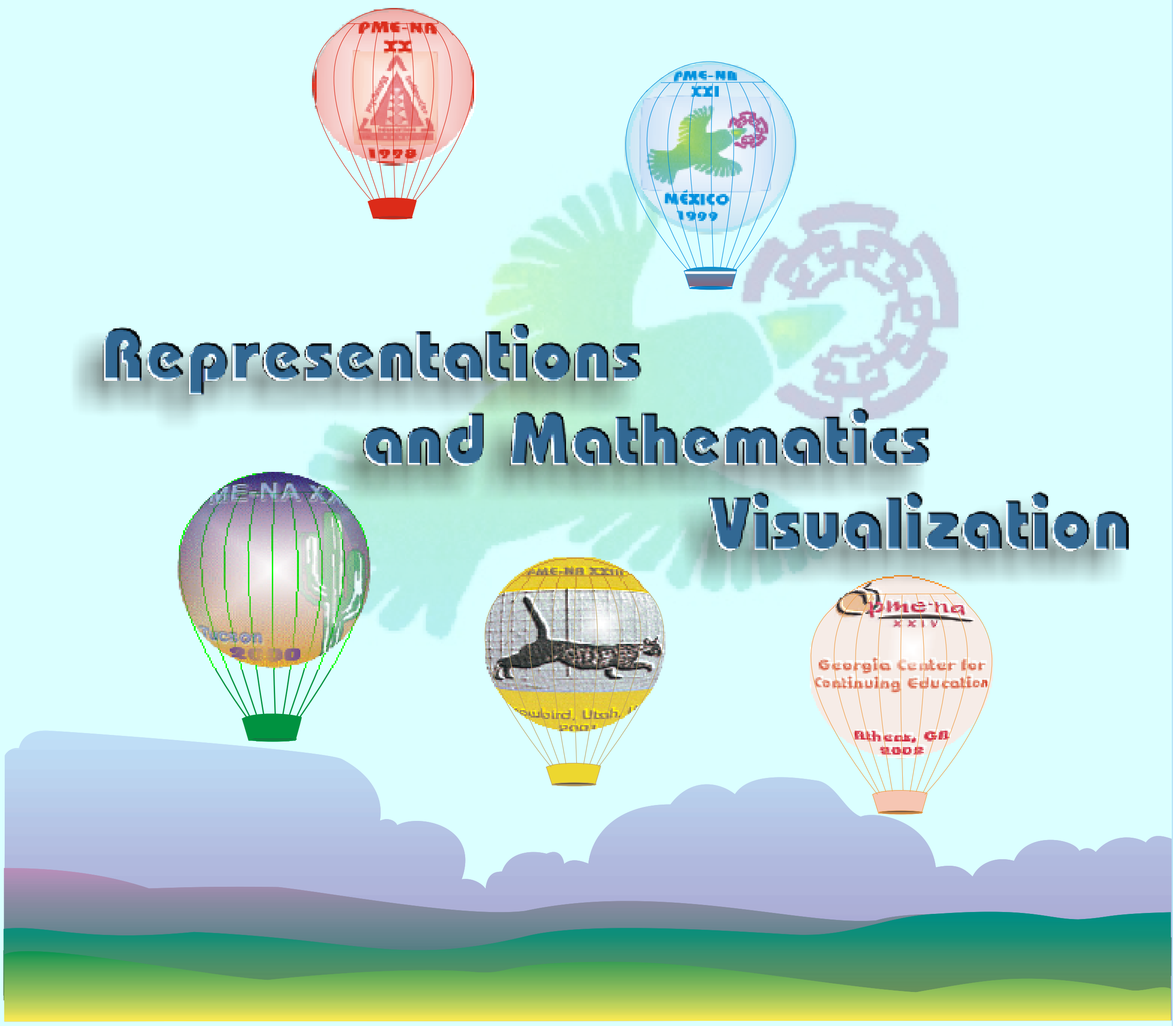
Representations and Mathematics Visualization



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Fernando Hitt

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North American Chapter of the International Group for the Psychology of Mathematics Education

Cinvestav-IPN

Working Group Representations and Mathematics Visualization (1998-2002)

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Preface

There is no understanding without visualization
(Duval, this volume, p. 322)

Duval's statement, standing alone as I have quoted it, may seem extreme. But this statement is part of a cogent argument, starting on p. 320 of this volume, for the distinction between vision or perception and visualization in mathematical cognition and learning, and for the power of coordination of different registers of representation in these activities. Duval's previous sentence sheds light: "In a string of discrete units (words, symbols, propositions) not any organization can be displayed. Thus, inasmuch as text or reasoning, understanding involves grasping their whole structure, there is no understanding without visualization" (pp. 321-322).

This volume is an uneven collection of papers, as individual as their individual authors, but reflecting a collective apprehension, well expressed in Duval's chapter, that various forms of representation are an integral part of the doing of mathematics and thus also of its teaching and learning. For this reason, representation and mathematical visualization need to be taken seriously in mathematics education, and we still have much to learn about how these forms of metonymy operate in the individual cognition and affect of learners. How do different forms of representation and their articulation facilitate or impede the grasping of that whole structure that is so important in Duval's chapter? How can teachers facilitate the moving between different forms of representation, the coordination of registers, that frees learners from some of the limitations of individual forms of representation? How can technology serve this purpose, or at least contribute to it? Many of these questions, or some aspects of them, are addressed in individual chapters. The result is a collection of papers that, taken together, are powerful in focusing attention on key aspects of the roles of representation and visualization in mathematical cognition. Pedagogical and didactical issues have received less attention than cognitive ones in the volume as a whole, and the affective issues have hardly been addressed at all. However, there are chapters that provide strong theoretical analyses of various aspects of representation, using different theoretical frameworks, and the abstract nature of some of these chapters is offset by others that give enticing and fascinating examples of forms of representation from actual mathematical problems in various content areas. Thus the book as a whole makes a valuable contribution to understanding of the roles of representation and visualization in mathematics, representing some powerful thinking and discussion in the Working Group in which most of these papers were presented in a four-year period.

The structure of the book is such that chapters with similar themes or foci are grouped together in three different parts, preceded by an introduction and one chapter by the organizer of the Working Group, Fernando Hitt, who has set the scene with a series of questions that are addressed in the volume, and with an excellent identification of important foci in this field. The four chapters in Part I of the book are replete with mathematical problems to whet the appetite of the reader for what is to come. I enjoyed doing the problems presented by Alfinio Flores in his chapter on the use of geometric representations in the transition from arithmetic to algebra. Although he does not provide full research justification for his claims (pp. 12-14), his advocacy for the use of geometrical representations in algebraic and numerical problems is reinforced by the power of his examples. These are illustrations of how useful geometric representations can be. They depend on the principle that one concrete case may be represented in an illustration in such a way that this case may metonymically stand for all such cases. Generality is implied in the particularity of the drawings. As he states, “In the examples shared here, geometric representations of particular numbers were used to make general statements tangible” (p. 26). A word of pedagogical caution is in order. In the first place, not all learners may comprehend this principle of the general in the particular. Secondly, even for those who do comprehend, this principle may be extrapolated by learners to confirm what many erroneously do in any case – try to prove a general principle by particular examples! This does not mean that there is no value in introducing these “proofs without words”, but merely that mathematics educators need to be aware of both the strengths and the possible pitfalls, and to help learners to understand these too. It has been my experience that the visualizers amongst mathematics learners find these illustrations particularly valuable – in fact invaluable. And for understanding of Duval's whole structure, *all* students may benefit. The remaining chapters in Part I are based on research studies that corroborate some aspects of Flores' advocacy of visual representations, and point to some of the intricacies. The three chapters combined describe research that spans the learning continuum, from elementary and secondary school (Stylianou & Pitta-Pantazi) to college-level learners, in rate-of-change problems (Presmeg & Balderas) and in covariation problems (Carlson).

Part II of the book groups together six papers that address in various ways the use of technology in mathematical representation. Kaput gives an interesting historical account of the shift from knowledge produced in inert, static ways, through "deep changes in the nature of the representational infrastructure" (p. 87), to dynamic, interactive computer systems that make new forms of knowledge possible. His excitement at the potential of the new forms is palpable. This is a visionary chapter, looking into the future, into what he calls "opportunity spaces" (p. 104) and outlining a new world, with need and opportunities for new research agendas in mathematics education. His chapter sets the scene for the other authors in this technology section of the book. Manuel Santos-Trigo in each of his chapters describes technology-based research studies and in the process gives the reader some interesting mathematical tasks to contemplate and work on. In both chapters, I enjoyed verifying the mathematical results without the use of the technology. There is some thrilling creative thinking in these chapters, for instance in the section (III), *Find and explore different conjectures* (p. 166 ff.). Both chapters may serve to convince the reader that computers and calculators are powerful tools in the teaching and learning of mathematics – not only in geometry, although that is striking here. More than that, they illustrate the role of the instructor in valuing students' ideas, as "starting point to construct robust solutions or explanations" (p. 124).

Ana Isabel Sacristán Rock describes some solid research on students' understandings of infinite processes in a computer microworld. She combines empirical and theoretical aspects to build a model of the processes involved, thus building on the theoretical ideas of others and making her own original contribution based on her empirical study. I had just one question. On the first page of her chapter, she wrote that "… some representations are of visual form (e.g. the graph of a function); others are purely symbolic or algebraic, lacking a graphical aspect."

Do representations form the strict visual-nonvisual dichotomy suggested here? I maintain that a visual-symbolic dichotomy such as that used in this chapter overlooks the visual aspects present in much algebraic symbolism. However, this small point does not detract from the value of the research reported. In the following chapter, too, Kay McClain describes a research study that is well grounded in theory. With her usual thoroughness, McClain gives the details of a teaching experiment and a teacher development experiment investigating the understanding of statistical data analysis. Arcavi and Hadas in their chapter are concerned with situations that surprise learners, and they illustrate one such situation dramatically in a computer environment. The power of the software is well illustrated here: it would have been difficult to create the surprising situation as effectively without the technology. A further strength of this chapter is the reflection of the authors on pedagogy that has the potential to create surprises and capitalize on these for the learning of mathematics. These six chapters taken together reinforce each other and create a persuasive picture of theoretical and practical benefits of the use of technology in mathematics education.

Part III is the meaty theoretical heart of the book. However, I was glad of the empirical and illustrative examples given in the first two parts, because without these some parts of the book might have appeared too abstractly theoretical. Many of the authors in the third section presuppose a reader's familiarity with some of the literature in this field of representation in mathematical cognition. I can imagine a fruitful dialogue between these authors – all of whom I am sure can provide their own examples, whether or not they have done so in these chapters – and the authors of the preceding chapters. The nine chapters in this section present various theoretical views of representation and visualization and associated issues. There is no unified voice, but it is the very diversity of theoretical lenses that may be the value of this section of the book.

Patrick Thompson, with his usual thoughtful attention to complexity, has a useful section on *intersubjectivity*, which he defines as "the state where each participant in a socially-ongoing interaction feels assured that others involved in the interaction think pretty much as does he or she" (p. 201). The importance of intersubjectivity in mathematical representation is argued in two significant points. Firstly, representations serve both an individual and a social function. "Representations, as personal constructs, are creative ways to remind ourselves systematically of ideas we had, connections we made, and operations we applied previously in the presence of operating now. Representations as social conventions are expressions of intersubjectivity (as I've defined it)" (p. 201). Secondly, following from the logic of the first point, it can be misleading to try to determine individuals' meanings through large-scale social interactions. These important clarifications are illustrated in an example drawn from his research. Such issues were hotly contested and debated through the 1990s, and the pendulum seems to have swung sufficiently for the need to be acknowledged for studies both of social processes in mathematics classrooms, and the meanings and mathematical cognition of individuals. There are clear implications that both kinds of studies are needed in research on mathematical representations.

With his intimate knowledge of the theoretical writings of Russian psychologists, Mourat Tchoshanov explains Vygotskian semiotic conceptions, and puts forward a post-Vygotskian model of representation. In the former, I wished that the notion of "sign" had been defined and problematized. I had difficulty understanding the significance of distinguishing between a *sign* and a *name*. Is a name not a sign? It seemed to me that a name should fall into this class of signifiers, as I understood the word "sign" to be used here (pp. 208-209). Going beyond the exclusive use of the *zone of proximal development* (ZPD), Tchoshanov describes a nested model of representation in which the ZPD is a middle category between the *level of actual development* (LAD), and the *zone of advanced/perspective development* (ZAD). While the ZPD involves inter-psychological development and leads to internalization, the ZAD is the zone of in-depth learning that is inter-psychological, resulting in generalization and externalization. The ZPD is a "guided zone" that leads to understanding; the ZAD with its externalization is both guided and private, and leads to creativity. This model resonates well with the need for both social and individual processes to be taken into account in research on mathematical representation, as expressed in Thompson's chapter. It seems to me that this theoretical framework has the potential to be a fruitful one in mathematics education research.

In his scholarly and deep chapter on the *object* of representations, Luis Radford explores the vital question "What is it which representations represent?" Kant, Plato and Aristotle, Piaget, Durkheim and Lévi-Strauss, and Wartofsky, are all invoked in this endeavor. "What is it which makes human cognition distinctive?" is answered by Wartofsky as "the ability to make representations" (p. 233). In this chapter the reader will find ontology and epistemology, politics and aesthetics, in a profound consideration of what it means to represent a mathematical idea.

The following four chapters, by Hitt, Presmeg, Saenz-Ludlow, and Maury respectively, are also theoretical formulations, but Hitt and Presmeg do give empirical examples to illustrate their approaches. In Hitt's framework, which resonates with that of Duval, I appreciated the insight that ownership of representations is important as students learn mathematics: "It was very important for Soath to produce her own semiotic representation to work and solve the given task" (p. 252). By articulating two registers, graphic and analytic, as she solved a problem involving numbers of taxis and motorcycles (often presented with two-legged and four-legged animals), Soath was in fact illustrating Duval's principle that articulation of two register is needed in order to construct a mathematical concept. Following Hitt's chapter, I articulated the need for, and presented, a triadic nested model that has been useful in empirical research on ways of facilitating use of outside-school activities in the teaching and learning of classroom mathematics.

I felt the need of examples and illustrations in Saenz-Ludlow's chapter, which nevertheless introduces some important theoretical ideas based largely on the writings of Peirce. Even with its social origins and contexts, human thought is individual (Thompson's chapter) and often specific to the individual. This specificity and individuality then seems to require that these characteristics be built into a model that purports to explain human cognition. Thus this description of processes in Peirce's formulation seems to need more specificity to bring it to life, and to illustrate its applicability in mathematics education. Saenz-Ludlow has done empirical work that has important implications for mathematics education, but neither her empirical research not its didactical consequences are treated in this chapter.

Maury bases her empirical and theoretical chapter on the works of Bertin, a specialist in graphic semiology. (I wondered if there is a substantial difference between *semiology* and *semiotics*, or if this is just a question of the terminology used. This issue is not addressed in this chapter.) Maury's interesting chapter stops very abruptly. However, the ideas lead very aptly into the amazingly dense and authoritative chapter by Duval. Even the abstract summarizing Duval's chapter is quite startling! I cannot do justice to Duval's work here: in fact I cannot even introduce it adequately. Duval's writing is full of quotable ideas. I shall content myself by giving just one such quotation, from the abstract, to give the reader the flavor of the introduction. After writing about "deceitful" studies, Duval adds, "In fact they use models for visualization, representation or, even the use of signs which aren't relevant, because these basic cognitive processes work quite differently in mathematics than in all the other fields of knowledge." This theme is developed throughout his chapter.

Finally, the collection of papers closes with a major paper by Michael Otte, which is introspective in the sense that he works through his original ideas about proofs involving the collinearity of the orthocenter, centroid, and circumcenter of a triangle (in the Euler line), generalizing the ideas and insights to prove several other theorems. This mathematical journey leads him to "a logic of abduction and generalization, which is firmly connected with our cognitive means and representational systems" (p. 342). I enjoyed working through the mathematical ideas with Otte – but these are only illustrative and are not the major goal of the chapter. This goal is finally stated (p. 356):

> To indicate the indispensable role of this complementarity of function or transformation on the one hand and of relation or objectively given law on the other in the development of mathematical cognition is the most important goal of this paper.

The pursuit of this goal results in a deep chapter, full of fascinating insights and replete with quotable sentences, such as one on the penultimate page of text that resonates with the insights of other authors in this book, "A mathematical concept does not exist independently of the totality of its possible representations, but must not be confused with any such representation, either" (p. 366).

This book may be the culmination of four years' of writing, presentation, and discussion of papers in the PME-NA Working Group on Representations and Mathematics Visualization (1998-2002), but in a sense it is an invitation to continue this work, informed by what has been accomplished. This accomplishment consists not only in the empirical studies described, but even more importantly in articulating the theoretical models that have been presented. As in Duval's coordination of registers, the need now is for the coordination of theoretical and empirical studies of representation and visualization for the benefit of mathematics education at all levels.

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Introduction

History of the Working Group: Representations and Mathematics Visualization (1998-2002)

The Working Group on “Representations and Mathematics Visualization” was formally constituted at the PME-NA XX conference in 1998 at the North Carolina State University. Our academic agenda focused on the need to examine the role that representations and mathematics visualization play in students' learning of mathematics.

An initial literature review was important to identify fundamental issues that were part of the working agenda. This functioned as a framework to prepare and organize each meeting held during the past 5 years. In particular, a set of research questions helped to focus the development of the sessions. In what follows, we present briefly those questions that were at the heart of our work in the different sessions.

> *Why do representations and mathematics visualization play an important role in the learning of mathematics? What types of theoretical frameworks are used to support that importance in students’ learning?*

The themes Representations and Mathematics Visualization have appeared in the recent literature as fundamental aspects related to the students’ construction of mathematical concepts and problem solving processes. To identify relevant aspects that involve representation and visualization it is important to recognize that there are different lines of research in which both theoretical and empirical approaches have been developed. The importance of discussing and contrasting present theoretical orientations is based on a well known fact that there are different theoretical approaches to investigate the learning phenomena. As a consequence, it may occur that the different perspectives are complementary but other times these perspectives are irreconcilable. It could also happen that one theoretical orientation offfers a better explanation of a phenomenon than other orientation. Hence, it is important to ask: (i) To what extent should one adopt or follow a theoretical orientation? (ii) Which orientation will help us better to explain a learning phenomenon?

In this trend of thought, we have encouraged the working group participants to discuss issues related to those questions.

What is the relevance in doing empirical research linked to representations and mathematics visualization?

Mathematics instructors, at all levels, traditionally have focused their instruction on the use of algebraic representations as a means to distinguish mathematical objects from their representations. Indeed, they often do not take into account geometric and intuitive representations. This is because they might think that the algebraic system of representation is formal and the others lack this property. Perhaps, some students' difficulties in the construction of concepts are linked to the restriction of representations in their learning experiences. Nevertheless, it is known by empirical research that the students' construction of a mathematical object is based on the use of several semiotic representations. The students' handling of different mathematics representations will permit ways of constructing mental images of a mathematical concept. The richness (or lack of) of their concept image will depend on the students' handling of the representations used. However, the tendency to remain in an arithmetic or algebraic system of representation is well documented in related bibliography review. This tendency might explain the "naïve" students' behavior in problem solving activities.

There is a wide range of theories explaining how mathematical concepts are learned and taught, but we know much of this knowledge is context bounded, and there is little explanations on how this knowledge could be used in processes that involve proofs or recognition of patterns in a given situation different from those where knowledge was originaly constructed. One hypothesis is that there must exist some internal mathematical structures (systems of schemata), partially independent of the other structures that promote students' transfer of knowledge when facing mathematical problems. In this context, it becomes important to discuss questions like: How can one recognize and encourage students' construction of mathematical abstraction and generalization and students' recognition of common mathematical structures across different contexts?

Why is it important to promote articulation among the different representations of a concept?

Some researchers claim that understanding is related to the use of different representations of a mathematical concept. That is, students construct a mathematical concept through the use and manipulation of the different representations of that concept.

Can students' learning difficulties be explained in terms of their abilities to operate basic transformation within the same semiotic representational system or their ability to examine the concept across various systems of representation? That is to say, difficulties faced by the students could be explained as a lack of coordination between representations? A plausible answer seems to be that not all the difficulties could be explained in terms of a lack of articulation between representations. There may be epistemological obstacles that are explained in other ways, for example through the evolution of mathematical ideas.

Why do we need theoretical approaches on representations and mathematics visualization?

Some authors think that it is important to understand the role of the semiotic representation in the construction of mathematical objects. The need of theoretical approaches is related to our understanding how students construct mathematical concepts. Why cannot students solve nonroutine problems? Why is visual thinking fundamental in problem solving?

a) What is the role of mathematical intuition in this context?

b) To what extent has mathematics visualization to deal with certain abilities related to the conversion of representations from one semiotic system to another?

c) Is the comprehension of a mathematical concept related to the use of different semiotic representations of the concept in question?

d) Should conceptual knowledge be taken as an invariant of multiple semiotic representations? Would some changes of semiotic representations of mathematical objects contribute to the development of mathematical thinking, communication and understanding?

e) Does cognitive knowledge, mediated semiotically, bring us a culturalized vision?

f) What is the nature of the interaction between external and internal representations?

g) How do individuals construct internal representations? Is a social approach important to the constructions of mathematical concepts? What are situations that encourage students to invent their own symbol systems? What are those examples of student-generated symbol systems? How can students link the symbols they construct to those created by other students? How can students link the symbol systems they construct to standard mathematical symbolic systems? How can students develop symbolic fluency with the classical or conventional symbol systems in mathematics? What are the circumstances that allow students to listen effectively to another student's arguments?

h) How do we infer internal representations?

The above questions were relevant to propose a framework to approach:

- Theoretical aspects of the learning of mathematics which take into account the role of the semiotic representations in the construction of mathematical concepts.
- Theoretical aspects of the learning of mathematics related to generalization and abstraction. What is the nature of mathematical abstraction and generalization, particularly in classroom settings?
- Theoretical aspects related to semiotic representations dealing with a social epistemology of mathematical knowledge with applications to didactical situations in the classroom.
- An analysis of the mathematical ideas related to a concept through the history of mathematics to detect epistemological obstacles.

System of mathematical representations, register of mathematical representations. Their use and construction in mathematical activities.

In recent literature not only the role of a system of mathematical representation has been studied, related to the construction of concepts, but the idea of register of mathematical representation appeared as fundamental to understand students' constructions of mathematical concepts (see Duval in this volume), and with their use we explain some of the partial constructions of a mathematical concept or a lack of it. How does social interaction play a role in the construction of mathematical concepts? How could individual reflection about a mathematical situation generate a construction of a mathematical concept? In what ways do multiple representations of mathematical ideas influence students in their thinking and construction of mental images?

What type of curriculum implications emerge from our theoretical approaches?

By taking into account different systems of representations, we can identify specific variables related to cognitive contents, and in this way, organize didactical proposals to promote the students articulation of different representations.

What is the role and importance of a technology-based multiple linked representation in the students' construction of mathematical concepts?

How can we develop new external systems of representations that foster more effective learning and problem solving? Can technology help in this trend? How does the nature of representational tools used in the classroom (technology included) influence the ways in which students construct mathematical meaning? How do different tools lead students to think differently about their mathematics?

We kept in mind all these questions during the development of the meetings.

At the first session of our working group there were five presentations followed by corresponding discussions. Fernando Hitt gave a general introduction about the learning of mathematics and the role of the representations and mathematics visualization. Manuel Santos' presentation focused on the use of technology in mathematical problem solving, James Kaput addressed issues on multiple linked representations and co-ordinated descriptions, Luis Radford challenged traditional conceptions of representations and proposed something related to rethinking representations. And finally, Norma Presmeg spoke on visualization and generalization in mathematics.

For the PME-NA XXI conference in Mexico, we invited the international community to focus on the importance and the role of representations in the learning of mathematics. On that occasion, we invited some scholars to give short presentations in our working group and to share with us their extensive work in this area. Abraham Arcavi, presented a study he did with Nurit Hadas on "Computer mediated learning: An example of an approach," Raymond Duval focused on one part of his plenary talk: "Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning," Adalira Saenz-

Ludlow introduced us to the theoretical approach that relies on the work of Peirce: "Interpretation, representation, and signification: A Peircian perspective," and finally, Pat Thompson spoke on "Some remarks on conventions and representations".

It was in Mexico, in PME-NA XXI conference, were we begun to think about a special issue to publish the product that emerged from our Working Group. At the same time we began to ask all the participants to contribute with short presentations in the following meetings. From this call, we had some responses, in PME-NA XXII, Tucson, Arizona (2000), we had the opportunity to discuss special issues presented by Manuel Santos on the potential of the use of technology in students' development of conceptual systems, Kay McClain on "Computer-based tools for data analysis: Support for statistical understanding," Norma Presmeg and Patricia Balderas on "Graduate students' visualization in two rate of change problems," Alfinio Flores on "Geometric representations in the transition from arithmetic to algebra," Anita H. Bowman on "Using concept representations for the construction of frameworks for research, curriculum design, instruction, and student reflection", Ana-Isabel Sacristán on "Coordinating representations through programming activities: An example using Logo", and finally Fernando Hitt on "Construction of mathematical concepts and cognitive frames."

The members of our working group were very enthusiastic and we arrived at PME-NA XXIII conference in Salt Lake City (2001) with different approaches to enrich the discussion of the working group. This time we had the presentations of Marilyn P. Carlson who focused on "Physical enactment: A powerful representational tool for understanding the nature of covarying relationships", Jean-Marie Laborde and Barbara J. Pence on "Representation with DGS," Norma Presmeg on "A triadic nested lens for viewing teachers' representations of semiotic chaining," and finally Mourat A. Tchoshanov on "Representation and cognition: Internalizing mathematical concepts."

Something important in all this process was the electronic version we put in the web to promote discussion. We mantained permanent communication with some researchers from different countries that were interested in following our work. In this process, we not only had the opportunity to share our work every year but also, we could spread the discussion outside of the formal meeting programmed once a year. As a consequence, we decided to invite the community to share its research ideas related to our work. As a result we expanded our vision and ways to investigate themes related to our working group agenda; in this trend, Sylvette Maury has written about “A look at some studies on learning and processing graphic information, based on Bertin’s theory”, Michael Otte has written about “Proof-analysis and the development of geometrical thought”, also Despina A Stylianou and Demetra Pitta-Pantazi have written about “Visualization and high achievement in mathematics: A critical look at successful visualization strategies.”

Finally, we would like to thanks to all the members of the working group that made the sessions more interesting with their reflections and discussion of their ideas. Also, we would like to acknowledge the support received by the Steering Committee of the North American Chapter of the International Group for the Psychology of Mathematics Education and the Centro de Investigación y Estudios Avanzados del IPN, México.

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Working Group on Representations and Mathematics Visualization (PME-NA XX North Carolina, 1998)

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ABSTRACT. Recent theoretical developments in mathematics learning have pointed to the importance of the role of representations in the construction of mathematical concepts. The first goal of the working group Representations and Mathematics Visualization is to establish, through open discussions, a broad background regarding mathematical representation; and, the second goal, is to promote experimentation and reflection on this matter to provide us with in-depth explanations about mathematics learning and the role of representations in this process of learning.

Introduction

The themes Representations and Mathematics Visualization have appeared in the recent literature as fundamental aspects to understand students' construction of mathematical concepts and problem solving (diSessa, 1994; Dubinsky, 1994; Duval, 1995; Eisenberg & Dreyfus, 1990; Dreyfus, 1991; von Glasersfeld, 1991; Janvier, 1987; Kaput, 1994; Presmeg, 1986; Steinbring, 1991; Vergnaud, 1987; Vinner, 1989; Zimmermann & Cunningham, 1990;…). It is interesting to observe that there are different lines of research in which both thoretical and empirical approaches have been developed.

An important objective of our working group on Representations and Mathematics Visualization is to promote an open discussion of the relevant theoretical orientations endorsed by different authors, and their influence in empirical research that intend to improve our understanding of the learning of mathematics. Particularly, there is interest in discussing how these research results can be interpreted and finally applied into classroom settings.

The importance of discussing and contrasting present theoretical orientations is based on a well known fact that often authors follow different theoretical approaches to tackle the learning phenomena. Here, it may occur that the different perspectives are complementary but other times these perspectives are irreconcilable. It could also be that one theoretical orientation permits a better explanation of a phenomenon than another orientation. Hence, it is important to ask: (i) To what extent one should adopt or follow a theoretical orientation? (ii) Which orientation will help us to better explain a learning phenomenon? The well organized discussion of these types of questions during the working group sessions might be useful for us to study in depth what happens in the students' process of learning.

One aspect that the group should include is the discussion of research related to problem solving in terms of searching what kind of mathematical representations students use when solving a problem and the role of mathematics visualization in this problem solving process.

We consider important in general that the working group should focus on the discussion on four relevant aspects:

- The relevance of doing empirical research linked to representations and mathematics visualization.
- The importance of the theory in pursuing research on representations and mathematics visualization.
- The application of research's results that links the students' learning of mathematics and the use of multiple representations within a theoretical frame.
- The influence of technology-based multiple linked representation in the students' construction of mathematical concepts.

The relevance of doing empirical research linked to representations and mathematics visualization

Mathematics instructors, at all levels, traditionally have focused their instruction on the use of algebraic representations with the intention to avoid confusion between mathematical objects and their representations, they normally do not take into account geometric and intuitive representations. This is because they think that the algebraic system of representation is formal and the others they are not. Perhaps, some students difficulties in the construction of concepts are linked to the restriction of representations when teaching. Nevertheless, it is known by empirical research that the students' construction of a mathematical object is based on the use of several semiotic representations. The students' handling of different mathematics representations will permit ways of constructing mental images (a concept image in Vinner and Tall's sense) of a mathematical concept. The richness (or lack of) of their concept image will depend on the students' handling of the representations used. However, the tendency to remain in an arithmetic or algebraic system of representations is well documented in the current bibliography. This tendency will produce errors in problem solving situations. In this context, Santos (1996) shows us an example in which *a tennis ball problem* was given to some students (How many tennis balls do you need to fill your classroom?), Santos says:

> For the tennis balls problem, the students [35, grade 10] experienced difficulty in estimating the dimensions of the classroom and the tennis balls. All the students asked the interviewer to provide the dimensions and when they were asked to estimate, some of the students asked for a meter stick. The most common approach was to divide the volume of the classroom by the volume of a tennis ball… (p. 275)

Why are these children not worried about the answer? Why did they remain in the arithmetic system of representation?

Also related to this part, Goldenberg (1995) quotes:

> We cannot expect to understand that understanding if we look only at the student's facility with one representation, or even the quality of a student's handling of each representation in isolation. But, as we observe students juggling the interaction among representations, we get a glimpse of the rich internal models they construct in their attempt to understand the bigger picture. (p. 155)

Are the students' errors a product of a deficient handling of a representation when transforming it in the same semiotic system of representation or when converting it to another system? That is to say, could the difficulties faced by the students be explained as a lack of coordination between representations? A plausible answer seems to be that not all the difficulties could be explained in terms of a lack of articulation between representations. There exist epistemological obstacles that are explained by other ways, for example through the evolution of mathematical ideas (see Hitt related to functions, 1994); however, the identification of students' errors when handling different representations could give us a glimpse of the concept image they have.

The importance of the theory in pursuing research on representations and mathematics visualization

There are several questions that need to be addressed here. For example, how can we explain the lack of success of first year university students in Selden et al. studies (1989, 1994) when solving calculus nonroutine problems? In this context, Zimmermann (1991), states that:

> Conceptually, the role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject. This is especially true if the course is intended to stress conceptual understanding, which is widely recognized to be lacking in many calculus courses as now taught. Symbol manipulation has been overemphasized and … in the process the spirit of calculus has been lost, (quoted by Aspinwall et al., 1997, p. 301).

Why do some authors consider so important the study of the semiotic system of representations on the construction of mathematical concepts? For example, Duval (1993) quotes:

> …we are then in front of what we could call a cognitive paradox of the mathematical thinking: On one hand, the **apprehension** of the mathematical objects can only be possible as a conceptual apprehension, and on the other hand, only mediated by semiotic representation is an activity on the mathematical objects possible. (p. 38).

Other authors emphasize the relations of symbols and ideas: Radford & Grenier (1996) assert:

> …the relation between symbols and ideas can not only be considered as an interaction which consists in putting in contact an object (or an idea), changeless and external to the individual, with the representations of that object, … Far away from this, the interaction between the symbols and the ideas must be, we think, seen as a system of relations constructed by the individual himself in his intellectual path, at the same time socially and individually.

These authors address their discussion to provide us, under their theoretical approach, on explanations of the deep processes of understanding and mathematics learning. For example, from Duval's point of view, as we never deal with mathematical objects, but only with their representations, representational activity is fundamentally semiotic in nature. Duval focusses on the role of the semiotic representations of a mathematical object in the construction of concepts.

The application of research's results that links the students' learning of mathematics and the use of multiple representations within a theoretical frame

By taking into account different systems of representations, we can identify specific variables related to cognitive contents and in this way, organize didactical proposals to promote the students articulation of different representations.

In this context, Eisenberg & Dreyfus (1990, p. 25) state that: *although the benefits of visualizing mathematical concepts are often advocated, many students are reluctant to accept them; they prefer algorithmic over visual thinking…Indeed, with respect to problem nine* [Given: f a differentiable function such that $f(-x) = -f(x)$. Then, for any given a: A) $f'(-a) = -f'(-a)$ B) $f'(-a) = f'(a)$ C) $f'(-a) = -f'(a)$ D) none of the above] *one typical calculus teacher (who also happens to have authored a calculus textbook) wrote:* $f'(-a) = (f(-a))' = (-f(a))' = -f'(a)$. It might be that, this teacher is totally convinced of his/her algebraic process and considers irrelevant a mental construction of one example of a function and the derivatives [for example, the graph of $f(x) = x^3$] that he/she may realize that the proposed solution is not correct. Similarly, as the case related to the tennis ball problem, mentioned before, Why did it happen that individuals get attached to the algebraic system of representation?

Dreyfus (1991, p. 42) quoted: *Mathematics educators seem to have recognized the potential power and the promise of visual reasoning; but in spite of this, implementation is lagging: Students tend to avoid visual reasoning.* It seems that teachers continue emphasizing their instruction on nonvisuals methods.

The influence of technology-based multiple linked representation in the students' construction of mathematical concepts

Related to this point, Kaput (1994) states that:

> More subtle examples of notation modification include such strategies as enabling students to act on traditional mathematical notations in more natural ways, as when in a computer environment, for example, one uses a pointing device and graphical interface to act directly on coordinate graphs by sliding,

bending, reflecting, and so forth (as with Function Probe,…). This is a subtle exploitation of the rich knowledge based in kinesthetic experience to act on mathematical notations, and hence to effect mental operations on mathematical objects, that is, functions. Another example is the direct manipulation of algebraic objects used in Theorist… Yet another example, reflecting the dynamic, interactive properties of the computer medium, is offered by CABRI-Géomètre… (p. 387).

Althought the authors of the papers in this book do not claim to have final answers to the questions raised in this introduction, many of the issues they address have bearing on these questions.

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PART I

Geometric Representations in the Transition from Arithmetic to Algebra

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ABSTRACT. This chapter describes the use of geometrical representations of interesting numerical relationships as one way to help students make the transition between arithmetic and algebra and develop their algebraic thinking. First he importance of visualizing and geometric representations for algebra is discussed, as well as how geometrical representations of numeric relations can be a means to explore algebraic ideas. Then 12 examples of interesting numerical relationships that can be illustrated with two- and three-dimensional geometric representations and that teachers can use in their classrooms in the middle grades are presented.

Prelude

Some striking numerical results (the reader may want to use a calculator).

1) Take the square of a whole number, add the number, and add the next number. The result is the square of the next number.

$$5^2 + 5 + 6 = 6^2$$
$$10^2 + 10 + 11 = 11^2$$

2) Take any odd number, square it, and subtract one. The result is divisible by 8.

$$7^2 - 1 = 8 \times 6$$
$$9^2 - 1 = 8 \times 10$$

3) Consider the following triangle of odd numbers. Add the numbers in each row. The result in each case is a cubic number.

1	1	1^3
3 + 5	8	2^3
7 + 9 + 11	27	3^3
13 + 15 + 17 + 19	64	4^3

4) Take a number, raise it to the third power, and subtract the number. The result is divisible by six.

$3^3 - 3 = 24 = 6 \times 4$
$4^3 - 4 = 60 = 6 \times 10$
$5^3 - 5 = 120 = 6 \times 20$

5) Multiply four consecutive numbers. The result differs by one from a perfect square.

$1 \times 2 \times 3 \times 4 + 1 = 5^2$
$2 \times 3 \times 4 \times 5 + 1 = 11^2$

Introduction

The continuities and discontinuities between arithmetic and algebra that students face are complex and need to be addressed from multiple perspectives. There are several aspects that teachers need to take into account in order to help students make the transition from arithmetic to algebra, and to develop algebraic skills with understanding (see for example Kieran & Chalouh, 1993; Wagner & Parker, 1993, Lodholz, 1990). This chapter focuses on one possible way to help students develop their algebraic thinking, using geometrical representations of interesting numerical relationships. The numerical examples provided are of the kind that students find striking or amazing (Mulligan, 1988). Using geometric representations for these numerical relations can help students go from statements about particular numbers to the corresponding generalized statements using variables. These representations provide a way to shift students' attention, from the purely procedural approach to numbers, to considering the terms and operations involved in a numerical relationship as entities that are worthwhile to pay attention to. Geometrical representations can provide a context where students

- learn to extract pertinent relation from problem situations and express those relations using algebraic symbols;

- make explicit the procedures they use in solving arithmetic problems;
- consider strings of numbers and operations as mathematical objects, rather than processes to arrive at an answer;
- gain explicit awareness of the mathematical method that is being symbolized by the use of both numbers and letters;
- focus on method or process instead of on the answer;
- pose and compose problems;
- write conjectures, predictions, and conclusions.

Geometric representations of relationships among numbers go back to antiquity. For example, Euclid, in book II includes the following statement and diagram (Figure1), "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments" (Euclid, 1956, p. 379).

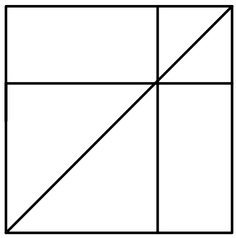

Figure 1. Squares on a segment

Some students also spontaneously use geometric representations to convince themselves of the truth of a statement. Krutetskii (1976, p. 325) gives the example of a sixth grade student who drew a fiigure to convince herself and better understand the formula for the square of a sum of numbers $(a + b)^2 = a^2 + b^2 + 2ab$.

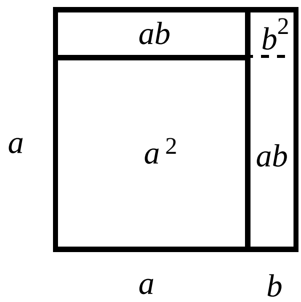

Figure 2. $(a + b)^2 = a^2 + b^2 + 2ab$.

The importance of the use of geometrical representations for algebra has been recently reiterated (Algebra Working Group, 1998; Bass, 1998). One of the components of the geometry standard (National Council of Teachers of Mathematics, 2000) emphasizes that instruction should enable students to use visualization, spatial reasoning, and geometric modeling to solve problems. For the grade band 6-8, it recommends that students "use geometric models to represent and explain numerical and algebraic relationships." (p. 232). Arcavi (1999) recommends to "bring geometry to the aid of what seem to be purely symbolic / algebraic properties." (p. 60)

These recommendations are supported by research on how humans learn. According to Fischbein (1987), visualization "is the main factor contributing to the production of the effect of immediacy" (p. 103). In mathematics as in many other fields, "the concreteness of visual images is an essential factor for creating the feeling of self-evidence and immediacy" (Fischbein, 1987, p. 104). Presmeg (1999) found that "visual methods were effective when they were used in ways that supported generalization" (p. 152). Botsmanova (1972a) explains why the use of pictorial visual aids helps to bridge the gap between the concrete situation reflected in the problem's subject and its abstract side—the mathematical structure. According to her, "any pictorial representation combines features of the abstract and the concrete. A pictorial representation is the abstracted and generalized expression of a rule, and at the same time it translates the solution of a problem from the abstract-verbal level into a concrete plan." (p. 105). Furthermore, Botsmanova (1972b) states that the

> interlacing of the abstract and the concrete in a graphic expression is clearly what permits the use of a graphic diagram in problem solving both in the transition from abstract to concrete and the reverse transition, from the concrete or visual to the abstract. The ability to operate with a diagram can permit a pupil to discover mathematical relationships more easily, and can become, under certain conditions, a generalized method of analysis and synthesis. (p. 119)

Geometrical representations can help focus the attention of the students on the relations in general and help them use letters to stand for given quantities; use algebra as a tool for proving rules governing numerical relations; use the equal sign to express a symmetric and transitive relation; use letters to represent givens in the generalization of number patterns; and use letters to specify a range of values in representing numerical relationships.

Geometrical representations of numeric relations can be a means to explore algebraic ideas. With them students can think about the relations, discuss them explicitly using ordinary language, and learn to represent them with letters. These representations can serve as a guide for students as they learn to use algebra as a tool for generalization and justification. Geometrical representations can serve as a scaffolding as students develop proficiency with algebra as a language for actions on quantities and relationships among quantities. They can provide a semi-concrete step towards the more abstract concepts that arise with the use of letters as variables. They can help students avoid over generalizing and help students not to judge algebraic expressions based on superficial characteristics. They can help students to develop a more complete mental construct of variable by considering a more ample range of cases of the concept of variable, by using them to represent given quantities and not just as abbreviations or unknowns.

Geometrical representations can help students see that multiplying and factoring are inverse operations by using area or volume models of multiplication. They can also serve to illustrate cases when more than one letter in a parenthesis are dealt with together, as a single entity, a necessary ability for algebra which is sometimes hampered by experiencing only procedural ways to deal with quantities in parenthesis, such as “do what is in parenthesis first” or “clear the parenthesis first.” As Warner and Parker (1993) point out, the use of pictorial representations together with concrete numerical cases should help students apprehend the equivalence of algebraic identities in symbolic form. Visual models and numerical examples can reinforce structural relationships in algebra.

Kieran (1990) states that “the power of the symbolic language is that it removes many of the distinctions that the vernacular preserves, thus vastly expanding its applicability. The cost is that the symbolic language is semantically extremely weak” (p. 97). Geometrical representations can provide in some cases a semantic support for dealing with the symbols.

The purpose of the examples that follow is to give some ideas for teachers how to help students make the jump from the visual and concrete to the symbolic and abstract. In algebra and beyond, students need the ability to deal with symbols for variables rather than just symbols for numbers. In addition, algebraic notation will help them reason about statements that apply to all numbers or to numbers in a specified set, rather than about statements that apply to particular numbers.

Teachers in the middle grades or in the first year of high school can help students develop their ability to deal with symbols, as students use concrete representations of relations among ordinary numbers to serve as a bridge to symbolic notation. As students write equations representing several cases and compare them, they will see the pattern and will be able to express the equations using variables.

An important step in understanding is to describe what the figures and the equations represent using plain language. Verbalizing the steps and using drawings or other concrete representations can also help students make the connection with the symbols. It is important to see how parts and terms of the equations are represented by parts of the figures. The drawings and particular numbers used will help students attach meanings to the different terms in the algebraic formula used to describe the general situation.

Although the figure itself only illustrates a particular case, the figure contains the elements that will facilitate general reasoning.

Of course, students can benefit from the use geometric representations years before the transition to algebra. For example, an area model can make explicit the use of the distributive property in a common procedure to multiply two-digit numbers (Figure 3).

$$\begin{array}{r} 23 \\ \times\ \underline{12} \\ 46 \\ \underline{230} \\ 276 \end{array} \qquad\qquad \begin{array}{r} 20 + 3 \\ \times\ \underline{10 + 2} \\ 40 + 6 \\ \underline{200 + 30\phantom{{}+6}} \\ 200 + 70 + 6 \end{array}$$

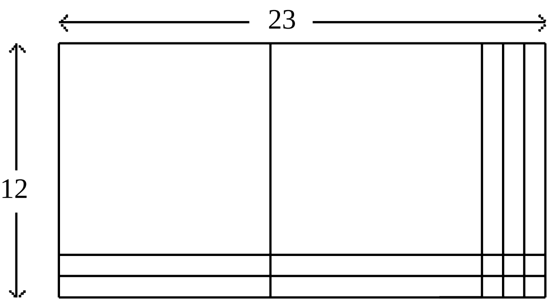

Figure 3. Area model for 12×23.

In the following sections we will provide some examples that can be used in the classroom in the middle grades. For more examples of geometric representations of interesting numerical or algebraic relations, the reader can consult Nelsen (1993), as well as previous published work by the author (Flores 1992; Flores Peñafiel 1999, 2000a, 2000b).

Examples for the classroom

Example 1. Consecutive squares

Students can see why starting with the square of a number, and then adding the number, and the next number does always give the square of the next number by using a drawing (Figure 4). They can realize that the drawing, although made for a particular number, permits reasoning that is general and does not use special properties of the particular numbers (4 and 5).

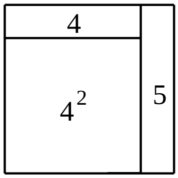

Figure 4. $4^2 + 4 + 5 = 5^2$.

Using this as their starting point they may later use a more general diagram (Figure 5). Of course, once students have developed skills in algebraic manipulation the equation can be easily verified. The point, however is that the geometric representation can give meaning to the different terms of the equation, $n^2 + n + (n+1) = (n+1)^2$ and students can use it to make sure the symbol manipulation is correct.

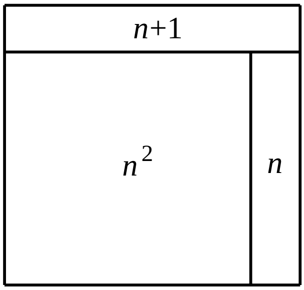

Figure 5. $n^2 + n + (n + 1) = (n + 1)^2$.

Example 2. Sums of consecutive numbers.

The sum of consecutive numbers 1 + 2 + 3 + 4 can be represented by columns of squares of increasing heights (Figure 6). Adding another number would mean adding another column to the staircase. If n numbers are added, then the highest column would have n squares. The sums of consecutive numbers are called triangular numbers. Thus 10 = 1 + 2 + 3 + 4 is the fourth triangular number.

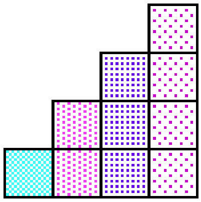

Figure 6. 1 + 2 + 3 + 4.

Students can make another exact copy of their stair, and use the two copies to form a rectangle (Figure 7). In this case it is a 4 by 5 rectangle, but students can realize that in general it will be a rectangle in which the lengths of the sides are two consecutive numbers. The shorter side will indicate how many numbers are being added. If they add the first n numbers, it will be an $n \times (n+1)$ rectangle.

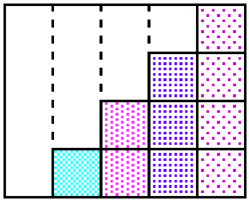

Figure 7. A 4 by 5 rectangle.

Because the rectangle has two copies of the triangular number, we need to divide its area by 2. Therefore, students can see that $1 + 2 + ... + n = \frac{n\times(n+1)}{2}$. The expression $\frac{n\times(n+1)}{2}$ can be algebraically transformed into $\frac{n^2}{2}+\frac{n}{2}$. Students canattach meaning to the terms of this expression using a big triangle and n half squares (Figure 8).

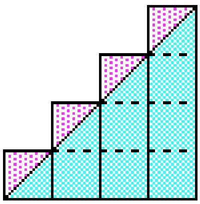

Figure 8. $\frac{n^2}{2}+\frac{n}{2}$.

Example 3. The square of an odd number minus one.

Ask students to take any odd number, square it, subtract one, and then divide the result by 8. They will be surprised to see that for all the odd numbers they chose, the division does not leave a remainder. Students can try different odd numbers in a systematic way, and organize data as in Table 1. This will provide quite some empirical evidence that the result is true for any odd numbers. However, empirical evidence is not enough in mathematics; students need to understand why it works. Some students may recognize the triangular numbers in the last column of Table 1. Based on this observation they may predict what the next result will be.

Table 1. Squares of odd numbers, minus one

Odd number	squared	minus one	divide by eight
1	1	0	0
3	9	8	1
5	25	24	3
7	49	48	6
9	81	80	10
11	121	120	15

A geometric representation can help students understand why when we subtract one from the square of an odd number the result is always divisible by eight Students are given a square (with a side of odd length) that is missing a unit square (Figure 9a). In general, because odd numbers are of the form $2n+1$, we can describe the square minus one unit as $(2n+1)^2 - 1$.

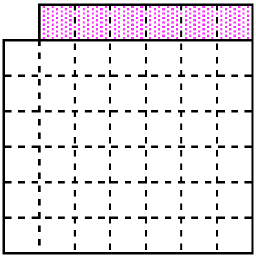

Figure 9a. $7^2 - 1$.

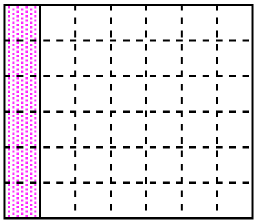

Figure 9b. 6×8.

By rearranging the colored strip, students can form a rectangle with the same area. Because the original number is odd, both sides of the rectangle will be even numbers, one side two units longer that the other (Figure 9b). In general, we can describe this rectangle as $2n \times (2n+2)$. Therefore, each side can be divided into two equal parts and giving rise to four rectangles (Figure 10a); the length of each side of the smaller rectangle will be a whole number. We can express the area of the big rectangle as $4 \times n \times (n + 1)$. The sides of the small rectangles differ by one unit. That is, they are the kind of rectangles we obtained by putting two triangular numbers together. Students can see that eight triangular numbers are hidden in the rectangle, (Figure 10b).

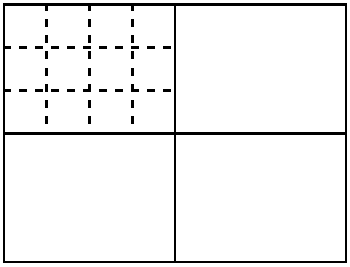

Figure 10a. $4 \times (3 \times 4)$.

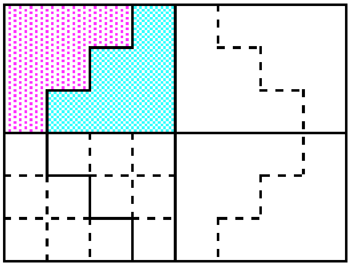

Figure 10b. $8\frac{3\times 4}{2}$.

Therefore we can conclude that $(2n+1)^2 - 1 = 8\frac{n\times(n+1)}{2}$. Another way to see that $\frac{n\times(n+1)}{2}$ is a whole number is because either n or $n + 1$ is an even number.

Example 4. The cube of a number minus the number.

We can represent raising a number n to the third power by forming an actual cube with n^3 unit cubes. Subtracting the original number from this cube can be done by deleting one column of n unit cubes (Figure 11a). One of the incomplete slices can be rearranged to form a rectangular brick (Figure 11b). Its dimensions are $(n - 1)$ by n by $(n + 1)$. Therefore $n^3 - n = (n - 1) \cdot n \cdot (n + 1)$. We see that the numbers on the right side are three consecutive numbers. Therefore one of them has to be divisible by 3, and at least one has to be divisible by 2. The product is therefore divisible by 6.

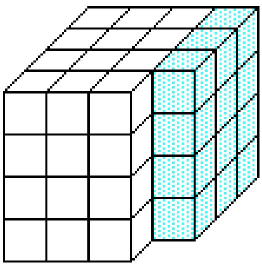

Figure 11a. $n^3 - n$.

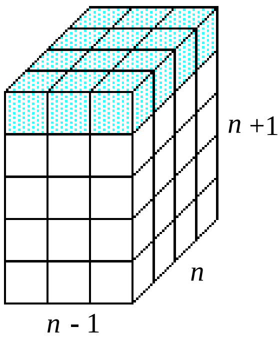

Figure 11b. $(n - 1) \cdot n \cdot (n + 1)$.

Example 5. Sums of odd numbers and sums of cubes.

Students can look at the triangle of odd numbers in table 2. They can compute the sum on each line, and verify that the sum is equal to a cubic number. We can represent the sums in each row in table 2 with the rectangles in Figure 12. Students can identify which rectangle corresponds to what row and explain why. They can predict what the next row will be. They can draw the next rectangle. They can relate the numbers in each row with different parts of the corresponding rectangle.

Students can express verbally the suggested relationship. Each rectangle represents, on one hand, the sum of consecutive odd numbers (each odd number is a thin straight or L shaped strip within each rectangle). On the other hand, each rectangle is the cube of a number, because it has n squares of area n^2. Each rectangle has a growing number of strips, 1 the first, 2 the second, 3 the third, and so on. Thus, the total number of strips up to a given rectangle is the corresponding triangular number ($1 + 2 + ... + n$). For example, the first two rectangles have 1 + 2 strips, therefore the third rectangle will start with the fourth odd number $7 = 2 \times 4 - 1$, which can also be written as $2 \times 3 + 1$, and it includes three consecutive odd numbers. In general, the n-th rectangle is the sum of n odd numbers, starting with $2(1+ 2 + ...+ (n - 1)) + 1$ and finishing with $2(1+ 2 + ...+ n) - 1$.

Table 2. Sums of odd numbers

1	1	1^3
3 + 5	8	2^3
7 + 9 + 11	27	3^3
13 + 15 + 17 + 19	64	4^3

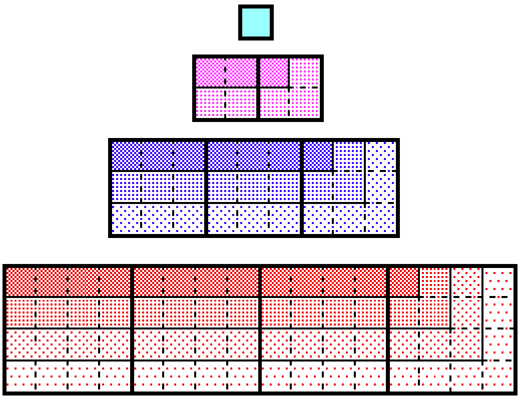

Figure 12. Sums of odd numbers and sums of cubes.

Example 6. Products of three consecutive numbers.

Students may notice that if we have three consecutive numbers (for instance 3, 4, 5), the product of the first by the third ($3 \times 5 = 15$) differs by one from the square of the middle number ($4^2 = 16$). Students can represent this by a rectangle and a square, and overlapping them as in Figure 13.

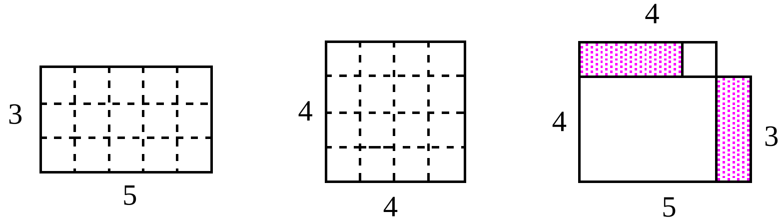

Figure 13. $3 \times 5 + 1 = 4^2$.

Students can express the number relations in general terms. The product of two numbers that differ by two, plus one, is equal to the square of their average. Then they can express the relation by using algebraic symbols. If m, $m + 1$, and $m + 2$ denote the three consecutive numbers, then $m(m+2)+1=(m+1)^2$. Figure 14 can help both to see why this general relation is true, and as a guide to give meaning to the intermediate steps when the equation is verified algebraically. For example, as students expand $m(m + 2)$, they can see that the terms $m^2 + 2m$ correspond to a big square and two strips.

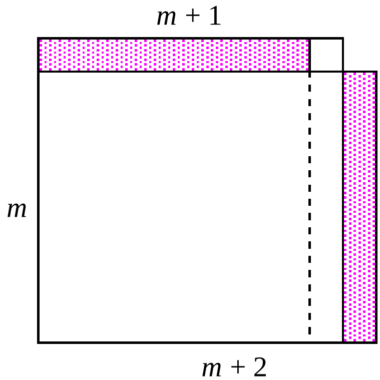

Figure 14.

Example 7. Products of four consecutive numbers.

In a way similar to the example before, if we have four consecutive numbers (for instance 3, 4, 5, 6), and pair the first and the fourth numbers on one hand (3, 6), and the second and the third on the other (4, 5), and form two rectangles whose sides are the pairs of numbers, we will see that their areas differ by two units (Figure 15). Students can try other examples.

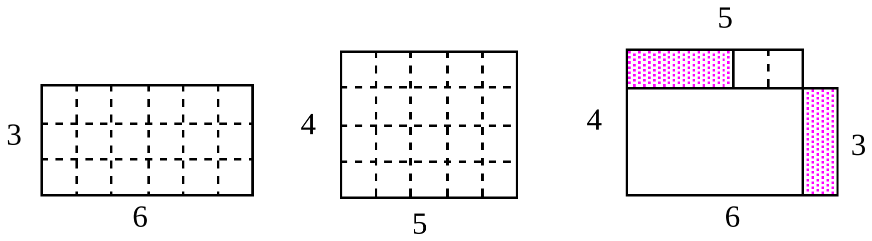

Figure 15. $3 \times 6 + 2 = 4 \times 5$.

Students can express the relations among the products of consecutive numbers in general. If we have four consecutive whole numbers, the product of the two middle terms is two more than the product of the first and the fourth. Then they can express the relation in algebraic terms by $n(n+3)+2=(n+1)(n+2)$. They can see why this result is true in general from Figure 16. The parts of Figure 16 can also provide meaning to the terms in the intermediate steps when the equality is proved algebraically. For instance they can identify $n^2 + 3n$ as one square and three strips.

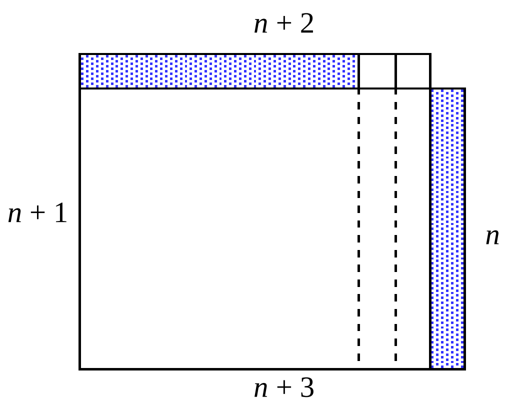

Figure 16. $n(n+3)+2=(n+1)(n+2)$.

Example 8. The product of four consecutive numbers

Let us consider the product of four consecutive numbers. For instance

$2 \times 3 \times 4 \times 5 = 120$. The product differs by one from a square number, $121 = 11^2$. Students can relate this to the previous two examples. As we saw in Example 7, the product of the first and fourth numbers differ by two from the product of the second and the third. That is, if we let $n(n+3) = m$, then $(n+1)(n+2) = m+2$. From Example 6, $m(m + 2) + 1 = (m + 1)^2$, or using the products $n(n + 3)$ and $(n + 1)(n + 2)$, and rearranging the terms, we have $n(n+1)(n+2)(n+3)+1 = [n(n+3)+1]^2$. So students can see why the product of four consecutive numbers plus one is a perfect square. Of course, with today's technology, students could simple ask a calculator with algebraic capabilities to factor $(x \cdot (x+1) \cdot (x+2) \cdot (x+3)+1)$ and see that it is indeed equal to $(x^2 + 3x + 1)^2$, but the gained insight of why it is true would be lost.

Example 9. An inequality

Inequalities can also be represented geometrically. In Figure 17 the length of each of the shaded rectangles is a, and their width is b. The two squares represent a^2 and b^2. Clearly, the two squares cover more area than the two rectangles. Therefore $a^2 + b^2 = 2ab$. The side of the remaining white square is $a - b$, therefore $(a - b)^2 = a^2 + b^2 - 2ab$.

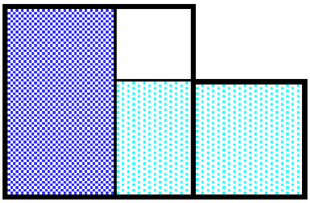

Figure 17. $a^2 + b^2 = 2ab$.

Example 10. Sums of squares $1^2 + 2^2 + 3^2 + ... + n^2$

We can represent square numbers by slices that have a square base and one unit height. We can also represent the sum of square numbers by stacking such slices into an echelon building (Figure 18). Students can make copies of this building with interlocking cubes (make sure the links do not protrude). If students work in teams, they can easily construct six copies. Each one represents the same sum of square numbers (Figure 19).

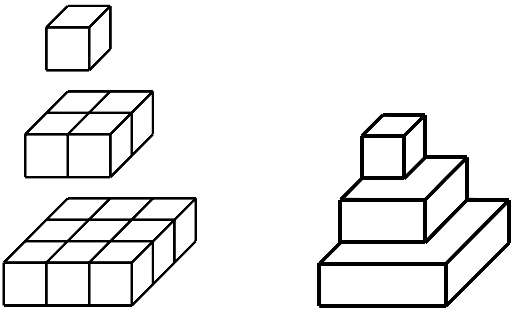

Figure 18. Sum of square numbers 1 + 4 + 9.

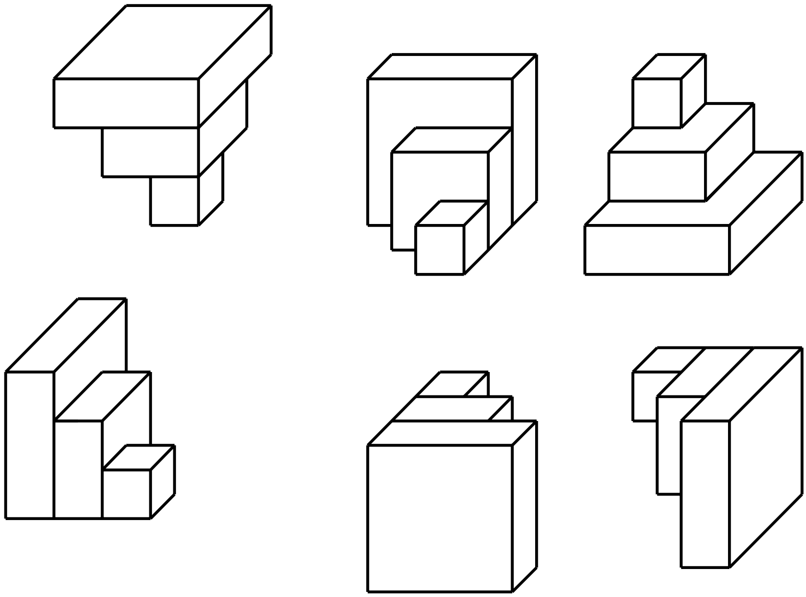

Figure 19. Six sums of square numbers.

Students can now arrange the six copies into a block (Figure 20). In this case it will have dimensions 3 by 4 by 7. In general, if we add six echelon buildings representing each the sum $1^2 + 2^2 + ... + n^2$, we will form a rectangular block of dimensions n by $(n + 1)$ by $(2n + 1)$. Therefore, the formula to add the first n square numbers is given by $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$.

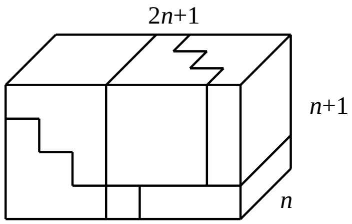

Figure 20. $6(1^2 + 2^2 + ... + n^2) = n (n + 1) (2n + 1)$

Example 11. Sum of triangular numbers

Triangular numbers can also be represented with stairs of cubes. We can use interlinking cubes to represent sums of triangular numbers as shown in Figure 21. (Again, make sure that the links do not protrude.)

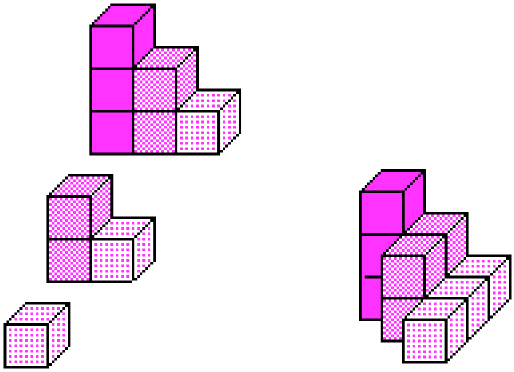

Figure 21. Sum of triangular numbers

Students can make two more copies of the building, one of which is a mirror image of the original (see central part of Figure 22). If T_n denotes the n-th triangular number, the sum of the first n triangular numbers is $T_1 + T_2 + ... + T_n$. By joining the three buildings to form a stair we can see that three times the sum of triangular numbers is equal to $(n + 2) \cdot T_n$ (left part of Figure 22). By arranging the three building to form a wider stair, we can see that the total is given by $n \cdot T_{n+1}$ (right part of Figure 22). Because $T_n = \frac{n \times (n+1)}{2}$, we have in each case that $3(T_1 + T_2 + ... + T_n) = \frac{n(n+1)(n+2)}{2}$. Therefore, a formula for the sum of the first n triangular numbers is given by $T_1 + T_2 + ... + T_n = \frac{n(n+1)(n+2)}{2}$.

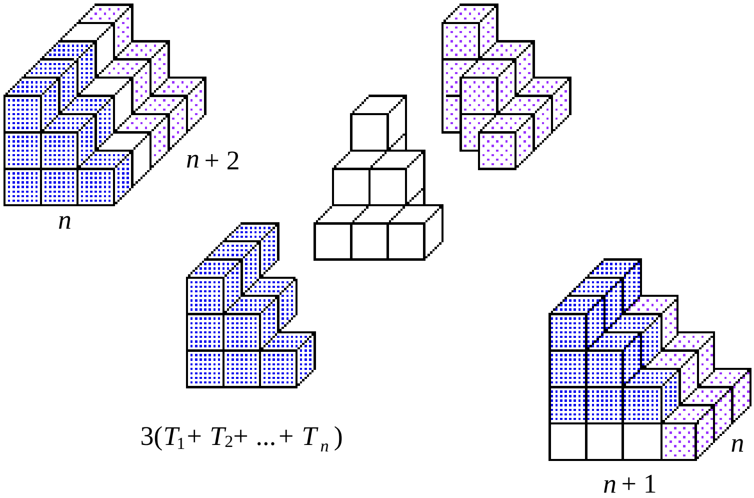

Figure 22. $3(T_1+T_2+\ldots+T_n) = (n+2)T_n = n\,T_{n+1}$

Example 12. Sums of triangular numbers and sums of special products.

By joining one building that represents the sum of triangular numbers with its mirror image, we can describe the sum of triangular numbers as the sum of special products. Notice that the number of cubes in each level of the joint building is given by 1×2, 2×3, 3×4, and in general by $n \times (n + 1)$. We can now use three copies of these double buildings (six sums of triangular numbers), to form a rectangular block. In this case the dimensions of the block will be 3 by 4 by 5, and in general n by $(n + 1)$ by $(n + 2)$. Thus we obtain the same formula for the sum of triangular numbers, or a formula for the sum of special products $1\times 2\times 2\times 3+\ldots+n\times(n+1)=\dfrac{n(n+1)(n+2)}{3}$.

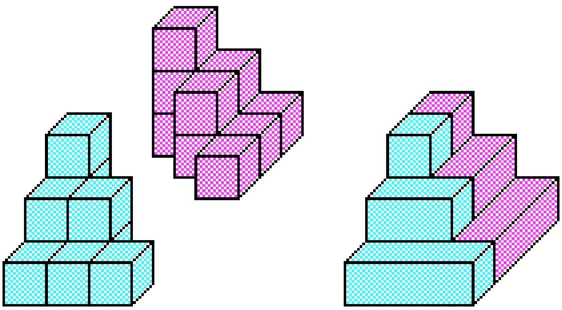

Figure 23. $2(T_1+T_2+\ldots+T_n)=1\times 2+2\times 3+\ldots+n\times(n+1)$

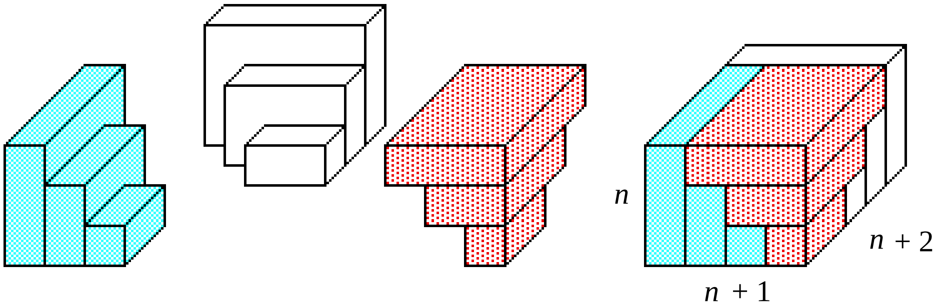

Figure 24. $T_1 + T_2 + \ldots + T_n = \dfrac{n(n+1)(n+2)}{6}$.

Conclusion

All too often students are asked to make the transition from arithmetic to algebra too quickly. Students need time, opportunity, and support to make the transition. Concrete representations of numbers and their relations in the form of manipulative materials, puzzles, and visual displays can help students give meaning to the symbolic expressions. In addition, by using concrete displays and particular numbers, students can explore relations that can be generalized. By using and extending the patterns they observe among numbers, they will find it easier to use variables later to generalize the patterns. Polya (1962) points out that, “abstractions are important; use all means to make them more tangible” (p. 102). In the examples shared here, geometric representations of particular numbers were used to make general statements tangible. The representations can be used to support reasoning in general.

The use of variables implies a reasoning that is general. However, many students are not prepared to simultaneously do general mathematical reasoning and use algebraic symbols. They may need to do the reasoning first with particular numbers. By using geometrical representations, students can develop their abilities to reason in general terms before they use symbols for variables. At the same time, by using geometrical representations, they will develop meanings for the algebraic terms that will be used later.

For students, geometric representations can provide fresh meanings and ways to see at a glance why the relations hold. Furthermore, these representations can help students to establish a connection with the algebraic notation. Geometric representations can help students see what each of the terms in an algebraic equation represents. At the same time, the geometric figures can guide students as they do the symbolic manipulation.

Algebraic notation has the power to carry much of the weight of thinking when deriving or proving mathematical results. Its abstractness allows us to forget the meaning of the terms as we manipulate the symbols. However, in the beginning, while students develop their skills with the symbols, it is important that they connect meaning to what the manipulation of symbols represent, so that their manipulation does not become senseless. Students need to avoid what Sowder and Harel (1998) have described as treating "the symbols as though they have a life independent of any meaning or any relationship to the quantities in the situation in which they arose" (p. 5). Geometrical representations can provide guidance and understanding as to why each step or term in a chain of algebraic manipulations is correct. Later, when students have developed skill in manipulating symbols and their algebraic (or symbol) sense, geometric representations can still be helpful. They can be additional sources of discoveries and inspiration.

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Visualization and High Achievement in Mathematics: A Critical Look at Successful Visualization Strategies

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ABSTRACT. This study investigated characteristics of visual representation use that underlie successful problem-solving across ages and levels of mathematical knowledge. We re-examined critically some of our previously conducted studies and our analysis brought to the surface some re-occurring patterns in low achievers' and high achievers' visual representation use. Low-achieving students appear to focus on descriptive images of real objects and actions and have difficulty viewing these as mathematical objects on which they can act. High-achieving students, on the other hand, appear to focus on finding connections between visual images and symbols, and use these as triggers for further thought and experimentation. The same patterns appear to hold in the work of experienced users of mathematics who consciously attempted to isolate and mathematize given visual representations.

Introduction

The role of visualization in mathematical problem solving remains an active question in educational research. For centuries, visual tools such as diagrams, graphs, and sketches were considered to be indispensable in the work of mathematicians (Rival, 1987). Reports on the work of mathematicians have provided anecdotal evidence of the use and value of diagrams and visual tools in research work. Among the most famous proponents for the use of visual representations in mathematics has been Pólya who argued that the use of visual representations is an essential element in problem solving and advised his students to use visual representations in their own problem solving: "Even if your problem is not a problem of geometry, you may try to draw-a-figure. To find a lucid geometric representation for your non-geometric problem could be an important step toward the solution" (1945, p.108).

In recent years, calls for mathematics instructional reform have explicitly focused on the importance of multiple representations (including analytic, graphic, and symbolic). For example, Principles and Standards for School Mathematics (2000), includes a representation standard and attests to the importance of multiple representations in mathematics teaching and learning in pre-K-12 classrooms. A similar theme has permeated recommendations for college mathematics instructional reform.

But, despite the general observation that visual representations can simplify and assist problem solving, there is also a substantial body of literature that suggests that visualization is not always associated with mathematical accomplishment. In fact, a well-documented finding in the literature suggests that students face difficulties when using visual representations in problem solving and are often reluctant to use visualization to process mathematical information (Eisenberg & Dreyfus, 1986, 1991; Vinner, 1989). However, little empirical work has been done towards the better understanding of the processes related to the use of visual representations, and specifically, in understanding what underlies the successful use of visual representations.

Further research, of qualitative nature, regarding the use and content of visual representations is warranted. Our work, whose purpose includes an investigation of students´ reasoning with visual representations in solving mathematical problems, aims to provide insight into this issue. Specifically, this study investigated patterns or characteristics of visual representation use that underlie successful problem-solving across a broad spectrum of mathematical sophistication, ranging from the early understanding of arithmetic to the advanced level of problem solving exhibited by expert mathematicians. Some studies related to the debate around visualization are discussed prior to providing the details regarding the study reported here.

Visual representations and mathematical accomplishment – not always found together

As noted above, there exist many anecdotal accounts of the central role and importance of visualization in the mathematical activity. Nevertheless, some research conducted in recent decades suggests that the relationship between visualization and mathematical accomplishment is actually not so clear, or at least that there may be a need for greater clarity about the nature of the relationship.

Lean and Clements (1981) looked for relations between students' preferred mode in processing mathematical information and students' performance on mathematical tests and concluded that "there is a tendency for students who prefer to process mathematical information by verbal-logical means to outperform other students" (p. 296). Difficulties associated with the use of visual representations were also documented at the advanced levels. Eisenberg and Dreyfus (1991) suggest that even advanced, college-level students demonstrate a reluctance to use visual representations. Students often tend to use algebraic paths of information processing over visual ones, even when the former are more complicated – a tendency which often leads to disastrous results since these students fail to give a complete analysis of the problems (Eisenberg & Dreyfus, 1991; Vinner, 1989).

Presmeg (1986, 1997) found that high school students who were identified by their teachers as outstanding students in their mathematics achievement were almost always non-visualizers, while students who showed a preference for the use of visualization often experienced difficulties in mathematics. Presmeg suggested that many students who have a tendency to visualize face difficulties in transcending the one-case concreteness of an image or a diagram–a difficulty of which their teachers are not aware. Presmeg's observations (1986) that students often get trapped by "the one-case concreteness of an image or diagram [which] may tie thought to irrelevant details, or may even introduce false data" (p. 44) provide specific evidence for the existence of cognitive obstacles related to visualization.

Presmeg's claims appear to be in accord with Krutetskii's (1976, p. 178) who argues that the differences in mathematical performance depend on how individuals process information. In his studies of children of varying performance in mathematics, he found that gifted children remembered general strategies rather than detailed ones, curtailed their solutions to focus on essentials, and were able to provide alternative solutions. Average children remembered specific detail, shortened their solutions only after practice involving several of the same type, and generally offered only a single solution to a problem. Low-achieving children remembered only incidental, often irrelevant detail, had lengthy solutions, often with errors, repetitions and redundancies, and were unable to begin to think of alternatives.

Drawing from the studies described above one can argue that the answer to the association between mathematical achievement and visualization may lay on the different use and content of visualization and not whether visual mental representations are more beneficial than verbal-logical representations, a question that guided most of the research conducted in this field. Therefore, the different ways in which individuals process visual information and the content of these visual representation at a given time can be either beneficial or severely compromising for their current and future development. This is what we now turn to.

The purpose of our work

The study reported here was designed in response to the need for further research on the role of visualization in mathematical problem solving. The accumulation of evidence regarding students' difficulties in the use of visual representations, and the certitude that advanced mathematicians use of visual representations in their mathematical problem solving, point to the need in examining in detail patterns in visual representation use across the entire range of mathematical activity. Specifically, we chose to examine the patterns that underlie successful use of visual representations across ages and levels of mathematical knowledge or sophistication. We hereby re-examine critically some of our previously conducted studies with the goal to learn more about successful use of visual representations. This involved new analyses of problem solving performance by students and professionals which included evidence of visualization. Our goal was to identify similarities in the behavior of the successful visualizers and contrast them with unsuccessful ones. We first discuss elementary school students´ attempts to use visual representations while solving mathematics problems. The patterns in elementary school students´ visual representation use appear to reoccur in the work of secondary school students while working in a different mathematical context. Finally, we argue that these patterns appear solidified in the work of experienced problem solvers.

Elementary school level – Use of visual representations at the outset of mathematical thinking

Our report on the use of visual representation by younger students (eight to twelve year-old) is based on a re-examination of our previous work on children's use of images in arithmetic (Grey & Pitta, 1999; Pitta & Gray,1999). Participants were chosen among both low and high achieving students in mathematics, as these were assessed by their teachers and by their scores in standardized mathematics examinations (SAT). During the first phase of the study (free-context), young students were presented with numerical words, for example “5”, “7”, “33”, “99”, “half”, “three-quarters”, “fraction” and “number” and asked to talk about these words, while, during the second phase of the study (arithmetical), students were asked to solve mental arithmetical tasks of simple addition and subtraction. In both phases, students were asked to report what comes to mind.

Children’s responses were classified into two categories in each phase based on the role that visual representations played in the response. In the free context phase the two categories identified as “descriptive” and “relational”, while in the arithmetical phase they were described as “figural” and “abstract”.

Descriptive and figural visual images. Images that emphasized the surface characteristics either of the object or of a context where the item in question was an integral component were classified as “descriptive”. For example, “I saw a picture of 99 like squiggly and misty”(Y5-, 99), “7 had patterns, spots.”(Y3-, 7) include some of the descriptive images projected by students in the first phase of this study. Figure 1 illustrates the way that some of the children illustrated their responses to the notion of five, as related to a birthday budge (i) more frequently as a bubble number (ii) or even as a three dimensional block (iii). A child who was very concerned about her spelling ability wrote out the word slowly (iv).

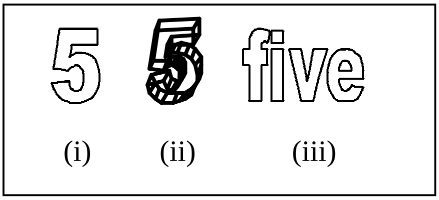
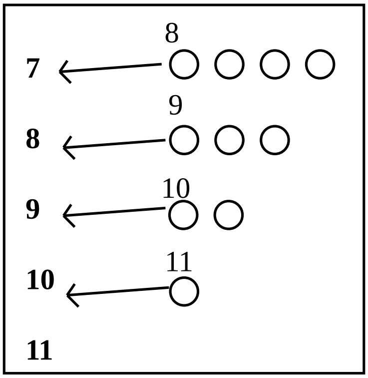


Figure1. Descriptive visual representations of number five.

In other instances children expanded their description to a scene or a series of scenes where the word in question was an integral part. For example “5 on a birthday budge”, “A number on a bus, on a door, on a sum, on a book” (Y4-, 7), “If your birthday is 7, if you are 6 and your birthday is tomorrow you will be 7…6 comes first then 7” (Y4, 7). “Number 5, I think of a row of numbers and a light shines on the number five. The light goes along and stops over the number five.”(Y5-, 5)

Descriptive visual images characterized low achievers’ responses to the free-context phase tasks. Overall, children who were pre-selected as ones who tend to perform low in mathematics, reported “seeing” descriptive visual images of numerical words, often embellished with a non-mathematical context. Similarly, during the arithmetic phase of the study, low achievers tended to project "figural" visual images. By figural visual images we mean that in the absence of actual items children created mental visual analogues of the items which were used to carry out a procedure. Figure 2 below illustrates a figural counting procedure:



"First a "black" seven appeared with "four white balls". One of the balls had an eight written above it and the eight written and the eight moved back to take the place of the seven, which disappeared. There were now three white balls the one nearest the eight having a nine written over it. This now moved to take the place of the eight, and so on."

