555292	−.5319428	−.1513164	−1.4371500

n	$u_1(n)$	$u_2(n)$	$x_1(n)$	$Q_Y(n)/x_o$	$R(n)$
131	.0118158	.8027564	−2.8171779	−.1453396	−1.4564793
132	.7456086	.4737788	.1256681	−.1392552	−1.4753034
133	.3579262	.1878796	1.3256648	−.1330653	−1.4936100
134	.4626054	.5578332	−.4413322	−.1267716	−1.5113877
135	.6459118	.8331244	−.8103280	−.1203763	−1.5286238
136	.7540406	.1004348	.4433272	−.1138815	−1.5453078
137	.0308702	.8378316	−2.2458848	−.1072893	−1.5614277
138	.7858534	.5926172	−.3815810	−.1006020	−1.5769728
139	.1155278	.8596524	−1.6037189	−.0938218	−1.5919316
140	.6223126	.5098108	−.0600006	−.0869510	−1.6062935
141	.2730942	.6100236	−1.0271840	−.0799921	−1.6200477
142	.5708614	.3774812	.7369748	−.0729475	−1.6331838
143	.9630638	.6615404	−.2330583	−.0658196	−1.6456914
144	.2912246	.7083068	−1.5171801	−.0586111	−1.6575605
145	.3181982	.5332556	−.3139155	−.0513246	−1.6687814
146	.8688294	.0420252	.1384049	−.0439626	−1.6793442
147	.4189198	.3219884	1.1864906	−.0365280	−1.6892394
148	.7975766	.0703228	.2876013	−.0290235	−1.6984581
149	.8317822	.2923276	.5855960	−.0214519	−1.7069912
150	.6749574	.6678492	−.7711650	−.0138161	−1.7148298
151	.9614958	.7081964	−.2706210	−.0061190	−1.7219655
152	.7941686	.4982588	.0074272	.0016363	−1.7283902
153	.7514462	.9480396	−.2424499	.0094470	−1.7340958
154	.2084454	.5285532	−.3160103	.0173100	−1.7390746
155	.0771918	.5513644	−.7178531	.0252222	−1.7433191
156	.4698006	.7825148	−1.2036219	.0331803	−1.7468220
157	.9267902	.0571916	.1371281	.0411813	−1.7495768
158	.7924934	.1297372	.4963946	.0492218	−1.7515765
159	.2204078	.1266924	1.2427530	.0572987	−1.7528150
160	.8692726	.1614908	.4495705	.0654085	−1.7532860
161	.5594142	.1925836	1.0084550	.0735479	−1.7529839
162	.3343014	.0490012	.4486076	.0817136	−1.7519021
163	.9735438	.5333804	−.0482130	.0899021	−1.7500387
164	.6133846	.0815868	.4849238	.0981100	−1.7473857
165	.4429182	.8630156	−.9677559	.1063337	−1.7439396

n	$u_1(n)$	$u_2(n)$	$x_1(n)$	$Q_Y(n)/x_o$	$R(n)$
166	.4050694	.5759452	−.6174449	.1145698	−1.7396961
167	.4069998	.5746284	−.6059431	.1228146	−1.7346516
168	.0189366	.1572028	2.3512551	.1310646	−1.7288022
169	.0029022	.8332876	−2.9607997	.1393161	−1.7221451
170	.9199974	.1521692	.3336217	.1475656	−1.7146768
171	.6391758	.2376364	.9432710	.1558094	−1.7063954
172	.6187286	.3307388	.8564802	.1640436	−1.6972982
173	.1369662	.8441996	−1.6548154	.1722648	−1.6873836
174	.4182854	.6112732	−.8496997	.1804690	−1.6766499
175	.5964718	.7736044	−1.0054256	.1886526	−1.6650962
176	.0815606	.2325948	2.2255571	.1968118	−1.6527214
177	.8547102	.7325516	−.5569801	.2049428	−1.6395255
178	.9431334	.2188572	.3356612	.2130418	−1.6255077
179	.9732878	.3777324	.1616970	.2211049	−1.6106688
180	.5322326	.7411708	−1.1213703	.2291284	−1.5950095
181	.7177342	.9511436	−.2461035	.2371084	−1.5785304
182	.5217414	.5125212	−.0896486	.2450409	−1.5612331
183	.3920238	.4692204	.2630179	.2529223	−1.5431194
184	.2715446	.7428668	−1.6130788	.2607486	−1.5241901
185	.0796382	.4887756	.1585186	.2685160	−1.5044513
186	.2453094	.9418652	−.5988315	.2762207	−1.4839026
187	.7630798	.5712684	−.3184028	.2838586	−1.4625473
188	.8962966	.0920828	.2558835	.2914260	−1.4403913
189	.1260222	.1902476	1.8935711	.2989190	−1.4174366
190	.9490374	.3084892	.3018445	.3063337	−1.3936882
191	.8448558	.7910764	−.5614384	.3136664	−1.3691509
192	.8192886	.7712188	−.6257746	.3209132	−1.3438299
193	.7144862	.4763596	.1213514	.3280701	−1.3177307
194	.4921254	.0059932	.0448311	.3351335	−1.2908594
195	.0037518	.8998444	−1.9672033	.3420994	−1.2632223
196	.1893206	.2506748	1.8244359	.3489642	−1.2348259
197	.0146302	.4639116	.6534775	.3557241	−1.2056774
198	.6377734	.0599772	.3490188	.3623752	−1.1757840
199	.1741678	.0127724	.1498783	.3689139	−1.1451542
200	.8765046	.5585468	−.1846452	.3753365	−1.1137957