Figure 2.

An 11-year-old gave Figure 3 as a diagrammatic representation of what he saw in his head when subtracting 20-8.

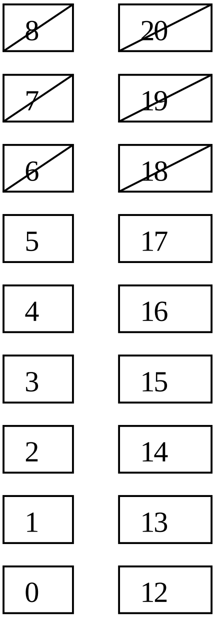


Figure 3.

He described the two "number tracks" as "two calculators going around in opposite ways in my head". The figures 8 and 20, were the first to appear, and then these were "crossed out" to be replaced by "7" and "19". This process continued until "0" and "12" were reached.

Of course, it was not always that their visual image involved analogues of real items, such as dots or calculators. In some instances the only items used were the mathematical symbols. However, these were retrieved in a number line form, which acted as an aid for their counting procedure. An eight year old described how she found the solution to 13-5: "I saw a line of numbers. It was 1, 2, 3 …13, 14, 15. After 10 the numbers got bigger. I counted from one until I got to 8." (Y6+, 13-5)

Relational and abstract visual images: During the free-context phase children on the other end of arithmetical achievement spectrum tended to give more relational visual images; images that emphasized the more intrinsic qualities of the item and its relationship with other objects. Examples, included a third-year and a sixth-year students´ attempts to compute mentally using derived facts: “You can do sums like 99+1=100 or something like that. You can do sums that make 99 for example 70 add 29 equals 99.”(Y3+, 99), and “Half way between 0-66 between 0-66…311, one third of 99”(Y6+, 33)

High achievers, on the other hand, when asked to respond to an arithmetical question, tended to create “abstract” visual images - a term that is based on the notion of Steffe et al. (1983). In this context not only did a child avoid the construction of countable units but the symbol was identified as the object of thought. The classification was most frequently associated with derived facts although in some instances it was related to children’s known facts. Typical abstract responses could be: “I saw 3 then I saw 9… first 3 then the 9” (Y3+, 6+3), and “Saw 45 and 57. This disappeared to see 102” (Y6+, 45+57).

Therefore seeing symbols was largely reported in two contexts: (a) Seeing the input or the solution as an expression or a transformation “in the mind” immediately following the verbal delivery of the combination by the interviewer and, (b) as seeing a visual image of the solution, as, for example, “I saw the answer before I told you”. Seeing the two, input and solution, at the same time was common for those children who indicated that they knew the solution to combinations.

Most often these visual images were associated with the formation of derived facts: “I saw 13-2 and thought I would split it up into different parts to make it easier. This stood out. Saw 13-3 then 10. Then I saw 10-2 this stood out. Then I saw 8. 13-3 and 10-2 stayed at the same time. I saw 8 on its own.” (Y6+, 13-5)

The relationship between visual images and verbal mental representations was one that clearly gave individuals the power to complete the combinations. This flexibility was becoming more apparent as tasks became more difficult. In a three digit addition and subtraction this is what a “high achiever” said: “I was working hard to remember the sum – but that got in the way of doing it. I saw 396 in my head and then saw 157. I saw some working out – that came and went because the numbers in the sum kept flashing on and off. When they flashed they changed a bit. I tried to remember what flashed but it was hard. If I didn’t see it, I thought it.

Once I had done the units – which were nine- I had to remember nine 10s was really eight” (Y4+, 396-157).

Two of the 8-year-old children reported using a mental image isomorphic to a written pencil and paper approach. The remaining children talked of “seeing the symbols being worked on”, seeing “the two numbers [to be added together] and then they went away and I remembered them” and “I saw the answer”. However, the common feature which emerged from the interviews with these children was the sense that they oscillated between seeing and thinking.

Secondary School Level – Some re-appearing patterns

Our report on the use of visual representation by secondary school students (twelve to eighteen year-old students) is based on a re-examination of some of our previous work on geometric problem solving (Stylianou, Leikin & Silver, submitted). As part of a broader study of students´ reasoning with two- and three-dimensional figures, we studied students´ work with nets (that is, diagrams of hollow solids consisting of the plane shapes of the faces so arranged that the cut-out diagrams could be folded to form the solids). Here we use students´ responses to one of the tasks in which they were asked to describe the net of a truncated cylinder as is shown in Figure 4 below.

The cylinder was cut as it is shown below.
Near the truncated cylinder draw its net.

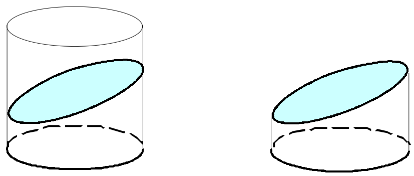


Figure 4.

Based on the centrality of the part-whole recognition in solving visualization tasks (Hershkowitz et al.,1989; Owens, 1999), a coding scheme was developed using the scheme described by Hershkowitz (1989). This consisted of two types of visual reasoning when introducing and understanding geometry concepts: (a) reasoning based on the whole figure and on non-critical attributes, and (c) reasoning based on critical attributes.

Reasoning with the whole figure and non-critical characteristics. A first step in solving problems involving visual representations is the isolation of characteristics or attributes of the image that are relevant or helpful in the problem situation. Not all secondary school students could isolate specific characteristics on the given image; A few of the participating students, when presented with the truncated cylinder problem, appeared to be at a loss, not knowing on what characteristics of the shape they should be focusing. The cylinder as an object was one concrete item whose parts were inseparable. These students failed in making any progress towards the solution of the task.

However, the majority of the participating students focused their efforts on finding identifying properties of the truncated cylinder. Among these differences the most profound appeared to be the cylinder´s curvilinearity (that is, the cylinder's curvy surface) and the shapes of the two bases (one being circular and the other elliptic). However, these characteristics are relatively unimportant for the construction of the net of the truncated cylinder. Hence, one cannot find a solution of the problem using these attributes of the cylinder, but is led to the drawing of inaccurate solutions as shown in figure 5a (the first of the five student-drawn figures).

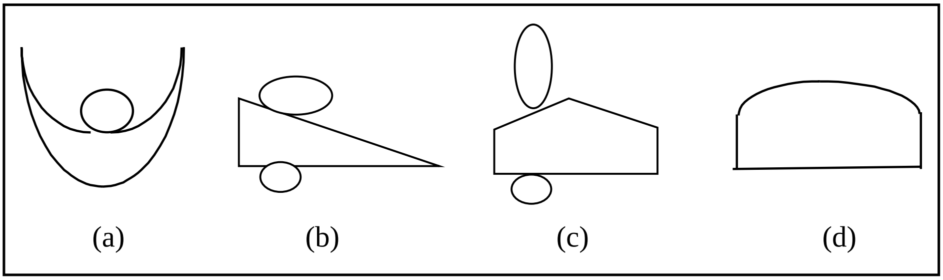


Figure 5.

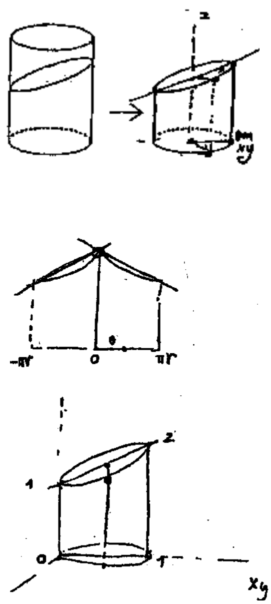
Reasoning based on critical characteristics. Only a subset of the cylinder´s attributes can be helpful in finding the solution to the problem. These include the changing height of the cylinder and the continuity in this change, symmetry, and "smoothness" and curvilinearity of the upper curve are the net.

A difference among incorrect responses was manifested by the number and quality of the critical characteristics of a truncated cylinder, which were accounted for while drawing its net. So taking into account only the change in height results in the construction of a net such as the one shown in Figure 5b, as is illustrated by Jonathan´s work and explanation ("the edge that goes along that base is still straight and the sides of it are still perpendicular to it, but they are not the same length: one is now shorter than the other") while, taking into account symmetry, as well, results in the construction of a net that has more accurate characteristics (though, still incorrect) as is illustrated in Figure 5c by Mark´s work and explanation ("the two sides of a net have to have equal length so that when you close the net the sides will meet each other perfectly"). The additional understanding of smoothness or differentiability is the necessary condition to create a correct net, while a partial understanding of it will, at least result to a net with a parabolic upper curve ("[a line will give us] a pointy ellipse on the top") as shown in Figure 5d.

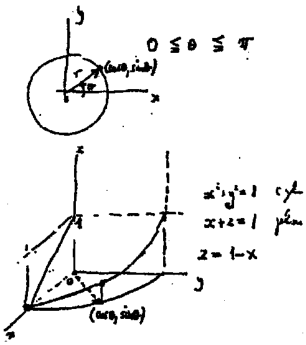
Advanced Level – Solidifying earlier patterns

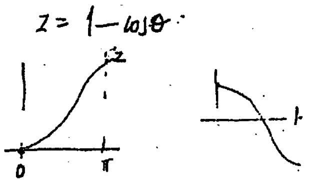
The truncated cylinder problem, was also used in a study that examined visual representation use in advanced mathematical problem solving (Stylianou, 2001). Here, we present the solution of a professional mathematician and we discuss the solution with respect to the strategies that were used in the solution:

E: I'm trying to figure out if this is linear or not, to find an argument that would convince… I can put some easy coordinates […] Down here we have the $x - y$ plane, and this is z. The radius will be fixed, so we depend on the angle θ . So, when I cut things, let me watch this point, and the corresponding point on the top. So, this length, if the radius is *r*, is the length of a circle which is $2\pi r$, so this 0 it will go all the way to πr , and this is $-\pi r$. I'm gonna use these coordinates with the angle θ . So, for $\theta = 0$ I'm here. And then as θ grows, I'm all the way to the end. Let's say θ grows from 0 first all the way to say π . So, where is this point here, 2? This point will be… this is *x* and this is *y*… and *z* is coming like that, so … this point here is the point ($\cos\theta$, $\sin\theta$). No. No, because it's actually, see this is the length of the arc, not the angle. So this is actually θ ! This is more difficult than I expected. So, how about this point here? On top. Of course the angle will be



θ , but the height is now whatever this… That circle… if we put the ellipse here, it means that… So, this is just one corner of the whole picture, you have the other corner… So, we use it as a coordinate, the length of the angle. We take this point. Let's find out where is that. I can find that height and that will be it! Umm… So, in order to do that, this point on the base is cos θ , sin θ . To find that point to get the equation of the plane and this is here and this is z, *xy*, This big cylinder is $x^2 + y^2 = 1$, No *z*. Let me take any plane, for example so that plane is the plane $x + z = 1$. That is the plane. So, this height is $z = 1 - \cos\theta$. So, the actual graph of that function will be something like… this is π this is 0… the maximum will be here, the sine for 0 is 0, for π , you get 2. That will happen, because this is 1 you will have 2. And 1-cos θ , well, it's like that…."





In a sense, the mathematician, similar to the more successful students, progressed in his solution by immediately shifting his attention from the cylinder as object to its characteristics, and, furthermore, within the wider set of the cylinder's characteristics, he focused on a narrower set of critical characteristics. Notice, however, that in the final segment in his investigation his intention was to "mathematize" or symbolize the critical characteristics. The mathematician made a strong effort to transfer the critical characteristics into a "symbol system" (Nemirovsky, 1994), that is, an environment in which he could manipulate these objects in a deductive manner while using familiar mathematical concepts, facts, and, especially, rules. The mathematician could operate on and experiment with the critical characteristics very comfortably within this rule-governed set with which he was familiar. Thus, he chose to move around one point on the rim and "watch" how this movement affected the height of the cylinder. Once he realized that the movement could be described by a trigonometric function he knew he had the solution to the problem; he could now use his vast previous experience with trigonometric functions to visualize and describe this new function.

Conclusions

Halmos (1987), speaking of what it takes to be a mathematician, asserted that “to be a scholar of mathematics you must be born with the ability to visualize” (p. 400). This study aimed to examine possible patterns that underlie successful use of visual representations across ages and levels of mathematical knowledge or sophistication.

The first part of this study focused on the work of elementary school children and their use of visual representations. Despite the fact that these students were asked to solve problems that are not geometrical or viewed as “visual”, that is, mostly problems of routine, our findings suggested that students often use visualization when attempting to solve the problems, or even when bringing mathematical concepts to mind. Our analysis brought to the surface some re-occurring patterns in low achievers’ and high achievers’ visual representation use. Children who face difficulties in mathematics appeared to translate mathematical symbols to visual images of real object and actions with them (descriptive and figural images). The objects in the mind were real things and actions were analogues of those that children may carry with real things. Even when mathematical symbols were represented mentally these were either embellished with surface characteristics or incorporated in a larger mental context which involved other items. Low-achieving children seemed unable to detach themselves from the search of substance and concreteness. The mental analogues of the objects and actions associated with these appear to be essential for solving the problems. High-achieving children, on the other hand, seem to interpret mathematical symbolism and use their visual representations differently. Their visual images are dominated by numerical symbols which are used as triggers for thought. Symbols appear to flash as memory reminders momentarily coming to the fore as new transformations or precursors of verbal comments. The detail both of the object and the action is filtered out to give a focus on the most abstract, “mathematized”, qualities of the symbol.

The second part of the study reviewed the work of secondary school children and their use of images or visual representations. In this case, a geometry context was used to examine students´ use of visual representations. But, despite the different context, the patterns that emerged from our analysis oscillated with and extended the patterns in younger students´ performance. Students who faced difficulties in solving the given tasks reported viewing the cylinder (the object of the given task) as a whole. This approach oscillates with younger low achievers´ images of mathematical objects. Both young and older low-achieving students reported a difficulty in detaching or isolating an object´s relevant characteristics. Rather, low-achieving secondary school students were “trapped” in a figure and its perceptual (non-critical) details. Successful secondary school students on the other hand, similar to younger high-achieving students, were able to extract the figure´s critical characteristics and use those flexibly to solve the given problems.

These observations corroborate with and extend earlier findings of Brown and Presmeg (1993) who investigated types of images that students form. In their study, Brown and Presmeg found that concrete images were prevalent in the problem-solving processes of high school students. These consisted primarily of a single, non-moving but often highly detailed picture. Use of concrete visual images in mathematical reasoning often led to many difficulties, as students either tended to overgeneralize. The type of imagery students used was found to be related to students’ relational understanding of mathematics; students with a greater relational understanding of mathematics tended to use more abstract forms of imagery, while students with less relational understanding tended to rely on concrete images (Brown & Presmeg, 1993).

Finally, the mathematician´s use of visual representation came to extend and solidify the patterns we observed in younger students´ problem solving. Indeed, the mathematician was quick to put aside the actual figure and to look for the characteristics that could easily symbolized, or “mathematized”.

Our interpretations can be further discussed in the context of the framework on concept image and concept definition proposed by Tall and Vinner (1981). Concept image is described as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It built up over the years through experiences of all kinds” (p.152, ibid.). At the same time, a formal concept definition is the (minimal) set of conditions which are necessary and sufficient to specify that concept. Purely formal deduction in problem solving involves reference to a formal concept definition. However, when a person chooses to operate with concepts that are not formally defined at all, but, are based upon her/his experience only, then the response is considered to be intuitive and makes reference to a concept image.

With respect to the work presented here, we propose that successful use of visual images can be closely related to a person’s ability to form and subsequently utilize successfully the concept definition of a mathematical object. It follows from the above theoretical view that low-achieving students, tend to have incomplete and concept images without any reference to the defining conditions of the concepts that are under investigation, in a way that compromises their problem-solving abilities. Furthermore, when presented with a problem, low achievers' elaborate and perceptual images prevent them from isolating those characteristics or attributes that may help them form a concept definition of the mathematical object with which they work. High-achieving students' reasoning indicated an attempt to locate and use critical characteristics in the mathematical situations they were investigating –an attempt to begin forming a concept definition for the situation. This became apparent in the work of the mathematician, who consciously attempted to isolate and mathematize the problem characteristics, thus leading to a complete concept definition of the task which he could use to give a formal deductive solution of the task.

In summary, while our work here is a not a systematic study of the development of mathematical thinking, we have identified some of the patterns that appear to underlie successful visualization representation use. It appears that the answer to successful mathematical understanding either of geometrical or non-geometrical concepts does not lay in the use or absence of visualization. It appears to be related to two things, (a) the individual's ability to select what needs to be visualized and what to be talked about in the mind, (b) the oscillation of the visualized and verbalized components and how these complement each other.

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Graduate Students’ Visualizations in Two Rate of Change Problems

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ABSTRACT. The research reported in this paper is part of a larger study to investigate the presence, role, extent, and constraints of visual thinking in the problem solving processes of graduate students as they solve nonroutine problems. This report gives details of the solving of two rate of change problems by three students. As evidenced by three descriptors, namely, drawing, verbal report, and gesture, visual imagery was used by each of the three students for both problems. Visual imagery was reported even in instances where no diagram was drawn and the solution appeared to be purely algebraic. The roles of visualization were investigated in four main moments of the solution processes, which we have called preparation, solution, conclusion, and hindsight. The types of imagery and their roles in these moments cause us to differentiate between use of imagery to make sense and to solve, as two distinct aims of visualization. Affect, base knowledge, spatial reasoning, and metaphors that may enable or constrain, all played a role in the graduate students’ use of visualization.

Introduction

Construction and use of imagery of any kind in mathematical problem solving are processes that challenge research approaches because of the difficulty of apprehending these processes without changing them. Visual imagery used in mathematics is frequently of a personal nature, not only related with conceptual knowledge and belief systems, but often laden with affect (Presmeg, 1997, Goldin, 2000). But it is these very personal aspects that may enable or constrain the mathematical solution processes of an individual (Aspinwall et al., 1997; Wheatley, 1997; Presmeg, 1997), and thus it is important to investigate these issues. Because there has been little research on the use of visualization by adult learners who are not research mathematicians, we decided to investigate this use in the solving of word problems by students enrolled in a doctoral mathematics education research course at a large university in the USA.

An approach was needed that could differentiate the presence, role, extent, and constraints of imagery used in mathematical problem solving. These four aspects might be influenced by the kinds of problems, the way that they are administered, the structure of the problem batteries, the solvers' base knowledge, their mathematics conceptions, and their metacognitive resources, amongst other influences. Thus, in order to investigate whether or not a solver visualizes, when he or she does so, why he or she does so, and what kinds of imagery are used, it is necessary to develop a deep inquiry into his or her problem solving processes. A naturalistic approach with case study format is the most suitable research methodology for investigating these issues (Merriam, 1998).

Theoretical Considerations

The theoretical framework that underlies this study is rooted in Krutetskii's (1976) position that all mathematical reasoning relies on logic, whether or not diagrams and mental imagery are employed. This use of visual imagery with or without drawing diagrams is called visualization in this study. Krutetskii's position paves the way to viewing the use of visualization as an axis separate from, and orthogonal to, an axis that represents the strength of the logical component of mathematical problem solving. If the strength of the logic in mathematical thinking is metaphorically represented on the positive y axis (the stronger the logic, the higher the y value), then the quantity of visual thinking might be placed on the positive x axis. Krutetskii (1976) took the position that there is no mathematics without logic, while visual thinking may be useful but it is not essential. His own case studies of gifted students confirmed that the amount of "visual supports" needed by a student determines the type of thinking employed, while the strength of the logic underlies the ease and elegance of solution.

Notwithstanding these theoretical considerations, we did not want to typecast individuals (as Krutetskii had done), nor to work only with "visualizers" who prefer to solve mathematical problems by visual methods (as Presmeg did in her 1985 study). We wanted to choose problems that would be rich enough to facilitate diverse ways of mathematical thinking, and then to allow our interviewees carte blanche to pursue their own strengths, preferences, and beliefs about mathematical problem solving, with our interpretations of their activities unfettered by prior typecasting.

Two Problems

As part of the larger study, the doctoral students were tape recorded as they solved problems "aloud" in task-based interviews. In this paper we report some of the data for Mr. Silver, Mr. Gold, and Mr. Green (pseudonyms). The following two problems were selected from the six problems in section C (intended for high school mathematics teachers) in Presmeg's (1985) preference for visuality instrument. These problems had been demonstrated to be capable of solution by many different methods, some of which involved visual thinking and some of which did not.

C-3. A boy walks from home to school in 30 minutes, and his brother takes 40 minutes. His brother left 5 minutes before he did. In how many minutes will he overtake his brother?

C-6. A train passes a telegraph pole in $\frac{1}{4}$ minute and in $\frac{3}{4}$ minute it passes completely through a tunnel 540 meters long. What is the train's speed in meters per minute and its length in meters?

Both researchers were present at all interviews, one in the role of interviewer, the other in an observer's role, taking field notes. Interviewees were given one problem at a time, typed on an unlined sheet of paper on which they could write. They were asked to solve the problem by the method that came most naturally, expressing their thinking aloud if possible. All interviews were tape recorded and subsequently transcribed. Thus there were three sources of data, namely, interview transcripts, field notes, and written solutions of interviewees. The three methods of data collection allowed for triangulation (Merriam, 1998). The two researchers repeatedly discussed their interpretations of the data during the analysis, and there was also a degree of respondent validation.

When and how did these students visualize?

All three of these graduate students drew unsolicited diagrams (indicating the presence of visual imagery) for both of these problems. To facilitate a deeper understanding of the issues of when, how, and with what effect solvers used imagery in this study, the following analysis is reported.

Mr. Silver teaches college level mathematics. His solution processes in both problems could be analyzed in terms of the four moments of our framework, namely, preparation, solution, checking, and hindsight. However, the complexity of these processes helped us to become aware of some of the cognitive demands of these problems: the solution processes were not straightforward.

In problem **C-3**, Mr. Silver started with a spontaneous one-dimensional diagram: two line segments represented the journeys of the boy and his brother, whom Mr. Silver called A and B respectively. Then, still in the preparation stage, some confusion arose whether the line segments were representing times or distances. Mr. Silver decided they were representing the boys' times, but he proceeded to draw another two-dimensional diagram on which time was represented on the horizontal axis and distance on the vertical. Because he set A's starting point at zero, and B had started five minutes earlier, the question of negative time arose. He decided that it was acceptable to have a negative time. A representation of his graph of the relationships is given in Figure 1.

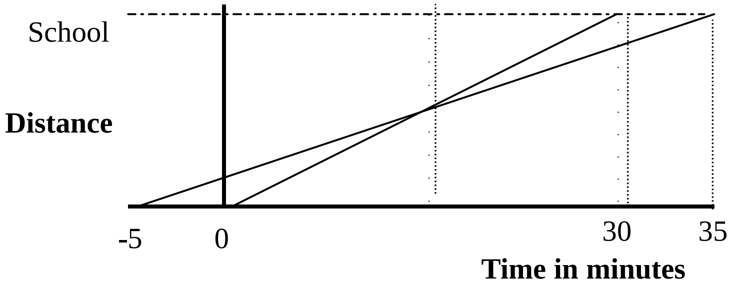

Figure 1. Mr. Silver's diagram for C-3

Mr. Silver understood the structure of the relationships based on his diagram, and in fact it would have been an easy matter to conclude from the symmetry of the relationships that A overtook his brother exactly halfway. But the cognitive demands introduced by the negative time obscured this method of proceeding, and in the next phase of the activity, Mr. Silver introduced symbols ("Let x be the distance") and worked with the rate formula ("distance equals rate times time") to establish an equation that could be solved. Before he had fully solved it, he returned to his diagram. The interviewer, in probing, asked if he had seen the symmetry of the relationships. He had not, but this question caused him to speculate that A must have overtaken his brother after 15 minutes. In a hindsight phase, he remarked that he would not have thought of the symmetry by himself, and even after he had seen it, he needed his method of symbols and equations, “to make sure”. The cognitive demands of the problem seemed to reside in the sliding reference point of the different starting times of the two brothers. Mr. Silver’s use of visualization occurred mainly in the preparation phase, to make sense of the relationships, and he returned to the diagram in the checking and hindsight phases, but his main confidence lay in his algebraic solution processes.

In problem **C-6**, the intricacies of the relationships reside in being able to synchronize the information that the train passes a pole in one fourth of a minute, with the information that it passes completely through the tunnel in three fourths of a minute. Mr. Silver reported spontaneously that he had an image of an actual train. He wrestled briefly with a question about the width of the pole (“Okay! We’re gonna disregard the width of the pole.”) Then he decided he would be more comfortable if he changed the fractions of a minute to 15 seconds and 45 seconds respectively (“And then I can divide”). At that point he needed to pause for reflection (“Okay. Can I think a little while?”). After a pause, he continued:

Mr. Silver: I think the tunnel is three times as long as the train … that sounds reasonable? … Okay, this is the relation, 15 seconds here … passes this pole …now this is the tunnel … and … fifteen …[longer pause] … It passed completely through!

This moment seemed to be crucial in Mr. Silver’s understanding of the relationships. As he spoke, he was drawing a starting point followed by four larger circular points, as in figure 2.

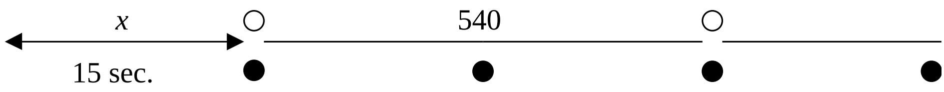

Figure 2. Mr. Silver’s diagram for C-6.

Mr. Silver: Because … say the tunnel is twice as long as the train … here is the front of the train and 15 seconds it will be here … and in 15 more seconds the front of the train will be here … and then completely through. So it will be twice as long as the train. Yeah! I would say the train is 270 meters long.

The visual reasoning of the preparation phase slid easily into numerical reasoning, still using the diagram, to give Mr. Silver a process of solution. To find the speed of the train, he initially wanted to divide by four instead of by one fourth (“So that’s 270 divided by 4 meters per minute.”). He paused, corrected himself, wrote 270 over one fourth, and commented, "Oh! 270 per one fourth of a minute … so it will be 1080 meters per minute … my final answer!"

In the hindsight phase, he explained that the relationship between the times caused him to think initially that there were three "pieces" in the tunnel ("one quarter and three quarters"). He continued, "But I was taking into account the extra dimension. I had to count out the times" [in order to arrive at the insight that the tunnel is twice as long as the train]. He used no algebraic symbolism in this solution, and seemed calmly aware of his own visual and numerical reasoning.

The cognitive demands of both problems were confirmed in Mr. Gold's and Mr. Green's diverse solution processes.

Mr. Gold, who holds a McKnight Fellowship for his graduate studies, also teaches college level mathematics courses. In solving both problems, his metacognitive awareness was strong, and he seemed confident and enthusiastic in spite of moments of doubt and uncertainty in certain moments of the processes.

After reading **C-3** aloud, he stated, "I'm going to look at it in terms of, ah, doing, distance, which is velocity times time [writing d = vt] and use that for approaching these problems. Now, I'll also use a graphical method … a visual method [drawing a line segment], and draw that diagram." He continued as follows.

Mr. Gold: I'll call one of the brothers A, the brother that can walk in 30 minutes, I'll use A with the subscript 30 [writing A_{30}] to represent him, and B with the subscript 40 $[B_{40}]$ to represent him.

H and S, for home and school respectively, were marked on his line segment. After a pause, he continued.

Mr. Gold: So I'll take B_{40}, he left walking … and the distance that he traveled … this distance I will put here … that distance is equal to his velocity which is 40 … no the time is 40 minutes. So, looking at the problem again … I have to … I have to find … some variables to represent this distance from home to school. So I think this variable will be x, to represent this distance from home to school. So his velocity would be x over 40 … which is the distance from home to school divided by the time that he takes to get there.

Still in this preparation phase, and "thinking aloud" as he proceeded, Mr. Gold assigned "x over 30" as A_{30}'s velocity. Then he seemed to strike a cognitive obstacle and paused ("I'd wish to check this"). He continued.

Mr. Gold: An easy part …distance … let's put one mile in here, for the distance from home to school [referring to his diagram]. So if it takes this brother 40 to walk a mile … 40 is two thirds of an hour … he's walking at 60 miles per hour.

Mr. Gold appeared to see nothing unusual in this unrealistic hypothesis. However, as he continued, there appeared to be a great deal of confusion, and neither of the researchers could apprehend his thinking is this phase. After a long pause, and referring to his diagram, he seemed to start afresh, as follows.

Mr. Gold: Okay! … He'll go one mile and it'll take 40 minutes to get there, and that's two thirds of one hour. So, and I'm trying to get his velocity which is distance over time, distance is one, time is two thirds of one hour. So this would be [writing] three over two …this for B, and for A, one over one half … so that is two miles per hour.

After this long preparation phase, marked by false starts and some confusion, Mr. Gold moved into a solution phase in which he worked with fractions of an hour for the times ("Five minutes would be one twelfth in terms of hours … so that's one twelfth of an hour … he left five minutes before.") He recognized that the distances traveled by the two boys to their meeting point would be the same. This phase involved quantitative reasoning using his diagram. After introducing another variable, t, for the time taken for B_{40} to reach this meeting point, he was able to construct an equation ("I equated these, one point five t is equal to two times, t minus one twelfth"). Solving this equation involved some uncertain moments, but at last yielded that the time for B_{40} was 20 minutes. In a hindsight phase, Mr. Gold explained,

Mr. Gold: The diagram helps me to visualize in my mind, because it seemed chaotic, the processes in my mind. I had a diagram to relate to all these equations.

Interviewer: It looks like a good mixture of the visual and working with equations.
Mr. Gold: That's right.

Interviewer: But in the end it was the equation that gave you the solution?

Mr. Gold: And this is how I approach everything that I do in math. I'll see, sort of scratch my way doing those rough notes, until I get an equation.

For **C**-6 Mr. Gold required two diagrams and a fair amount of visual reasoning involving his diagrams before he could construct and solve an equation. His first diagram depicted a stylized train passing the pole. ("The front of this train … crosses this point … and the back of the train passes it in a quarter of a minute.") His second diagram represented the train and the tunnel in a similar fashion. Then using his familiar formula, velocity equals distance over time, rather mechanically, in a false start he divided the length of the tunnel, 540 meters, by the time, three quarters of a minute, in trying to find the train's velocity. In a checking phase he realized that he had not taken the train's entry and exit of the tunnel into account. Calling the length of the train "L" meters, he constructed two expressions for the velocity of the train from his two diagrams ("L over a quarter, and 540 plus L all over three quarters"). He could equate these expressions because the velocity must be the same in both cases. His solution of this equation yielded the length of the train as 270 meters. Going back to his first expression for the velocity of the train, he multiplied 270 by four to obtain the velocity of the train. Hindsight confirmed that he had gone back and forth between the preparation phase, false solution, preparation phase again, and also the movement between visual and symbolic solution processes.

Mr. Gold's transcripts confirmed the complexity of the thinking that may underlie problem solving processes, and the usefulness of a combination of visual and symbolic methods, particularly in the preparation phase. These considerations also underscore that a written solution alone gives a very incomplete picture of the thinking that lies behind that solution.

Mr. Green also teaches college level mathematics courses. He appeared to be nervous throughout the interview, did not talk much, in spite of the injunction to please "think aloud", and in a hindsight phase confirmed his meta-affect (Goldin, 2000), i.e., feelings about his own emotional state: "Oh, I don't know why I'm so nervous … I've never been studied before like that! [nervous laugh]."

Although it might have been implicit in the thinking of Mr. Silver and Mr. Gold, Mr. Green was the only solver of the three, who mentioned the variable concept right at the beginning of the preparation phase for problem **C-3**. After reading the problem, he stated, "Okay, I'm going to introduce variables right away. Because without their names I'm getting confused, so I'm going to give … the boy and his brother like different variable names. And try and keep them straight in my head … ahhh … Can I write anything?" He proceeded to call the boy x and his brother y, and to draw a line diagram, accompanied by the formula $d = rt$. After designating the two boy's rates as r_x and r_y, he apparently struck an obstacle and exclaimed, "Oh, man! It doesn't work today! Should be easy." After he read the problem statement again to himself, there was a long pause:

Interviewer: How are you thinking?

Mr. Green: Ahh … I'm using an equation … ahh …I'm finding out what the distance is. The way the question is asked, it is the same. So, one thing that makes it difficult is the times, with the five minutes. … I visualize the clock for some reason … Ahhhm … Oh boy! [Long pause.]

Interviewer: Any inspiration?

Mr. Green: No, I'm thinking …

Interviewer: I think here you used a diagram to help you make sense initially. [He had drawn a simple line segment.]

Mr. Green: Right!

Interviewer: Could you use a different kind of diagram?

Mr. Green: Oh yeah! I'm just having in mind a clock. I was thinking suppose the ten … I was thinking, ah, if he left five minutes before he did … so let's assume he left at ten oh five, and his brother left at ten o'clock. His brother I assume gets there at ten forty minutes and the boy gets there at ten thirty five.

Interviewer: Okay. Can you put that down on the paper. Write it down, for us both, or just for me to see it.

Mr. Green: For both! I have some pictures right away.

Mr. Green proceeded to draw four clock faces, to indicate starting and ending times for the two boys. In the process of drawing the "pictures", his reference point of zero shifted from that of the brother to that of the boy. So now he explained, using his clock face diagrams, "If t is equal to zero … it being ten o'clock, time equals zero, then the brother, who walks slower, is going to have left five minutes before … so that is nine fifty five. And the boy is going to leave at ten o'clock which has been put equal to zero." This was the end of the preparation phase, and Mr. Green then made an equation ("The distance to overtake is the same.") and solved the equation to obtain $t = 20$. The confusion over the zero point in the times persisted throughout a checking phase: "Let's see if that makes sense. … So I guess if … the zero is ten o'clock I'm saying he will overtake in ten twenty if the boy left at ten." Later, in a hindsight phase, he still did not seem to realize that he was using a shifting point of reference.

Mr. Green: "*t*" equals zero is ten o'clock as before, so he's gonna leave first and the brother … five minutes before he did.

Interviewer: So, ah, the boy is x and he leaves …

Mr. Green: Five minutes later than his brother.

Interviewer: Yes, so actually "t" is your time on the clock … It's not the time taken …

Mr. Green: Right! Right!

Interviewer: Yes. Whereas before … "*t*" was the time taken, wasn't it? [Pause.]

Mr. Green: This "*t*" here … which starts with the boy … would be how long it takes him to overtake his brother. In this case, it's complex, a little bit …[murmers].

He proceeded to re-do his equations, writing $r_x(t+5) = r_y t$. He did not seem to realize that he was taking "*t*" to have sliding meanings. A long hindsight phase ensued, in which the interviewer and Mr. Green negotiated at length the meaning of "t". Mr. Green, in a phase reminiscent of the "folding back" described by Pirie and Kieren (1994) in their model, returned to his drawings of the four clock faces, and put in the hands of the clock, which had been implicitly represented by marks on the circumference previously. A long conversation about the symmetry of the clock faces led to Mr. Green's insight, "Oh, I think I see what you're saying. The question is asked in terms of the boy; I did my answer in terms of the brother. That was confused at the beginning. Okay, we're talking in different ways … I feel better."

Interviewer: Now can we just look at it from the point of view of the imagery. The impression that I got is that your algebra is firm and that is what you like to use most of the time. Is that right?

Mr. Green: Ahh … no. And it might be the same, it depends … If I were working it out for myself first I'd do it based on what I did here, visual vision. If I were teaching algebra then I'd like to go back to different ways of teaching and explaining. So, I don't know if it's what I prefer, but … I don't know if I've just grown comfortable with it over the years.

Interviewer: For the sake of teaching …

Mr. Green: I like to look out for many different ways … yeah, as many ways as I possibly can.

Interviewer: You might not feel the need to do that yourself, but for the sake of the students, you would do that.

Mr. Green: Right! Right!

These final comments resonate with Presmeg's (1985) finding that several of the high school teachers in her study who were in her "nonvisual" group with regard to their use of visualization in her preference test, nevertheless used visual methods when teaching. She found a rank order correlation coefficient of 0.4 between "mathematical visuality" and "teaching visuality" in her original study, suggesting that some of these experienced teachers were taking the perceived needs of their students for more visual methods into account in their pedagogy.

Mr. Green's image of a clock came into play again in his preparation phase for the solution of **C-6.** After reading the problem aloud, apparently referring to the tunnel, he commented, "Passes completely through, so … There's the start of the train, and it's gonna start there [drawing a line segment] … The clock starts … and I guess, the end of the train passes through and then stops in three quarters of a minute." His words seem to indicate a clear link between his images of a clock and the train, starting, both moving, and then coming to a stop. The metaphor here seems to be that of a stopwatch, rather than the clock faces of problem C-3.

After that he turned to the first part of the problem statement, and questioned, "The train passes a telephone, telegraph pole? I've never seen a telegraph pole, it seems like a … like a regular pole? [Interviewer replied in the affirmative.] Okay." He proceeded to draw a second line segment for this part of the problem statement, commenting, "Beginning to end it takes a fourth of a minute … mmm … it's like a single point." Then he wrote the formula $D = rt$. After some puzzlement, he returned to his first line diagram of the tunnel. Focusing on the back of the train, he commented as follows.

Mr. Green: Let's start there … for this … it'd be there when it starts, with the clock taking three quarters of a minute, and it doesn't stop so that it's over here … ahh …so I can look at the end of the train. So, the distance it travels is the length plus 540 meters, I don't know the rate, and the time is three fourths of a minute. Okay! Now I'll get an equation. How can I do this? The telegraph pole is … in one fourth of a minute, that's just the length of the train has to pass. So that's gonna be the length, I know that, and …"

He wrote the following two equations as he spoke:

$$\tfrac{3}{4}r = 540 + L$$

$$\tfrac{1}{4}r = L$$

In a few minutes, he eliminated L by substitution, solved for r ("1080 meters per minute") and hence found the length of the train to be 270 meters. In comparison with the extended preparation phase, this solution phase was very short, as was the checking phase too ("Let's see if they make sense … 270 meters … yeah! Okay, that makes sense."). During part of the hindsight phase, the interviewer asked Mr. Green to attempt solving the problem without using algebra.

Interviewer: I'm interested in whether just from your initial diagrams, without using your symbols and equations, you'd be able to get it; because here, you had very good diagrams that made sense for you, both of the telegraph pole and the train … the back of the train entering the tunnel. In fact you ignored when the back was out of the tunnel, that didn't concern us.

Mr. Green: Right!

Interviewer: Now can we synchronize these two diagrams to help you get the solution without any algebra?

Mr. Green: Nooo.

Interviewer: Thinking of the times here, which are not written on the diagram? It's one fourth of a minute …

Mr. Green: And this is three fourths of a minute.

Interviewer: How can you relate that to this?

Mr. Green: Ahh … then I can take one fourth of a minute, taken from here to here … on the back of the train, and it takes the remaining two fourths of a minute to go from here to here … So in half a minute it will cover 540, which is … visual reasoning. Then, 540 is equal to r times a half [writing r = 1080].

Interviewer: Okay. Which gave it directly and immediately just by multiplying 540 by two [Mr. Green affirms]. Could you get the length directly from it? If it takes one fourth for this and another two fourths for 540?

Mr. Green: Ahhh … it takes … yeah! This would be half as long as the tunnel.

In conclusion, Mr. Green commented: I make sense of it with pictures, but I don't have the whole … so many things in my head. I write it down.

Interviewer: Yes, and then go to the algebra …

Mr. Green: Right! I guess, mainly because of the training … it's so many years, you know … the school stresses the algebra.

Discussion

Mr. Green's final comment relates to several points that we wish to make. We considered three cognitive aspects of visualization, namely, base knowledge, spatial reasoning, and metaphors that enable or constrain. With regard to the first of these, base knowledge, Mr. Green's comment highlights the role that pedagogy at all levels may play in whether or not a student will choose to use visual methods in solving a mathematical problem. "The school stresses the algebra", he said. In the United States, the term "school" is used to designate colleges and universities as well as the K-12 school experience. The stressing of algebra at several levels may well be the reason for the "reluctance to visualize" at tertiary level discussed by Eisenberg and Dreyfus (1991). However, it is abundantly clear in our data that when university graduates strike cognitive obstacles in their attempts to solve nonroutine problems, they resort to visual imagery, often expressed in a diagram, in their attempts to break the impasse. The attempts to coordinate the relationships in problems C-3 and C-6 resulted in diagrams for all three students, as well as further use of other imagery in four instances. Most of this use of visual methods took place in the phase of preparing to solve the problems. However, resonating with Pirie and Kieren's (1994) "folding back", students frequently examined their diagrams again during the solution phase if any doubt arose (e.g., Mr. Gold's solution of C-6). Their diagrams and imagery also played a large role in making sense of their solutions during the checking phase. In both problems C-3 and C-6, the difficulties occasioned by sliding reference points (the boys' starting times in C-3, the relationships between the train's movement past the pole and through the tunnel in C-6) occasioned repeated folding back, even when algebra was the first choice for solution, following use of the formula "distance is rate times time". The power of alternating visual and symbolic modes of cognition is illustrated in these examples with graduate students, confirming the efficacy of this alternation suggested in the spiral model of Zaskis et al. (1996), who worked with college students, and also suggested in Presmeg's (1985) data with senior high school students.

This leads us to raise issues relating to the second aspect of visualization as we have conceived it, the processes involved in reasoning mediated by geometrical and spatial concepts, dynamic imagery, and time references. It was the latter in the references to the boy's time of leaving and his brother's in C-3, that caused both Mr. Gold (with his "chaotic" solution), and Mr. Green (with his sliding meaning for "t" on his clock faces) to experience not only cognitive obstacles, but also a great deal of discomfort and negative affect. Like Mr. Silver, both of these students felt initially that an algebraic solution would be appropriate and easy (Mr. Green: "Oh man! It doesn't work today! Should be easy"). Both manifested the perseverance to continue, possibly reinforced by the confidence of their long experience in teaching college level mathematics courses. This experience with algebraic and analytical processes may have been an obstacle as well as a booster of their confidence. Without the strong belief that an algebraic solution was required, they might have been more open to see the symmetrical relationships in the times of the boy and his brother, leading directly to the insight that the boy must have passed his brother exactly halfway to school. Mr. Silver, too, did not apprehend the symmetry until the interviewer asked him about it. He reported later that he would not have seen this by himself, and even after he had seen it, he needed symbols and equations, "to make sure". Also in problem C-6, the dynamic relationships of the train passing the pole and entering the tunnel could have been synchronized by imagining the pole at the entrance to the tunnel (or alternatively at the exit). Visual reasoning then yields the insight that two train lengths make up the length of the tunnel, based on the time relationships. Of the four students interviewed, Mr. Silver came the closest to apprehending this relationship directly after he carefully "took" the train through the tunnel on his diagram.

In conclusion, these problems were rich sources in the sense that our analysis of the data reported here could continue: for instance, we have not yet reported the enabling and constraining roles that metaphors played in the students' representations and problem solving. However, the data reported in this paper show convincingly that visualization may play a useful role in the problem solving strategies even of teachers of college level mathematics.

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Physical Enactment: A Powerful Representational Tool for Understanding the Nature of Covarying Relationships?

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ABSTRACT. Representation of a system that involves two quantities that change in tandem has been shown to be complex for students. As students are asked to physically manipulate the system while attending to the changing nature of the system, they appear to become more comfortable in using dynamic imagery to transform the system. As they become more comfortable in engaging in mental actions to transform the objects of the system, they are better able to make generalizations about the subtle behaviors of the system. The ability to create a mental image of the system and to mentally transform the objects of the system appears to be foundational for the meaningful representation and interpretation of the system in more abstract representational contexts, such as graph and formulae.

Introduction

Investigations of students' developing notion of function have consistently revealed that students have difficulty translating from one function representation to another (e.g., Carlson, 1998; Hitt, 1998; Leinhardt, Zaslavsky & Stein, 1990; Monk, 1992). Even high performing university students have been observed manipulating symbols without meaning and constructing graphs by applying memorized rules and procedures (Carlson, 1998). Other studies have reported that students' function conceptions appear to have a strong influence on a student's ability to move fluidly among the various representations of functions (Breidenbach, Dubinsky, Hawks & Nichols, 1992). A view of function as a 'process that accepts input and produces output' is essential for the development of a mature function conception

(Breidenbach, Dubinsky, Hawks & Nichols, 1992; McGowen, 2000). This view has also been shown to be foundational for coordinating images of two variables changing in tandem and for attending to the ways in which they change in relation to each other (Carlson, 1998; Monk, 1992; Kaput, 1994; Nemirovsky, 1996; Thompson, 1996),

Undergraduate students have been reported to have difficulty modeling functional relationships of situations involving the rate of change of one variable as it continuously varies in a dependency relationship with another variable (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Cottrill, 1996; Monk & Nemirovsky, 1994; Thompson, 1994a). Curricular materials are not broadly available to promote the longitudinal development of these abilities. However, research has shown that the ability to coordinate two variables as they change in tandem is not only essential for interpreting models of covarying relationships (Kaput, 1994), but it is also foundational for understanding major concepts of calculus (Carlson, Larsen & Jacobs, 2001; Kaput, 1994; Thompson, 1994a; Zandieh, 2000).

Covariational reasoning refers to the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other (Carlson et al., 2002). This reasoning ability is needed to interpret and represent covarying relationships. For the purpose of this chapter the term *physical enactment* refers to the physical movement of one's body and the physical movement of an object (e.g., walking, pulling a ladder away from a wall, moving two cars on a track). Me*ntal enactment* refers to dynamic imagery that involves imagining the transformation of a system (imagining physical movement of one's body, thinking about a ladder falling down a wall, imagining a bottle being filled with water).

In reflecting on trends in my own research I have observed that university students have exhibited better reasoning patterns when performing a physical enactment of an event or system (Carlson et al., 2002). When working within a physical representational context, undergraduate students have more consistently engaged in meaningful mathematical discussions, and have demonstrated more effective reasoning patterns to analyze patterns of change. The body of this paper elaborates on this claim.

The paper provides a brief description of a framework for describing and analyzing covariational reasoning. This description is followed by findings that support the claim that the use of physical enactment may be useful for promoting covariational reasoning and building internal structures to support meaningful constructions in other representational contexts. A curricular intervention designed to promote covariational reasoning in first semester calculus students is described and select results from implementing this curriculum are provided.

The Covariation Framework

Students' covariational reasoning abilities are important for interpreting and representing the changing nature of dynamic function events. Examples of covariational reasoning include mental actions of sliding a tic mark back and forth on the *x*-axis while concurrently forming an image of the varying magnitude of the height of the graph; it also involves thinking about changing amounts of water in a bottle while concurrently constructing an image of the varying magnitude of the height of the water in the bottle. The five mental actions of the covariation framework are described and select behaviors that have been associated with each mental action are provided (Table 1).

Table 1
The Mental Actions of the Covariation Framework

Mental Action	***Description of Mental Action***	***Behaviors***
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• constructing an increasing straight line • verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• plotting points/constructing secant lines • verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of- change of the function with uniform increments of change in the input variable.	• constructing contiguous secant lines for the domain • verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function	• constructing a smooth curve with clear indications of concavity changes • verbalizing an awareness of the instantaneous changes in the rate-of-change for the entire domain of the function (direction of concavities and inflection points are correct)

A close look at students' covariational reasoning in the context of a graph reveals that students who exhibit behaviors supported by MA1 typically recognize that the value of the *y*-coordinate changes with the changing *x*-coordinate. (Typically, the *x*-coordinate plays the role of the independent variable, although we have observed students treating the *y*-coordinate as the independent variable). This initial coordination of the variables is commonly revealed by a student labeling the coordinate axes of the graph, followed by utterances that suggest that students are able to recognize that as one variable changes the other variable changes. Attention to the direction of change (in the case of an increasing function) involves the formation of an image of the *y*-values getting higher as the graph moves from left to right (MA2). The common behavior displayed by students at this level has been the construction of a line that rises as one moves to the right on the graph or utterances that suggest an understanding of the direction of change of the output variable while considering increases in the input variable (e.g., as more water is added the height goes up). MA3 involves the coordination of the relative magnitudes of change in the *x* and *y* variables. In this context, students have been observed partitioning the *x*-axis into intervals of a fixed lengths (e.g., x_1, x_2, x_3, x_4), while considering the amount of change in the output for each new interval of *x*. This behavior has been commonly followed by the student constructing points on the graph (the points are viewed as representing amounts of change of the output while considering equal amounts of the input) and is followed by the construction of lines to connect these points. Activity at the rate level involves recognition that the amount of change of the output variable with respect to uniform increments of the input variable expresses the rate of change of the function for an interval of the function's domain. This is typically revealed by the student sketching secant lines on a graph or by carrying out the mental computation or estimation of the slope of a graph over small intervals of the domain (the sketching of these lines would result from the student imagining and adjusting slopes for different intervals of the domain). It is noteworthy that mental actions 3 and 4 may both result in the construction of secant lines; however, the type of reasoning that produces these constructions is different (i.e., MA 3 focuses on the *amount* of change of the output (height) while considering changes in the input; and MA4 focuses on the *rate of change* of the output with respect to the input for uniform increments of the input). Attention to continuously changing *instantaneous rate* (MA5) is revealed by the construction of an accurate curve and includes an understanding of the changing nature of the instantaneous rate of change for the entire domain, and an understanding that the instantaneous rate of change resulted from examining smaller and smaller intervals of the domain. This images supports the understanding that a concave up graph conveys where the "rate of change" is increasing and the inflection point relates to the point on the graph where the "rate of change" changes from increasing to decreasing, or decreasing to increasing.

The mental actions of the covariation framework provide a means of classifying behaviors that are exhibited as students engage in covariation tasks; however, an individual's covariational reasoning ability relative to a particular task can only be determined by examining the collection of behaviors and mental actions that were exhibited while responding to that task. A student is given a *Level Classification* according to the overall image that appears to support the various mental actions exhibited in the context of a problem or task. The Covariation Framework contains five distinct levels (Carlson et al., 2002). A student who is classified as exhibiting Level 5 (i.e., Instantaneous Rate Level) covariational reasoning, relative to a specific task, exhibits behaviors indicative of mental action 5. He/she is able to coordinate images of the continuously changing rate while imagining the input variable changing continuously. He/she also demonstrates the ability to unpack that mental action to reason in terms of the other mental actions of the framework; as needed, the student is able to describe the changing nature of the event in terms of MA3 and MA4 (note that MA3 included MA1 and MA2). This image of covariation (i.e., reasoning Level 5) supports behaviors that demonstrate the student understands that the instantaneous rate-of-change resulted from smaller and smaller refinements of the average rate-of-change and that an inflection point is where the rate-of-change changes from increasing to decreasing, or decreasing to increasing. For additional description of the Covariation Framework, see Carlson et al. (2002).

Findings from previous covariation studies

Observed trends from my previous studies (Carlson, 1998; Carlson et al. 2002) suggest that successful calculus students have difficulty constructing images of continuously changing rate and are generally unable to represent and interpret "increasing rate" and "decreasing rate" for a dynamically changing situation (MA5). However, when performing a physical enactment to transform the system (i.e., examine the different states of the system or situation), these students were more often able to observe patterns in both the changing magnitude of the output variable

(MA3) and the changing nature of the instantaneous rate (MA5). Nonetheless, their difficulty in unpacking MA5 to explain instantaneous rate as the result of imagining smaller and smaller refinements of the average rate-of-change persisted. This limitation (an inability to unpack MA5) appeared to create difficulties for them in accurately interpreting and understanding graphical function information; in particular in explaining the meaning of an inflection point and in providing a rationale for the construction of a smooth curve. Their difficulties were also found to be more prevalent when attempting to represent covarying relationships using formula.

Results from the Race-Track Problem

As an illustration, when first semester calculus students were asked to provide a response to the Race-Track Problem (Figure 1) (not previously reported) most provided an increasing straight line, while fewer than 5% of the students completing traditional calculus were able to construct an accurate graph. Even when probed to attend to the ‘shortest distance from the starting line’ students appeared either unwilling or unable to graphcially represent the changing nature of the distance (of the runner from the starting line) while imagining the passage of time.

> *The Race-Track Problem*
> Construct a rough sketch of a graph of the shortest distance between Joni and the starting line (as a function of time) for a half-mile race that Joni ran on a quarter-mile oval race-track (see diagram of track below). Due to Joni’s excellent conditioning and her running start, she was able to maintain a constant speed during the entire race.

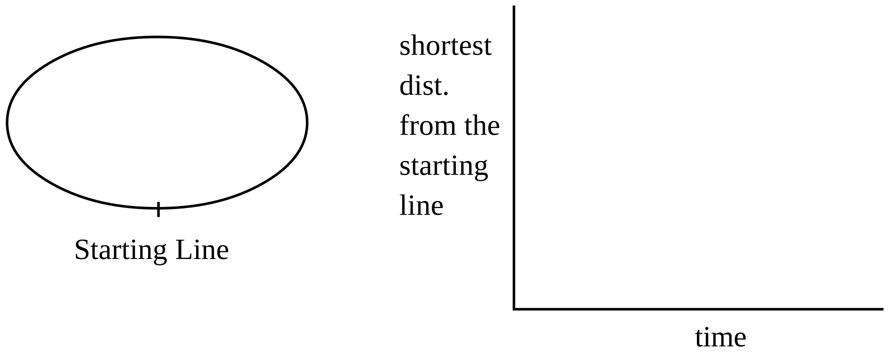

Figure 1.

During follow-up interviews, students were generally able to discuss the changing position of the runner around the track and the changing nature of the 'shortest distance' of the runner from the starting line. However, when asked to represent this information in the context of a graph, most students made very little progress. Similar results have also been reported for other covariation tasks (e.g., the bottle problem, the sliding ladder problem) (Carlson, 1998; Carlson et al. 2002) for these same populations of undergraduate students.

Results from the Ladder Problem

When confronted with a scenario of a ladder leaning against a wall in an upright position and asked to determine the speed of the top of the ladder as the bottom is pulled away from the wall, 10 out of 16 high performing second semester calculus students indicated that the speed of the top of the ladder was increasing. When conducting follow-up interviews, the successful students frequently devised a model, and used this representational context to experiment with the behavior of the ladder. When prompted to explain how they attained their answers, students expressed a wide range of responses, with most indicating that they had observed the varying amount or speed by which the top of the ladder was dropping while moving the bottom of the ladder away from the wall (MA5). These responses suggest that students were able to coordinate images of two variables changing in tandem; however they appeared unable to apply this reasoning in the context of a graph or formula. One student leaned his pencil against a book and repeatedly pulled the pencil out at the bottom by equal increments until it fell to the desk. Another student placed her pen against her hand, as if she was letting her hand represent the wall, and then appeared to simulate the motion of the ladder by pulling the base of her pen away from her hand at a constant rate. When further prompted to explain their solutions, these students frequently attempted to relate their intuitions to an algebraic representation of the situation. However, most of these attempts were abandoned. They appeared unable to construct a meaningful formula or accurate graph of the situation. In fact one student initially indicated that he could see the top of the ladder moving faster as it fell to the floor, but when he attempted to justify his intuition using the Pythagorean relationship, he was unable to determine how to associate the variables with the model he had constructed.

Curricular interventions

Building on the insights from these past studies, I have devised curricular activities that focus on developing students' informal intuitions and using these intuitions to describe and represent the changing nature of covarying relationships in the contexts of both graphs and formulae. Initial tasks engage students in physical enactment and include prompts that promote the use of mental actions described in the *Covariation Framework*. Follow-up tasks have been designed to promote transfer of the reasoning patterns exhibited during physical enactment to other representational contexts. The sequence of tasks and activities attempt to development well-formed internal structures by exploring the subtleties in covarying relationships, first by physical enactment of a system, then in other representational contexts (e.g., graphs, tables, formulae).

These curricular materials were devised, then piloted and refined twice, and will no doubt undergo several more refinements prior to dissemination. They consist of a module for major conceptual strands of first semester calculus (e.g., function, covariation, limit, derivative, accumulation). Whenever possible, the modules' initial activities include prompts for students to model in real-time various dynamic events. Specific prompts are included to both assist students in overcoming the conceptual obstacles that have been previously reported (Kaput, 1992; Thompson, 1994; Monk & Nemirovsky, 1994; Carlson, 1998; Saldanha & Thompson, 1998) and to engage in the reasoning patterns described in the Covariation Framework.

One of the module's initial worksheets containes 8 different graphs with prompts to describe a walking strategy to create each graph. Students initially used motion detectors to attempt to move in patterns to generate a particular graph. Once each student successfully moved in a pattern to generate (with the real time construction of the graph displayed on a calculator screen) the distance/time graph on the worksheet, each group negotiated one written response to both explain their walking strategy and to justify why their strategy worked. Additional prompts were included to assess their ability to connect the changing rate of the graph with the motion they had walked, with particular emphasis placed on their ability to represent and interpret changing rate-of-change information across a function's domain.(e.g., Explain why the generated graph is smooth; In your own words, explain what this graph conveys about the changing rate of this situation; Discuss the nature of the changing shape of the graph in the context of the dynamic event). When observing students attempt to walk the graph, many students initially moved in patterns that were inconsistent with the motion needed to model the distance/time graph. However, each student was encouraged to continue making adjustments in their motion until he/she was successful in modeling each of the graphs. (Observations of students while making these adjustments suggest that this activity was useful for assisting students in confronting their misconceptions and making immediate adjustments to some of their fundamental ideas about rate-of-change). Each student was subsequently required to complete a take home activity that included prompts to reinforce the in-class worksheet. The final task (also completed individually at home) (see Motion Activity) was designed using the Five Principles of Model Eliciting Activities (as defined by Lesh, 2001). These activities were designed to encourage students to make sense of meaningful situations, and to invent, extend, and refine their own mathematical constructs. The resultant student products reveal students' thinking and provide both teachers and researchers with a powerful lens for viewing students' reasoning and concept development.

Motion Activity

Your physics professor has announced that you will have an exam about motion. She has indicated that you will be required to use a motion detector to reproduce by walking a collection of distance-time graphs and has asked that you produce a strategy guide in advance to assist you in quickly producing each of the graphs. Each strategy should also include a brief explanation of why the strategy works. Please type your strategy guide (make sure it is carefully written to avoid confusion during the exam).

Preliminary observations

Prior to completing the modules each student was given a written pretest assessment to determine the student's function knowledge, with a primary emphasis on assessing the student's ability to apply covariational reasoning in the context of a dynamic function event. Each group of students were video taped while attempting to model the graphs, and when completing select written group activities during class. Post-instruction written assessments were administered to all students. Follow-up clinical interviews were conducted with 7 students that were somewhat representative of the diverse understandings exhibited by the class.

Analysis of the pre-instruction data revealed that the overwhelming majority of the students initially had weak function knowledge and impoverished covariational reasoning abilities. They exhibited many of the same weaknesses that have been reported in other investigations of student's function knowledge (e.g, inability to view a function as a process that accepts input and produces output, weak covariational reasoning abilities, difficulties using and understanding algebraic symbols) (Carlson, 1998, Carlson & Larsen, 2002, Carlson et al., 2002).

As revealed by the students' responses to the written activities and post-instruction interviews, positive shifts have occurred in these first-semester calculus students' ability to utilize the reasoning patterns exhibited in the covariation framework. Further, it appears that the use of dynamic imagery to transform the system was useful for promoting effective reasoning in these students.

Race Track Results

The Race-Track Problem (See Figure 1) was administered for the first time as a post-instruction task, with follow-up interviews conducted with seven students. 73% of the students (19 of 26) provided a mostly correct response to this task. The follow-up interviews with 7 students revealed that five of these interview subjects provided responses that revealed strong covariational reasoning abilities. Of the two students who had not provided a correct response on the written post-test, one quickly overcame his block and proceeded to use covariational reasoning to think through the solution. The remaining student provided discussions indicating the construction of a coordinated image of the distance changing, while imagining the passage of time and the runner moving around the track. However, he had not interpreted the question accurately. His increasing straight line was justified by the interpretation that he was being asked to represent, not the shortest distance from the starting line, but the shortest distance from the starting line as the runner moved in one direction away from the start of the race.

The reasoning used by students who provided a correct response revealed that two of these students appeared to be thinking about the changing length of the shortest distance (the line that they constructed). By physically manipulating the system, they were able to observe the changing distance of the runner from the start. They transformed the system by constructing lines from the start to various points along the track and proceeded to sweep one end of the line around the track, as if they were simulating the runner moving around the track. Concurrently, the students engaged in discussions of the relative amounts by which the distance from the start changed, while the runner was at various positions on the track. The other three students who provided a correct response, without hesitation, indicated that the rate-of-change of the distance (length of the line) would increase, first at a constant rate; then as the runner moved into the first curve, the rate of change of the length would slow down even though the length of the line would continue to increase. According to Jeff, " it would slow down as the runner moved around the first curve, while it would still be increasing…then when the runner comes out of the first curve the length would begin to get smaller and smaller until it reached the shortest distance which is directly across from the starting line." Jeff appeared to be engaging in mental action 5 as he explained the rationale for the construction of his correct graph. He appeared to coordinate, in a continuous fashion, the rate of change of the line, as he repositioned the line that represented the distance of the runner from the starting line (the line was positioned with one end at the start and the other end repositioned around the track, as if to represent the runner moving around the track in a continuous fashion).

In the context of The Bottle Problem (i.e., students were asked to construct the graph of height as a function of volume for a spherical bottle with a cylindrical neck), most students fluently discussed the reasoning they used to construct their graph and provided responses that suggested their imagining the instantaneous rate-of-change of the height (with respect to volume) while imagining the volume changing continuously. In addition, these students' explanations of the reasoning they used to determine the shape of their graphs (how they knew to construct an inflection point, why they constructed a smooth curve) were encouraging.

The early analysis has revealed that most students emerged from this instructional unit with strong covariational reasoning abilities. These students were better able to construct and represent images of the "slope/rate" and "changing slope/changing rate" while imagining continuous change in the domain and had little difficulty constructing a graphical representation of various covarying situations.

It appears that physical enactment was a powerful representational tool for promoting covariational reasoning and a useful foundational context for developing internal conceptual structures. The high level of interest and persistent efforts in 'sense making' exhibited by students when responding to these tasks appeared to contribute to the number of connections and strength of the internal structure that emerged. Investigations of this claim and refinement of the curricular activities are ongoing.

Concluding remarks

Pressmeg (1997) has suggested that mathematical generalization may be facilitated by the use of dynamic imagery in which the dynamic images become the bearer of abstract information. Thompson (this volume) has also advocated that an individual's actions are both predicated on and textured by current understandings and imagery. My investigations support these claims. As students become comfortable in creating mental images of a system and subsequently engage in mental actions to transform the physical objects of the system, they begin to make generalizations about the subtle behaviors of the system. Further, I suggest that the use of physical enactment may provide a powerful context for promoting the use of dynamic imagery. As students learn to manipulate dynamic systems, both physically and mentally, they appear to be more comfortable in using dynamic imagery to represent the system in more abstract representational contexts such as graphs and formulae.

The Covariation Framework has proved to be useful for guiding the development of curricular activities. It defines the mental imagery needed to effectively transform a system involving two covarying quantities. This information has been shown to be valuable both for informing the development of curriculum and as a lens for interpreting and describing students' reasoning patterns.

It is my view that the mental actions involved in applying covariational reasoning are characteristic of *transformational reasoning* as described by Simon (1996). I claim that the "actions of coordinating the change in one variable with changes in the other variable, while attending to how they change in relation to each other" involves a mental enactment of the operation of coordinating on two objects (these objects are different dependent upon the mental action in the framework). The mental actions are examples of transformational reproductive images (i.e., the problem solver is able to visualize the transformation resulting from an operator). When engaging in mental action 3 (MA3) the student is able to visualize the transformation of a dynamic situation (e.g., a ladder falling down the wall, a bottle filling with water) by performing a mental enactment of coordinating two objects (the amount of change in one variable with an amount of change in another variable); while mental action 5 involves a mental enactment of coordinating the instantaneous rate-of-change in one variable with changes in the other variable. In both cases the mental enactment on the objects results in a transformation of the system (e.g., the ladder is envisioned as falling, the bottle is envisioned as filling with water). Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or continuum of states are generated"(p. 201). It is this ability that appears to promote the development of important reasoning patterns in students.

It is my claim that meaningful interpretation and representation of covarying relationships is reliant on the existence internal cognitive structures that support the dynamic mental enactment required for transforming a system. And further, meaningful engagement of students in tasks involving physical enactment (movement of one's body or other objects) is an effective way to initiate the development of these internal cognitive structures. Although some may claim that involvement in physical enactment is not necessary for all students, it is my claim that most (and maybe all) students benefit from engaging in tasks that have this component. Engaging students in meaningful tasks, that include prompts to promote the exploration of the system, provides unique opportunities for students to confront and overcome conceptual obstacles that are inherent in learning a concept. It is also my claim that the experiential nature of physical enactment has unique sensual and emotional qualities that promotes the development of internal structures with many entry points and strong internal connections. Although many have reported that knowledge transfer is not immediate, it is my claim that students are much more likely to spontaneously and/or subconsciously draw on internal structures that were acquired by, or linked to, some experiential event.

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PART II

Implications of the Shift from Isolated, Expensive Technology to Connected, Inexpensive, Diverse and Ubiquitous Technologies

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ABSTRACT. We examine the long term history of the development of fundamental representational infrastructures such as writing and algebra, and how they were physically implemented via such devices as the printing press and computers, in order to (1) gain insight into what is occurring today both in terms of representational infrastructure change and in physical embodiments, (2) obtain clues regarding what to do next, and (3) determine the kinds of questions that research will need to answer in the coming decade if we are to make optimal use of new diverse and connected classroom technologies.

Placing Current Changes in Historical Perspective: The Evolution of Representational Infrastructures and Their Material Technologies

I suggest that we need to have a sense of some of the major changes of the past in order to understand the technology-related transformations of the past few decades and their trajectory through this coming decade—especially those changes having to do with the representational infrastructures with which we think and communicate. I will briefly examine the evolution of writing systems and then the printing press, as well as glance briefly towards the histories of arithmetic and algebra, in order to gain a perspective on what is happening today. In no way are the historical abstracts below intended to be definitive or at all complete, since the formal study of these matters is well beyond the scope of this paper. Further, while

I will be drawing general parallels between the early development of basic representational infrastructures and the physical means by which they could be made available to wider populations, historical analogies are fraught with difficulties, especially in the details. But it is exactly in these complicating details where we can get a sense of what needs to be done to exploit the apparent representational advantages of the computational medium in mathematics education. In particular, we will, in the third part of this paper, examine some specific examples of new directions, and in the fourth part of the paper examine the kinds of open questions needing exploration in the coming few years.

The Evolution of Representational Infrastructures in Static Inert Media

The Evolution of Written Languages—Alphabetic, Phonetic Systems Across the past half-dozen or so millennia of human history since the gradual emergence of writing —the primary means by which humanity extended its biological mind— several major changes in representational infrastructure have occurred. Over several thousand years and in several societies in the Middle East, and at various later times in other places around the world (Woodard, 1996), writing began as an ideographic, non-phonetic system for expressing ideas. In the Middle East the information was often quantitative information (Schmandt-Besserat 1978; 1981; 1988; 1992; 1994) expressed for economic purposes. These latter systems, with roots in the impressing of physical tokens in clay, made large demands on human memory and interpretive skill, and hence were laboriously learned and used only by specialists —scribes. For example, approximately 15% of all the 100,000 existing cuneiform tablets were used to train scribes. This writing system used hundreds of non-phonetic symbols in a highly nonlinear and context-dependent way (Walker, 1987). Note also that the complex non-phonetic system and the lexical lists used to train scribes during the 3rd millennium B.C. remained essentially unchanged for more than 600 years—a hint that the conservative nature of education is not a recent development. Indeed, the time scale of evolution of writing systems is on the order of thousands of years, which suggests that the achievements did not come easily and were driven and constrained by many factors beyond the inventiveness of the scribes.

Over a period of almost 3000 years, the systems of writing in the Middle East, including Egyptian hieroglyphs, gradually evolved into more phonetic systems (although the early hieroglyphs were more pictorial than early cuneiform writing). This allowed the users to tap into the meaning-carrying and meaning-making resources of spoken language, which had evolved over the previous 300,000 years and hence had deep biological support in the muscular and neurophysiological structure of all normal humans (Deacon, 1997). However, there was never a direct map onto the sound-stream of speech, but rather a far more complex relationship that typically involved decoding the meaning of a symbol sequence prior to being able to specify the sounds associated with it (Davies, 1987). Similar evolutions occurred with many other writing systems across the world (Woodard, 1996).

So, while the move to a phonetic system improved expressiveness of the writing systems, it did not solve the problem of learnability. The solution, which appeared gradually, took the form of a small, efficient alphabet. From two to four thousand years ago, across the Middle East and Mediterranean basin, various Semitic scripts developed single-consonant syllabaries (none for vowel sounds) enabling increasingly consistent encoding of symbols for sounds that in turn carried the meanings expressed by spoken language. As happened elsewhere, these evolved over many hundreds of years in different Middle East locations to become the Arabic, Hebrew, Aramaic, and Phoenician alphabets, which provided genuinely phonetic mappings onto the sound stream—achieving a mapping of time onto space (Ong, 1982).

During the period between 700–1100 BC, the 22 consonants of the Phoenician alphabet were adopted by the Greeks, where they were extended to encode their Indo-European language by converting certain unused consonants to vowels, to take into account the sounds of their Indo European language, which gives more prominence to vowel sounds. So by reinterpreting certain Phoenician consonants as vowels, and adding three more consonants, the Greeks produced a version of the alphabet that, in various forms, we use in virtually all Western languages today. Hence, by using the sound-mapping rules of the language at hand, every written combination of these symbols can now be spoken aloud, whether or not it corresponds to the words of that language, and any idea that can be spoken in that language can be written down! As with the development of spoken language, this extraordinary achievement changed the nature of what it means to be human by changing cognition, culture and the societies in which writing ensued (Donald, 1991). As put by Goody & Watt (1968, p.9) and cited in Haas, (1996, p.11): “the notion of representing a sound by a graphic symbol is itself so stupefying a leap of the imagination that what is remarkable is not so much that it happened relatively late in human history, but that it ever happened at all” (Havelock, 1982).

Of course, there are logograms that carry meaning independent of speech sounds, including the combinations of Hindu-Arabic numerals that express numbers(whose pronunciation varies according to the phonetic language used), and various visual icons such as the circled picture of a cigarette crossed by a diagonal line that is internationally used as a "no smoking" sign. But these are at the boundary of an extraordinarily efficient symbol system that uses roughly two dozen alphabet marks with an amazing, almost infinite, range of expressiveness.

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However, it was not until writing was able to tap into human speech capability *efficiently* via a highly compact alphabetic system that writing could have a chance to become universal (although the coding/decoding processes are decidedly complex and change quite radically as learning occurs, e.g., Langer, 1986). It became a fundamental representational infrastructure, learnable by most humans by the age of eight or nine, if given the opportunity. And it changed the means by which humans constructed their world individually (Nelson, 1996) and culturally (e.g., Cole, 1997; Donald, 1991). Humans became able to communicate, build, and accumulate knowledge (and all that comes with knowledge—including power and control) across time and space.

The Next Step: Making the Representational Infrastructure Widely Available via the Printing Press—An Evolutionary, Not Revolutionary, Impact

In the West, the next 1500 years saw the availability of the writing infrastructure limited by two apparently linked factors, the physical scarcity of written materials due to the lack of inexpensive reproduction technology and social conditions that limited the availability of literacy instruction primarily to elite males—elite by virtue of belonging to the ruling class or by virtue of belonging to the academic (religious) class (Kaestle, 1985; Kaufer & Carley, 1992; Resnick & Resnick, 1977). The Kaufer & Carley study shows how the widely used argument for the "revolutionary" impact of the printing press provided by Eisenstein's massive and widely accepted historical account (Eisenstein, 1979) ignores the evolutionary nature of the change that actually seems to have occurred across the three hundred years following Gutenburg. For example, for the first 100 years the traditional manuscript/scribe structure existed in parallel to the printing system, and it was not until the much faster (by 1–2 orders of magnitude) steam-driven press was available during the Industrial Revolution two hundred years later that printed materials became widely available for the kinds of documents (e.g., newspapers and political statements) that would have large social impact. Indeed, as pointed out by Haas (1996), Eisenstein's own arguments and detailed historical accounts help make the argument that the cultural, social, and political changes that accompanied the spread of printed materials—especially in vernacular languages—were gradual and related to multiple factors beyond the innovation of moveable type (which actually was used by the Chinese, along with paper, at least 500 years earlier). Furthermore, as argued in detail by Clanchy (1979), many preconditions seemed to be necessary that involved shifts in government and business needs from oral to written modes, acceptance of written records of events, distribution and communication channels, availability of paper (as opposed to expensive parchment), and other factors.

Hence, the physical technology of the printing press *did* have profound effects on society, *did* make the democratization of literacy possible, *did* lead to a standardization of language, and *did* participate in the deep cultural changes of the Renaissance leading to Modernism as outlined by Eisenstein, but its effects were intimately linked to a variety of other changes, both prior to and following its advent in the 15th century, including changes in the printing technology itself (Kaufer & Carley, 1992).

One important distinction should be kept in mind—the distinction between a change in representational infrastructure, such as alphabetic writing, and a change in the material means by which that infrastructure can be embodied, such as the printing press and inexpensive paper—which participates in a different kind of infrastructure, a combined technological and social infrastructure.

Two Additional Representational Infrastructures—The Operative Symbol Systems of Arithmetic and Algebra

The histories of arithmetic and algebra are well-known, and we will not recount them here. See (Hoyles, Kaput, & Noss, in press) for a more detailed account. However, it is worth noting that the evolution of each was a lengthy process, covering thousands of years before the achievement of an efficient symbol system upon which a human could operate. Unlike written language, which supported the creation of fixed records in static, inert media, the placeholder system of arithmetic that stabilized in the 13th–14th centuries supported rule-based *actions* by an appropriately trained human upon the physical symbols that constitute quantitative operations on the numbers taken to be represented by those symbols. This system and the algorithms built on it, seems to be optimal in an evolutionary sense similar to the way the alphabetic phonetic writing systems seem to be optimal. Each has remained relatively stable for many centuries and has spread widely across the world. The arithmetic system, although initially a specialist's tool—for accounting purposes—came to be part of the general cultural tool-set as needs for numerical computation arose in Western societies. Interestingly, the early algorithms developed for accounting in the 14^{th}–15^{th} centuries and that appeared in the first arithmetic training books at that time have remained essentially unchanged to this day, and continue to dominate elementary school mathematics (Swetz, 1987).

Algebra began in the times of the Egyptians in the second millennium BC as evidenced in the famous Ahmes Papyrus by using available writing systems to express quantitative relationships, especially to "solve equations"—to determine unknown quantities based on given quantitative relationships. This is the so-called "rhetorical algebra" that continued to Diophantus' time in the 4^{th} century of the Christian era, when the process of abbreviation of natural language statements and the introduction of special symbols began to accelerate. Algebra written in this way is normally referred to as "syncopated algebra." By today's standards, achievement to that point was primitive, with little generalization of methods across cases and little theory to support generalization.

Then, in a slow, millennium-long struggle involving the co-evolution of underlying concepts of number, algebraic symbolism gradually freed itself from written language in order to support techniques that increasingly depended on working with the symbols themselves according to systematic rules of substitution and transformation—rather than the quantitative relations for which they stood. Just as the symbolism for numbers evolved to yield support for rule-based operations on symbols taken to denote numbers, where attention and mental operations guide actions on the *notations* rather than what they are assumed to refer to, the symbolism for quantitative relations likewise developed. Bruner (1973) refers to this as an "opaque" use of the notations rather than "transparent" use, where the actions are guided by reasoning about the entities to which the notations are assumed to refer. In effect, algebraic symbolism gradually freed itself from the (highly functional) ambiguities and general expressiveness of natural language in order that very general statements of quantitative relations could be very efficiently expressed.

However, the more important aspects of the new representational infrastructure are those that involve the rules, the syntax, for guiding operations on these expressions of generality. These emerged in the 17th century as the symbolism became more compact and standardized in the intense attempts to mathematize the natural world that reached such triumphant fruition in the "calculus" of Newton and Leibniz (more on this below). In the words of Bochner (1966, pp. 38–39):

> Not only was this algebra a characteristic of the century, but a certain feature of it, namely the "symbolization" inherent to it, became a profoundly distinguishing mark of all mathematics to follow. ... (T)his feature of algebra has become an attribute of the essence of mathematics, of its foundations, and of the nature of its abstractness on the uppermost level of the "ideation" a la Plato.

Beyond this first aspect of algebra, its role in the expression of abstraction and generalization, he also pointed out the critical new ingredient:

> ... that various types of 'equalities,' 'equivalences,' 'congruences,' 'homeomorphisms,' etc. between objects of mathematics must be discerned, and strictly adhered to. However this is not enough. In mathematics there is the second requirement that one must know how to 'operate' with mathematical objects, that is, to produce new objects out of given ones (p. 313).

Indeed, Mahoney (1980, p. 142) points out that this development made possible an entirely new mode of thought "characterized by the use of an operant symbolism, that is, a symbolism that not only abbreviates words but represents the workings of the combinatory operations, or, in other words, a symbolism with which one operates."

This second aspect of algebra, the syntactically guided transformation of symbols while holding in abeyance their potential interpretation, flowered in the 18th century, particularly in the hands of such masters as Euler, to generate powerful systems of understanding the world. But this operative aspect of algebra is both a source of mathematics' power, and a source of difficulty for learners. However, another learning-difficulty factor is the separation from natural language writing and hence the separation from the phonetic aspects of writing that support tapping into the many powerful narrative and acoustic memory features of natural language. Indeed, as well known via the error patterns seen in the "Student-Professors Problem", the algebraic system is in partial *conflict* with features of natural language (Clement, 1982; Kaput & Sims-Knight, 1983). For many good reasons, traditional character string-based algebra is *not* easy to learn.

Historical Analogies with Writing: Arithmetic and Algebra

Small, Elite Literacy Community. Thus, over an extremely long period, a new special-purpose operational representational infrastructure was developed that reached beyond the symbolic operational infrastructure for arithmetic. However, in contrast with the arithmetic system, *the algebra system was built by and for a small and specialized intellectual elite* at whose hands, quite literally, it extended the power of human understanding far beyond what was imaginable without it. Importantly, it was designed and used by specialists without regard for its learnability by the population at large. The effect of these learnability factors did not really become felt until the latter part of the 20th century when education systems around the world began to attempt to teach algebra to the general population. Prior to the middle of the 20th century, the algebra literacy community was quite small, quite analogous to the small literacy communities of specialists associated with early writing.

The Growth of Societal Need for Writing, Arithmetic and Now, Quantitative Insight. Early writing was a response to a social need, which grew over time to include broad expressiveness requirements to encode the rules and cultures of growing urban societies, which led to the phonetic system and eventually the alphabetic system. A similar growth occurred in the case of arithmetic, which was initially the province of specially trained "reckoners"—the accountants of the 15th and 16th centuries. (A needs-argument could be given for the initial development of the arithmetic system over the previous centuries.) But the need for arithmetic skill spread across the population over the next 200–300 year, which led it to become one of the core topics of universal schooling in most countries.

By the end of the 20th century, with the growth of the knowledge economies, the need for quantitative insight spread across the population of industrialized countries in a way analogous to the way it spread for arithmetic skill earlier. This general need combined with the politically driven need to democratize *opportunity* to learn higher mathematics, typically assumed to require knowledge and skill in algebra, has produced considerable tension in many democracies, especially the United States, where access to algebra learning has come to be seen as political right (Moses, 1995). The attempts to democratize access to traditional algebra in the Unites States have not been successful despite much work and energy. Algebra is in many ways analogous to early writing in its learnability and associated literacy community.

Deep Changes in the Nature of the Representational Infrastructure—Writing and Algebra Linking Into Existing Powerful Systems. However, just as writing gradually tapped into another, previously established human system of meaning making and communicating, and became radically reconstituted in the process as it became phonetic, algebra may likewise be on the verge of doing so. In this case, instead of the auditory-narrative system, it is the visuo-graphic system. Although it may not have been Descartes' (or Fermat's) intent, anticipated by Oresme (Clagett, 1968), he laid the base for tapping into humans' visual perceptual and cognitive capacities previously employed only by geometry. I am reminded of Joseph Lagrange's comment "As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company, they drew from each other fresh vitality and thenceforward marched on a rapid pace towards perfection," cited in Kline, (1953, p. 159). From 350 years ago to our contemporary graphs of quantitative relationships, I see an analogy to the gradual transition that occurred as writing became more phonetic—the newer ways coexist with the old in various combinations as we graph algebraic functions which are defined and input into our graphing systems via character strings. And more recently, we have been able to hot-link these in the computer medium.

Will we be able to make the next transition that might make the representation and manipulation of quantitative relationships broadly learnable? In the case of writing, this required the invention of the alphabet. Below, I shall propose an analogous step for algebra.

The Development of Material Means By Which Access to Learnable Representational Infrastructures Might Be Democratized—Analogy With the Printing Press. Anticipating the discussion below, I will suggest that, with the emergence of computers, we are involved in an extended process analogous to the evolution associated with the development of the printing press. The first stages involved expensive and hence rare central computers, what we have known as "mainframes" and "mini computers." Then came the microcomputer and networks—connectivity. More recently, we have seen the emergence of hand-helds and, even more recently, connectivity across device-types. For computation, we are heading towards the kind of ubiquitous access and full integration into life and work that was achieved by printed writing materials that eventually occurred by the time of the Industrial revolution. But, as our cursory examination of the printing press evolution suggests, the process will take time and will depend on many other changes taking place along the way.

The Birth of New Representational Infrastructures in Dynamic, Interactive Computational Media—the Case of Calculus

The Shift from Static, Inert Media to Dynamic, Interactive Media. The systems of knowledge that form the core of what was taught in schools and universities in the 20th century were built using some representational infrastructures that evolved (e.g., alphabetic and phonetic writing) and others that were somewhat more deliberately designed, mainly by and for a narrow intellectual elite (e.g., arithmetic and, to a greater extent, operative algebra). In all cases, they were instantiated in and hence subject to the constraints of the static, inert media of the previous several millennia. But, the *computational* medium is neither static nor inert, but rather, is dynamic and interactive, exploiting the great new advance of the 20th century, autonomously executable symbolic processes—that is, operations on symbol systems not requiring a human partner (Kaput & Shaffer, in press). The physical computational medium is based on a major physical innovation, the transistor, that, in turn, was the product of the prior knowledge system (Riorden & Hoddeson 1997). The longer term development of the computational medium is reviewed in Shaffer & Kaput (1999).

Reflecting the fact that we are in the midst of a huge historical transition, I see three profound types of consequences of the development of the new medium for carrying new representational infrastructures:

Type 1: The knowledge produced in static, inert media can become knowable and learnable in new ways by changing the medium in which the traditional notation systems in which it is carried are instantiated—for example, creating hot-links among dynamically changeable graphs equations and tables in mathematics. Most traditional uses of technology in mathematics education, especially graphing calculators and computers using Computer Algebra Systems, are of Type 1.

Type 2: New representational infrastructures become possible that enable the reconstitution of previously constructed knowledge through, for example, the new types of visually editable graphs and immediate connections between functions and simulations and/or physical data of the type developed and studied in the SimCalc Project—to be described briefly below.

Type 3: The construction of new systems of knowledge employing new representational infrastructures—for example, dynamical systems modeling or multi-agent modeling of Complex Systems with emergent

> behavior, each of which has multiple forms of notations and relationships with phenomena. This is a shift in the nature of mathematics and science towards the use of computationally intensive iterative and visual methods that enable entirely new forms of dynamical modeling of nonlinear and complex systems previously beyond the reach of classical analytic methods—a dramatic enlargement of the MCV that will continue in the new century (Kaput & Roschelle, 1998; Stewart, 1990).

Tracing any of these complex consequences is a challenging endeavor, particularly since they overlap in substantive ways due to the fact that knowledge is co-constituted by the means through which it is represented and used—it does not exist independently of its representation (Cobb, Yackel, & McClain, 2000). Hence, we will limit our discussion to a few cases close to our recent work in the SimCalc Project involving the Mathematics of Change & Variation (MCV), of which a subset concerns the ideas underlying Calculus. Thus we will be focusing on a Type 2 change.

Calculus

While the Greeks, most notably Archimedes—whose extraordinary computational ability compensated for the weaknesses of the available representational infrastructure in supporting quantitative computation—developed certain mainly geometric ideas and techniques, the Mathematics of Change and Variation leading to what came to be called "Calculus" evolved historically beginning with the work of the Scholastics in the 1300's through attempts to mathematize change in the world (reviewed in Kaput, 1994).

The resulting body of theory and technique that emerged in the 17^{th} and 18^{th} centuries, cleaned up for logical hygiene in the 19^{th}, is now institutionalized as a capstone course for secondary level students in many parts of the world, and especially in the United States. These ultimately successful mathematizing attempts were undertaken by the intellectual giants of Western civilization, who, in so doing, also developed the representational infrastructure of algebra, including extensions to infinite series and coordinate graphs, as part of the task. Their work led to profoundly powerful understandings of the different ways quantities can vary, how these differences in variation relate to the ways the quantities accumulate, and the fundamental connections between varying quantities and their accumulation. These efforts also gave rise to the eventual formalization of such basic mathematical ideas as function, series, limit, continuity, etc. (Boyer, 1959; Edwards, 1979).

Over the past two centuries this community's intellectual tools, methods and products—the foundations of the science and technology that we utterly depend upon—were institutionalized as the structure and core content of school and university curricula in most industrialized countries and taken as the epistemological essence of mathematics (Bochner, 1966; Mahoney, 1980) as noted above. This content has also been taken as the subject of computerization. That is, it has been the target of Type 1 reformulation.

As already noted, the SimCalc Project has been engaged instead in a Type 2 reformulation of the core content associated with Calculus, which we review briefly before turning to newer technological issues.

Summary of SimCalc Representational Infrastructure Changes

We summarize the core web of five representational innovations employed by the SimCalc Project, all of which require a computational medium for their realization. The fifth—not discussed below in detail—is mentioned for completeness, actually falls into Type 3. In order to save space, these innovations are illustrated in later examples, which will refer to this list. Cross-platform software, Java MathWorlds for desktop computers can be viewed and downloaded at

http://www.simcalc.umassd.edu

and software for hand-helds can be examined and downloaded from

http://www.simcalc.com.

- **Definition and direct manipulation of *graphically defined* functions, especially piecewise-defined functions**, with or without algebraic descriptions. Included is "Snap-to-Grid" control, whereby the allowed values can be constrained as needed—to integers, for example, allowing a new balance between complexity and computational tractability whereby key relationships traditionally requiring difficult prerequisites can be explored using whole number arithmetic and simple geometry. This allows sufficient variation to model interesting situations, avoid the degeneracy of constant rates of change, while postponing (but not ignoring!) the messiness and conceptual challenges of continuous change.
- **Direct connections between the above representational innovations and simulations**—especially motion simulations—to allow immediate construction and execution of a wide variety of variation phenomena, which puts phenomena at the center of the representation experience, reflecting the purposes for which traditional representations were designed initially, and enabling orders of

magnitude tightening of the feedback loop between model and phenomenon.

- **Direct, hot-linked connections between graphically editable functions and their derivatives or integrals**. Traditionally, connections between descriptions of rates of change (e.g., velocities) and accumulations (positions) are usually mediated through the algebraic symbol system as sequential procedures employing derivative and integral formulas—but need not be. In this way, the fundamental idea, expressed in the Fundamental Theorem of Calculus, is built into the representational infrastructure from the start, in a way analogous to how, for example, the hierarchical structure of the number system is built into the placeholder system.
- **Importing physical motion-data via MBL/CBL and re-enacting it in simulations, and exporting function-generated data to drive physical phenomena** LBM (Line Becomes Motion), which involves driving physical phenomena, including cars on tracks, using functions defined via the above methods as well as algebraically. Hence there is a two-way connection between phenomena and mathematical notations.
- **Use of hybrid physical/cybernetic devices embodying dynamical systems,** whose inner workings are visible and open to examination and control with rich feedback, and whose quantitative behavior is symbolized with real-time graphs generated on a computer screen.

The result of using this array of functionality, particularly in combination and over an extended period of time, is a qualitative transformation in the mathematical experience of change and variation. However, short term, in less than a minute, using either rate or totals descriptions of the quantities involved, or even a mix of them, a student as early as 6^{th}–8^{th} grade can construct and examine a variety of interesting change phenomena that relate to direct experience of daily phenomena. And in more extended investigations, newly intimate connections among physical, linguistic, kinesthetic, cognitive, and symbolic experience become possible.

Importantly, taken together, these are not merely a series of software functionalities and curriculum activities, but amount to a reconstitution of the key ideas. Hence we are not merely treating the underlying ideas of Calculus in a new way, treating them as the focus of school mathematics beginning in the early grades and rooting them in children's everyday experience, especially their kinesthetic experience, but *we are reformulating them in an epistemic way.* We continue to address such familiar fundamentals as variable rates of changing quantities, the accumulation of those quantities, the connections between rates and accumulations, and approximations, but they are experienced in profoundly different ways, and are related to each other in new ways.

These approaches are not intended to eliminate the need for eventual use of formal notations for some students, and perhaps some formal notations for all students. Rather, they are intended to provide a substantial mathematical experience for the 90% of students in the U.S. who do not have access to the Mathematics of Change & Variation (MCV), including the ideas underlying Calculus, and provide a conceptual foundation for the 5–10% of the population who need to learn more formal Calculus. Finally, these strategies are intended to lead into the mathematics of dynamical systems and its use in modeling nonlinear phenomena of the sort that is growing dramatically in importance in our new century (Cohen & Stewart, 1994; Hall, 1992; Kaput & Roschelle, 1998; Stewart, 1990).

In terms of our historical perspective, we see this current work as part of a large transition towards a much more broadly learnable mathematics of quantitative reasoning, where both the representational infrastructure is changing as well as the material means by which those more learnable infrastructures can be made widely available. Taken together, it may be that the kinds of representational innovations outlined above and illustrated in the next part of the paper constitute the development of a new "alphabet" for quantitative mathematics which might do for mathematical representation what the phonetic alphabet did for writing, particularly if coupled to the right kinds of physical implementations, which are examined below.

Affordances & Constraints of Diverse, Connected, and Ubiquitous Computing Devices

Revisiting the Analogy with Changes Made Possible by the Printing Press As we saw above, it is one thing to have a powerful and learnable representational infrastructure, such as alphabetic writing, and it is entirely another matter to have broad access to that infrastructure. We traced briefly the gradual three-century impact of the printing press on the democratization of literacy. It is a major hypothesis of this paper that a similar change is now underway in the 21st century relative to the new representational infrastructures made possible by the computational medium. We have traced three types of changes and introduced an example of one of them targeting the Math of Change & Variation (MCV) as developed in the SimCalc Project. Others could be put forward as well—for example, Dynamic Geometry™ or dynamically manipulable data management and analysis systems for statistics.

However, it is one thing to instantiate representational infrastructure innovations on expensive and hence scarce computational devices. It is entirely another to render them materially available on inexpensive, ubiquitous devices well-integrated into the flow of life, work, and education. A message from the history of the printing press is that the change needed to democratize access to the new infrastructures will be slow and will complexly involve many aspects of our culture. Of course, "slow" in today's terms is relative to rates of change that are at least an order of magnitude faster than in previous centuries, so that 300 years may shrink to 30 or less—but not likely to 3. The integration of automobiles, telephones, and television each took about a generation to reach wide penetration in industrialized societies. Penetration of the world wide web into everyday life and intellectual work seems to have taken only about half that time.

In today's school climate, full-sized, desktop computers are a relatively costly and rare commodity, compared to, say, pencils or notepads. Therefore, most schools that have computers share their availability across many different uses and populations. Furthermore, despite falling costs for a given level of power and functionality, their maintenance cost, especially in network configurations, tends to be prohibitive for most schools to deploy them on a wide-scale basis except in the wealthiest communities. The school mathematics alternative has been the graphing calculator, which has been designed primarily to support Type 1 changes. It has typically taken the form of a full, open toolkit, isolated technologically from other computational devices, and independent of any particular curriculum, which has been supplied offline. This condition is now changing.

Illustrations of Representational Infrastructure Innovations across Multiple But Non-Networked Hardware & Software Platforms

We will now explore some of the changes that are occurring in the nature and configurations of the technologies that can support the new representational infrastructures that computational media make possible. In particular, we see a rapid growth in the availability of flexible, relatively inexpensive, and wirelessly networkable hand-held devices that can run independently produced curriculum-targeted software and hence support new kinds of teaching and learning opportunities. We will first provide a series of activity-examples that simultaneously illustrate certain of the representational innovations identified above, how they map onto radically different hardware systems, and second, how they can be extended to operate in a networked classroom.

Determining Mean Values—SimCalc Representational Infrastructure Innovations 1–3 Above. Figure 1 shows the velocity graphs of two functions, respectively controlling one of the two elevators on the left of the figure (graphs on the desktop software are color-coded to match the elevator that they control). The downward-stepping, but positive, velocity function, which controls the left elevator, typically leads to a conflict with expectations, because most students associate it with a downward motion. However, by constructing it and observing the associated motion (often with many deliberate repetitions and variations), the conflicts lead to new and deeper understandings of both graphs and motion. The second, flat, constant-velocity function in Figure 1 that controls the elevator on the right provides constant velocity. It is shown in the midst of being adjusted to satisfy the constraint of "getting to the same floor at exactly the same time." This amounts to constructing the average velocity of the left-hand elevator which has the (step-wise) variable velocity. This in turn reduces to finding a constant velocity segment with the same area under it as does the staircase graph. In this case the total area is 15 and the number of seconds of the "trip" is 5, so the mean value is a whole number, namely, 3. We have "snap-to-grid" turned in this case so that, as dragging occurs, the pointer jumps from point to point in the discrete coordinate system. Note that if we had provided 6 steps for the left elevator instead of 5, the constraint of getting to the same floor at exactly the same time (from the same starting-floor) could not be satisfied with a whole number constant velocity, hence could not be reached with "snap-to-grid" turned on.

The standard Mean Value Theorem, of course, asserts that if a function is continuous over an interval, then its mean value will exist and will intersect that function in that interval. But, of course, the step-wise varying function is *not* continuous, and so the Mean value Theorem conclusion would fail—as it would if 6 steps were used. However, if we had used imported data from a student's physical motion, as in Figure 3, then her velocity would necessarily equal her average velocity at one or more times in the interval. We have developed activities involving a second student walking in parallel whose responsibility is to walk at an estimated average speed of her partner. Then the differences between same-velocity and same-position begin to become apparent. Additional activities involve the two students in importing their motion data into the computer (or calculator) *serially* (discussed below) and replaying them *simultaneously*, where the velocity-position distinction becomes even more apparent due to the availability of the respective velocity and position graphs alongside the cybernetically replayed motion.

Note how the dual perspectives of the velocity and position functions, both illustrated in Figure 1, show two different views of the average value situation. In the left-hand graph, we see the connection as a matter of equal areas under respective velocity graphs. In the right-hand graph, we see it through position graphs as a matter of getting to the same place at the same time, one with variable velocity and the other with constant velocity. Depending on the activity, of course, one or the other of the graphs might not be viewable or, if viewable, not editable. For example, another version of this activity involves giving the step-wise varying position function on the right and asking the student to construct its velocity-function mean value on the left. This makes the slope the key issue. By reversing the given and requested function types area becomes the key issue. Importantly, by building in the connections between rate (velocity) and totals (position) quantities throughout, the underlying idea of the Fundamental Theorem of Calculus is always at hand.

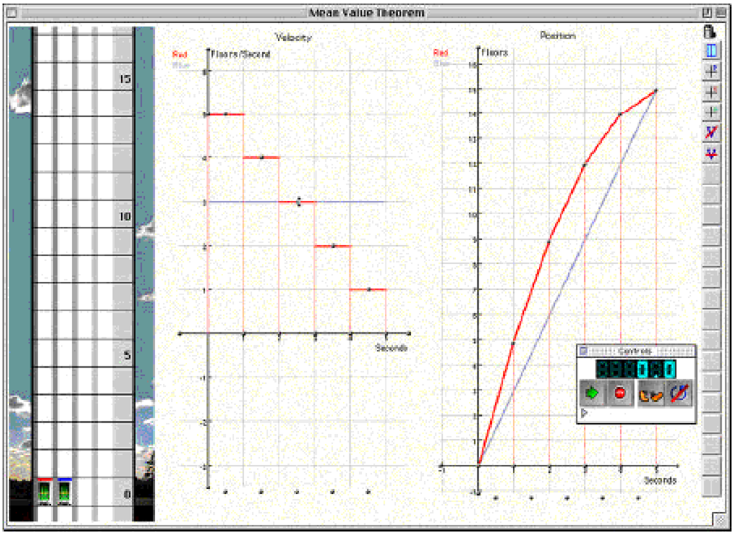

Figure 1. Averages from Both Velocity and Position Perspectives

Parallel Software and Curricula for a Graphing Calculator. Now, in the left two pictures of Figure 2 below are partially analogous software configurations—two elevators controlled by two velocity graphs. Instead of the clicking and drag/drop interface of the desktop software, most user interaction is through the SoftKeys that appear across the bottom of the screen which are controlled by the HardKeys immediately beneath them. The left-most screen depicts the Animation Mode, with two elevators on the left controlled respectively by the staircase and constant velocity functions to their right. The middle screen depicts the Function-Edit Mode, which shows a “HotSpot” on the constant-velocity graph. The user adjusts the height and extent of a graph segment via the four calculator cursor keys (not shown), and can add or delete segments via the SoftKeys. Other features allow the user to scale the graph and animation views, display labels, enter functions in text-input mode, generate time-position output data, and so on—very much in parallel with Java MathWorlds, but without the benefits of a direct-manipulation interface. The right-most screen shows a horizontal motion world with both position and velocity functions displayed (hot-linkable if needed, as with the computer software).

We have developed a full, document-oriented Flash ROM software system for the TI-83+ and a core set of activities embodying a common set of curriculum materials that parallels the computer software to the extent possible given the processing and screen constraints (96 by 64 pixels!). The parallelism is evident in the Calculator MathWorlds screens shown. We have also developed a prototype version of MathWorlds for the PalmPilot Operating System (see http://www.simcalc.umassd.edu to download it to a Palm device).

Figure 2. Calculator MathWorlds

Classroom/Homework Flexibility and Cognitive Flexibility. We are currently completing an Algebra and Pre-Calculus course for academically weak first year college students in which we are using a common set of activities that run on parallel versions of software for desktop computers and TI-83+ graphing calculators. The course takes place in a classroom with two computers per student and an overhead display panel for each kind of device. And each student also has a graphing calculator. Much of the classroom discussion uses the computers although some switching takes place—simply by exchanging the panel that sits on top of the overhead projector. Homework is usually assigned for the calculators, although frequently, the homework is an extension of what was begun in class on the computers. That is, the students might do the first few problems on the computer in class and then do the remaining ones with the graphing calculator at home. We imagine that there are likely to many situations where computers are only occasionally available, or where only the teacher has one, but where the students have access to hand-held devices. Hence parallel software *and* curricula can substantially expand the usability of curriculum-oriented technology in the classroom (as opposed to open tools, which likewise are useful and continue to be available in these dual-device scenarios).

Importantly, we have seen full student flexibility in switching between the two device types, despite the radically different interfaces. Careful analyses of students' discourse and gestural activity reveals that, when discussing problem solutions and difficulties, the language is primarily about the mathematical objects and relations rather than about the interface or device. Hence they refer to "the velocity graph" or "I need to increase the slope" or "I need to extend the domain" rather than, dragging a HotSpot, or pushing a certain key, etc. We have a suspicion, not justified at this point, that the crossing between interfaces may help in exposing the mathematical structure. After all, when a student only sees one device and one interface for working on a mathematical domain, we have every psychological reason to expect that, without reason to do otherwise, they will link their experience of that mathematics with the interface through which they learned it.

We have also found subtle perceptual carryovers from the computer to the calculator environments that may provide guidance on how to exploit the visual detail possible on the computer screen to compensate for limited screens of hand-helds. For example, despite the hard to read grid of the calculator screen, the students, who were sometimes presented activities using graph printouts based on the computer screens, seemed to treat the calculator screen as having visual attributes that were present only on the computer software. These kinds of potentially important phenomena need to be studied and documented in more detail, as do potential interference effects across the different environments.

Illustrations of Representational Infrastructure Innovations Across Multiple NETWORKED
Hardware and Software Platforms

Simple Pedagogical Supports—Doing Old Stuff Better. Increasingly rich interactions are possible as connectivity increases between a teacher's computer or hand-held device, and a classroom of hand-helds. For example, a teacher can download sets of "documents" for homework or quizzes, and more interestingly, the students can upload their solution-documents as well as other data, which can then be aggregated in a variety of ways on the teacher's computer. With easy data-flow, teachers can ask diagnostic questions to 10 groups of 3 students each, such as the following (imagine the right-most part of Figure 2 above with its graphs hidden):

> The top car starts at 0 meters/second and accelerates to 3 meters/second in 6 seconds. Send me a position function for the bottom car so that it's motion matches the top car's motion exactly.

The teacher then shows all ten graphs on the same axes and then runs them. Each group's function is alive on the screen, so the diagnostic question illuminates everyone's understanding. Furthermore, patterns become evident, e.g., several groups might create a constant slope position function. Then, of course, there's a natural follow-up question—can you get your group's car to the same endpoint at exactly the same time, but at constant velocity?

New Learning Opportunities—Doing Better Stuff By Investing Individuals in a Collective Object. For example, groups of students can act out or choreograph a collective motion, say a dance, collectively, and then sit down to plan the coordination of their individual motions as mathematical functions that they will produce on their hand-held. They then upload their individual synthetically defined functions to the teacher's computer where the independently produced motions are aggregated into a simultaneously executed dance to be viewed by the entire class as in Figure 3. This kind of activity can quite literally pull students towards a parameter-based description of their motions because the motions differ in a quantitatively systematic way.

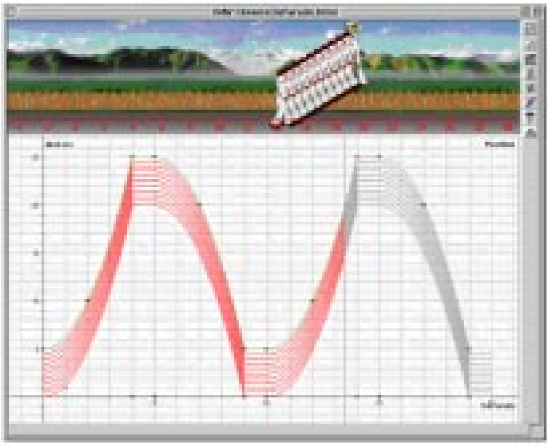

Figure 3. A Clown Parade—Staggered Initial Positions

Variations of this kind of aggregation activity can use CBL motion data input as well, where one character has its motion based on a student's actual physical motion imported into MathWorlds and where the other students create motions to participate in the parade. A wide variety of other aggregation and target activities is possible, for example, where each character's motion is imported from a serially produced dance. The kind of planning required for this kind of activity is exactly the kind of thinking that one wants in defining functions describing change. Another example involves building a "wave" action via delayed starting times as in Figure 4 where the dots hit the far "wall" at 18 meters and reflect backward. How could you make the same reflected wave motion with staggered starting positions?

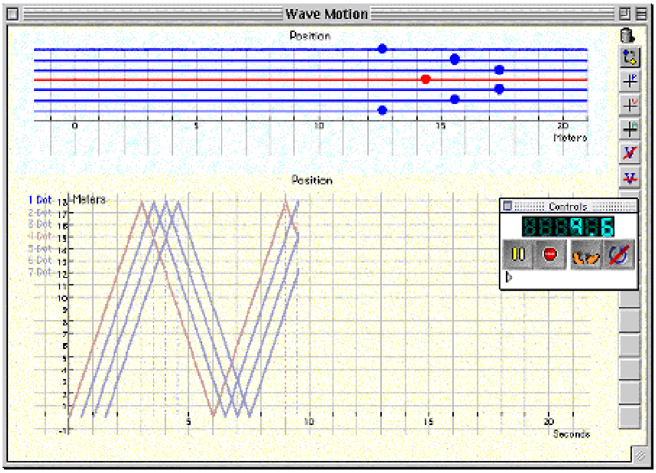

Figure 4. Wave—Staggered Starting Times

More Traditional Topics Using Participation in Shared Mathematical Objects. A standard student difficulty is in appreciating what it really means for a point to be on a line defined by an equation or other constraint. In the activity depicted in Figures 5 & 6, I "personalizes" this idea in a networked classroom where each student can send data to the teacher's computer (or calculator) as follows. First, students count-off in class, so each takes a number, which will serve as their x-coordinate. Then they are asked to make a point with their personal number as the x-coordinate but with whatever y-coordinate they wish, and send it up to the teacher's display. This results in the scattered points in Figure 5(a). Next, they are asked to make their y-coordinate *double their x-coordinate* and send this up. The result is the transformation depicted in Figure 5(b), where all the points assume their position on a line—which, of course, is the line we come to call "y = 2x". Naturally, any errors will show up as outliers.

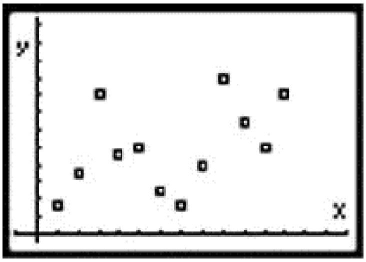

Figure 5(a).

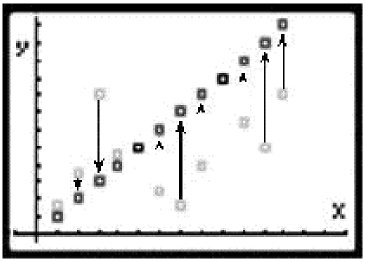

Figure 5(b).

The sequence is repeated in Figure 6, where a new scatter of points appears in (a) and then the students are asked to make the second coordinate of their point to be the square of half their number. Here, the order of halving and squaring is important, and anyone who squares first and then takes half will not lie on the line. Indeed, the issue of order of operations turns into the issue of the identity of the appropriate line. Then, when the line is finally determined, we discover that one person, Damien, who's number is 8, is on both lines! Why? This becomes a lead-in to the matter of simultaneous equations, where Damien satisfies both constraints—the double of her number is the same as half of her number, which is then squared. The fact that this is true only for her and nobody else (adopting zero and negative numbers comes a bit later) is a reflection of the fact that she and only she satisfies the equation x/2 = (x/2) 2. We can then ask, what could we do to the rule so that Jeri is on both lines? And how much algebra can be done in this kind of connected classroom?

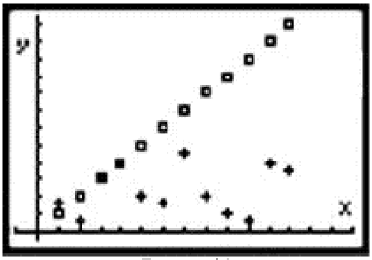

Figure 6(a).

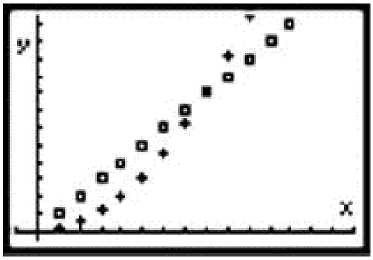

Figure 6(b).

We are currently examining, along with others, e.g., Stroup and Wilensky, how best to exploit the affordances of networked classrooms. The Participatory Simulations Project led by Stroup and Wilensky has been pursuing the opportunities for modeling emergent phenomena in dynamical systems, especially agent-based systems. For example, students can play the roles of predator or prey in a dynamic population model, or players in an economic model, and so on (Wilensky & Stroup, 2000). This work as well as that of another participatory simulations project based at the MIT Media Lab led by Resnick and Colella exploits parallel processing software that enables each participant to be represented as an independently controlled agent in the system.

Reflections on the Examples—Network-Based Activity Frameworks that Integrate Social Structure and Mathematical Structure

The above illustrations are only a small sliver of the possible range of activity structures in a networked classroom. Below is a further elaboration of the possibilities, limited due to space considerations.

Student-Teacher Activity Structures Using the Teacher's Computer as Publicly Viewable Common Ground. (A) Students create functions on their own device and publicly upload them to a shared, publicly displayed object on the teacher's computer for a variety of purposes, including:

- ♦ Students contribute to an emergent object, where the properties or identity of the object are not well understood beforehand, and where the determination of the properties and identity is at the heart of the learning opportunity as in the last example involving emergent lines and intersections.
- ♦ Students contribute to an object of their own design where the design work is at the heart of the learning opportunity as in the parade and marching band examples.

- ♦ Students contribute to an object where the identity and properties of the object are known in advance and where they act as the scaffolding for learning of something else, as might occur when the parade design and motions are well established, but where one half the class is defining the motion of their actors via position functions and the other half is using velocity functions. In this case the target is well-defined, and the heart of the learning opportunity is in defining the means by which the target is reached.

- ♦ Students upload survey or other data (e.g., probability trials) to a common data-set on the teacher's computer that is then aggregated and downloaded as a data object subject to further analysis by the students on their local devices. This could take place inside the classroom or, depending on connectivity, engage students elsewhere. Indeed, this is a possibility in almost all the activity structures, although some would be more convenient if the students were close-at-hand.

(B) Target Activities between teacher and students where students upload responses to classroom challenges, where challenges and responses are shown on the teacher's display.

- ♦ Define a function to fit this data, or an equation for this curve, or a polynomial that has these roots, etc. How many of these Gray Globs can you hit with one quadratic function? (Dugdale, 1982). Define a velocity (or position or acceleration) function to match this motion, … .

(C) Miscellaneous teacher-directed activities that will utilize the teacher's existing repertoire of classroom moves, e.g., pool solutions to an open ended problem and investigate solutions for generalities, optimality, etc.

Student Target Activities Between Students or Small Groups of Students. Essentially all of the whole-classroom activities have student-student analogs, either between single students working in pairs or, more likely, between small groups of students.

We have only had a brief glimpse into the learning opportunity space opened up by classroom connectivity. Time will be needed for the technology to be tuned to the possibilities and to the realities of such classrooms. But even more challenging is the need to understand how the social structures of the classroom and the mathematical structures can be made to interact in fruitful ways. At this point, experience is very limited, and we inherit a long tradition of three modes of activity: (1) Teacher-centered classroom activity, (2) Small-group activity, and (3) Individual activity. In all three cases, the communication among participants is biased towards oral communication, typically of an indirect nature about the mathematical objects and relations being used or studied. The new connectivity obviously offers much more direct communication.

We are currently pursuing research that includes several private sector partners to examine the affordances and constraints of networked mathematics classrooms employing mixes of hardware and software platforms.

Looking Ahead—Research Questions and Agendas

Introduction: The Bigger Picture

After approximately a generation of growing computer use in the world of business, LAN and WAN connectivity *coupled with the integration of computation into all aspects of business practice* has paid off in surprising increases in economic productivity during the past decade, now approximately 4% annually in the U.S. And, of course, the connectivity embodied in the WWW has led to even more startling impacts on the world outside of education. Indeed, this wider connectivity has changed the conditions of innovation in ways that compound and accelerate change (Bollier, 2000). We are poised to begin a comparable application of connectivity in education. The missing ingredients are at-hand computation (see, for example, Becker, et al., 2000) and connectivity at the epicenter of teaching and learning, the classroom. But of course, as was the case in business, and as our brief analysis of the printing press suggested, many changes must take place across many different dimensions before a new representation infrastructure delivery technology can have full impact. The classroom connectivity ingredient can pay off only if coupled with the integration of computation into educational practice.

The illustrations in Part III focused primarily on student learning, so we need also to address what connectivity among diverse inexpensive computing devices might mean, not only for learning, but for teaching, classroom management and assessment—the broader ingredients of teaching practice. Each is a complex matter, and we cannot delve into detail here, but will simply offer some major issues needing investigation.

Research Issues and Opportunities in Assessment, Learning & Teaching: Three Opportunity Spaces

We need to understand how new configurations and applications of connected devices can support or perhaps impede potentially profound progress in three opportunity spaces at the communicative heart of mathematics education in real classrooms:

Diagnostic assessment and evaluation: We need to study how teachers can use connectivity and analytic tools to exploit what we know about student thinking and learning in order to actively diagnose and efficiently respond to student thinking on a regular basis in the classroom.

Student learning and activity structures: We need to study classroom affordances and constraints of new activity structures that exploit the ability of students to design and pass structured mathematical objects (e.g. functions) and representations (e.g. graphs) among themselves and to the teacher—as illustrated briefly above.

Teaching and the classroom management of information flow: We need to study the classroom management implications of wirelessly connected hand-helds, with particular attention to teacher-specific tools to help organize the flow of the vast amount of information available to them (e.g. what each student is doing on their individual hand-held), decide among alternative actions (e.g. send every student an identical graph vs. call students' attention to a projected graph on a central display), and set policies on network communication (e.g., Can students send each other text messages? Only within their group? Only to an assigned partner?)

Specific Kinds of Concrete Research Questions Needing Investigation

Across these opportunity spaces, we need to gather and analyze data addressing the following broad kinds of questions in ways that span technology-specifics.

- What uses of mathematical notations and representations, when shared across devices, lead to deep, intense or efficient content-oriented interaction and meaning-making among students and between teacher and students?
- Which characteristics of networked, hand-held devices (e.g. screen size, lack of color, availability of stylus, ease of beaming data, ability to move about the room) strongly enable or impede the ease, comfort, and effectiveness of mathematical conversations in the classroom?

- In what ways do networked, hand-held devices most strongly engage learners' cognitive strengths and solve practical, important teaching problems, or conversely, distract learners from the task at hand and impose new burdens on the teacher?

Importantly, given the novelty of these environments and technologies, we cannot trap ourselves into study of phenomena that will disappear when the technologies become better established. Hence we need to analyze how the answers to the above questions change as both the teachers' experience with connectivity, and the technologies themselves, mature. These research questions reflect the belief that hand-held, networked devices will not necessarily have a simple causal effect upon learning outcomes. As has been the case historically, introducing technology into schools will not necessarily change practice (Cuban, 1986), and it is likewise well appreciated that many technological tools simply reinforce existing practice (Marx, et al., 1998; Means, 1994). Similarly, as argued by Roschelle & Pea (1999) regarding WWW connectivity with resources outside the classroom, the promise may be ill-understood or near-sighted.

Hence, we need to begin building a framework that can adequately describe *uses* that are likely to distinguish effective from ineffective practice. Likewise, we cannot take for granted that manufacturers specifications (processor speed, communication speed, screen size) are the device characteristics that necessarily enable or impede use, and should seek to build a *conceptual analysis of device characteristics* that directly relates to observable classroom behaviors.

Finally, widespread impact from such devices is only likely if we identify the strongest ties to learner's strengths, solve difficult teaching problems, and introduce no serious *new* difficulties. Thus we need conceptual and analytic frameworks to clearly identify how these affordances and constraints play out in realistic classroom settings, and thereby to guide iterative design that magnifies the unique benefits and minimizes the newly introduced impediments. This will in turn inform the design of appropriate teacher development and support structures so that connectivity becomes a pedagogical support tool. As we begin to understand the classroom issues and technological issues we must immediately employ these understandings in the development of teacher development and support systems.

Last Words: Recognizing the Depth of the Changes that Are Underway

We are early in an exciting new era for technology in mathematics education. Both the representational infrastructures are changing and the physical means for implementing them are changing. We are seeing new alphabets emerging, new visual modalities of human experience are being engaged, and new physical devices are emerging—all at the same time. Much work needs to be done.

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Students' Use of Technological Tools to Construct Conceptual Systems in Mathematical Problem Solving

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ABSTRACT. To what extent the use of technology helps students develop powerful ways to represent and explore mathematical tasks? What type of mathematical features distinguish students' technological approaches from traditional ones? In this paper, we document what grade 10 students exhibited during a problem solving course that enhanced the use of technology. The set of activities used during the sessions illustrates that some of their representations attained via a technological tool favor mathematical processes that involve looking for patterns, making conjectures, providing arguments and communicating results.

Introduction

Students develop conceptual systems to make sense and interact with a variety of mathematical experiences (including mathematical problems). Those conceptual systems include a set of beliefs about the discipline and ways to access and use distinct mathematical resources and strategies to solve problems. Students express their conceptual systems via the use of particular models which involve the use of representations, notations, and distinct lines of mathematical thinking (Lesh, in press). The framework proposed by the NCTM (2000) emphasizes the importance of organizing students' learning activities around lines of mathematical content (numbers, geometry, algebra, measurement, probability and data analysis) and processes that appear in the practice of doing mathematics (problem solving,

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reasoning, communication, connections, and representations). It is also recognized that the use of calculators and computers is important to promote features of mathematical thinking which are consistent with the practice of the discipline.

> Electronic technologies –calculators and computers- are essential tools for teaching, learning, and doing mathematics. They furnish visual image of mathematics ideas, they facilitate organizing and analyzing data, and they compute efficiently and accurately. They can support investigation by students in every area of mathematics… When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving (NCTM, 2000, p.24).

When students are encouraged to utilize technology systematically in their learning experiences, it becomes important to document what type of methods and strategies or models appear as fundamental in their problem solving approaches. The term model is introduced to characterize ways in which students identify and employ ideas, concepts, representations, operations, and relationships to solve problems. So the construction of models is an ongoing process that involves constant exchange and refinements of students' ideas. Thus, the goal of instruction is that students construct powerful models to understand and solve problems. In particular, some questions that need to be discussed in terms of what students show in their learning experiences include: What type of mathematical thinking can be enhanced or constructed via the use of technology in students learning of the discipline? To what extent is students' thinking compatible or consistent with approaches based on paper and pencil? What features of mathematical proof are privileged via the use of technology? In this paper, we report examples of learning activities that were used in a problem-solving course with grade 10 students during one semester. Specifically, we are interested in characterizing the type of tasks used during the course and documenting what aspects of mathematical thinking emerged from implementing them during the development of the course. The calculator that students used was the TI92 and they also had access to use dynamic software (Cabri-Geometry) from a computer.

Background to the Course

We designed and organized a series of tasks around mathematical activities that involves (i) generalization and formalization of patterns, (ii) representation and examination of mathematical situations through the use of algebraic symbols, (iii) the study of functions that involves variation or change in different context, and (iv) the use of models to represent and analyze quantitative relationships. Kaput

(1999) recognizes that these types of activities are crucial in promoting students' algebraic reasoning and should be addressed throughout the mathematical curriculum.

Through the use of designed tasks and those proposed by students themselves, we encouraged students to both re-examine their previous knowledge with the lenses of technology and develop ways of thinking that might not be necessary present when dealing with task only via the use of paper and pencil. Another theme of interest that appeared during the development of the activities was that the use of technology helped students to search and explore connections among different ideas. For example, a task that initially might include the use of basic algebraic resources (slope, equation of a line, and coordinate system) could be transformed into a platform to discuss ideas of variation and display the power of other mathematical representations. During the implementation of the task, we followed a structure that includes several phases:

- **(i) Presentation of the tasks**. The instructor introduces the task and provides general information regarding the development of the class and students' participation.
- **(ii) Students' participations**. Three common students' interactions appear during the development of the sessions: (a) Students spend some time working individually, (b) later students are asked to work and share their ideas within a small group of three, and (c) each small group presents its ideas to the entire group.
- **(iii) Students' extension of the tasks**. Students are asked to search for extensions, connections, or changes of the original problem, and
- **(iv) Students' conclusions**. Students are asked to report in writing their approaches to the task and to identify themes and goals that appear during their interaction with the task.

It is also important to mention that students throughout the development of the sessions were encouraged to share their ideas, to propose questions, to listen to other students, and to communicate their approaches in writing and orally (Schoenfeld, 1998). An advantage in the use of calculator is that students could show their approaches to the entire class (using a viewscreen or computer projector) and share files with other students. Each task aimed at motivating students to reflect on various aspects mathematical practices and we observed that the actual implementation of some tasks favored the study of particular content and mathematical processes. Thus, we chose to present four examples that are representative of the collection of tasks that were implemented during one semester. We do not offer a detailed analysis of students' work, instead, we describe the tasks, illustrate their development and add some students' observations that emerged during the sessions.

Searching for invariants, Generalization and Formalization of Patterns. In this group, we characterize tasks or situations in which students examine particular cases to detect mathematical features associated with each task.

An exploration of π. Students were asked to construct a square and measure its perimeter and the segment that joins its center with one vertex. They calculate the ratio of the perimeter to length of the segment. What do you observe when the length of the side of the square changes? (Figure 1).

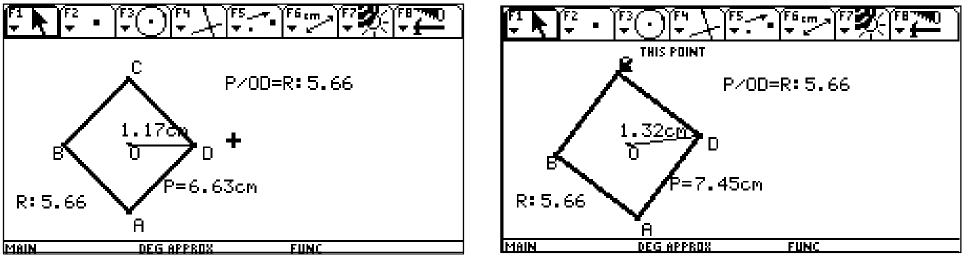


Figure 1. Examination of two particular cases.

Students observed that when they changed the length of the side of the square then the perimeter and segment OD changed but the ratio remains constant. Here, they were surprised that for any length of the side, the ratio between the perimeter and the segment that joins the center and one vertex was always the same number.

Students were asked to symbolize both the perimeter and the segment of a square with side **s**, (Figure 2).

$p = 4s$ $OD = \dfrac{\sqrt{s^2 + s^2}}{2} = \dfrac{\sqrt{2}s}{2}$ $\dfrac{P}{OD} = \dfrac{4s}{\frac{\sqrt{2}s}{2}} = \dfrac{8}{\sqrt{2}} \approx 5.66$	Some students explained that the value of $\dfrac{P}{OD}$ would not change because in the expression the symbol **s** was

Figure 2. The use of symbols.

What does it happen if we examine other regular polygons?

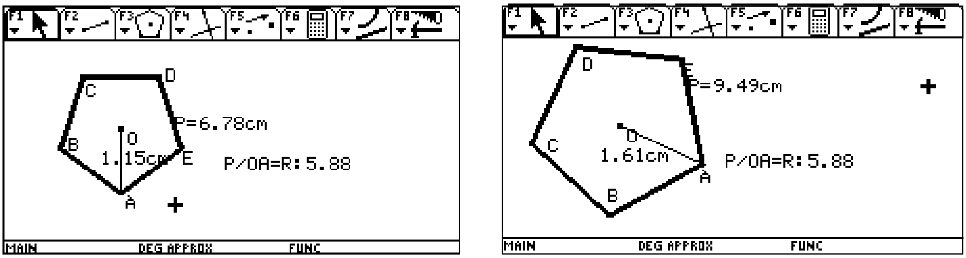


Figure 3. Increasing the number of sides.

They noticed that when the number of sides of the regular polygon increased the value of the ratio was closer to 2π. For example, they drew a regular polygon with 17 sides and observed that the ratio was 6.25 which gives an approximation of π = 3.12 (Figure 4).

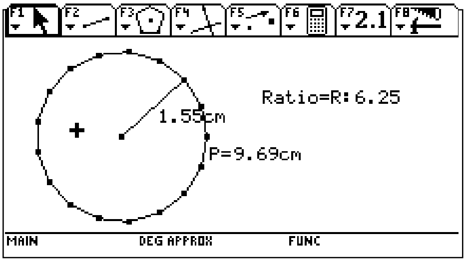


Figure 4. Polygon with 17 sides.

Other students drew a regular polygon with thirty sides and reported that the ratio was close to what they were told the value of π was, (Figure 5).

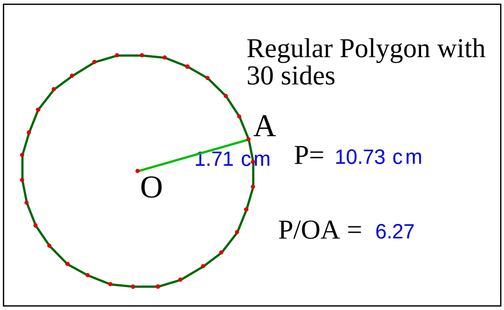


Figure 5. Polygon with 30 sides.

Making Connections:

The above task was part of the students' activities during two sessions of the course. Students were asked to identify in the literature expressions that approximate the value of π. A student brought into the class discussion an expression proposed by John Wallis (1616-1703).

$$\frac{\pi}{2}=\frac{2\times2\times4\times4\times6\times6\times8\times8\times\cdots}{1\times3\times3\times5\times5\times7\times7\times9\times9\times\cdots}$$

Based on this expression students were asked to examine the behavior of this product. They observed that when the number of terms in the product increases, the result shows an approximation of π (Figure 6).

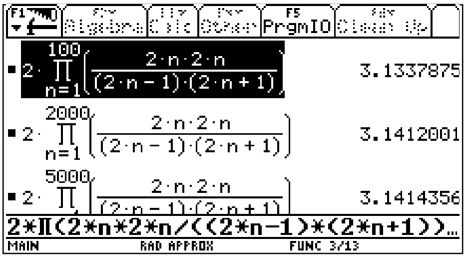


Figure 6. Increasing number of the terms in the expression.

Here, it is important to mention that in order to introduce Wallis' expression into the calculator they needed to transform it into a compact form. Thus, they identified patterns involved in the expression and ways to represent it. Similarly, students also had opportunity to examine other expressions to approximate the value of ***e*** as the next formula:

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots$$

Again a first important step in introducing this expression into the calculator was to write it in a condensed form. Students eventually wrote the expression as: $\sum_{k=0}^{\infty}\frac{1}{k!}$ And explored the behavior of partial sums (Figure 7).

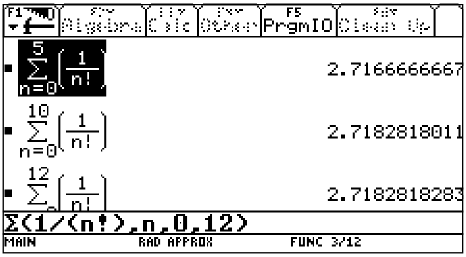


Figure 7. Exploration of the sum.

Mathematical themes and processes that appeared during the implementation of this task include the concept of perimeter, ratio, the idea of limit, notion of convergence, the use of symbols, looking for patterns and invariants and the use of basic properties to operate symbols. In addition, students realized that the constant π has particular meaning and also that observed the behavior of a formula to calculate it, via analyzing partial sums.

Representation and Examination of Mathematical Situations through the Use of Algebraic Symbols. An important problem solving strategy that students used in a particular series of tasks was to examine the behavior of situations via symbolic representation. Calculators functioned as a powerful tool in finding general expressions of phenomena that include certain type of regularity. For example students were asked to work on problems similar to the next examples:

(i) Determine a formula for the product $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)$. What is the sum of the series $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{n}{n\times(n+1)}$? A first goal for students, in this type of tasks, was to represent each expression in a condensed form. This process demands from students to observe the involved pattern and to transform, in this case, the product and sum into an expression that can be worked out with the calculator. The corresponding expressions and results are shown next. It is important to mention that students verified their formulae by calculating and comparing results from the original expressions and formulae of various values of *n* (Figure 8).

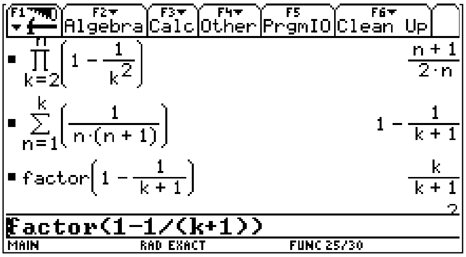


Figure 8. Finding a formula.

(ii) Another example in which it is possible to achieve a representation that is easy to analyze it in terms of criteria of divisibility is "show that the expression n^5-5n^3+4n is divisible by 120" for all positive integers n". Here, students used their calculator to factor the expression and think of an argument to support their response. For example, they observed that the factors of the expression were five consecutive numbers (Figure 9).

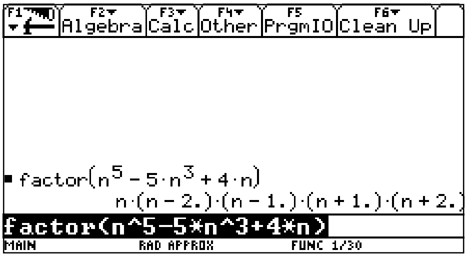


Figure 9. Searching for factors.

Another example in which students needed to provide an argument to support their response rather than reporting what they got via the calculator involves "prove that 999,991 is not a prime". Here, they observed that this number can be expressed as 1000,000 – 9 which is also written as (1000 - 3)(1000 + 3) = (997)(1003). Thus, they relied on using certain properties rather than reporting what they got on their calculators (Figure 10).

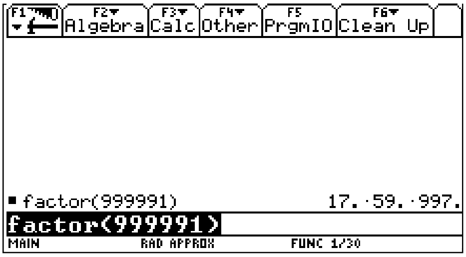


Figure 10. Direct factorization.

Modeling through the use of various representations. Figure 11 represents a fixed bar AB of 10 cm. A piece of string of 4cm is tied to A and on the other extreme to a movable pulley (the pulley can be moved along arc described y AQ. Other string of length equal to the fixed bar is tied to B and passes by the pulley and hangs a weight on its other extreme[2]. It is observed that when the pulley is moved along the arc described by AQ, the distance PR varies.

[2] This description corresponds to a problem proposed by l'Hôpital (1661-1704) to study motion and optimization.

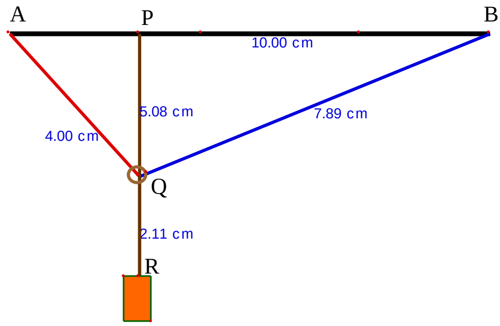
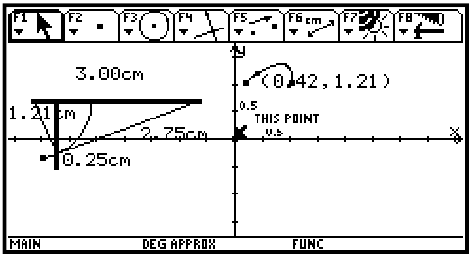


Figure 11. A fixed bar, a pulley and a string.

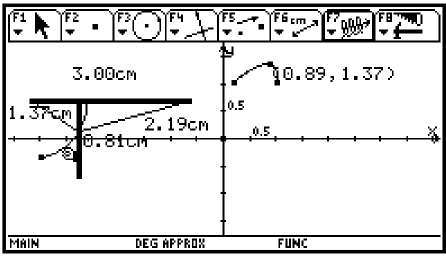
With the help of a calculator, students could model the situation and analyze the variation of segment PR when Q is moved along the arc formed by segment AQ (A is a fixed point).

Students identified key components associated with the situation and thought of ways to represent them in a dynamic form. In particular, they chose to observe the phenomenon and its corresponding graph simultaneously.



Students examined the representation of the situation and its corresponding graph. They observed that when the point moved along the arc the length of the segment varied.

Figure 12. Linking the situation to a graph.



They explained the variation of the segment in terms of approaching to one of the extreme of the fixed bar. They mention that the approximate maximum value appears when the distance between one of the fixed point and extreme of the segment is about 0.89 units and its length there is 1.37.

Figure 13. Quantification of variation.

Two key ideas appeared during the students' interaction with the task: the quantification of variation and the use of a graph representation to examine specific parameters. In addition, students compared different forms to represent it and discussed advantages and limitation of each approach.

The Study of Functions that Involves Variation or Changing Phenomena

An important component in using technology (dynamic software) is that some problems can be represented via a function and analyze its behavior directly without expressing the function explicitly. A task discussed during two sessions of the course was:

> In the figure, point P(5, 2) is a fixed point; a line passing by P intersects both axes, a triangle with vertices the origin and points of intersection of the line with axes is constructed. When moving the line a family of triangles will appear. Find the triangle with the minimum area? (Figure 14).

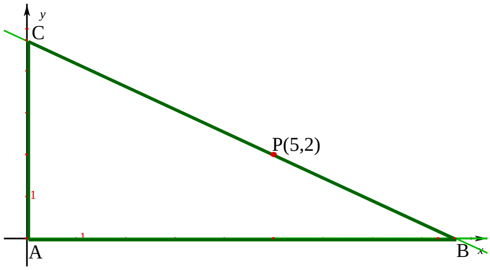


Figure 14. A line generating a family of triangles.

When students worked on this task initially they mainly chose an algebraic approach. They recognized that the triangle was a right triangle and that its area could be expressed as half of the product of its legs; however, they experienced difficulties in expressing the area in terms of only one parameter. Ideas from students themselves were used during the whole group discussion that eventually led them to connect important information of the problem.

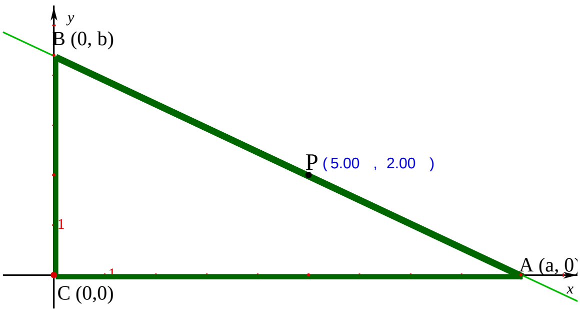


Figure 15. Identification of elements.

Students identified legs of triangle, slope of line AB, and point P as key components of the problem to express the triangle area. In this context, part of the students' discussion focused on finding ways to calculate or express such area in terms of the slope and point P (Figure 15). When finding the slope they discussed the meaning of positive and negative slope and eventually they utilized and combined the following expressions:

The area was expressed as $A = \frac{ab}{2}$ and the slope of line AB as $m = -\frac{b}{a}$ and expressed the equation of the line as $y = -\frac{b}{a}x + b$. They observed that Point P was part of the line and they used the equation to verify this condition. That is, $2 = -\frac{5b}{a} + b$ from this expression they found that $b = \frac{2a}{a-5}$ and with this value they represented the area as $A = \frac{2a^2}{2a-10}$. Here, the problem was reduced to find the value of *A* for which the area gets its minimum value. Here, they used their calculator to graph the area expression and found the corresponding value of ***a*** in which the area reaches its minimum value (Figure 16). It is important to mention that when they selected the point to find the minimum they discussed the domain of the function in terms of what happens to the triangle when the line gets perpendicular to *x*-axis.

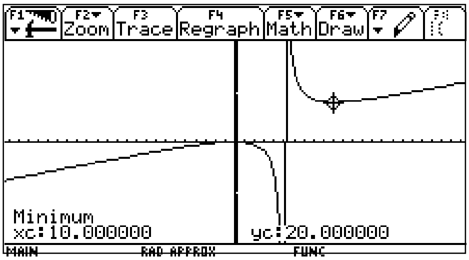


Figure 16. Graphic solution.

When discussed the graph, they explained that when the value of one leg was 10, then the triangle reaches its minimum area of 20 squared units. With this information they calculate the value of *b* = 4 and construct the desired triangle (Figure 17).

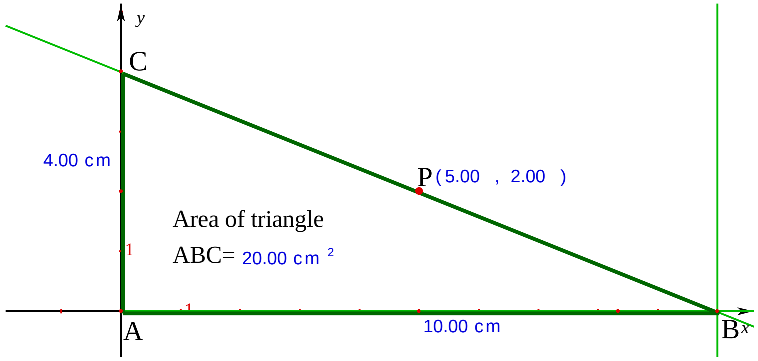


Figure 17. The triangle with the minimun area.

The discussion went beyond the time assigned to regular sessions and it was decided to continue with this task the next class. Here, students were asked to look for other approaches to deal with the task. A small group used a functional approach to solve it. It involves drawing point *P*, line *PB* and quantifying the area of triangle *ABC*. By moving point *B* along *x*-axis, a family of triangles will appear. Hence, it is possible to examine the behavior of their areas through a graph (Figure 18).

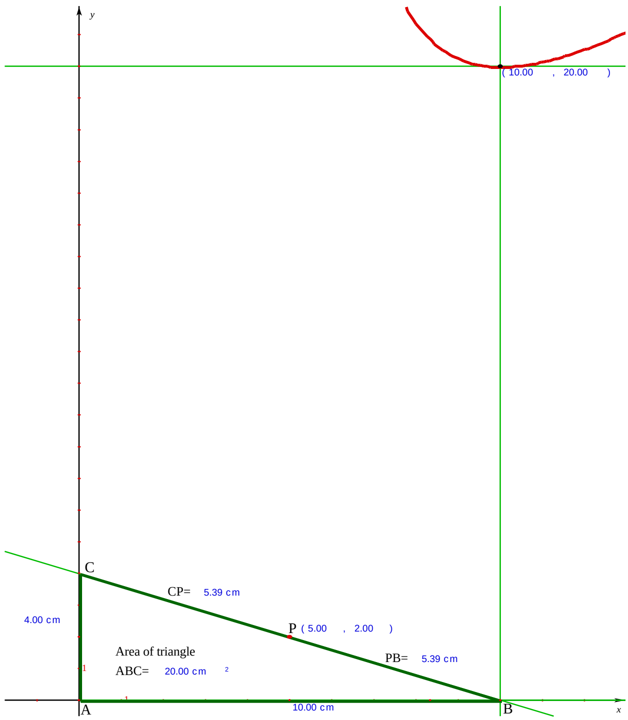
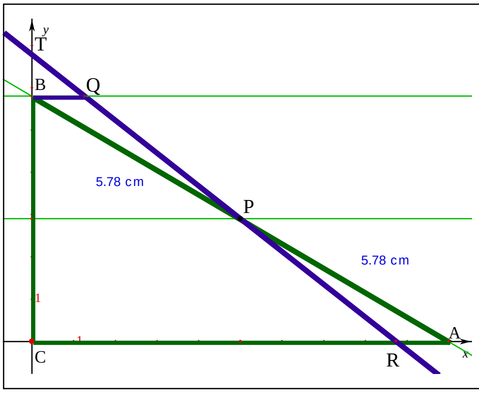


Figure 18. A dynamic solution.

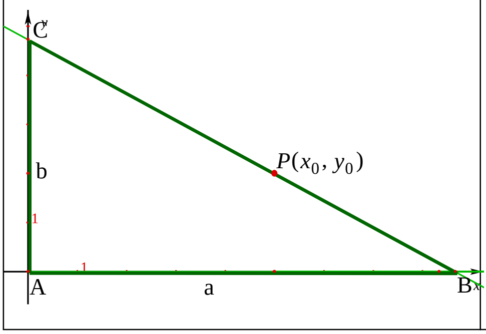
It was observed that the dynamic approach provided another window for students to explore and solve the task. For example, without using algebraic resources that appeared in the first approach, students observed that the minimum area of a triangle is generated when P is the midpoint of segment *AB*. Later they provided a geometric argument to support this observation.



Students drew line *RP* (which is different from the line that includes segment *AB* where *P* is the midpoint). They also drew *BQ* perpendicular to *CB*. Here they argued that triangles BPQ and APR are congruent and concluded that triangle *ABC* has less area than triangle *CRT*. This is because triangle *CRT* includes an additional part (triangle *BTQ*).

Figure 19. Providing an argument.

Other approach suggested by the instructor was to analyze the general case related of the original problem. That is when point *P* was given by $P(x_0, y_0)$ (Figure 20).



The equation of *BC* can be expresed as:

$\frac{x}{a}+\frac{y}{b}=1$ which can be written as:

$bx+ay=ab$ at $P(x_0, y_0)$, we have that

$bx_0+ay_0=ab$, $a=\frac{bx_0}{b-y_0}$. We also have that: $A=\frac{ab}{2}$, $A=\frac{x_0 b^2}{2(b-y_0)}$.

Figure 20. General solution.

Now using the calculator, it is possible to find the derivative of A (Figure 21).

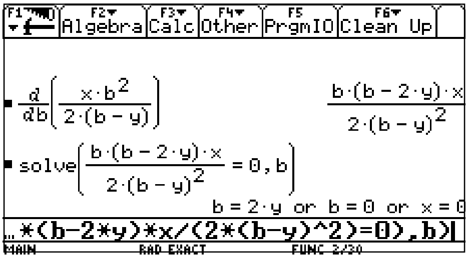


Figura 21. Use of a calculator.

Thus the information provided by the calculator means that $a = 2y_0$ and $b = 2x_0$.

Mathematical content and processes that appeared during the students' discussion include the concept of slope, equation of a line, properties or right triangles, solution of linear equations, graphs of functions and congruent triangles.

Remarks

The students' approaches that emerged during their interaction with the tasks show that the use of calculators/computer can become a powerful tool for students in their learning of mathematics. For example, students could easily examine a family of regular polygons to identify the behavior of particular relationships and explore meaning of some mathematical expressions. There is evidence that the use of dynamic representations offers clear advantages for students to connect the situation to a graph that helps them explain or interpret the behavior of key parameters attached to the tasks. In general, the use of technology provides another window for students to observe and discuss mathematical resources that not necessarily appear in paper and pencil approaches. However, it is important to mention that the role of the instructor becomes fundamental component in orienting students' discussion.That is, he/she should value students' ideas and take them as a starting point to construct robust solutions or explanations. In particular, it is important to encourage students to present and refine their ideas and the use of technology plays a fundamental role in achieving representations that can be examined and shared with other students.

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Coordinating Representations Through Programming Activities: an Example Using Logo

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ABSTRACT. The research literature has pointed to difficulties encountered by students in interpreting and establishing links between different types of representational registers. Here I present results from a study where a Logo-based computational microworld for the exploration of infinity and infinite processes, was meant to facilitate the construction and articulation of diverse types of representations of those infinite processes. I provide a couple of examples from case studies illustrating some of the ways in which students used and coordinated the elements of the exploratory medium to construct meanings for the infinite.

On the importance of constructing links between different representational registers

We can consider that two main representational registers are those that are often denoted as “visual” and “symbolic” (even though visual representations are also a form of symbolizing). That is, some representations are of visual form (e.g. the graph of a function); others are purely symbolic or algebraic, lacking a graphical aspect. Visual and symbolic representations are complementary, each representation holding a different form of interpreting the information. The visual (graphical) representation of a mathematical situation gives a global view (as explained for instance by Larkin & Simon, 1987), while, on the other hand, the symbolic representation involves more local analysis. An integration of both types of representations appears to be essential for constructing a richer meaning of the mathematical object.

The research literature, however, has pointed to the difficulties encountered by students in interpreting and establishing links between different types of representational registers. It has been found that pupils tend to prefer symbolic manipulation to visual interpretation, perhaps because the latter requires moving towards a higher cognitive level — for 'decoding' the visual information (see Dreyfus & Eisenberg, 1990; Eisenberg & Dreyfus, 1986; 1991). Many students have difficulties in reading diagrams, and one of the things that has been observed is that students do not easily make links between visual representations and analytical thought (see for instance Artigue, 1990; Presmeg, 1986; Hillel & Kieran, 1987; Dreyfus et al., 1990).

Findings such as these have led many researchers (e.g. Cuoco and Goldenberg, 1992) to advocate incorporating more representations and types of thinking, particularly visual ones, into school mathematics. The need for including more of the visual aspect in mathematics education is evident, particularly in contexts that link it to the numerical and symbolic aspects of mathematics, although it is clear that the mere presence of multiple representations does not guarantee that the learner will construct cognitive links between them. As Noss et al. (1995, p.191) have pointed out, "we should not take for granted that building links *between* representations is straightforward, or that the more representations which are available, the better it is for learning". On the other hand, these same authors (Noss & Hoyles, 1996) have explained that computational environments offer a setting where the objects and relationships can become *meaningful* through actions within the microworld where students can generate and articulate mathematical relationships that are general to the computational situation in which they are working.

We consider that it is the construction of cognitive links between representations and pieces of knowledge that facilitates the learning of a concept, and that this construction is in turn facilitated when there are more opportunities of *engaging* — *constructing* and interacting — with multiple modes of external representations of the concept. In fact, Wilensky (1991) uses this idea of building connections in his discussion of what makes knowledge abstract or concrete. He points out that the "formal is often abstract because we haven't yet constructed the connections that will concretize it" (ibid., p. 202). Thus, an abstract concept can become concrete by relating to it in as many ways as possible. As he puts it:

> The more connections we make between an object and other objects, the more concrete it becomes for us. The richer the set of representations of the object,

> the more ways we have of interacting with it, the more concrete it is for us. ... This view will lead us to allow objects not mediated by the senses, objects which are usually considered abstract — such as mathematical objects — to be concrete; provided that we have *multiple modes of engagement with them and a sufficiently rich collection of models to represent them.* (ibid., pp. 198-199; emphasis added)

Our theoretical approach thus follows the 'constructionist' paradigm (see Harel & Papert, 1991). We adopt the position that the construction of meanings involves the use of representations; that representations are tools for understanding; and that the learning of a concept is facilitated when there are more opportunities of constructing and interacting with, as diverse as possible, external representations of a concept.

A study centering on the infinite: using the computer to provide an interaction between diverse representations of infinite processes

For many years we have been studying students' conceptions and difficulties in the learning of concepts related to the mathematical infinity (e.g Sacristán, 1993). The concept of infinity is central to calculus: infinite processes form the basis for the concept of limit; it is also present in other important areas of mathematics. This concept, however, is recognized as difficult and has historically been the origin of paradoxes and confusions. Furthermore, the areas of mathematics where infinity occurs are those that have traditionally been presented to students mainly from an algebraic/symbolic perspective, which has tended to make it difficult to link formal and intuitive knowledge.

There is evidence, however, that in a context that combines numerical and geometrical contexts through the use of algebraic language, some of the obstacles observed in single-representation situations in the reasoning related to infinite processes and sets, seem to disappear (Waldegg, 1988). This is an important finding which supports the idea that by building connections between different types of representations (in this case through algebraic language) some of the difficulties that arise when working in a single context can be diminished. We thus took it upon ourselves to create situations in which the learning of the infinite infinity could be facilitated by incorporating the use of the computer and the representational systems it provides, even though we are aware that attempts to use the computer for the learning of the concept of limit have pointed to difficulties (e.g. Monaghan, Sun & Tall, 1994).

We postulated that at least some of the infinite processes found in mathematics, could become more accessible if studied in an environment that facilitates the construction and articulation of diverse types of representations, including visual ones. Based on this premise, we built a computational set of open tools (diSessa, 1997) — a microworld — using the Logo[1] programming language, for the study and exploration of infinity and infinite processes. Being aware that the conceptions that students could develop would be conditioned by this environment, the main focus of the research (see Sacristán, 1997) was to investigate, through detailed case-studies of 8 subjects, these students' developing conceptions of infinity and infinite processes as *mediated* by the Logo-based microworld. Part of this research was to investigate how students articulated and coordinated the different representations of the microworld in order to construct meanings for the infinite, which is the focus of this paper.

Description of the microworld

The microworld (see Sacristán, 1999) was a programming[2] environment where infinite processes —infinite sequences and series— were explored through the construction (using Turtle Geometry) of different visual models for representing these processes: such as "unfolding[3]" spirals and fractal figures, with a complementary numerical analysis (e.g. students constructed tables of values of, for instance, the distances forming the figures at different levels of the construction process). Thus, the infinite processes were explored through three types of representations:

- the symbolic code,
- different types of visual models (such as unfolding spirals; and fractals), and
- numeric representations, to complement and validate the visual observations

[1] The rationale behind the use of Logo is discussed elsewhere (see Sacristán, 1997), but one of the reasons for choosing this medium was the built-in visual interface (through Turtle Geometry) that Logo has, which can be very helpful for the requirements of incorporating the visual representations and creating the interaction with the symbolic code. Also, because figures have to be created linearly in Turtle Geometry this is helpful in the observation of the process as it unfolds (it adds movement and dynamism).

[2] In the programming environment students wrote, created, modified and explored procedures that represented, in different ways, the infinite processes under study.

[3] The visual models under study could be seen in a dynamic way as they were constructed (as they "unfolded") on the screen: this allowed for the observation of the evolution in time and behaviour of the underlying processes, eliminating the limitation of only observing the final state (the result of the process).

(taking advantage of the computing capabilities of the computer for reaching high terms in numerical, or other types of, sequences).

The activities were divided into two sections:

a) explorations of classical infinite sequences and their corresponding series, through geometric models such as spirals, bar graphs, and staircases.

b) fractal explorations.

For the explorations of infinite sequences and their corresponding series, we began with geometric models such as spirals, since they seemed to be a straightforward way of translating arithmetic series into geometric form. For instance, in the 'spiral' type of representation each term of the sequence is translated into a length, visually separated by a turn, so that the total length of the spiral corresponds to that of the sum of the terms (the corresponding series). Thus, for instance for the sequence $\{1/2^n\}$, the visual process and added lengths of the spiral would represent the series: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$, a notation which is descriptive of the process involved — the ellipsis points indicating an indefinite continuity of the process (a *potentially* infinite process). On the other hand, in the symbolic computer code, the same series can also be represented by a notation corresponding to $\sum_{n=1}^{\infty}\frac{1}{2^n}$, which is an object in itself (an *actual* infinite object[4]). This could serve to illustrate how the same (mathematical) object can be represented both as a (complete, integral) *object* as well as in terms of a *process*.

The activities included programming by the students of visual models such as spirals[5], bar graphs, staircases, and straight lines of sequences such as $\{1/2^n\}$, $\{1/3^n\}$, $\{(2/3)^n\}$, $\{2^n\}$ etc., and then $\{1/n\}$, $\{1/n^{1.1}\}$,... , $\{1/n^2\}$. The Bar graph, Steps and Line models are illustrated in figures 1, 2, and 3, respectively.

Through the observation of the visual (and numeric) behavior of the models, students were able to explore the type of convergence, or divergence, of a sequence and that of its corresponding series, and predict the behavior at infinity. The different

[4] This is independent of the convergence or divergence of the series, although when there is convergence it is easier to think of the series as a "complete" object, e.g. when $\sum_{n=1}^{\infty}\frac{1}{2^n}=1$.

geometric models for the same sequence provided different perspectives of the same process. But an aspect that was considered important for this, was that the students carried out the transformations of the models themselves by changing the computer code. It was intended that this involvement would help build links between the symbolic representation (in the code) and the different models.

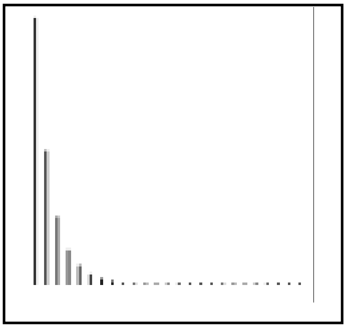

Figure 1. Bar graph corresponding to the sequence $\{1/2^n\}$.

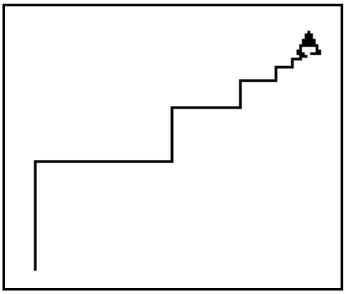

Figure 2. Steps model corresponding to the sequence $\{1/2^n\}$.

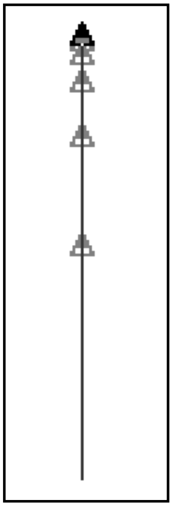

Figure 3. Line model corresponding to the sequence $\{1/2^n\}$.

Because the production of the graphical and numerical representations was carried out through the construction of the Logo symbolic programming code, the different types of representations were explicitly linked one with the other through the first: the procedural code. In fact, the code can act as an *isomorphism* between the different (visual) models, as well as acting as the link between all the representations, and the subject.

In the second part of the microworld, the exploration of fractal figures centered mostly on the study of the recursive structures of the Koch curve and snowflake (Figure 4) and the Sierpinski triangle (Figure 5), and involved the encounter of some apparent paradoxes such as that of a finite area bounded by an infinite perimeter. Through these sequences and fractals activities we intended to confront

[5] I would like to acknowledge the books on infinity by Mason (1988), in particular, and Hemmings & Tahta (1984), which served as inspiration for some of the geometric models used in the study presented here.

students with the idea of "what happens in the infinite" by allowing them to visualize an infinite process through the computer-based approximations.

An essential aspect of the functioning of the microworld was the programming activity on the part of the students. All of the students therefore wrote their own procedures using the Logo programming language, although the activities were suggested by the researchers (who also served as guide). Programming the computer to draw on a computer screen can be thought of as a process of interaction between contexts, from the symbolic code to the visual and conversely. The symbolic code of the computer language can serve, within the computer context, to "explain", model, or represent the process and it encapsulates the structure behind the process. Thus, the programming activities emphasized the interaction between different types of (re-)constructed representations of the infinite processes.

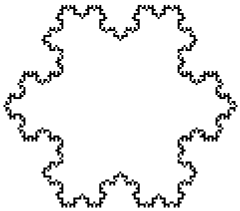

Figure 4: The Koch snowflake.

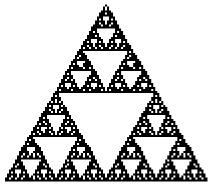

Figure 5: The Sierpinski Triangle.

Examples of the ways in which students coordinated the different representations within the microworld: linking the symbolic code and the visual output

Here, I present a few examples from the activities with the microworld to illustrate some of the ways in which students used and coordinated the elements of the exploratory medium *to construct meanings for the infinite*. They illustrate how the microworld gave the students means to *make sense of what they saw on the screen via the programming code*: the interactions between the code and its outputs. Emphasis is also placed on the role of the *structure* of the procedures, particularly the *iterative or recursive* structure (which was present in all the procedures for constructing the visual models of infinite sequences), and its relationship to the visual structure. There are two facets to the phenomenon:

(a)The link of the endlessness of the process represented on the screen, with the iterative structure of the code; and

(b) The use of the symbolic recursive structure of the code to *visualize the self-similar visual behavior*.

a. Endless movement and the link with the recursive (iterative) structure of the code. An example of the interaction from visual to symbolic, with a return from the symbolic to the visual.

In the first microworld activity, the subjects were given the following procedure, and asked to predict the output and its behavior:

```
TO DRAWING :L
PU
FD :L
RT 90
WAIT 10
DRAWING :L/2
END
```

This procedure makes Logo's turtle walk (without leaving a trace ¾ the Pen is up) through a spiral with arms each having half the length of the previous one (see Figure 6). It should be noted that as there is no stop condition, the procedure continues indefinitely[6]. It was designed to induce students to reflect on the behavior of the turtle and the procedure itself.

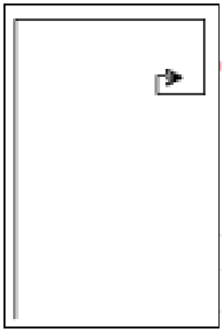

Figure 6. Spiral output of the initial DRAWING procedure (representing the sequence $\{1/2^n\}$).

This initial activity produced interesting results: in particular, the students had to visualize the actual pattern without relying on the computer drawing; it induced students to try to make sense of the relationships between the code and the graphical output: For instance, most students did not expect to see the turtle endlessly spinning without leaving a trace. In order to explain to themselves this unexpected behavior and make sense of why the turtle was endlessly spinning, the students had to re-examine the procedural code. Victor was one student who immediately remarked that the procedure would never stop because the recursive structure of the code represented an infinite process. He explained it was because the procedure called itself without anything telling it to stop, so it never would; the process of turning and walking half the previous distance would continue repeating itself and would never stop. By analyzing the code Victor was able to connect to it the behavior of the visual output (in this case the *movements* of the turtle) since he correctly predicted the outcome and was able to justify that visual behavior through the code. He *linked the recursive structure of the code with the infinitude of the process.*

A modified procedure (with the Pen down) produced an inward spiral with the turtle then turning endlessly in its center. Victor and his partner, Alejandra, pointed out that although the turtle seemed to be just turning in the same spot, in reality there was “a variation”. There were two factors here: a) The turtle kept turning and b) the turtle turned at same spot. The first factor could have served as an indicator that the process continued, but it was the fact that the students seemed to be able to disregard the *visual appearance* of the turtle — spinning in apparently the same spot — that suggests that they understood that the underlying (mathematical) process continued, and that they were able to link the output with the code and the process.

Later in the activities, when the students modified the procedure to give out numeric values, they would confirm the continuation of the process by still getting an output of values, even when the turtle seemed stuck:

Alejandra: Apparently it is stopping on the screen, but it is still walking because we are still getting the values.

It was thus that students were able, via a process of experimenting backwards and forwards from code to figure, to make sense of the behavior of the turtle, which seemed to be spinning on the same spot realizing that the amount that the turtle moved each time was halved. The key point here is that the analysis of the code allowed them to:

- recognize in the recursive structure a potentially infinite process; and
- to quantify the movement, to explain that although the turtle seemed to be turning without moving forward, in reality there *was* a variation.

Thus by coordinating the visual and symbolic — in the order visual to symbolic to visual — and later complementing it through numerical explorations, their understanding of the process became integrated and potentially misleading visual appearances could be ignored.

[6] Later, when the students became aware of the recursive structure, all of them eventually added a stop condition, which also served as an important investigation tool for making sense of the relationship between code and figure (see Sacristán, 1997).

This interplay between the code and its outputs, which led students to make sense of what they observed by linking all the elements, is illustrated in Figure 7:

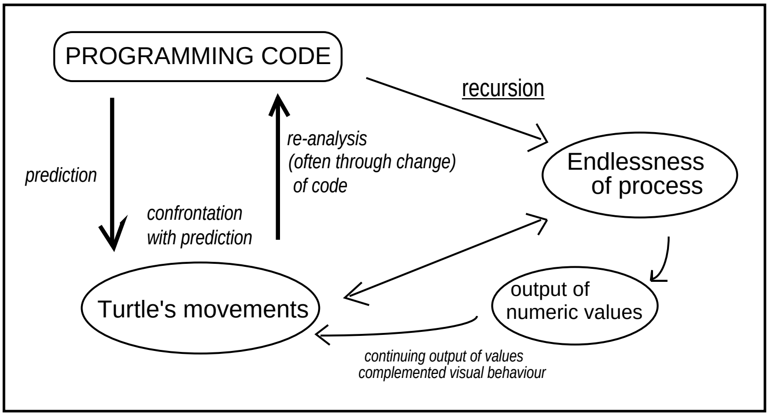

Figure 7. The interplay between the code and its output to make sense of the endless movement: the graphical image gains meaning from the symbolic representation.

b. Using the *recursive* structure of the code to predict a self-similar visual structure: the code 'encapsulates' the process. *Visualizing the output from the symbolic code: an example of the interaction from symbolic to visual.*

Whereas the recognition of the (tail) recursive structure in the code explained the endlessness of the process, which involved going from the visual to the symbolic, this recursive structure also served to predict and *visualize* the figure produced by the code — a process from symbolic to visual. For instance, when the students were unable to see the deeper levels in the visual representation of the sequence under study (e.g. they noticed that the center of the spiral model looked like a point), some students blamed this on the resolution and were able to compensate for the deficiencies of the screen by using the information provided in the symbolic structure of the code to visualize those levels. Victor, for example, in the activity described earlier, explained that even though the center of the spiral looked like a point, the figure did *not* become a point, and would always have the same spiral shape, even at its center. Another student, Martin, explained this same point as follows:

Martin: What happens is that there is a part that our eyes can no longer perceive. Inside [the spiral] it continues the same way, because it is the same process that continues... If we used a magnifying glass and looked at that little square there, we would see like all this part [the full spiral].