n	$u_1(n)$	$u_2(n)$	$x_1(n)$	$Q_Y(n)/x_o$	$R(n)$
201	.2111926	.0488508	.5328300	.3816392	−1.0817174
202	.9480542	.4857036	.0293007	.3878185	−1.0489282
203	.5331814	.9680412	−.2236924	.3938706	−1.0154374
204	.0185038	.8690604	−2.0705787	.3997920	−.9812549
205	.8657046	.4921468	.0264889	.4055792	−.9463910
206	.4283582	.0105356	.0861354	.4112285	−.9108557
207	.7895494	.3397852	.5809287	.4167365	−.8746604
208	.2871598	.7119084	−1.5346653	.4220997	−.8378161
209	.0296566	.6749628	−2.3631776	.4273147	−.8003344
210	.4011422	.9632076	−.3096823	.4323782	−.7622276
211	.1620774	.3368092	1.6309022	.4372868	−.7235076
212	.3785358	.7685164	−1.3844528	.4420371	−.6841876
213	.9958486	.6196988	−.0623146	.4466259	−.6442802
214	.6840062	.4445196	.2976963	.4510503	−.6037993
215	.8299654	.9127132	−.3183015	.4553067	−.5627583
216	.0990318	.3300844	1.8839432	.4593922	−.5211716
217	.3930806	.6367548	−1.0349719	.4633038	−.4790536
218	.6065502	.8512716	−.8042689	.4670384	−.4364189
219	.2764134	.8530972	−1.2787976	.4705932	−.3932831
220	.6230478	.4318124	.4041316	.4739653	−.3496610
221	.5061526	.8845308	−.7743125	.4771518	−.3055693
222	.4503742	.3962636	.7761981	.4801499	−.2610233
223	.7686214	.6155612	−.4816808	.4829571	−.2160399
224	.6529838	.1329004	.6844315	.4855707	−.1706358
225	.9958646	.1294268	.0661390	.4879882	−.1248281
226	.6890782	.0282956	.1526259	.4902069	−.0786340
227	.4377894	.9697052	−.2431820	.4922246	−.0320714
228	.7792398	.3965484	.4274518	.4940388	.0148416
229	.0190166	.7058428	−2.7074605	.4956474	.0620873
230	.0282622	.7521676	−2.6704187	.4970480	.1096469
231	.9591174	.4371292	.1111923	.4982385	.1575019
232	.0402158	.5699564	−1.0787863	.4992171	.2056326
233	.7484086	.6761788	−.6808847	.4999816	.2531074
234	.2455262	.3486796	1.3639723	.5005301	.3026449
235	.8318054	.5314332	−.1190830	.5008610	.3514867

n	$u_1(n)$	$u_2(n)$	$x_1(n)$	$Q_Y(n)/x_o$	$R(n)$
236	.6823118	.4643244	.1943601	.5009725	.4005260
237	.2928406	.1908348	1.4601925	.5008528	.4497422
238	.8304702	.4946316	.0205559	.5005305	.4991147
239	.2590534	.7982172	−1.5687536	.4999742	.5486232
240	.1199278	.0348524	.4474116	.4991923	.5982464
241	.0171126	.0482108	.8508720	.4981838	.6479629
242	.4246942	.2828236	1.2809949	.4969472	.6977519
243	.6280614	.6550812	−.7979846	.4954817	.7475916
244	.0954638	.6607404	−1.8354558	.4937861	.7974606
245	.2620246	.4547068	.4595062	.4918596	.8473365
246	.0617982	.1420556	1.8373850	.4897013	.8971974
247	.5900294	.0316252	.2027737	.4873105	.9470213
248	.0393198	.0251884	.4009469	.4846866	.9967858
249	.4643766	.9847228	−.1187095	.4818291	1.0464685
250	.2073822	.1571276	1.4802762	.4787375	1.0960468
251	.7401574	.8094492	−.7222561	.4754116	1.1454976
252	.6298958	.5953964	−.5423985	.4718510	1.1947986
253	.6769686	.7406588	−.8818026	.4680557	1.2439266
254	.5990462	.7888396	−.9823453	.4640257	1.2928587
255	.8976454	.0621532	.1769028	.4597611	1.3415719
256	.5535918	.7025644	−1.0395538	.4552621	1.3900428
257	.4886006	.7129148	−1.1644928	.4505288	1.4382483
258	.8863902	.9939916	−.0185361	.4455619	1.4861650
259	.9856934	.0953372	.0957188	.4403617	1.5337699
260	.4648078	.2218924	1.2185920	.4349288	1.5810393
261	.3440726	.3398908	1.2338928	.4292642	1.6279502
262	.0710142	.7453836	−2.2989782	.4233684	1.6744789
263	.5115014	.2866012	1.1274481	.4172426	1.7206023
264	.1459438	.8525804	−1.5683091	.4108877	1.7662969
265	.2641846	.2679868	1.6212204	.4043048	1.8115393
266	.7465182	.9518156	−.2279749	.3974953	1.8563063
267	.6462694	.7255452	−.9233776	.3904606	1.9005749
268	.8673998	.9978284	−.0072776	.3832021	1.9443215
269	.9657366	.3116028	.2445264	.3757214	1.9875229
270	.1385022	.7780876	−1.9575166	.3680203	2.0301561