This is a key issue: the visual self-similar structure is a reflection of the recursive structure of the code (and of the infinite iterative nature of the process). And the programming code thus serves for "visualizing" beyond the visual image. The programming code can thus be said to embody or "encapsulate" the entire process. An infinite process would, of course, take infinitely long to be generated; but the symbolic code (which generates it) holds the entire process in *latent* form, as would a mathematical formula, and its structure reflects the structure of the process.

This connection between the structure of the code and that of the figure was of course more obvious in the fractal explorations. For example, the construction of the Koch curve procedure was based on the theoretical structure of the visual figure — the structure of the procedure mirroring the way the process would be (visually) constructed. Then as the Koch curve was generated, the students would make sense of the figure (the fact that each part contains the whole) by relating it back to the structure of the code (which calls itself). The same correspondence between code and figure was found for all the fractal figures investigated, and students were able to predict a recursive visual structure from the observation of the structure of the code.

It is interesting to point out that students explicitly recognized the value of having a recursive code that defined and therefore was connected to the process both visually and in general. As they pointed out, the self-similar/recursive structure of the code allowed them to have an idea of what was going to happen subsequently, particularly since the figure can only have so much resolution. This was best explained by one student, Jesús, who pointed out that it was the recursive structure of the procedure, which helped him realize (and reflected the fact) that the figure would repeat itself in a self-similar way, adding that it was the procedure (i.e. the code), which helped to understand what happens at infinity:

Jesús: I would say that our most powerful weapon is recursion, which is what allows us to be aware of the details...

In those shapes that repeat themselves, it is the same part, which is the basis...

...

And because we understand the language, it was the recursion which helped us understand better, and do a better analysis... going from the figure to the procedure, or from the procedure to the figure...

In any case, once I knew that the procedure was recursive, I more or less had an idea of what would happen later, because the drawings by themselves do not have enough resolution...

> The Logo procedures give support and help define what happens at infinity. That is, they include the notion of infinity, and that is really helpful. They help convince us, or confirm the ideas we may have....

Summary: A sketch of the microworld

There were three main representational elements involved in the microworld: symbolic (the programming code), visual (geometric figures), and numeric (numerical values). A mathematical process could be represented in each of these complementary forms: symbolically in the code, and visually or numerically by running the code. Thus, the structure of the microworld was such that the process and its different representations were all linked through the computer code (see Figure 7). All of the above elements, including the process, and their characteristics are described in more detail below:

The mathematical process.

All the microworld activities involved a process of construction of an infinite sequence (this includes the fractal activities, since fractals are constructed through *geometric* sequences). In the case of sequences, the process was first described as an iterative action, involving repeated operation on a variable: e.g. the action of halving an element and repeating this for the resulting process. It was later defined as a symbolic function formula — a Logo function in the programming code —, which encapsulated the process.

The programming activity and code.

The programming activity and the code were the means that integrated all the elements and acted as a “control structure”.

The visual/graphical representations.

These representations had several relevant characteristics during the explorations:

(i) First, the visual element was an entry point for the explorations. That is, although the visual figures required the computer procedure, in most cases the realization of what the procedural code was describing and how it operated did not emerge until after the visual figures were generated and an attempt was made to relate the observed phenomena on the screen with the code that produced it.

(ii) These representations served to “visualize” the behavior of the processes in two ways:

On the one hand, they provided a *global* view of the processes: the process is synthesized in the figure.

On the other hand, the computer provided the added benefit that these visual images *gradually* unfolded, so that each element of the figure (e.g. the visual representations of the terms of the sequence) could be specially observed in relation to the previous ones, giving the sense that a *process* was taking place. This allowed for a local analysis, particularly since the computer simultaneously generated the numeric representations which quantified the process.

(iii) The element of *movement* also provided information that is difficult to access. For example, the turtle turning in the same spot conveyed the idea that the process was continuing even when there was no longer any visible change.

(iv) The visual representations had a *self-similar geometric structure,* which resulted from the recursive structure of the code and could be related to the latter, forming an additional link between the two.

(v) Visual models could be *transformed* into others, through the use of the same programming code. Each model had particular characteristics that helped in the understanding of the behavior of the other models. The different models are *complementary,* each adding its own particular perspective on the process: However, since they were all produced by the same code, the connection between them could be made apparent, facilitating the integration of the information. Particular visual appearances could therefore become less dominating.

(vi) Additionally, a valuable exploration tool for the visual representations was the definition of a scale variable in the code. In a sense the scale acted as a “zoom” function: for instance, it sometimes enabled the students to look “deeper” into the graphical representation (e.g. into the spiral model or the Koch curve). Through this activity they could, for example, appreciate the self-similar characteristic of fractal figures. The use of the scale variable had another value in that its relationship with the figure had to be made explicit through the programming activity. In this sense it could be considered to have an additional function to that of a mere “zoom button”.

The numeric values

The numeric output provided a means for working with the visual and symbolic representations. It was another representation of the mathematical process that added numeric *precision* to both the terms of the sequences and to the limit values.

Furthermore, the students could "visualize" *in the numeric values* the limit value of the process or its divergence. The numeric output also provided an additional connection between the visual and the symbolic code acting as a means for quantifying the visual models and confirming the accuracy of the code.

Because the values were generated *simultaneously* with the visual representation as it unfolded, these values had a concrete connection, rather than being abstract numbers whose meanings could be obscure.

Figure 8 is a schematic of a typical paradigm case of the use of the representational tools of the microworld; in it we see the following (refer to numbers in diagram): (1) The mathematical process is symbolically defined in the code. (2) By running the programming code, a visual representation, which models the entire process (3), is produced. Additionally, the visual representation is produced by the *movements* of the turtle, gradually unfolding: this allows the process to be observed sequentially and for its characteristic *behavior* to be highlighted. (4) There are different visual models that can be produced, each providing a different visual perspective on the process. The switch between models is done through the code whose structure, and the process it defines, remain invariant: the models are *isomorphically* constructed. (5) Numeric values can also be produced through the same code; this links them to the visual representations and both (6) simultaneously gradually unfold. (Numeric values are also produced through complementary procedures — also representing the same process). (7) The numeric representations serve to add precision and give confirmation of the observed behavior of the process.

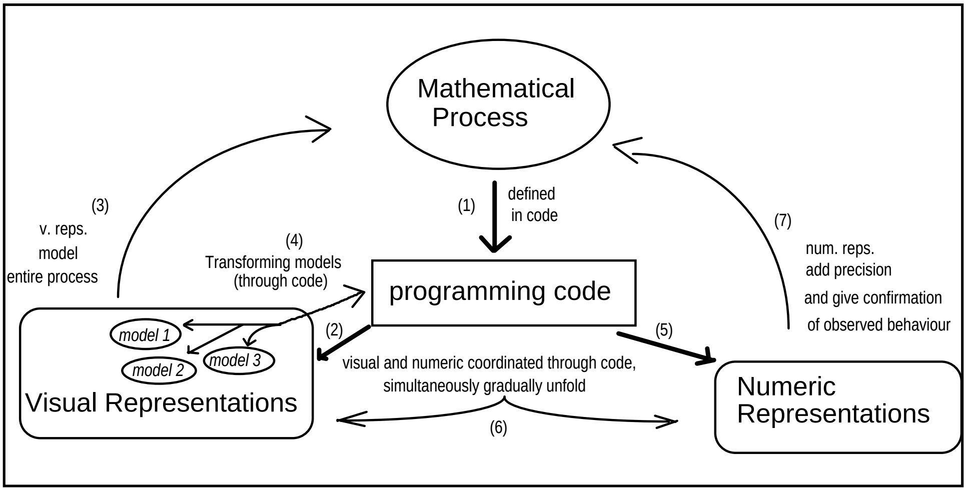

Figure 8. The representational elements of the microworld and their interactions.

I would like to emphasize the way in which the representational tools were used and compare the structure of this microworld with the use of multiple representations illustrated in the work of other researchers (where, although the different representations are linked in the sense that they work simultaneously, the inner workings of the connections between them are not available for the student to manipulate and re-construct). In the study presented here, through interacting with the programming code, the students themselves created and controlled the way in which the multiple representations worked. It is in this sense that — unlike other multiple representation environments (e.g. Kaput, 1995) where the tools are fixed — this microworld is what diSessa (1997) would call an *open tool set:* the students were able to reconstruct or redesign the tools, and the links between them, and express themselves through the programming activity.

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Computer-based Tools for Data Analysis: Support for Teachers' Understanding

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ABSTRACT. This chapter presents an analysis of a teacher development experiment (cf. Simon, 2000) focused on supporting a cohort of middle-school teachers' understanding of statistical data analysis. The collaborative effort conducted with the teachers built from a twelve-week classroom teaching experiment conducted in the fall semester of 1998 with a group of seventh-grade students (age twelve). Part of the research efforts associated with the classroom teaching experiment included the development of an instructional sequence and two accompanying computer-based tools for data analysis. The design of the instructional sequence and the tools was focused on the big idea of distribution and the importance of multiplicative reasoning. This chapter documents the role of the tools and instructional tasks developed in the course of the seventh-grade teaching experiment in supporting teachers' understandings of these big ideas as they relate to statistical data analysis.

Introduction

In the analysis presented in this chapter, I describe the role of an instructional sequence and two accompanying computer-based tools in supporting teachers' developing understandings of statistical data analysis. The data reported in this paper are taken from a collaborative effort conducted between a cohort of seventeen middle-school teachers (grades six through eight) and myself during the 2000-2001 academic year. The collaboration built from a classroom teaching experiment conducted with a group of twenty-nine American seventh-grade students (age twelve) in the fall semester of 1998. During the twelve weeks of the classroom teaching experiment, the research team's[1] primary goal was to investigate ways to proactively support middle-school students' ability to reason about data while developing statistical understandings related to exploratory data analysis. An integral aspect of that understanding entailed students coming to view data sets as distributions. Inherent in this understanding is a focus on multiplicative ways of structuring data. The research team's interest was motivated by current debates about the role of statistics in school curricula (cf. Burrill, 1996; Burrill & Romberg, in press; Cobb, 1997; Lipson & Jones, 1996; National Council of Teachers of Mathematics, 2000; Shaughnessy, 1992; Shaughnessy, et al., 1996). The guiding image that emerged for the research team as we read and synthesized the literature was that of students engaging in instructional activities in which they both developed and critiqued data-based arguments (cf. Wilensky, 1997).

The teacher collaboration reported in this paper is part of a larger research effort that is focused on supporting the development of professional teaching communities.[2] This effort builds from the premise that, in order to be effective, teachers must develop a deep understanding of the mathematics that they will teach (cf. Ball, 1989; Ball, 1997; Bransford, Brown, & Cocking, 2000; Grossman, 1990; Ma, 1999; National Research Council, 2001; Schifter, 1995). To this end, the initial goal of the collaboration was to engage the teachers in analyzing tasks from the statistics instructional sequence. The purpose was to support a deepening understanding of statistical data analysis while supporting a focus on students' diverse ways of reasoning.

In the following sections of this chapter, I begin by describing the instructional sequence and computer-based tools for analysis that the research team developed while planning for and conducting the seventh-grade teaching experiment. Against this background, I present analyses of episodes that occurred during the teacher work sessions. The purpose of this analysis is two-fold. First, it highlights the diverse ways the teachers reasoned on tasks. Second, it points to the importance of the computer-based tools in supporting the teachers' emerging understandings.

Instructional Sequence

As the research team began to design the instructional sequence to be used in the seventh-grade classroom, we attempted to identify the "big ideas" in statistics. Our plan was to develop a single, coherent sequence and thus tie together the separate, loosely related topics that typically characterize middle-school statistics curricula.

In doing so, we came to focus on the notion of distribution. This enabled us to treat concepts such as mean, mode, median, and frequency as well as others such as “skewness” and “spread-outness” as characteristics of distributions. It also allowed us to view various conventional graphs such as histograms and box-and-whiskers plots as different ways of structuring distributions. Our instructional goal was therefore to support students’ gradual development of a single, multi-faceted notion, that of distribution, rather than a collection of topics to be taught as separate components of a curriculum unit. A distinction that we made during this process which later proved to be important is that between reasoning additively and reasoning multiplicatively about data (cf. Harel & Confrey, 1994; Thompson, 1994; Thompson & Saldanha, 2000). Multiplicative reasoning is inherent in the proficient use of a number of conventional inscriptions such as histograms and box-and-whiskers plots.

In our development work, we were guided by the premise that the integration of computer tools was critical in supporting our mathematical goals. Students would need efficient ways to organize, structure, describe, and compare large data sets. This could best be facilitated by the use of computer tools for data analysis. However, we tried to avoid creating tools for analysis that would offer either too much or too little support. This quandary is captured in the current debate about the role of technologies in supporting students’ understandings of data and data analysis. This debate is often cast in terms of what has been defined as expressive and exploratory computer models (cf. Doerr, 1995). In one of these approaches, the expressive, students are expected to recreate conventional graphs with only an occasional nudging from the teacher. In the other approach, the exploratory, students work with computer software that presents a range of conventional graphs with the expectation that the students will develop mature mathematical understandings of their meanings as they use them. The approach that we took when designing computer-based tools for data analysis offers a middle ground between the two approaches. It introduces particular tools and ways of structuring data that are designed to fit with students’ current ways of understanding, while simultaneously building toward conventional graphs (Gravemeijer, et al., 2000). As such, the computer tools we designed were intended to support students’ emerging mathematical notions while simultaneously providing them with tools for data analysis.

The first computer tool was designed to facilitate students' initial explorations of univariate data sets and provide them with a means to manipulate, order, partition, and otherwise organize small sets of data. Part of our rationale in designing this tool was to orient the students to analyze data as measures of an aspect of a situation rather than simply to "do something with numbers" (cf. McGatha, Cobb, & McClain, 1999). When data were entered into the tool, each individual data value was shown as a bar, the length of which signified the numerical value of that single data point (see Figure 1).

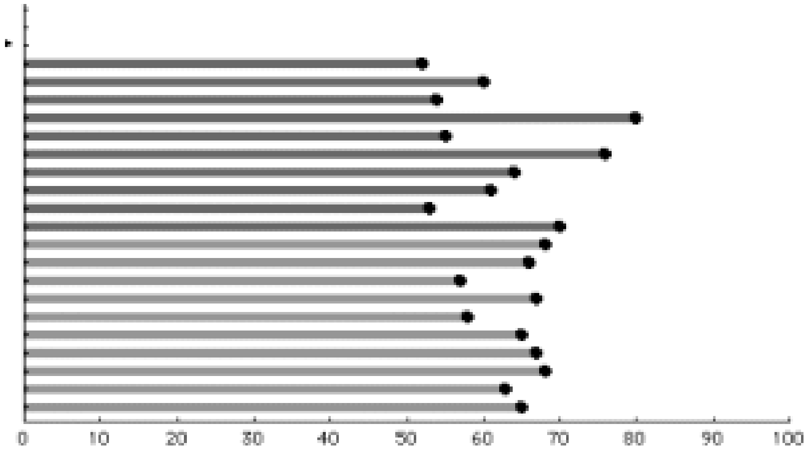

Figure 1. Data displayed in first computer tool

The bars could be either pink or green, allowing for two data sets to be easily compared (as shown in Figure 1). Our choice of this relatively elementary way of inscribing individual data values reflected our goal of ensuring that students were actually analyzing data. To this end, the initial data sets the students analyzed were also selected so that the measurements made when generating the data had a sense of linearity and thus lent themselves to being inscribed as horizontal bars such as the braking distance of a car or the longevity of batteries or light bulbs. A data set was therefore shown as a set of parallel bars of varying lengths that were aligned with an axis as shown in Figure 1. The tool also contained a value bar that could be dragged along the axis to partition data sets. In addition, there was a range tool that could be used to determine the number of data points within an interval. The use of the computer tool in the classroom made it possible for the teacher to support shifts in the students' activity such that they came to act on data in a relatively direct way.

The second computer tool was introduced during the eighth week of the seventh-grade teaching experiment. It can be viewed as an immediate successor of the first in that the endpoints of the bars that each signified a single data point in the first tool have, in effect, been collapsed down onto the axis so that a data set was now shown

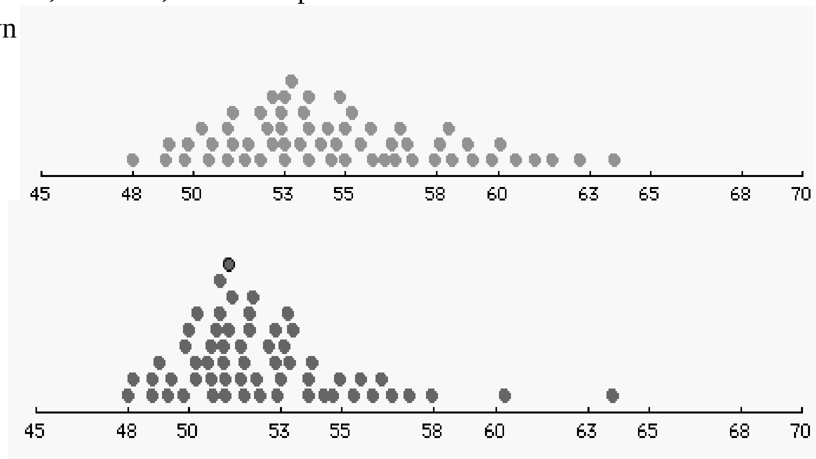

Figure 2. Data displayed in second computer tool.

The tool offered a range of ways to structure data. Two of the options can be viewed as precursors to standard ways of structuring and inscribing data. These involve organizing the data into four equal groups so that each group contains one-fourth of the data (precursor to the box-and-whiskers plot) and organizing data into groups of a fixed interval width along the axis (precursor to the histogram). However, the three other options available to students do not correspond to graphs typically taught in school. These involve structuring the data by (1) making your own groups, (2) partitioning the data into groups of a fixed size, and (3) partitioning the data into two equal groups. The first and least sophisticated of these options simply involves dragging one or more bars to chosen locations on the axis in order to partition the data set into groups of points. The number of points in each partition is shown on the screen and adjusted automatically as the bars are dragged along the axis.

The activities the students completed when the second computer-based tool was first introduced involved analyzing data sets with an equal number of data points. The way of reasoning with the second computer tool that emerged involved using the various options to partition the data sets (cf. Cobb, 1999; McClain, Cobb, & Gravemeijer, 2000; McClain & Cobb, 2001). For example, in the first instructional activity in which this tool was used, the students were given data on the speeds of sixty drivers on a particularly busy highway in the local area both before and after a police speed trap (e.g. issuing a large number of speeding tickets by ticketing anyone who exceeds the speed limit by even 1 mph) had been in place (see Figure 2). The task was to determine if the speed trap was effective in slowing the speed of the traffic and thus reducing accidents. A variety of solutions were discussed including 1) partitioning the data at the speed limit by using the create your own groups feature on the computer tool and comparing the number of drivers going above the speed limit before and after the speed trap was introduced, 2) using the fixed interval feature and comparing numbers of drivers in corresponding intervals (e.g. the number of drivers going between 50 and 55 mph both before and after the speed trap), and 3) envisioning the data as "hills" and reasoning about the shift in the hill. We viewed this last solution as an important first step in seeing the data set in global, qualitative terms by referring to its shape in terms of qualitative relative frequency (cf. Cobb, 1999). The last series of activities in the instructional sequence also involved the second computer tool, but the data sets had different numbers of data points. Having students investigate these activities provided means for supporting shifts in the students' reasoning toward multiplicative ways of structuring the data since direct additive comparisons were insufficient in these situations (cf. Cobb, 1999; McClain, Cobb, & Gravemeijer, 2000; McClain & Cobb, 2001).

As we began to develop tasks, we reasoned that students would need to encounter situations in which they had to develop arguments based on the reasons for which the data were generated. In this way, they would need to develop ways to analyze and describe the data in order to substantiate their recommendations. We anticipated that this would best be achieved by developing a sequence of instructional tasks that involved either describing a data set or analyzing two or more data sets in order to make a decision or a judgment. The students typically engaged in these types of tasks in order to make a recommendation to someone about a practical course of action that should be followed. The students' investigations were grounded in the use of the tools. As the sequence progressed, students began to develop inscriptions to support their arguments. The development of these inscriptions can be traced to the emergence of practices that evolved from the students' use of the computer-based tools. As such, the tools offered means for supporting students' developing ways of reasoning statistically.

Analyses

The teacher cohort that is the focus of analysis participated in monthly work sessions designed to support their understandings of effective ways of teaching statistical data analysis in the middle grades. Fundamental to this effort was support of the development of the teachers' content knowledge. The intent was to build from the mathematical practices that emerged in the course of the seventh-grade classroom teaching experiment (cf. Cobb, 1999). The hypothesis was that the same general learning trajectory (i.e., a parallel progression of conceptual development) could serve as a basis for guiding the mathematical development of the teachers. This trajectory served as a conjecture about the learning route of the teachers and the means of supporting their development. During the collaboration, the conjecture was continually being tested and refined in the course of my interactions with the teachers. The trajectory therefore offered a conjectured route through the mathematical terrain. This conjecture included not only taking the classroom mathematical practices that emerged in the course of the seventh-grade teaching experiment as a basis for the learning route of the teachers, but also taking the accompanying *means of support* as tools for supporting the emergence of the mathematical practices. These tools included the choice of tasks, the use of computer-based tools for analysis, the use of the teachers' inscriptions and solutions, and the norms for argumentation.

One of the initial activities in which the teachers engaged involved their analyzing data on the braking distances of ten each of two makes of cars, a coupe and a sedan in order to determine which one was safer. I introduced the task by first talking through the data creation process with the teachers. In the course of this discussion, the teachers clarified what procedures would be necessary to generate data needed to make a reasoned decision. After this rather lengthy discussion of the data generation, I then presented the data by giving the teachers paper copies of the data inscribed in the first computer tool as shown in Figure 1. I asked the teachers to work at their tables to decide which of the two makes of car they thought was safer, based on this data. My decision to use printouts of the data was based on my own experiences working with students on these tasks. I had noticed that when students were asked to make initial conjectures based on informal analysis of the printouts, their activity on the computer tool seemed more focused. They used the features on the tool to substantiate their preliminary analysis instead of to explore the structures that resulted from the use of the features. In addition, they focused more on features of the data sets such as clusters. I was also curious to see if the tools we had designed offered the teachers the means of analyzing data that fit with their initial, informal ways of analyzing the data.

As the teachers began their analyses, most of them initially calculated the mean of each set of data. They subsequently judged that measure to be inadequate for making the decision and proceeded to find ways to structure the data that supported their efforts at analysis. In this process, they used vertical lines drawn in the data to create cut-points and to capture the range of each data set. As an example, one teacher noted that all of the coupes took over 55 feet to stop whereas four of the ten sedans were able to stop in less than 55 feet. Other teachers focused on the "bunched-up-ness" of the coupes and reasoned that a consistent braking distance was an important feature.

I found these ways of reasoning significant for two reasons. The first was that the ways of structuring that they were creating with the drawn lines paralleled the features that we had designed on the tool. This implied that the tool would be a useful resource in supporting their analyses. The second was that their ways of reasoning about the data were consistent with the methods that we saw emerge in the seventh graders' activity. It therefore appeared that the conjectured learning trajectory could guide the mathematical development of the teachers.

As the teachers discussed the results of their analysis in whole-group setting, I introduced the first computer tool as a resource for sharing their ways of structuring the data. I used a projection system to make the data sets visible to the group and as the teachers explained their analysis, I used the features on the tool to complement their explanations. As an example, as the teachers talked about the "bunched-up-ness" of the data sets, I activated the range tool so that they could identify the extreme values in each data set. The teachers found this support helpful and were easily able to use the features on the tool to mirror their earlier activity with drawn lines.

In the remainder of the session, the teachers analyzed a second set of data on the longevity of two brands of batteries. Their analyses were again focused on cut points and clusters. These ways of reasoning parallel what we found in analysis of the seventh graders' activity. In particular, Cobb (1999) notes that in the seventh-grade classroom "[t]he characteristics of data sets that emerged as significant in this discussion and in the subsequent classroom sessions in which the first computer tool was used included the range and maximum and minimum values, the number of data points above or below a certain value or within a specified interval" (p. 17). The significant difference between the seventh-grade students and the teachers was that the teachers were able to talk about the number of data points above or below a cut point in terms of percentages (or ratios) of the whole. Further, they could reason probabilistically such as arguing that "you have a 30% chance of getting a bad battery" with a Brand A. Their arguments therefore appeared to be multiplicative in nature.

A shift to the second computer tool began with the introduction of the speed trap task. After a lengthy discussion of the data creation process, the teachers were shown printouts of data on the speeds of two sets of sixty cars (see Figure 2). The first were recorded on a busy highway on a Friday afternoon. The speeds were recorded on the first sixty cars to pass the data collection point. The second set of data was collected on a subsequent Friday afternoon after a speed trap had been put in place. The goal of the speed trap was to slow the traffic on a highway where numerous accidents typically occur. The task was to determine if the speed trap was effective in slowing traffic.

As the teachers worked on the printouts of the data, most of them created cut points at the speed limit and reasoned about the number of drivers exceeding the speed limit both before and after the speed trap. They used a range of strategies including ratios and percentages. Further, none of the teachers calculated the mean. One teacher focused on the shape of the two data sets and noted that at first it "looked like a Volkswagen Beetle" and then it "flattened out like a large Town Car." I found this particularly significant because it was the first occasion where a teacher found a way to describe the shape of the distribution. I capitalized on her solution and recast it in terms of "hills." My reason for doing so was grounded in my earlier work in the seventh-grade classroom where a similar incident had occurred with this same data set. For the students, the notion of the hill was pivotal in shifting their reasoning from viewing data sets as collections of points to distributions (cf. Cobb, 1999; McClain, Cobb, Gravemeijer, 2000; McClain & Cobb, 2001).

The significance of this shift became apparent in the next session when I introduced a task containing data sets with unequal numbers of data points. In the task, two sets of AIDS patients were enrolled in treatment protocols — a traditional treatment program with 186 patients and an experimental treatment program with 46 patients. T-cell counts were reported on all 232 patients (see Figure 3).

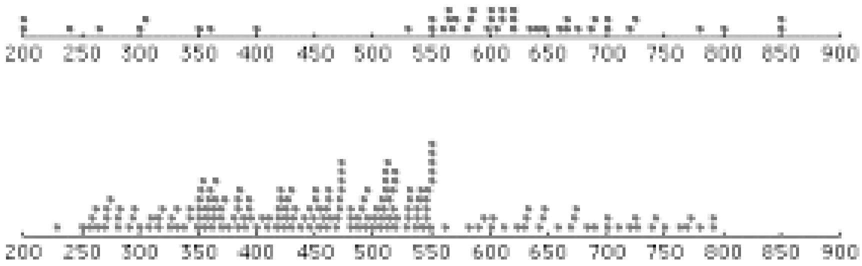

Figure 3. AIDS data displayed in the second computer tool.

As the teachers worked on their analysis, they initially noted that the clump, cluster, or hill of the data shifted between the two groups. This was a significant aspect of the distributions and one on which they focused as the continued their analysis using the second computer tool. In their work at the computers, they found ways to characterize this shift including creating cut points and reasoning about the percentage of patients in each group with T-cell counts above the cut point. They, like the seventh grade students, noted that that cluster of T-cell counts in the traditional treatment program was below the cut point whereas the cluster of T-cell counts in the experimental was above. Arguments such as these indicate their ability to reason about characteristics of distributions.

Conclusion

Throughout the analysis, I have documented the development that occurred as the teachers and I together engaged in a series of tasks from the statistics instructional sequence. In doing so, I have pointed to the teachers' developing ways understanding their own and others' mathematical activity. A focal point of this activity was the teachers' use of the computer-based tool for analysis. In particular, the analysis documents that the features on the computer-based tool fit with the teachers' informal ways of analyzing data and therefore supported shifts in their ways of reasoning. In addition, the computer-based tool enabled the teachers to easily manipulate and structure the data in a manner that fit with their informal analyses. In this way, the tool provided a *means of support* not only for the learning of the teachers, but also for the attainment of the goals of the collaboration.

Notes

1. Other members of the research team were Paul Cobb, Koeno Gravemeijer, Maggie McGatha, Cliff Konold, Jose Cortina, and Lynn Hodge.
2. Paul Cobb is Principal Investigator on the grant and the author is Co-Principal Investigator. Other members of the research team include Teruni Lamberg, Jose Cortina, Chrystal Dean, Qing Zhao, Lori Tyler, and Jason Silverman.

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Enhancing Students' Cognitive Systems Via the Use of Technology in Mathematical Problem Solving

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ABSTRACT. It is well recognized that the presence of technological tools in our society has changed what students need to learn in terms of curriculum goals. In particular, dynamic representations (achieved via technology) of mathematical data or phenomena offer students the possibility to identify and examine relationships and their associated properties. In this process, students need to develop ways to communicate and support what they observe during their interaction with those phenomena or problems. The goal of this paper is to illustrate how the use of technology can help students develop problem solving processes that are inherent of the mathematical practice.

Introduction

Recent mathematics curriculum reforms have pointed out the relevance of using technology in the learning of mathematics. Indeed, the use of graphic calculator or particular software such as Dynamic Geometry[2] has produced changes not only in the type of tasks and questions that students examine during their solution processes; but also in the role played by both teachers and students throughout the development

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[1] I completed this paper while I was a Visiting Professor at Purdue University and I would like to thank Dick Lesh for his hospitality and academic support during my stay.

[2] This type software includes primitive objects (points, lines, circles, rays, etc.), tools like drawing parallel or perpendicular lines which allow the construction of other objects or configurations. It is also possible to apply transformations (reflection, rotation, dilatation) and measure parts of those configurations. A particular feature called "dragging" allows the user to move certain elements of the configuration and to observe the effect in other parts of such configuration.

of the class. The National Council of Teachers of Mathematics (2000) identifies the use of technology as one of the key organizer principles of Pre-K-12 curriculum. A remarkable feature that distinguishes the use of technology is that it allows students to experiment and examine mathematical relationships from diverse angles or perspectives. Balacheff & Kaput (1998) identify two dimensions in which the use of computers makes an impact in students' mathematical experiences:

(1) Symbolic, by changing the representational medium in which mathematics is expressed; and
(2) Interactivity, by changing the relationships between learners and the subject matter and between learners and teachers –by introducing a new partner (p. 495).

To what extent does the use of representations (achieved via technology) help students in the process of understanding and solutions of tasks or problems? What is the role of teachers/instructors in an enhanced technology class? Is it possible to identify what aspects or features of students' mathematical learning can be enhanced by the use of technology? To what extent do mathematical arguments or ways to approach problems vary from traditional approaches (paper and pencil)? These questions are used as frame to discuss different approaches in using technology as a tool in students' learning of mathematics. In particular, we document the use of Cabri Geometry software and the TI-92 Plus calculator as a means to:

(i) Search for mathematical constructions and meaning,
(ii) Work on nonroutine problems,
(iii) Formulate and explore conjectures, and
(iv) Determine general patterns of recursive functions.

In addition, it is shown that the use of this type of technology helps students visualize the problem or phenomenon from various representations that include the use of table, graphs and algebraic forms. These representations become important for students to identify and explore diverse mathematical qualities attached to the solution process. Duval (1999) recognizes that the coordination of representations plays a fundamental role in solving problems or learning mathematical concepts. This coordination requires a set of operations carried out by students within each representation and explicit connections or translations across them. In fact, the NCTM (2000) recognizes that the use of representations plays a crucial role in students' learning of mathematics.

> Representations should be treated as essential elements in supporting students understanding of mathematical concepts and relationships; in communicating mathematical approaches; arguments, and understanding to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling (p. 67).

Searching for mathematical constructions and meaning. To explore and show the potential in using technology, a task that involves the construction of a parabola that eventually generates ellipses, hyperbolas, within the same construction is presented. This initial example illustrates features of mathematical thinking (conjecturing and proving) that permeate all the technology approaches to problems. In particular, when students move part of certain configuration via the software, they often will be surprised by the type of results that appear as a product of this operation or action. That is, serendipity seems to be a key component attached to tasks' explorations that constantly students would encounter and need to pay attention to during their learning experiences.

Dynamic Software such as Cabri Geometry provides an environment in which students can investigate the behavior of different relationship by moving part of certain configurations or objects. It is common that by experimenting with situations, students eventually identify new mathematical relationships. For example, high school students were asked to work on the following task:

(i) Draw a line L and select point S on that line and draw a perpendicular line to L (ST) that passes by S. Construct a point F and draw the perpendicular bisector of segment SF. This bisector interecepts line ST at M (Figure 1).

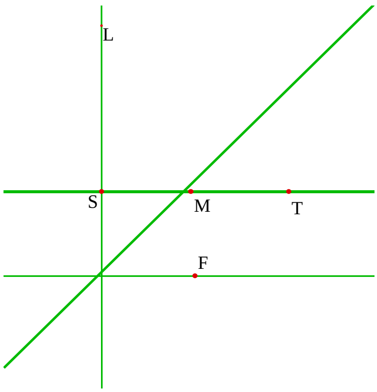

Figure 1. Initial configuration.

(ii) What is the locus of point M when point S moves along line L? Students with the help of the software can easily find that the locus corresponds to a parabola. Indeed, the F is the focus and L is the directrix and it is evident that each point on the parabola satisfies that FM = MS (Figure 2).

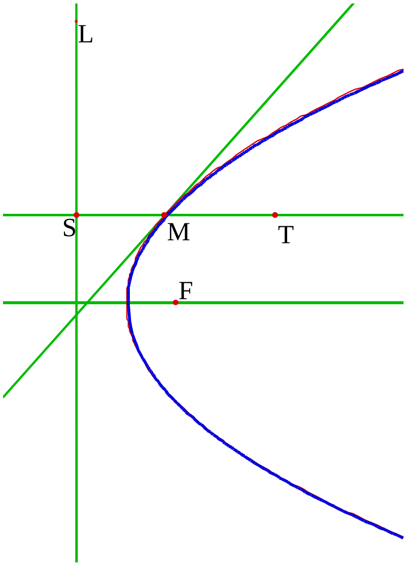

Figure 2. What is the locus of point M when point S moves along L?

(iii) Draw a perpendicular line to L that passes by F and select a point P on this line. Identify point Q on the parabola and draw its reflection point Q’ with respect line PF (Figure 3).

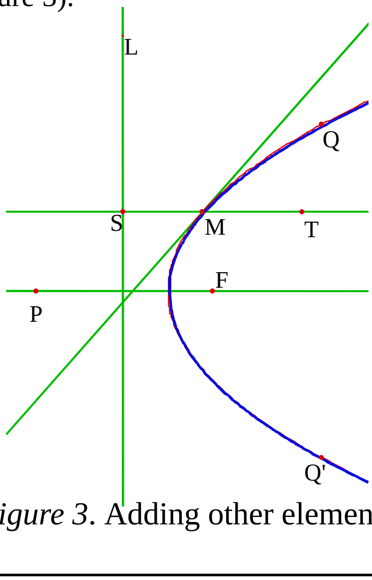

Figure 3. Adding other elements.

iv) Draw line PQ and line Q'F. Identify the intersection point of these two lines as R (Figure 4). What is the locus of R when point Q is moved along the parabola?

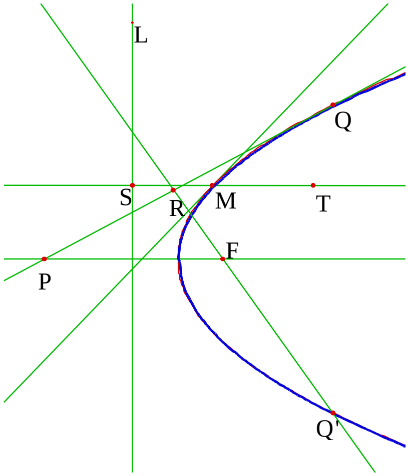

Figure 4. Identifying point R.

It may be difficult for students (and people in general) to imagine and describe the locus of point R when point Q is moved along the parabola with focus F. However, with the help of the software students can carry out this task easily. Thus, by applying the command "Locus" to point R when point Q is moved along the parabola, the following figure appears: (Figure 5).

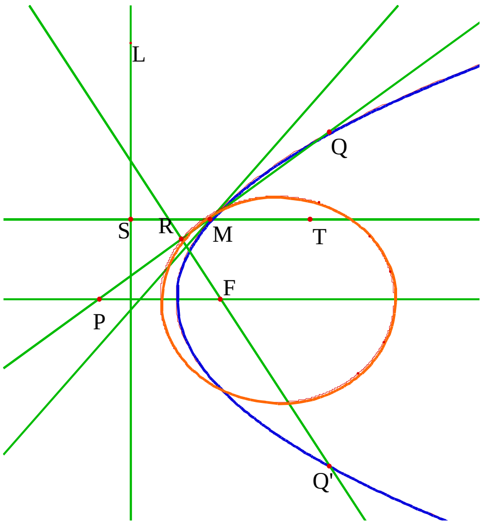

Figure 5. What is the locus of point R when point Q is moved along the parabola?

(iv) Now by moving point P along line FP, it is observed that the locus changes into other figures. For example, when P gets close to point F the following figure appears: (Figure 6).

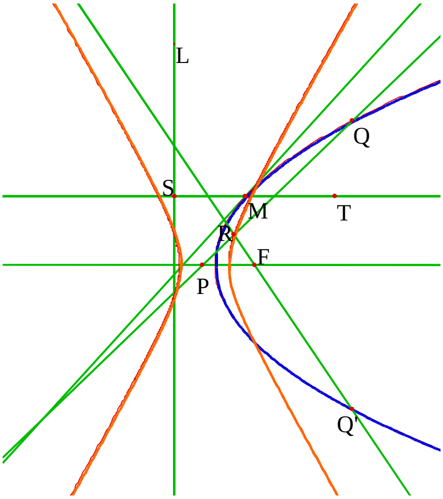

Figure 6. Changing the position of P.

It is also observed that when point P becomes the intersection point of line L and the perpendicular to L that passes by F the Figure 7 will be the original parabola.

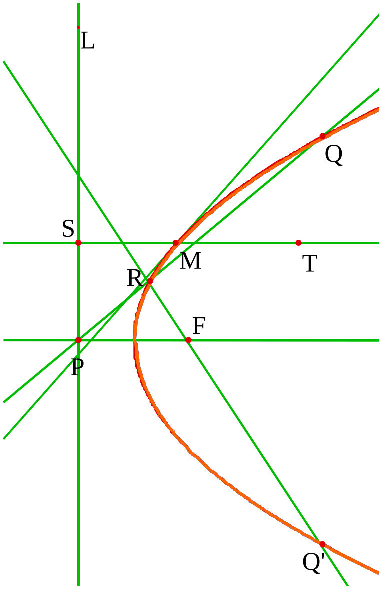

Figure 7. Generating the original parabola.

Students at this stage realized that by moving point P along the perpendicular line to L, the locus produces figures that seem ellipses, hyperbolas, parabolas, or circles. The next task was to argue that the shown figures meet properties that define those figures. Indeed, the software became again a powerful tool to identify main components of each figure (vertex, focus, directrix, center, etc). That is, it was necessary to explain via specific arguments what they observed while moving part of the configuration. In general, students introduced a coordinate system and selected point O as the origin, as a first step to support their conjetures.

During this process, students compared measurements and eventually formulated diverse conjectures to identify each relevant component of the figures. Throughout this task it was clear that the software became a powerful tool for students to verify whether a particular locus held the definition they have studied for that figure. For example, an ellipse is the set of points in a plane whose distances from two fixed points (foci) in the plane have a constant sum. Thus, since it was also easy to assign measurements to particular parts of the figure, then the task offered them an opportunity to verify properties of those figures that they have previously studied. Besides, they were surprised that from a simple construction they could generate all the figures that they have examined in an entire course.

It became clear that a powerful tool to identify and explore mathematical relationships is to trace the locus of any specified object. Thus, when students were asked to find the locus, they showed clear disposition to find and explain the rationale behind each figure described by the point's path as it travels. Nevertheless, they seldom relied on the constructed figure as the only means to explain or prove that, for example, what they saw as an ellipse, they often rushed to say that it was indeed an ellipse. Here, The role of the teacher was to discuss with students the need to present arguments to support what they saw during their interaction with the task. Furthermore, there were cases in which students used and express clear arguments (verbally) to support their conclusions, but they showed serious difficulties to communicate them in writing. So, it was evident that what students achieve via the software (in terms of finding and explaining mathematical relationships) not necessarily gets reflected in their writing.

The above task shows that the use of technology plays an important role in observing and exploring mathematical relationships. However, it is clear that for students to observe and identify, in this case, ellipses, hyperbolas, etc. it seems necessary to have previous knowledge or experiences regarding these figures.

Indeed, the appearance of those figures triggered them a series of fragmented knowledge that they remembered about properties, or definitions of those figures. That is, the software became a tool to examine and contrast ideas held by those students. Here again the teacher's role was to bridge students' previous experience with an "efficient" use of the software. What follows is an illustration of the type of other students' approaches that emerged from introducing technology in a mathematical problem-solving course with the same high school students.

Access basic mathematical resources to work and solve non-routine problems

Several non-routine problems that require powerful mathematical resources for their solution with traditional approaches can become accessible to a variety of students when are approached via technology. We have utilized the TI92 as a means to examine particular or simpler cases associated with various problems. This process leads students to eventually access basic resources that help them to approach them. To illustrate this idea we select a problem suggested by Polya (1945) and later used by Schoenfeld (1985) in his problem solving course. Although the essence of solution process rests on the use of a particular strategy (relaxing the original conditions of the problem and examining particular cases), it is clear that the use of technology offers advantages to visualize and evaluate cases that eventually lead to the solution.

> Inscribe a square in a given triangle. Two vertices of the square should be on the base of the triangle, the other vertices of the square on the two other sides of the triangle, one on each (Figure 8).

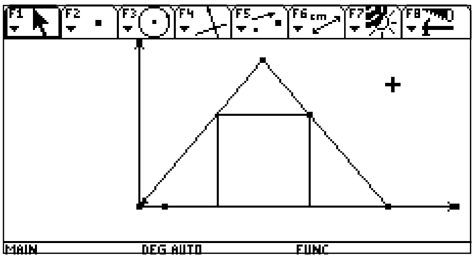

Figure 8. Inscribed square.

This problem was used in a problem solving session with high school students. The students' initial approach was organized through an open discussion of the following questions:

What is the given information? What does it mean to have a triangle? Does it mean that it is possible to know its vertices, or the length of its sides and the measure of its angles? What is the question or task? What are the conditions? Is it possible to quantify properties (areas, perimeters, slopes) of the shown figures and observe a particular relationship? Eventually, students suggested that an important problem solving strategy that can be useful to consider when the problem involves various conditions is to reduce them and explore their behavior through the analysis of particular cases (Polya, 1945).

In this problem a key condition is that the four vertices of the square should be on the perimeter (sides) of the triangle. Hence, a student suggested to picturing some squares with only three vertices on the perimeter (three sides of the triangle) as the Figure 9.

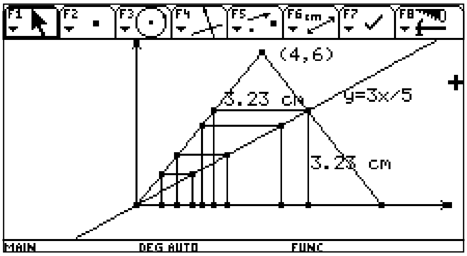

Figure 9. Relaxing original conditions.

The figure shows three squares with three vertices on the sides of the triangle. Here, students observed that the fourth vertex for each square (the one that is not on the sides) seemed to rest on a line. Indeed, cabri geometry helped them check easily whether the points are collinear. It was also easy to verify that the intersection of this line with the side of the triangle is the fourth required vertex (Figure 10).

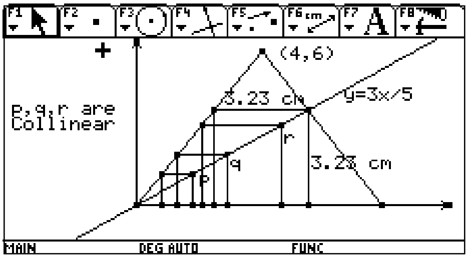

Figure 10. Identifying the inscribed square.

An important question here is: Is it possible to inscribe a square in any given triangle? Although the example shown above only represents a particular case, it is important to observe that the software permits to move vertex (4, 6) on the plane and check that for all the new triangles, that appear while moving such vertex, will maintain the inscribed square. That is, point *p*, *q*, & *r* will be collinear and the intersection of the line pq with the triangle side becomes the fourth vertex of the inscribed square. Indeed, these ideas can be applied to treat the general case in which (a_1, b_1); (a_2, b_2) & (a_3, b_3) are the vertices of the given triangle.

Students might also check the collinear property of points *p, q,* and *r* by using the distance criterion or slope condition. Is $pq + qr = pr$? Or is the slope of *pq* equal to the slope of *qr*.

Some students also recognized that they could use another feature or tool provided by the software (locus) in which they only needed to draw one square. That is, they drew one particular case, as before, and then asked by the locus of the fourth vertex when moving or changing one side of the square (Figure 11).

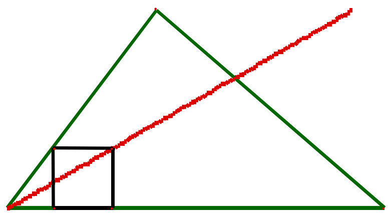

Figure 11. Locus of the fourth vertex.

Students stated that the key feature in this problem was to relax its original conditions. The software functioned as tool to visualize what was not evident in the figure (the alignment of the fourth vertices). Again, what did it inspire students to think of a problem solving strategy? An initial response to this question is that the core of the activities implemented during the course emphasized both the students' development of a set of problem solving strategies and the use of mathematical content normally studied in high school curriculum.

Find and explore different conjectures. The use of technology offers great potential for students to search for invariants and propose corresponding conjectures. Here, we illustrate an example in which students build dynamic environments to represent problems that eventually lead them to propose conjectures. The software becomes a tool for students to look and document the behavior of objects and relationships.

(a) Two congruent squares are placed so that vertex M of square MOVE lies at the center of square STAY. Square MOVE can rotate freely as it pivots about point M. When is the area of the shaded region the largest? Explain clearly and completely (Figure 12).

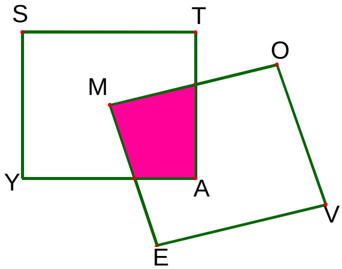

Figure 12. Two squares.

By using Cabri Geometry it is possible to construct both the fixed and rotating squares. The software allows students to explore cases in which they can quantify and record the value of area or perimeter of the shaded region. Again, a salient strategy here is to explore particular cases.

How does the intersecting area behave when square MOVE rotates around its vertex M (Center of the fixed square)? How can we calculate the intersecting area? Students realized that they could record information of several particular cases. For example, the figures below represent two cases in which the intersecting area is 1/4 of the area of one square (Figures 13 and 14). Indeed, these two cases include the analysis of 8 special cases in which the shaded area is always 1/4 of the area of one square.

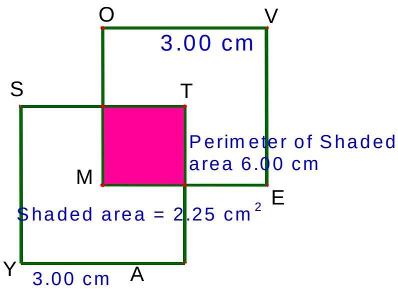

Figure 13. A particular case.

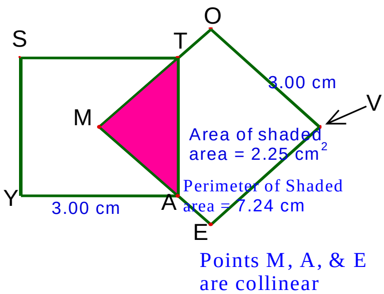

Figure 14. Another special case.

The software also shows that for the case (Figure 15) the value of the intersecting area remains constant. Here, a conjecture emerged from the analyzed cases “**the intersecting area of the two squares is always 1/4 of the area of one of the square**”. To prove this conjecture, some students examined the symmetry of the figure by proving that triangles KMW & RMV are congruent (Figure 16).

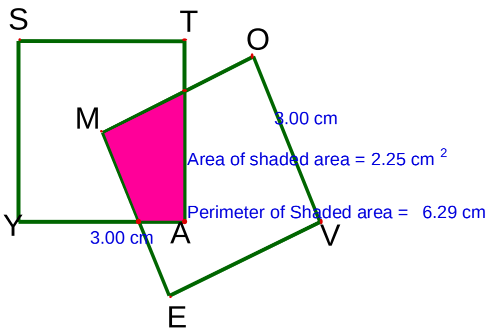

Figure 15. Shaded area is one fourth of square area.

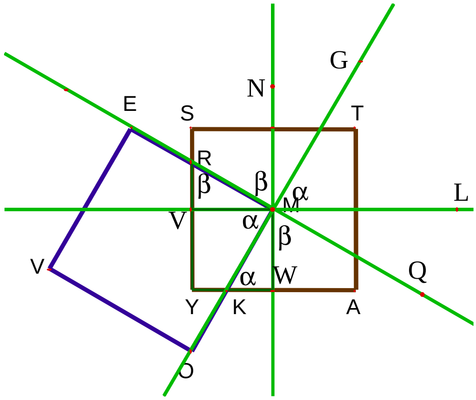

Figure 16. Supporting the conjeture

Other related questions that emanated from the interaction with this problem included: How does the perimeter of the intersecting region behaves? When does it reach its maximum/minimum value?

(b) Given an equilateral triangle ABC and a random point P is placed inside of that triangle (Figure 17). If from P a perpendicular segment is drawn to each side of that triangle, then explore the behavior of the sum of the three perpendicular segments (Figure 18).

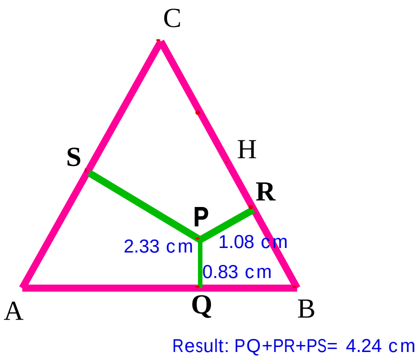

Figure 17. An equilateral triangle and an interior point.

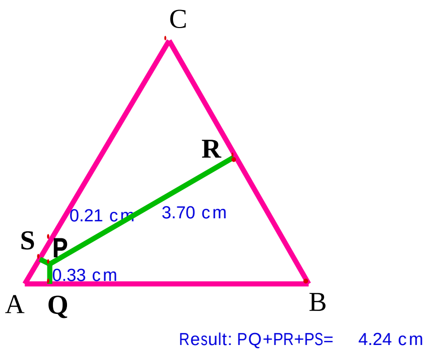

Figure 18. Exploring the sum of the three perpendiculares.

A conjecture that some students proposed from moving P inside triangle ABC is that **the sum of the perpendicular segments drawn to each side is equal to the length of the altitude of the triangle**. This was based on observing that when point P gets close to any vertex, then two of the perpendicular segments get decreased significantly and the constant sum gets close to the length of the third perpendicular (the altitude). Students were not able to prove this conjecture on their own. However, when the teacher proposed to visualize the triangle in terms of three triangles and asked them to express the area of the original triangle as the sum of the three areas, they verified such conjecture.

In order to prove this conjecture we drew segments from P to each vertex of triangle ABC (Figure 19). Here, three triangles can be identified: ΔAPC, ΔAPB & ΔBPC . Thus, the area for each of these triangles can be expressed by (AC)(PS)/2; (AB)(PQ)/2 and (BC)(PR)/2 respectively (PS, PQ, and PR are perpendiculars segments to each side). Since the triangle is equilateral then AC = AB = BC = ℓ. The total area of the original triangle can be expressed by (ℓ)h/2, where h is the altitude of the triangle. It is observed that (ℓ)(h)/2 = ℓ (PS + PQ + PR)/2. Here we have that h = PS + PQ + PR.

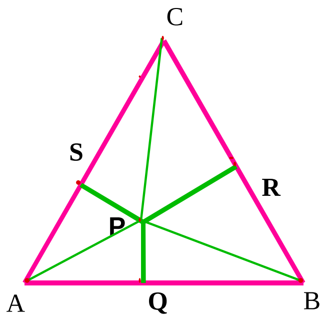

Figure 19. Providing an argument.

An important problem solving strategy used in this problem is that “it is often useful to express important information of the task in two different ways”.

Generalize mathematical patterns using defined and recursive functions. An important goal during learning mathematics is that students develop different strategies to find and analyze the behavior of mathematical relationships that emerge from the consideration of particular cases. For example, we illustrate that the TI-92 can be used to identify a formula or general expression for data that represent a recursive phenomenon (evolution of a medical treatment). This task was discussed with the entire class during two sessions (three hours). The teacher focused the discussion on problem solving phases that include understanding the statement and key information of the task, ways to represent and plan the solution, and qualities associated to solutions achieved via table, graphic and algebraic approaches (Santos, 2002).

Treatment Description: The patient will take some tablets and receives the following information:

i. Doses or amount of active substance of the tablet: 16 units

ii. When the patient takes the tablet, his organism begins to assimilate the active substance and 10 minutes after; his body will assimilate the total amount of 16 units. That is the patient organism will assimilate the doses 10 minutes after he took the tablet.

iii. When the patient organism assimilates the total doses of the tablet, his organism begins to eliminate the medication. The patient must take the next tablet when the previous doses are reduced to a half of that amount. The physician tells the patient that this reduction takes place every four hours. Thus, the patient will take the second tablet when the previous amount (16 units) assimilated from the first tablet is reduced to 8 units. This doses reduction will occur four hours later since the patient takes the first tablet. Thus when the patient registers 8 units, he will take the second tablet and his body will assimilate again the 16 units ten minutes later, here the amount of units stored in his body will be 8 + 16. Here, again his body commences the elimination period and when it reaches half of this amount (12 units after four hours), then the third supply takes place, and so on.

iv. The patient will follow this treatment during a week.

Based on the treatment description there is interest to analyze the amount of medication stored by the patient organism at different stages or moments of the treatment. In particular, What amount of medication is stored by the patient when he takes each tablet/ and 10 minutes later? Is it possible to discuss the evolution of the patient treatment in terms of mathematical resources?

Understanding the task. In order to respond the above questions, students addressed the need to identify relevant information associated with the treatment. Data that they judged to be relevant included the amount of active substance of the tablet (16 units), the time in which the patient assimilates the active substance (10 minutes); frequency of dosage, amount of active eliminated by the patient during each drug admission.

Use of a table. Two ways were proposed to organize the information provided in the task, a systematic list and a table initially. Later, students decided to use a table to present the information. An important aspect to decide was to determine the entries that could help display the behavior of the relevant information during a period of time. Thus, the entries of the suggested table included frequency of dosage, total time (hours) elapsed from the beginning of the treatment at each take, amount of active substance at each supply and ten minutes after each supply.

Supply # (every four hours)	Elapsed time (hours)	Amount of active substance at each supply	Amount of active substance 10 minutes after each supply
1	0	0	16
2	4	8	24
3	8	12	28
4	12	14	30
5	16	15	31
6	20	15.5	31.5
7	24	15.75	31.75
8	28	15.875	31.875
9	32	15.9375	31.9375
10	36	15.96875	31.96875

Based on this table all students were able to describe the amount of active substance stored in the patient's body at the moment of each supply and ten minutes after. They notice that in both cases the amount of substance gets stable in the patient's body.

(b) Graphic Representation. By using the data shown in the table, it is possible to present a visual approach or graphic representation of relevant information. Here, students were asked to represent the data shown in the table into a graph. A student proposed to use Excel (Figure 20).

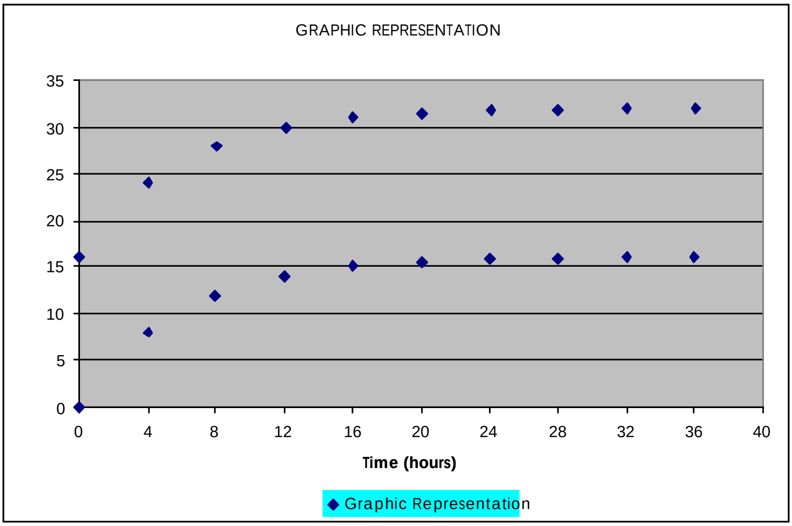

Figure 20. Graphic of behavior of the active substance.

What differences do you see between data represented in the table and data displayed in the graphic representation? Students again observed that both representations show that the amount of active substance at each taking and 10 minutes after converge to 16 and 32 respectively.

(c) An Algebraic Approach. Is it possible to analyze regularities observed in the two above representations via the use of algebra? Here, the information of the table will be used to examine the pattern.

Supply No.	Amount of active substance stored in the patient's body at each supply
1	0
2	$\frac{0+16}{2}=\frac{16}{2}$
3	$\frac{\frac{16}{2}+16}{2}=\frac{16+2\times16}{2^2}$
4	$\frac{\frac{16+2\times16}{2^2}+16}{2}=\frac{16+2\times16+2^2\times16}{2^3}$
5	$\frac{\frac{16+2\times16+2^2\times16}{2^3}+16}{2}=\frac{16+2\times16+2^2\times16+2^3\times16}{2^4}$
6	$\frac{\frac{16+2\times16+2^2\times16+2^3\times16}{2^4}+16}{2}=\frac{16+2\times16+2^2\times16+2^3\times16+2^4\times16}{2^5}$

$$C_n = \frac{16 + 2\times 16 + 2^2 \times 2^3 \times 16 + \cdots + 2^{n-2} \times 16}{2^{n-1}} =$$

$$= 16\left(\frac{1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-2}}{2^{n-1}}\right)$$

$$= 16\left(1 - \frac{1}{2^{n-1}}\right)$$

Students were asked to obtain the above formula via the TI-92 calculator. Here it was important to introduce the general expression C_n in a condensed form. By using the commands expand and factor from the Algebra menu, the general formula was easily obtained (Figure 21).

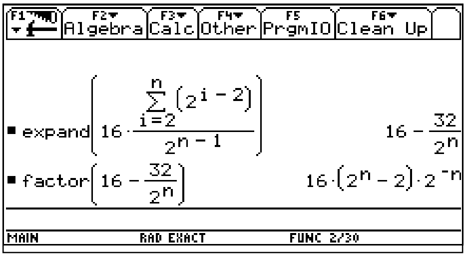

Figure 21. Determining a formula

At the first glance they did not see the equivalence between the two formulae; however, when they transformed the calculator formula via simple algebraic operation, they recognized that it was the same formula.

They also realized that the expression that described the amount of active substance stored by the patient 10 minutes after each supply was:

$$A_n = C_n + 16 = 16\left(1 - \frac{1}{2^{n-1}}\right) + 16 = 16\left(2 - \frac{1}{2^{n-1}}\right).$$

The group discussion focused on the advantages provided by the algebraic approach. They recognized that the formula was useful to calculate the amount of active substance at any supply. They also mention that it was important to work with the rational representation of data to detect a pattern or formula.

Final Remarks. Different aspects of the mathematical practice are displayed while using technology to approach problems. In particular, representing data seems to be an important step for accessing basic mathematical resources that become crucial to approach the tasks. Working with dynamic environments helps students identify and examine particular conjectures. It is also shown that the process of working with the problems via technology introduces a natural environment for posing and pursuing related questions. Indeed, seeing different dynamic representations of the problems helps students visualize connections and realize other approaches to the problem (Santos & Díaz Barriga, 2000). It is also evident that even when the problems or tasks may not call for the use of the computer directly, the software could become a vehicle for examining their mathematical qualities. As a consequence, students get engaged in the process of utilizing diverse representation as a means to approach the problems and explore other related questions.

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Computer Mediated Learning: An Example of an Approach[1]

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ABSTRACT. There are several possible approaches in which dynamic computerized environments play a significant, and possibly unique, role in supporting innovative learning trajectories in mathematics in general, and particularly in geometry. These approaches are influenced by the way one views mathematics and mathematical activity.

In this paper we briefly describe an approach based on a problem situation and our experiences using it with students and teachers. This leads naturally to a discussion of some of the ways in which parts of the mathematics curriculum, classroom practice, and student learning may differ from the traditional approach.

Key Words: dynamic geometry, representations, functions, explanations, hypothesizing.

Rationale

Dynamic computerized environments constitute virtual labs in which students can play, investigate and learn mathematics. The following are some of the characteristics in which such labs have the potential to nurture, provided they are accompanied by suitable curriculum materials and classroom practices.

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Visualization

"Visualization generally refers to the ability to represent, transform, generate, communicate, document, and reflect on visual information" (Hershkowitz, 1989, p. 75). As such it is a crucial component of learning geometrical concepts. Moreover, a visual image, by virtue of its concreteness, "is an essential factor for creating the feeling of self-evidence and immediacy" (Fischbein, 1987, p.101). Therefore, visualization "not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution." (ibid.) Dynamic environments not only enable students to construct figures with certain properties and thus visualize them, but also allow the user to transform those constructions in real time. This dynamism may contribute towards forming the habit of transforming (either mentally or by means of a tool) a particular instance, in order to study variations, visually suggest invariants, and possibly provide the intuitive basis for formal justifications of conjectures and propositions.

Experimentation

Besides visualization, playing with dynamic environments allows students to learn to experiment, and "to appreciate the ease of getting many examples..., to look for extreme cases, negative examples and non stereotypic evidence..." (Yerushalmy, 1993, p. 82). They can do so not only by looking, but also by measuring, comparing, changing (or even distorting) figures and making supporting constructions rather easily. The information obtained in this way can be a step towards stating generalizations and conjectures, which in turn should serve as the basis for the next important characteristic.

Surprise

It is unlikely that students will direct their own experimentation fruitfully from the outset. Curriculum activities, such as problem situations, should be designed in such a way that the kinds of questions students are asked can play significant roles in the depth and intensity of a learning experience. One significant type of question to accompany the experimentation, is to require students to make explicit and thoughtful predictions about the outcome of a certain phenomenon or action they are about to undertake. Making such predictions explicitly a) nudges students to be clearer about how they envision the situation they are working on, b) brings students to the position of "prediction owners" and thus they are likely to be more careful in the way they think about it, and as a consequence, more committed to the situation, and c) create expectation and motivation for the actual experimentation. The challenge is to find situations in which the outcome of the activity is unexpected or counter-intuitive, such that the surprise (or puzzlement) generated creates a clear disparity with explicitly stated predictions. (Students working on such activities are described in Hadas and Hershkowitz, 1998 and 1999). This can be the trigger for nurturing the students' own need for re-inspection of their knowledge and assumptions, establishing opportunities for meaningful learning.

Feedback

Surprises of the kind described above arise from a disparity between an explicit expectation from a certain action and the outcome of that action. The feedback is provided by the environment itself, which re-acted as it was requested to do. It is the "dry" consequences of the student action that are to be confronted. Such direct feedback is potentially more effective than the one provided by a teacher, not only because of its affective underpinnings (lack of value judgment), but also because it may engage motivation to re-check, revise the prediction and induce the need for proof. The assumption is that there exists appropriate previous knowledge to appreciate such feedback as meaningful, and that this would serve as the basis for reflection. Moreover, we maintain that the problem situations to be designed for computerized environments (an example of which we propose in this paper) and the teacher implementation of them, should support students to appreciate and formulate "conflicts" or inconsistencies and to find ways to resolve them.

Need for proof and proving

Dreyfus and Hadas (1996) discuss and exemplify how one can capitalize on such student surprises in order to instill and nurture the need for justification and proof. Following a surprise, many students may require a proof, maybe not explicitly, but by demanding from others or from themselves an answer to their 'why' (or 'why not').

The engineering of successful tasks should take into account something else as well. If possible, the proof, namely the answer to the 'why', should arise from the observations and the re-visions of the experimentation process itself. In other words, the experimentation-feedback-reflection cycle should provide the seeds for the argumentation which helps to explain and prove an assertion. In this way the dynamic environment really supports the 'closing of the circle'.

In the following we analyze a problem situation which is "…sufficiently concrete - 'well connected to what the learner already knows' - as well as interesting and soluble -" (Noss and Hoyles, 1996, p. 67) with the potential to encourage students to generate genuine and spontaneous questions for themselves. The general theoretical perspective which inspired the design of the problem situation is based on the main ideas by Duval (1999, for example). Briefly stated, the theory claims that an essential component of learning mathematics is the coordination of different representations of a same idea or concept. Such a coordination implies manipulations within a certain representation and translation across representations. The problem situation, we present and analyze below is just one example of many, which we believe illustrates and substantiates the claims and processes described above. The description includes the problem situation itself, its implementation with a particular geometry system (Geometry Inventor, 1994), and anecdotal data from classrooms (9th and 10th grades, ages 15-16) and from teacher workshops in Israel. The background knowledge of the students (and of the teachers) included: the concept of function in general, symbolic, tabular and graphical (Cartesian) representations of a repertoire of functions, and a substantial amount of Euclidean geometry. The data come from several trials. However, in the following, we neither analyze protocols nor bring verbatim quotations. Rather we attempt a) to describe the flow of the activity and the way it interweaves different ideas across different representations, b) to propose the intermediate prompts or leading questions (which we refined after several trials) and a sample of interesting responses (not necessarily correct), and c) to convey the general spirit of the special characteristics of the activity, as an example of the pedagogical and cognitive potential of the use made of computerized technologies. We intersperse comments to analyze and highlight the mathematical, cognitive and pedagogical issues our experiences raise. Finally, we reflect on possible implications.

The Problem Situation - First Phase

The first step consists of building on the "drawing board" (of the Geometry Inventor), two segments of length 5 with a common end point. Joining the two other end points produces an isosceles triangle (Figure 1). The dragging of, for example, the vertex C, yields many possible isosceles triangles whose equal sides are 5 (as in Figure 2).

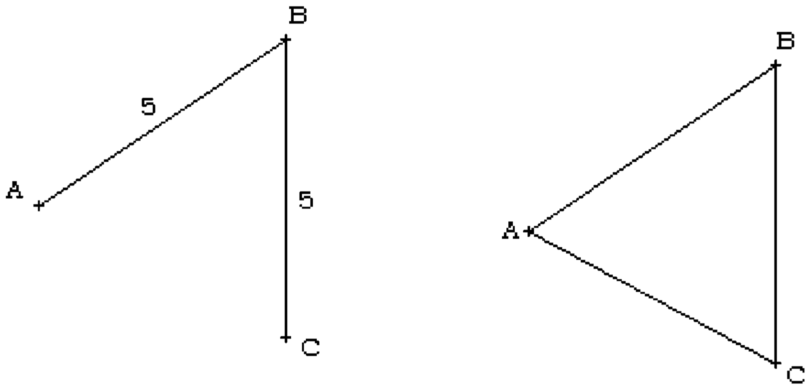

Figure 1. Constructing a dynamic isosceles triangle.

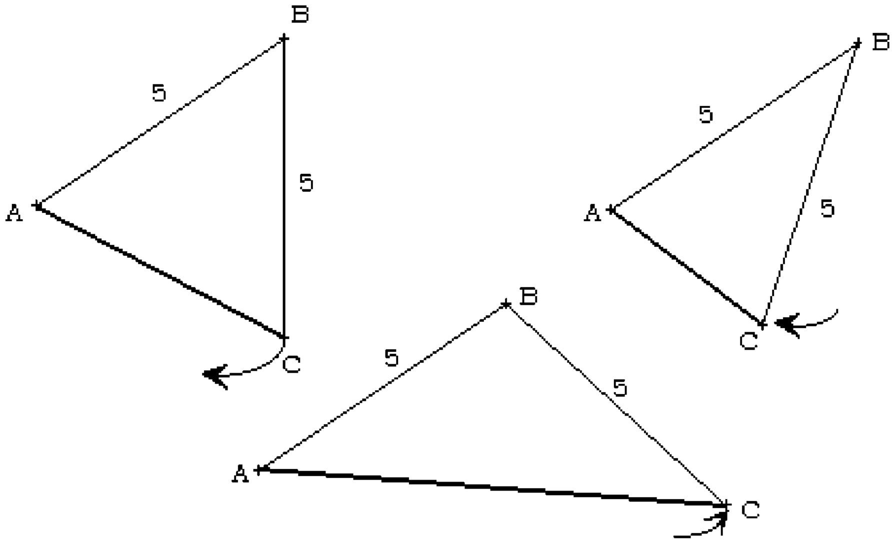

Figure 2. The dynamic change of the triangle.

The effect of the “continuous” and dynamic variation sets the scene for our first question: what changes and what stays the same? Students usually pointed to the given equal sides as constant (some even mentioned the sum of the internal angles) and they mentioned the most obvious variable: the third side AC. When required to find more variables, they also mentioned the angles and the area of the triangle. We proposed to study the variation of the area as a function of AC. By dragging the vertex C, the domain of variation of AC (between 0 and 10) became evident (Figure 3).

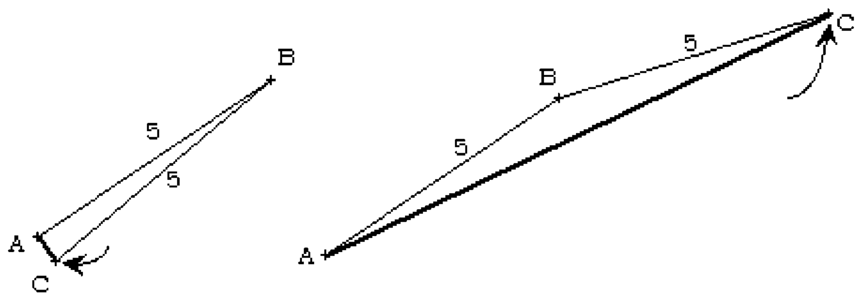

Figure 3. Visualizing the domain of the function.

Also the range (or co-domain) became evident from the dynamic changes: the minimum area is 0 (when AC is 0), it then increases, but at some point it starts to decrease until it reaches 0 again when AC becomes 10.

Our next question followed quite naturally: predict when the area reaches its maximum value.

One answer consisted of pointing at one or more triangles which look like having a maximum area and saying “the largest triangle is probably one of these”. Another answer, more frequent among teachers, was that the equilateral triangle (AC = 5) has the largest area. Given the medium, it is natural that most predictions were geometrical characterizations of the triangle with maximum area, rather than a numerical value for AC or the area.

All predictions were recorded (the correct one was not the most common). Then we directed the work back to the “drawing board” for further checks and feedback.

At this stage, we suggested using measurements, which change in real time with the changes in the triangle. Measurements play a significant role in providing useful feedback (and especially counterexamples).

Figures 4a and 4b illustrate how the screen might look in two discrete shots when measurements are introduced.

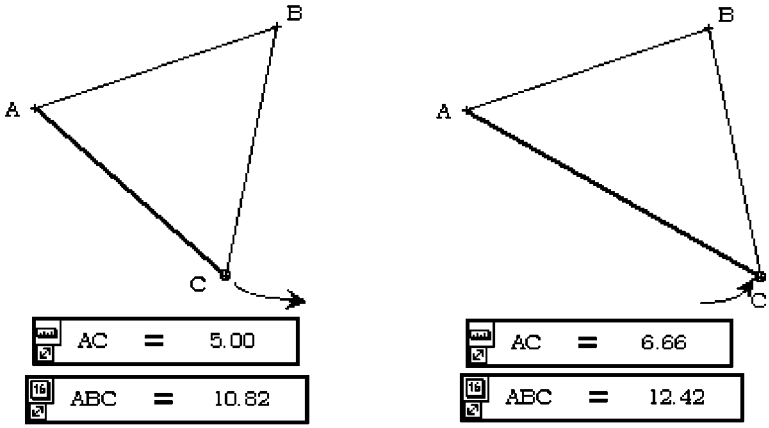

Figure 4. Numeric and figural representations.

Dragging C "beyond" the equilateral triangle (as shown in the transition from Figure 4a to 4b), usually helps to dismiss the conjecture that it is the one with maximum area: the area values increase beyond 12, towards 12.4, 12.5 and then begin to decrease. The surprise caused by the empirical (numerical) refutation of the "equilateral triangle conjecture" sets the scene for the geometrical work on the 'why'.

Recalling the half-base-times-altitude formula was usually the next step. However, when students or teachers chose AC as the "base", the difficulty was immediate: how to establish the maximum value for the product of two variable quantities (AC and the corresponding altitude)? Thus, someone in the class suggested (and in some classes we did so) changing the perspective and looking for another base-altitude pair in which one of these values remains constant; for example, AB and the corresponding altitude, as shown in Figure 5.

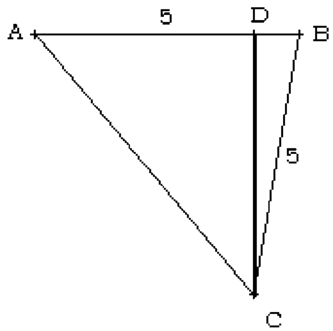

Figure 5. Change of perspective.

By dragging C again, some were immediately able to "see" not only the correct answer, but also its geometrical justification: the area will be maximum, when the altitude is the largest, namely when DC coincides with BC (Figure 6 illustrates the visually strong dynamic increase of the altitude's size until it reaches its maximum).

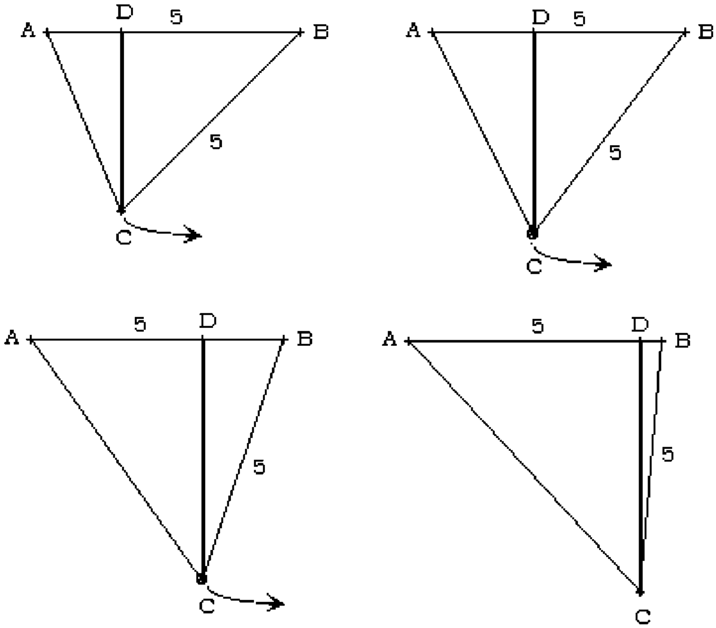

Figure 6. Representing the justification by the dynamic change.

Thus the maximum is reached when ABC is right-angled. This finding is coherent with the 12.5 value for the maximum area (half the base times the altitude is $\frac{5 \times 5}{2}$), already observed previously.

To summarize so far, the exploration began with a family of isosceles triangles with equal sides of fixed length changing dynamically, observing variation and making predictions. In most cases, the predictions lead to a surprise, which caused the "why". The answer to that why was induced by further experimentation, leading to the right answer and its justification (the variable altitude DC is always a leg of the right-angle triangle BDC and thus shorter than the hypotenuse of length 5. This is true except when DC coincides with the hypotenuse BC, in which case it reaches the maximum length). Such argument is grounded on a visual-geometrical-dynamic view of the static symbolic formula for the area. Note that the formula was not invoked, nor used, in an operational sense, rather it qualitatively-visually guided the search for the triangle with the maximum area, based on qualitative geometrical arguments.

Usually, at this point, it seemed that the situation had been fully explored, and that there were no more interesting issues to pursue. However, we decided to make use of the of a feature of the software by which one can draw (after matching variables to axes and setting appropriate scales) a Cartesian graph of the variation in real time, as the figure is being dragged. But first, we asked for predictions of the form of the graph of the variation of the area of ABC as AC changes (i.e. as a function of AC). A large majority of participants (students and teachers alike) predicted a parabolic graph, supporting their prediction by saying that the initial area is 0 when AC is 0, it increases to a maximum and then decreases to 0, when AC is 10.

Schwarz and Hershkowitz (1999), report that many students tend to think in terms of two prototypical functions: linear and quadratic. In this particular problem situation, this manifested itself explicitly: a graph which describes an increase from 0 towards a maximum and then back to 0, must be a parabola. Possibly, as we shall see later, there was also an implicit expectation of symmetry as well. In order to check the prediction, we asked students to make use of the software to draw the graph. Figure 7 shows two static snapshots of what happens dynamically on the screen. Dragging C further, Figure 7 turns into Figure 8.

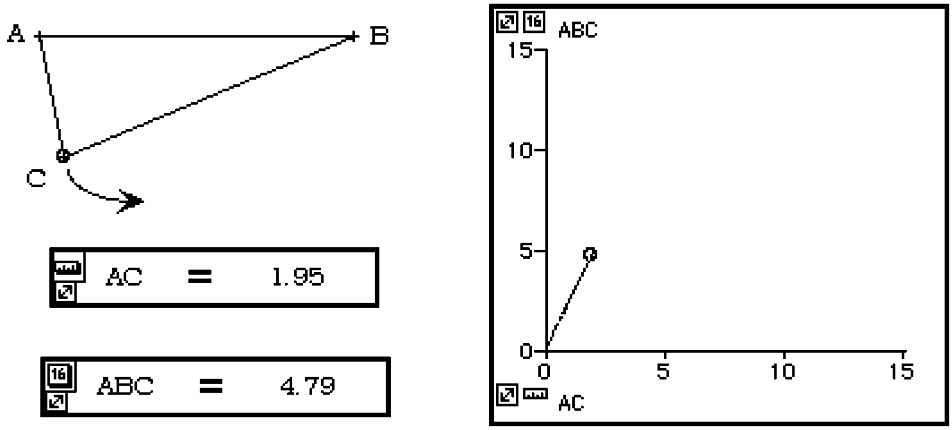

Figure 7. One incipient snapshot of the first graph.

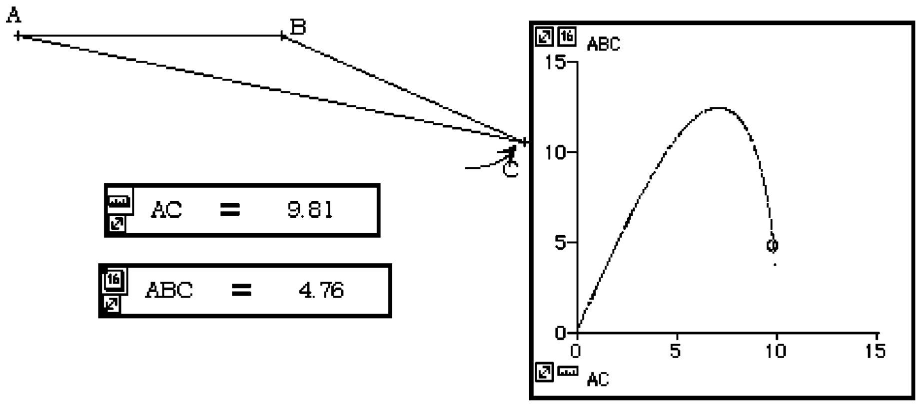

Figure 8. The asymmetry of the graph.

At this point, most were surprised to notice the apparent asymmetry of the graph (with respect to a vertical axis), against their explicit expectation of a parabola, and posed the question “why?” or “why not (symmetrical)?”

In our trials, the realisation (and subsequent explanation) that the maximum area is reached when the triangle is right-angled, was not explicitly connected to the possible shape of a graph (and why should it be?). However, against the visual impact of the asymmetry of the graph and the need to explain it, and (in some cases) following our questions (“what would it mean for the graph to be symmetrical? Where would its maximum be?”), attention was redirected to the values of AC for which that maximum is reached. As a result, some produced the answer: among other things, symmetry of the graph would have implied that the maximum is in the middle of the domain (AC = 5, namely when the triangle is equilateral), but that was already discarded. The maximum was somewhere else - beyond the middle of the domain- when the triangle is right-angled, for which the value of AC is $\sqrt{5^2+5^2} \approx 7.07$ (found using Pythagoras).

For some, this explanation ended the matter. For others (especially teachers), in spite of understanding the explanation of why the graph cannot be symmetrical, it did not. They felt and expressed discomfort, because their intuitive feeling of symmetry, which now became explicit, was still strong. Perhaps, it was even reinforced by what they have found about the maximum area being for the right-angled triangle, since this triangle may appear to be “half way” between the extreme values of the domain when one makes the dynamic changes. We thought that we needed to address this issue, even if we had not planned to do so. We therefore suggested for discussion the proposal that maybe there is indeed something symmetrical going on here. The discussion and the reasoning went more or less as follows: for the graph to be symmetrical, it is necessary that the x-value for the maximum area splits the domain of the ‘independent variable’ into two halves. But, the maximum is obtained for a right-angled triangle, where the right angle is between the two constant sides. That angle varies between 0° and 180°; therefore if the independent variable were to be that angle, the x-value of the maximum is indeed the mid-value of its domain. Thus if we graph the area as a function of the angle between the equal constant sides, it might be symmetrical! An empirical confirmation of symmetry was obtained by taking the angle ABC (instead of the side AC) as the independent variable to draw the graph of the area (as a function of the angle). We did not pursue this further to show that the graph corresponds to $y = 12.5 \sin x$, which although it is not a parabola, is indeed symmetrical.

In this case, we encouraged the examination of the graphical representation to enable the re-viewing of the phenomenon. This led to an awarness of properties which went unnoticed for many (or even counter-intuitive for some) when merely observing the phenomenon itself. Moreover, the realisation of these properties can be taken as a trigger to re-analyze the sources of a strong intuition (of some), which was shown to be “wrong” in some sense and “right” in another.

So far, the exploration of the situation, which took advantage of the special features of the computerized environment, was based on

- the dynamic manipulation of the situation itself (i.e., dragging one vertex of the triangle and noticing/conjecturing properties);
- measurements (which change in real time as the triangle changes and thus enable quantification of the visual phenomenon observed);
- the graph (interpreting the graph as another descriptor of the situation helped to identify properties previously unnoticed); and
- questions, discussions and reflection, based to a large extent on making sense of outcomes which were at odds with those predicted.

“Absent“ from the activity until now, was the symbolic representation of the function describing the variation of the area of ABC as a function of the variable side AC.

The symbolic representation of the problem is certainly compact, precise and general ($A = \frac{x\sqrt{100 - x^2}}{4}$, where x is the length of AC and A the area), and it encapsulates all the information we found above (domain, co-domain, asymmetry etc.). However, if this problem were to be investigated by first creating the symbolic representation, we would have distanced ourselves from the phenomenon itself, by trying to decode the somehow cryptic symbols.

Consider, for example, finding the maximum symbolically. Firstly, it may imply knowledge and proficiency in algebraic techniques (possibly calculus). Secondly, such a treatment directs attention and mental energy to rather syntactic issues, relegating or "forgetting" for a while the reference situation, and in the best of cases, postponing meaning until a result is obtained. Such a trajectory does not always allow for raising and discussing 'why' questions. Thirdly, the symbolic solution, by its very static and general nature, seems to engulf, and thus dissociate itself from, the visual and dynamic views of the problem and the nuances thereof as described above.

In our trials, we introduced the symbolic representation *after* the investigation (described above) took place. In the spirit of developing "symbol sense" (Arcavi, 1994), the emphasis then became not on learning about the situation from the symbols, but mostly to decode and trace how the information, we already know and expect, is expressed in symbols. As we shall see below, in the next phase of the activity, symbols, when brought at the end of the exploration, can also play a further, more explanatory, role.

The area of the triangle was explored as a function of AC (the variable side) and we also made reference to the area as a function of the angle (between the two equal sides). Therefore, it was natural to propose exploration of the area of the triangle, this time as a function of yet another variable: its altitude to the variable side (from B to AC). What would the graph look like, symmetrical or asymmetrical?

Playing with the software and looking at the measurements, suggested to many that the graph may look similar to the graph of the area as a function of the base. We proposed drawing the two graphs in juxtaposition, for which the screen looks like Figure 9 (1).

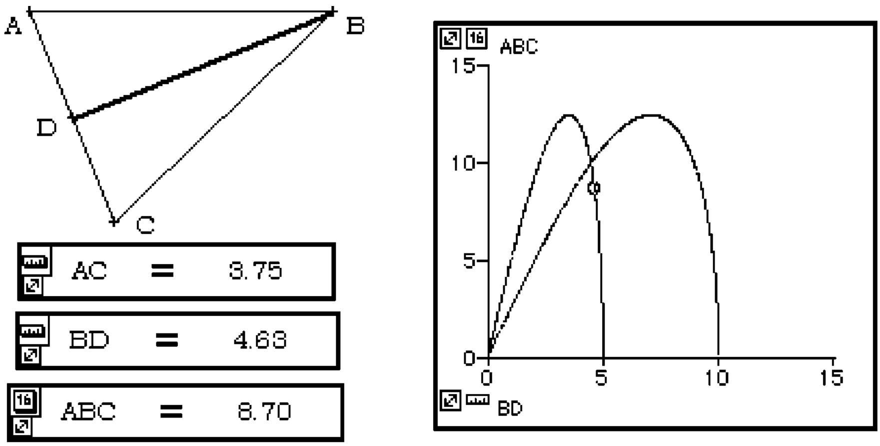

Figure 9. The two graphs in juxtaposition.

Some asked why is the graph of the area as a function of the altitude "thinner" than the other? And we added other issues to consider such as: the graphs appear to reach the "same height" - do they? If so, why, otherwise, why not? What is the meaning, in terms of the geometrical situation, of the point of intersection of the two graphs?

Again, the observation of a certain representation, in this case the graph, posed questions whose answers should be looked for in the phenomenon. We skip here the details of the discussion of these questions, in order to focus on the flow of the problem situation towards what we believe was its "climax".

The problem situation - second phase

Following again the "what if?" approach, the question became: "So far, we explored isosceles triangles where the equal sides have a fixed given value. What would happen if we make a 'small' change, so that the triangle is not isosceles, but "close" to being one?"

The software allows for changes which propagate to the whole construction to be made easily. Thus, using this facility, we proposed changing only one of the fixed sides of length 5 to length 4 (leaving the other with length 5). Exploring how the area varies as a function of the variable side leads to the graph in Figure 10.

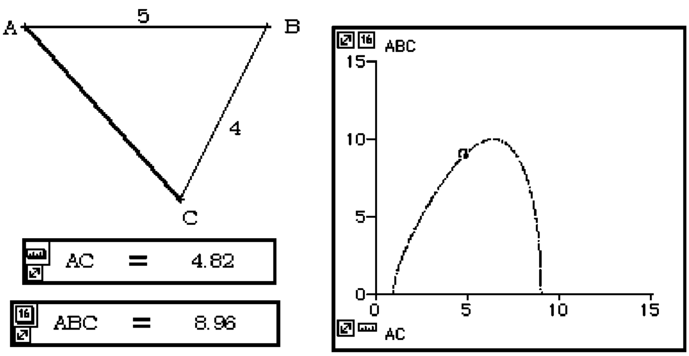

Figure 10. The area as a function of the variable side.

The small surprise which we explored here was that the graph "does not start" at the origin. However, the greatest surprise came in the next step. Here again we proposed to investigate the area as a function of the altitude to the variable side. The following (Figure 11) is a series of snapshots of the graph as it is drawn by the computer when the vertex C is dragged (when starting with triangles for which the altitude falls inside).

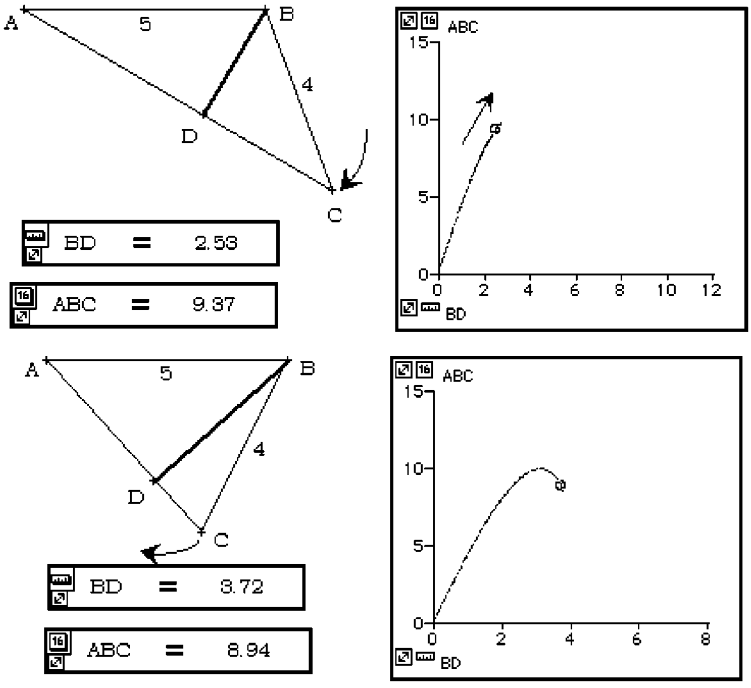

Figure 11. Incipient snapshots of the area as a function of the altitude.

The incipient graph seemed to be very similar to those previously obtained. In some classes, we did the dragging in front of the class. The general expectation was for a similar shape to those previously obtained (as in Figure 9). However, further dragging yields what we see in Figure 12 [(2)].

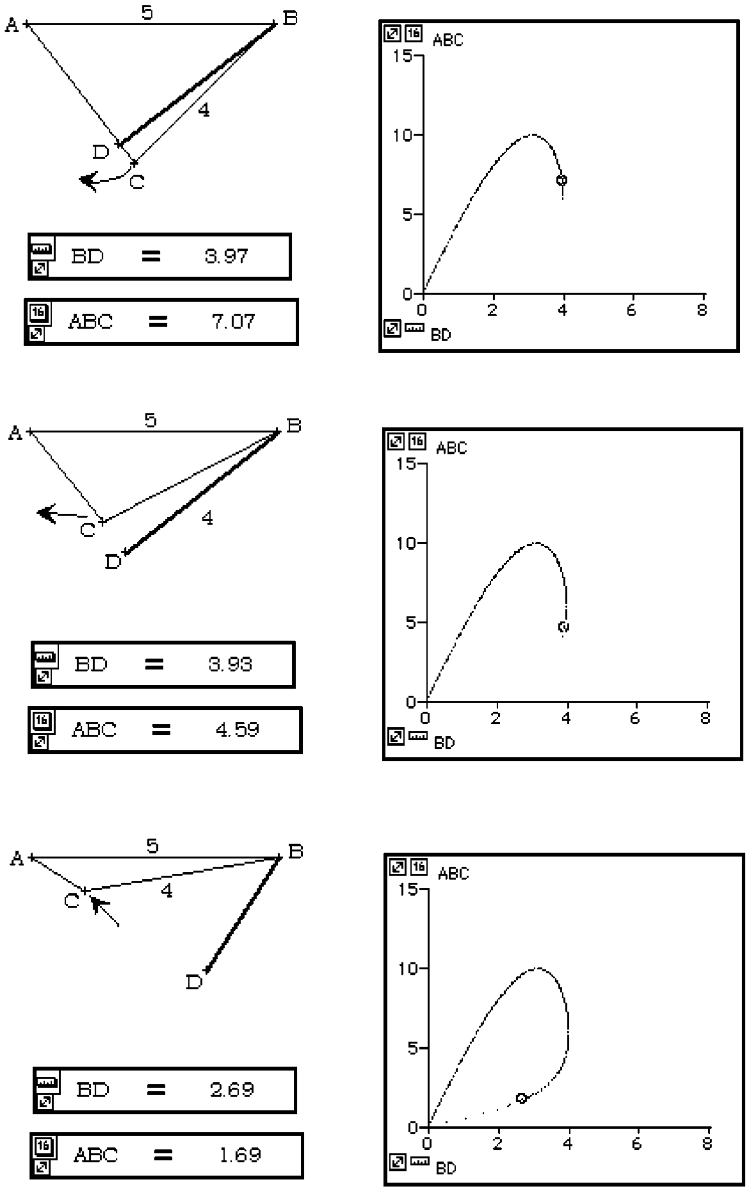

Figure 12. The surprising graph.

The great majority of teachers were more surprised than the students, and the most frequent spontaneous exclamation we heard was: "…but it is not a function!". For many teachers, the visual "vertical line test" to determine whether a graph is a function or not is a tool much used and invoked, and graphs which fail the test are somehow suspect (see, for example, Even and Bruckheimer, 1998). For others, the graph "going backwards" was the most surprising feature.

The immediate question (which we did not need to raise) was "what is going on here?".

The first step many participants took was to replay the situation, and to notice that, a certain stage, the altitude "escapes" outside the triangle (or jumps from outside to inside the triangle). We took the opportunity to discuss why this had not happened in the isosceles triangle. The altitude escaping outside was suspected to be somehow linked to the surprising graph type. Others linked that observation to their further noticing that the altitude does not increase in all its domain, it also "shrinks" in some parts of it, which explains why the graph "goes backwards". Others also observed that the increasing and then decreasing values of the altitude (the independent variable) creates two different values of the area for the same value of the altitude: if the altitude is outside the triangle the 'base' is smaller and thus the area is smaller, but the same altitude can be inside the triangle, and this happens for a larger 'base', in which case the value for the area is larger (see Figure 13). We found these kinds of explanation especially interesting because: a) they emerged after playing and replaying the geometrical situation dynamically, b) they are closely related to perceptually experiencing the variability which is at the core of the idea of a function as depicting a dynamic situation, and c) although they emerge from empirical observations, they serve as stepping stones to produce more general, deductive arguments which seemed to be pleasantly convincing.

At this point, we decided to devote some time to the creation and analysis of the symbolic representation, to 'replay' with teachers and students what happened to us when we invented and played with this situation. We asked (as we asked ourselves then): "how would symbols depict this situation?".

For those who produced/followed the qualitative geometrical explanation, the process of modeling this situation symbolically added another layer of meaning. Firstly, it formalized the verbal explanation into $A = \frac{BDAC}{2} = \frac{BD(AD \pm DC)}{2}$.

This formula helped to re-view in symbols how for the same altitude BD, one gets two different values for the area (which was the graphical surprise): once, the area is calculated as the sum of two triangles with this altitude as a common side, and again, when the altitude falls outside, the area is calculated as the difference between two such triangles (see Figure 13).

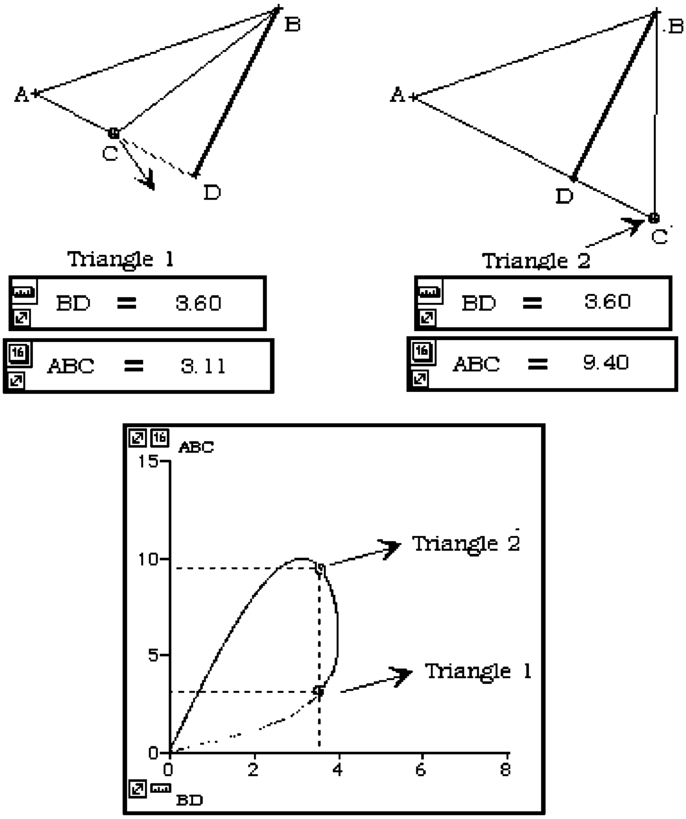

Figure 13. Figural explanation of the surprise.

Secondly, by re-writing the formula in terms of the variable altitude (x = BD) to get A = $\frac{x(\sqrt{25-x^2} \pm \sqrt{16-x^2})}{2}$, it became clearer that, in fact, what we have is two functions depicting this situation, for different subdomains. The graphs of these two functions 'smoothly join' when the altitude coincides with the side of length 4 (which is precisely the maximum value of the altitude). The substitution of that value in the expression provides a further realization that, in this case, there is no sum or difference of two triangles, which were already seen graphically as "collapsing" into a single triangle.

The issues with which we dealt here consisted in comparing, contrasting and moving back and forth from the geometrical situation, its graphical representation and the symbolic expressions. For example, why is 4 the maximum value for the altitude to AC and how do the situation, the graph and the symbols express this? Or, what is the geometrical meaning of the functions joining 'smoothly'? Or, what does it mean that the graph has always (with two exceptions) two x values for each y value and vice versa? Or, what would happen if we were to consider, for example, the area of ABC as a function of the altitude to AB (the side of length 5)? And so on.

We see this activity in a similar way in which Dennis & Confrey (1996, p. 35) describe their views of evolution of knowledge in the history of mathematics: "as the coordination and contrast of multiple forms of representation. … often one sees a particular form of representation as primary for the exploration, whereas another may form the basis of comparison for deciding if the outcome is correct. The confirming representation should be relatively independent from or contrasting to the primary exploratory representation. It must show contrast as well as coordination for the insight to be compelling." They claim that, the sense for certain representations is lost and distorted in most curricula. For example, "the Cartesian plane is treated predominantly as a means of displaying algebraic equations". Instead, in our case, the Cartesian graph was directly expressing characteristics of the phenomenon which required explanations. Some of those explanations were produced verbally by re-viewing the phenomenon. These and others were refined and re-produced when the symbolic representation was introduced to contrast, verify and extend the insights.

Reflection

The problem we proposed and its exploration is a case in point which raises the following issues.

The role of computerized tools

The existence of the computer poses to mathematics educators the challenge of designing activities which take advantage of those features with the potential to support new ways of learning. In the specific case we report here, it was the possibility to represent graphically and dynamically in real time the variation of a geometrical phenomenon which seemed promising.

We played for a while until we came up with a collection of problem situations similar to the one described (Arcavi & Hadas, 1999). Thus, in our case, the trigger to design exploration projects was the availability of the tool and the advantage to be gained from its capabilities. We sensed that we have a powerful means to promote student learning via mathematical investigations and meaning making, which in our view is strongly linked to producing explanations to unexpected phenomena, relying on the phenomena themselves and different mathematical representations thereof.

For us, the encounter with the tool preceded our realisation of its need. On the contrary, it was a starting point to inspire us to harness it in consonance with our views of learning, and thus searching for and designing problem situations accordingly.

Mathematics and mathematical activity

We believe that the problem situation described above and the way we implemented it can be considered as an example of doing some mathematics in a rather new way, instead of using the tool to put "old approaches" into just another dress.

Imagine this situation presented in a more traditional paper and pencil setting; e.g. "Which of the isosceles triangles whose equal sides are of length 5 has the largest area?" Or even in a more open way: "Investigate the area of isosceles triangles with equal sides of length 5." It is almost certain that symbols would be the first, if not the only, medium to be invoked (possibly with some sketchy drawings to support the modeling process). In our experiences, the graph was produced and used *before* its algebraic representation. Both the situation and its graph are looked at dynamically, all information gathered is intimately related and expressed in terms of the original situation, and is put at the service of better understanding it. It is in the translation between the graph and the situation that more subtleties of the phenomenon unfold. The initial absence of the algebraic representation, does not seem to impede genuine and deep mathematical reasoning. Quite the contrary, we would claim that its absence helps to keep in mind almost all the time the phenomenon modeled by the graphs, and prevents us from being distracted by symbol manipulation which may distance us from the original meanings. But, when symbols are finally introduced, the "algebraic expression … comes alive… " (Noss and Hoyles, 1996, p. 245), as it is inspected for the ways in which a) it expresses the information already found, and b) it may add insight to the analysis (as in the last phase above).

Thus the activity illustrates new ways of doing mathematics, since:

– Empirical explorations of geometrical phenomena take the form of looking at many particular cases and the dynamic transitions among them. Observation may not only help to reveal patterns, it may also be the source of insight and meaning, and serve as the basis for proving and further exploration.

– Making sense of a geometrical situation while playing with the situation itself first, and then by interpreting its representations (graphs, symbols), seems to enhance both the understanding of the situation and of the representations. The concept of function, which models the geometrical situation, recovers its dynamism, as a genuine model for change and variation since its graphical representation is being created in real time describing the phenomenon as it occurs. Moreover, the interplay between the global (the graph as a whole, the formula as an object) and local views of a function (pointwise view of the function as a procedural connection between two quantities) becomes apparent. Thus graphs and functions are used both as objects and processes (e.g., Moschkovich et al., 1993) with which to think about aspects of the situation being mathematized.

– The teacher, who brings this problem situation to the classroom with students or with fellow teachers, becomes a guide who poses the appropriate questions at the appropriate moments. For example, the teacher

  a) requests predictions which encourage students to take a stance on the problem and serve as a background against which to deal with unexpected results (e.g. “Predict when the area reaches its maximum”, “Predict the form of the graph”);

  b) requests to be explicit about the why (or why not) of what they see (e.g. “Why the graph turned out to be not symmetrical”);

  c) helps to make explicit and to deal with intuitions or knowledge which may underlie an ‘incorrect’ prediction (e.g. “What would it mean for the graph to be symmetrical? Where would its maximum be?”),

  d) leads the discussion, poses new questions, and promotes the coordination between different representations.

  In other words, the technological tool in itself is of little value if it is not accompanied by problem situations which make meaningful use of it.

However, any curriculum also requires the skillful implementation of the teacher roles described above in order to be faithful to its spirit. In our case, it implies to engage student knowledge, make it explicit, build on it, and help students coordinate seemingly disconnected pieces and conciliate apparent inconsistencies.

– In tasks as the above, the boundaries between mathematical sub-disciplines get blurred: functions depict geometrical phenomena, geometric explanations emerge to explain features of the graphical and symbolical representation of the function, and vice versa.

A new way of thinking?

Noss and Hoyles (1996, p. 240 *ff*.) describe three completely different approaches (produced by a group of mathematics education researchers) to solve the same problem, two of which are "computerized". They report that those who solved the problem were surprised by the "diversity of knowledge, style and solution" involved in each approach. They claim that the choice of medium to solve a problem "mediate the range of meanings and connections which are likely to structure the interaction, and which are likely to emanate from it" (p. 245). Such choice depends upon familiarity, suitability and expertise developed over years of working in a medium, which gradually determine a preferred approach to problems.

In retrospect, we see that that is precisely what we have experienced. Our own continued practice in a certain computerized environment, playing with situations as the one described above, resulted in the development of an approach with which we began to envision a whole class of problems. We found ourselves using Cartesian graphs to represent certain geometrical situations as a way to inspire proofs and to gain insights. This way of thinking was further reinforced during the search for and design of geometrical problem situations in the process of creating the collection we developed.

The rationale for the design was based on what we believe the development of sound mathematical understandings is. But, more subtly, we realise that we are advancing a certain approach, a certain viewpoint, which we developed after having many experiences with it, and which we happen to like. Although an important component of education is about intervention and about establishing a set of values to influence points of views and practices, several questions remain open. Are we making rational use of the decision power we, as teachers or designers, have to advance a certain practice over others? Or, should we advance several approaches and let students choose? Could it not be that the new ways seem exciting and meaningful to us, designers, because we can enjoy them against the background of what we already know, but could be less fruitful for many students in the long run?

We propose these issues for feedback and reflection on the basis of the problem we analyzed, and as a basis for further research.

Notes

(1) Since this is a static picture, it highlights the latest graph drawn: the area as a function of the altitude (the "moving point" is on it and the variable appearing on the x-axis is BD).

(2) Some students or teachers who did it by themselves, generated the graph in the "opposite direction". For them, the final result was no less of a surprise.

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PART III

Some Remarks on Conventions and Representations

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ABSTRACT. There is an essential difference between radical constructivism and socioculturism. It is that the former takes social interaction (specifically, communication) as a phenomenon needing explanation, whereas the latter takes it as a constitutive element of human activity. This difference expresses itself most vividly in the types of explanations coming from radical constructivists and socioculturists. The former tend to focus on human discourse as emanating from interactions among self-organizing, autonomous individuals. The latter tend to focus on the collective activity in which individuals participate. That is, from a radical constructivist perspective, what we take as collective activity is constituted by interactions among individuals each having schemes by which they generate their activity and by which they make sense of other's actions. From a sociocultural perspective, collective activity and social interaction are given, predating any individual's participation in it. The individual accommodates to social meaning and practice.

Introduction

Social interaction, from a radical constructivist perspective, is constituted by individuals' mutual interpretation of what each perceives as other-oriented action. These interactions, collectively and over time, constitute social activity. If we also assume that individuals reflect on their actions, then it follows that each individual's participation changes as she becomes aware of, elaborates, and interiorizes her activity and her understanding of its repercussions. A radical constructivist perspective on the constitution of collective activity is similar to points of view originating in complexity theory and chaos (Mainzer, 1994; Sandefur, 1993). In complexity theory, the intent is to model complex phenomena by attempting to identify elementary process that, through large numbers of interactions over sufficient amounts of time, regenerate the phenomena. The elementary processes, from a radical constructivist perspective, to account for collective activity are intersubjective operations within individuals and interactions among groups of individuals. This, combined with the facts that people have memories and use them, and that interactions often produce artifacts that people use both informationally and practically, engender social activity from a complexity theory point of view.

The elementary processes of social interaction, from a radical constructivist perspective, are mutual interpretation, mutual accommodation and personal action textured by each. This is similar to symbolic interactionism (Cobb, Jaworski, & Presmeg, 1996; Miller, Katovich, Saxton, & Couch, 1997; Prus, 1996), specifically those forms that treat the fact of communication problematically. From this symbolic interactionist perspective, people do not communicate meanings *per se*. Rather, to say "one person communicates a meaning" to another means that a listener attributed meanings to utterances he perceived, where the production of those meanings are both enabled and constrained by their own understandings and by the image they've built of, or impute to, the speaker (see Figure 1).

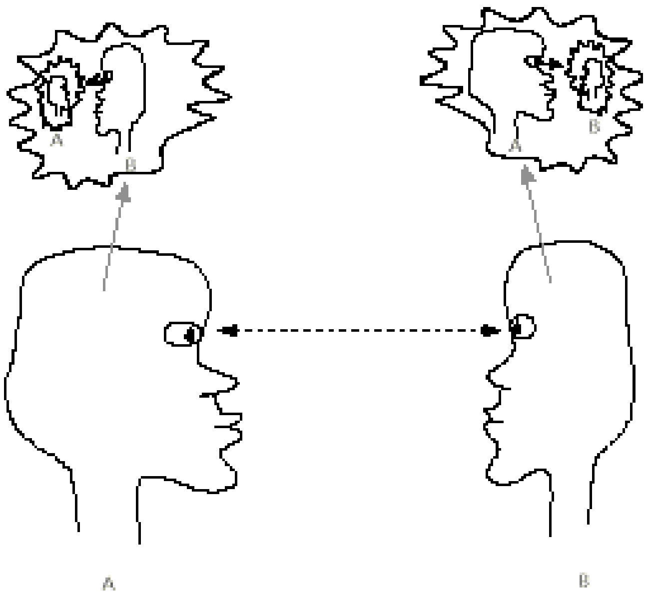


Figure 1. A symbolic interactionist view of one moment in human communication.

Figure 1 portrays a symbolic interactionist understanding of two people in communicative interaction, as opposed to physical or asocial interaction. We see the two of them acting, but we understand each to be predicating his actions on an image of the other and upon those action's implications. Each person acts reflexively, and yet acts with consideration of others, much as Baursfeld (1980) described. Figure 1 makes this point a bit too strongly, in that it portrays interactors as thinking this through quite carefully. Instead, I mean that each person imagines and acts, sometimes reflectively but always through imagery built from past interactions. Individuals' actions are textured by current understandings and imagery as much as predicated on them.

As von Glasersfeld (1995) notes, to say two people communicate successfully means no more than that they have arrived at a point where their mutual interpretations, each expressed in action interpretable by the other, are compatible – they work for the time being. *Intersubjectivity* is the state where each participant in a socially-ongoing interaction feels assured that others involved in the interaction think pretty much as does he or she. That is, intersubjectivity is *not* a claim of identical thinking. Rather, it is a claim that no one sees a reason to believe others think differently.

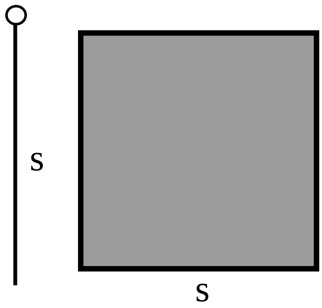
The significance of these considerations in regard to issues of representation is twofold:

- It is clear that, from a constructivist perspective, claims that representations are socially mediated or that representations are mediators of social processes are claims about the surface features of large scale, complex social interactions (as I've described them). Representations, as personal constructs, are creative ways to remind ourselves systematically of ideas we had, connections we made, and operations we applied previously in the presence of operating now. Representations as social conventions are expressions of intersubjectivity (as I've defined it).
- Features of large-scale social interactions can be terribly misleading when trying to determine the nature of individuals' participations in the interactions and meanings that individuals attribute to socially-shared "representations."

An example from a research study (Thompson, 1994) with 4th-year mathematics majors and 1st-year math education graduate students illustrates the second point. The study was on their understanding of rate of change and the Fundamental Theorem of Calculus.

The setting for this example (unreported in the 1994 article) is that we had worked for some time rebuilding their notion of average rate of change. To raise the issues I foresaw as central to making the Fundamental Theorem a discussible idea students needed to realize that constructing a coherent interpretation of formulas and graphs was more problematic than they realized. The class session I describe here lasted for approximately 90 minutes and centered around interpreting a formula that defined an "average rate of change" function in regard to the variation in a square's area expressed as a function of the square's side length (Figure 2). I will describe this session in some detail.

A(s) = 1.633 square inches
s = 1.278 inches



s	A(s)
1.000	1.000
1.111	1.235
1.278	1.633

Figure 2. Square region whose side length and area varies. Table shows various side lengths and associated areas.

A theme of the course up to this session had been to develop a consensual understanding of average rate of change of one quantity (A) with respect to changes in another (B). The "consensual meaning" by this time was "that constant rate of change of A with respect to B which will produce the same change in A as actually happened in conjunction with the actual change in B" (Bowers & Nickerson, in press).

The instructional segment began with students watching the square in Figure 2, displayed in Geometer's Sketchpad, as I increased the length of each side of the square by dragging the endpoint of the vertical segment labeled *s*. After each change in the side length I asked, "What was the average rate of change of the square's area with respect to the change in its side length?" For each computed value (e.g., .235/.111 for the first entry) I asked, "What does it mean that the average rate of change of area with respect to side length was 2.117 in this step?" In this case we proceeded after establishing that it means *if the area were to increase at the constant rate of*

2.117 in^2 per inch increase in side length as the square's side length increases from 1 inch to 1.111 inches, then the area would increase precisely as much as it actually did. We repeated this discussion for several more values of *s* and A(*s*).

We observed that the side of a square, measured in inches, is increasing in length and that the square adjusts appropriately as the side's length increases. The area of the square, in square inches, was given at all times by $A(s) = s^2$. We also established that each value of the function

$$r(s) = \frac{A(s+0.25) - A(s)}{0.25}$$

gave the average rate of change of area with respect to change in side length as the side-length changes from *s* inches to *s*+0.25 inches. This means that for each side-length *s*, if the area were to increase at the constant rate of change given by *r*(*s*) for the next 0.25 inch increase in side length, then the change in the constantly-changing area will be precisely the same as the change in *A*(*s*).

I displayed the graphs of *A*(*s*) and *r*(*s*) (see Figure 3) and said "Here is a point on the graph of *r* (clicking on a point of *r*'s graph, displaying coordinates (.7667,1.635)). What does this point having coordinates (0.7667,1.635) represent?" I then told students to form groups of two and three and directed each group to arrive at a consensus statement of what is represented by the point (0.7667,1.635). After all were finished, each group reported the result of their discussion. Everyone in every group stated his or her satisfaction that their spokesperson represented their group's interpretation, and everyone stated his or her satisfaction that the groups arrived at essentially identical interpretations. I then asked each person to write his or her answer to the question, "What is represented by the point on *r*'s graph having coordinates (0.7667,1.635)?"

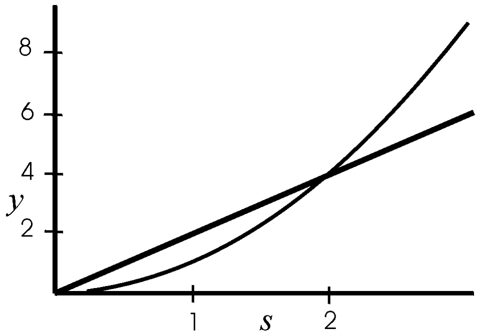


Figure 3. Graphs of *A*(*s*) and *r*(*s*).

Eleven of 19 students responded to the question. Eight stated that they couldn't remember what their group had said before, that they couldn't reconstruct it, or they couldn't come up with an interpretation. Responses from the other 11 students are given in Table 1.

Table 1. Responses to "What is represented by the point (0.7667,1.635)?"

1. That point represents the average rate of change in the area with respect to the change in length of the side at .7667 inches.

$$\text{Ave. rate of change} = \frac{1.635\ sq\ \text{in}}{\Delta \text{ in } s \text{ in inches}}$$

2. Given side length *s* of .7667 in., the <u>avg rate of change</u> in area with respect to side length is 1.635 sq/in.

3. *c* =.7667 is the rate change of side length at this point. *y* = 1.635 is the average rate of change to respect of change of *x*.

4. *r*(*x*) represents avg rate of change. Therefore, the specific point represents rate of change from area of sq with side *s* to (*s*+.25) where side *L* = .7667. Avg. rate of change here is 1.6.

5. *x* = .7667, *y* = 1.635. At point *x* = .7667, if the rate of change of *r* were to be constant thereafter, the area would increase by 1.635 every time *x* increased by 1.

6. What does the point x=.7667 and *f*(*x*) = 1.635 mean? The average rate of change in the area in sq in of a square with side .7667 in as it is increased by .25 in.

7. The point represents the area of the square with a side of .7667 using the average rate of change for a change of .25 inches in the side.

8. At *s* = .76 we find the average rate of change between a square of length .76 and a square of length approximately 1.

9. *r*(*s*) represents the rate of change of the area when the side changes from about 0.75 to 1.00.

10. What does the point *s*, *r*(*s*) stand for when *s* = .75?

$$\frac{r(.75+.25)-r(.75)}{.25}$$

$$\frac{rate\ \ of\ \ change\ \ in\ \ area}{change\ \ in\ \ s}.$$

11. What does $\begin{pmatrix} x \\ y \end{pmatrix}$ represent? An average rate of change of 1.635 units^2 for side length of .7667 units increased by .25 inches.

Two aspects of students' responses are striking. First, none of the responses is internally consistent. Five are relatively close (numbers 2, 6, 8, 9, and 11). Six responses are conceptually incoherent, entailing internally conflicting meanings. Second, no interpretation even remotely resembles those that they spent 50 minutes developing and to which they each expressed satisfaction that they had said what they intended.

The two aspects together, lack of internal coherence in their interpretations and lack of agreement between private and publicly stated interpretations, points to a matter worth considering. When we claim that agreement has been reached on a relatively complex idea because disagreement hasn't been expressed, we must consider the possibility that students haven't analyzed their own or others expressions sufficiently to detect severe inconsistencies. Therefore, when incorporating an emphasis on "representations" in conceptually-oriented discussions, it may be prudent to exercise caution before concluding that a consensus in meaning or interpretation has been reached. It may also be prudent to be cautious about concluding what individuals understand even when public agreement seems certain. What individuals understand may be expressed as something stable in the way they interact, but the extent to which interaction-as-stable-pattern reflects individuals' understandings may be uncertain at best.

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Representation and Cognition: Internalizing Mathematical Concepts

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ABSTRACT. NCTM-2000 standards document highlights the importance of developing students' abilities to use multiple representations as problem solving and reasoning tools. Students should be fluent in choosing and applying representations to communicate mathematical ideas more effectively. Following Krutetskii (1976) we consider representational flexibility, reversibility and generalization as an indicator of students' conceptual understanding and advanced learning in mathematics. One can find a very deep theoretical analysis of a role of sign/semiotic representations in child's cognitive development in works of such Russian psychologists as Vygotsky, Leont'ev, Gal'perin, Salmina, etc. We will consider key ideas of Vygotskian Semiotic Conception. We will also examine the role of representations in constituting a cognitive change based on Vygotskian idea of zones of cognitive development. In order to promote cognitive changes in child's development one should analyze differences between low and high level of cognitive representation as well as evaluate a potential of cognitive representation in formation of mathematical understanding. Finally, the method of cognitive symmetry will be described as a mechanism of interrelationship between external and internal representations.

Vygotskian Semiotic Conception

The main idea of Vygotskian semiotic conception is that "sign operations are not simply invented by children or transmitted from adults, they appear from something which initially is not a sign operation and which becomes a sign operation after a sequence of qualitative transformations…" (Vygotsky, 1984, p. 66). Vygotsky claimed that the sign operation is the result of a complicated developmental process, which somehow reflects "the natural history of a sign". By the latter, Vygotsky means a genetic connection between the history of high psychological functions and the development of a child's natural forms of behavior such as play. Vygotsky argued, "… at the beginning of its development every high psychological function inevitably carries out the character of external activity. At the beginning, a sign represents, as a rule, an external auxiliary stimulus, an external mean of auto stimulation" (p. 72). Moreover, "in order to become a sign of the thing (word), the stimulus must have a support in qualities of a signified object" (ibid, p. 69). The structure of the stage of a child's primitivism (natural history of a sign) as a relationship between an object/thing, its meaning, and sign is shown in Figure 1.

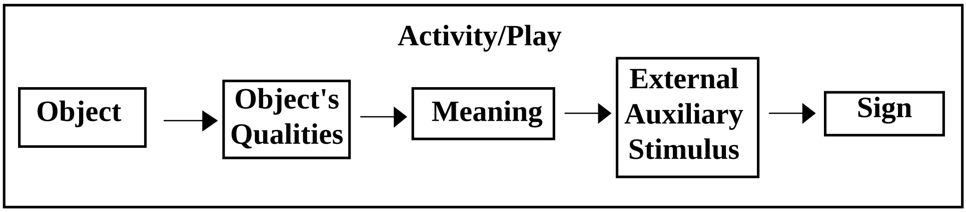

Figure 1. Stage of child's primitivism or natural history of a sign.

Thus, through a sequence of qualitative transformations brought about through activity with the object children develop connections between an object, its meaning, and a sign that represent the object (Figure 2).

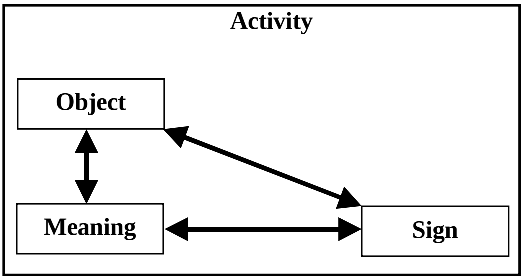

Figure 2. The relationship between an object, its meaning, and a sign.

According to Vygotsky, the next step in the "natural history of a sign" is "naming" or the development of a connection between an object, its meaning and name (Figure 3). Naming is a very difficult psychological act for a child. To illustrate the stage of a child's primitivism Vygotsky gives the following marvelous example:

> Thus, for the child, a word is connected to the thing through its qualities and it is included along with the thing into a common structure. That is why in our experiments the child doesn't agree to name a floor "a mirror" (because he couldn't walk on a mirror), however he transforms a chair into a train using its qualities, i.e. he manipulates with the chair as he would with the train. The child refuses to name a lamp "a table" and vice versa because "you can't write on the lamp, and a table can't give light". For the child, to replace a name means to replace quality. (p. 70).

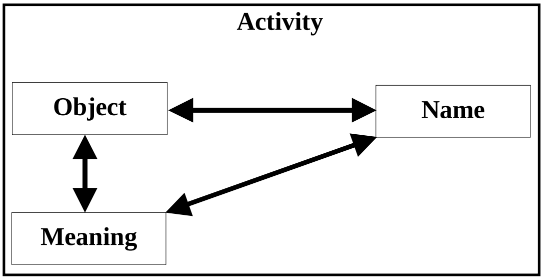

Figure 3. The relationship between an object, its meaning, and a name.

Whole semiotic chain contains connections between an object/signified, its meaning, a designated name, and a sign/signifier (Figure 4). An informal type of activity such as play is a critical part of a child's semiotic development. During the play and social interaction a child transforms external patterns of semiotic behavior into internal structures. "The internalization of cultural forms of behavior involves the reconstruction of psychological activity on the basis of sign operation" (Vygotsky, 1978, p. 57). The sign becomes a mediator between external social activity and a child's internal individual cognition.

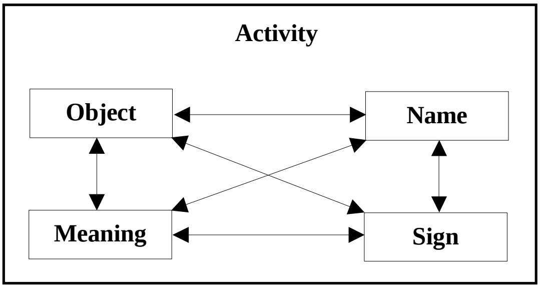

Figure 4. Semiotic chain.

Representation and Cognitive Change: Post-Vygotskian model

Vygotsky noted that the possibilities of genuine education depends not so much on the already existing student's knowledge and experience (level of actual development) as on the characteristics that are in the zone of proximal development. He wrote: "Pedagogy should be oriented not toward yesterday, but toward tomorrow in child development. Only then will it be able to create, in the process of education, those processes of development that are at present in the zone of proximal development" (Vygotsky, 1996, p. 251). Thus, the Zone of proximal development may be defined as the distance between what the child knows and his or her potential for knowing with the help of "more knowledgeable other". It is necessary to stress that in Western pedagogy the main attention is paid to the ZPD (zone of proximal/ potential/nearest development) though Vygotsky considered ZPD as only one domain between the lowest and highest levels of cognitive development. "We always should determine lowest threshold at which instruction may begin. But it is not the end of the deal: we should be able to determine the upper threshold of instruction as well. Only between these thresholds instruction might be fruitful" (Vygotsky, 1996, p. 251). The lowest threshold is the level of actual development (LAD) which contains the student's actual knowledge, skills and experience. Following the LAD is the zone of proximal development (ZPD) which has as its focus cognitive change connected with the guided development of student's understanding. "The ZPD is the locus of social negotiations about meanings, and it is, in the context of school, a place where teachers and pupils may appropriate one another's understandings" (Newman, Griffin, Cole, 1989, Foreword by S. White). There is one more zone after the ZPD. When Vygotsky wrote about the upper threshold, he didn't mean that it is equal to the ZPD. As we have noted elsewhere (Tchoshanov, 2001), till now few scholars have paid attention to this important fact in Vygotsky's work. This zone goes beyond the development of understanding and is a zone of formation of student's in-depth learning. Whereas in the ZPD the functions of comparison, reproduction, assimilation, and coping are of primary importance, in a new zone the functions of construction, generation, and creation are most important. This upper threshold of instruction and cognitive development we call a zone of advanced/perspective development (ZAD). If ZPD is the interpsychological dimension where social activity and interpersonal dialog is taking place, ZAD is the intrapsychological dimension where advanced individual activity and intrapersonal dialog occurs.

Activity cannot be understood as simple internalization of ready-made standards and rules. Rubinshtein (1973) stressed that human activity presupposes not only the process of internalization but also the process of externalization when humans create new standards and rules. Therefore, if the psychological outcome of ZPD is internalization, then the psychological outcome of ZAD – is externalization. According to Vygotsky guidance is crucial in helping a student move from LAD to ZPD. We cannot say the same about student's transfer from ZPD to ZAD. In other words, if ZPD is a domain of guided cognitive change (understanding), ZAD is a zone of student's individual (independent) activity. Therefore, we consider ZAD as a domain of higher cognitive achievement and creativity, which the student may reach in the process of intense individual studies (Figure 5).

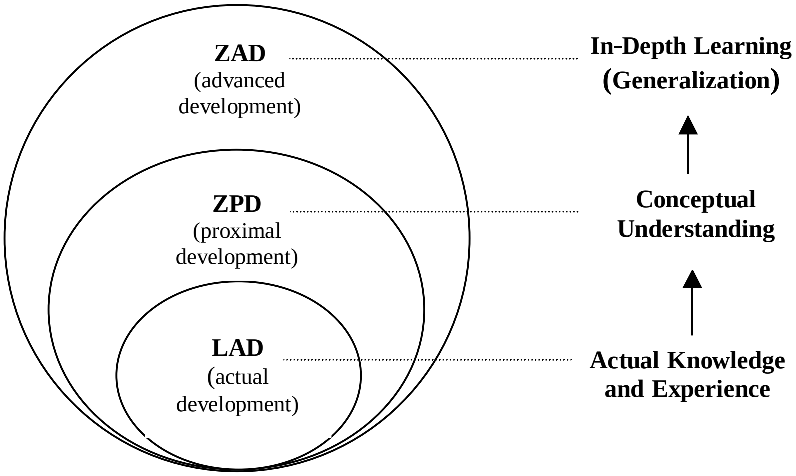

Figure 5. Zones of Cognitive Development

Model of Cognitive Representation

The post-Vygotskian semiotic conception and the model of zones of cognitive development require the design of a theoretically appropriate representational model. Models such as Bruner's (1966) learning model, Lesh, Post, and Behr's (1987) model of representation systems, Pape & Tchoshanov's (2001) model of the interrelationship between external and internal representations reflect different levels of abstractness of representation: enactive/concrete, iconic/pictorial, symbolic/abstract (including written and spoken language). Paivio (1971) suggested a three functional model of imagery that contains the following pairs of distinctions:

concrete-abstract, static-dynamic, and holistic-sequential. A new model should reflect a multidimensional relationship between key cognitive characteristics of representations. Based on the analysis of Post-Vygotskian perspectives we consider the following key components of a representational model: abstractness (different modes of representation), generalization (different levels of cognitive demand), and conventionality (relationship between idiosyncratic and standard representations). The holistic process of the relationship between key cognitive components of representation, its internalization and externalization, we call *cognitive representation*. Below we briefly describe levels of each component along with the 3-D model of cognitive representation (Figure 6).

Representations on the Concrete to Abstract continuum reflects different modes of representation:

- Concrete (real object, physical model, manipulative) (A1)
- Pictorial (photograph, picture, drawing, sketch, graph) (A2)
- Abstract (sign, symbol, written, verbal language) (A3)

Representations on the Illustrative to Generalized continuum reflects different levels of cognitive demand:

- Illustrative representations (procedural representations without connections) (G1)
- Conceptual representations (heuristic representations with connections)(G2)
- Generalized representations (extended representations of big ideas) (G3)

Representations on the Idiosyncratic to Conventional continuum reflects different levels of conventionality:

- Idiosyncratic representations (students generated low level representations) (C1).
- Non-standard representations (students generated medium level semi-conventional representations) (C2).
- Conventional representations (student "discovered" high level standard representations) (C3).

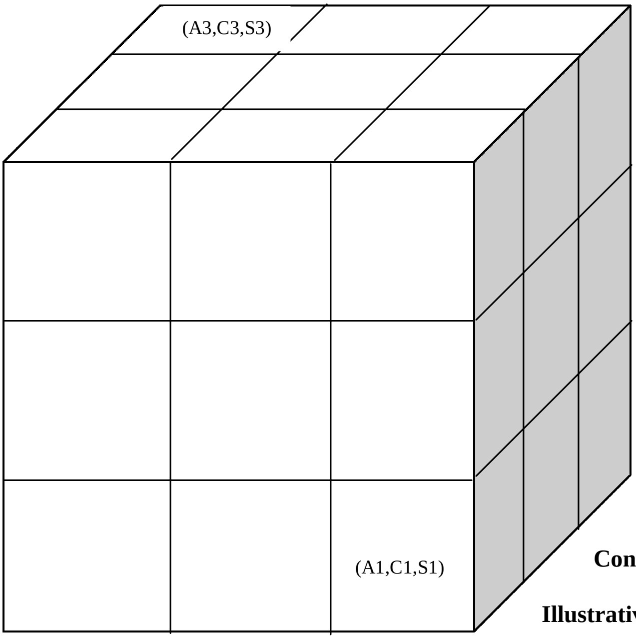

Figure 6. Model of Cognitive Representation

In order to describe the differences between low level (concrete-illustrative-idiosyncratic - A1, C1, S1) and high level (abstract-generalized-conventional - A3, C3, S3) cognitive representation we will use the following table (Table 1).

Table 1. High vs. Low level of Cognitive Representation

High Level of Cognitive Representation	Low Level of Cognitive Representation
• Leads problem solving and proof • Gives a whole picture • Makes easy heuristic encoding and decoding of information • Answers questions “How” and “Why” • Aims on generalization • Goes beyond ZPD • More dynamic • Emphasizes connections between multiple representations	• Follows problem solving and proof • Focuses on details • Requires memorization • Answers the question “What” • Aims on concrete level • Stays on LAD/ riches ZPD • More static • Emphasizes one mode of representation over other modes

Method of Cognitive Symmetry

In understanding the nature of internalization in the process of cognitive representation, we adhere to the Vygotskian conception of mediation. Vygotsky and his supporters argue that determination of individual cognition might be presented by the following scheme: collective (social) activity – culture signs/ symbols – individual activity – individual cognition (Vygotsky, 1996; Leont'ev, 1978). Vygotsky argued that "every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level" (Vygotsky, 1978, p. 57). In the framework of the conception of cognitive representation we consider the Vygotskian scheme as a basis for designing a cognitive cycle. The importance of the method of cognitive symmetry is that it gives a clue for designing the structure of externalization process based on the scheme of the internalization one. So, the holistic cycle, as we have written elsewhere (Tchoshanov, 2000), of cognitive representation includes the sequence of stages. To illustrate these stages we used *a representational dialog* as a method of studying the process of co-construction of shared cognitive representation. The dialog might start with posing a problem which requires a use of a particular representational mode. For instance, novice-1 gets a card with a following instruction: Draw a picture for the problem "You have ¾ of a pizza left. If you give 1/3 of the leftover pizza to your friend, how much of a whole pizza will your friend get?" so your partner (novice-2) can recognize what operation is behind your picture. Novice-2 doesn't see the instruction and problem. His task is to recognize mathematical operation behind the picture that is presented by novice-1.

First, potential representation should be interpreted (e.g., by expert) in order to become a standard/conventional representation for the students. It still might be a not-shared (standard) representation for the students. A representational dialog and negotiation of meaning should take place in order to help the students to transfer a not-shared standard representation to an internal representation. After students construct their internal representation through negotiation of meaning they might be able to express it externally as non-standard/idiosyncratic representation. To do so they need to "conduct" a private dialog and make their internal representation "stable" enough to express it externally. Finally, through reflection and abstraction they create a shared (negotiated) cognitive representation (Figure 7). We must make it clear that if internalization is primarily a guided zone, externalization might combine both socially interactive/guided and private/independent domains of students' activity. Furthermore, if internalization aims at understanding (e.g., seeing, comprehension, interpretation, etc.), externalization tends toward creativity (e.g., construction, generalization, abstraction, etc.).

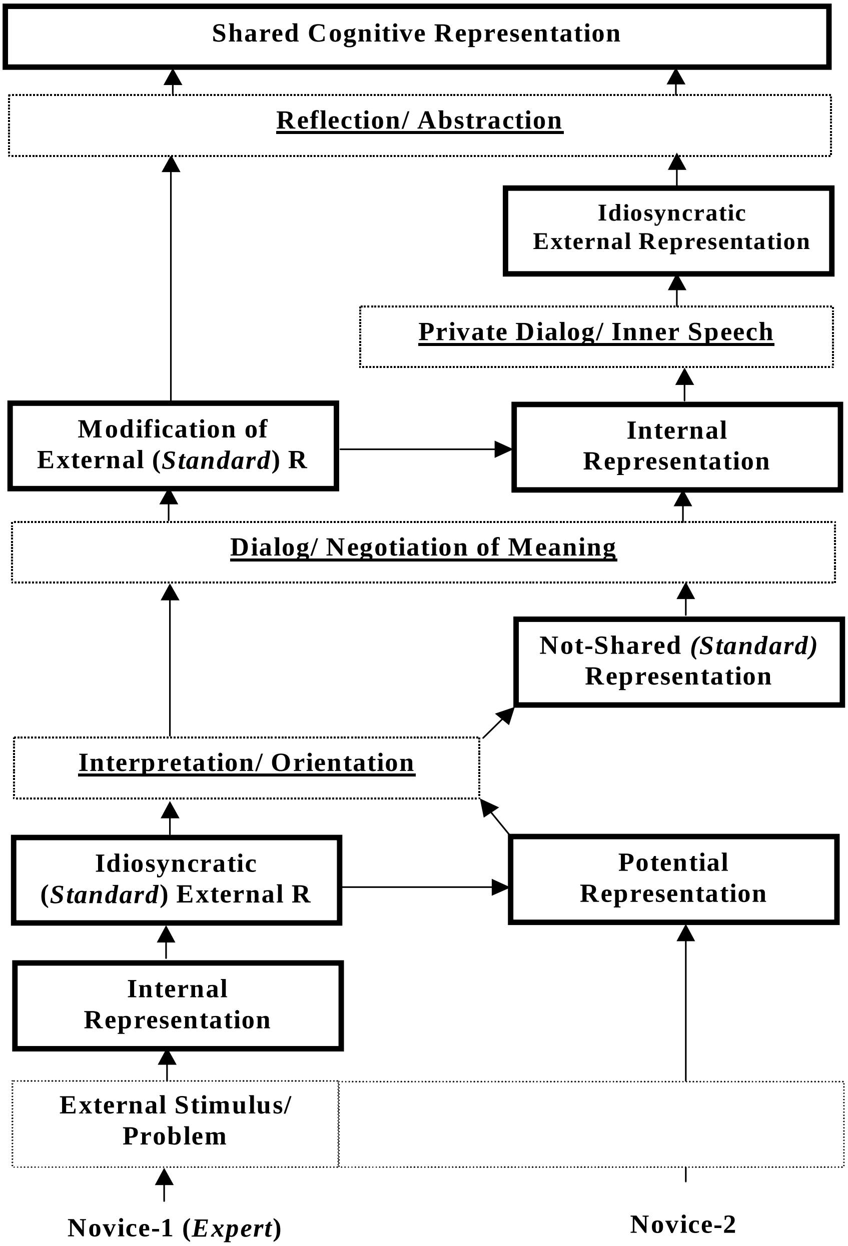

Figure 7. Structure of the Representational Dialog.

Final Remarks

Summary of the main ideas of this paper contain the following list:

- Interrelationship between an object, its meaning, a designated sign and a name is a key component of a signification process
- Flexibility, reversibility, and generalization in using multiple representations is an indicator of student's conceptual understanding and advanced learning in mathematics
- Ultimate goal of using multiple representations is a development of student's abstract mathematical thinking
- Assessment model for cognitive representation based on the following criteria: abstractness, generalization, and conventionality.
- Method of cognitive symmetry describe the mechanism of interrelationship between external and internal representations.

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The Object of Representations: Between Wisdom and Certainty[1]

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ABSTRACT. What is that which representations represent? The goal of this article is to discuss the answer to this question provided by two influential contemporary epistemological theories –Piaget's and Wartofsky's. Our discussion is motivated by a practical concern: new technologies of semiotic activity in the classroom (e.g. artefacts, calculators, dynamic software) offer new forms of knowledge representation and text production that are radically different from the time-honoured formats of written traditions based on a sequential organization of sentences. Nevertheless, the use of new forms of knowledge representation (both discursive and artefactual) requires us to better understand how representations in general relate to their conceptual contents. In scrutinizing Piaget's and Wartofsky's epistemologies, it becomes apparent that the answer to the aforementioned question depends not only on our conceptions about how the objects of knowledge become known by the individuals but also on ontological stances that we take about the nature of mathematical objects.

Introduction

The idea of representation became one of the essential conceptual constructs of 20th Century psychology of learning. It would be virtually impossible to find another so embracing and powerful concept in this field. Perhaps the reason of its spectacular success was rooted in the perplexing situation that we all *think*, although we still do not know exactly how. Capitalizing on the long-lasting venerated opposition

[1] This paper is part of a research program funded by the Social Sciences and Humanities Research Council of Canada.

between body and mind cultivated by Western thought, the idea of representation appeared as a means to inform us about what is occurring in our heads. Thus, traditional cognitive psychology proposed a distinction between external representations (symbols, graphics, language, etc.) and internal representations (mental forms of information processing). However, the point in studying internal representations was not so much to crack the “black box”. Cognitive psychology was not so ambitious. Rather the point was to get a glimpse of our internal psychological processes. And this, of course, seduced many of us.

Nevertheless, conceptions of thinking have changed substantially in the past few years. Current discourse in psychology tends to include more and more the role played by social interaction and the use of artefacts in the representations that an individual comes to form. The progressive abandonment of the Cartesian cogitator model and the solitary mind has led us to a point in which we need to reconceptualize the concept of representation and its psychological and epistemological roles. It is, nonetheless, beyond the scope of this paper to offer the elements for such a reconceptualization. My aim is much more modest. I want to focus on *that* which a representation is about, that is, its *object*: the represented as represented by the representation.

I do not take this question as a fancy philosophical one. On the contrary, the advent of sophisticated technologies in past decades has made available new means to *objectify* knowledge (Radford, 2000a, in press-1, in press-2). The means of objectification of new technologies (e.g. dynamic software in geometry) are different from those cast in the time-honoured formats of written tradition, based on the classical linear organization of sentences, and have put on the table pressing questions about the legitimacy of traditional versus new modes of knowledge representation. Of course, the controversies that have arisen from mathematical computer-based proofs are a good example. But there are simpler ones, much closer to school mathematics. For instance, in a recent article, Rodd discusses the role of *figures* in mathematics and asks whether visualization can be an a-linguistic mathematical warrant (Rodd, 2000, p. 237). Questions like this cannot be answered without adopting an ontological and epistemological position. For the answer will depend on what we consider the ‘nature’ of the object to be (i.e. the objects’ ontological status) and on the kind of relationship between the representation (e.g. a diagram or a figure) and its object (i.e. the epistemological status of the representation).

In this article I want to discuss some key conceptualizations that influenced the elaboration of contemporary views on the object of representations. Since this is a very broad problem, to narrow it, I will focus on two of the more influential 20th century theories: Piaget's genetic epistemology and Wartofsky's historical epistemology. In order to understand Piaget's and Wartorfsky's theoretical approaches, I will also consider the problem of the object of representation as it was formulated by Kant, Plato and Aristotle, on the one hand, and by the anthropologists Durkheim and Lévi-Strauss, on the other hand.

The Limits of Sense and Reason: The Kantian Doubt

The period between 1770 and 1780 is known as the "Silent Decade" in Kant's intellectual life. It was the period in which Kant was struggling with one of the problems that have hunted philosophers and mathematicians of all centuries, namely, the problem of the relationship between representation and the represented object. In a letter sent to his friend and former student Marcus Herz, dated February 21, 1772, Kant said: "What is the ground of the relation of that in us which we call "representation" to the object?"[2]

Kant had at his disposal two solutions, provided by two different theories but he was not satisfied with either of them. The first one was provided by empiricists such as Hume and Locke. The empiricists argued that our representations, concepts and beliefs derive from the information that we receive through our senses. The second one was offered by rationalists like Descartes and Leibniz. According to the second, the human mind has a stock of innate concepts and principles through which we generate and construct the representations of the world. One of the main differences between these views was, as Kant noticed, to be found in the active role with which the mind is endowed in the rationalist account. Indeed, instead of receiving sensuous impressions and being affected by them in a passive way, in the rationalist theory, the mind is conceived of as playing an active role in bringing up the representations.

In his letter to Herz, Kant summarized first the empiricist solution:

> If a representation is only a way in which the subject is affected by the object, then it is easy to see how the representation is in conformity with this object, namely as an effect in accord with its cause, and it is easy to see how this

[2] The letter was published in Zweig, 1970, pp. 32-38.

> modification of our mind can *represent* something, that is, have an object (Kant in Zweig, 1970, p. 33).

Then, Kant summarized the rationalist solution:

> If that in us which we call "representation" were active with regard to the object, that is, if the object itself were created by the representation [...], the conformity of these representations to their objects could be understood (Kant in Zweig, 1970, p. 33).

Nevertheless, Kant contended, each one of these positions has its own problems. As for empiricist theory, the crucial difficulty lies, as Kant remarked, in the assumption that representations are reduced as the effects of objects through our senses. It may be fair to say that

> The passive or sensuous representations have an understandable relationship to objects, and the principles that are derived from the nature of our soul have an understandable validity for all things insofar as those things are supposed to be objects of the senses (Kant in Zweig, 1970, p. 33).

But, Kant asked, how according to this view can we be affected by objects that cannot be given through the senses? How can we be affected by conceptual objects? In this point Kant found the crucial difficulty of empiricism.

As for the rationalist account, Kant said:

> If such intellectual representations depend on our inner activity, whence comes the agreement that they are supposed to have with objects –objects that are nevertheless not possibly produced thereby? (Kant in Zweig, 1970, p. 34)

In other words, the difficulty in rationalism is its impossibility to explain how the products of inner activity can be said to coincide with those of the external world. In his letter, Kant confided to Herz that he was planning to write a work that might have the title "The Limits of Sense and Reason", a book that would lead the difficult problem of representation and the represented object beyond the point in which his *Inaugural Dissertation* of 1770 had left it. Talking about the *Dissertation*, Kant said: "I silently passed over the further question of how a representation that refers to an object without being in any way affected by it can be possible." (Kant in Zweig, 1970, p. 34). "Now", he continued in another passage, "I am in a position to bring out a "Critique of Pure Reason" (*op. cit*. p. 35).

In the *Critique* (published in 1781 and re-published with modifications in 1787) the dilemma was synthesized as follows[3]. Either the object alone makes the representation possible, or the representation alone makes the object possible. In considering the first case, Kant said that the relation between object and representation would only be empirical. In the second case, Kant observed that the representation cannot produce the objective existence of the object, for what the representation does produce is a *representation* of the object but not the *object itself*. (see B 125, Kant, 1781/1787/1996, p. 147)

To attempt to provide a new solution to this problem, Kant drew on the aforementioned empiricist and rationalist viewpoints and came up with something different. In conformity with the former, he maintained that all knowledge starts *with* experience. But, in agreement with the latter, he added, "that does not mean that all of it arises *from* experience" (A1, Kant, 1781/1787/1996, p. 44). He built a system in which he stressed the human faculty or capacity of receiving or being affected by the particular objects of sensual experience. He termed this faculty *sensibility* (from *sense*). According to Kant, in being affected by objects, we form a kind of representation that he called *intuitions*. The epistemological role of intuitions is a corner stone in Kant's theoretical edifice. Intuitions are the gates through which objects have to enter in order to become known. As he said in the *Critique of Pure Reason*, "no object can be given to us in any other manner than through sensibility" (B33/A19, Kant, 1781/1787/1996, p. 72).

Evidently, sensibility and intuition are tokens of the empiricist influence in Kant's thought. But there were also other less crude sensualistic considerations. Kant observed, as Russell noted, "that the geometers of his day could not prove their theorems by unaided argument, but required an appeal to the figure" (Russell, 1919, p. 145). Kant was hence led to realize that our inferences always require the support of something tangible and came to the conclusion that, even in the case of mathematical objects, we need the support of intuitions, that is, of particular representations of the objects.

But the faculty of sensibility, Kant contended, cannot organize by itself the intuitions of sensual experience. This organization or synthesis is done by what Kant called *understanding*, another faculty of the mind. In his *Anthropology*, one of his later writings, Kant says that the power of intuition is limited to objects in

[3] Passages from the edition of 1781 of the Critique of Pure Reason are usually referred to with the letter A followed by the corresponding pages of the original edition, and those of the amended edition of 1787 with the letter B and its corresponding pages. I will conform to this tradition in Kantian studies but will add the pages according to Pluhar's unified translation (Kant, 1781/1787/1996).

their *singularity*, whereas understanding represents things to us by *concepts* thorough the unification of sensuous intuitions (Kant, 1797/1974, pp. 68-69). The difference between sensibility and understanding is hence that the former does not think, it is rather a receptive faculty. The latter, on the contrary, is the faculty of thinking, that is, of judging in accordance to pure concepts or categories to which the intuitions are presented (see Figure 1).

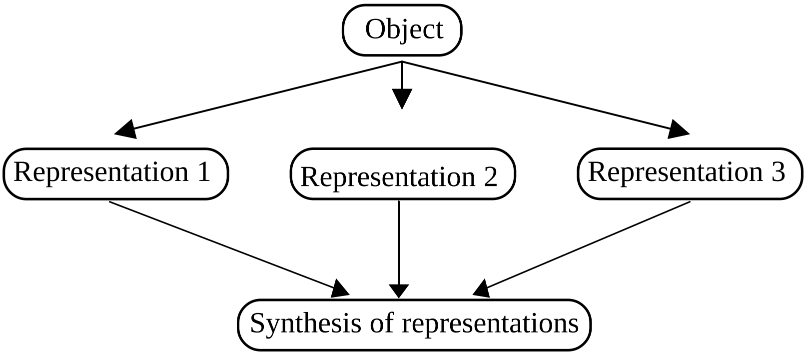

Figure 1. The diverse intuitions (representations) of the object are made possible by the faculty of Sensibility. The concepts of the faculty of Understanding ensure the synthesis of intuitions.

Evidently, the pure concepts and the faculty of understanding are a token of the rationalist influence in Kant's thought. Pure concepts are, according to Kant, prior to experience; they are part of the architecture of the mind and serve to guide experience. But neither the empiricist idea of sensibility nor the rationalist idea of understanding can lead, when separated from each other, to knowledge. Kant insisted that sensibility and understanding need to work together. Without sensibility, the understanding would not be able to proceed to the synthesis of particular representations and to proceed to the unity necessary to accomplish knowledge. This is why, Kant said, sensibility is more necessary and indispensable (Kant, 1797/1974, p. 69). Within this context Kant was able to formulate the two conditions for knowledge to be possible:

> The first condition is *intuition* [i.e. sensible representation –L.R.]; through it the object is given, though only as appearance. The second condition is the *concept*; through it an object is thought that corresponds to this intuition. (A93/ B125, Kant, 1781/1787/1996, p. 147).

What was, then, Kant's final answer to the question of the adequacy between object and representation? How did he resolve the problem that he mentioned to

Herz in his letter? He drew a clear border limiting what can humanly be known. The object of representations is not 'the object-in-itself', that is, what he also termed the 'object = *x*' or 'transcendental object', but the object as it appears in the phenomena and is constructed through sensual experiences by the acting individual. He gave a clear example using rain drops. Suppose, he said, that we take something empirical like a rain drop. This empirical intuition is a mere appearance in which nothing whatever belonging to some thing in itself is to be found. However deeply we explore the objects, we deal with nothing else but appearances:

> Not only are these drops mere appearances; rather, even their round shape, and indeed even the space in which they fall, are nothing in themselves. They are, rather, mere modifications, or foundations, of our sensible intuition. The transcendental object, however, remains unknown to us (Kant A45-46, Kant, 1781/1787/1996, p. 97).

To sum up, for Kant the object of representation is not the thing-in-itself but the thing as it becomes known through our faculty of sensibility, through the intuitions that we form of it and the way these are synthesized in a coherent unit by the faculty of understanding. In accordance to the rationalist trend, the representation, he maintained, determines the object, but, in accordance to the empiricist trend, he added, the representation determines the object in the sense that only through the sensual representation can the object become knowable. As Wolff noted, prior to Kant

> the object was assumed to exist, the problem being to explain how it could be known. […] Now, however, Kant proposed to reverse this order, making the realm of existing objects dependent upon the subjective conditions of knowledge. The *a priori* representations determine what can and cannot be known as an object, and hence what can and cannot be considered to exist. The realms of being and knowledge are co-terminous, and even more significantly, the latter defines the former. Wolff, 1963, pp. 97-98).

In drawing the border between the thing in itself and its appearance, Kant accomplished something really new, a revolution that, short of modesty, he compared to the one Copernicus accomplished in Sciences.

Plato and Aristotle

According to Kant, Plato was misled on two counts. First, since, according to Plato, senses confuse our ideas, the sensual realm was seen by him as something to be distrusted. Here Plato committed an *excess of prudence*. Second, Plato thought that the real objects were intelligible. Here, in Kant's eyes, Plato committed an *extravagance*. Plato showed a lack of prudence concerning the limits of the intelligibility of objects. Plato's epistemological presuppositions lead obviously to a different perspective on representations. It is true that being first of all a moralist, Plato did not deal with representations in a systematic way. However, it is clear that he assumed a kind of *resemblance* between representations and their objects. In a very often quoted passage of *The Republic*, Plato says:

> You know too that they [the geometers] make use of and argue about visible figures, through they are not really thinking about them, but about the originals which they resemble. (*The Republic*, Book 6, 510 d).

Aristotle saw the problem differently. While the link between object and representation appears in Plato' ontology as a kind of assumed *iconicity*, Aristotle envisaged it as an act performed by the individuals in their commerce with things. At the centre of this act lies the human possibility of *separating* in thought, the physical and the conceptual. He wrote:

> The mathematician is able to study surfaces, volumes, lengths, and points in isolation from their physical instantiations because (in some way which needs to be explained) he is able to separate the two in thought. (Aristotle, *Physics* B2, 193b33 ff.)

Aristotle's distinction between bodily and conceptual objects led him to an interesting point: conceptual objects can be managed in ways that their bodily counterparts cannot. Thus, in a conceptual rectangle we can imagine diagonals that were *not* in the concrete object (e.g. in a bronze rectangle). With the help of representations we can also separate the conceptual rectangle into two triangles, make visible new properties, and so on. For Aristotle, the link between object and representation appears hence ensured by the fact that arithmetic and geometry were conceived of as a conservative extension of physical theory (see Lear, 1982, p. 188). And, according to him, that which makes the extension possible is *abstraction*. Piaget also insisted on the role of abstraction in the link between object and representation. However, for Piaget, abstraction does not occur by retaining some features of the object and mentally omitting other features of it. The ground of abstraction for Piaget is the *actions of the individual*. Naturally, the theoretical shift from the Aristotelian emphasis on material objects to the Piagetian focus on actions leads to a different idea of conceptual objects. In the next section we will look at representations and the nature of their conceptual object in Piaget's genetic epistemology.

Piaget's dilemma

Piaget's epistemology was certainly influenced by Kant's work. But he had to modify some of Kant's basic assumptions. Piaget was dissatisfied in particular with Kant's rationalist ideas of "a priori categories" (e.g. causality) and "pure concepts", that is, concepts prior to *all* experience. He said:

> For my part, I consider myself to be profoundly Kantian, but of a Kantianism that is not static, that is, the categories are not there at the outset ; it is rather a Kantianism that is dynamic, that is, with each category raising new possibilities, which is something completely different (Piaget in Piattelli-Palmarini, 1980, p. 150)

Piaget also had to deal with a problem that Kant left beyond discussion, namely, the problem of the *origin* of knowledge. The Copernican revolution accomplished by Kant is, as Derrida noted (Derrida in Husserl, 1978, pp. 39-40), a "revelation" for the first geometer as opposed to a "construction". "The [Kantian] a priori nature of [the geometrical] concept within which we operate precludes all historical investigation whatever about its subject matter." (*op. cit*, p. 41). Piaget, hence, had to theoretically thematize the abstraction of the actions in such a way that the origin and development of knowledge could become clear. And he did so against the background of two influential theories of knowledge development of his time, namely, empiricism and preformation theory[4]. In the famous debate with Chomsky, held in France in 1975, after considering these theories, Piaget concluded that "an epistemology conforming to the data of psychogenesis could be neither empiricist nor preformationist, but could consist only of a constructivism, with a continual elaboration of new operations and structures." (Piaget in Piattelli-Palmarini, 1980, p. 23)

What can be said, within this context, of the conceptual object in Piaget's epistemology? What is the relation between representations and their objects? The conceptual object, in Piaget's epistemology, arises neither from the impressions

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[4] Preformation theory, which was widely known in the 18th Century, contended that human development was the unfolding or growing of *preformed* structures. It re-emerged in the 20th Century under a biological form of innatism. For a more detailed discussion see Furinghetti and Radford 2002.

that we receive from a material object nor from what we discover by separating in thought the physical and the mental. They arise from the actions that the individual carry out in his/her physical experience with the material object:

> The acquisition of knowledge … results from an abstraction, which we must consider as starting from these actions, since the properties discovered in the objects are the very ones which the actions have introduced to begin with. (Beth and Piaget, 1966, p. 232)

Piaget contended that, through abstraction, the actions become transformed into operations. These operations, he added, "can sooner or later be carried out symbolically without any further attention being paid to the objects which were in any case 'any whatever' from the start." (Beth and Piaget, 1966, p. 238). The relationship between representation and the represented object appears thereby linked in a strong manner to the operations to which the abstracted concrete actions lead.

There is, of course, a generous dose of Kantianism in the manner in which the conceptual object is conceived in Piaget's epistemology. But to what extent Piaget followed Kant? In particular, what can we say about the ontological status of the conceptual *object* in Piaget's epistemology?

The question of the ontological status of the conceptual object in Piaget's epistemology was brought up by René Thom and Norbert Bischof during the debate Piaget/Chomsky as a question concerning the difference between construction and reality. Referring to Piaget's approach, Thom said that either the space exists as a universal framework prior and exterior to the individuals, as a space where all reality would be subsumed or, in contrast, the space is a Kantian projection of an internal (a priori) structure of the individuals. The problem for Piaget was, as Thom put it, to reconcile a genetic approach of subjective constructions, where what the object *is* depends on the individual's action, with a realist ontology where what exists is beyond the will and actions of the individual. Thom was hence urging Piaget to explain how the individuals' *construction* of space can coincide with the reality of space. (Thom in Piattelli-Palmarini, 1979, p. 503).

Piaget had in fact dealt with this question in a book written more than ten years before the debate with Chomsky. In his book, co-authored with the logician Beth, Piaget himself asked: "in what sense can we then speak of a pre-established harmony between deduction and experience to explain the accordance of mathematics with reality?" (Beth & Piaget, 1966, p. 284). The answer was given in terms of “a common origin” to be found in “the laws of organic co-ordination” and “the physico-chemical environment”. (*op. cit*). In addressing Thom’s question, Piaget did not present any opposition to Thom’s suggestion that “Piaget seems indeed to adopt the realist thesis of an external existence of space” (Thom in Piattelli-Palmarini, 1980, p. 362). Instead, drawing on the idea of a “common origin” elaborated the previous years, Piaget stated that the space constructed by the individual is in accordance with the exterior one: “both exist without conflict and converge without merging.” (Piaget in Piattelli-Palmarini, 1980, p. 369). And that what ensures their convergence is the fact the individual is a physicochemical and spatial object, the individual “starts from neurological and biological sources whose laws are those of reality” (*op. cit*. p. 369).

But even assuming an ontologically structured reality, a common origin of actions and the psycho-chemical environment seems not good enough to ensure the convergence of the individual’s construction of space with the space of reality. We may still be confusing two different ontological layers, as Bischof suggested in his intervention during the debate. Bischof indeed introduced a distinction between *critical realism* and *naïve realism*. Critical realism presupposes an independent reality, “the so-called “objective world,” the structure of which remains whether or not there are organisms who perceive it correctly, or perceive it at all.” (Bischof in Piattelli-Palmarini, 1980, p. 233). Furthermore, critical realism distinguishes between the aforementioned objective world and the “phenomenal world”, that is, the world in which the individuals live, make their experiences and produce their conceptual constructs. There may be a corresponding relation between them, but they are ontologically different. Naïve realism confounds both. Bischof’s major worry was that the products of our thought (Piaget’s conceptual objects) are manifested in the course of experiences and that these products may present to us as the “irresistible evidence” of a solution (in the phenomenal world) that, in the end, presents itself as “a good form” only. Thus, given this distinction between phenomenal and objective world, how to ensure, in the Piagetian account, the convergence between knowledge and reality?

Piaget came back to this point in his text *Afterthoughts* and commented again on the relation between knowledge and reality, which now he expressed in terms of divergence. He said:

> As for critical realism, which Bischof analyzes so well, and the characteristics of optimal (but never complete) adaptation of knowledge, and so on, I think that I generally agree with Bischof‘s considerations, with the exception of

> perhaps one important point. Bischof, if I understand him correctly, thinks as I do that the object is never completely attained; it remains a limit, and he speaks in this regard of an asymptotic progression. But the problem is to know whether this limit allows for convergence or divergence. I believe that the progression is divergent in the sense that the object is transformed as knowledge closes in. Now, this makes me differ even more from the “naïve realism to which Bischof fears that the utilization of mechanisms of equilibrium might lead me. (Piaget in Piattelli-Palmarini, 1980, pp. 283-284)

We see then that in his answer to Thom, Piaget talked about convergence. Now, in his comments on Bischof he talks about divergence. I do not take these answers as contradictory, but as an additional precision that Piaget was bringing about his epistemology. A constructivist approach which adopts a realist ontology cannot say anything about how close we are from reality. The answer to Thom reflects, I think, the fact that in such a framework one *expects* the individual’s constructions to be close to reality. The answer to Bischof expresses the awareness that the *abyss* between reality and the individual’s conceptual constructions made in the phenomenal world is essentially impossible to fill. Knowledge and reality, as Kant would have said, remain separated.

But Piaget’s convergent/divergent stance towards reality is also an indication of the tension that his epistemology born as a result of adopting a realist ontology. All in all, Piaget was confronted with the following dilemma:

- either to follow Kant’s ontology and to do what Radical Constructivism did a decade later, that is, to work within the boundaries of a Kantian nominalism and consistently claim that the individuals’ constructions do not inform us about the things in themselves
- or to hold a kind of realistic ontology and to try to explain the gap between the subjective construction of reality and objective reality[5].

Piaget preferred the latter. He did so because he valued logical necessity and because of his commitment to rational thought as modeled by sciences and mathematics. The first option is *more philosophico*. The second one is *more mathematico*. To prefer the first option would amount to embrace *wisdom* only. To choose the second option is to stick to *certainty*. Although Piaget said that he turned

[5] Piaget even went on to say that constructivism “in fact, only differs from Platonism in that it does not speak of the universe of possibles as if it were already achieved or “existing”. But constructivism retains the Plationist belief that this universe is accessible, through the procedure common to all schools of thought: that of effective construction” (Beth & Piaget, p.303).

away from Kant's a priori categories, he indeed also departed from Kant's ontology. Instead of the "wisdom" of philosophy, he opted for the "certainty" of sciences[6].