n	$u_1(n)$	$u_2(n)$	$x_1(n)$	$Q_Y(n)/x_o$	$R(n)$
271	.9051974	.6537692	−.3671941	.3601005	2.0721985
272	.9475758	.2448364	.3279989	.3519640	2.1136268
273	.3815286	.6131388	−.9058025	.3436129	2.1544181
274	.9445662	.3649996	.2533327	.3350492	2.1945502
275	.4274854	.7048732	−1.2516581	.3262752	2.2340004
276	.5128718	.4448044	.3927883	.3172933	2.2727462
277	.5803606	.0029948	.0196280	.3081060	2.3107652
278	,9743102	.9493516	−.0713846	.2987158	2.3480356
279	.8563334	.9444572	−.1904454	.2891255	2.3845352
280	.4576878	.3929324	.7790584	.2793379	2.4202427
281	.0870326	.5595708	−.8079153	.2693557	2.4551361
282	.5893342	.3839436	.6851714	.2591822	2.4891944
283	.8189414	.7101212	−.6123111	.2488203	2.5223960
284	.6044238	.1084204	.6319328	.2382733	2.5547207
285	.6023446	.3692668	.7371672	.2275447	2.5861473
286	.9432382	.0575756	.1209931	.2166377	2.6166555
287	.0065094	.2514652	3.1730366	.2055559	2.6462253
288	.0634798	.7144684	−2.2899286	.1943031	2.6748370
289	.1230966	.4864828	.1736317	.1828828	2.7024706
290	.0216222	.2150476	2.7026187	.1712991	2.7291067
291	.8542374	.1700892	.4920493	.1595558	2.7547269
292	.7932558	.9182764	−.3343220	.1476569	2.7793120
293	.4620886	.0936188	.6894867	.1356067	2.8028438
294	.4820862	.6771596	−1.0836782	.1234093	2.8253043
295	.8213254	.6595932	−.5288978	.1110691	2.8466756
296	.3601518	.0910444	.7736766	.0985906	2.8669406
297	.1681206	.8610748	−1.4468938	.0859782	2.8860823
298	.2942302	.9607116	−.3822277	.0732366	2.9040840
299	.2709734	.5455772	−.4564757	.0603705	2.9209295
300	.8985678	.9479724	−.1485123	.0473847	2.9366031
301	.8459926	.5072508	−.0263394	.0342841	2.9510896
302	.1796542	.7985036	−1.7675705	.0210736	2.9643733
303	.9503814	.1256412	.2264993	.0077584	2.9764404
304	.2709038	.8282604	−1.4246798	−.0082690	2.9862545

TABLE 16.
CALCULATION OF THE SUMS OVER n OF THE PRODUCTS $x_1(n)Q_Y(n)/x_o$ AND $x_1(n)R(n)$.

N	$\sum_{n=0}^{N} x_1(n)Q_Y(n)/x_o$	N	$\sum_{n=0}^{N} x_1(n)R(n)$
10	.0939377	10	−6.7214615
20	.2532940	20	−9.3297653
30	.2830836	30	−9.1443151
40	−.1044807	40	−7.5678072
50	−.9524426	50	−4.6857731
60	−.4131410	60	−5.6820003
70	−.0507954	70	−6.3972300
80	−.6131496	80	−6.5065263
90	−1.7920221	90	−7.4149483
100	−2.2029405	100	−7.9924522
110	−1.7108395	110	−6.7500003
120	−2.4196728	120	−10.3377010
130	−2.3582360	130	−9.8003370
140	−1.6046544	140	.1159521
150	−1.5155529	150	1.5962746
160	−1.4489933	160	2.2974635
170	−1.5980787	170	3.2557601
180	−1.7645042	180	4.3010455
190	−1.7715406	190	4.5247972
200	−1.7850229	200	4.9845704
210	−3.9894607	210	9.4378826
220	−4.3266594	220	9.4712318
230	−6.6992265	230	9.0280998
240	−6.6217173	240	9.1742093
250	−4.7925534	250	12.686858
260	−6.6131011	260	8.0079055
270	−7.6577445	270	2.5933738
280	−8.0373681	290	.5358483
290	−7.1832264	290	12.6671250
300	−7.3637752	300	5.6213579
304	−7.3883893	304	−3.2764056

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