Representations and their objects as cultural constructs

Piaget's genetic epistemology never denied the role of culture in his accounts of intellectual development. The problem, he said, is to determine whether the intervention of inter-individual factors is necessary to explain the development of thinking (by which he meant *logical* thinking) (Piaget, 1967, p. 155; Piaget 1968, p. 9). Piaget conceived of inter-individuality (and social co-operation at large) as a *system of actions*, and attempted to show that inter-individuality was also governed by the same formal structures as the logical one. Thus, he concluded that social cannot explain the intellectual and vice-versa, because both are framed by the same dynamics of evolving structures, namely, by the laws of equilibration:

> Social co-operation is also a system of action, inter-individual actions and not simply individuals, but actions nevertheless, and consequently under the effect of all the laws that characterize them. Hence, one can say that the social actions that result in the co-operation are also governed by the laws of equilibrium. (Études Sociologiques, p. 158, my trans.)[7]

The cultural tradition that emerged from the work of the anthropologists of the early 20th century offered a different perspective. Thus, for Durkheim and the sociological tradition inspired from his work, the objects of our representations are related to collective cultural experiences. The concepts with which we think, he claimed, are those consigned in the vocabulary of a culture. This vocabulary expresses collective experiences beyond the always limited space of any individual's activity. He said:

> There are scarcely any words among those which we usually employ whose meaning does not pass, to a greater or less extent, the limits of our personal experience. Very frequently a term expresses things which we have never

[6] The opposition between wisdom and certainty was elaborated by Piaget in his book *Insights and Illusions of Philosophy*. In this book he said that "philosophy ends in a wisdom and not [in] a mode of knowledge" (Piaget, 1971, p. 116). For knowledge demands that verification (in the sense of the science) be possible. "What is important" he wrote in the postscript to the second edition of his book, "is the trilogy reflection x deduction x experiment, the first term representing the heuristic function and the other two cognitive verification, which is alone constitutive of 'truth'." (*op. cit*. p. 232) (see also Bkouche, 1997, pp. 36-39).

[7] A more detailed analysis of Piaget's conception of inter-individuality as actions can be found in Radford, 2000b.

> perceived or experiences which we have never had or of which we have never been the witnesses. Even when we know some of the objects which it concerns, it is only as particular examples that they serve to illustrate the idea which they would never have been able to form by themselves. Thus there is a great deal of knowledge condensed in the word which I never collected, and which is not individual. (Durkheim, 1915, p. 483).

For Durkheim the object of representations was a collective object. Of course, within this context, the Kantian problem of the relationship between the represented object and its representation cannot have a solution in terms of the standards of objectivity that started with Galileo and that became adopted and expanded in the 18^{th} Century philosophy of Enlightenment. It has a solution, nevertheless, but it requires conceiving science in a different manner. It requires seeing science as a one of the various modes of comprehension of the world, as a kind of “more perfect form of religious thought” (Durkheim, 1915, p. 477). The relationship between representation and its objects is hence to be found in the kind of *ideality* that societies construct, for, as Durkheim noted, “a society can neither create itself nor recreate itself without at the same time creating an ideal” (op. cit. p. 470). Lévi-Strauss held a similar position, although he did not agree with the ‘vulgar thesis’ according to which magical or religious thinking would be a lesser stage in the evolution of scientific thinking. Instead, he saw both as lying on their own structures and suggested that they should be placed in parallel (Lévi-Strauss, 1962). We may very well understand Piaget’s refusal to accept Lévi-Strauss’ position according to which all cultural groups have a similar logic –a logic based not on structural features of operations but on complex and subtle oppositions with which we organize our world (Lévi-Strauss, 1962). Piaget maintained that the so-called primitive cultures have reached the stage that he identified as concrete operational thinking (Piaget, 1967, pp. 147-148) and that the disagreement with Lévi-Strauss could only be resolved experimentally. Thus, in an interview, Piaget said:

> I spend a lot of time during every discussion I have with Lévi-Strauss repeating that neither he nor I can decide the matter deductively. It is not the ethnographer’s enquiries which will decide the issue for us. What is need are studies in the field by psychologists used to our methods of enquiry who will question adults. (Grinevald 1983a, p. 75)

To which Lévi-Strauss answered:

> What I do ask, and I formulate this question rather naively in ethnological terms, is whether Piaget’s research techniques aren’t rather artificial in

> character. His experiments are set up in advance, prefabricated, which does not seem to me to be the best way to understand the mind in all its spontaneity. (Grinevald 1983b, p. 84)

In an important sense, Lévi-Strauss' plea was an attempt to invite epistemology to consider in a decisive manner cognition and representations as consubstantial of context and culture. As an anthropologist having been in contact with numerous tribes, he held the conviction that it is impossible to circumscribe all of them in a same Universal History where Reason would unfold stage after stage. Of course, in such a perspective, not only the object of the representation but also the problem of the relationship between representations and the represented objects change. But how can this relation be theoretically thematized? Lévi-Strauss did not offer a straight answer. In the next section I will sketch the historic epistemology of Marx Wartofsky, which constructs the idea of representation in an intimate relation to its cultural context.

Wartofsky on Representations

Wartofsky started with a question that underpinned Vygotsky and Luria's work: what is it which makes human cognition distinctive? Wartofsky's answer was: "the ability to make representations". Representations for him included not only tables, drawings, formulas but also artefacts, objects and ideas. In their broader sense, representations are culturally and historically constituted modes of action; they are mediating agents in human activity, concrete and theoretical artifacts of our forms of perception and cognition:

> Theoretical artifacts, in the sciences, and pictorial or literary artifacts, in the arts, constitute the *a priori* forms of our perception and cognition. But contrary to the ahistorical and essentialist traditional forms of Kantianism, I propose instead that it is we who create and transform these *a priori* structures. Thus, they are neither the unchanging transcendental structures of the understanding, nor only the biologically evolved *a priori* structures which emerge in species evolution (as, for example, Piaget and the evolutionary epistemologists suggest). Piaget's dynamic, or genetic structuralism is important here, of course. His dictum, "no genesis without structure, no structure without genesis", suggests the dialectical interplay of the practical emergence and transformation *of* structures with the shaping of our experience and thought *by* structures. But the domain of this genesis I take to be the context of our social, cultural and scientific practice, and not that of biological species-evolution alone (…) In a sense, then, our ways of knowing are themselves artifacts which we ourselves

> have created and changed, using the raw materials of our biological inheritance. (*Op. cit*, p. xxiii)

Piaget, as we saw in the previous sections, considered objects and artefacts as accessorial epistemological elements. He shifted the Aristotelian emphasis from artefacts to actions. Wartofsky subsumed the actions into their modes of social praxis and, in addition, he put the artefact back into the cognitive scene and stressed its epistemological role.

The genesis of representations, he argued, has to be examined in the two fundamental forms of human activity: in our making of things (i.e., is to say, in modes of production) and in our interaction with individuals (i.e., in modes of social relations):

> Tools and language (…) become the basic artifacts by means of which the human species differentiates itself from its animal forebears; and it is therefore in an analysis of these basic artifacts that a theory of the genesis of representation needs to be developed. (Wartofsky, 1979, p. xvi)

Of course, these forms of human activity are not to be conceived as functioning separately. Neither should one of them be considered the functional complement of the other. There is a strong (although not necessarily visible) relationship between them. An artefact can be an element of (material or intellectual) production. But it bears the very form of the individuals' activity:

> Thus, spears and axes are not only made for the sake of hunting and cutting, but at the same time represent both the method of their manufacture and the activities of hunting animals or chopping wood. (*op. cit*, p. xiii-xiv)

Reciprocally, individuals' activities are generally organized in terms of artifacts. Thus, we will organize our actions differently when building a rhombus using a pencil-and-paper technology or when using computer software.

In Wartofsky's epistemology the Kantian question of "how is representation possible?" is replaced by the new question "by what means, and in the course of what activity does [representation] take place?" (see *op. cit*., xvi). How, however, can the problem of the objectivity of knowledge be dealt with here?

> What I am proposing here, only programatically and sketchily, is what I would characterize as an historical-materialist theory of the genesis of theory, or of theoretical cognitive praxis. The apparent instrumentalism of truth in such a

> view, as I set forth here, is then replaced by a realist emphasis: As a representation in some symbolic form of a mode of action or practice, the theoretical formation has its truth-value in the adequacy of the representation: i.e. in its practical conformity to a successful mode of activity, which it institutionalizes, so to speak, in the representation. It is *not*, therefore, an unmediated representation of some external state of affairs, but rather one which is mediated by the practice or mode of action which it represents. The sheer externality of a state of affairs become 'objective' for us, then, only as it is mediated by our practice. What we can know is therefore always conditioned by the way that we come to know it. In a sense, our knowledge of the 'external' world is a knowledge of what this externality is amendable to, in our incursion upon it and intervention in it. (Wartofsky, *op. cit*, p. 136)

We see then that, in Wartofsky's epistemology, the individuals are seen as placed in an independently existing environment. Nevertheless, this environment appears to the individuals as mediated by representations[8]. The object of representations is thereby clearly distinguished from the thing-in-itself, the 'real' object that hits our senses. Our forms of perception –or our faculty of sensibility, as Kant would have put the matter– create an unbridgeable gap between the thing-in-itself and what we know of it. Certainly, in taking this theoretical position, the historical epistemology becomes close to Kant's epistemology and the one adopted by Radical Constructivism and moves away from Piaget's genetic epistemology. However, Wartofsky's epistemology departs from Kant's, in that the faculty of sensibility was taken by Kant as a rigid part of the architecture of the mind. For Wartofsky, on the contrary, the human faculty of sensibility (i.e. our modes of perception) is historically and culturally constituted. With our physical organs we produce representations which come to be integrated in our forms of perception and cognition to the extent that what we perceive is already a world tainted with the historical color of human needs and intentions.

Thus, instead of a "sentimentalist relativism" –to use Goody's splendid term (Goody, 2000, p. 137), or instead of a Radical Relativism, where 'everything goes' (Putnam, 1981), objective knowledge and truth are seen as embedded in culture, in the modes of production and in the social forms of interaction. The 'truth' and 'objectivity' proposed by Wartofsky's epistemology are not transcendental entities.

[8] As he said in another passage, "the 'objects of perception' are taken to be independent of perception, though they are mediated by the activity of perception, in that they are perceived *by means of our representations of them.*" (op. cit. p. 193; emphasis as in the original).

They find their support in the adequacy of representations and in the conformity and success of the latter in terms of institutionalized activities[9]. The verification to which knowledge and its representation have to be submitted finds now its normativity in the cultural institutions serving as the background of the individuals' activities. The normativity is no longer one of the laws of biological equilibrium and of constructive conceptual autoregulations leading to deductive necessity. Normativity has to be understood as a flexible, situated and historical construct. Unfortunately, Wartofsky did not develop this point further. And it is evident that we need to reflect much more on it. In particular, we need to better understand how cultures dynamically produce normative elements and how, in the space opened by the normative elements, scientific and mathematical discourses become constituted. At any rate, in front of the same dilemma to which Piaget was confronted, Wartofsky had also to make a choice. His was not certainty. It was wisdom.

Concluding Remarks

In this article I dealt with different conceptualisations about the object of the representations. The question was motivated by a practical concern: the spreading of new forms of knowledge representation associated with the increasing use of new technologies of semiotic activity. The time-honoured format of sentence-based mathematical text of written traditions is now becoming part of larger processes of objectification of mathematical knowledge (Radford, 2000, in press-1, in press-2) where other kinds of knowledge representation are being made possible (e.g. discursive and artefactual). Nevertheless, changes in modes of knowledge representation do not go without raising a delicate problem, namely the problem of *legitimacy*. We need to understand it as a problem having two different albeit related aspects, a *political* one and an *epistemological* one.

By a political aspect (from the Greek *polis*) I mean here an aspect related to social actions conveying a distinction between what is good and what is not. Thus, the technology of the written was condemned by the Pythagoreans, who favoured the technology of the word. In a similar vain, not long time ago the question of the pertinence of a calculator in the classroom was still a polemical issue; now the

[9] Naturally, this point is a highly developed form of one of the succinct remarks made by Marx in his *Theses on Feuerbach*: "The question whether objective truth can be attributed to human thinking is not a question of theory but is a *practical* question […] The dispute over the reality or non-reality of thinking which is isolated from practice is a purely *scholastic* question".

same question is no longer under discussion[10]. The written in Plato's time and the calculator in ours were highly controversial in terms of their legitimacy to represent and objectify knowledge.

By an epistemological aspect I mean the manner in which the objects of knowledge become known. But, as Kant noticed, how an object becomes known depends on the manner in which the object is *sensed* or *presented* to us, how it affects us. To express this affectation of our sensibility by the object he used the word *aesthetics,* not as relating to the beautiful but in its etymological sense, *aisthetá*, that is, as pertaining of sensuous perception. We can see, then, that in dealing with the problem of knowledge representation it is not possible to dissociate the political from the epistemological. But what this article wanted to explore was something that underlies the problem of knowledge representation and that often remains implicit, namely, that which we assume to be the nature of the mathematical objects. In other words, their *ontological* status. We cannot deal with the problem of the representation of knowledge without looking into the ontological dimension.

I began with Kant because it is impossible to understand Piaget and Wartofsky without understanding the manner in which Kant posed and tried to solve the problem of knowledge. Of course, Piaget's *épistémologie génétique* has had a remarkable influence in mathematics education. It is one of the best achieved contemporary epistemological accounts[11]. In the course of the article we saw how Piaget drew on, and departed from, Kant. The interpretation that I have offered of Piaget's ontology is different from von Glasersfeld's, who said that "Piaget's position is a bit ambiguous. Despite the important contributions that he has made to constructivism, he has always a tendency for metaphysical realism." (1988, p. 27; my trans.). I think that Piaget was showing much more than an unfortunate or inopportune penchant for realism.

Wartofksy drew from Kant and from Piaget but he departed from both. In thematizing Piaget's basic notion of *action* from a historic and cultural viewpoint, and in resituating the artefact at the center of human activity, Wartofsky's epistemology offers a rich avenue to theorize the problem of knowledge

[10] Thus, The Ministry of Education and Training of Ontario (Canada), for instance, has provided High School students with access to graphic calculators.

[11] I have to say, nevertheless, that to narrow my discussion I had to resist the temptation of dealing with another interesting ontological problem with which Piaget was confronted –that of logical necessity.

representation. To carry out this theorization however involves a change of perspective concerning the ontological status of mathematical and conceptual objects in general. And, as the section on Durkheim and Lévi-Strauss intimated, this change of perspective is not easy in that, in the end, it implies changes that are rooted in basic assumptions that are taken for granted. In this respect, Piaget's commitment to scientific method, which for him delimitated a clear border between philosophy and sciences and a distinction between wisdom and certainty, is most revealing. I am not contending, however, that to find the answer to the question with which we started this article, namely, "What is that which representations represent?", we need to avoid any recourse to basic assumptions. Thinking and understanding are contingent upon assumptions[12]. This is why there is not a unique answer to the aforementioned question for the answer will depend on ontological matters which are in turn supported by basic assumptions. A critical stance towards the object of representations does not require us to necessarily remove assumptions; rather it requires of us an awareness of the assumptions that we are making.

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This article resulted from the discussions that I had with José Guzmán of Cinvestav, Mexico, and Michele Cerulli of University of Pisa, during their stay at Laurentian University the Spring 2002. It became apparent that in my previous work on the objectification of mathematical knowledge the term *object* required more explanation and further theoretical elaboration. As Durkheim said, the vocabulary that we use is embedded in cultural traditions and my use of the term object may have sounded, I have to admit, too much colored of Platonism. I do not think this article answers Cerulli and Guzmán's questions concerning the nature of mathematical objects and the relation of the objects to the way we represent them. But it may help to understand better what an object may be in a cultural-historical epistemology and how it differs from other epistemologies.

[12] As Adorno (2001, p. 13) noted "[i]f you refuse to make any assumptions … then you will understand nothing." To further become aware of the role of assumptions in the ontology of mathematical objects one only needs to read hot debates between e.g. Hilbert and Brower or Poincaré and Couturat surrounding the foundation of mathematics.

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Construction of Mathematical Concepts and Cognitive Frames

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ABSTRACT. The development of students' abilities to carry out operations and understanding in mathematics has been featured as separate entities in the past. Today, they are studied from an overall perspective as to the construction of concepts. The scheme concept and the internal and external representations of concepts are not recent in the literature. Nevertheless, possibly due to the technological development of graphs and figures representations, such concepts came again under analysis. Current theoretical aspects such as representational semiotic systems and their implications in the articulation of internal representations may profit from previous research (Duval 1993, 1995). As far as the construction of concepts and their implications in certain tasks are concerned, the notions of "conceptual knowledge" and "procedural knowledge" (Hiebert & Lefevre, 1986) have gained relevance when studying learning phenomena. Furthermore, we see the need of constructing structures strongly associated with understanding –which are more general than specific to the construction of a mathematical concept. According to Perkins and Simmons (1988), and Perkins and Salomon (1989), these cognitive structures they labeled as content, problem solving, epistemic and inquiry structure, they are important to the learning of mathematics.

Introduction

Research in Mathematics Education has –on its strive to unfold the way knowledge is built- motivated the study of the role of semiotic representations and processes in the construction of knowledge. This has entailed revisiting papers concerning the role of sign, and the consideration of new theories involving the construction of internal networks in individuals who participate in the construction of such knowledge.

What we present here has to do with a certain evolution of the notions of sign, semiotic representation, understanding, construction of internal networks, the idea of transference, the notion of cognitive obstacle, and the construction of cognitive structures, which are key component to achieve better understanding. We attempted to collect research work on the above-mentioned notions, and tried to focus them under the same light.

Saussure's idea of sign and the construction of concepts

Research needs to explain how the role of representations brings in to discussion the importance of sign in the construction of knowledge. Then, before introducing the idea of semiotic system of representation and the construction of mathematical concepts, we would like to review ideas of sign expressed by Saussure related to linguistic contexts. Saussure's standpoint (1900) in relation to the sign is:

> *The linguistic sign is, then, a two-faced psychic amount that may be represented by the figure* [left]. *Ambiguity disappears if we assign names to the three current notions* (See Figures 1 and 2)

Figure 1.

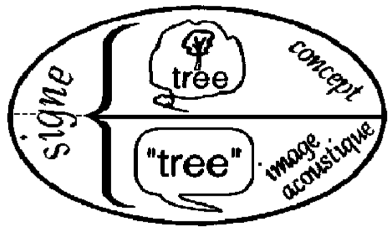


Figure 2.

Saussure's idea of sign is indissoluble, and has been classified as a dyadic type of representation. It is important to remember that Saussure's theory of sign is restricted to linguistic context. In this trend a lot of research has been done to explain the construction of knowledge under the paradigm in connection with the construction of "concepts raising from daily life". Lacan and Walkerdine's research (See Whitson, 1997), among others, have found limitations of Saussure's approach. We will contrast it with others related to the construction of mathematical concepts.

The scheme notion and the construction of concepts

Skemp (1962) to introduce Piaget's idea of "scheme", he used a list of arbitrary symbols to construct a meaning related to them with a population. In this study, we can verify that the symbols he used (see annex) do not constitute a mathematical sign system. That research conducted by Skemp (1962) is far from a mathematics education point of view. Probably, the reflection on his preliminary idea of scheme related to the construction of mathematical concepts made him to have a better approach. We can see in what follows that the paradigm of constructing concepts from a point of view of "daily life", continued in his research program ten years after this publication of 1962.

We now present the scheme notion developed by Skemp (1971) as it appears in his book *"The Psychology of Learning Mathematics"*. Skemp wrote about the construction of concepts in general and the construction of mathematical concepts specifically. He wrote:

> …We shall consider how concepts fit together to form conceptual structures, called schemas, and examine some of the results which follow from the organization of our knowledge into these structures. (p. 19)
>
> **Abstracting and classifying**. At a lower level we classify every time we recognize an object as one which we have seen before,… From these varying inputs we abstract certain invariant properties, and these properties persist in memory longer than the memory of any particular presentation of the object [see Figure 3 related to the concept chair]. (pp. 19-20)

Figure 3.

> **abstracting** is an activity by which we become aware of similarities (in the everyday, not the mathematical, sense) among our experiences.
> **classifying** means collecting together our experiences on the basis of these similarities.
>
> A concept therefore requires for its formation a number of experiences which have something in common. (p. 21)

> The distinction between a concept and its name is an essential one for our present discussion. A Concept is an idea; the name of a concept is a sound, or a mark on paper, associated with it… (p. 22)

As we said before, this approach is mainly related to "concepts raising from real life". The example given by Skemp is about the concept of "chair". We would like to contrast this approach with others where the researchers made a change of paradigm. This approach was followed during the 60s to the beginning of the 80s. Skemp (idem, p. 35 y 45) adds to the scheme idea:

> …each of these by its very nature is embedded in a structure of other concepts. Each (except primary concepts) is derived from other concepts and contributes to the formation of yet others, so it is part of a hierarchy. (p. 35)
>
> … To understand something means to assimilate it into an appropriate schema. (p. 43)

Mental representations and mathematical concepts

The research conducted on representations on the 70's and beginning of the 80's was centered to understand the mental representations students had of a mathematical concept trough the analysis of the answers given by them when asked for a particular concept (see for example, Tall and Vinner, 1981; Vinner, 1983). Then the notion of "Concept image and concept definition" was developed by those authors (Tall and Vinner, 1981, p. 151). They defined those notions as follows:

Concept image

A concept name when seen or when heard is a stimulus to our memory. Something is evoked by the concept name in our memory. Usually, it is not the concept definition, even in the case the concept does have a definition.

> The concept image is something non-verbal associated in our mind with the concept name. It can be a visual representation of the concept in case the concept has visual representations; it also can be a collection of impressions and the experiences associated with the concept name can be translated into verbal forms. But it is important to remember that these verbal forms were not the first thing evoked in our memory. They came into being only at a later stage.

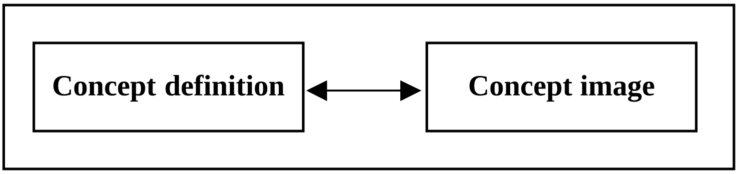


Figure 4.

Although Tall and Vinner's approach focuses on the importance of mental representations, their main problem was to clarify why students had no connections between their concept image of a concept and the definition of that mathematical concept. For example, related to the concept of function Vinner (1983) designed a questionnaire to show this disconnection among the population.

It was until the end of the 80's when explicitly researchers put the problem of representations on the table of discussion. We can find in Janvier (1987) in the book "Problems of representation in the teaching and reasoning of mathematics" different ideas and definitions about representations. We think, that after the publication of this book and the controversies they raised (see for example Pimm, 1990), researchers looked for a more suitable role of the representations on the constructions of mathematical concepts (see for example, Kaput, 1987 and Kaput, 1991). Kaput focused, firstly (1987), on a theoretical approach to explain the use of mathematical symbols. But in 1991, Kaput stressed the importance of representations as mental objects (Idem, p. 59):

> "I intended to express [Figure 5] relationships between "notation A and referent B" where each (and perhaps even the correspondence) is expressible in material form -but where the actual referential relationship exists only in terms of the mental operations of members of a particular consensual domain".

We can see that the emphasis was on the mental representations.

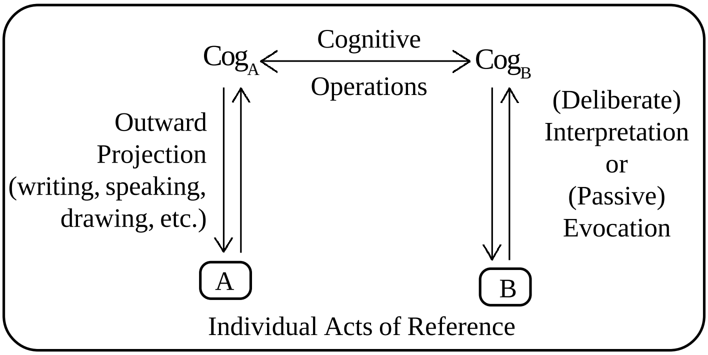


Figure 5.

Semiotic representations as a paradigm on the construction of mathematical concepts

The construction of concepts trough the manipulation of semiotic representations was at the hart of Duval's research (1988, 1993, 1995, 2000). From Duval's point of view, he stresses the fact that mathematical objects cannot be directly accessed by the senses, but only through semiotic representations. Then, his approach is not that one related to constructions of concepts as those related to "daily life".

Duval's experimental studies identified two related aspects -manipulation of representatives within a mathematical sign system, on the one hand, and the conversion of representations between two or more systems of the same mathematical object, on the other- he has generated a new notion, the representation register, totally linked to the essential functions of any cognitive activity. Duval (1993, 1995, 2000a) features a semiotic system as a representational system as follows:

A semiotic system may be a representational register if it allows for three cognitive activities associated to semiotics:

1) The presence of an identifiable representation…

2) The treatment of a representation which is the transformation of a representation within the same register where it was formed…

3) The conversion of a representation which is the transformation of the representation into one of different register which preserves the totality or part of the meaning of the initial representation…

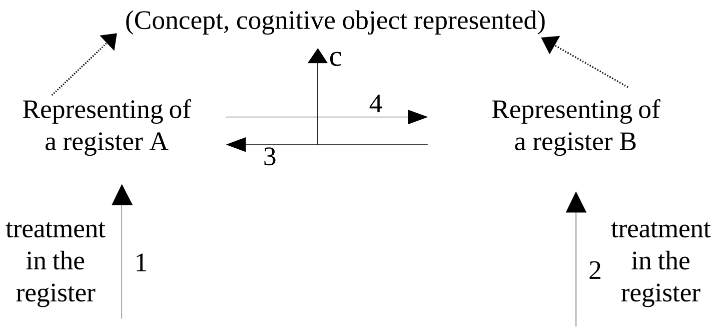


Figure 6. Representational model based on the objectivization function. Arrows 1 and 2 correspond to the internal transformations of a register. Arrows 3 and 4 correspond to the external transformations, that is, to the register change conversions. Arrow C corresponds to what we call the integrative comprehension of a representation: it presupposes a two-register coordination. The dotted arrows correspond to a classic distinction between the representative and the represented. Naturally, this scheme considers the simplest case of coordination between two registers.

As regards mathematical objects, Duval (idem, pág. 46) asserts that given the fact that every representation is partial to what it represents, we must consider the interaction among those different representations of the mathematical object as being absolutely necessary to construct the concept. In other words, student's acquisition of a concept will be achieved at the moment when the student is capable of coordinating, free of contradictions, different representations of the mathematical object. We must highlight that from this point of view the concept always is under construction.

Stressing the importance of semiotic representations, and more important, the register of representations and the task to convert from one representation to another, that lead researchers to analyze carefully the characteristics of each register and the cognitive problems to convert from one representation to another. Let's take an example, the graphic register, and let us suppose that we want to analyze the problems students experience when converting from a graphic representation of a linear function to its algebraic representation. The work of Duval (1988) explains the difficulty from a graphic representation of an equation to the algebraic expression (the work of Duval involves lines in the plane and equations); this difficulty deals with the development of students' ability to distinguish visual variables that play a central role in the solution of that kind of exercise. Duval (idem) distinguishes 18 visual variables related to the position of the line in the Cartesian plane. The idea of visual variable was worked it out by Duval (1988, 1999, 2000b) when passing from a graph to the algebraic expression, he describes two general variables:

– *The distinction of the two relevant parts of a figure plays a fundamental role, that is, one figure embedded in another figure* [for example, Cartesian axes and a curve]
– *The form of the task: the sketch made, that delimit or not a zone, it is straight or curve, it is open or closed.*

And three alternative visual variables: (a) the sense of the slope of the line (b) the angles of the line with the axes (c) the position of the line with regard to the origin of the vertical axe. Taking into account all these general variables, we can detect 18 visual variables related to the position of the line in the Cartesian plane. As we can see, the study of the difficulties when converting one kind of representation to another, can drive us to a deep particular study about the construction of a single concept: The straight line.

Schoenfeld et al (1989) also documented visual aspects considered by a student when constructing one equation from a straight line sketched on a computer's screen. In this work they mention that during the clinic interview, the student considered as essential the intersection of the line with the x-axe, she said that this is because this information provided an element for the construction of the algebraic expression of the form $y = mx + b$. The work of Schoenfeld et al (idem) tell us about the internal structure that this young girl had constructed and that it is not the one we can expect in the educative context she studied.

What we want to stress here, is that every concept can be analyzed under the perspective of the register of representations and if a student makes some errors, we can try to explain these errors within this theoretical approach. Then, related to the task of converting from a graphic to an algebraic expression, the analysis of the visual variables involved is determinant to achieve or complete the task.

As we can see, the theoretical aspects on the role of the representations and their interpretations had been evolving and little by little we have different approaches to explain phenomena related to learning mathematics. New approaches appeared under the influence of Vygotsky and Luria's work, related to "concepts of daily life", to explain the construction of mathematical concepts, we are not discussing this approach in this document.

Mental representations and web-connected knowledge

Hiebert & Carpenter (1992) explain what is understanding based on the idea of a frame of networks formed by internal representations that, in turn, were generated by the manipulation of external representations. They quote (idem, p. 67) the following:

> **Defining Understanding.** We begin by defining understanding in terms of the way information is represented and structured. A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections.

On the same trend of though, when you need to remember a fact, a procedure, or any kind of information needed in problem solving situations, the idea of web-connected is coming to explain the process, Hiebert and Carpenter (idem, p. 74), quoted in the following paragraph:

> One advantage of the inclination to create connection between new and existing knowledge is that web-connected knowledge is remembered better (Baddeley, 1976, Bruner, 1960, Hilgard, 1957). There are probably two explanations for this. First, an entire network of knowledge is less likely to deteriorate than an isolated piece of information. Second, retrieval of information is enhanced if it is connected to a larger network. There simply are more routes to recall.

The authors focus on the discussion of mental representations and web-connected knowledge. They are not taking into account in deep the role of the semiotic representations on the construction of the web-connected knowledge. From Duval's point of view, the coordination of registers of representations is fundamental. In other words, the articulation among representations related to a mathematical concept, that is generated because the conversion tasks among semiotic representations, is essential on the construction of that concept.

Then, on the one hand, authors were focusing on the role of the mental representation in the learning of mathematics without putting at the same level of importance the semiotic representations. On the other hand, the authors who were putting more attention to the semiotic representations, they are stressing the importance of semiotic representations as they appear in textbooks or those produced by teachers on the blackboard. They are focusing on the reproduction of those particular representations when the students are solving problems. For example, Hiebert and Lefevre's (1986), when describing "conceptual and procedural knowledge" they state that:

> Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge.
> Procedural knowledge is made up of two distinct parts. One part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks. … In summary, procedural knowledge of mathematics encompasses two kinds of information. One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. (pp. 3-6)

From our point of view, it is important to analyze the representations students are generating when dealing with a mathematical problem, because the representations they use are not necessarily similar as those we find in textbooks or those reproduced by teachers. In what follows we will discuss this approach.

Representations in problem solving situations

On the concept of function we found different kinds of obstacles (see Hitt, 1994, 1995 and 1998). In that study, through a questionnaire that included applications of functions, we found in one teacher the following performance that constitutes a different cognitive obstacle to those documented before. Given some graphs, the teachers were asked to draw a container associated with a physical phenomenon (filling a container with some liquid). In one question the teacher's answer was:

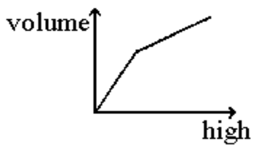
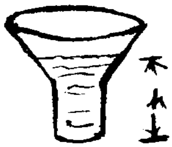
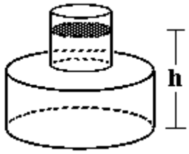
Graph given to the teachers	*A teacher's answer*	*Example of a correct answer*
		

Figure 7. Conversion task.

The primacy (in Fischbein's sense, 1987, p. 53) of a global intuition on the analytic thought did not permit the teacher to achieve suitable reasoning to give a correct answer. On another questionnaire (in the same study) the same teacher gave the following answer to the following question: Given a container like that in Figure 8, draw a graph showing how the area of the liquid at the top varies according to the high of the liquid.

If we look at the answer related to the question about drawing a container, we could say that this teacher is unable to undertake analytic reasoning to answer the question. But, analyzing the other question (see Figure 8) where the teacher was asked to draw a graph, we see he was able to develop an analytic reasoning of the situation in order to provide a correct answer. We see that *the articulation between representations may follow paths that depend on how the subject recalls his previous knowledge*. In one case, his intuition played a major role than his analytic reasoning, leading to a wrong answer.

Transcription of the teacher's work

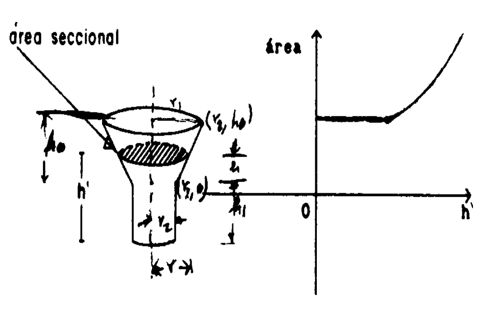
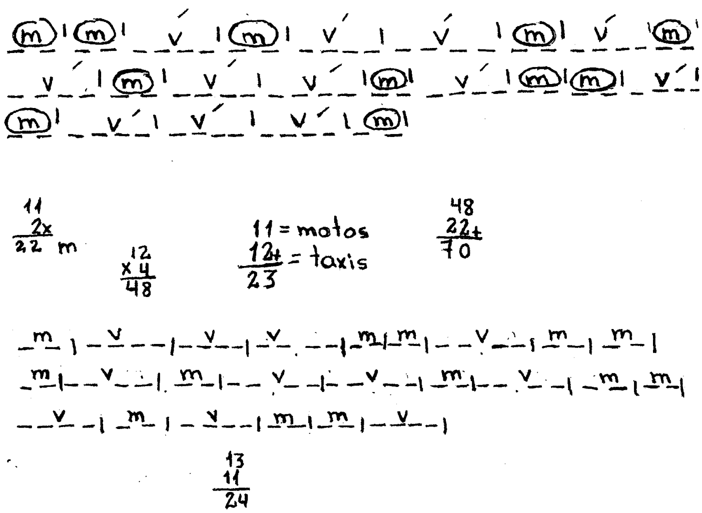
Teacher's algebraic process	Graphic representation
$h=\frac{h_0-0}{r_1-r_2}(r-r_2)=\frac{h_0}{r_1-r_2}(r-r_2)$ $h=m(r-r_2)=mr-mr_2$ $r=\frac{h+mr_2}{m}=\frac{1}{m}h+r_2$ $Area=S=\pi r^2=\pi(\frac{1}{m}h+r_2)^2,\ h\in[0,h_0]$ $h'=h_1+h=h_1+mr-mr_2$ $r=\frac{h'-h_1+mr_2}{m}=\frac{1}{m}h'+\left(\frac{mr_2-h_1}{m}\right)$	

Figure 8. Conversions among representations.

Let us see another example that shows the production of semiotic representations in a problem-solving situation. You will see that those representations could be different to those treated in the classroom by teachers or those you can find in textbooks. In a Colombian study (see Benitez & Santos, 2000) the researchers gave several problems to be solved by different students in different grades. Soath (girl of 11 years old) utilized a specific representation to respond: *"In a racing Manuel counted 25 vehicles, and Carlos counted 70 wheels. Among the vehicles there were taxis and motorcycles. How many taxis and motorcycles were there in the racing?*



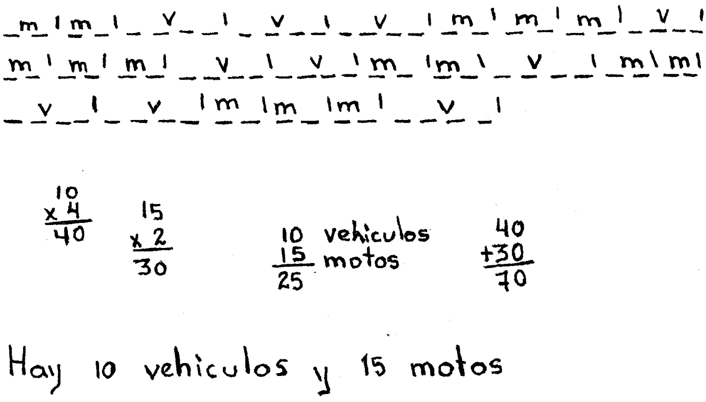


Figure 9.

We can appreciate that the girl used different representations that permitted her to take absolute control of the situation to arrive to a correct answer. She took control about the number of wheels (70), but using this first particular representation gave 23 vehicles. It seems that in her second drawing she is thinking of a general case. It did not matter how the taxis and motorcycles arrived, it was important to control the number of wheels, on the one hand, and the number of taxis and motorcycles on the other. Indeed, she changed one taxi by two motorcycles, thinking to arrive to the right answer. But, it was her final drawing which permitted her to obtain the right answer.

According to the theoretical approach we are dealing with, it was very important for Soath to produce her own semiotic representations to work and solve the given task. From our point of view the representations generated first in her mind and then putting them on her notebook, show us a dynamic idea of representation in a functional context.

We would like to compare the performance of Soath with the one developed by Alvaro (14 years old), who was taking an algebra course (see Figure 10).

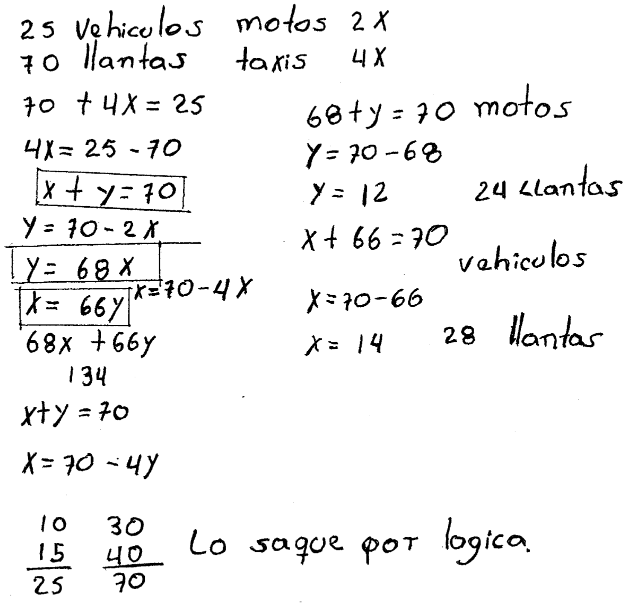


Figure 10

Probably, Alvaro thought that “2*x*” represents the “number of wheels used by the motorcycles” and “4*x*” as “number of wheels used by the taxis”. The problem is that the unknown “*x*” designates at the same time the “number of motorcycles” and the “number of taxis”. Then, what is wrong here? His mental representations are not well connected to the unknowns? That is, the way he identified the number of taxis and motorcycles seems not appropriate, since he used the same unknown “*x*”. Immediately, it is observed that he gave an algebraic expression and explained (via an interview) that 4*x* represented the vehicles! It seems that what this student showed, did not include fundamental information attached to the situation.

This student did not develop a concrete referent to support his work. He made algebraic and arithmetic errors, and showed serious contradictions in handling the information, and it seems he was not aware of them. The correct answer given at the end, he said in the interview that he copied it from his classmate.

It is important to stress here that this student did not exhibit different semiotic representations, different from the usual algebraic representations, to help him in his thinking when facing the problem. Alvaro’s performance was restricted to the algebraic register. Why he did that way? It seems that in the teaching of mathematics, the majority of teachers continue privileging the algebraic system of representation without considering that research issues about learning, are supporting the equilibrium that must be present during the use of different representations in the construction of mathematical concepts and problem solving. This tendency, possible unconscious, seems that has influenced the students learning and that is why they do not have enough support to construct connections. In Alvaro's case, the student was not able to make connections in this problem-solving situation. Hiebert and Lefevre (1986) explain that that kind of difficulties students show is because their knowledge is context-bound, they state: *Context-bound knowledge is not looking for relationships outside the immediate context.*

Transfer of knowledge

If knowledge is acquired in one context, then, how does transfer of knowledge occur when a student face a mathematical problem? Is it easy to document transfer of knowledge in an empirical study?

Hiebert and Lefevre (1986, p. 11) on the same subject, said:

> … There also is reason to believe that in addition to enhancing memory for procedures, linking conceptual and procedural knowledge facilitates the effective use of procedures. This may occur in at least three different ways. Of conceptual knowledge is linked to procedures it can; (a) enhance problem representations and simplify procedural demands; (b) monitor procedure selection and execution; and (c) promote transfer and reduce the number of procedures required.

Analyzing Soath performance, we can say that the representations she generated when dealing with the problem helped her to understand and take control of the situation she was engaged. Her representations were functional, guided her to the solution of the mathematical problem, trough the integration of actions in a coherent heuristic approach. Under the theoretical aspects we are treating, then, the notion of transfer is relevant; it seems essential in problem solving situations.

Thus, when reading a statement we need to create some associations to understand the statement and to create a special kind of representations. Then, possible, those representations will help us to make connections between what is called conceptual and procedural knowledge. And something important is that some times those representations that are helping us to understand the statement are dynamic representations that are playing a major role in a problem-solving situation. They are inducing the production of semiotic representations and ways to handle them throughout the solution process involved when reading the statement. On this trend, Hiebert and Carpenter (idem, pp. 76-77), said:

> The framework we propose retains the notion of internal representations. In fact, we propose that the way internal representations are connected helps to explain the potential for transfer. But, we also suggest that the problem situations in which students engage influence the nature of the internal representations and their connections to other representations. In other words, the situation or context influences the amount of transfer that actually occurs.

Epistemic frame, problem solving frame and cognitive conflict

From Perkins y Simmons' (1988) point of view, the students' misconceptions in the learning of mathematics deals with a lack of cognitive frames that do not permit the student to connect with other knowledge when solving problems; and to go further to construct new knowledge. Perkins & Simmons (1988, p. 323) call our attention to four cognitive frames: *Content knowledge, also knowledge in problem-solving, epistemic, and/or inquiry*. They said (idem, pp. 319-320) that mathematics instruction focuses on the construction of the content frame and lately with the problem-solving frame. And not promoting the students' construction of the epistemic and inquiry frames: *We have argued that learning with understanding depends on all four frames and their interrelationships …The problem-solving and content frames are the focus of most classroom instruction.*

Coming back to Alvaro's performance, it seems that he has not developed a structure that permits him to take control of situations that arises during problem solving processes, a structure in the sense of Perkins and Simmons (1988, p. 305), they said:

> Epistemic frame: This frame incorporates domain specific and general norms and strategies concerning the validation of claims in the domain. Within a well developed domain, the 'facts' in the content frame are valid by the measure of the norms in the epistemic frame…

Then, from Perkins and Simmons point of view, it seems important not only the construction of knowledge related to mathematical content, but also, the epistemic frame and the use of that knowledge in problem solving.

We can find examples related to this from history of mathematics. For example, the famous controversy related to Cauchy's theorem (1821) seems important to discuss, in general, here. Abel (1826) gave a non-example of Cauchy's theorem.

But Seidel (1847) found the error in Chauchy's "proof", this finding conducted him to a new mathematical notion (González, 1992). What I am trying to say is that we can develop a structure about a general idea that every time we are facing a "statement and a proof", if we have doubts, then we need to construct a non-example, or to examine the proof. Here, I am not talking about a logical contradiction from a mathematical point of view, instead, I am talking about a "cognitive conflict" that is coming at the moment you realize there is a contradiction, or when seems to be (see for example, Hitt & Codina, 2000; Rogalski & Rogalski, 2001).

Let us see another example related to a research carried out in 1978, that seems important in this theoretical context. In this study regarding sensitivity of mathematical contradiction (cognitive conflict), which involved 300 middle-school students (see Hitt, 1978), we asked them to solve an inequality see Figure 11.

$$(0.2)\,[0.4x + 15\,] - 0.8x \le 0.12$$

a) *Find the values of x where the inequality is true.*

b) *Verify that the inequality is true when x = 10.*

Figure 11.

The preliminary analysis presented different incorrect approaches followed by students when solving the exercise (see Figure 12). The first item of the exercise mentioned above was designed to induce errors in students' answers and the second item was supposed to provide a warning sign to the students. Our purpose was to measure how many of the students who made an error on the first item, providing an incorrect answer, could change their procedure when solving the second one. We wanted to analyze the idea of cognitive conflict throughout these students' performances. We already knew from pilots' studies that some students could follow some procedures as show in Figure 12.

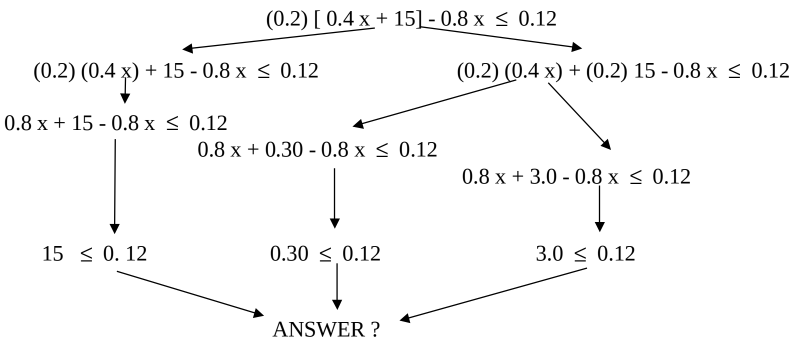


Figure 12.

In what follows we analyze the work shown by one student (Figure 13).

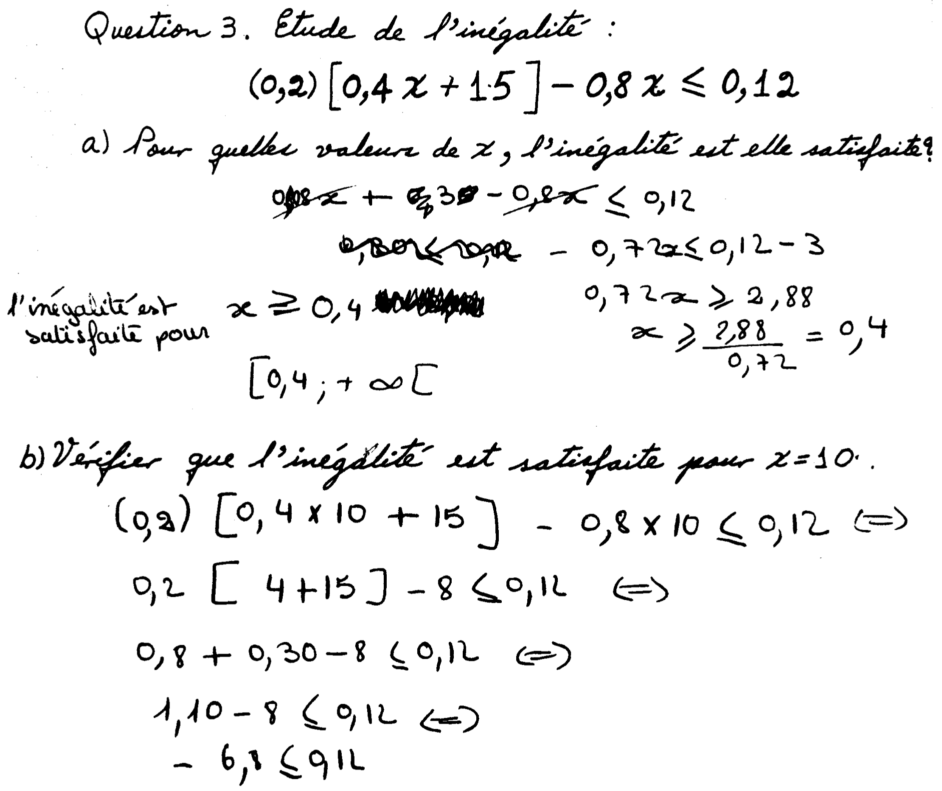


Figure 13.

Table 1

	Interpretation of the student's process
(0,2) (0,4 x + 0,15) – 0,8 x ≤ 0,12 0,8 x + 0,30 – 0,8 x ≤ 0,12 0,30 ≤ 0,12	Original expression Errors when multiplying decimals. Elimination of the terms in x. Reading second item b) and detection of a contradiction (see a point on one side of the numeral 10). Possible the student solved the item b) and then returned to the item a).

There were some errors made by the student in his first try. He showed a cognitive conflict because he reviewed his process solution. He modified some of the initial errors, but he committed others again. However, his final answers to the two items are coherent in the way that the cognitive conflict disappeared (but not the mathematical contradiction). We consider that those students, like this one, that were sensible to a contradiction have developed an epistemic frame (in a sense pointed out by Perkins and Simmons) that allowed them to understand mathematics.

We can infer from Duval's approach with the registers of representations that one part on the construction of concepts can be explained under this point of view, but it seems that we need to develop in our students another kind of cognitive structure more general that monitor their actions when solving a problem that is different from the one related to the specific concept.

A need to construct cognitive frames to promote understanding

The construction of knowledge that involves students' erroneous behaviors, might produce, permanent schemata or networks that blockade the coherent construction of a mathematical concept. Then, our opinion is that those schemata are not going to disappear even if students construct an alternative scheme. In some cases, given a task, those incoherent schemata will show up in students' work and will induce error. It seems to be the case when the student is in front of, a sort of, a complex task. For example, in a study about the concept of function, the following task: "calculate $(5^x + \tan(x))^3$", was given to 29 secondary teachers of mathematics. Three cases among the 29 teachers, made an error when using a wrong scheme that they surely constructed when they were young, that is using that $(a+b)^3 = a^3 + b^3$.

When students face problem-solving situations their knowledge is in constant examination. As we pointed out, it is possible that students had constructed two incompatible schemata, and when they are in front of a complex situation one scheme might show up and in some cases produce misleading performance. The instruction may play an important role in the construction of a better scheme and/or to construct an alternative scheme that could fight an improper one. But, incoherence between schemata is something that the student could solve little by little. Some schemata have to do with an epistemological obstacle (Brousseau's[1] sens) and it is a need to

[1] Brousseau (1976; 1983; 1997, p. 84): *An obstacle is thus made apparent by errors, but these errors are not due to chance. Fleeting, erratic, they are reproducible, persistent. Also, errors made by the same subject are interconnected by a common source: a way of knowing, a characteristic conception, coherent if not correct, an ancient "knowing" that has been successful throughout an action-domain. These errors are not necessarily explainable.*

promote ways to overcome that obstacle. The proper construction has to deal with a coherent and wider network of knowledge. That amplitude has to deal with the articulation among representations of the mathematical concept involved. That, does not mean that students can delete their improper scheme at once, but it might persist and stay forever in the mind of the student. Also, it is not easy to detect when the student has two incoherent schemes, because the student usually is not conscious of such incoherence; and it is not easy either to detect when he/she has constructed a new better one or when he/she has solved the incoherence. These internal processes seem not to be detected directly. They develop in periods of time larger than we usually think and as a consequence, surely the students develop them outside the classroom.

Another example in this context is the following. A couple of students (a girl and a boy) were discussing the expression: . The student was explaining to his friend some algebraic facts:

Boy (17 years old): $(x-3)^2 = x^2 - 9\,x$

Girl (17 years old): Incorrect, Do you know the formula $(a+b)^2 = a^2 + 2ab + b^2$ *? If you do not remember the formula there are several ways to calculate it, for example:* $y^2 = y * y$ *, that is,* $(x-3)(x-3) = x^2 - 3x - 3x + 9$ *, or the usual multiplication,* $\begin{array}{r} x-3 \\ \times\ x-3 \\ \hline \end{array}$ *. Which one do you prefer?*

Boy: The formula is easier.

This example shows the difference between the content frame (*"…do you know the formula…"*) and the epistemic frame (*"…there are several ways to calculate it…"*). It seems that the boy has not constructed an epistemic frame as the one the girl is showing. Related to the boy's performance, we can quote Hiebert y Lefevre (1986, p. 18) that have mentioned that there is a tendency in the students' approaches to compartmentalize knowledge: *"A third general factor that seems to impede the construction of relationships between units of knowledge is that knowledge just acquired often is context bound."*

Discussion

The research carried out by Duval takes into account the idea of register of representation and shows the importance of conversion tasks among representations of a concept to construct it. This approach takes into account that one representation of a mathematical object is partial about what it represents. That means, that is absolutely necessary to take into account the conversion tasks at least between two registers of representations, the promoted articulations will provide the student a support for the construction of the concept in question.

From Duval's perspective, most students' frequently errors appear when manipulating a representation within a system of representations. Another kind of error can be generated when transforming one representation from one system into another representation of the other system. Or, when solving a problem, we can choose an inappropriate representation that can guide us to a wrong result. Among the explanations we can provide from this theoretical approach is the lack of articulations among different semiotic registers of representation. Duval's work emphasizes the importance of the semiotic representations.

From our point of view, in problem-solving situation, there may exist some students' productions of semiotic representations that are not similar to those used by teachers or in textbooks. For instance, there are some mental representations used by the students when solving a problem that produce semiotic representations that are not usual. As we saw some students use representations in a dynamic functional way when solving a problem, mental and semiotic representations are tied in the solution process.

Hiebert & Lefevre (1986) stress the importance of the construction of a network of knowledge that can permit the students' interaction between the conceptual and procedural knowledge. Hiebert & Carpenter (1992) highlighted the importance of frame of networks (internal representations) related to understanding in mathematics. Their approach is more related to mental representations than the semiotic one.

The discussion we made to the problem of the construction of mathematical concepts and the construction of cognitive frames, show that one aspect that is under study and difficult to grasp is that of transfer. This notion is important to be analyzed to better understand the construction of knowledge. A possible branch of research is the one highlighted by Perkins and Simmons (1988) and Perkins and Salomon (1989) related to the construction of frames (content, problem-solving, epistemic and inquiry). It seems important to this approach to include the study of the notion we called "cognitive conflict".

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A Triadic Nested Lens for Viewing Teachers' Representations of Semiotic Chaining

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ABSTRACT. Over a period of about seven years, starting in 1993, teachers enrolled in a course on Ethnomathematics taught by the author were required to construct mathematical ideas from their own personally meaningful cultural experiences. For the purpose of deepening the mathematical ideas produced, after 1996 the teachers were taught to use semiotic chains based on a dyadic model of signifiers and signifieds. Explanations are given of the basic structure of these chains, based on Lacan's inversion of Saussure's dyadic model. Examples from teachers' projects illustrate the usefulness of Lacan's inversion in mathematical semiotic chaining of signifiers, and also the limitations of this model, leading to an alternative Peircean triadic model in which signs are nested, analogously to the nesting of Russian dolls. From a representational point of view, the examples are then analyzed through the lens of this triadic nested model involving object, representamen, and interpretant, for each of the nested signs. Some of the representations involved may be characterized as iconic, some as indexical, and some as symbolic. The need for each successive link in the semiotic chains to be interpreted suggests that a Peircean nested triadic model, which includes a new interpretant for each successive link, is a model with fuller explanatory power than the dyadic one, and a potential tool for teachers to use in linking mathematics of everyday practices of their students with the concepts of the mathematics curriculum that they are required to teach.

Introduction

My first introduction to semiotic theories was through conversations with a colleague, Marcy Driscoll, at The Florida State University in the early 1990s. At the time, I was pioneering a graduate course in *Ethnomathematics*, and searching for ways to avoid trivializing the potential for students to construct mathematical ideas based on cultural practices that were meaningful to them (Presmeg, 1998b). Shortly after being introduced to Peircean semiotics through Driscoll's work in instructional design (Driscoll, 1994), I attended Whitson's (1994) plenary address at the 16th Annual Meeting of the International Group for the Psychology of Mathematics Education, in which he outlined Saussure's dyadic model and Lacan's inversion of it, enabling the chaining of signifiers that Valerie Walkerdine (1988) had used in her post-structuralist analysis of activities of mothers and their daughters. It is not my intention to describe the details of these threads. (The reader may go directly to Walkerdine's, Driscoll's, and Whitson's publications for those details.) But I shall trace the influences in my own thinking that led from the use of Lacanian chains of signifiers linking real-world cultural practices to abstract mathematical ideas in a series of steps, to the need for a more powerful model that took various interpretations of the links into account. Such a model was constructed from Peirce's ideas.

Triadic and dyadic models of semiosis

Peirce's (1992) formulation of a semiotic model is complex, and the following account is offered as a mere introduction. There are three basic components:

- Firstness is a primary *object* (o), "existing independently of anything else" (Driscoll, 1994, p. 1).
- Secondness involves a relation between the object and some *sign* which represents it. This sign is called the *representamen* (r) in some of Peirce's writings (1992). (In this case the *sign* refers to the entire triad.)
- Thirdness is the interpretation of the sign, involving relationships among the object, the representamen, and a third component, called the *interpretant* (i). Note that thirdness thus refers to all three components in this triad, and the relationships among them (and this completeness is not fully captured in Figure 1). Peirce sometimes referred to the representamen as the sign, and sometimes the whole triad. In this paper, the latter usage will be adopted; thus thirdness connotes all the components of the sign (o, r, and i) and their relationships.

By using two examples, namely, learning how to sew, and understanding the ramifications of the question, “Is the majority always right?”, Driscoll (1994) made it clear that the *object* (firstness) may be an abstract idea *or* a physical object. Driscoll’s two examples and Whitson’s (1994) example of the possibility of rain are outlined in Figure 1. Whitson (1994) pointed out that the object is not known or apprehended directly, but through the mediation or interpretation of r, the representamen.

Firstness	**Secondness**	**Thirdness**
object o	**representamen r**	**interpretant I**
1. Sewing machine itself (design, function, mode of operation)	picture of sewing machine or part of it	apprehension of picture to reveal its function
2. Idea, “Is the majority always right?”	multiple signs, e.g., verbal description in textbook, example of a person whose minority religious views are a source of discrimination	interpretation that the majority is not always right
3. Likelihood of rain	falling barometer	decision to take an umbrella

Figure 1. Examples of three components of Peirce’s triadic model.

Pierce’s triadic model in the United States (Peirce, 1992) had its counterpart in the Swiss structural approach of Saussure, who defined the sign as a combination of a “signified” together with its “signifier” (Saussure, 1959; Whitson, 1994, 1997). This constitutes a dyadic model, but “thirdness” seems to be implicit in the interpretation of the sign. Lacan inverted Saussure’s model, which gave priority to the *signified* over the signifier, to stress the *signifier* over the signified, and thus to recognize “far ranging autonomy for a dynamic and continuously productive play of signifiers that was not so easily recognized when it was assumed tacitly that a signifier was somehow constrained under domination by the signified” (Whitson, 1994, p. 40). This formulation allows for a chaining process in which a signifier in a previous sign combination becomes the signified in a new sign combination, and so on. I see this chaining process as involving metonymy - as indeed all signifiers are metonymic in a semiotic model, since they are “put for” something else - and also reification, since each signified in turn is constructed as a new object (Sfard, 1991) that is represented by the new signifier. Chaining thus casts light on both processes as they are implicated in the construction of mathematical objects. Two examples are given in the next section.

Ethnomathematics in the classroom

In 1996 I started using chains of signifiers in the *Ethnomathematics* course to ensure the deepening of the mathematical content we were addressing (Presmeg, 1997b, 1998a). As I learned later, at this same time, Cobb et al. (1997) were using the same model independently as a lens for interpreting events in a first grade classroom. In this section, two examples of chains of signifiers from the *Ethnomathematics* course are presented, together with arguments for the inadequacy of the model, and a rationale for constructing a nested triadic model as a lens and a tool for mathematizing everyday practices.

One component of the course required each student to select a cultural practice that was personally meaningful to him or her, to research this practice, and to construct mathematical ideas based on the investigation. Students wrote reports, and also shared the results of their investigations with the class. Amid many interesting and inspiring examples of mathematical ideas emerging from personally meaningful activities (Presmeg, 1998a), it also became apparent that in some cases the activities took on a life of their own and the mathematical ideas that were constructed were initially quite trivial. Recognizing that this could be a problem, one of many, faced by teachers who attempted to connect their mathematics pedagogy with the home activities of their students, I started to make semiotic chains to model the successful ethnomathematics projects, and later taught the students in the course to make their own chains. The process resonates with what Hans Freudenthal called horizontal and vertical mathematizing (Treffers, 1993). The starting point of constructing mathematics from a cultural practice is horizontal mathematizing, while the need to operate on the mathematical ideas constructed and to develop these further results in vertical mathematizing. The following chains are from projects in the class. The first is one that I constructed, based on part of the investigation of "mountain bike mathematics" by Vivian Knowles, an elementary school teacher. Vivian started with her own GT Rebound mountain bicycle. In the second part of her project (after investigating the geometry of the bicycle), she drew a diagram showing that the 18 gears resulted from the pairing of three chain rings on the pedal with six sprockets on the back wheel (this diagram is signifier 1; representing relationships in the bicycle itself). Next followed a table linking the gear ratios with the "development" – the distance the bicycle would travel with one turn of the pedal (signifier 2, representing relationships from the gear diagram and the bicycle). Finally, Vivian drew a graph depicting the development plotted against the gear ratio – a hyperbola! This third signifier represented the relationships in the table, and hence everything that preceded it in the chain.

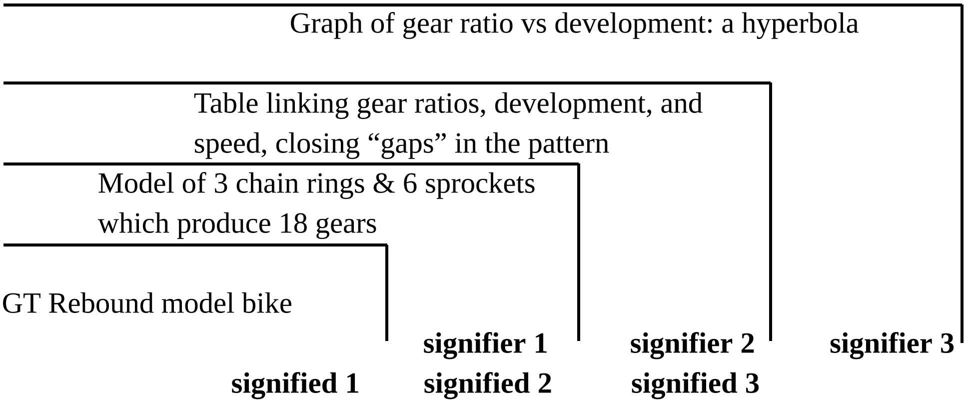

Figure 2. Chaining of signifiers in “Mountain Bike Mathematics”, by Vivian Knowles.

In analyzing the components of Vivian’s project, it is useful to think in terms of metonymy and Vivian’s interpretation of the context (Otte, 2001). In the second link of the chain, Vivian thought of the relationships between the chain rings and the teeth on the sprockets of the rear wheel as depicted in Figure 3.

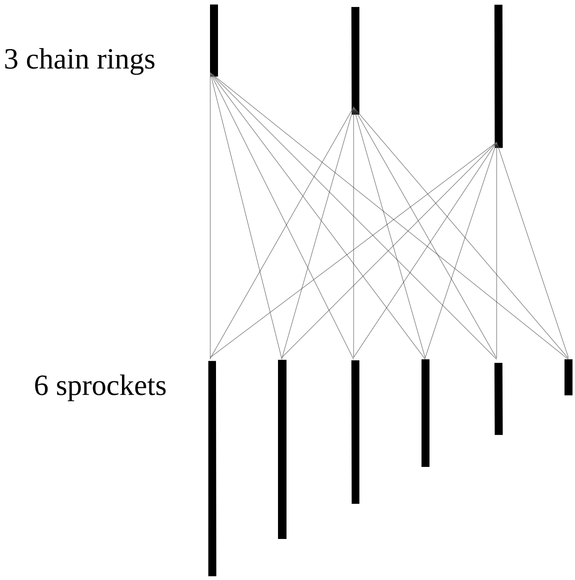

Figure 3. Vivian’s 18 bicycle gears.

This diagram is a metonymy, in that it stands for, or represents, the actual relationships of the chain rings and sprockets. But there is also an element of interpretation involved in this link and in subsequent links, which is not captured in the dyadic chaining model shown in Figure 2. Because 48 is the highest numeral that appears in this diagram, Vivian is led in subsequent links to conceptualize her gear ratios as a constant, 48, divided by each of a series of numbers. She fills in the missing values to complete the following table. (The gear number is the gear ratio multiplied by the diameter of the back wheel, that is, 26 inches; the development is the distance in feet that the bicycle will travel for one turn of the pedal, that is, the gear number multiplied by π and divided by 12 to convert the inches to feet.)

	Gear Ratio	**Gear Number**	**Development**
n	**48/2n**	**(48/2n)**x**26**	**(48/2n)**xπ x(**26/12)**
1	48/2	624	163.28
2	48/4	312	81.64
3	48/6	208	54.43
4	48/8	156	40.84
5	48/10	125	32.66
6	48/12	104	27,21
7	48/14	89	23.32
8	48/16	78	20.41
9	48/18	69	18.14
10	48/20	62	16.33
11	48/22	57	14.83
12	48/24	52	13.61
13	48/26	48	12.56
14	48/28	44	11.66
15	48/30	41	10.89
16	48/32	39	10.21
17	48/34	37	9.60
18	48/36	35	9.07
19	48/38	33	8.59
20	48/40	31	8.16
21	48/42	30	7.77
22	48/44	28	7.42
23	48/46	27	7.01
24	48/48	26	6.80

Figure 4. Vivian's table of development for each gear ratio

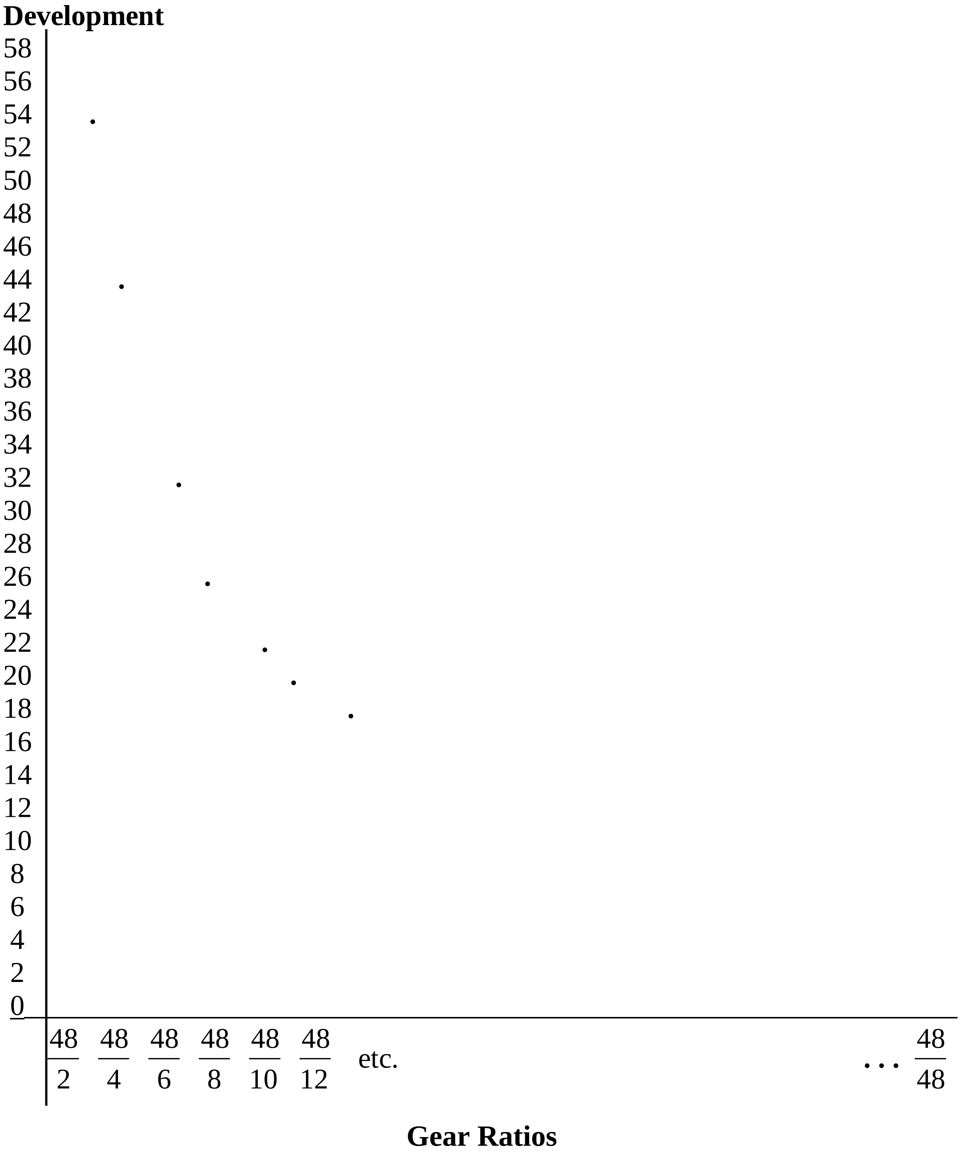

Figure 5. Vivian's axes for her graph of development and gear ratio

Figures 4 and 5 show clearly that it is Vivian's interpretation that creates the resulting hyperbola, y = 48/x. Higher gears make the bicycle go further for each turn of the pedal than do lower gears: the proportion is direct, not inverse. But because Vivian has reversed the order, starting with the highest gear ratio and using this particular symbolism, a hyperbola results.

An example of a different kind concerns the chain of signifiers constructed by a doctoral student, Derek Smith, himself, to illustrate his analysis of the symmetries of a tennis court:

Symmetries of the tennis court

V = vertical reflection R_{180} = rotation through 180°

H = horizontal reflection R_{360} = rotation through 360°

Binary operation of composition: o

e.g. R_{180} o V = H

Operation table

O	**R_{360}**	**R_{180}**	**V**	**H**
R_{360}	R_{360}	R_{180}	V	H
R_{180}	R_{180}	R_{360}	H	V
V	V	H	R_{360}	R_{180}
H	H	V	R_{180}	R_{360}

The set $\{R_{360}, R_{180}, V, H\}$ under the operation of composition forms an Abelian group, a dihedral group of order 4, that could be denoted D_2.

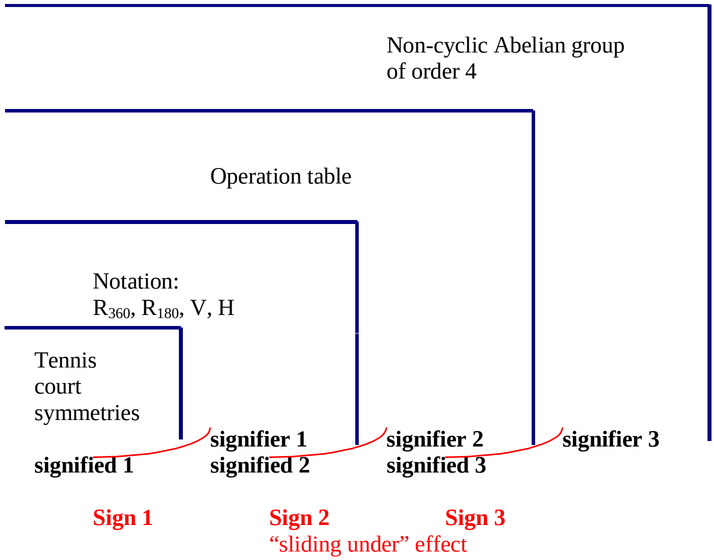

Figure 6. Derek Smith's chaining of signifiers in a progression of generalizations from symmetries of the tennis court to a dihedral group of order 4.

Derek's example illustrates the chaining of signifiers in vertical mathematizing, but the horizontal mathematizing is less strong than in Vivian's mountain biking chain. The tennis court seems incidental: the chain could have started with any rectangle. Nonetheless, the power of chaining to take the mathematical ideas to new levels is illustrated. Each link in the chain involves reification of processes as objects (Sfard, 1991) to produce a new signifier that represents all the previous links.

These examples illustrate the usefulness of dyadic chains in "moving" mathematical ideas in the direction of abstraction and generalization. On the one hand, Vivian could have stopped her investigation after any of the links in the chain, as could Derek. On the other hand, any chain could, at least theoretically, be extended by an indefinite number of links. When the students in the *Ethnomathematics* class were required to make chains of at least three links for their projects, there were fewer trivial examples (e.g., "this basket I made has mathematical shapes such as a square and a triangle, so I am doing mathematics"). But the limitations of the dyadic chain as an explanatory lens are correspondingly evident.

Chaining using this model also proved to be a useful tool in enabling elementary school teachers to link activities from the lives of their students, in a series of steps, with the mathematics of the classroom (Hall, 2000). But as in the *Ethnomathematics* class, this conceptual model was not completely adequate as an explanatory lens. Hall's dissertation research grew from his participation in my *Ethnomathematics* course and attempts in that course to use this model of semiotic chaining to link everyday practices in a series of steps to formal, abstract mathematics. However, when Hall (2000) in his dissertation research taught two elementary school teachers to construct and use semiotic chains in their own mathematics classrooms, starting from activities in which their students were engaged, it became apparent in his analysis of the data that a more complex model would have provided a better lens for understanding the processes involved. In one instance there was the phenomenon of a sign within another sign (two related and nested signifiers for the same signified), which did not fit the pattern of the dyadic chain.

In my work, too, as illustrated in Vivian's interpretation, the chains sometimes appeared to be too simplistic to adequately convey the structure of the processes involved. So I played with the idea of a triadic model in which the signs were nested, like Russian dolls. This model is described in the following section.

A Peircean nested model

In a dyadic model of chaining of signifiers each new signifier in the chain *stands for* everything that precedes it in the chain. The previous signifier, as well as *everything that it represents*, is now the new signified. Thus the new sign, consisting of the new signifier and signified, comprises everything in the entire chain to that point. It is this nested quality that gives chained signifiers their power for mathematics education (Cobb et al., 1997; Presmeg, 1997b; 1998a). A chain is not the best metaphor for this model, because the links in a chain do not exhibit this nestedness, which is better represented in a diagram attributed to Lacan by Whitson (1997, p. 111). In Figure 5, I have attempted to adapt such a diagram to capture this nested quality of chained signifiers as Peirce might have represented it, using his terminology.

Key: O = Object (signified); R = Representamen (signifier); I = Interpretant

These three components together constitute the Sign, thus the three nested rectangles represent Signs 1, 2, and 3 respectively

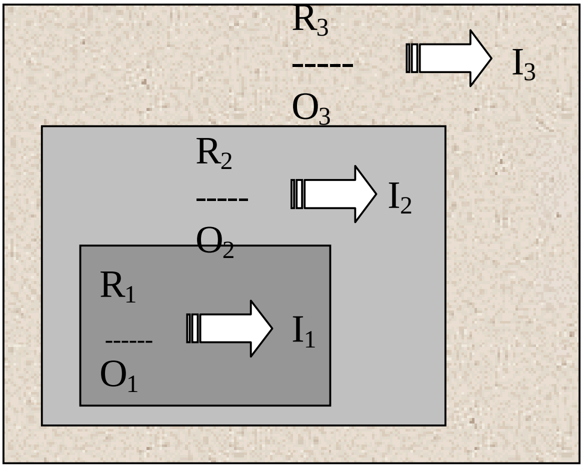

Figure 7. A Peircean representation of a nested chaining of three signifiers.

Each of the rectangles in Figure 7 represents a sign consisting of the triad of object, representamen, and interpretant, corresponding roughly to signified, signifier, and a third interpreted component, respectively. This interpretant involves meaning making: it is the result of trying to make sense of the relationship of the other two components, the object and the representamen. Note that the entire first sign with its three components constitutes the second object, and the entire second sign constitutes the third object, which thus includes both the first and the second signs. Each object may thus be thought of as the reification of the processes in the previous sign. Once this reification occurs, this new object may be represented and interpreted – or rather, resonating with the cyclic nature of the processes involved, the construction of symbolic notation and its interpretation also inform the creation of this object.

Going back to the chain from Vivian's project, this triadic model allows room for the interpretations that occurred at each step. It is important to note that already in the original chaining model, the nestedness and the interpretation were implicit – but a triadic nested model makes these relationships more explicit. The bicycle is the first object. The gear diagram is interpreted, forming the first sign $(O_1,\ R_1 \text{ and } I_1)$. Next, the table (R_2) reifies the entire first sign (O_2), and is in turn interpreted (I_2). Finally, the graph (R_3) reifies everything that precedes it (O_3) and is itself interpreted as a hyperbola (I_3).

In a nested model, Vivian's chain might appear as in Figure 8.

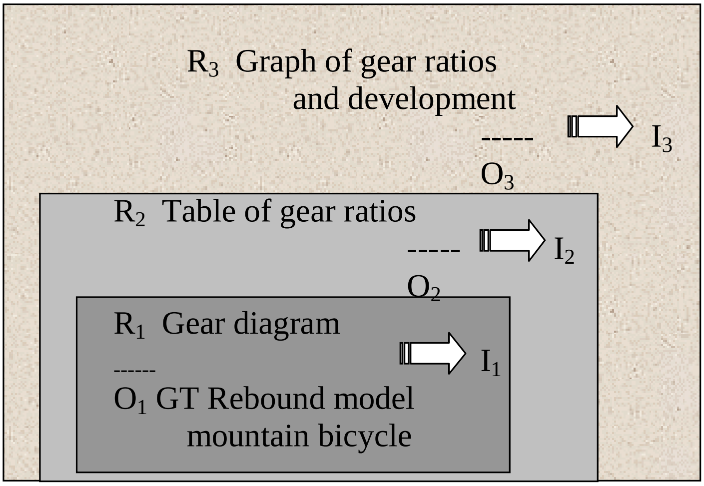

Figure 8. Vivian's mountain bicycle chain viewed through a nested semiotic lens.

In Peirce's (1992) categorization of signs as icons, indexes, or symbols, the signs in Vivian's project appear to be indexical, because they partake of the context at each stage. Thus this model may be thought of as a nesting of metonymies, because, as Otte (2001) pointed out, metonymies operate contextually while metaphors with their direct comparisons of similar elements in two disparate domains operate iconically.

Derek's chain may also be represented in a nested triadic model, with each new representamen (signifier) interpreted in constructing the new object. However, in this case the signs appear to be symbolic rather than indexical. There is an element of habituation in his mathematical formulation, resonating with Peirce's characterization of a symbol as a representamen whose representative nature resides in its being a rule that will determine its interpretant (Otte, 2001).

Viewed in a nested model, Derek's chain might appear as in Figure 9.

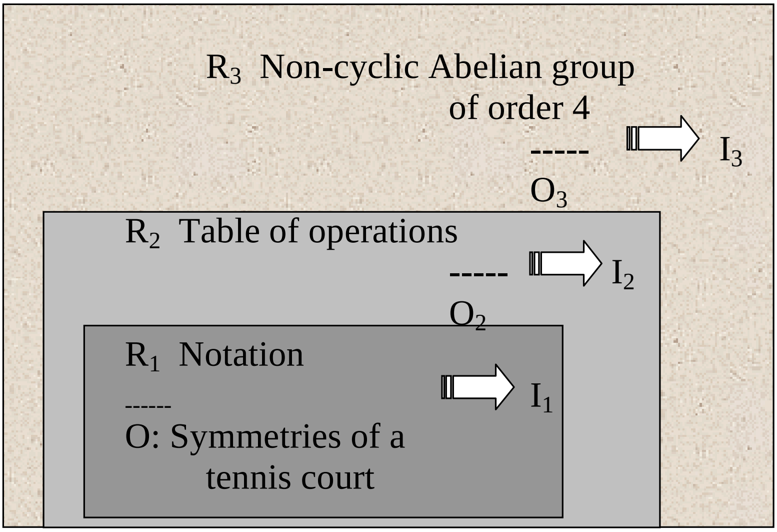

Figure 9. Derek’s Tennis Court Chain viewed through a nested semiotic lens.

Finally, this model has the potential to constitute a *web* of signs (as suggested by Willibald Dörfler in a conversation in July, 2001). One nested component of such a web may be related metaphorically or metonymically with another such component. Research using this model is ongoing; thus far, the model has the versatility to represent the complexity of the processes identified as teachers use semiotics to link the experiences of their students with the mathematical concepts they are learning.

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Sign as a Process of Representation: A Peircean Perspective

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> One selfsame thought may be carried upon the vehicle of English, German, Greek, or Gaelic; in diagrams, or in equations, or in graphs: all of these are but so many skins of the [same] onion, its essential accidents. Yet that thought should have some possible interpreter, it is the very being of its being.
>
> (Peirce, CP 4.7)

ABSTRACT. The paper focuses on symbols as special kinds of signs that when interpreted engender continuous processes of representation. From the Peircean standpoint, signs are dynamic triadic entities (object-representamen-interpretant), which elements coexist in a synergistic relation. This relation among the elements of a sign contributes to the emergence and expression of thought. The Peircean perspective on signs connects the different definitions on representation used in mathematics education (internal representations, external representations, and representation systems) and it seamlessly reveals pedagogical implications.

Introduction

The individual represents in so far as he creates symbols to stand for objects, ideas or events, sets up relationships among them, and operates with them at the conceptual level. Wilder (1968) calls this ability to create and use symbols *symbolic initiative* and differentiates it from *symbolic reflex*, which is the ability of only manipulating and reacting to symbols. Wilder argues, from the standpoint of the history of mathematics, that the evolution of mathematical concepts and the invention of algorithms were in fact the result of symbolic initiative. In contrast, he contends that the memorization of algorithms and the manipulation of symbols and mathematical representations, without understanding their deep meanings, drop the cognitive activity of the individual to the level of *symbolic reflex*. The mere reaction to a symbol is, for Wilder, a reflex response in much the same way as one reacts to an environmental stimulus. Thus, the individual appears to be able to perform within a wide spectrum of symbolic activity that goes from the higher level of symbolic initiative (creating, using, and relating symbols) to the lower level of symbolic reflex (manipulating and reacting to symbols). Symbolic initiative requires awareness (a) of the object, idea, or event to be represented, (b) the medium chosen to express the object (representation), and (c) the effect of that representation on some interpreter in virtue of which it becomes, for him, a representation of the object in question.

Since we adopt the Peircean theoretical framework, we would like to reserve the word symbol for the more sophisticated kind of sign in his triadic classification of signs concerning their relationship to the objects they represent (icon, index, and symbol). Most of the time we will refer to signs in general; however, we will refer to symbol as we see fit given that this word is extensively use in language and literature in mathematics education. For Peirce, a sign is a triadic entity (object-representamen-interpretant) accounting for the thought processes that emerge from the interpreting efforts of the individual.

Semiosis is the process in which something functions as a sign. Morris (1938), who analyzes Peirce's notion of sign, defines semiosis as a "mediated-taking-account-of" in which *the mediators* are representamens/sign vehicles, *the taking-account-of* are interpretants; *what is taking account of* is the class of objects which the sign can denote and *the agents* of the process are interpreters. That is, a sign is a sign only through a process of interpretation or semiosis. This process involves the representamen/sign vehicle, the object it represents, and the interpretant it generates in the mind of the interpreter (creator or user) of the sign. That is, the process of semiosis is an intentional process of representation and interpretation. Higher processes of semiosis take place when signs produced/used by an interpreter are employed as a means of gaining information about the interpreter; for instance, analyzing students actions through the signs that they produce and use is a higher process of semiosis.

A sign is not an isolated entity and it usually comes into being within an already existing system of signs in a particular area of knowledge. The representamen of a sign relates to other representamens (a syntactical dimension); the representamen, as a mediating entity, refers to the class of objects it represents (a semantical dimension); and the representamen also has an ascribed meaning and as such it has an interpretant in view (a pragmatical dimension) (Morris, 1938). While the interpreter (producer or user of the sign) invests himself in the semantical and syntactical dimensions of semiosis, his own processes of interpretation (the pragmatical dimension) escape to him. These processes withdraw, so to speak, into the background awareness of the interpreter's own actions and only in retrospect he becomes aware of them in much the same way as an actor becomes aware of his own actions and thoughts as he watches the records of those actions.

Within the realm of the English language, it is difficult to talk about representation without talking about symbols in a generic way and vice versa. Such circularity has social and linguistic explanations in the evolution of language. The first section of the paper explains such circularity shedding light on the conceptual complexity of representation and symbols. The second section is dedicated to the Peircean notion of sign as an explanatory theoretical framework that accounts for the dual nature of the sign as a process and as a product. The third section of the paper focuses on the notion of semiosis as an essential cognitive process through which signs transform themselves into new signs. The fourth section analyzes definitions of representation in mathematics education under the lens of the Peircean triadic sign.

Evolution of the Meanings Given to Representation and Symbolization

Williams (1983) gives a social and cultural history of the English words *represent, representation, and symbol.* The socio-historical account of this evolution sheds light on the variety of meanings that these words have as objects and as processes. According to Williams the word *present* and *represent* appeared with related meanings. When the word *represent* appeared in English in the 14th century, the word *present* already existed as a verb with the meaning of *making present* but, later in the same century, *present* also acquired the sense of *offering something*. Soon, the word *represent* also acquired a range of meanings: (a) of making present in the physical sense; (b) of presenting oneself or another to some person of authority; (c) of making present in the mind; and (d) of making present to the eye as in painting or in plays. The meaning of the word *represent* was extended when it was used in the sense of *symbolize* or *stand for*. Since then, there has been an overlap between the meanings of the words *represent* and *symbolize,* in the sense of (a) making present to the mind and (b) standing for something that is not actually present.

In the 19th century the word *representation* developed a sense of description of real objects and situations, and eventually it was widely used as an identifying element of *realism* or *naturalism*. Later in this century or probably at the beginning of the 20th century, an old meaning of representation (the visual embodiment of something) became specialized in the sense of *accurate representation*, and in this sense, produced the distinctive category of *representational art* (Williams, 1983). Yet, there was nothing in the general sense of *represent* or of *representation*, Williams argues, that made this specialization inevitable and, in fact, it was rather opposite to the initial sense of representation as symbolization.

Today, the word *represent* conveys an entire range of accumulated meanings that still conserves an overlap with the meaning of the word symbol. For example, the American Heritage Dictionary and the Random House Thesaurus give the following meanings that can be classified as:

Process	*To represent clearly to the mind* *To symbolize* *To be a substitute for* *To signify* *To stand for something other than itself*
Agency	*To be an agent for* *To be deputy for* *To be in position of speaking and acting for* *To act in place of* *The proxy for*
Product	*A depiction* *A visual embodiment of a specific quality*

Von Glasersfeld (1987) notes that the German language has different words for the variety of meanings that are given to *represent* in the English language. In German, *vorstellen* means a representation to the mind; *bedeuten* means standing for something else; *vertreten* means serving as agent for others; and *darstellen* means something that depicts reality (p. 216). In contrast to the English language, the existence of these words in the German language facilitates communication and makes the interpretation of the meanings of representation more accurate.

Williams (1983) also points out that the word *symbol* took on an ambiguity that is comparable to the ambiguity of the word *representation.* The word symbol undertook different connotations from the early senses of a mark or a token, to the intermediate sense of something which represents something else, to the late sense of something significant (not a representation but an *image)*, which indicates something deliberately not defined in its own terms. This last connotation is embedded in Peirce's description of symbol: "Symbol is a sign naturally fit to declare that the set of objects which is denoted by whatever set of indices may be in certain ways attached to it is represented by an icon associated with it" (CP 2.295). He further explains that "a symbol cannot indicate any particular thing; it denotes a kind of things. Peirce was well aware of the different meanings of the word symbol and he did not want to add to the confusion. He says, "the word symbol has so many meanings that it would be an injury to the language to add a new one. I do not think that the signification that I attach to it, that of a conventional sign, or one depending upon habit (acquired or inborn), is so much a new meaning as a return to the original meaning" (CP 2.297).

Given that one of the meanings of representation is to stand for something other than itself, it is natural for Peirce to include the notion of representation in his definition of sign:

> Representation [in Peirce's triadic definition of sign] is defined as the act or relation in which one thing stands for something else to the degree that it is taken to be, for certain purposes, that second thing by some subject or interpreting mind. Because the representation substitutes or is regarded as substituting for the object, the interpreting mind acquires knowledge about the object by means of experience of the representing sign. (Parmentier, 1985, p. 27)

Peirce (1974/CP, vols. 2 and 4) considers a *symbol* to be a sophisticated kind of sign that best exemplifies decontextualized semantic meaning. In addition, he sees *representation* to be the most essential mental operation without which the notion of *sign* would make no sense (Peirce, 1903). It is important to note that, at some point, Peirce blended the notions of sign and representation by making no difference between *sign* and *representamen* (Nöth, 1990). The word *representamen* was coined by Peirce because of his awareness of the importance of representation as a general cognitive process and the need for new words to separate specialized meanings (Peirce, 1903; CP 1.542). He specialized the meaning of the word *representamen* to stand for the vehicle of the sign or representation of the object, kept the word *sign* for his triad object-representamen-interpretant, and reserved the word *representation* to refer to the general cognitive process of representing.

In conclusion, three points are worth to keep in mind with respect to *representation* and *symbol*. First, since one of the meanings of *representation* is that it can stand for something other than itself, the use of the words *representation* and *symbol* have been intermingled for centuries. Second, it seems to be impossible to talk about *symbol* without being aware of the concomitant mental operation of *representation*. Third, *representation* and *symbol* have an existence that is totally dependent on the interpreter. That is, they do not have existence in themselves unless they are interpreted. The above points press the realization that there is no *symbol* without a process of *representation* and therefore a symbol has the dual nature of being a process and a product.

Up to now we have used the words sign and symbol interchangeably and in general ways, but from here on, we would like to use the word *sign* to refer to all signs in general and to reserve the word *symbol*, as Peirce did, to refer to the most sophisticated sign in his classification of signs into icons, indexes, and symbols. "Symbols afford the means of thinking about thoughts in ways in which we could not otherwise think of them. They enable us, for example, to create Abstractions, without which we should lack a great engine of discovery" (Peirce, 1906a, p. 251).

Sign as a Process of Representation

Signs, in the broadest sense, are semiotic entities that serve as vehicles to trigger thought, to facilitate the expression of thought, and to embody original and conventional thought. As Perice (1868a) says,"Only by external facts can thought be known at all. The only thought, then, which can possibly be cognized is thought in signs. But thought, which cannot be cognized, does not exist. All thought therefore must be in signs" (p. 49).

Leibniz and Peirce, both mathematicians, dedicated a great deal of time and effort not only on obtaining an accurate description of signs but also on generating a coherent theory to better understand thought, both in the communication of the sciences, in general, and in mathematics, in particular. Leibniz describes a sign as "*that* which we now perceive and consider to be connected with something else, by virtue of our or someone else's experience" (Dascal, 1987, p. 181). Peirce describes a sign as "something that stands to somebody for something in some respect or capacity" (CP 2.228). In this description, he leaves implicit the *interpretation processes* of the *Interpreter* (creator or user) of the sign. This interpretation process is emphasized by Peirce in more extended descriptions of signs:

> **A Sign, or Representamen,** addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the **Interpretant** of the first sign. The sign **stands for something**, **its object**. It stands for that object, not in all respects, but in reference to a sort of idea, which I have sometimes called the ground of the representation. (CP 2. 228; emphasis added)

> By a **Sign** I mean anything whatever, real or fictile which is capable of a sensible form, is applicable to something other than itself…and that is **capable of being interpreted** in another sign which I call its **Interpretant** as to communicate something that may have not been previously known about its **Object**. There is thus a triadic relation between any **Sign** [representamen], and **Object**, and an **Interpretant**. (MS 654. 7) (Quoted in Pamentier, 1985; emphasis added).

> A sign stands **for** something **to** the idea that it produces or modifies. Or, it is a **vehicle** conveying into the mind something from without. That for which it stands is called its **object**; that which it conveys, its **m*eaning***; and the idea to which it gives rise, its **interpretant**. (CP 1.339)

Peirce explicitly calls either the *mental idea* or the *physical object* that the sign *stands for* the *object* of the sign; the idea the sign produces in the mind of the Interpreter its *interpretant*; and the material or mental vehicle of the sign its *representatmen* (see Figure 1). The representamen could be a mental image or a physical object that stands for the object of the sign. For Peirce, the sign has a triadic nature and inherent dyadic relations among its three elements, namely, the relation between the object and the representamen, the relation between the representamen and the interpretant, and the relation between the interpretant and the object. Peirce (1906b) calls this tri-relative relation semiosis or semiotic activity.

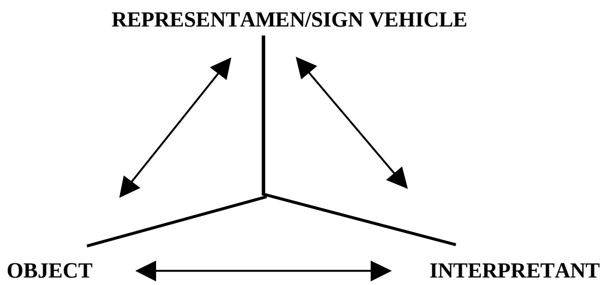

Figure 1. Peirce’s triadic sign.

In Peirce's triadic sign, the interpretant is passive to (or shaped by) the representamen, in the same way that the representamen is passive to (or shaped by) the object, but the interpretant becomes active with respect to both the object and the representamen generating an abstraction of the initial object (Deely 1994). This tri-relative relation cannot be resolved into three separate dyadic relationships (object-representamen, representamen-interpretant, object-interpretant) but its existence has to be taken in the concomitant relationship among the three elements of the sign.

Hence, when one focuses on only one of the entities of the sing triad (object-representamen-interpretant), the other two continue to be simultaneously present and synergistically active. In other words, in semiosis, object, representamen, and interpretant are neither static nor isolated entities. On the contrary, they are changing interrelated entities that come into existence as a result of the processes of interpretations and representation of the *Interpreter* (creator or user) of the sign. That is, at each step of the representation and interpretation process the interpretant functions as a new representamen that stands for a newly generated and more sophisticated interpretation of the initial object. This interpretant represents the initial object in the "same respect" and with the "same meaning" as the first representamen did although it is more highly determined. The sequence of triads, namely, object-representamen-interpretant (O_1, R_1, I_1), representamen-object-interpretant (R_2, O_2, I_2), and so on, is at the heart of the sign as a process of representation (see Figure 2). The power of the interpretant to create a more decontextualized object is what Peirce calls "hypostatic abstraction". This power is the key to the interpretant's capacity to fulfill its original charge of representing the same object with the same meaning that the first representamen does. This continuous process of representation is essential in the evolutionary construction and refinement of initial objects and conceptions.

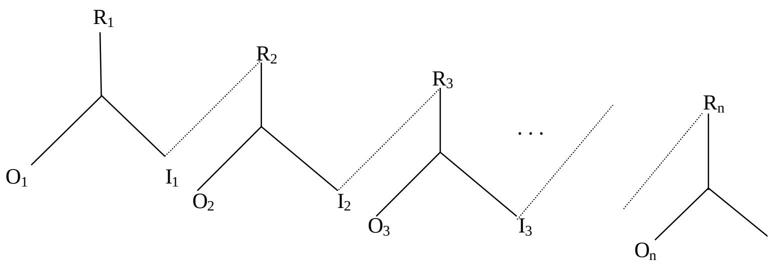

Figure 2. Semiosis: A continuous process of representation and interpretation.

Figure 2 can be better explained in Peirce's own words when he considers object, meaning, and interpretant as a general processes of representation:

> The **object** of a representation can be nothing but a representation of which the first representation is the interpretant. But an endless series of representations, each representing the one behind it, may be conceived to have an absolute object at its limit. (CP 1.339)
>
> The **meaning** of a representation can be nothing but a representation. In fact, it is nothing but the representation itself conceived as stripped of irrelevant clothing. But this clothing never can be stripped off; it is only changed for something more diaphanous. (CP 1.339)
>
> There is a special variety of esthetic goodness that may belong to the representamen, namely, *expressiveness*....The mode of being of a *representamen* is such that it is capable of repetition…This *repetitory* character of the representamen involves as a consequence that it is essential to a representamen that it should contribute to the determination of another representamen distinct from itself. (Peirce, 1903, p. 203) (emphasis added).
>
> The power of the representamen is in his capacity to determine some interpretant that is a representamen of the same object. (Peirce, CP 1.542)

Peirce considers the possibility of a *final interpretant* as the limit of an infinite sequence of interpretants that is nothing else than the product of a sequence of *representamens* associated to a sequence of *objects* that are each time more descontextualized and abstract. That is, the *interpretant* is a dynamic and generative constituent of the sign that makes the *interpreting process* a dynamic process of representation (see Figure 2). This recursive sequence of concatenated objects, representaments, and interpretants conveying a sequence of more and more "diaphanous" meanings is a continuous process of representation in the mind of the Interpreter. The result of this process is the construction of a decontextualized object or idea on the part of the Interpreter.

The representamen not only plays an essential role in the expressive nature of the sign but it also satisfies a particular relation with the object. Peirce recognizes that one of the possible modes of relation between the object and the representamen is determined when the representamen partake the character of the object so that in some respect they are practically interchangeable. He calls this relation an *iconic relation*. A second possible relation is when the individual existence of the representamen is connected with the individual object. He calls this relation an *indexical relation*. A third possible mode of relation that Peirce recognizes between representamen and object is one that transcends both the realm of common quality and the realm of common context. In such a relation, representamen and object are related only because the interpretant represent them as related and he calls it a *symbolic relation*.

In general, a sign is a continuous process interpretation and representation on the part of the interpreter (user or creator) of the sign that is guided by the relation between the representamen and its object, the relation between the representamen and its interpretant, as well as the relation between the interpretant and its object. In other words, a sign, as conceived by Peirce, is a constitutive process of representation and interpretation.

Semiosis as a Process of Sign Interpretation

We adopt Morris' (1938, 1946) definition of semiosis as a process in which a "mediated-taking-account-of" takes place. This process has syntactical, semantical, and pragmatical dimensions. The focus on the potential or actual relations of signs with other signs, the relations of signs with the objects they denote, and the relationships of signs with their interpreters constitute different branches of semiotics, namely syntactics, semantic, and pragmatics. Syntactics is the consideration of signs and signs combinations in so far as they are subject to the syntactical rules of formation and transformation. Semantics deals with the semantical rules that relate the signs to their object(s). Morris (1938, p. 23) defines a semantical rule as follows: "the sign vehicle '*x*' designates the conditions *a*, *b*, *c*, …under which it is applicable; the statement of those conditions gives the semantical rule for '*x*'. When any object or situation fulfills the required conditions, then it is denoted by '*x*'". Icons, indexes, and symbols have different semantic rules for their use. The icon characterizes that which can denote by exhibiting itself the properties of its object. The index designates what it directs attention to but does not characterize its object since it needs not be similar to it. In contrast, the semantical rule for the use of a symbol must be stated in terms of other symbols whose rule of usage are not in question, or by pointing out specific objects which serve as models (and so as icons), the symbols in question then being employed to denote objects similar to the models.

A simple illustration of the application of a semantic rule occurs when an interpreter is asked to solve the following equations $x^4 + 2x^2 + 4 = 0$, $x^2 + 8 = 3x$, and $x^{1/2} - 3x^{1/4} = 4$. Prior to getting involved in the solution of the equations, the interpreter has to recognize that all of them are of the form $ax^2 + bx + c = 0$. That is, the interpreter has to be aware that they are elements of the class that satisfy the conditions of the formula (or model); that is, they are elements of $\{ax^2 + bx + c = 0 | a, b, c$ are real numbers$\}$.

Pragmatics considers the relations of the signs to their interpreters and therefore it deals with the biotic aspects of semiotics, that is, with all psychological, biological, and sociological phenomena, which occur in the functioning of signs. While syntactical rules determine the sign relations between sign vehicles/representamens, and semantical rules correlates sign vehicles/representamen with their objects, the pragmatical rules state the conditions in the interpreters under which the sign vehicle/representamen is a sign.

Morris' (1938, 1946) stresses a concept as a semantic rule determining the use of characterizing signs (icons, indexes, and symbols). He expresses the view that a concept is not an entity but a way in which certain perceptual data function representatively and that such mental functioning, instead of being a mere contemplation of the world, is a highly selective process in which the organism gets indications as how to act with reference to the world in order to satisfy its needs and interests. That is, the individual's processes of representation and interpretation are essential in his mental functioning and therefore the product of semiosis.

Nothing is intrinsically a sign but it can become one in so far as it permits someone to *take-account-of* something through its mediation. Potentially, if not actually, every sign has relations to other signs and prepares the interpreter for *what* to take-account-of because that *what* con only be stated in terms of other signs. That is, semiosis does not occur in the isolated domain of one sign but in the realm of sign systems.

In summary, the conceptualization of an object (a physical object or an idea) is a dynamic process of semiosis that permits the Interpreter to think of signs in terms of signs. As Peirce puts it, "every sign is a living thing…the body of the sign changes slowly, but its meaning inevitably grows, incorporates new elements and throws *off* old ones". (CP 2.222)

Semiosis, then, is that continuous process of signification *intentionally* initiated by the Interpreter (user or creator) that involves the cooperation of the three elements of the sign (object, representamen/sign vehicle, and interpretant), in addition to the relation of signs with other signs and with the interpreters. Through semiosis the continuous process of representation goes hand in hand with the processes of abstraction. It is then not surprising that Peirce and Piaget, who invested a great deal of their theoretical efforts to understand the process of abstraction, describe such process in a somewhat similar manner. Peirce calls it *hypostatic abstraction* and Piaget *reflective abstraction.*

> That wonderful operation of *hypostatic abstraction* by which we seem to create *entia rationis* that are, nevertheless, sometimes real, furnishes us with the means of turning *predicates from being signs that we think or think through, into being subjects thought of.* We thus think of the thought-sign itself, *making it the object of another thought-sign*. Thereupon, we can repeat the operation of hypostatic abstraction, and from this second intention derive third intentions. (CP 4.549, emphasis added)

> With the construction of operations on operations, *the role of objects is modified bit by bit and more and more profoundly*. Because [the objects] cannot change physically but can only be put in different forms, a succession of stages must be distinguished. At level n, *objects* are the content of the form that is applied to them; at level $n + 1$, the form of level n becomes the content and *the objects* only constitute a content of content. At level $n + 2$, the form of $n + 1$ becomes a content of the new form while at the same time being a form of forms. At this level and succeeding levels, objects play an even less significant role. We see, therefore, why sooner or later it becomes easy for the subject *to replace concrete objects with symbolic ones* and to take the path that in the end will lead to formalization. (Piaget, 1985, p. 51, emphasis added)

What Peirce refers as the transformation of thought-signs into thought-signs is what Piaget refers to as the transformation of forms into other forms. They both agree that in the process of abstraction objects are decontextualized in so far as they take on more symbolic meanings in a necessary sequential order of precedence. Signs that we think through (or forms at a particular level) become signs that are the object of thought (form of forms). These two compatible definitions of abstraction take us back to the sign action (i.e., semiosis) represented in Figure 2.

The continual transformation of signs into more abstract signs in the process of semiosis reinforces Peirce’s commitment to the Kantian principle of continuity that “focus on the relation between concepts and its representations” (Otte, 1998, p. 448). For Kant the “continuity principle” is based on the synthesizing agency of the mind that brings heterogeneous material together under a series of synthetic judgements (Corrington, 1993). Likewise for Peirce, the human mind follows a logical procedure of deductive and inductive forms of inference aided by perceptual judgments and forms of abduction to shape and order innate temporal structures and the materials of sensation through a continuous momentum linking antecedent and consequent (Corrington, 1993, Otte, 1998). In summary, semiosis for Peirce is a continuous process of representation and therefore a continuous process of interpretation.

Representation in Mathematics Education

The continuous and dynamic nature of the sign *interpreting process* is considered by Peirce as a general *process of representation*. Representations of objects, ideas, or events (i.e., representamens) are of different nature according to their relationship with their objects, ideas, or events. These relations go from actual physical resemblance with the object (iconic), to some kind of connection with the object (indexical), to an ascribed resemblance with the object (symbolic).

- [Representamens] whose *relation to their objects* contains a resemblance in some way, and these *representations* may be termed *Likeness or Icons*.
- [Representamens] *whose relation to their objects* consists in a correspondence in fact, and these *representations* may be termed *Indices*.
- [Representamens] *whose relation to their objects* is an imputed character, which are the same as general signs, and these *representations* may be termed *Symbols*. (Perice, 1868b, p.7; emphasis added)

Mathematical notations fall in the category of symbols and they have neither a relation of similarity nor a connection with the objects they represent. Symbols have been defined in a variety of ways in mathematics education as being manifestations or physical expressions of thought. Mason (1987) describes symbols as the “artifacts resulting from recording perceptions on paper, mere vestiges of a complex inner experience” (p. 75). Leibniz considers symbols as *things* that mediate and facilitate thinking as well as instruments of human cognition to abbreviate complex ideas (Dascal, 1981, p. 181). Hersh (1979) argues that “symbols are aids to thinking just as musical scores are used as aids to music” even though “the score can never be a full embodiment of the musical thoughts of the composer” (p. 40). In these definitions, the natural expressiveness of symbols functions as external representations of something else (and idea, a quality, or a condition) and makes them essential tools for thinking in mathematics.

Symbols have a natural internal component (interpretant) and an expressive component (representamen). It is, then, not by accident that mathematics educators refer to internal and external representations. Goldin and Kaput (1996), for example, use the term *internal representations* for all “possible mental configurations of individuals that are not directly observable” (p. 399). They also use the term *external representation* for those representations that “are physically embodied and observable configurations such as words, graphs, pictures, equations, or computer microworlds which are *in principle accessible to observation* to anyone with suitable knowledge” (p. 400; emphasis added). For Peirce, all these representations are also representamens (sometimes internal and sometimes external) capable of generating interpretants; for Leibniz’s these representamens are instruments of human cognition; and for Hersh these representamens are embodiments of thought. Goldin and Kaput’s suggestion that external representations could be “accessible to observation to anyone with suitable knowledge” seems to indicate that a representation is not a representation unless it is interpreted and any interpretation is not independent of some kind of external representation. Therefore, the sharp division between “internal” and “external” representations seems to beg for a linkage rather than for a detachment.

Each science has established its own representational systems (sign systems) and mathematics is not an exception; in fact, mathematics is one of the sciences that expresses itself in symbols almost to the point of excluding natural language. A symbolic mathematical representation is not a representation for everybody; it is only a representation for those willing to interpret it, and such interpretation usually depends on other representations within the representational systems of mathematics. In mathematics concepts have different representations expressing the same idea through different means. For instance, “a mathematical object such as a function, does not exist independently of the totality of its possible representations, but it is not to be confused with any particular representation, either” (Otte, 1998, p. 442).

Duval (1999, 2000, 2001) calls a “register” a system of representations with not only the purpose of reproduction and communication but also with the purpose of objectification. In each register (system of representations) it is possible to modify a representation *within* that register according to its syntax (*treatment of the representation*) as much as it is also possible to transform representations from one registers into another (*conversion of represenations*). This *conversion* of representations is what Janvier (1987) calls a *translation process*. Further, the totality of the coordinations among representations (treatments, conversions/ translations) on the part of the interpreter is the syntactical dimension of semiosis that contributes to for the construction and refinement of mathematical concepts.

Morris define syntactics as the dimension of semiotics that

- study signs and sign combinations in so far as they are subjected to syntactical rules (Morris, 1938, p. 14),
- study the way in which signs of various classes are combined to form compound signs (Morris, 1946, p. 367), and
- the study the formal relations of signs to one another (Morris, 1938, p. 6).

Morris definition of syntactics accentuates Duval's notions of *treatment* within a register (system of representations or system of signs) as well as the notion of *conversion* between registers. The fact that in the registers the focus is on syntactics, the semantical dimension of semiotics is also present because syntactics presupposes semantics.

In mathematics education, we can see how the three dimensions of semiotics (syntactics, semantics, and pragmatics) have been indirectly used to describe representations and systems of representations. For instance, Goldin and Janvier (1998) synthesize various descriptions of representation and systems of representation in reference to the teaching and learning of mathematics:

(1) *An external, structured physical situation, or structured set of situations in the physical environment*, that can be described mathematically or seen as embodying mathematical ideas.

(2) *A linguistic embodiment, or a system of language*, where a problem is posed or mathematics is discussed, with emphasis on syntactic and semantic structural characteristics.

(3) *A formal mathematical construct, or a system of constructs, that can represent situations through symbols or through a system of symbols*, usually obeying certain axioms or conforming to precise definitions including mathematical constructs that may represent aspects of other mathematical constructs.

(4) *An internal, individual cognitive configuration, or a complex system of such configurations*, inferred from behavior or introspection, describing some aspects of the process of mathematical thinking and problem solving". (pp. 1-2, emphasis added).

In the first three descriptions, the interpreter (agent or learner) who is representing and interpreting is left hidden in expressions like "can be described mathematically or seen as embodying mathematical ideas" (from (1)); "where a problem is posed or mathematics is discussed" (from (2)): and "can represent situations through symbols or through symbol systems" (from (3)). Although these three descriptions leave implicit the pragmatical dimension of representational systems, they acknowledge the syntactical and semantical dimensions. Only description (4) explicitly acknowledges the existence of the interpreter (agent or learner) and his representational activity; that is, the pragmatical dimension of representational systems is accounted for. As Peirce (NEM, 1976) and von Glaserseld (1987) contend, a representation is a representation only when the individual interprets it. This indicates that the pragmatical dimension of sing systems or systems of representation is of essence.

In summary, none of the above descriptions takes into account, at the same time, the syntactical, semantical, and pragmatical dimensions of representational system. Peirces' semiotics and its triadic perspective on the nature of signs as well as Morris' dimensions of Peirce's semiotics shed light on the nature of representation (signs) and representational systems (system of signs) in such a way that, when one focuses on one of the dimensions, it is still possible to retain awareness of the other two.

There is a subtle but important difference between a "system of representation" as used in mathematics education and a "semiotic system of representation". In a semiotic system of representation the cognitive activity of the interpreter (creator or user) of the sign is always acknowledged as an integral part of a representation. That is, the pragmatical dimension is acknowledged along the syntactical and semantical dimensions. From the Peircean semiotic perspective, a representation is seen as a sign standing for an object (mental or physical) and expressed through a representament (mental or physical) that generates an interpretant in the mind of the interpreter. The tri-relative nature of the Peircean sign accounts for its process-product nature mediated by processes of interpretation and representation. This process-product nature of the sign has implications for the teaching of mathematics. A sign could be already a product for the teacher abbreviating his cognitive processes of interpretation and representation. In contrast, for the student, a sign could be a simple material object with the potential of being interpreted and of abbreviating his cognitive processes of interpretation and representation; that is, with the potential of becoming a product that when needed can be unfolded into a process.

In addition to Peirce's triadic model of the sign that is grounded on representation, Vergnaud (1987) offers, for mathematics education, a triadic model of representation that encompasses all physical and cognitive aspects of signs and representations. He discusses the triad referent-signifier-signified taken from linguistics and defines each element of the triad from the mathematics education perspective.

> [The *referent*] is the real world as it appears to the subject along his experience. The world is changing and the subject acts in and upon it to produce events and effects that please him or that are in accordance with his conscious or unconscious expectations and representations. (p. 229)
>
> [The *signifier*] consist of different symbolic systems (representation schemes for Kaput) that are differently organized. The syntax of algebra is different from the syntax of graphs, from the syntax of diagrams, and from the syntax of tables. Above all, the syntax of natural language is different, more varied and more complex than the other symbolic systems, more ambiguous, and also more powerful.…It is essential to recognize that symbols used in communication lie at the signifier level, whereas the meanings lie at the signified level. (p. 229)
>
> [The *signified*] is at the heart of the theory of representation, in the sense that it is the level in which invariants are recognized, inferences drawn, actions generated, and predictions made. That is the level of conceptions (Janvier), phenomenological primitives (di Sessa), and presentation (von Glasersfeld). (p. 229)

From a Peircean perpective, Vergnaud's *referent* stands for the realm of objects as experienced by the individual; the *signifier* stands for the realm of representamens and the process of representation; and the *signified* stands for the realm of interpretants and the process of conceptualization. Although Vergnaud offers a triadic model for representation, this model leaves implicit the interpreter (or agent) that carries out the interpretational and representational processes while in Peirce's model the intentional interpretation of the individual is of essence.

In conclusion, the descriptions of "representation" used in mathematics education take into account only one or two dimensions of semiotics at a time. By adopting the Peircean triadic we gain the connection and consistency among our different definitions of representation for at least four reasons. First, Peirce conceptualizes signs under the archway of the essential process of representation in human cognition. Second, internal and external representations can be seen in a dialectical relation of determination and representation in which external representations determine internal representations but at the same time internal representations interpret external representations and vice versa. Third, Peirce's definition of sign takes into account three concomitant representations: Object, representamen, and interpretant. While the object and the repesentamen could be at times external representations and at times internal representations, the interpretant is always an internal representation. The continuous dialectical interaction between the elements of the sign accounts for the transformation of signs into new signs through a process of interpretation. This sign interpretation seamlessly reveals the syntactical, semantical, and pragmatical dimensions of semiosis and it encompass the Kantian notion of continuity in thinking and learning. Fourth, the process of sign interpretation or semiosis is a process of internalization in Piaget's (1885) sense, of encapsulation in Dubinsky's (1991) sense, of reification in Sfard's (1994) sense, and also of internalization in Vygotsky's sense (Kozulin, 1990). In the different interpretations of "internalization" representamens are determined by and represent their objects while at the same time they determine and represent new interpretants that are transformed into new representamens standing for transformed objects that are more decontextualized and therefore more general and abstract.

As we consider signs as processes of representation, we can see why symbols could convey straightforward meanings to the teacher and some students; for them, signs are not only processes but also products. For other students, those meanings remain hidden waiting to be interpreted and they manipulate signs only at the reflex level; for them, signs are only objects instead of being products of processes of representation and interpretation.

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A Look at Some Studies on Learning and Processing Graphic Information, Based on Bertin's Theory

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ABSTRACT. In the first part of the paper, we present and compare several analytical frameworks on the processing of graphic information. We focus on the theoretical analysis of Bertin, relating to the work of other authors: mainly Kosslyn, Duval and Janvier.

In the other parts of the paper, we illustrate the interest of Bertin's theory for educational studies by considering studies conducted by our research team. Although very diverse in their methodology, these studies all investigate graphic learning and graphic information processing by children at different school levels.

Introduction

Within the past few years, our research group has conducted a number of studies which, although highly varied, have a common ground: they all deal with graphic representations. This symbolic representation mode is highly useful in mathematics, particularly for representing functions, functional dependencies among several variables, and statistical data. Our studies address a wide variety of issues related not only to graphic information processing, but also to the role of graphs in the acquisition of certain concepts and the learning of graphic representation modes. The school levels examined are also varied, ranging from preschool to high school (not to speak of our work on adult populations). The methodologies are just as diverse, and include individual interviews, experimental methods, and teaching experiments in the classroom.

In all of our research, as diverse as it is, we have been consistently led to analyze graphic representations as systems of signs that translate a thought or a piece of information. Our major references on the matter are found in Bertin's work. Of course, being a specialist in graphic semiology, Bertin does not supply a didactic or psychological framework for analyzing and designing experiments. A semiological analysis, in our minds, is an indispensable step nonetheless, and its importance will be stressed in the present chapter. To this end, it seemed worthwhile to devote a preliminary section to recalling some of the key points in Bertin's work, and to describing the links we establish between his work and that of other authors. The rest of the chapter presents a very brief summary of some of the studies conducted by our research team. The studies chosen for this rapid presentation are ones in which the graphs or diagrams in question are among the easiest to analyze, and in which graphic learning or graphic information processing is under study.

A Few Lines of Analysis

In a series of remarkable books, Bertin (1967, 1977, 1983) proposes an effective method for constructing and using graphs, based largely on facts about human visual information processing. The author's basic idea can be broken down into two points: (1) a piece of information is a series of correspondences within a set of "components" (which one might also call "variables"), and (2) the correspondences define a shared notion which he calls the "invariant". Let us illustrate with one of the simplest examples proposed by Bertin (1967, p. 16):

The information here has two components :
- -Number of francs
- -Time (broken down into days)

The invariant is the definition common to all observed points, namely, "the price of a share of X at the Paris Stock Exchange".

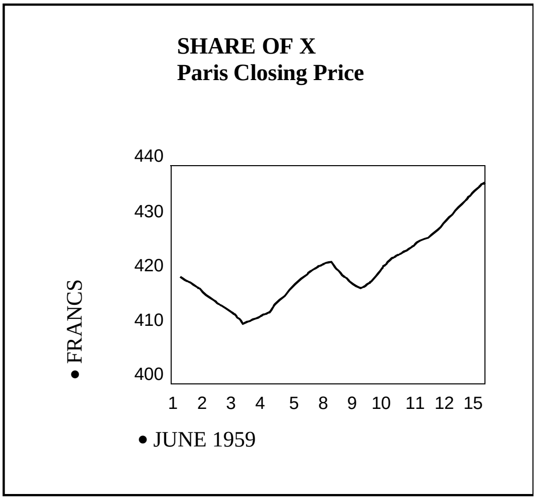

Figure 1.

Bertin distinguished two steps in graph reading. The first step, called “external identification”, is where the reader relies on titles, labels, and legends to identify the components and the invariant; the second step, called “internal identification”, is where the reader recognizes the visual variables associated with the components. These two identification steps constitute a sort of prerequisite to “reading” graphic information (Bertin’s typology of reading levels will be described below). Note that the time taken to carry out these steps varies with the complexity of the graph and the clarity of the labels, titles, and legends.

Kosslyn (1980, 1985, 1989, 1995), an American psychologist and renowned specialist of mental images and human processing of graphs, presents an analysis that is quite similar to Bertin’s.[1] It too is based on perception and memory research, but it also includes a three-level analysis of graph processing: syntactic, semantic, and pragmatic. For each level, Kosslyn states a number of «acceptability» principles whose violation is a major source of error. For example, at the semantic level, Kosslyn proposes representativeness, availability, and congruence as principles that guide graph and chart interpretation. These principles, he tells us, «were derived partly through a review of the literature on how people spontaneously describe visual displays, and partly by generalizing from psycholinguistic phenomena» (1987, p. 203). It is interesting to note, as Kosslyn himself underlines, that Bertin relies consistently on the same principles, even if in an implicit way: «Although Bertin never summarizes these principles in one place, he makes effective use of them throughout the book» (1985, p. 506).

Thus, the work and proposals of these two great specialists of graphic information processing, Bertin and Kosslyn, are fully convergent. This validates our use of Bertin’s tools for analyzing the graphic tasks performed by pupils in our experiments. In particular, one of the factors we often use in our analyses is what Bertin calls “reading levels”. This author distinguishes different hierarchical levels in graph reading: (1) the basic reading level,[2] which serves to answer questions about the correspondence between a single element of one component and a single

[1] Note that in a review (Kosslyn, 1985) of the major work done in the area of graphic semiology, including Bertin’s book (1983), Kosslyn points out the immense richness of Bertin’s work: «It is an understatement to say that this book is ‘meaty’. I especially recommend this book to anyone considering doing research on graphs.»

[2] An example of a question that refers to the basic reading level (for the graph in Figure 1) is: «On what day was the price of a share of X equal to 430 francs at the Paris Stock Exchange?»

element of the other component; (2) the intermediate reading level,[3] where the questions concern a group of elements, and (3) the advanced level,[4] where the questions pertain to an entire component. In many of our experiments, where we have witnessed the difficulties actually encountered by pupils, we were able to judge the relevance of this «a priori» hierarchy proposed by Bertin.

In the work of another author, Duval (1993, 1999 p. 50-52) — who in fact treats issues that are very different from those examined by the investigators mentioned above since his main interest is learning — we find a typology that is quite similar, but not identical, to Bertin's. Duval sees three hierarchical modes for grasping a Cartesian graph: local via pointing, iconic, and global qualitative. While the first and third modes can be likened to Bertin's basic and advanced reading levels, respectively, the iconic approach does not correspond in any way to the intermediate reading level. This is not surprising if we take into account the different issues addressed by these two authors and the specifics of the situations to which they refer. Problems that require the intermediate reading level are very common in Bertin's chosen domain, the interpretation of statistical data, whereas they are less so in the graphic representations of functions studied by Duval. Furthermore, Duval's proposals are grounded in difficulties regularly observed in mathematics class, especially ones encountered in the interpretation of Cartesian graphs depicting functional dependencies, whereas Bertin looks into the kinds of questions one might reasonably ask a reader confronted with any "well-designed" graph, i.e., one that abides by the principles he advocates.

According to Duval, the iconic way of apprehending a graph is the one primarily and spontaneously used in many situations, and the goal — in a learning perspective — is to enable the learner to move from a local or iconic mode of apprehension to a global one.

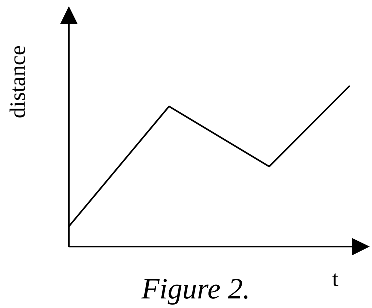

Figure 2.

[3] An example of a question that refers to the intermediate level is: «How did the price evolve in the first three days?»

[4] An example of an advanced-level question is: «How did the price evolve over the entire period?

Convergent results indicating spontaneous reliance upon iconic interpretations have also been found by other authors, in particular Bell and Janvier (1981) and Janvier (1993). For example, for the graph in Figure 2, these authors found that many pupils interpret the graph iconically by imagining an individual who starts climbing up a hill, then goes down a ways, and then finally continues climbing upwards. We mentioned above that the principle of congruence is among the principles whose violation is a major source of error in graph processing according to Kosslyn. Congruence plays a privileged role for Duval as well, in a framework which — importantly — goes far beyond that of graphs, encompassing other, non-graphic types of representation. Duval makes “non-congruence”, i.e., the violation of congruence, a major determinant of learning difficulties.

Note, however, that while the ways these two authors characterize congruence share a number of points (surface compatibility and ordering for Kosslyn; semantic correspondence between the meaningful units of two representations and the same possible order for grasping the units in the two representations for Duval), there are also several notable differences. First of all, unlike Duval, Kosslyn’s interest lies in the problem of understanding graphs and charts, without concern for learning-related issues. Second, Duval treats congruence between two representation “registers”, whereas for Kosslyn, congruence is envisaged between “the specifier” and what is being specified.

One of the results clearly demonstrated by Duval and his group in several experiments is that, whenever the questions involve a change in the semiotic representation system, congruence and non-congruence are very strong factors in determining success or failure, respectively. This led Duval to advocate that, in order to promote learning, pupils should be explicitly taught how to switch between different representational registers.

This last proposal has much in common with Janvier’s (1993) suggestion in terms of “translation”. This author, a specialist of didactics who is interested in learning-related questions, contends that when functions are at stake, teachers should avoid dwelling on exercises that bring only one mode of representation into play (formulas, for example, or the graphic mode). For Janvier, time should be spent instead on “translating” or converting from one mode to another, i.e., on identifying the meaningful units of two different registers and establishing the correspondence between them, in other words, on what Duval calls “register switching”.

One might add in regards to Janvier that, since his doctoral dissertation (1978), this author has published an entire series of studies containing a wealth of information on graphic representations. He demonstrates a number of interesting phenomena, such as difficulties caused by the erroneous semantic transfers pupils make between the "source mode" and the "target mode", which, for this author, more or less correspond to the referent and the signified concept. Our research team has obtained similar results on this topic (Maury, 1998; Lerouge, 1992).

Where We Look at Procedures for Statistical Graph Processing

Work on bar-diagram processing

In reference to Bertin's typology (1967, p. 199-202), we classify bar diagrams among graphs that relate a short, orderable component (the set of cities in the example in Figure 3) to a quantitative series (the size of the population in that example). These graphs are very common and are also among the simplest. However, one can predict that a major difficulty will arise in the processing of such diagrams because of the fact that proportionality — the concept required to establish the link between the numerical and graphic data[5] — is far from being acquired, even at the secondary school level. It is therefore legitimate to question the alleged simplicity of this task, and thus to attempt to detect the solving modes used to attack such problems. This is the issue we set out to resolve a few years ago when we began a series of experiments on this type of diagram. The present section talks about the first of these experiments; two others will be briefly examined in Sections 3 and 5.

Here are 5 towns whose
A: 000 inhabitants,
B: 33 000 inhabitants,
C: 22 000 inhabitants,
D: 55 000 inhabitants,
E: 44 000 inhabitants.

On the graph, de bar represent town A has already been labelled.
Under de other bars, write the name of the two towns represented.

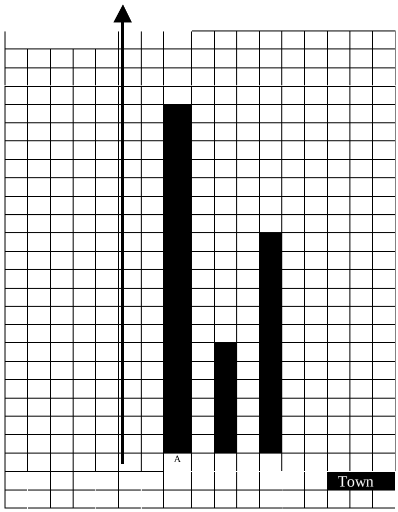

Figure 3.

[5] Proportionality in such graphs exists between the numerical data and the height of the bars.

The purpose of the first study (Maury, Janvier, & Baillé, 1990) was to determine and describe the procedures used by 6th to 9th grade students to solve a series of six problems analogous to the one shown in Figure 3. It was an experimental study in which 370 subjects were tested. We examined the effects of several factors, which will not be described here. Only the main results regarding the procedures used will be mentioned.

The results showed that in order to relate the measures represented in graphic form to the measures given in numerical form, the students implemented procedures suited to solving multiplicative problems involving isomorphic measures. But other more specific procedures — both qualitative and quantitative — were also employed, most likely due to the graphic nature of the task. Moreover, a high degree of variability in procedure selection was observed. This variability is a reflection of the fact that, in addition to operational devices specific to problem solving, these children had decision-making rules at their disposal that partially depended on task content.

Age pyramids (Maury & Vion, 1999)

In the diagram below, each of the two histograms — which when placed back-to-back form a pyramid of ages — can be classified as a two-component diagram in Bertin's typology. One component is obtained by dividing the ordered variable "age" into age classes; the other corresponds to the number of persons in each class. The pyramid presented in Figure 4 is thus a three-component diagram, and the depicted invariant is "the number of persons per age class".

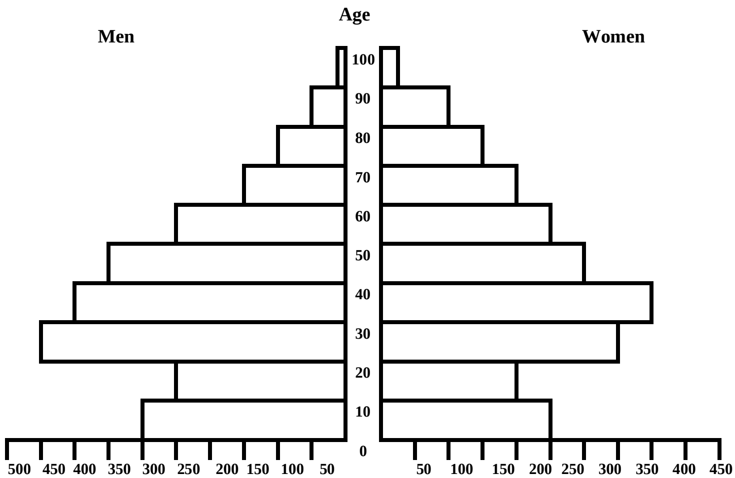

Figure 4. Number of people.

A priori, this type of graph is much more complex than the previous one. In particular, the numerical values are proportional this time to areas of rectangles rather than to lengths of bars as above.[6] In addition, the questions asked of the pupils, who had to solve a series of ten problems comparable to the one shown in Figure 4, were aimed at testing the different reading levels (e.g., the question «Are there more men or more women between the ages of 60 and 100 in this Indian tribe?» refers to the intermediate reading level).

The experiment was run on a sample of 280 pupils in the 4th, 5th, and 6th grades. The effects of a number of factors were studied, although they will not be described in this review. As a whole, the results indicated a wide variety of procedures, as well as substantial variability in procedure selection. Moreover, they confirmed the reading-level hierarchy proposed by Bertin.

Where We Compare the Reading and Construction Processes (Maury, Janvier, & Baillé, 1989)

This experiment was conducted using analogous material to that presented before in paragraph "Work on bar-diagram preocessing" (Figure 3, bar diagrams) and concerned a population of pupils in the same age range as those in the first experiment (6th to 9th grades). The sample tested consisted of 461 pupils, each of whom had to solve 15 problems. Half of the pupils were given graph-reading tasks and half were given graph-construction tasks. Again, the effects of various factors were tested but will not be described here.

An analysis of variance on the performance data pointed out a highly significant effect of school grade (at a significance level of .0001): performance improved with age. The type of task also had a highly significant effect (again, at .0001): reading was easier than construction. The massive character of these effects masks the complexity of the situation in this very brief overview, but in fact, a highly significant interaction was observed between certain factors.

Where Methods for Teaching Statistical Graphs Are Implemented

Introduction to graph reading in kindergarten (Elie-Auzé, 1999)

This study put into effect a teaching plan that lasted about three months. It combined sports practice in preparation for a running contest among several schools (8 sessions)

[6] Even if, in the example presented, equal size classes imply proportionality between the areas and lengths of the rectangles.

and sessions held in the classroom, where the numerical data obtained during practice was examined (the pupils' running times). During the classroom sessions (the only ones of interest to us here), the pupils (ages 5 and 6) were divided into groups of seven, with the same pupils in each group for all sessions. With the help of the teacher, each group gradually drew up a table containing the numerical running data and then constructed the corresponding graph. In each group, the seven curves (one per child) were drawn on the same graph (each in a different color). These graphs thus fell into the time-comparison category (in Bertin's typology), given that the component representing the practice sessions (numbered from 1 to 8), plotted on the x-axis, was common to all curves. Running time was plotted on the y-axis.

Graph construction by each group was done with the guidance of the teacher, who devoted herself to one group at a time and for relatively long periods. The pupils also had to solve various graph-reading problems, which were introduced throughout the sessions by questions pertaining to the different reading levels, asked by the teacher. For example: During what practice session did Paul take 1 minute 20 seconds (basic reading level)? Who made the best time on the third session (intermediate reading level)? Who improved (advanced level)? Note that this situation violates Kosslyn's semantic congruence principle — indeed, an improvement in performance across sessions corresponded to a decline on the graph.

One month after the teaching period, the children were called in individually to perform a control task. The experimenter showed the pupil a graph with the curves of three fictitious children (see Figure 5) and asked a series of questions corresponding to the different reading levels. Given the pupils' young age, the results were spectacular: most of the children attained the intermediate reading level, some even the advanced level; they were not disturbed by the semantic non-congruence. Note, however, that the question of transfer to problems unlike the one used in the learning phrase remains to be examined.

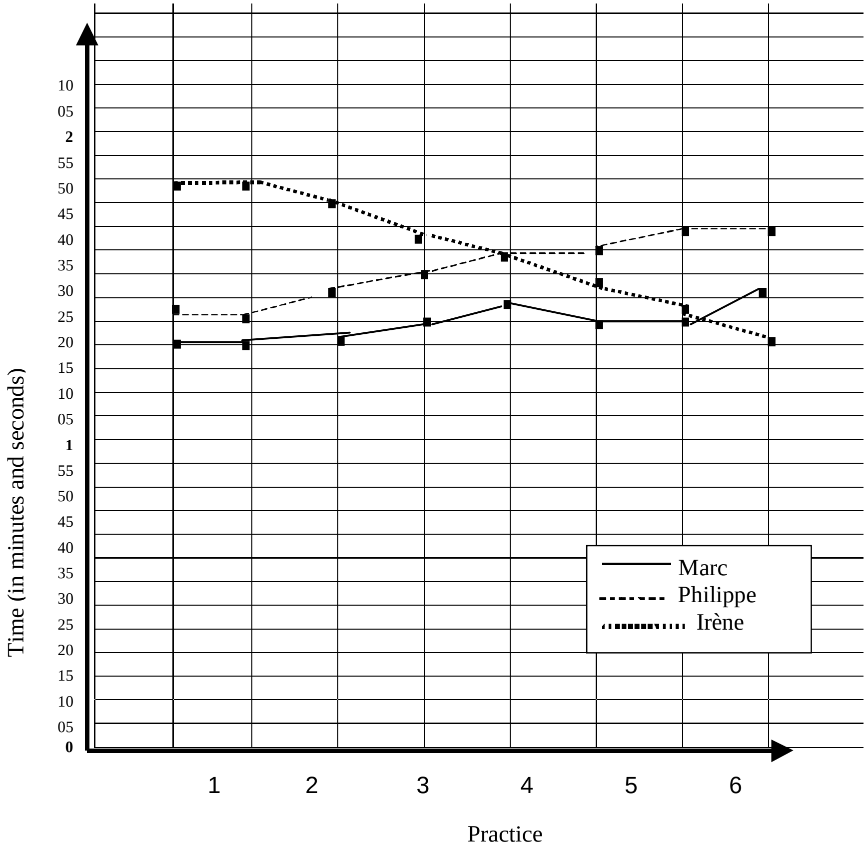

Figure 5.

Using graphic representations in the remedial classroom (Gauthier, 2001)

This experiment was conducted in three remedial classes of children with a minor or moderate mental handicap. The goal of such classes is to enable the children to return to an ordinary classroom; the class size is always small and the children, who are of different ages, are monitored by specially-trained teachers. In the three classes tested here, there were about eight pupils per class and their age range was 7 to 11 years. The lessons were given by the experimenter (herself a special education teacher) and took place over a period of three months. The teaching alternated between group sessions, individual work times, and pupil-teacher talks (recorded on tape). The group sessions were videotaped, and from time to time, a self-observation session based on video viewing by the whole class was held. The children were taught how to construct and read Cartesian graphs and bar diagrams representing a variety of situations (temperature readings, population growth in a country, etc.). After the teaching period, the pupils were called in individually to perform control tasks that were not isomorphic to the problems addressed in class. The control tasks tested the pupil's graph-reading ability, and the questions asked pertained to each of the different reading levels. Most of the pupils learned. They were able to achieve the various identification steps, and some managed to reach the intermediate or even advanced reading level.

Where We Raise the Question of Possible Learning Transfers from the Numerical Framework to the Graphic Framework (Baillé, Maury, & Janvier, 1998).

The experiment reported above in "Work on bar-diagram processing" showed that, although the difficulties observed in bar-diagram processing cannot be ascribed solely to proportionality, this concept constitutes one of the major obstacles to proper task management. Because proportionality is usually learned in a numerical framework, the question raised here was whether it is possible to transfer numerical-framework learning to the graphic framework. To answer this question, we devised an experiment aimed at studying the short- and long-term impact, on bar-diagram processing, of systematic and repeated teaching of the procedures used to solve proportionality problems.[7]

The experiment was conducted on four 7th grade classes containing a total of 103, 12- and 13-year-old pupils. Each class included three 50-minute sessions, taught by its regular math tacher. The four teachers had been fully informed of the goals of the experiment, and agreed to adhere strictly to a standardized procedure for presenting the work and conducting the sessions.

We assessed the pupils' performance, the types of procedures they chose, and their ability to switch back and forth between procedures. The results showed a number of complex learning effects. Cognitive flexibility, one of the essential characteristics of bar-diagram processing by pupils in the first few years of secondary school, was found to be enhanced by the observed learning effects, especially those related to procedure selection and switching.

[7] The tasks given to the pupils were comparable to those performed in the experiment reported in "Work on bar-diagram processing". For comparison, we also tested for potential effects on the solving of numerical problems isomorphic to the bar-diagram problems.

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Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. Basic Issues for Learning[1]

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ABSTRACT. There are many studies about visualization, representation and, also, the need of an semiotic approach in mathematics education. However, they are often deceitful and do not provide many helpful issues for the analysis of processes and learning problems in mathematics education. In fact they use models for visualization, representation or, even the use of signs which aren't relevant, because these basic cognitive processes work quite differently in mathematics than in all the other fields of knowledge. The key matter for research in mathematics education is to analyze these specific way of functioning. What require new distinctions and more discriminating concepts not only about processes of visualization and representation but also in semiotics. And most of them must lead to more relevant experimental and didactical variables. In this paper we introduced some issues concerning the framework for analyzing the cognitive functioning of mathematical thinking and conditions of its learning.

Introduction

Mathematics education has been very sensitive to needed changes over the past fifty years. Researches in developmental psychology, new technologies, new requirements in assessment have supported them. But their impact has been more effective on mathematics curriculum and on means of teaching than on the

[1] In F. Hitt & M. Santos (Eds.), *Proceedings of the Twenty-first Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico, Vol. I, 3-26.

explanations of the deep processes of understanding and learning in mathematics. Difficulties of such research stem from the necessity to define a framework within the epistemological constraints specific to mathematical activity and the cognitive functions of thought which it involves are not separated. That requires going beyond local studies of concept acquiring at each level of the curriculum and beyond a reference to general theories of learning and even beyond global description of student's activity in classroom.

Representation and visualization are at the core of understanding in mathematics. But in which framework can their role in mathematical thinking and in learning of mathematics be analyzed? Already in 1961, Piaget admitted the difficulty to understand what mathematicians call "intuition", a way of understanding which has close links with representation and visualization: "rien n'est plus difficile à comprendre pour un psychologue que ce que les mathematiciens entendent par intuition". He distinguished "many forms of mathematical intuition" (1961, pp. 223-241) from the empirical ones to the symbolizing ones. From a cognitive viewpoint, the question is not easier. Representation refers to a large range of meaning activities: steady and holistic beliefs about something, various ways to evoke and to denote objects, how information is coded... On the contrary, visualization seems to emphasize images and empirical intuition of physical objects and actions. Which ones are relevant to analyze the understanding in mathematics in order to bring out conditions of learning?

Our purpose is to focus on some main distinctions which are necessary to analyze the mathematical knowledge from a learning point of view and to explain how many students come up against difficulties at each level of curriculum and very often cannot go beyond. Studies about reasoning, proving, using geometrical figures in problem solving, reading of graphs... have made these distinctions necessary. They lead not only to emphasize semiotic representations as an intrinsic process of thinking but also to relativize some other ones as the distinction between internal and external representations. They lead also to point out the gap between vision and visualization. And from a learning point of view, visualization, the only relevant cognitive modality in mathematics, cannot be used as an immediate and obvious support for understanding. All these distinctions find accurate expression in different sets of cognitive variables. Within the compass of this panel we shall confine ourselves to sketching the complex cognitive architecture that any subject must develop because it underlies the use of representations and visualization in mathematics.

Three Key Ideas to Define a Framework to Analyze the Conditions of Learning

The first one is the paradoxical character of mathematical knowledge

On the one hand, the use of systems of semiotic representation for mathematical thinking is essential because, unlike the other fields of knowledge (botany, geology, astronomy, physics...), there is no other ways of gaining access to the **mathematical objects** but to produce **some semiotic representations**. In the other fields of knowledge, semiotic representations are images or descriptions about some phenomena of the real external world, to which we can gain a perceptual and instrumental access without these representations. In mathematics it is not the case.

On the other hand, the understanding of mathematics requires not confusing the mathematical objects with the used representations. This begins early with numbers, which have not to be identified with digits, numeral systems (roman, binary, decimal...). And figures in geometry, even when they are constructed with accuracy, are just representations with particular values that are not relevant. And they cannot be taken as proofs.

The second one is the ambiguous meaning of the term "representation"

This term is often used to refer to mental entities: image, something away or missing that is evoked and, finally, what subjects understand. In this context, "mental" representation is considered as the opposite of signs which should be only "material" or "external" signs. Semiotic, and therefore external representations, would be at first necessary for the communication between the subjects. But this is a misleading division (Duval, 1995b pp. 24-32) which brings about two very damaging confusions.

When it is applied to the representations, the distinction mental/external refers to their mode of production and not to their nature or to their form. In that sense, signs are neither mental nor physical or external entities. More specifically, there is not a term to term correspondence between the distinction mental/material and the distinction signified/signifiant, because the signifiant of any sign is not determined by its material realization but only by its opposite relations to the other signs: it is the number of possible choices what matters, as Saussure explained it. The binary system and decimal systems are very trivial examples of this semiotic determination of significance: the significance of any digit depends not only on its position but also on the number of possible choices per position. And, as for language, any use of a semiotic system can be mental or written (that is external). Thus, mental arithmetic uses the same decimal system like written calculation but not the same strategies because of the cognitive cost.

There are two kinds of cognitive representations. Those that are intentionally produced by using any semiotic system: sentences, graphs, diagrams, drawings... Their production can be either mental or external. And there are those which are causally and automatically produced either by an organic system (dream or memory visual images) or by a physical device (reflections, photographs...). In one case, the content of the representations denotes the represented object: it is an explicit selection because each significant unit results from a choice. In the other case, the content of representations is the outcome of a physical action of the represented object on some organic system or on some physical device (Duval *et al*., 1999, pp. 32-46). In other words, the basic division is not the one between mental representation and external representation, which is often used in cognitive sciences as though it was evident and primary, but the other one between semiotic representation and physical/organic representation. We cannot deal anyway with a representation without taking into account the system in which it is produced.

The third one is about the need of various semiotic systems for mathematical thinking

History shows that progress in mathematics has been linked to the development of several semiotic systems from the primitive duality of cognitive modes which are based on different sensory systems: language and image. For example, symbolic notations stemmed from written language have led to the algebraic writing and, since the nineteenth century, to the creation of formal languages. For imagery, there was the construction of plane figures with tools, then that in perspective, then the graphs in order to «translate» curves into equations... Each new semiotic system provided specific means of representation and processing for mathematical thinking. For that reason, we have called them «register of representation» (Duval, 1995b). Thus, we have several registers for discursive representation and several systems for visualization. That entails a complex cognitive interplay underlying any mathematical activity.

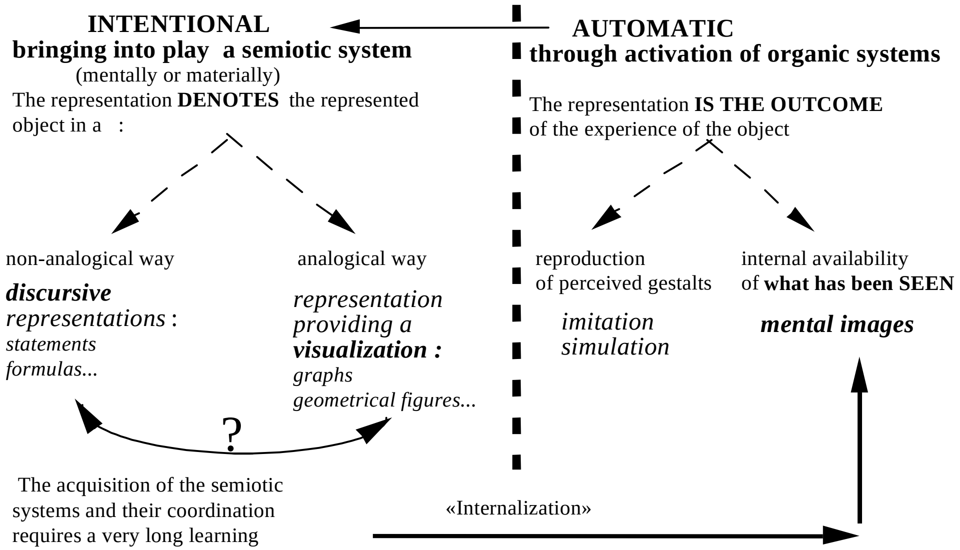


Figure 1. Cognitive classification of conscious representations. This classification can be expanded more and includes all kinds of representations. We can notice the existence of two heterogeneous kinds of “mental images”: the «quasi-percepts» which are an extension of perception (on the right) and the internalized semiotic visualizations (on the left). Actions like the physical ones (rotation, displacement, separation...) can still be performed on some quasi-percepts and their time cost can be measured by reaction times to comparison tasks.

Firstly, as well as for discourse (description, explanation, reasoning, computation...) as for visualization, we have two kinds of registers: the registers with a triadic structure of significance (natural language, 2D or 3D shapes representation) and registers with a dyadic structure of significance (symbolic notations, formal languages, diagrams...) (Duval, 1995b, pp. 63-64). Within a dyadic structure any meaning is reduced to an explicitly defined denotation of objects. Within a triadic structure, we have meanings playing independently of any explicit denotation of objects and one must take into account their interplay. We can even fall into a cognitive conflict between the meaning game, which is proper to the register, and the denotation set for the representation. For example, the complexity of geometrical figures stems from their triadic structure of significance. Secondly, mathematical thinking often requires to activate in parallel two or three registers, even when only one is externally used, or seems sufficient, from a mathematical point of view.

This need of various registers of representation gives rise to several questions that are important in order to understand the real conditions of learning mathematics. First of all, there is a question about the specific way of working in each register: what operations are favored, or are only possible, within each register? This question is not trivial, because there are several registers for visualization and because they cannot be the same. Then, there are questions about the change from one register to another one. Are these changes very frequent or necessary? Are they always easy or evident to make? ... At last, is there a register more convenient or more intrinsically suitable for the mathematical thinking than others? It is obvious that registers with dyadic structure are technically more useful and more powerful than registers with triadic structure. But natural language remains essential for a cognitive control and for understanding within any mathematical activity. These questions may appear unimportant from a mathematical point of view. Even more, very often a mathematician cannot see why these questions arise. But from a didactical point of view, they are those questions that the difficulties of learning pose.

How the Problems of Mathematics Learning Come to Light in this Framework

No learning in mathematics can progress without understanding how the registers work

Cartesian graphs are very common examples because they look visually easy to grasp. But many observations have shown that most 15-17 year old students cannot discriminate the equations $y = x + 2$ and $y = 2x$ when looking at these two graphs:

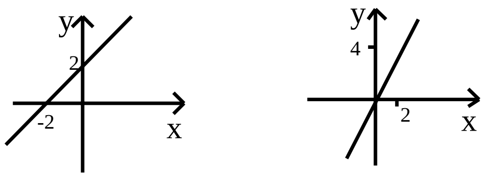


Figure 2. Visual discrimination of two elementary linear functions. This kind of discrimination presupposes that the qualitative values of two visual variables be distinguished: comparison between the angle with x-axis and the one obtained by bisection of xy angle, and position of crossing point with y-axis. Most often students confine themselves to the visual variable which is not relevant: how far some points are above x-axis (Duval, 1988).

Notwithstanding this kind of failure, students succeed in the standard tasks such as constructing the graph from a given equation or reading the coordinates of a point! This kind of failure means that graphs cannot be useful representations neither to control intuitively some calculations nor to organize and to interpret data in other fields. And we have similar observations for each register of representation, even those which look more natural, like geometrical figures, or which are very utilized, like the decimal system in which the position of digits determines the operative meaning (French National Assessment, 1992, 1996).

All these repeated observations show that semiotic representations constitute an irreducible aspect of mathematical knowledge and that wanting to subordinate them to concepts leads to false issues in learning. That amounts to forget the paradox of mathematical knowledge: mathematics objects, even the more elementary objects in arithmetic and geometry, are not directly accessible like the physical objects. Each semiotic register of representation has a specific way of working, of which students must become aware.

We must distinguish two kinds of cognitive operations in mathematics thinking: "processing" and conversion

Mathematical processes are composed of two kinds of transformations of representations. There are transformations that are made within the same register of representation, like arithmetical or algebraic computation. The semiotic possibilities of generating a new representation from a given representation are exploited. With the dyadic structure, these possibilities depend both on the semiotic system and on mathematical rules. The geometrical figures give also rise to the intrinsic gestalt transformations of configurations apart from any previous consideration of mathematical properties. These gestalt transformations are like the visual transformations that anamorphoses or jigsaws lead to bring into play. We have called "processing" this kind of transformation.

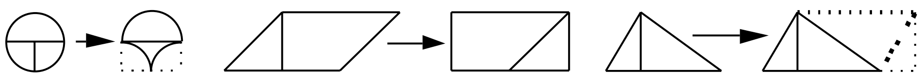


Figure 3. Visual change of configuration. The figurative units of any figure can be "reconfigured", mentally or materially, in another figure. For this kind of merely figurative transformation, neither hypotheses nor mathematical justification are required. Very often problem solving or explanations meant to convince students to resort to such transformations as if they were immediate and obvious for every student. Many observations show that this is not the case. There are factors that inhibit or trigger the «visibility» of such transformations. We can study them experimentally (Duval, 1995a).

And there are transformations that lay on a change of register: the representation of an object is «translated» into a different representation of the same object in another register. For example, when we go from a statement in native language to a literal expression. The transformation of equations into Cartesian graphs is another example. We have called “conversion” this kind of transformation.

One does not pay very close attention to the gap between these two kinds of cognitive operations that underlie mathematical processes. Nevertheless, if most students can learn some processing, very few of them can really convert representations. Much misunderstanding stems from this inability. But, very often, teachers attach more importance to the mathematical processes than to their application to daily life problems or to physical, or economic problems.

Conversion of representations is a crucial problem in the learning of mathematics

Mathematical activity, in problem solving situations, requires the ability to change of register, either because another presentation of data, which fits better an already known model, is required, or because two registers must be brought together into play, like figures and natural language or symbolic notations in geometry. From a didactical point of view, only students who can perform register change do not confuse a mathematical object with its representation and they can transfer their mathematical knowledge to other contexts different from the one of learning. Two facts show the great complexity of conversion operation.

— Any conversion can be congruent or non-congruent. When a conversion is congruent the representation of the starting register is transparent to the representation of the target register. In other words, conversion can be seen like an easy translation unit to unit. Very accurate analyses of the congruent or non-congruent character of the conversion of a representation into another one can be systematically done. And they explain in a very accurate way many errors, failures, misunderstandings or mental bloks (Duval, 1995b, pp. 45-59; 1996, pp. 366-367).

— The congruence or the non-congruence of any conversion depends on its direction. A conversion can be congruent in one way and non-congruent in the opposite way. That leads to striking contrasts in the performances of students, such as those summarized in Figure 4.

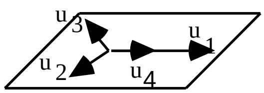
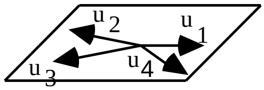
	Starting register	Target Register	144 students
(2D Rep.) **T ? G**	1 0 k p 0 1 m 0		**.83**
G ? T		1 0 a c 0 1 b d	**.34**

Figure 4. Elementary task of conversion (Pavlopoulou, 1993, p. 84).

Of course, the contrasts caused by the non congruence can be observed in a systematic way at all stages of the curriculum, from the more elementary verbal problems at primary school (Damm, 1992), to the university level.

It is surprising to see that this wide-ranging phenomenon is always ignored in the teaching of mathematics. Most teachers, mathematicians and even psychologists pay little attention to the difference of nature between processing and conversion. These two kinds of cognitive operations are grouped together in the unity of mathematical processes to solve a problem. And when a change of register must be introduced in the learning, one generally chooses one direction and the cases that are congruent. From a cognitive point of view, it is frequently a one-sided activity, which is proposed to students! There is something like an instinct to avoid the non-congruence situations that lead to real difficulties. But they are impossible to avoid especially when transfer of knowledge is required. Then failures and blocks are explained as conceptual misunderstanding, what is not a right explanation, since we have a contrast of successes and failures for the same mathematical objects in very similar situations. In reality the fact that students don't recognize anymore, when direction of conversion is changed, reveals a lack of co-ordination between the registers that have to bring into play together. The coordination of registers is not the consequence of understanding mathematics; on the contrary, it is an essential condition.

The learning of mathematics and the progressive coordination among registers

All these various and continual observations point out to a basic requirement that is specific for any progress in the learning of mathematics: the coordination among the registers of representation. This basic requirement is not fulfilled for most students, what is noticed in a global way often at the end of learning. For example, many teachers have, in one way or another, experienced what Schoenfeld (1986) described after a one yearlong study:

> [S]tudents may make virtually no connections between reference domains and symbols systems that we would expect them to think of as being nearly identical... the interplay occurs far more rarely than one would like (pp. 239-242)
>
> [T]he students did not see any connection between the deductive mathematics of theorem proving and the inductive mathematics of doing constructions... they fail to see the connections or dismiss the proofs as being irrelevant (pp. 243-244)... If students fail to see such obvious connections, they are missing what lies at the core of mathematics... (p. 260)

Schoenfeld characterized this splitting rightly like an "inappropriate compartmentalization" (p. 226). But, unlike Schoenfeld's analysis, the kind of operative connections we expect to be made when learning is not between deductive and empirical mathematics, proofs and constructions, nor between mathematical structures and symbol structures, but between the different registers of semiotic representation. These connections between registers make up the cognitive architecture by which the students can recognize the same object through different representations and can make objective connections between deductive and empirical mathematics. Learning mathematics implies the construction of this cognitive architecture. It always begins with the coordination of a register providing visualization and a register performing one of the four discursive functions (Duval, 1995b, pp. 88-94).

Vision and Visualization

From a psychological point of view, "vision" refers to visual perception and, by extension, to visual imagery. As perception, vision involves two essential cognitive functions.

— The first one consists in giving **direct access** to any physical object "in person". That is the reason why visual perception is always taken as a model for the epistemological notion of intuition. Nothing is more convincing than what is seen. In that sense, vision is the opposite of representation, even of the "mental images", because representation is something which stands instead of something else (Peirce). We shall call this function the ***epistemological function***.

— The second one is quite different. Vision consists of **apprehending simultaneously several objects** or a whole field. In other words, vision seems to give immediately a complete apprehension of any object or situation. In that sense, vision is the opposite of discourse, of deduction... which requires a sequence of focusing acts on a string of statements. We shall call it the ***synoptic function.***

In fact, visual perception performs in a very imperfect way the synoptic function. Firstly, because we are inside a three dimensional world: just one side of things can be seen, and complete apprehension requires movement, either of the one who is looking at it or of what is seen. In any case, this movement is a transformation of the perceived content: we have just a juxtaposition of successive sights which can be full-face, in profile, from above...Secondly, because visual perception always focalizes on a particular part of the field and can jump from one part to another one. There is no visual perception without such an exploration.

Now we can ask the following question that is decisive in the perspective of learning: are there cognitive structures that can perform both the epistemological and the synoptic function for the mathematical knowledge? The previous remarks lead us to answer this question negatively. More precisely, they lead to distinguish visualization from vision. Unlike vision, which provides a direct access to the object, visualization is based on the production of a semiotic representation. As Piaget, who has highlighted the synthetic inability of 3-5 year old children for the drawing of geometrical gestalts, explained it:

> **Le dessin est une représentation,** c'est-à-dire qu**'il suppose la construction d'une image bien distincte de la perception elle-même**, et rien ne prouve que les rapports spatiaux dont cette image est faite soient du même niveau que ceux dont témoigne la perception correspondante (1972, p.65)

We have here the breaking point between visual perception and visualization. A semiotic representation does not show things as they are in the 3D environment or as they can be physically projected on a small 2D material support. That is the matter of visual perception. **A semiotic representation shows relations or, better, organization of relations between representational units.** These representational units can be 1D or 2D shapes (geometrical figures), coordinates (Cartesian graphs), propositions (propositional deductive graphs or "proof graph"), or words (semantic networks)... And these units must be **bi-dimensionally connected**, **because any organization requires at least two dimensions to become obvious**. In a string of discrete units (words, symbols, propositions) not any organization can be displayed. Thus, inasmuch as text or reasoning, understanding involves grasping their whole structure, there is no understanding without visualization. And that is why visualization should not be reduced to vision, that is to say: visualization makes visible all that is not accessible to vision. We can see now the gap between visual perception and visualization. Visual perception needs exploration through physical movements because it never gives a complete apprehension of the object. On the contrary, visualization can get at once a complete apprehension of any organization of relations. We say "can get" and "cannot get" because visualization requires a long training, as we shall prove it below. However, what visualization apprehends can be the start of a series of transformations, that makes its inventive power.

This difference between visual perception and visualization entails two consequences for the learning of mathematics.

Visualization refers to a cognitive activity that is intrinsically semiotic, that is, neither mental nor physical. Also such expressions as "mental image", "mental representation", '"mental imagery", are equivocal. They can only be the extension of visual perception. Accordingly, Neisser wrote:

> "[V]isual image" is a partly undefined term for something *seen* somewhat in the way real objects are seen when little or nothing in the immediate or very sensory input appears to justify it. Imagery ranges from the extremely vivid and externally localized images of the eidetiker to the relatively hazy and unlocalized images of visual memory. (Neisser, 1967; p.146)

Experiments on mental rotation of three-dimensional objects, since Shepard and Meztler (1971), are in the line of this conception of mental image as an extension of visual perception. But "mental imagery" can also be a mere visualization, that is, the mental production of semiotic representations as in mental calculation. Thus in "mind", we find the split into two kinds of representation back (Figure1). By resorting to mental images one does not avoid the difficulties arising from the paradoxical character of mathematics.

The way of watching is not the same in vision than in visualization. Two phenomena are confusing this issue. First, when they are graphically produced, semiotic representations are subject to visual perceptive apprehension. In that sense, visualization is always displayed within visual perception or within its mental extension. Second, some semiotic representations, like drawings, aim at being "iconic" representations: there is a relating likeness between the representation content and the represented object, so that one recognizes it (a tree, a car, a house...)

at once, without further information. Iconic representations refer to a previous perception of the represented object, from which to their concrete character. In mathematics, visualization does not work with such iconic representations: to look at them is not enough to see, that is, to notice and understand what is really represented.

The use of visualization requires a specific training, specific to visualize each register. Geometrical figures or Cartesian graphs are not directly available as iconic representations can be. And their learning cannot be reduced to training to construct them. This is due to the simple reason that construction makes attention to focus successively on some units and properties, whereas visualization consists in grasping directly the whole configuration of relations and in discriminating what is relevant in it. Most frequently, students go no further than to a local apprehension and do not see the relevant global organization but an iconic representation.

To sum up, visualization, which performs only the synoptic function, is not intuition but representation. In that sense, there are several possible geometrical registers for visualization. Visualization in mathematics is needed because it displays organization of relations, but it is not primitive, because it is not mere visual perception. In this respect, there is learning from the geometrical registers. Is there any vision that could perform the epistemological function? That is a philosophical question. From a cognitive view, the essential fact is the paradoxical character of the mathematical knowledge, which excludes any resort to mental representations as direct grasping of mathematical objects, at least in the didactical context.

How Visualization Works Toward Understanding

We have characterized visualization as a bi-dimensional organization of relations between some kinds of units. Through visualization, any organization can be synoptically grasped as a configuration. In this way, we have as many kinds of visualization as kinds of units: geometrical configurations where units are 1D or 2D shapes or Gestalts, Cartesian graphs where units are couples {point, coordinates}, propositional graphs where units are statements... For the visualization of each register of representation there are some rules or some intrinsic constraints to produce units and to form their relations. Thus, geometrical configurations can be constructed with tools and according to mathematical properties of the represented objects. One does not draw a pentagon as an oak-leaf or as a flower. There lies the point where visualization leads away from any iconic representation of a material object. In the perspective of learning, three problems have to be taken into account about visualization: the problem of discrimination, the problem of processing and the problem of coordination with a discursive register.

How can the relevant visual features be discriminated?

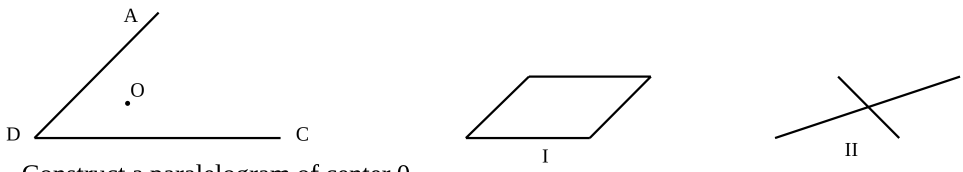
Unlike iconic representations, visualizations used in mathematics are not sufficient to know what are the denoted objects. Very early, young children learn quickly to recognize by themselves images of physical objects, perhaps because schematizations of frequently perceived outlines are automatically developed. But learning visualization in mathematics is not quite so easy and successful as it is for physical objects and real environment.

In front of simple Cartesian graphs, most students only have a local apprehension confined to the associations of points with coordinates. They do not get a global apprehension of all visual variables, which enables them to discriminate visually between the different graphs of functions such as

$$y = 2 - x, \;\; y = 2x, \;\; y = x + 2.$$

In other words, Cartesian graphs do not work visually for most students except for giving the naïve holistic information: the line goes up or down, like a mountain road. But that can be misleading when they have to compare the graphs of two series of observations. And Cartesian graphs can perform anyway a checking or a heuristic function in the tasks of formulae computation or interpretation. No connections can be made between the different graphs and the definitions, descriptions or explanations that are displayed in other registers.

Some simple 2D geometrical figures are taught at the primary level: triangle, circle, different quadrilateral polygons... But all these geometrical figures are equivocal representations. They can be hard iconic representations and they are nothing further than an herbarium of mathematical Gestalts. Or they can work as representations of geometrical objects and, in this case, they must appear as 2D organizations of 1D figural units. In other words, there are quite different apprehensions of the most elementary geometrical figures; the one which is according only to the spontaneous perceptive work, and the other which is «discursive» or anchored in some statements (definitions, theorems...), (Duval, 1998; pp.39-40). Thus, with the discursive apprehension, we can have several figures for the same geometrical object: for example, there are two typical figures to represent a parallelogram (I and II in Figure 5).



Construct a paralelogram of center 0, havining one side on DA, another on DC

Figure 5. Which of the two figures, I or II, can be useful to solve the problem? With the visual help of *Figure* I, one can only roughly make the drawing by successive attempts of measurements on DA and DC. With the visual help of *Figure* II, one easily succeeds by drawing the diagonal DOD'. Although they knew all the properties of parallelogram, most students failed as if they were confined themselves to visualization I (Dupuis, 1978, pp.79-81). In fact, I and II give a visual help only when one works with configurations of 1D figural units.

Such observations have been made many times for very simple problems (Schoenfeld, 1986, pp. 243-244). And these phenomena are all the stronger the geometrical figure appears as a joint of several Gestalts (triangles, parallelograms, circles, straight lines...). For most students, there is like a heuristic deficiency of geometrical interpretation to visualization. But the equivocal character of geometrical figures appears also when a figure is directly taken for proof and leads to reject any resort to deductive reasoning. In that case, the figure works as a true iconic representation which makes discursive apprehension meaningless.

All these observations, which can be made anytime and anywhere in curricula, reveal the intrinsic difficulties of mathematical visualization. The intricacy of mathematical visualization does not consist in its visual units — they are fewer and more homogeneous than for the images — but in the implicit selection of which visual contrast values within the configurations of units are relevant and which are not. Here is the representation barrier specific to learn visualization in mathematics. Is it really taken into account in teaching?

Very often one believes that to learn how to construct graphs or geometrical figures is enough to learn visualization in mathematics. Moreover, in this kind of task students get satisfactory results. But any such a task of construction requires only a succession of local apprehensions: one needs to focus on units and not on the final configuration. In other words, a student can succeed in constructing a graph or a geometrical figure and being unable to look at the final configurations other than as iconic representations. That is easy to observe and to explain.

Constructing a graph requires only to compute some coordinates and to plot a straight line, a curve: one goes ever from data tables, or from equations, to graduated axis. But visualization requires the opposite change: one must go from the whole graph to some visual values that point to the characteristic features of the represented phenomenon or that correspond to a kind of equation and to some characteristic values within the equation. Therefore, visualization causes the anticipation of the kind of equation to find out. And this gap between local apprehension and global apprehension that can exist to the end of the construction is more important for geometrical figures than for graphs. The reason is that from a geometrical figure we have not one but many possible configurations or subconfigurations. And the relevant configurations or subconfigurations in the context of a problem are not always those highlighted at first glance. What we called above a heuristic deficiency is like an inability to go further from this first glance. What reason is it due to? Teaching or some cognitive way of working?

Visualization and figural processing

In order to analyze any form of visualization there is a key idea: the existence of several registers of representation provides specific ways to process each register. Thus, if geometrical figures depend on a register, that is, on a system of representation, we must obtain specific visual operations that are peculiar to this register and that allow to change any initial geometrical figure into another one, while keeping the properties of the initial figure. What are these visual operations?

Three kinds of operations can be distinguished according to the way of modifying a given figure (Duval, 1988, pp. 61-63; 1995a, p.147):

— The mereologic way: you can divide the whole given figure into parts of various shapes (bands, rectangles...) and you can combine these parts in another whole figure or you can make appear new subfigures. In this way, you change the shapes that appeared at the first glance: a parallelogram is changed into a rectangle, or a parallelogram can appear by combining triangles... We call «reconfiguration» the most typical operation.

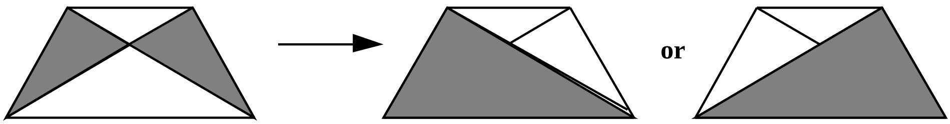


Figure 6. Figural processing by reconfiguration. Apprehension of this transformation within the only starting figure can be inhibited by the visual difficulty of double use of one sub-figure. But the starting figural frame is not changed like in the examples in Figure 3.

— The optic way: you can make a shape larger or narrower, or slant, as if you would use lenses. In this way, without any change, the shapes can appear differently. Plane figures are seen as if they were located in a 3D space. The typical operation is to make two similar figures overlap in depth (Duval, 1995 b, p.187): the smaller one is seen as it was the bigger one at the distance.

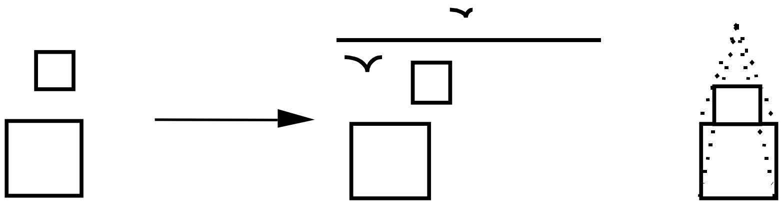


Figure 7. Figural processing by overlapping in depth of two similar figures.

— The place way: you can change its orientation in the picture plane. It is the weakest change. It affects mainly the recognition of right angles, which visually are made up of vertical and horizontal lines.

These various operations constitute a specific figural processing which provides figures with a heuristic function. One of these operations can give an insight to the solution of a problem. We call it the operative apprehension of a given figure. It is different both from perceptual and discursive operation.

Operative apprehension is different from perceptual apprehension because perception fixes at the first glance the vision of some shapes and this evidence makes them steady.

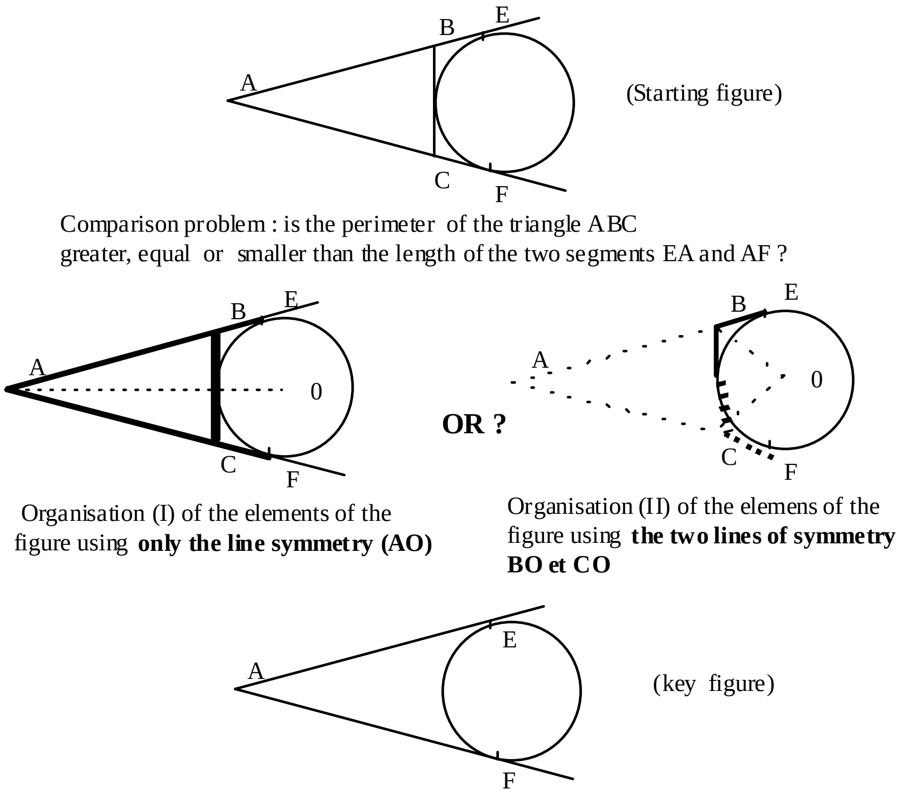


Figure 8. The perception of the starting figure highlights the shape organization (I) and makes it steady. But solving the problem requires the apprehension of the shape organization (II). Changing the perceptual apprehension of (I) into a perceptual apprehension of (II) constitutes a non-natural jump, because the symmetry axis AO sets forward the triangle within which the side BC is like an indivisible visual unit. Changing (I) into (II) requires looking at BC as a configuration of two segments! Moreover, the starting figure can be constructed without having to take into account the shape organization (II) with BO and BC as symmetry axis. Less than fifty per cent of 14-year-old students succeeded such a jump. And key figure does not help them for that (Mesquita, 1989, pp. 40, 68-69; Pluvinage, 1990, p. 27). However, by changing just a little the problem statement, and therefore the starting figure, all students can succeed: by naming I the point of intersection between AO and BC and by asking them to compare BI and IC, students are led to look at BC as a configuration of two segments. In that case, the statement of the problem becomes a congruent description of the subconfiguration (II), and

geometrical visualization is reduced to an illustration function (Duval, 1999). But the learning problem is bypassed. A true didactical approach requires to embrace the whole range of variations of the conditions of a problem and to bring out the various factors that make them clear. It is only on the basis of students' knowledge that teachers can organize learning sequences.

In operative apprehension, the given figure becomes a starting point in order to investigate others configurations that can be obtained by one of these visual operations. In this respect, operative apprehension can develop several strings of figures from a given figure. According to the stated problem, one string shows an insight to the solution. The ability to think of drawing some units more on the given figure is one of the outward sign of operative apprehension. Now we can pose well the problem of heuristic deficiency: why perceptual apprehension does not ever lead to operative apprehension? For each operation, we were able to identify visual variables that trigger or inhibit the visibility of the relevant subfigure and operation within a given figure. And we were able to define the conditions of their influence on operative apprehension. Even the use of key figures in problem solving depends on these visual variables. Therefore, it would be naïve to believe that providing students with key figures would help them in problem solving. At the least change in the starting figure, most students do not recognize the correspondence with the key figure anymore. The visual variables must be taken into account in teaching. Their study opens an important field of research in order to understand the way cognition works for visualization in geometry (Duval, 1995a, pp.148-154; 1998, pp. 41-46).

Operative apprehension is independent of discursive apprehension. Vision does not start from hypotheses and does not follow from mathematical deduction. Otherwise, geometrical figures would not perform a heuristic function but only an illustrative function (Duval, 1999). That is the blind spot of many didactical studies. They do not differentiate between visualization and hypotheses, which depend on two heterogeneous registers of representation, and they subordinate the way of working of visualization to the way of working of deduction or of computation. In fact, shape recognition is independent of shape size and of perimeter magnitude. For example, when hypotheses include numbers as measures of sides or segments, operative apprehension is neutralized and the figure fulfils only an illustrative or support function. We can have even a conflict between the figure and the measures leading to a paradox. The most well known case is the reconfiguration of an 8 x 8 square into a 5 x 13 rectangle, within which a parallelogram is perceptively reduced to a diagonal.

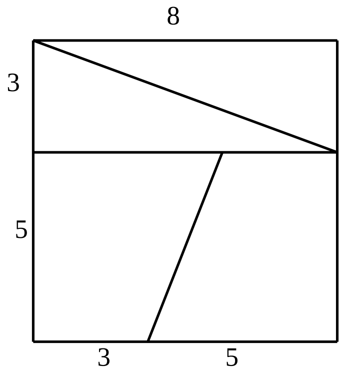


Figure 9.

Visualization consists only of operative apprehension. Measures are a matter of discursive apprehension, and they put an obstacle in the way not only for reasoning but also for visualization. Usually, the introduction of «geometrical figures» runs against this fact. Mathematical tasks are conceived as if the perceptual, discursive and operative apprehensions were inseparable! And the general outcome for most students is the inhibition of operative apprehension and a lack of interplay between perceptual and discursive apprehension.

Transitional visualization and development of the coronation of registers of representation

There is an introspective illusion that often distorts the analysis of mathematics learning processes. What is simple from a mathematical point of view appears also simple from a cognitive point of view when we are becoming experts. In fact, more often than not, what is taken as mathematically simple becomes cognitively evident only at the end of learning (Duval, 1998c). That is why assuming these simple-evident conditions cannot be taken as a starting point for learning and teaching. As I said above in "The learning of mathematics and progressive coordination among registers", learning mathematics implies the construction of this cognitive architecture that includes several registers of representation and their coordination. Thus geometrical figures used to solve problems involves some ability in operative apprehension and awareness of how deductive reasoning works. Students do not come into such apprehension and awareness by themselves. Moreover, some coordination is required between operative apprehension, discursive apprehension and deductive reasoning. In other words, geometrical activity requires continual shifts between visualization and discourse. In order to achieve such coordination another kind of visualization is required.

The introduction of graphs in proof learning is well known since their use with computer tutor (Anderson *et al.*, 1987). This example is interesting because it shows the hidden cognitive complexity of any visualization. In front of that use we must ask two questions:

– Firstly, what can be visualize from any propositional graph?
– Secondly, what kind of task makes the students able to understand proof by means of visualization?

The answer to the first question seems easy. «Proof graphs» display the whole deductive organization of propositions like a tree structure. But from that, one does not visualize how such organization works. The essential point is not visible on a graph: each connection is only based on the status of the connected propositions, and we have three kinds of deductive status. And in order to be able to become aware of this point, one must succeed at least once in constructing a whole proof graph. That concerns to the second question, we find two kinds of task: to construct oneself the whole graph or to find out forward and backward paths from hypotheses to the to-be-proven statement, which are already given at the top and at the bottom of screen.

In Anderson's research, proof graph was used to provide heuristic help «during problem-solving». Hence the second kind of task was chosen. As to what graph is expected to visualize, it is mainly «a hint in the form of suggesting the best nodes from which to infer» (Anderson *et al.*, 1987, p.116). In other words, proof graph must focus attention on the new step to find out in order to progress. This way of using a graph turned out to be disappointing. And it is easy to know why. On the one hand, a graph cannot perform a heuristic function in geometry problem solving: that depends on figural processing. On the other hand, if the goal is to understand how deductive organization of propositions works, the task has the crucial point bypassed. In fact, proof graph becomes a helpful visualization for the students only when they have to construct it by themselves according to rules explaining how to shift the status of propositions into visual values. Then proof graph can visualize **not a particular proof of the to-be-proven statement, but how any proof works** (Duval, 1989, 1991). To understand how a mathematical proof works and why it does not work, as other language reasoning is the necessary condition for being convinced by a mathematical proof. We are there on the crucial threshold of learning in mathematics. Under very specific conditions, proof graph is a kind of visualization that allows one to explore and to check our own understanding of deductive reasoning. Once this threshold is crossed, proof graph becomes useless and interplay can start between deductive reasoning and geometrical figures. Proof graph is a transitional visualization that furthers register coordination.

It may more evident for proving than for any other mathematical activity, that what is mathematically simple is cognitively complex and can be understood only at the end of learning. Heterogeneous ways of working, specific to each register, must be first learnt in parallel. Is it possible to lead frontally all the training that this requires? For experimental reasons, our researches have aimed separately at each register and we have identified some conversion problems. But, recently, an attempt to join all the aspects involved in proof activity has been made within a computational environment (Luengo, 1997). And this attempt seems to be promising.

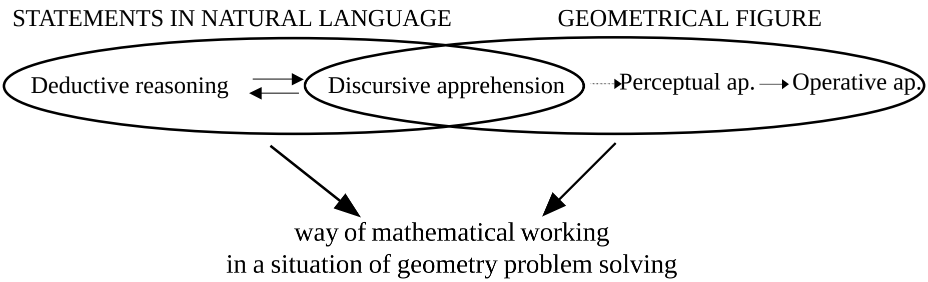


Figure 10. Skills and coordinations to be developed in mathematics education. Most often students confine to perceptual apprehension and reduce discursive apprehension to simple denomination of recognized shapes.

Conclusion

There is no direct access to mathematical objects but only to their representations. We cannot compare any mathematical object to its representations, as we can compare a model with its photo or its drawn image. This comparison remains attached to epistemological patterns to analyze knowledge (Platon, *Res Publica* VI 510 a-e, X 596 a-e), and it cannot be relevant in mathematics and in teaching of mathematics. We can only work on and from semiotic representations, because they are the means of processing. At the same time we have to be able to activate in parallel two or three registers of representations. That determines the three specific requirements in learning of mathematics: to compare similar representations within the same register in order to discriminate relevant values within a mathematical understanding, to convert a representation from a register to another one; and to discriminate the specific way of working in order to understand the mathematical processing that is performed in this register. This is not the familiar way of thinking. And it is the reason why an anchorage in concrete manipulations or in applications to real situations is often pursued in order to make sense of the activity proposed. But that comes often to a sudden end, because it does not provide means of transfer to other contexts. Besides, representation becomes usable in mathematics only when it involves physical things or concrete situations. We find the same problems with visualization use, whatever the register be, it focuses on a synoptic way, organization of particular units and it does not show objects as any iconic representation. One does not look at mathematical visualization as one does at images.

Mathematical activity has two sides. The visible or conclusive side is the one of mathematical objects and valid processes used to solve a given problem. The hidden and crucial side is the one of cognitive operations by which anyone can perform the valid processes and gain access to a mathematical object. Registers of semiotic representation and their coordination set up the cognitive architecture which anyone can perform the cognitive operations underlying mathematical processes. That means that any cognitive operation, such as processing within a register or conversion of representation between two registers, depends on several variables. To find out what these variables are and how they interact is an important field of research for learning mathematics. Indeed, from a mathematical point of view only one side matters, from a didactical point of view the two sides are equally essential. In concrete terms, any task or any problem that the students are asked to solve requires a double analysis, mathematical and cognitive: the cognitive variables must be taken into account in the same way as the mathematical structure for “concept construction” (Duval, 1996). But for that, teachers must know themselves these variables and take them into account as didactical variables. They must be able to analyze the function that each visualization can perform in the context of a determined activity (Duval *et al.*, 1999). We are here in front of an important field of research. But it seems still often neglected because most didactical studies are mainly centered on one side of the mathematical activity, as if mathematical processes were natural and cognitively transparent. There is no true understanding in mathematics for students who do not “incorporate” into their “cognitive architecture" the various registers of semiotic representations used to do mathematics, even those of visualization.

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Proof-Analysis and the Development of Geometrical Thought

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ABSTRACT. Mathematical demonstration cannot be a merely logical and thus tautological process and cannot consist in causal compulsion or reaction, but must rather be conceived of as a semiotic or representational process, because otherwise a paradox of proof arises (how can by means of demonstrations or proofs new knowledge be reached if these consist in merely reducing the new to the old and already known). The essential features of the activity of proving consist in seeing an *A* as a *B*, or representing *A* as *B*: *A* = *B*. Thus the "equation" *A* = *B* is constituted by resemblance or analogy. It is a representation or transformation, whose significance is in general merely analogical and open to failure. There is no necessity involved in this process, as Piaget believes, although Piagets description of its stages, as intrafigural, interfigural and transfigural or structural are useful as an overall framework.This claims will be illustrated by means of an analysis and proof of the theorem about the "Eulerline" of a triangle and its generalizations, finally demonstrating that this theorem is a special case of Desargues theorem.

Piaget's characterization of intrafigural, interfigural and transfigural thought in the historical development of geometry

Piaget characterizes the historical development of geometry as a succession of three periods of *intrafigural, interfigural*, and finally, *transfigural* or *structural* thought. Describing the three developmental stages Piaget writes:

> Geometry begins with EUCLID – with a period during which the object of study is geometrical properties of figures and solids seen as internal relations

> between elements of figures and solids. No consideration is given to space as such, or consequently, to the transformations of these figures within a space that contains them all. We shall call this period *intrafigural* – an expression already used in developmental psychology to account for the development of geometrical concepts in the child.

The following period is characterized by efforts to find relationships between the figures. This manifests itself specifically in the search for transformations relating the figures according to various forms of correspondence. However, these transformations are not yet subordinated to structured sets. This is the period where projective geometry predominates. We shall call this period *interfigural*.

Following next is a third period, which we call *transfigural*. It is characterized by the predominance of structures. The work most characteristic of this period is the *Erlanger Programm* of Felix Klein" (Piaget/Garcia 1989, chapter III, conclusions, p. 109)

We shall discuss first of all an example to illustrate Piaget's conception of geometrical development and to provide a particular interpretation of it. The example concerns Euler's theorem, stating that the concurrency points of the perpendicular bisectors, the medians and the altitudes of any triangle were collinear.

> *Theorem 1 (Euler 1765): The orthocenter O, centroid CG and circumcenter M of any triangle are collinear. The line passing through these points is called the Euler line of the triangle. The centroid divides the distance from the orthocenter to the circumcenter in the ratio 2:1.*

By analyzing the proofs of this theorem as presented by textbooks of elementary geometry (see for example: Coxeter/Greitzer 1967, p. 18), one might hit upon the idea that the theorem is not at all about the relations between different properties of a single triangle, but rather is an affirmation about the relation between on and the same property (namely the location of the orthocenter) of two different triangles (the original one and its medial triangle, the triangle formed by joining the midpoints of the sides of the given triangle). In this manner we proceed from the intrafigural to the interfigural perspective (Figure 1).

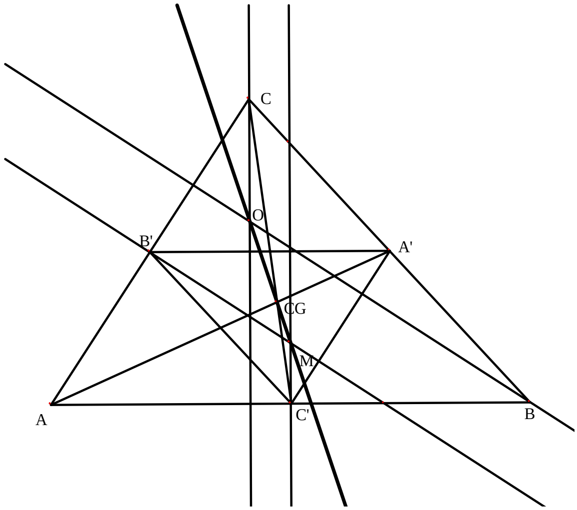

Figure 1.

Now these two triangles are related to one another by means of a rotation of 180 degrees about the centroid of the given triangle and an additional shrinking of the rotated triangle towards the centroid to half its size. Thus the image point X' of any point X of the plane under this transformation lies on the line that contains X and the centroid, the center of the transformation, to the other side of the centroid and half the distance from it. We shall call such a transformation a DST, just for the sake of convenience.

Let us pause for a moment and reflect about what has happened. The argument of our new transformational proof rests on analogy or on some principle of continuity according to which similar things in the givens are mapped on to similar things again. Such a view immediately opens the doors to further generalization. Whereas the traditional synthetic or "Euclidean" proofs, for instance, use all the premises of the theorem in the most intricate and ingenious manner (another such proof is presented in Figure 7) the new proof does not. It must be considered to be a general proof scheme, rather than a particular proof. Our interfigural perspective, in fact, not only yields a proof of the original theorem, but also proofs of some other ones. The theorem about the nine-point center, i.e. the fact that the center F of the Feuerbach circle, or nine-point circle, also lies on the Euler line, can immediately be established in a like manner (Figure 6) by observing that F is just the circumcenter of the transformed triangle A'B'C'.

Finally the proof also provides a first generalization of our original Theorem 1 (Figure 2), because the consideration of the intersection point of any cevians of the given triangle, and not just the orthocenter, leads to a similar theorem.

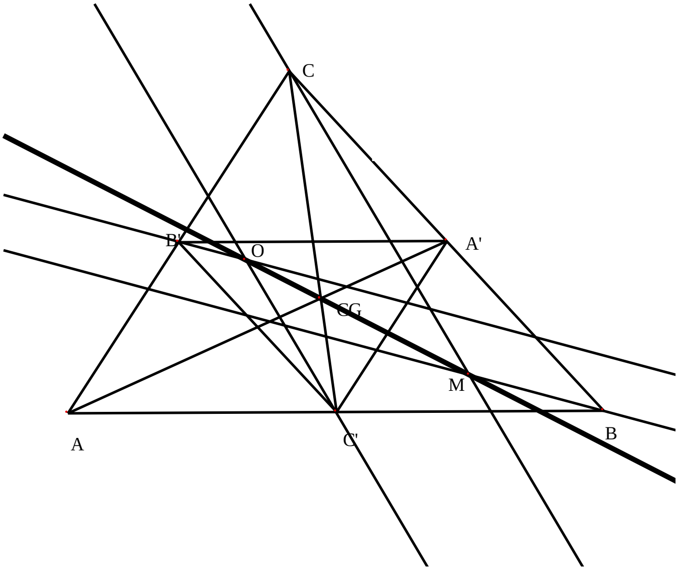

Figure 2.

Euler's theorem 2: Take any two cevians and their point of intersection (for the sake of visual clarity, we shall in the sequel use only two lines of each type, but two already determine the important collinear points) and construct parallels to these lines through the midpoints of the corresponding opposite sides of the given triangle as well as their intersection point. Then the line through these two intersection points contains also the centroid of the triangle.

Looking once more at our proof by means of a geometrical transformation of type DST we realize that the factor 2 can also be substituted by any other number, that is the two similar triangles in question need not necessarily be related in the ratio 2:1. This means that the centroid CG of the given triangle which is also the center of the DST may be substituted by any other "center of gravity", as long as the cevians CC' and BB' passing through it intersect the sides of the triangle at points C' and B' such that the line B'C' through these points remains parallel to the third side BC of the given triangle (Figure 3). Stated differently, the two similar triangles should have parallel sides. Are we not already now taking a glimpse of a Desarguen configuration?

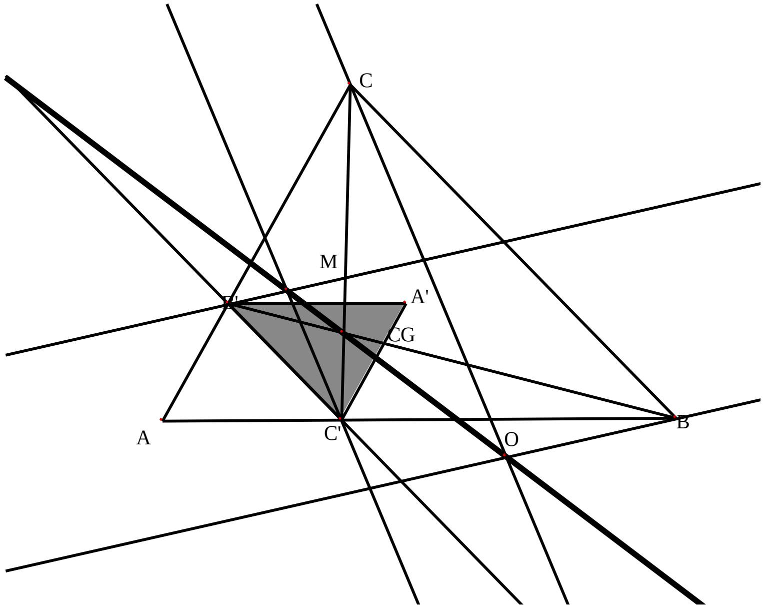

Figura 3.

Our configuration essentially consists of three pairs of parallel lines. Mark two points on both lines of the first pair (C and B respectively C’ and B’) and let the other two pairs of parallel lines pass through these points respectively (Figure 4). The pairs of parallel lines thus are determined by AC and A’C’; AB and A’B’; AC and A’C’. We thus arrive at a Desarguen configuration, were the respective sides of the two triangles ABC and A’B’C’ in perspective intersect on the line at infinity, that is remain parallel. The existence of the “Eulerline” AA’ is now guaranteed by the inverse of Desargues theorem, stating that if the intersections of corresponding sides of two different triangles ABC and A’B’C’ (or the prolongations) lie on the same line, then the lines through corresponding vertices all pass through the same point CG. Now in our case this same line is the line at infinity, but a simple transformation of the coordinate system gives the general statement, thus generalizing once more our original theorem.

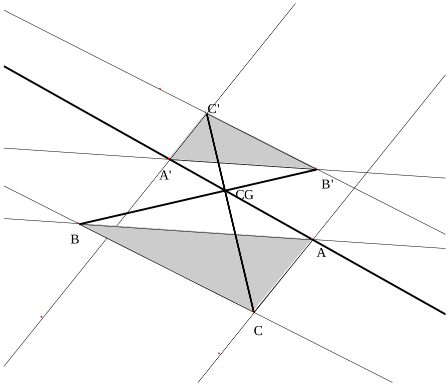

Figura 4.

We gain also, interpreting Figure 1 in the light of these new insights, another proof of our original theorem. The triangles CBO and C'B'M have parallel sides. Define CG as the intersection of the lines CC' and BB' then Desargues' theorem says that the line joining the third vertices of our triangles, namely O and M, also passes through this point of intersection CG (Figure 5). There certainly might exist extremely talented persons, who would have immediately hit upon this new proof idea, thereby shortcutting the whole process of generalization. But such a thing might not happen frequently and our proof and the diagram on which it was based, marking a more natural path rather impedes such a radical idea. That means that there exists a logic of abduction and generalization, which is firmly connected with our cognitive means and representational systems.

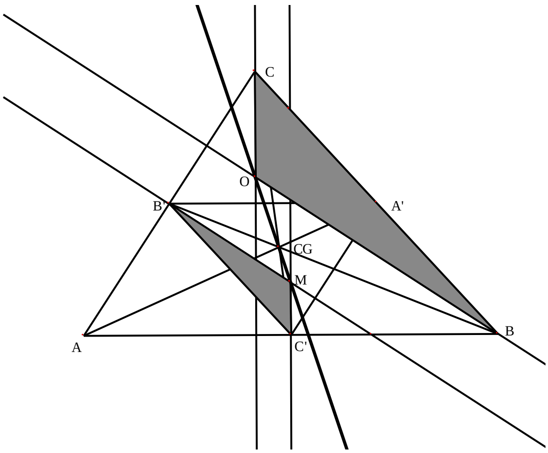

Figura 5.

Means and objects of activity in the development of cognition

Means and objects of activity are fully differentiable by their respective moments on individual cognitive activity, but they play a completely symmetric part in the development of cognition. This complementarity (difference and unity) of objects and means accounts for the emergence and dynamism of pure mathematics since the 17th century. This thesis seems to combine well with Piaget's affirmation that reflexive abstraction (abstraction reflechissante) makes up for the very mechanism of mathematical development. Only that Piaget interprets the objectivity of cognitive means exclusively in terms of formal structure.

It follows from this that there are neither absolute foundations nor universal justification processes for mathematics. Looking from different perspectives (i.e. different choices of coordinates at one and the same thing) and viewing different objects from one and the same point of view become methodologically indistinguishable approaches, as in the fusion of analytical geometry with linear algebra. Linear mappings and coordinate transformations are both described by means of matrices (coordinates). The realization of this symmetry, we believe, characterizes the Piagetian transfigural stage. But does Piaget himself see things this way?

Before entering into a description of Piaget's genetical epistemology, it seems appropriate to add several observations at this point.

It is not the problem or the geometrical situation as such which is to be classified as intrafigural, interfigural or transfigural, but rather the perspective taken and the means by which it is treated. Indeed, the transfigural proof of Figure 5 is based on exactly the same diagram as the original proofs of Theorem 1. These perspectives possibly coexist within much of geometry. A possible perspective to be taken obviously depends on prior knowledge and mathematical experience. Piaget derives from this the affirmation that the evolution of activity is based on a supposed logic. But what kind of logic is this? The generation and further analysis of proofs depends very much on how things are represented and on experimentation on these representations. Contingent factors will always influence this experimental activity. Such a logic, as demanded by Piaget, therefore depends on continuity, on a continuity, which avoids great gaps as well as radical and too daring leaps.

Our commitment to the logical over the figural and the basic reasons that logical proofs are so much preferred and are considered so important to mathematical knowledge result from the importance of the machine metaphor as well as from the fact that first-order predicate logic provides exactly the right model for our modern computer systems. Learning, development and education, require, however, not only proofs that prove but also rather proofs that provide insight. Therefore the cognition as perception metaphor is so relevant and DGS may be helpful here. In a sense visual perception must be conceived of as such a continuous series of small generalizations or abductive inferences. Perception is interpretative. “If the percept or perceptual judgment were of a nature entirely unrelated to abduction, one would expect that the percept would be entirely free from any characters that are proper to interpretations, while it can hardly fail to have such characters if it be merely a continuous series of what, discretely and consciously performed, would be abductions”, writes Peirce.

But its interpretations normally do not roam about at random because we perceive continua or general qualities first. Wandering around we might see a small brown animal and only at second hindsight we might identify it as a rabbit or a cat or a squirrel, or whatever. We perceive the general context first, and determine the details only later. All the familiar examples, which are presented in order to demonstrate the failings of perception and intuition do show that perception is relational or contextual (see for example Kline 1985 chapter IV). To perceive a fact, that means something in variant in the variation of representation, would require a sort of comparison (like it is provided by measurement, for instance). Obsession with facts then discards the figurative and relational.

Perception although depending on difference and particularity always is directed towards the continuum of the possible, towards the general and it hence essentially depends on our classifications of the phenomenological world. Mathematical ontology in particular can only be conceived of as an inexhaustible continuum of real possibilities of relations. And which of these possibilities become actualized in a certain context and at a certain point in time depends on our goals and on the means of representation and knowing.

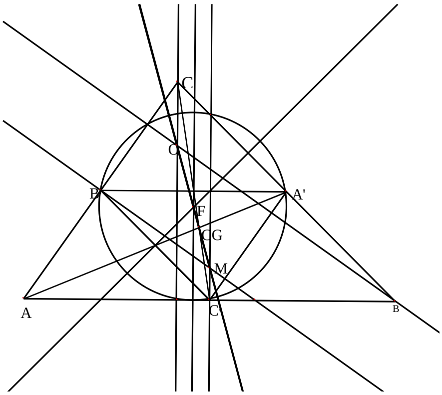

Figure 6.

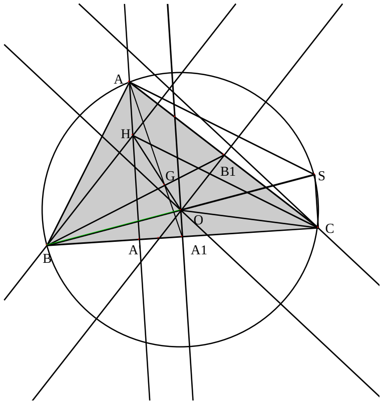

Figure 7.

Kant's fundamental insight vs. Piaget's dynamical version of Kant's espistemology

Piaget was a Kantian, he adhered to Kantianism, as he often affirmed, but to a Kantianism "that is not static, that is, the categories are not there at the outset, it is rather a Kantianism that is dynamic that is, with each category raising new possibilities, which is something completely different. I agree that the previous structure by its very existence opens up possibilities, and what development and construction do in the history of mathematics is to make the most of these possibilities, to convert them into realities, to actualize them" (Piaget in: Piattelli-Palmarini (Ed.) 1980, p. 150).

Piaget lost, however, also something of the Kantian heritage. Kant had realized, in contrast to all rationalists and empiricists of the 17th/18th centuries that we have no immediated access neither to our inner mental world nor to objective reality. Kant discovered the objectivity of the subject. Piaget appreciated this and his "dynamical" version of Kant's epistemology is nothing but a particular further elaboration of Kant's fundamental insight. Only that Piaget conceives of this objectivity in terms of formal structure and logical necessity. Let us comment on this a little further.

Kant has become famous among logicians for his affirmation that mathematics is (like metaphysics or exact natural science) synthetic, albeit *a priori*. That means mathematical judgments are essentially apodictic, like statements of fact, but differently from the latter are at the same time universal. Hume (1777, p. 76), who preceded Kant, accepting the usual view as to what makes knowledge *a priori*, discovered that in many cases which had previously been supposed analytic, and notably in the case of cause and effect, the connection was actually synthetic. But he maintained that pure mathematics is analytic or based on “relations of ideas” rather than on matters of fact. Still mathematical affirmations require, he affirmed, “a train of reasoning”, whereas “to convince us of this proposition, *that where there is no property, there can be no injustice*, it is only necessary to define the terms, and explain that injustice is a violation of property” (D. Hume, Enquiries, Section XII, Part III).

Kant, in contrast to Hume, understood that the foundational problems of mathematical knowledge presented themselves in exactly the same terms as those of the empirical sciences, because the “train of reasoning” about which Hume speaks and which makes mathematical judgments far from obvious, must as Kant believes, be represented and demands a constructive effort and an intuition. Mathematical reasoning is, according to Kant, to an essential part constructive because quite a number of predicates cannot be linked, as Hume already had pointed out, to a concept without providing additional context. For instance the idea of a triangle does not in itself contain the fact that the sum of its angles amount to two right angles. The philosopher would try, Kant writes, to analyze the concept of triangle, but: “He may analyze the conception of a straight line, of an angle, or of the number three as long as he pleases, but he will not discover any properties not contained in these conceptions. But, if this question is proposed to a geometrician, he at once begins by constructing a triangle” (A 716/B 744).

What seems essential in Kant’s description of mathematical construction is the fact that it does not proceed from concepts alone, but rather has to rely on particular instantiations of these as well. One argues, for instance, that line A is parallel to line B, or intersects it at point C etc. etc. Hintikka seems so far to have been the only author to observe this point. He writes: “...Kant’s characterisation of mathematics as based on the use of constructions has to be taken to mean merely that, in mathematics, one is all the time introducing particular representatives of general concepts and carrying out arguments in terms of such particular representatives, arguments which cannot be carried out by the sole means of general concepts.” (Hintikka 1992, p. 24).

Now it is in exactly in this manner that (semiotic) space and the ostensive indication of points in space become important. Without the “Here” and “Now” of spatio-temporal locations neither the problem of mathematical existence could have been solved nor would relational geometrical thinking have developed towards Piaget's "transfigural" stage, because it would have been difficult to conceive of one's own actions as of objective entities. Mathematics thus typically proceeds by constructing (algebraic or geometric) diagrams and by observing the relations in them, rather than by analyzing the meaning of mathematical concepts. Mathematics is a sort of semiotic activity. This diagrammatic thinking and the employment of the notion of space represent a penultimate preliminary stage of the axiomatic or structural conception of geometry. But Kant is ambiguous in his conception of mathematics.

Kant could be interpreted as describing to us the method of mathematics providing insight into the character of mathematical proofs as mere "if then"-connections. If Euclid's parallel postulate is valid, that means, if the line parallel with the opposite side of the triangle exists, then the sum of angles in the triangle is 180 degrees. This is exactly what the diagram exhibits. On the other hand Kant thinks that mathematics deals with objects that derive their generality from the rules according to which they have to be constructed. In geometry, for instance, we are concerned with the meaning of concepts like "triangle" or "straight line" in general, only in as much as we realize them constructively. As Kant says, "We cannot cogitate a line without drawing it in thought" (B 154), that is, without imagining an activity. Mathematics depends on intuitive thinking because it is an "objective" activity, an activity on objects and nevertheless it is no mere calculation.

But how can we, Bolzano asks, construct or imagine an infinite line? Kantian philosophy, Bolzano says, takes those "intuitions which shall be a quite particular addition to mathematical definitions of concepts to be nothing but an object subordinated to the definition of a concept in geometry, an object, which our productive imagination is to add to the definition provided". It has to be said, Bolzano continues, "what is demanded here may well apply to many but by no means all concepts pertaining to geometry. So, for example, the concept of an infinite straight line is also a geometric concept, which therefore also has to be explained geometrically. And nevertheless the productive imagination can certainly not create an object, which corresponds to this concept. For we cannot draw an infinite line by means of any imagination, but we can and have to think it by means of reason only" (Bolzano 1975, p. 76).

This may be true enough. Still we must use icons to determine the properties of such an object like a straight line. Take, for example, writes Kant, "the proposition: 'Two straight lines cannot enclose a space and with this alone no figure is possible' and try to deduce it from the conception of a straight line and the number two', … "(B 65). The properties of mathematical objects, like straight lines, are relational properties, in our example expressible by the affirmation that two points determine exactly one straight line, and therefore are to be constructed in intuition. An infinite straight line itself must be a symbol or sign and needs a context to be so conceived of. The infinite line itself as a relatum can only be indicated as part of a theoretical structure of which it is a part. We may use this symbol as an index, referentially, rather than attributively, only as part of a system or structure (perhaps the structure of a diagram in semiotic space; remember the method of "diagram chasing" of homological algebra).

Mathematical truths are established by means of proofs. But mathematical proofs are, according to the Kantian tradition, kind of thought experiments. Ian Mueller believes that the major obstacle to an acceptance of the interpretation of mathematical arguments as thought experiments "is the belief that such arguments cannot be conclusive proofs. In particular, one might ask how consideration of a single object can establish a general assertion about all objects of a given kind. Part of the difficulty is due, I think, to failure to distinguish two ways of interpreting general statements like 'All isosceles triangles have their bases angles equal'. Under one interpretation the statement refers to a definite totality [...] and it says something about each one of them. Under the other interpretation no such definite totality is presupposed, and the sentence has much more conditional character - 'If a triangle is isosceles, its two base angles are equal.' A person who interprets a generalization in the second way may hold that the phrase 'the class of isosceles triangles' is meaningless because the number of isosceles triangles is absolutely indeterminate" (Mueller 1969, pp. 299-300).

That means that according to this understanding there are no absolute existence claims involved neither in the relational characterization of mathematical objects nor in mathematical proofs. Mathematics operates with relative existence only. It is not by accident that the view of mathematics which is expressed by the formula that mathematics consists of "if-then" assertions, that means that mathematical activity typically consists in establishing proofs, correlates with the conviction that it is <u>not</u> reference to specific mathematical objects what distinguishes mathematics from other sciences “just as botany is distinguished from marine biology by the difference in the objects studied” (Putnam 1975, p. 2). Mathematics according to this view is characterized by a certain style of reasoning; that style of reasoning is not connected with a definite ontology but gains its objectivity from its intended applications.

Yet mathematics is an activity and there is no activity without objective context. Mathematical knowledge as long as it is the knowledge of objects, however, depends on intuition. In intuition other than in discursive knowledge something is immediately present, that means an object is provided about which one might reflect. The strength of the intuitive is to be seen in its emphasis on the acquaintance with the object, since a content has to be given from whence we can advance to knowledge. As Kant said: “In the absence of intuition all our knowledge is without objects, and therefore remains entirely empty” (A 62). “But by intuition a thing is only given to us, not apprehended“, writes Moritz Schlick (Schlick 1925/1979, p. 146).

These considerations might suggest to a realist that the object is both the starting point as apprehended by intuition, as well as the goal of cognition and that the problematic thus is situated in the developmental process itself, for instance, in the relationship between knowledge and the application of knowledge. Intuition then is a means rather than a result of cognition. This view combines with the fact that all our access to objects is mediated by representations. A mathematical object or objective relationship is given then by a class of equivalent representations. As we have seen from our proof of Euler’s theorem this kind of objectivity develops by means of generalization. What develops is our perception of objective relationships more or less implicitly given by various representations. Mathematical objects cannot be defined nor pointed at. They are universals and the universal or general is, as Peirce says, the indeterminate. It is a possible relationship between various representations. A mathematical concept, such as the concept of triangle or function, does not exist independently of the totality of its possible representations, but it is not to be confused with any such representation, either. A mathematical theory does not exist independently of the entirety of its axiomatic characterizations, and it is still not to be confused with one of them. A formal system can be represented in various ways, and still the theorems have to be invariant in their truth content with regard to changes of the representation. However, here, too, this does not mean, “there is a hypostatized entity called a formal system which exists independently of any representation” (Curry 1970, p. 30).

Now to adapt the notion of activity itself to such a view one has to understand activity also as system of means-objects relations rather than as a mere process. There is no activity without means and without objective context Mathematical cognition is essentially situated cognition. Piaget, in contrast, conceives of activity independently of any domain of objectivity and he is also neglecting the symbolic means and the role of representation in mathematics. There exists, as he states, a "radical autonomy of operational development. From the level of logico-mathematical experience, where the first pre-scientific mathematical concepts appear ... the operations are constructed by abstraction starting from the subject's general actions, independently of specific physical objects and of the subjective characteristics of the actions of individuals as such" (Beth/Piaget 1966, p. 244).

Piaget's dynamism breaks down, we believe, because of the emphasis he puts on necessity and because of his negation of objective context in consequence of this preference. Piaget in a sense realizes the problematic of directionality when he uses Gödel's incompleteness theorem to propose that the foundations of knowledge lie in the future, attempting to define an internal directional law or a gradient of the trajectory of cognitive development. Gödel's theorem, in fact, provided Piaget with an internal criterion of hierarchical order of structures or theories. The rejection of any notion of context makes "necessity" an important concept in Piaget's account of cognitive development, in which the necessary is a result of development rather than its beginning.

"The chief difficulty of genetic interpretation consists, in fact, in explaining why the constructions progressively succeed one another and in particular, why they achieve new forms. A higher structure is derived from a lower structure by means of abstraction of elements starting from the latter, but this abstraction assumes that these elements are reflected by means of new operations, which reconstruct whilst transposing them. We then have to explain how these operations are at once new and in advance determined by the lower structure. The answer is that as this structure is limited, its lacunae demand a construction, which can complete them. But there is an infinity of ways of completing an incomplete structure, and we have to explain why the one, which seems the simplest and most probable is chosen. Now Gödel's results suggest a first reply to these questions: The construction continues indefinitely because no system is self-sufficient, not as regards any other, but because it lacks sufficient internal coherence to assure its own non-contradiction. Every system must therefore proceed in the direction in which its own consistency can be reinforced" (Piaget in Beth/Piaget 1966, pp. 274-275).

Logico-mathematical knowlege and mathematica proof's

There exists a criticism of Piaget claiming that Piaget's whole structuralist explanation of logico-mathematical knowledge is wrongheaded because it ignores the central role of proof and communication in mathematics. What is argued is that Piaget' characterization of mathematics and its creation is limited by its misunderstanding of "the nature and status of proof" (Rotman 1977, 151) What Piaget seems to confuse, it is said, is the discovery and invention of structure with the proving of assertions about structure. Brian Rotman, for example, who concentrates on the semiotic and social aspects of mathematics, writes:

"What we shall argue is that Piaget's whole structuralist explanation of logico-mathematical knowledge is at fault because it ignores the central role that proof plays in mathematics. ... The central error of Piaget's structuralism is the belief that it is possible to explain the origin and nature of mathematics independently of the non-structural justificatory questions of how mathematical assertions are validated." (Rotman 1977, p. 141 and p. 144)

Rotman denies on principle that mathematics is mainly concerned with the creation of structures. And he laments that Piaget's writings contain no theory of proof while claiming that "necessity", as emphasized by Piaget, is not essential to mathematical development. Replacing a discussion of mathematical proof by an analysis of necessity is misleading, according to Rotman. Rotman then again attacks Piaget's structuralism in which structure, proof and assertion are artificially separated due to the false assumption that mathematics does not function as a language (p. 156). The problem, he asserts, is that in Piaget's constructivism it is the individual who constructs mathematics without any influence of the social environment or of language. "The most serious counter theory, in fact, to Piaget's progressive case, one that he is well aware of, is the claim that the origin and laws of thought are social and cultural products and not, in any simple sense, evolutionary ones" (Rotman 1977, p. 120).

Now the perspective of argument and proof, or so it appears, inevitably turns mathematics into a collection of statements. Proof belongs to meta-mathematics and seems to be an exercise in logic and logic has nothing to say on anything that is not a statement. According to the linguistic consensus or received view, mathematics consists of theories and theories are sets of propositions, which can be organized on an axiomatic basis, and which appear as intransparent mere *faits accomplis*. The teaching of proof in our schools treats its problem in the context of propositional logic not of mathematics. A statement, for the purposes of propositional logic, is something that is either true or false. Propositional logic is built up entirely from this one fundamental rule. Its power is also its weakness because it does not support guessing and experimentation. From a mathematical point of view the generation of fertile hypotheses and the tentative formulation of (in general only half correct) conjectures seems more important than any proof and this demands proceedings which include experimentation with objects and diagrams.

Propositional logic does not concern itself with objects. According to logicians like Frege the judgment, rather than object or concept, is the fundamental unit of knowledge. “It is enough if the proposition taken as a whole has a sense; it is this that confers on its parts also their content” (Frege 1884, § 60).

When, however, by the modern conception of axiomatics this point of view was driven to the extreme by claiming that logical consistency is sufficient for existence, Frege no longer agreed to it, refusing to accept the formalist interpretation of Hilbertian axiomatics. But Frege failed to provide a definite alternative solution, however, and when in his treatment of classes, he tried to avoid the logical complexity involved in the hierarchy of logical types by supposing that a concept or propositional function of any level determines a ground-level object, called its extension, thus collapsing the type distinctions, he ran into paradox. The assumption that the extension exists for every formula is contradictory, as Russell had informed Frege. The paradoxes of set theory arose from a conflation of two incompatible notions: set-as-one vs. set-as-many. Is a forest just a set of trees or is it something more?

If one is concerned primarily or exclusively with method and formal proof one might emphasize the importance of knowledge and propositions, rather than that of activity and objectivity. Out of such a perspective mathematics seems not to have changed since the days of Euclid and Euclid’s *Elements* have in fact preserved their exemplary status, at least with respect to mathematical method. This is particularily true with respect to number: Euclid’s proof for example that the square root of 2 cannot be a fraction is valid in exactly the form Euclid gave to it. “The mathematical method, as presently used, probably would not appear strange to the Greeks” writes S. Ulam (1969, p. 2), and he continues, “However, the objects to which mathematical thought is devoted today have been vastly diversified and generalized”. It obviously follows from these statements, in contrast, that any genetical perspective on mathematics has to be concerned with mathematical objects and with the development of mathematical content.

Problems do, however, not produce the means to their solutions out of themselves. Modern mathematics even obtains its own dynamics in no small part from applying theorems and methods which at first glance have nothing to do with the problems at hand. Such that, what becomes an object of mathematics is certainly conditioned by internal as well as external forces, by the methods as well as goals or motives of mathematization. Proof-analysis may show that a proof really does not depend on the totality of characteristics of the objects involved and thus may lead to a weakening of the premises. Proof is not concerned with the objects as such, but rather with relationships between certain characteristics of possible objects. The new formulation of the situation in question may suggest a different theorem. Thus proof analysis leads to reconstruction and generalization, as well as to new proofs, which again may trigger another generalization etc. We have already seen this in the first part of this paper.

On this account mathematical objectivity resides in the deductive organization of mathematics. It is proof that cements communication in mathematics. But proof as such does not produce any dynamics independently of the dialectics of problems and means as exemplified in proof analysis. If we ignore, as Kant warns us, the actual proof activity and the process of proving, we may easily become convinced that mathematics has no objects at all as well as that deductive method and proof make up its essential characteristics. This received view is responsible for the continuous and static impression of mathematics and for the exemplary character of Euclid's Elements. But even Euclid's geometry is mainly constructive, and it is so in the service of learnability and intelligibility (Lachterman 1989, p. 110).

Mathematics viewed as an activity has changed tremendously, and this is aptly indicated by Piaget's three levels. But Piaget conceives of mathematical activity only in abstract or formal terms because he seems obsessed with necessity. Activity is relevant to him only in so far as actions form structures, which may be characterized formally. Piaget believes "that evidence develops in parallel with the emergence of mathematical structure, that is, with the recognition of abstract relations independently of the particular objects between which the relations hold" (Castonguay 1972, p. 87).

According to Rotman, "the claim that an assertion is necessary cannot be separated from how the assertion gets its justification" (p. 145). Piaget would deny that, and in fact, he shies away from proof because of the conventional and a-logical aspects that inevitably pervade linguistic behavior. Piaget's emphasis on necessity is in its turn, due to the influence of Kant, Rotman claims. Piaget writes about the epistemic subject, which may be seen as a mere deviant of Kant's "transcendental Subject" and furthermore admits that decentration of this individual construction is necessary through the cooperation with others.

But Rotman is not satisfied by Piaget's invocation of the epistemic subject. "It is just as reasonable to suppose, in effect, that co-ordination of viewpoints is a matter of explicit justified argument about public entities and not, as Piaget insists, a question of the inner necessities operating within an individual mind" (Rotman 1977, p. 154). Rotman worries about Piaget's conception of social generality, about the idea that "what is common to all subjects rests ultimately on the assumption that mathematics results from individuals and then becomes, in some sense, social" (p. 136).

There seem several quite different issues involved here

First, the affirmation that mathematical generality and necessity is something foreign to individual construction, something that has to be brought about by an extra effort based on logic and language. We have already implicitly dealt with this allegation when characterizing mathematical proof as a sort of thought experiment. It has also to be observed here that no social theory of mathematics can be developed without extensive exploration of the interaction between mathematics and technology and technological construction. Furtheron, movements and constructive actions are per se general, that means, not specific traits of particular individuals. The action itself and the mere perception or imagination of that action activate, for instance, the same parts of the brain. There exists in neurophysiology a theory of "mirror neurons", which have the characteristics that there is a "strict congruence between observed action effective in triggering the neural visual response and the executed action effective in driving the motor response. In other words, the observed action performed by another individual evokes in the observer the same neural pattern that occurs during the active execution of that action" (Gallese, in Th. Metzinger (Ed.) 2000, p. 327).

Secondly Rotman seems to believe that coordination and unification, rather than intellectual pluralism and variety of perspectives is the most important goal of mathematics and science. One might, in opposition to such a view, claim that pluralism of perspectives rather than unification and strict coordination is the basis of scientific and mathematical progress. Modern mathematics and science are a child of specialization and social division of labor. Pure mathematics in particular arose from an explosive growth of mathematical activity which occurred around 1800 and that, in its sources, may be summarily characterized by stating that for the first time in the history of mathematics a great number of connections between apparently very different results and problems was discovered.

But Rotman is right with respect to one central point, namely he is right when claiming that constructivism is not sufficient to build mathematical theories as for this it is always necessary to understand that seemingly different representations do essentially mean the same. Rotman himself indicates the example of Descartes' invention of analytical geometry "which linked the previously separated subjects of algebra and geometry ... Thus any proof of isomorphism, for example, establishes a relation between two different descriptions of a structure" (p. 157).

Mathematical knowledge consists in chains of "if-then" statements and consequently considers objects only with regard to the consequences they might have for the process of mathematical reasoning. Hence, mathematics is foremost concerned with how objects could be introduced into mathematical reasoning or theory. Mathematical objects are intensional objects in the first place, that is objects whose criteria of individuation are to be seen in the specific way they are introduced into theory by means of certain representations. Two mathematical objects can be extensionally identical, but intensionally different by being presented differently. Just like other sciences, however, mathematics is interested in obtaining objective insights, and hence in extensional objects. This is why mathematical theorems as a rule have the form of equations $A = B$. One might even be justly led to defining the essential features of an act of imaginative creation in exactly these terms, by stating that they consist in seeing an *A* as a *B* : $A=B$, or "all *A* are *B*", or "*A* represents *B*" etc.

It may be difficult indeed to prove a theorem, that is to discover such an equality, a fact that had already caused Hume to concede that mathematical statements are not straightforward analytic in the sense that the predicate flows immediately from the meaning of the subject term and that had motivated Kant to call mathematics synthetic knowledge *a priori*. This difficulty implies that it is essential to mathematics to be able to represent one and the same in ever new ways. The progress of mathematical knowledge as a rule depends on a suitable, possibly quite unexpected interpretation or representation of a given problem. The pluralism of perspectives and the variety of representations are essential to a special extent for the development of mathematics, and have been emphasized as a characteristic of mathematics again and again (see for instance Putnam 1975, p. 45).

There is neither a direct manner nor a formal theory – apart from logical or syntactical rules – that guarantees that two different representations refer to the same. To establish these “equations” $A = B$ demands a construction and a synthesis as present in the discovery of the relation between electricity, magnetism and light, which were all found to be different aspects of the very same thing, which we call the electromagnetic field today. Or synthesis as exhibited by the greatest discovery of the Industrial Revolution, namely the relation between heat and motion, expressed by the theorem of energy-conservation.

Nevertheless we must accept $A = B$ as an objective relation in the first place, and hence accept also that there are mathematical objects, even thought these are generated by what Piaget had called “reflective abstraction”. There is no objectivity without objects although these objects are not given as such independently of a symbolic representation. The characterization of “mathematical objects” in semiotic terms answers a difficulty Piaget observes, namely the pluri-functionality of mathematical objects, which caused Piaget to deny that they be really existing. Mathematics no longer consists “des sortes d’objets idéaux donnés une fois pour toutes en nous ou audehors [...] (Piaget 1970, p. 88).

Now a representational “equation” $A = B$ holds, and thereby it differs from the identity $A = A$, besides the identical which is indicated by the equals sign, something different as well, suggested by the different symbols A and B. If we want to emphasize this second aspect we would perhaps better interpret the relation between A and B as a <u>transformation</u>, rather than as equality. Our problem of the Eulerline provides a very pertinent example indeed, as the proof consists in seeing that the circumcenter M of the triangle ABC is exactly the orthocenter of the triangle $A'B'C'$. We understand now that the idea of transformation in this sense is essential for passing from the intrafigural to the interfigural stage. To this Piaget would certainly agree. He would however, or so goes the critique, not adopt the other point of view namely seeing functions or transformations also as equalities or identities.

> To indicate the indispensable role of this complementarity of function or transformation on the one hand and of relation or objectively given law on the other in the development of mathematical cognition is the most important goal of this paper.

This complementarity reflects the necessity of the two different ways according to which we have to understand our concepts - attributively as well as referentially, and it is established by the complementarity of space and structure, which serve as universal cognitive metaphors. Piaget goes much too far when asserting that the very notion of space can be reduced to structure as it is a mere construction. Referring to axiomatic group theory he writes, for instance: “The associative law of the transformation groups is fundamental for the coherence of space, because if termini in group theory did vary with the paths traversed to reach them (if they were conceived of as intensional objects, my insertion M.O.), space would lose its coherence; what we would have instead would be a perpetual flux, like the river of Heraclitus” (Piaget 1970, p. 20). This is true enough, but it is the coherence of space and the possibility it offers of employing indexical and iconic representations, which provides mathematics with the idea of objective knowledge, rather than the other way around. Piaget’s constructivism, Thom, for instance, believes, is “hopelessly enmeshed in difficulties linked to the following problem: how can geometrical continuity arise from a discrete “dust” of psychological states or processes?”

The complementarity of transformation and relation applies to all areas of mathematical education. Pupils, as a rule, have difficulties with algebraic equations because they have interpreted and learned the equality sign in the sense of “yields”. This “input-output”-interpretation represents a direct understanding of the equation. The concept of equation has not yet been transformed into an object of mathematical reflection. This functional view has a strong affinity to certain standardized situations of application, which could be characterized as producing a new object out of given objects by executing given rules; or, more generally, the correct transformation of a given initial state into a desired final state. Even elementary tasks, however, also require a different interpretation of an equation, an interpretation that treats the equation as an independent concept. How would one possibly treat $x+3 = 8$ as a function?

Cognitively the continuity principle has always played an important role in establishing this complementarity of relation and transformation. This principle has after all been introduced into mathematical reasoning by people, like Desargues or Poncelet, who tried to mathematize perspective drawing. The continuity principle is always dependent on the representation at hand and it helps in discovering the “laws” governing it. It is, for instance, with respect to our example of the Eulerline not sufficient to observe that this theorem, being about the collinearity of three points, belongs to the context of projective geometry. One has rather to start from the observation of concrete diagrams and from some concrete (proof) idea, which inevitably will at first deal with the objects as described in the statement of the theorem. Development must be continuous, that is, has to proceed by sufficiently small steps, the choice of which seems not totally determined. An idea can only be affected by another idea in continuous connection with it. All reasoning is based on continuity although in general not all the infinitesimal steps are to be spelled out explicitly.

And, there is no necessity and no guarantee of success. As Peirce states it: "The principle of continuity is the idea of fallibilism objectified. For fallibilism is the doctrine that our knowledge is never absolute but always swims, as it were, in a continuum of uncertainty and of indeterminacy. Now the doctrine of continuity is that all things so swim in continua. … But fallibilism cannot be appreciated in anything like its true significancy until evolution has been considered. … Once you have embraced the principle of continuity no kind of explanation of things will satisfy you except that they grew." (CP 1, pp.171-175).

Now Dynamic Geometry Systems (DGS) are apt to make the principle of continuity operative and thus to foster the growth of fertile hypotheses. Representational systems like DGS, having revitalized this principle, play a very important role in cognitive development because they realize an intimate and indissoluble interaction between observation and reasoning.

Descartes' idea of analysis and proof

Piaget points out the importance of the concept of transformation for the development of geometrical thought, and he understands Descartes algebraization of mathematics as the essential force behind it. "It was to require a long period of uninterrupted work in algebra and infinitesimal calculus [...] to finally come to a conceptualization of the very idea of geometrical transformation without going through algebra or analysis" (Piaget/Garcia 1989, p. 106). Piaget seems to pay no attention to the fact, however, that at least infinitesimal analysis and the function concept essentially depend on the very idea of space and the principle of continuity as well. The concept of mathematical function or transformation has a double root, algorithm and objective relation, as exemplified by the regularities of nature. As we have stated as a thesis in the last section, this complementarity might be fundamentally important for the transition to the interfigural and structural stages of development.

To understand mathematical functions means to understand the complementarity of formula and relation, as well as the self-referentiality that governs its evolution, as became apparent in Cauchy's definition of a continuous function. In the mathematics of the 17th/18th centuries, discontinuous functions could not be represented, because a function was an analytical law. A geometrical curve, on the other hand, was called continuous if it could be represented by $a(n)$ (analytical) function (Euler 1748, vol. II). But this characterization proved to be incoherent.

Cauchy, after having demonstrated the inconsistency of these efforts (Grattan-Guinness, 1970), revised the whole approach on the basis of the principle of continuity, transforming mathematics into extensional theory. A function in Cauchy's or Dirichlet's sense can be seen as an equivalence class of analytic expressions or formulae, where the equivalence relation is based on the axiom of extensionality. This switch from an intentional to an extensional view has made it possible since Cauchy to single out sets of functions by certain of their properties, and in general to reason about them without representing them explicitly. For instance, instead of giving a linear function directly by $f(x) = ax$, Cauchy proves that a continuous function having the property $f(x+y) = f(x) + f(y)$ can be represented as above (Cauchy 1821, pp. 99-100). Now this kind of reasoning on the mathematical object itself came to dominate mathematics at the very time when proof-analysis became its basis.

Still, strictly speaking we cannot operate on the object or the concept as such, because it has to be represented anyway to become accessible. A concept is not to be conceived as a completely isolated and distinct entity in Platonic heaven, but must not on the other hand be confused with any set of intended applications. Two predicates or concepts or functions (or functions of functions) are to be considered as different even if they apply to exactly the same class of objects because they influence mental activity differently and may lead to different developments. The extensional view on mathematics ignores these facts completely.

Piaget himself accepts the idea that setting up correspondences on the one hand and operational constructions on the other might be two processes "common to all fields of knowledge" (Piaget/Garcia 1989, p. 11). It seems indeed to be important to be aware of the fact that the Cartesian innovation already had a twofold nature from the very beginning, represented by the combination of number and variable on the one hand, and of space and quantity on the other; "for the extension in length breadth and depth, which constitute space is plainly the same as that which constitutes body", says Descartes (Garber 1992, p. 134). Cartesian mathematics is not algebraic in our sense, it is not "a science of pure structure", but is based on an interaction of number and geometric visualization. This duality which is, in our view, the basis of a complementarist understanding of algebra proper, (i.e.

understanding algebra as a system blending those two lines) becomes crucial as soon as we consider relations between bodies, or in Piagetian terms, if we pass from the intrafigural to the interfigural.

How mathematics was practiced and understood by Descartes “can be captured in two interconnected expressions, both to be found at the deepest stratum of the ‘Cartesian’ soul . . . : one, that mathematics is essentially occupied with the solution of problems, [...] two, that mathematics is most fertilely pursued as the ‘construction of problems or equations’ - that is, as the transposition of mathematical intelligibility and certainty from the algebraic to the geometric domain, or from the interior forum of the mind to the external forum of space and body” (Lachterman, 1989, VIII).

In the opening paragraph of his *Geometry*, Descartes describes his constructive program as follows:

> Tous les problèmes de géométrie peuvent facilement se réduire à des termes tels qu’il n’est besoin par la suite que de connaître la longueur de quelques lignes droites pour les construire. (Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain lines is sufficient for its construction).

Geometry is based on insight, and this involves two procedures, analysis and synthesis. Synthesis means construction and construction is, as in Euclid, defined in terms of means of construction (ruler and compass in the case of Euclid, or Descartes’ proportionality-compass, for example; see Figure 9), not as an algorithm or a precisely specified procedure. Therefore analysis was essential to complete the solution of mathematical problems, especially in cases were construction failed.

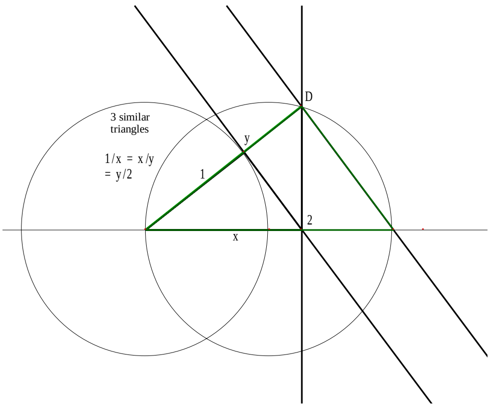

Figure 8.

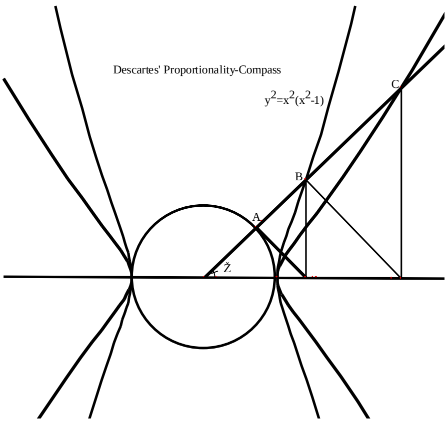

Figure 9.

Descartes entire analysis and analytical geometry was derived from his examination of proportionality. The discovery of one or more mean-proportionals through the appropriate association of known terms with unknown terms is the paradigm of ingenious discovery and construction in terms of figurative representation that Descartes explains in the *Regulae*, in particular in the second part, beginning with Rule 14. Descartes believed that applying figures to all kind of problems would be fruitful because similarity or analogy as expressed in the mathematics of proportional relations is the only way of discovering new truths.

“If someone is blind from birth, we should not expect to be able by force of argument to get him to have true ideas of colors just like the ones we have, derived as they are from the senses. But if someone at some time has seen the primary colors, though not the secondary or mixed colors, then by means of a deduction of sorts it is possible for him to form images even of those he has not seen, in virtue of their similarity to those he has seen” (Descartes 1985, Rule 14). Thus we know something either by direct perceiving it or by means of comparison.

This also shows that we know the truth in all deductive reasoning by comparison only. Thus begins the explanation of Rule 14. The reader should realize, Descartes continues, “that every knowledge which is not obtained through simple and pure intuition of a single solitary thing” results from a “comparison between two or more things”. This comparison must be constructed if it is not simple and straightforward. “The reason why preparation is required for other sorts of comparison is simply that the common nature in question is not present equally in both, but only by way of other relations or proportions which imply it. The chief part of human endeavor is simply to reduce these proportions to the point where an equality between what we are seeking and what we already know is clearly visible” (Descartes 1985, a.a.O.).

The process of knowing can come to a proper end only if there are self-evidencing truths, phenomena that reveal their meaning simply in terms of themselves. What we perceive, and what we therefore can imagine, are always only relations, which must be conceived as parts of chains of figurative reasoning, on the one hand, or simple facts on the other.

Analysis was revolutionary, and it was Descartes’ greatest achievement, as it motivated him to enlarge the class of constructive means and generalize the notion of constructability itself, thus becoming able to solve problems that the Greeks could not solve by means of ruler and compass alone. The famous “*x*” of algebra transforms the yet unknown itself into an object of activity. Cartesian mathematics proceeds from the unknown as if it were known, to its possible antecedents until arriving at premises we recognize to be true, proven, or known. But it does this by calculation and construction, rather then by mere analysis, the means of this construction being the very conditions or relations the unknown has to meet. According to Viète or Descartes, “analysis” was nothing but the first part of a method to solve geometrical problems; these problems were translated in terms of algebraic equations in which either known or unknown “quantities” occur; the solutions of such equations provided an algebraic expression of unknown “quantities” in terms of the quantities known. Such an expression had to be interpreted as a geometrical relation permitting, in the second part of the method - the “synthesis”,- to construct the unknown entities starting by data.

Thus, for instance, the so-called Delian problem, the duplication of the cube (while preserving the cube’s shape) demanded by Apollo in the oracle - a central construction problem of Greek geometry - cannot be solved with ruler and compass alone, whereas it appears as a solvable problem in Descartes because, now, in contrast to antiquity, conic sections have been accepted as legitimate means of construction (cf. Figures 10 and 11).

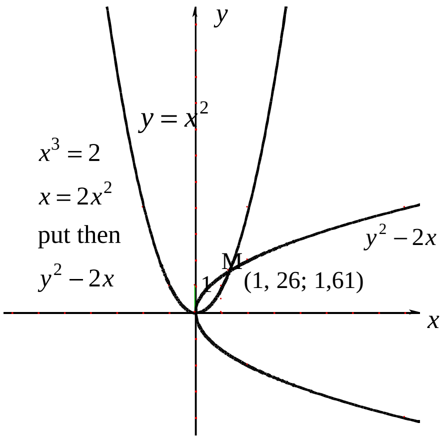

Figure 10.

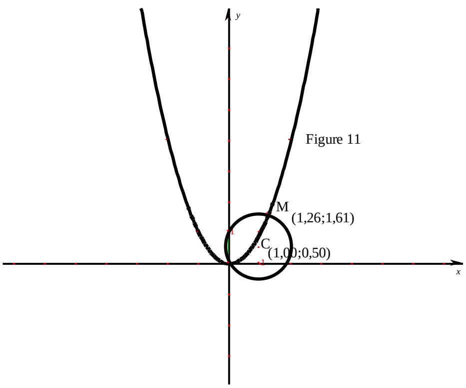

Figure 11.

It was obvious to Descartes, just as it had been obvious to the Greeks that what is to be constructed the side of the cube, actually exists. It was from there that the very impetus sprang which led to an expansion of the concept of construction. If we look at the diagram of Figure 8 – provided by Hippocrates of Chios and used in Antiquity to analyze the Delian problem – we immediately realize how Descartes came to his proportionality-compass (Figure 9), and also how essential the principle of continuity is to construct this diagram which is intended to find the essential point D. Greek mathematicians banished any reference to movement from proof. A demonstration which contained involving a moving point and thus reference to the principle of continuity was thought defective and the Delian problem, known to be solvable by such means was thus held to be unsolved. Descartes considered it to be solved because he admitted new means of construction and oriented himself towards new problems, and not because he had redefined the notion of proof. Descartes was concerned with problems and constructions rather than with theories and proofs.

It has been conjectured that Descartes "doubted the existence of a curve corresponding to an equation unless he could supply a kinematic construction for it" (Boyer 1988, p. 88). Construction, however, furnishes existence proofs only in as much it is realized as embedded in space.

In his secular periodization of mathematical history, Pierre Boutroux' focuses, when describing mathematical development from Antiquity to the 17th century, rather more on the conservatism of the mathematicians of the classical age, on their preoccupation with method and means of mathematical activity.

> "Between the Greek conception of mathematics and the completely different conception of algebraic synthetists there was a notable similarity. Both assume some kind of pre-established harmony between the goal and the method of the mathematical science, between the objectives this science pursues and the procedures which permit it to attain these objectives" (Boutroux 1920, p. 193).

And, Boutroux continues, "Descartes assumes that once the principles of analytical geometry had been established, the consequences would have to follow naturally by way of algebraic transformation and combination" (Boutroux 1920, p. 109), thus that for the entire algebraic approach the objects considered are mere compositions or composites of elements, "containing no more and no less than the elements themselves; accordingly, the goal that one pursues will always be determined by the means applied [...]"

Descartes, for instance, asks us to gradually ascend, after we have algebraically studied the curves of second order (or conic sections), to ever more "complicated' curves of ever higher order. The problem thus posed had from the very outset been cast in the mould of the algebraic composition [...] Descartes, in sum, had been content with sketching a program; he had shown a path for mathematics; forthwith, it was sufficient to follow this path to arrive precisely at the procedures of calculus which developed at the end of the 17th century" (Boutroux 1920, 193, p. 111).

Even today it is possible to recognize, without too much reflection, Boutroux goes on, "that the difficulty encountered was not new and basically the same which had so much occupied and vexed the Greek geometers. The confrontation the algebraic functions with the non-algebraic functions (transcedental functions) raises a problem comparable in all respects to that posed by the theory of irrational numbers as soon as it appeared. Now how was this latter problem solved? By taking recourse to the methods of approximative calculus, more precisely by appealing to the idea of the *arbitrarily large approximation* or of *convergence*. It is this idea which is at the basis of the method of exhaustion as we find it in the writings of Euclid or Aristotle (Boutroux 1920, p. 105).

Pierre Boutroux thinks that the analytic conception of mathematics emerges not before the 19th century, inasmuch this is the first point where mathematics begins to become purely conceptual, seeming no longer to be constrained by its means.

Now Boutroux seems to be both, right and wrong concerning the conception and historical role of Descartes. He seems to be wrong in claiming that Descartes mathematics is completely synthetic and totally constrained by its means of representation. In principle, it is true, Descartes continued to view the problem of mathematical construction within the frame of antiquity. Properly speaking, he did not extend this frame but *quantitatively*. Descartes' conception of method and objectivity, however, was not so much inspired by Euclid's deductive procedure, but by the example of invention and discovery in geometry and the mechanical arts.

Differently from Fermat, Descartes did not regard an algebraic equation as an adequate definition of a curve, but rather used algebra only as a means for classification and ordering of curves because Descartes was not interested in curves as such, as objects of investigation, but wanted them rather as means of construction. To construct these curves, he made use of various mechanical devices. He would admit only such curves as can be described by "a regular and continuous motion". Descartes also wished to systematize geometry on a higher level so that there should be no limitation on the degree or dimensionality of a problem. Descartes, in fact, desired to achieve something the Greeks had not attained, that is to introduce a common perspective into the totality of mathematical knowledge, and to create the basis for further generalization by this systematization.

Descartes uses the structure of arithmetic, and in particular the fact that “the entire arithmetic contains but four types of calculations”, to classify all the problems of geometry and to present them consistently. After having explained the parallelism of geometric construction and of arithmetic operation by means of some diagrams, and after having stressed that “all problems of ordinary geometry can be constructed by exclusive application of the few things contained in the figures explained”, he proceeds to criticize the “Ancients” for obviously not having realized this, “as they otherwise would have spared the effort to write so many voluminous books on it, books in which the order of their theorems alone shows that they were not in possession of the true method which supplies all these theorems, but merely had picked up those which they had accidentally come across” (Descartes’ *Geometry*, our translation).

In this role, algebra functioned like a logic, an idea that Leibniz, rather than Descartes, because of his own emphasis on the priority of form over the concrete content, was to develop. “Algebra is specifically a matter of getting rid of some content. Hence in virtue of Descartes’ discovery, geometrical proof can be conceived as purely formal. Leibniz thought that Descartes had stopped short, and did not see his way through to a completely general abstract Universal Characteristic” (Hacking 1980). Leibniz wanted his characteristic because he thought that truth is constituted by proof. Descartes did not believe this. “Descartes wanted good ways to find out the truth and was indifferent to the logical status of his methods” (Hacking 1980). But Descartes decided to proceed like he did in order to bring analysis, the method of discovery, closer to synthesis or proof. He might have done this simply by “admitting into geometry all curves given by algebraic equations, but he preferred a kinematic basis” (Boyer 1988, p. 88).

Boutroux is completely right, however, in his affirmation that Descartes, like Leibniz, did not think conceptually and structurally in mathematics, but rather remained confined within the boundaries of representationalism and synthetic or constructive method.

What does this mean? The geometers of the 19th century, like Poncelet, Grassmann, Möbius, Plücker etc. complained a lot about the artificial character of Cartesian coordinates, and they set themselves the goal “to calculate with the things themselves”. They wanted to operate on the full concept, that is, with the form or structure itself, as it is represented by a class of equivalent diagrams, rather than with a specific representation. A formal expression, Grassmann says, attains a concrete meaning "by looking for all such expressions that are equal to the one given". Through this procedure, "we get a series of concrete representations of that formal connection and the class these possible representations looked upon as a unity, like the species of a genus (not like the parts of a whole) would display the concrete concept to our eyes" (Grassmann 1844, p. 108).

A mathematical concept does not exist independently of the totality of its possible representations, but must not be confused with any such representation, either. Now this class or totality of possible representations obviously is not a determinate class at all, as the extensional or set-theoretical conception of mathematics believes. In geometry one uses the principle of continuity as a sort of heuristic orientation. Klein's Erlanger Programm represents another way of defining this class. On this account, to operate on the concept itself in geometry then means to look for those aspects which are invariant under certain transformations. This structural conception of the mathematical object is expressed in new forms of proof, as exemplified above when exhibiting the theorem about the Eulerline as a special case of Desargues' theorem and thus deriving it from the axioms of projective geometry.

Conclusion

Piaget's three stages of cognitive development - from the consideration of individual objects, to the orientation towards actions and transformations and finally to structures - seems *grosso modo* correct. Piaget, however, makes too radical a distinction between acting and perceiving and between empirical and reflective abstraction. The reason lies exactly in his structuralism.

Piaget started from the fundamental observations that operations on any set of objects can be combined to form structures in a very natural manner, whereas the objects themselves seem completely isolated from one another. For instance, the transformations of any set of objects onto itself form a mathematical group structure. Thus structures of actions, for instance, the structure of a group of transformations, are general, whereas the actions themselves remain individual, concrete entities. This "self-organization" of actions into structured wholes, by which they are generalized, certainly does not occur without perceiving the "empirical" effects of these actions. Structural thinking in terms of relations is impossible without the perception of very concrete effects. Hence the idea of concrete transformation or movement, as it arose from algebra and since Descartes was gradually introduced into geometry remains essential.

Piaget is right in emphasizing the fundamental importance of the notion of structure in our times. We do not only think in structural terms, we perceive and visualize structures, we paint and compose and we write guided by structural principles. But Piaget neglects representation, perception and language, and he forgot that structures do not come, so to speak, naked, without flesh or clothing. The structures are only skeletons, which can neither evolve nor sustain themselves on their own.

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